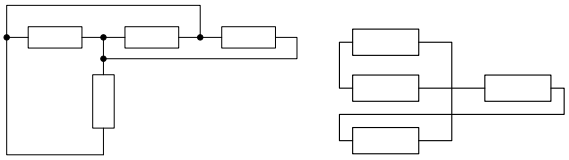
E1.1 Circuit Analysis

Problem Sheet 1 (Lectures 1 & 2)

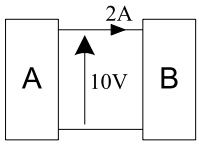
Key: [A]= easy ... [E]=hard

1. [A] One of the following circuits is a series circuit and the other is a parallel circuit. Explain which is which.

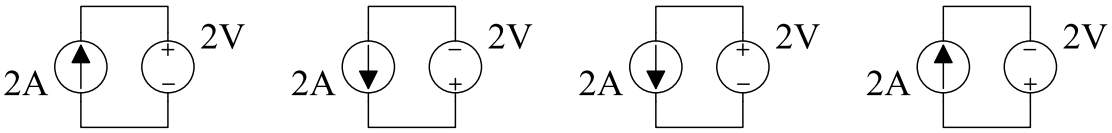


(a) (b)

2. [B] Find the power absorbed by by each of the subcircuits A and B given that the voltage and current are 10 V and 2 A as shown.

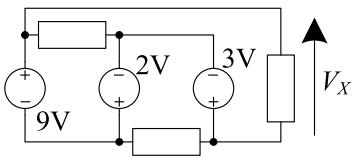


3. [B] For each of the four circuits below, find the power absorbed by the voltage source (P_V), the power absorbed by the current source (P_I) and the total power absorbed ($P_V + P_I$).

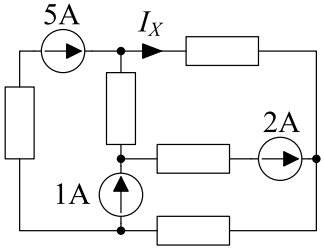


(a) (b) (c) (d)

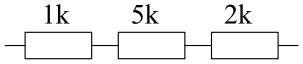
4. [B] Determine the voltage V_X in the following circuit.



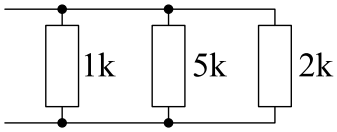
5. [B] Determine the current I_X in the following circuit.



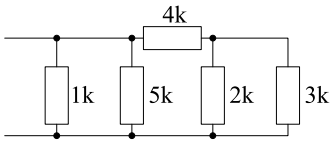
6. [B] What single resistor is equivalent to the three resistor sub-circuit shown below?



7. [B] What single resistor is equivalent to the three resistor sub-circuit shown below?

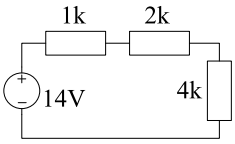


8. [C] What single resistor is equivalent to the five resistor sub-circuit shown below?

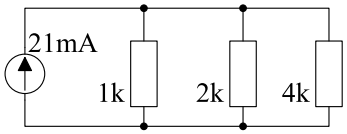


9. [A] If a resistor has a conductance of $8\,\mu$S, what is its resistance?

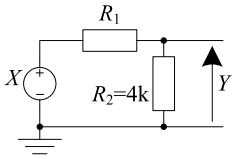
10. [B] Determine the voltage across each of the resistors in the following circuit and the power dissipated in each of them. Calculate the power supplied by the voltage source.



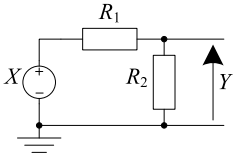
11. [B] Determine the current through each of the resistors in the following circuit and the power dissipated in each of them. Calculate the power supplied by the current source.



12. [B] Determine R_1 so that $Y = \frac{1}{4}X$.



13. [B] Choose R_1 and R_2 so that $Y = 0.1X$ and $R_1 + R_2 = 10\,\mathrm{M}\Omega$.



14. [D] You have a supply of resistors that have the values $\{10, 12, 15, 18, 22, 27, 33, 39, 47, 56, 68, 82\} \times 10^n\,\Omega$ for all integer values of n. Thus, for example, a resistor of $390\,\Omega$ is available and the next higher value is $470\,\Omega$. Show how, by combining two resistors in each case, it is possible to make networks whose equivalent resistance is (a) $3\,\mathrm{k}\Omega$, (b) $4\,\mathrm{k}\Omega$ and (c) as close as possible to $3.5\,\mathrm{k}\Omega$. Determine is the worst case percentage error that might arise if, instead of combining resistors, you just pick the closest one available.

E1.1 Circuit Analysis

Problem Sheet 1 - Solutions

1. Circuit (a) is a parallel circuit: there are only two nodes and all four components are connected between them.

   Circuit (b) is a series circuit: each node is connected to exactly two components and the same current must flow through each.

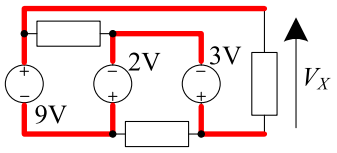
2. For subcircuit B the voltage and current correspond to the passive sign convention (i.e. the current arrow in the opposite direction to the voltage arrow) and so the power absorbed by B is given by $V \times I = 20\,\mathrm{W}$.

   For device A we need to reverse the direction of the current to conform to the passive sign convention. Therefore the power absorbed by A is $V \times -I = -20\,\mathrm{W}$.

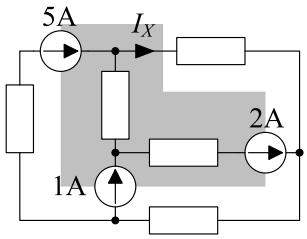
   As must always be true, the total power absorbed by all components is zero.

3. The power absorbed is positive if the voltage and current arrows go in opposite directions and negative if they go in the same direction. So we get: (a) $P_V = +4$, $P_I = -4$, (b) $P_V = +4$, $P_I = -4$, (c) $P_V = -4$, $P_I = +4$, (a) $P_V = -4$, $P_I = +4$. In all cases, the total power absorbed is $P_V + P_I = 0$.

4. We can find a path (shown highlighted below) from the bottom to the top of the V_X arrow that passes only through voltage sources and so we just add these up to get the total potential difference: $V_X = (-3) + (+2) + (+9) = +8\,\mathrm{V}$.



5. If we add up the currents flowing out of the region shown highlighted below, we obtain $I_X - 5 - 1 + 2 = 0$. Hence $I_X = 4\,\mathrm{A}$.



6. The three series resistors are equivalent to a single resistor with a value of $1 + 5 + 2 = 8\,\mathrm{k\Omega}$.

7. The three series resistors are equivalent to a single resistor with a value of $\frac{1}{1/1+1/5+1/2} = \frac{1}{1.7} = 0.588\,\mathrm{k\Omega}$.

8. We can first combine the parallel 2 k and 3 k resistors to give $\frac{2\times3}{2+3} = 1.2\,\mathrm{k}$. This is then in series with the 4 k resistor which makes 5.2 k in all. Now we just have three resistors in parallel to give a total of $\frac{1}{1/1+1/5+1/5.2} = \frac{1}{1.39} = 0.718\mathrm{k\Omega}$.

9. The resistance is $\frac{1}{8\times10^{-6}} = 125\,\mathrm{k\Omega}$

10. [Method 1]: The resistors are in series and so form a potential divider. The total series resistance is 7 k, so the voltages across the three resistors are $14 \times \frac{1}{7} = 2\,\text{V}$, $14 \times \frac{2}{7} = 4\,\text{V}$ and $14 \times \frac{4}{7} = 8\,\text{V}$. The power dissipated in a resistor is $\frac{V^2}{R}$, so for the three resistors, this gives $\frac{2^2}{1} = 4\,\text{mW}$, $\frac{4^2}{2} = 8\,\text{mW}$ and $\frac{8^2}{4} = 16\,\text{mW}$.

    [Method 2]: The total resistance is is 7 k so the current flowing in the circuit is $\frac{14}{7} = 2\,\text{mA}$. The voltage across a resistor is IR which, in for these resistors, gives $2 \times 1 = 2\,\text{V}$, $2 \times 2 = 4\,\text{V}$ and $2 \times 4 = 8\,\text{V}$. The power dissipated is VI which gives $2 \times 2 = 4\,\text{mW}$, $4 \times 2 = 8\,\text{mW}$ and $8 \times 2 = 16\,\text{mW}$. The current through the voltage source is 2 mA, so the power it is supplying is $VI = 14 \times 2 = 28\,\text{mW}$. This is, inevitably, equal to the sum of the power disspipated by the three resistors: $4+8+16 = 28$.

11. [Method 1]: The resistors are in parallel and so form a current divider: the 21 mA will divide in proportion to the conductances: 1 mS, 0.5 mS and 0.25 mS. The total conductance is 1.75 mS, so the three resistor currents are $21 \times \frac{1}{1.75} = 12\,\text{mA}$, $21 \times \frac{0.5}{1.75} = 6\,\text{mA}$ and $21 \times \frac{0.25}{1.75} = 3\,\text{mA}$. The power dissipated in a resistor is I^2R which gives $12^2 \times 1 = 144\,\text{mW}$, $6^2 \times 2 = 72\,\text{mW}$ and $3^2 \times 4 = 36\,\text{mW}$.

    [Method 2]: The equivalent resistance of the three resistors is $\frac{1}{1/1+1/2+1/4} = \frac{4}{7}\,\text{k}\Omega$. Therefore the voltage across all components in the parallel circuit is $21 \times \frac{4}{7} = 12\,\text{V}$. The current through a resistor is $\frac{V}{R}$ which gives $\frac{12}{1} = 12\,\text{mA}$, $\frac{12}{2} = 6\,\text{mA}$ and $\frac{12}{4} = 3\,\text{mA}$. The power dissipated in a resistor is VI which gives $12 \times 12 = 144\,\text{mW}$, $12 \times 6 = 72\,\text{mW}$ and $12 \times 3 = 36\,\text{mW}$. The power supplied by the current source is $12 \times 21 = 252\,\text{mW}$ which as expected equals $144 + 72 + 36$.

12. The resistors form a potential divider, so $\frac{Y}{X} = \frac{4}{R_1+4}$. So we want $\frac{4}{R_1+4} = \frac{1}{4} \Rightarrow R_1 + 4 = 16 \Rightarrow R_1 = 12\,\text{k}$.

13. The resistors form a potential divider, so $\frac{Y}{X} = \frac{R_2}{R_1+R_2}$. So we want $\frac{R_2}{R_1+R_2} = \frac{1}{10}$ and $R_1 + R_2 = 10\,\text{M}\Omega$. Substituting one into the other and cross-multiplying gives $10R_2 = 10\,\text{M}\Omega \Rightarrow R_2 = 1\,\text{M}\Omega$. Substituting this into the simpler of the two initial equations gives $R_1 = 10 - 1 = 9\,\text{M}\Omega$.

14. (a) $3\,\text{k} = 1.5\,\text{k} + 1.5\,\text{k} = 3.3\,\text{k}||33\,\text{k}$, (b) $4\,\text{k} = 3.9\,\text{k} + 100$, (c) $3.488\,\text{k} = 3.9\,\text{k}||33\,\text{k}$.

    To make an exhaustive search for creating a resistance of R, you need to consider two possibilities: (i) for two resistors in series, the largest of the two resistors must be in the range $[\frac{1}{2}R, R]$ or (ii) for two resistors in parallel, the smallest resistor must be in the range $[R, 2R]$. In both cases there are at most four possibilities, so you need to consider up to eight possibilities in all. So, for example, for $R = 3.5\,\text{k}$, we would consider the following possibilities: (i) $1.5\,\text{k} + 1.8\,\text{k} = 3.3\,\text{k}$, $1.8\,\text{k} + 1.8\,\text{k} = 3.6\,\text{k}$, $2.2\,\text{k} + 1.2\,\text{k} = 3.4\,\text{k}$, $2.7\,\text{k} + 0.82\,\text{k} = 3.52\,\text{k}$ and (ii) $3.9\,\text{k}||33\,\text{k} = 3.488\,\text{k}$, $4.7\,\text{k}||15\,\text{k} = 3.579\,\text{k}$, $5.6\,\text{k}||10\,\text{k} = 3.59\,\text{k}$, $6.8\,\text{k}||6.8\,\text{k} = 3.4\,\text{k}$. The choice with least error is the one given above.

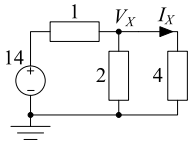
    Since we are interested in % errors, we need to consider the ratio between resistor values. The largest ratio between successive resistors is the series is $\frac{15}{12} = 1.25$ (this includes the wraparound ratio of $\frac{100}{82} = 1.22$). The worst-case percentage error will arise if our target resistance is the mean of these two values, 13.5. The percentage error in choosing either one is then $\frac{1.5}{13.5} = 11.1\%$.

E1.1 Circuit Analysis

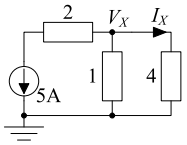
Problem Sheet 2 (Lectures 3 & 4)

Key: [A]= easy ... [E]=hard

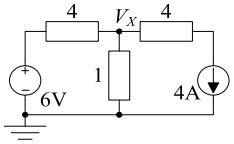
1. [B] Calculate V_X and I_X in the following circuit using (a) nodal analysis and (b) simplifying the circuit by combining parallel resistors.



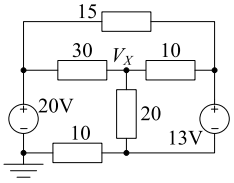
2. [B] Calculate V_X and I_X in the following circuit using (a) nodal analysis and (b) simplifying the circuit by combining parallel resistors.



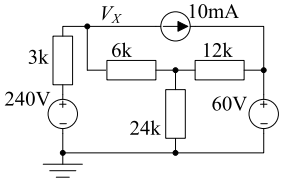
3. [C] Calculate V_X in the following circuit using (a) nodal analysis and (b) superposition.



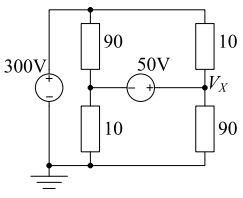
4. [C] Calculate V_X in the following circuit.



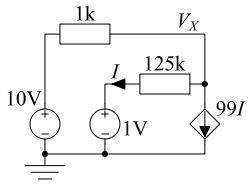
5. [C] Calculate V_X in the following circuit.



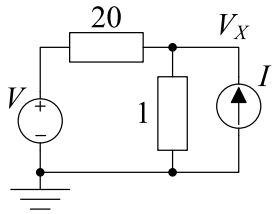
6. [C] Calculate V_X in the following circuit.



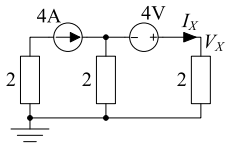
7. [C] Calculate V_X in the following circuit. The value of the dependent current source is 99 time the current flowing through the 1 V voltage source.



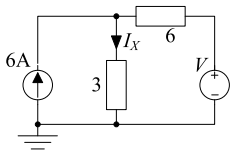
8. [C] In the following circuit calculate V_X in terms of V and I using (a) nodal analysis and (b) superposition.



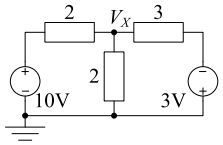
9. [C] Calculate V_X and I_X in the following circuit using (a) nodal analysis and (b) superposition.



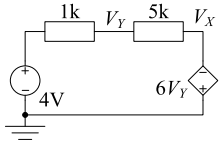
10. [C] Determine an expression for I_X in terms of V in the following circuit. Determine the value of V that will make $I_X = 0$.



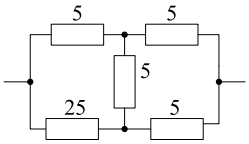
11. [C] Calculate V_X in the following circuit using (a) nodal analysis and (b) superposition.



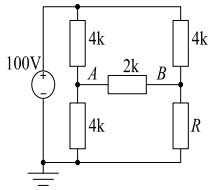
12. [C] Calculate V_X in the following circuit which includes a dependent voltage source.



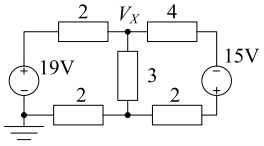
13. [C] Find the equivalent resistance of the network shown below.



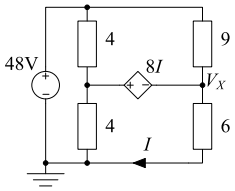
14. [D] Prove that if $V_{AB} = 0$, then $R = 4\,\text{k}\Omega$ in the following circuit. The circuit is used to detect small changes in R from its nominal value of $4\,\text{k}\Omega$. Find an expression for V_{AB} as a function of R. If changes in V_{AB} of $10\,\text{mV}$ can be detected, what is the smallest detectable change in R .



15. [D] Calculate V_X in the following circuit. You can either use nodal analysis directly or else simplify the circuit a little to reduce the number of nodes.



16. [D] Calculate V_X in the following circuit which includes a floating dependent voltage source.



E1.1 Circuit Analysis

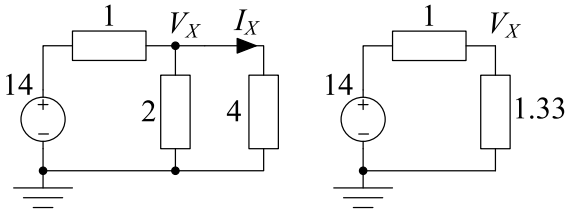
Problem Sheet 2 - Solutions

Note: In many of the solutions below I have written the voltage at node X as the variable X instead of V_X in order to save writing so many subscripts.

1. [Nodal analysis] KCL at node V_X gives $\frac{V_X-14}{1}+\frac{V_X}{2}+\frac{V_X}{4}=0$ which simplifies to $7V_X-56=0$ from which $V_X=8$.

   [Parallel resistors] We can merge the $2\,\Omega$ and $4\,\Omega$ resistors to make one of $\frac{2\times4}{2+4}=\frac{4}{3}\,\Omega$ as shown below. Now we have a *potential divider*, so $V_X=14\times\frac{1.33}{2.33}=8\,\mathrm{V}$.

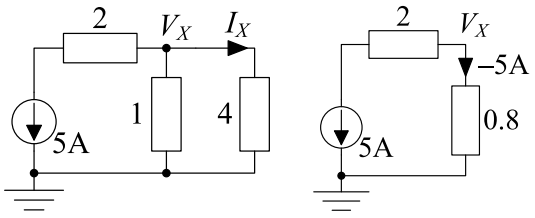
   In both cases, we can now calculate $I_X=\frac{V_X}{4}=2\,\mathrm{A}$. Note that when we merge the two resistors, I_X is no longer a distinct current on the diagram.



Original Simplified

2. [Nodal Analysis] KCL at node V_x gives $5+\frac{V_X}{1}+\frac{V_X}{4}=0$ which simplifies to $20+5V_X=0$ from which $V_X=-4$.

   [Parallel Resistors] We can combine the $1\,\Omega$ and $4\,\Omega$ resistors to make one of $\frac{1\times4}{1+4}=\frac{4}{5}\,\Omega$ as shown below. Now we have $V_X=-5\times0.8=-4\,\mathrm{V}$.



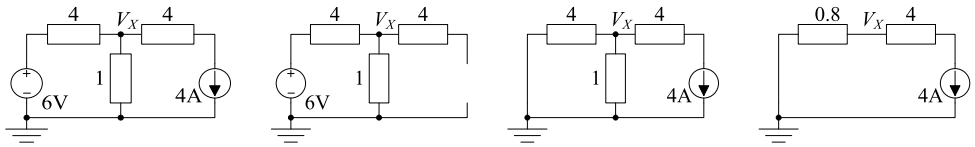
Original Simplified

   In both cases, we can now calculate $I_X=\frac{V_X}{4}=-1\,\mathrm{A}$.

   In this question, you have to be a bit careful about the sign used to represent currents. Whenever you use Ohm's law, you must be sure that you use the passive sign convention (with the current arrow in the opposite direction to the voltage arrow); this is why the current through the $0.8\,\Omega$ resistor is $-5\,\mathrm{A}$ rather than $+5\,\mathrm{A}$.

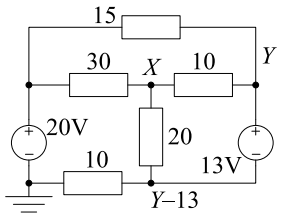
3. [Nodal Analysis] KCL at node V_x gives $\frac{V_X-6}{4}+\frac{V_X}{1}+4=0$ which simplifies to $5V_X+10=0$ from which $V_X=-2$.

   [Superposition] (i) If we set the current source to zero, then the $4\,\Omega$ resistor connected to it plays no part in the circuit and we have a potential divider giving $V_X=6\times\frac{1}{5}=1.2$. (ii) We now set the voltage source to zero and then simplify the resultant circuit as shown below. Being careful with signs, we now get $V_X=-4\times 0.8=-3.2$. Adding these two values together gives a final answer of $V_X=-2$.

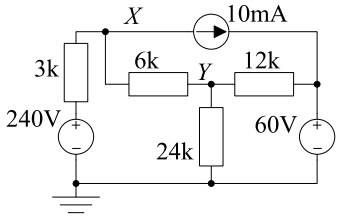


Original $I=0$ $V=0$ $V=0$ simplified

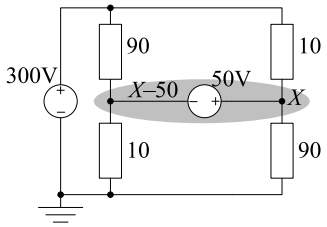
4. We first label the nodes; we only need two variables because of the floating voltage source. KCL at X gives $\frac{X-20}{30}+\frac{X-(Y-13)}{20}+\frac{X-Y}{10}=0$ which gives $11X-9Y=1$. KCL at the supernode $\{Y,Y-13\}$gives $\frac{Y-20}{15}+\frac{Y-X}{10}+\frac{(Y-13)-X}{20}+\frac{(Y-13)}{10}=0$ which gives $-9X+19Y=197$. Solving these two simultaneous equations gives $X=14$ and $Y=17$.



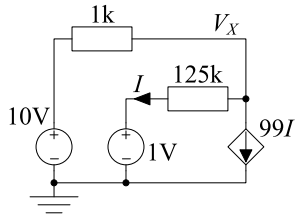
5. We first label the nodes; there are only two whose voltage is unknown. Working in mA and kΩ, KCL at X gives $\frac{X-240}{3}+\frac{X-Y}{6}+10=0$ which gives $3X-Y=420$. KCL at Y gives $\frac{Y-X}{6}+\frac{Y}{24}+\frac{Y-60}{12}=0$ which gives $-4X+7Y=120$. Solving these simultaneous equations gives $X=180$ and $Y=120$.



6. We first label the nodes; since the two nodes having unknown voltages are joined by a fixed voltage source, we only need one variable. We write down KCL for the supernode $\{X,X-50\}$ (shaded in the diagram) which gives $\frac{(X-50)-300}{90}+\frac{(X-50)}{10}+\frac{X-300}{10}+\frac{X}{90}=0$ which simplifies to $20X=3500$ or $X=175\,\text{V}$.

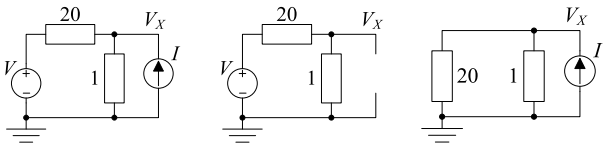


7. There is only one node with an unknown voltage, namely X. However, there is a dependent current source, so we need to express its value in terms of the node voltages: $99I = 99 \times \frac{X-1}{125}$ where we are expressing currents in mA. So now we can apply KCL to node X to obtain $\frac{X-10}{1} + \frac{X-1}{125} + 99 \times \frac{X-1}{125} = 0$ which simplifies to $225X = 1350$ from which $X = 6$.



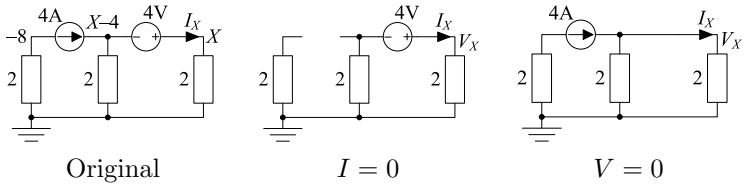
8. [Nodal Analysis] Using KCL at node X gives $\frac{X-V}{20} + \frac{X}{1} - I = 0$ which we can rearrange to give $X = \frac{1}{21}V + \frac{20}{21}I$.

   [Superposition] If we set $I = 0$ then we have a voltage divider in which $X = \frac{1}{21}V$ (see middle diagram). If we set $V = 0$ then (see right diagram) we can combine the two parallel resistors as $\frac{20\times1}{20+1} = \frac{20}{21}\,\Omega$ and it follows that $X = \frac{20}{21}I$. By superposition, we can add these two expression together to give $X = \frac{1}{21}V + \frac{20}{21}I$.



9. [Nodal Analysis] We can easily see that the 4 A current flowing through the leftmost resistor means the top left node has a voltage of -8 (although actually we do not need to calculate this because of the isolating effect of the current source). Using KCL at the supernode $\{X-4, X\}$ gives $-4 + \frac{X-4}{2} + \frac{X}{2} = 0$ which we can rearrange to give $2X = 12$ or $X = 6$. It follows that $I_X = \frac{6}{2} = 3\,\text{A}$.

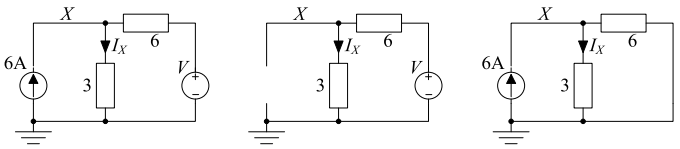
   [Superposition] If we set $I = 0$ then we have a voltage divider (since the resistors are in series) in which $X = 4 \times \frac{2}{2\times2} = 2$ and $I_X = 1\,\text{A}$ (see middle diagram). If we set $V = 0$ then (see right diagram) we can combine the two parallel resistors as $\frac{2\times2}{2+2} = 1\,\Omega$ and it follows that $X = 4$ and, by current division, that $I_X = 2\,\text{A}$. By superposition, we can add these two expression together to give $X = 6\,\text{V}$ and $I_X = 3\,\text{A}$.



10. [Nodal Analysis] We first label the unknown node as X. Now, KCL at this node gives $-6+\frac{X}{3}+\frac{X-V}{6}=0$ which rearranges to give $3X = V + 36$ from which $X = \frac{1}{3}V + 12$. Now $I_X = \frac{X}{3}$ so $I_X = \frac{1}{9}V + 4$.

    [Superposition] If we set $I = 0$ (see middle diagram) then $I_X = \frac{V}{9}$. If we set $V = 0$ then (see right diagram) we have a current divider in which the 6 A current divides in proportion to the conductances. So $I_X = 6 \times \frac{1/3}{1/3+1/6} = 6 \times \frac{2}{3} = 4$. Adding these results together gives $I_X = \frac{1}{9}V + 4$.

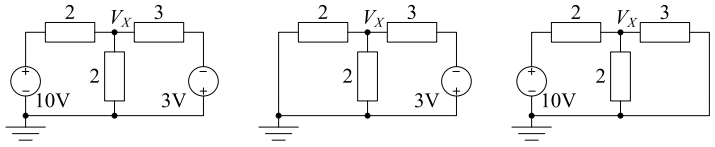
    If $V = -36$ then $I_X = 0$.



Original $I = 0$ $V = 0$

11. [Nodal Analysis] KCL at node X gives $\frac{X-10}{2} + \frac{X}{2} + \frac{X-(-3)}{3} = 0$ which rearranges to give $8X = 24$ from which $X = 3$.

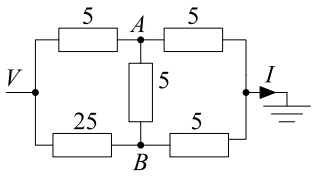
    [Superposition] If we set the left source to zero (see middle diagram) then, the two parallel 2 Ω resistors are equivalent to 1 Ω and so we have a potential divider and $X = -3 \times \frac{1}{4} = -0.75$. If we set the other source to zero (see right diagram) we can combine the 2 Ω and 3 Ω parallel resistors to obtain $\frac{2\times3}{2+3} = 1.2\,\Omega$. We again have a potential divider giving $X = 10 \times \frac{1.2}{2+1.2} = 3.75$. Adding these together gives $X = -0.75 + 3.75 = 3\,\text{V}$.



Original $U_1 = 0$ $U_2 = 0$

12. KCL at node Y gives $\frac{Y-4}{1}+\frac{Y-X}{5}=0$ which rearranges to $-X+6Y=20$. We also have the equation of the dependent voltage source: $X = -6Y$. We can conveninetly eliminate $6Y$ between these two to give $-2X = 20$ and so $X = -10$.

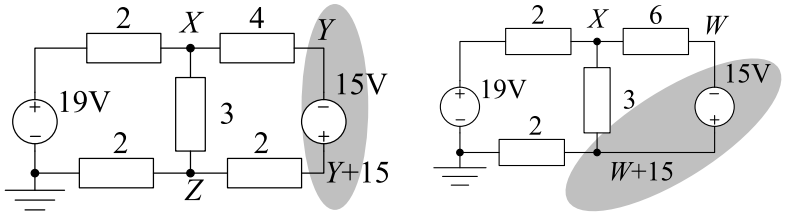
13. We first pick a ground reference at one end of the network and label all the other nodes. The equivalent resistance is now $\frac{V}{I}$. We assume that we know V and then calculate I. KCL at node A gives $\frac{A-V}{5} + \frac{A-B}{5} + \frac{A}{5} = 0$ from which $3A - B = V$ or $B = 3A - V$. KCL at node B gives $\frac{B-V}{25} + \frac{B-A}{5} + \frac{B}{5} = 0$ from which $11B - 5A = V$. Substituting $B = 3A - V$ into this equation gives $33A - 5A = 12V$ or $A = \frac{12}{28}V$ which in turn gives $B = 3A - V = \frac{8}{28}V$. The current I is the sum of the currents throught the rightmost two 5 Ωresistors: $I = \frac{A}{5} + \frac{B}{5} = \frac{1}{7}V$. So the equivalent resistance is $\frac{V}{I} = 7\,\Omega$.



14. If $V_{AB} = 0$ then no current flows through the 2 k resistor, so the two vertical resistor chains form potential dividers. In the leftmost chain, $V_A = 100 \times \frac{4}{4+4} = 50\,\text{V}$. Since $V_{AB} = 0$, $V_B = V_A = 50 = 100 \times \frac{R}{4+R}$ which implies that $R = 4\,\text{k}$.

    KCL at nodes A gives $\frac{A-100}{4} + \frac{A}{4} + \frac{A-B}{2} = 0$ which gives $4A - 2B = 100$ or $B = 2A - 50$. KCL at node B now gives $\frac{B-100}{4} + \frac{B}{R} + \frac{B-A}{2} = 0$ into which we can substitute the expression for B to get $\frac{2A-150}{4} + \frac{2A-50}{R} + \frac{A-50}{2} = 0$ from which $A = \frac{100+125R}{4+2R}$. Substituting this into $B = 2A - 50$ gives $B = \frac{150R}{4+2R}$ and hence $A - B = \frac{100-25R}{4+2R}$. If we can detect a value of $A - B = 10\,\text{mV} = 0.01$ then the corresponding value of R is the solution to $\frac{100-25R}{4+2R} = 0.01$ which gives $25.02R = 99.96$ from which $R = 3995.2\,\Omega$ which is a change of $4.8\,\Omega$ or 0.12%.

15. We label the nodes as shown below (using X instead of V_X for ease of writing). Note that when we have labelled the upper node of the floating voltage source as Y we can label the lower node as $Y+13$ and do not need another variable. KCL at node X gives $\frac{X-19}{2} + \frac{X-Z}{3} + \frac{X-Y}{4} = 0$ which gives $13X - 3Y - 4Z = 114$. KCL at node Z gives $\frac{Z}{2} + \frac{Z-X}{3} + \frac{Z-Y-15}{2} = 0$ which gives $-2X - 3Y + 8Z = 45$. Finally, KCL at the supernode $\{Y, Y+15\}$ gives $\frac{Y-X}{4} + \frac{Y+15-Z}{2} = 0$ from which $X - 3Y + 2Z = 30$. Solving these three simultaneous equations gives $X = 11$, $Y = -1$ and $Z = 8$.

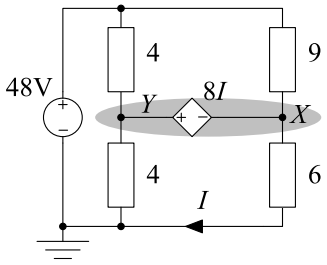


Original Simplified

    An alternative approach is to notice that the three rightmost components are in series and so you can reorder them without affecting the rest of the circuit to give the simplified circuit shown above. Now, we only have two unknowns and hence only two simultaneous equations to solve. KCL at node X gives $\frac{X-19}{2} + \frac{X-W-15}{3} + \frac{X-W}{6} = 0$ which gives $6X - 3W = 87$. KCL at the supernode $\{W, W+15\}$ gives $\frac{W-X}{6} + \frac{W+15-X}{3} + \frac{W+15}{2} = 0$ from which $3X - 6W = 75$. These equations are easily solved to give $X = 11$ and $W = -7$.

16. Since the floating voltage source is a dependent voltage source, we need to label its two ends with separate variables (see below). We now write down a KCL equation for the supernode shown shaded: $\frac{Y-48}{4} + \frac{Y}{4} + \frac{X-48}{9} + \frac{X}{6} = 0$ which simplifies to $10X + 18Y = 624$.

    We also need to express the voltage source value in terms of nodal voltages: $Y - X = 8I = 8 \times \frac{X}{6} = \frac{4}{3}X$ which rearranges to give $Y = \frac{7}{3}X$. Substituting this in the previous equation gives $10X + 18 \times \frac{7}{3}X = 624$ which simplifies to $52X = 624$ from which $X = 12$.

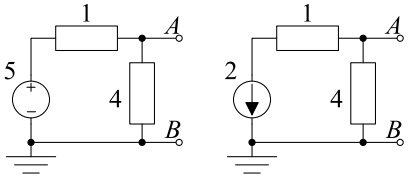


E1.1 Circuit Analysis

Problem Sheet 3 (Lectures 5, 6, 7 & 8)

Key: [A]= easy ... [E]=hard

1. [B] Calculate the Thévenin and Norton equivalent networks at the terminals A and B for each of the following.



(a) (b)

2. [B] Use nodal analysis to calculate an expression for A in Fig. 2 in terms of I and then rearrange this to give I in terms of A. Show how these expressions are related to the Thévenin and Norton equivalent networks at the terminals A and B.

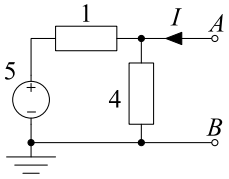


Fig. 2

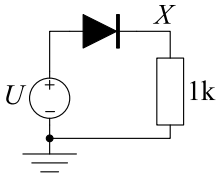


Fig. 3

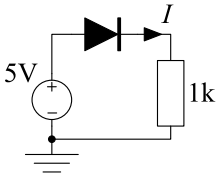
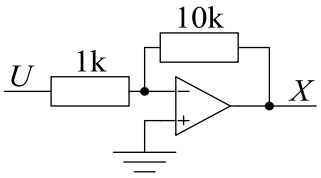


Fig. 4

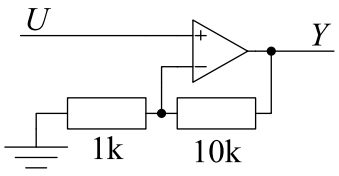
3. [B] Determine X in Fig. 3 when (a) $U = +5\,\mathrm{V}$ and (b) $U = -5\,\mathrm{V}$. Assume that the diode has a forward voltage drop of 0.7 V.

4. [B] In Fig. 4, calculate I and the power dissipation in the resistor and in the diode. Assume that the diode has a forward voltage drop of 0.7 V.

5. [B] Find the gains $\frac{X}{U}$ and $\frac{Y}{U}$ in the following circuits:



(a)



(b)

6. [C] Calculate the Thévenin and Norton equivalent networks at the terminals A and B in Fig. 6 in two ways (a) by combining resistors to simplify the circuit and (b) by using nodal analysis to express A in terms of I.

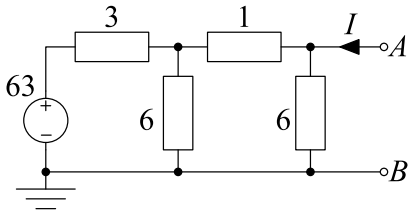


Fig. 6

7. [C] Find the current I in Fig. 7 in two ways: (a) by nodal analysis and (b) by combining the leftmost three components into their Thévenin equivalent.

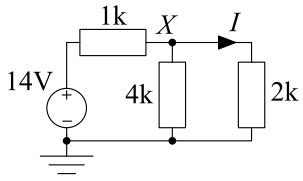


Fig. 7

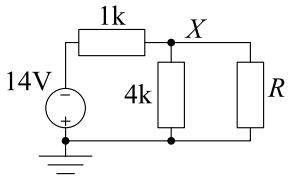
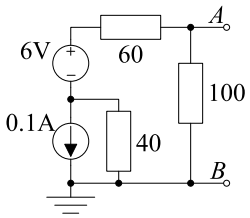


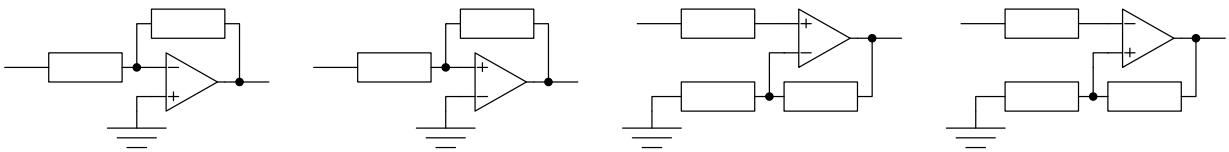
Fig. 8

8. [C] For what value of R in Fig. 8 will the power dissipation in R be maximized. Find the power dissipation in R in this case.

9. [C] Find the Thévenin equivalent of the circuit between nodes A and B in two ways: (a) by performing a sequence of Norton↔Thévenin transformations and (b) using superposition to find the open-circuit voltage and combining resistors to find the Thévenin resistance.

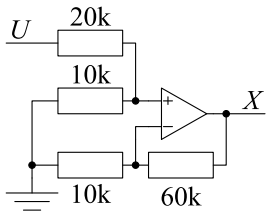


10. [C] State whether the feedback in the following circuits is positive or negative:

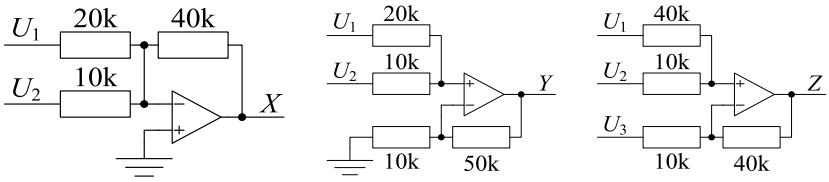


(a) (b) (c) (d)

11. [C] Find the gain $\frac{X}{U}$ in the following circuit:



12. [C] Find expressions for X, Y and Z in terms of U_1, U_2 and U_3.



(a) (b) (c)

13. [C] Choose values of R_1 and R_2 in Fig. 13 so that $X = 2U_2 - 3U_1$.

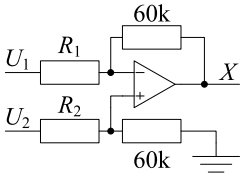


Fig. 13

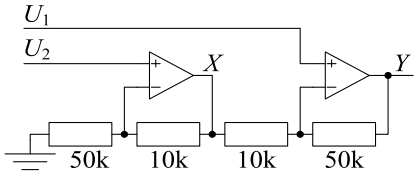
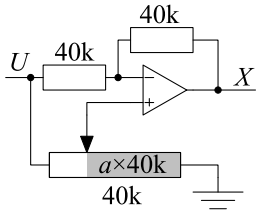


Fig. 14

14. [C] Find an expression for Y in Fig. 14 in terms of U_1 and U_2.

15. [C] In the circuit diagram, the potentiometer resistance between the slider and ground is $a \times 40\,\text{k}\Omega$ where $0 \leq a \leq 1$. Find the gain of the circuit, $\frac{X}{U}$ as a function of a. What is the range of gains that the circuit can generate as a is varied.



16. [C] By replacing the rightmost three components in Fig. 16 by their Thévenin equivalent, find X when (a) $U = 0\,\text{V}$ and (b) $U = 5\,\text{V}$. Assume that the diode has a forward voltage drop of 0.7 V. Determine the value of U at which the diode switches between operating regions.

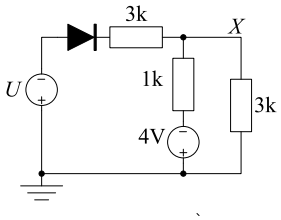


Fig. 16

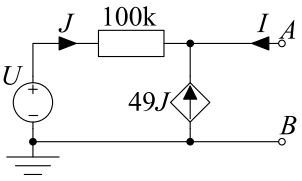


Fig. 18

17. [C] In the block diagram of Fig. 17, F, G, H represent the gains of the blocks. Find the overall gain $\frac{Y}{X}$.

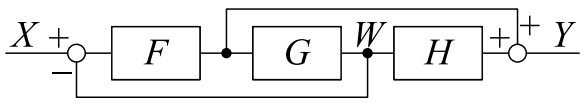


Fig. 17

18. [D] The circuit of Fig. 18 includes a dependent current source whose value is proportional to the current J. Find the Thévenin equivalent of the circuit by two methods: (a) use nodal analysis including the current I and express the voltage V_{AB} as a function of I and (b) assume $I = 0$ and find (i) the open circuit voltage, V_{AB} as a function of U and (ii) the short-circuit current (with A joined to B) as a function of U. Hence find the Thévenin equivalent of the circuit.

19. [D] Choose resistor values in Fig. 19 so that (a) $X = \frac{1}{2}U_3 + \frac{1}{3}U_2 + \frac{1}{6}U_1$ and (b) the Thévenin resistance between X and ground is $50\,\Omega$. Why would the question be impossible if it had asked for $X = \frac{1}{2}U_3 + \frac{1}{3}U_2 + \frac{1}{3}U_1$?

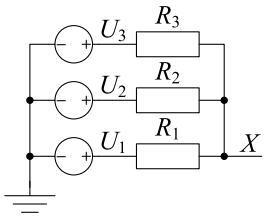


Fig. 19

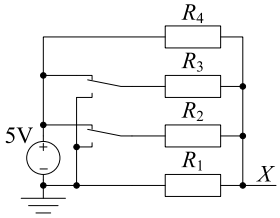
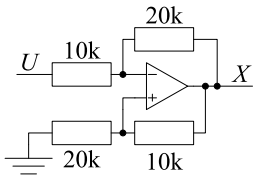


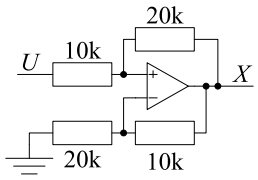
Fig. 20

20. [D] Choose resistor values in Fig. 20 so that (a) the Thévenin between X and ground is $50\,\Omega$ and (b) it is possible to set X to 1, 2, 3 or 4 V by setting the switches appropriately.

21. [D] State whether the feedback in the following circuits is positive or negative:



(a)



(b)

22. [D] Find an expression for Z in Fig. 22 in terms of U_1 and U_2.

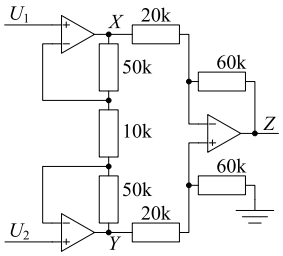


Fig. 22

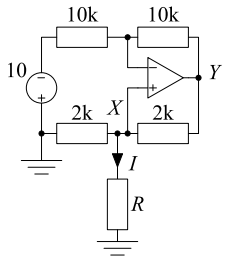


Fig. 23

23. [D] The circuit of Fig. 23 is called a Howland current source. Show that the circuit has negative feedback. Use nodal analysis to determine the current I and show that it does not depend on R.

24. [D] A non-linear device has a characteristic $Y = \sqrt{X}$ for inputs in the range $0 \le X \le 1$. To improve its linearity, the device is placed in a feedback loop using an op-amp with a gain of A as shown in Fig. 24. Determine an expression for Y in terms of U and A. Simplify the expression by using the Taylor series approximation: $\sqrt{v+w} \approx \sqrt{v}\left(1 + \frac{w}{2v} - \frac{w^2}{8v^2}\right)$ valid for $v \gg w$. Estimate how large A must be to ensure that $Y|_{U=0.5} = 0.5 \times Y|_{U=1}$ to within 1% of $Y|_{U=1}$.

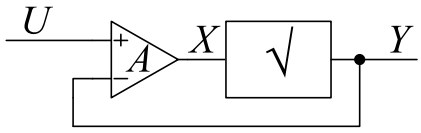


Fig. 24

25. [D] The diodes in Fig. 25 have a characteristic $I = k \exp \frac{V}{V_T}$ where $V_T = 25\,\text{mV}$.
    (a) For the circuit of in Fig. 25(a), find an expression for U in terms of X assuming that $X > 0$.
    (b) For the circuit of in Fig. 25(b), find and expression for Y in terms of X assuming that $X > 0$.

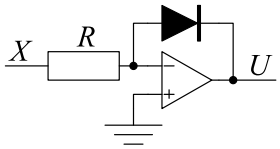


Fig. 25(a)

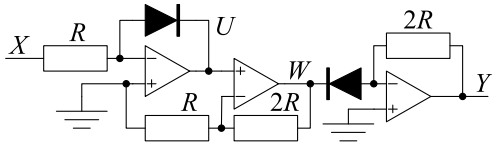
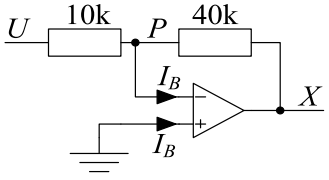
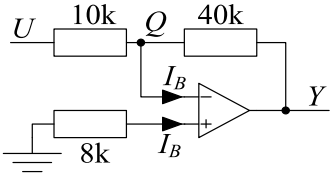


Fig. 25(b)

26. [C] We normally assume that the current at the inputs to an opamp are negligible. However this is not always true and this question investigates the effect of non-zero input bias currents. If $I_B = 100\,\text{nA}$, find expressions for $\frac{X}{U}$ and $\frac{Y}{U}$ in the circuits below. Explain the advantage of circuit (b) and derive a general principal that should be followed when using opamps having non-negligible input bias currents, I_B.



(a)



(b)

E1.1 Circuit Analysis

Problem Sheet 3 - Solutions

1. (a) Thévenin voltage equals the open circuit voltage is 4 V (from potential divider). To obtain the Thévenin/Norton we set the voltage source to 0 (making it a short circuit) and find the resistance of the network to be $1||4 = 0.8\,\Omega$. From this the Norton current is $\frac{4}{0.8} = 5\,\text{A}$. This may also be found directly from the short-circuit current of the original circuit.

   (b) The open-circuit voltage is $-8\,\text{V}$ (since the 2 A current flows anticlockwise). To obtain the Thévenin/Norton resistance, we set the current source to zero (zero current implies an open circuit), so the resultant network has a resistance of $4\,\Omega$. The Norton current is $\frac{-8}{4} = -2\,\text{A}$; this may also be found by observing that the short-circuit current (flowing into node A) is $+2\,\text{A}$.

2. KCL at node A gives $\frac{A-5}{1} + \frac{A}{4} - I = 0$ from which $5A - 20 - 4I = 0$ which we can rearrange to give $A = 4 + 0.8I = V_{Th} + R_{Th}I$. We can also rearrange to give $I = -5 + \frac{1}{0.8}A = -I_{Nor} + \frac{1}{R_{Nor}}A$.

3. (a) When $U = 5$, the diode is on and so $X = U - 0.7 = 4.3$. To check our assumption about the diode operating region, we need to calculate the current through the diode; this equals 4.3 mA which is indeed positive.

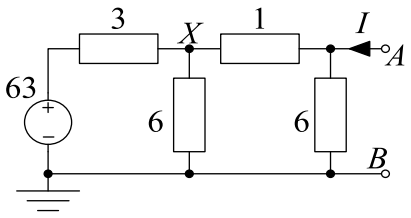
   (b) When $U = -5$, the diode is off, so the current through the resistor is zero and $X = 0$. As a check, the voltage across the diode is $U - X = -5$ which confirms our assumption that it is off.

4. As in part (a) of the previous question, $I = 4.3\,\text{mA}$. The power dissipation in the diode is therefore $V_D I = 0.7 \times 4.3 = 3.01\,\text{mW}$. The power dissipation in the resistor is $I^2R = 18.5\,\text{mW}$.

5. Circuit (a) is an inverting amplifier with gain $\frac{X}{U} = -\frac{10}{1} = -10$. Circuit (b) is a non-inverting amplifier with gain $\frac{Y}{U} = 1 + \frac{10}{1} = +11$. Another way to see this is to notice that, since the opamp inputs draw no current, the potential divider means that the -ve opamp input is at $\frac{Y}{11}$ and, since the negative feedback ensures the opamp terminals are at the same voltage, $U = \frac{Y}{11}$.

6. [Method 1 - circuit manipulation] To calculate the Thévenin equivalent, we want to determine the open-circuit voltage and the Thévenin resistance. To determine the open-circuit voltage, we assume that $I = 0$ and calculate V_{AB}. Since $I = 0$, we can combine the $1\,\Omega$ and $6\,\Omega$ resistors to give $7\,\Omega$ and then combine this with the $6\,\Omega$ resistor in parallel to give $\frac{42}{13}\,\Omega$. We now have a potential divider so the voltage at point X is $63 \times \frac{42/13}{3+42/13} = \frac{98}{3}$. This is then divided by the $1\,\Omega$ and $6\,\Omega$ resistors to give an open-circuit voltage of $\frac{98}{3} \times \frac{6}{7} = 28\,\text{V}$. The Thévenin/Norton resistance can be found by short-circuiting the voltage source to give $3\,\Omega$ in parallel with $6\,\Omega$ which equals $2\,\Omega$. This is then in series with $1\,\Omega$(to give $3\,\Omega$) and finally in parallel with $6\,\Omega$ to give $2\,\Omega$.

   [Method 2 - Nodal Analysis]. We can do KCL at node X (see diagram below) to get $\frac{X-63}{3} + \frac{X}{6} + \frac{X-A}{1} = 0$ which simplifies to $9X - 6A = 126$ or $3X - 2A = 42$. We now do KCL at A but include an additional input current I as shown in the diagram. This gives $\frac{A-X}{1} + \frac{A}{6} - I = 0$ from which $7A - 6X = 6I$. Substituting for $6X = 4A + 84$ gives $3A = 84 + 6I$ or $A = 28 + 2I$. This gives the Thévenin voltage as 28 and the Thévenin/Norton resistance as $2\,\Omega$.Hence the Norton current is 14 A.

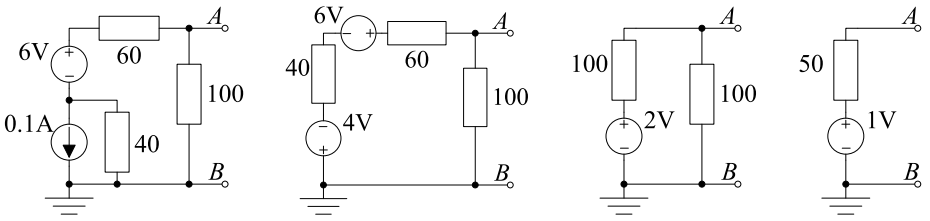


7. (a) KCL @ X gives $\frac{X-14}{1} + \frac{X}{4} + \frac{X}{2} = 0$ from which $7X = 56 \Rightarrow X = 8 \Rightarrow I = \frac{X}{2} = 4\,\text{mA}$.

   (b)Finding the Thévenin equivalent of the left three components: we consider the two resistors as a potential divider to give $V_{Th} = 14 \times \frac{4}{5} = 11.2\,\text{V}$. Setting the source to zero (short circuit) gives $R_{Th} = 1||4 = 800\,\Omega$. Hence $I = \frac{V_{Th}}{R_{Th}+2000} = \frac{11.2}{2.8} = 4\,\text{mA}$.

8. From question 7, the left three components have a Thévenin equivalent: $V_{Th} = 11.2\,\text{V}$ and $R_{Th} = 800\,\Omega$. It follows that the maximum power will be dissipated in R when $R = R_{Th} = 800\,\Omega$ (see notes page 5-8). Since the voltage across R will then be $\frac{1}{2}V_{Th}$ the power dissipation will be $\frac{1}{4R_{Th}}V_{Th}^2 = 39.2\,\text{mW}$.

9. (a) As shown in the sequence below, we first combine the current source with the $40\,\Omega$resistor, then combine the four series components into a single Thévenin equivalent and finally find the Thévenin equivalent of the simple network (using parallel resistors for R_{Th} and a potential divider for V_{Th}).

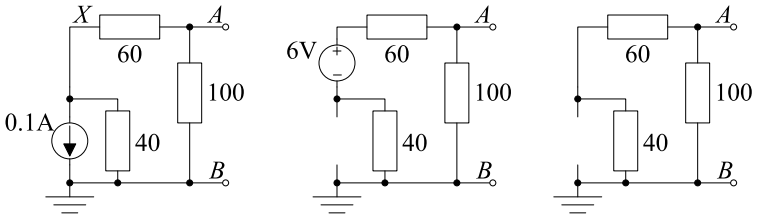


(b) Setting the voltage source to zero gives us the first diagram. Combining $40||(60+100) = 32$ so $X = -0.1 \times 32 = -3.2\,\text{V}$. it follows (potential divider) that $A = -3.2 \times \frac{100}{160} = -2\,\text{V}$.

Now setting the current source to zero gives the second diagram and we have a potential divider giving $V_{AB} = 6 \times \frac{100}{100+60+40} = 3\,\text{V}$.

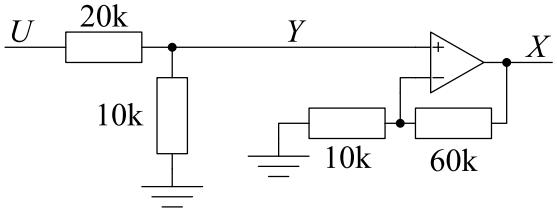
Superposition now gives us $V_{AB} = V_{Th} = -2 + 3 = 1\,\text{V}$.

To find R_{Th} we set both sources to zero and find the resultant resistance of $100||(60+40) = 100||100 = 50\,\Omega$.



10. (a) Negative, (b) Positive, (c) Negative, (d) Positive. In simple circuits like these, you can just see which terminal the output feeds back to.

11. The best way to think of this circuit is as a potential divider with gain $\frac{Y}{U} = \frac{1}{3}$ followed by a non-inverting opamp circuit with gain $\frac{X}{Y} = 1 + \frac{60}{10} = 7$. The combined gain is then $\frac{X}{U} = \frac{1}{3} \times 7 = \frac{7}{3}$.



12. (a) You can either recognise this a a standard inverting summing amplifier with gain $X = -\left(\frac{40}{20}U_1 + \frac{40}{10}U_2\right) = -2U_1 - 4U_2$ or else apply KCL at the +ve input terminal with the assumption that negative feedback will ensure that this terminal is at the same voltage as the –ve terminal i.e. 0 V. This gives: $\frac{0-U_1}{20} + \frac{0-U_2}{10} + \frac{0-X}{40} = 0$ from which $X = -2U_1 - 4U_2$.

    (b) The network connected to the +ve terminal is a weighted averaging circuit (page 3-7 of the notes) so $V_+ = \frac{1}{3}U_1 + \frac{2}{3}U_2$. The opamp circuit itself is a non-inverting amplifier with a gain of $1 + \frac{50}{10} = 6$. So, $Y = 6 \times \left(\frac{1}{3}U_1 + \frac{2}{3}U_2\right) = 2U_1 + 4U_2$.

    (c) [Superposition method] Following the method of part (b) above, if $U_3 = 0$, we have $Z = 5 \times \left(\frac{1}{5}U_1 + \frac{4}{5}U_2\right) = U_1 + 4U_2$. If, on the other hand, $U_1 = U_2 = 0$, then $V_+ = 0$ and so we have an inverting amplifier with a gain of $-\frac{40}{10} = -4$. Hence $Z = -4U_3$.

    Combining these gives $Z = U_1 + 4U_2 - 4U_3$.

    [Nodal analysis method] The top two resistors are a weighted average circuit so $V_+ = \frac{1}{5}U_1 + \frac{4}{5}U_2$. Now, assuming that $V_- = V_+$, we do KCL at V_- to give $\frac{\frac{1}{5}U_1 + \frac{4}{5}U_2 - U_3}{10} + \frac{\frac{1}{5}U_1 + \frac{4}{5}U_2 - Z}{40} = 0$ from which $U_1 + 4U_2 - 4U_3 - Z = 0$ giving $Z = U_1 + 4U_2 - 4U_3$.

13. We can use superposition. If $U_2 = 0$,then $V_+ = 0$ and we have an inverting amplifier with a gain $\frac{X}{U_1} = -\frac{60}{R_1}$. The question tells us that this must equal -3 so we must have $R_1 = 20$. Now, if $U_1 = 0$, the circuit consists of a potential divider with a gain of $\frac{60}{R_2+60}$ followed by a non-inverting amplifier with a gain of $1 + \frac{60}{R_1} = 4$. The combined gain must equal 2 (from the question) so the potential divider must have a gain of $\frac{1}{2}$ which means $R_2 = 60\,\text{k}\Omega$.

14. The first opamp is non-inverting with a gain $\frac{X}{U_2} = 1 + \frac{10}{50} = 1.2$. We can use superposition to find Y: If $U_2 = 0$,then $X = 0$ and we have a non-inverting amplifier with a gain of $\frac{Y}{U_1} = 1 + \frac{50}{10} = 6$. If, on the other hand, $U_1 = 0$, then we have an inverting amplifier and $\frac{Y}{X} = -\frac{50}{10} = -5$. It follows that $\frac{Y}{U_2} = \frac{Y}{X} \times \frac{X}{U_2} = -5 \times 1.2 = -6$. Combining both portions of the superposition, $Y = 6U_1 - 6U_2 = 6(U_1 - U_2)$. This is therefore a differential amplifier (whose output is $\propto (U_1 - U_2)$) that draws almost no current from either if its inputs. A better circuit is given in question 22.

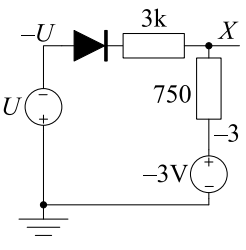
15. The potentiometer is a potential divider and so $V_+ = aU$. Assuming that $V_- = V_+$, we can do KCL at V_- to get $\frac{aU-U}{40} + \frac{aU-X}{40} = 0$ from which $X = (2a-1)\,U$. For $a = 0$, the gain is -1 and when $a = 1$, the gain is $+1$. So the circuit can generate any gain between these two extremes.

16. The Thévenin voltage is $-3\,\text{V}$ (potential divider) and the Thévenin resistance is $1||3 = 0.75\,\Omega$ as shown in the diagram below. Note that if you drawn the Thévenin voltage source the other way around (with "+" at the bottom) then the Thévenin voltage will be $+3\,\text{V}$; this is an equally valid solution.

    (a) If $U = 0\,\text{V}$, the diode is forward biassed and KCL @ X gives $\frac{X-(-3)}{0.75} + \frac{X-(0-0.7)}{3} = 0$ from which $X = -2.54$. The current through the diode is $\frac{X-(-3)}{0.75} = 613\,\text{mA}$ which is > 0 confirming our guess about the diode operating region.

    (b) If $U = 5\,\text{V}$, the diode is off and so $X = -3$ (since no current flows through the $750\,\Omega$ resistor). The diode forward voltage is $(-U) - X = -2$. This is < 0.7 confirming our operating region guess.

    The diode switches regions when both operating region equations are true: $I_D = 0$ and $V_D = 0.7$. The current equation implies $X = -3$ while the voltage equation (and the zero current through the 3 k resistor) implies $X = -U - 0.7$. Combining these gives $U = 2.3$. Extreme care with signs is needed in this question.



17. There is only one feedback loop in this circuit: from W back to the input adder. We can write $W = FG\,(X - W)$ from which $W = \frac{FG}{1+FG}X$. Then $Y = F\,(X - W) + HW$. From the first equation $(X - W) = \frac{W}{FG}$ so we can substitute this in to get $Y = \frac{W}{G} + HW = \left(\frac{1}{G} + H\right)\frac{FG}{1+FG}X = \frac{F+FGH}{1+FG}X$.

18. (a) [Nodal analysis including I] We have $J = \frac{U-A}{100}$, so the current source value (expressed in terms of node voltages) is $49J = 0.49\,(U - A)$. The KCL equation @ A is $\frac{A-U}{100} - 0.49(U - A) - I = 0$ from which $50\,(A - U) = 100I$ which gives $A = 2I + U = R_{Th}I + V_{Th}$. So the Thévenin voltage is U and the Thévenin resistance is $2\,\text{k}\Omega$.

    (b) In this method, we assume $I = 0$ and calculate the open-circuit voltage and short-circuit current. For the open-circuit voltage, we do KCL at A and obtain $\frac{A-U}{100} - 0.49(U - A) = 0$ from which $50\,(A - U) = 0$ so $A = V_{Th} = U$. For the short-circuit current, we join A and B and the current is then $\frac{V_{Th}}{R_{Th}} = \frac{U}{100} + 0.49U = 0.5U$. Hence $R_{Th} = 2\,\text{k}\Omega$.

19. From the notes (page 3-7) $X = \frac{U_1G_1+U_2G_2+U_3G_3}{G_1+G_2+G_3}$. The equations are made much easier to solve because we know that the Thévenin resistance must be $50\,\Omega$. The Thévenin resistance is just the parallel combination $R_1||R_2||R_3 = \frac{1}{G_1+G_2+G_3} = \frac{1}{20\,\text{mS}}$. Hence $X = \frac{1}{2}U_3 + \frac{1}{3}U_2 + \frac{1}{6}U_1 = 50\,(U_1G_1 + U_2G_2 + U_3G_3)$ where the first expression is given in the question and the second comes from substituting for $G_1 + G_2 + G_3$. Identifying the coefficients in this equation gives $G_1 = \frac{1}{300}$, $G_2 = \frac{1}{150}$ and $G_3 = \frac{1}{100}$ from which $R_1 = 300$, $R_2 = 150$ and $R_3 = 100$. In a weighted average circuit, the coefficients must sum to 1; thus $\frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1$ but $\frac{1}{2} + \frac{1}{3} + \frac{1}{3} \neq 1$ so the latter set of coefficients is inadmissible.

20. From the notes (page 3-7) $X = \frac{U_2G_2+U_3G_3+5G_4}{G_1+G_2+G_3+G_4}$. where U_2 and U_3 can equal 0 or 5. As in the previous question, we are told that $G_1 + G_2 + G_3 + G_4 = 20\,\text{mS}$. Hence $X = 50\,(U_2G_2 + U_3G_3 + 5G_4)$. The only way that we can obtain the required voltages is if switching U_2 causes X to change by 1 V and switching U_3 causes X to change by 2 V (or vice versa). Thus we need $X = \frac{1}{5}U_2 + \frac{2}{5}U_3 + 1$; it is easy to see that this satisfies the required output voltages. Now equating coefficients, we get $50G_2 = \frac{1}{5}$, $50G_3 = \frac{2}{5}$ and $250G_4 = 1$ from which $R_2 = \frac{1}{G_2} = 250$, $R_3 = \frac{1}{G_3} = 125$ and $R_4 = \frac{1}{G_4} = 250$. Finally, $G_1 = 0.02 - G_2 - G_3 - G_4 = 0.004$ so $R_1 = \frac{1}{G_1} = 250$.

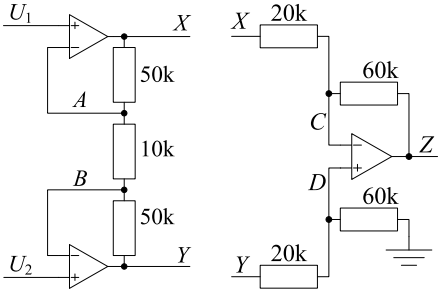
21. We need to determine how the differential input to the opamp $(V_+ - V_-)$depends on X. We are not interested in how it depends on U so it is convenient to set $U = 0$. So, with this assumption, we get potential divider equations:

    (a) $V_+ = \frac{20}{10+20}X = \frac{2}{3}X$ and $V_- = \frac{10}{20+10}X = \frac{1}{3}X$. Hence $V_+ - V_- = \frac{1}{3}X$ which, since the coefficient is positive means positive feedback.

    (b) $V_+ = \frac{10}{10+20}X = \frac{1}{3}X$ and $V_- = \frac{20}{20+10}X = \frac{2}{3}X$. Hence $V_+ - V_- = -\frac{1}{3}X$ which, since the coefficient is negative means negative feedback.

22. We can split the circuit up into two independent parts because the opamp outputs are voltage sources whose voltage is not affected by how many other things are connected to them. Note that all three opamps have negative feedback and so we can assume that $V_+ = V_-$.

    For the first part, we have $A = U_1$ and $B = U_2$. KCL @ A therefore gives $\frac{U_1-X}{50} + \frac{U_1-U_2}{10} = 0$ which gives $X = 6U_1 - 5U_2$. Similarly, KCL @ B gives $\frac{U_2-Y}{50} + \frac{U_2-U_1}{10} = 0$ which gives $Y = 6U_2 - 5U_1$. The second part of the circuit is a differential amplifier (see page 6-9 of the notes) for which $Z = \frac{60}{20}\,(Y - X)$. Substituting in the expressions for X and Y gives $Z = 3\,((6U_2 - 5U_1) - (6U_1 - 5U_2)) = 3\,(11U_2 - 11U_1) = 33\,(U_2 - U_1)$.



23. We can verify that the opamp has negative feedback since (if we set the 10 V source to zero), we have $V_- = \frac{1}{2}Y$ but $V_+ = \alpha Y$ where $\alpha = \frac{2||R}{2+2||R} < \frac{1}{2}$. So, we can assume that $V_+ = V_- = X$. KCL @ V_- gives $\frac{X-(-10)}{10} + \frac{X-Y}{10} = 0$ which gives $X - Y = -X - 10$. KCL @ V_+ gives $\frac{X}{2} + \frac{X-Y}{2} + I = 0$. Substituting $X - Y = -X - 10$ gives $\frac{X}{2} + \frac{-X-10}{2} + I = 0$ which simplifies to $I = 5$. Notice that this does not depend on R, so we have constructed a current source.

24. From the block diagram, we can deduce $Y = \sqrt{X} = \sqrt{AU - AY}$ from which $Y^2 + AY - AU = 0$. Solving this quadratic equation gives $Y(U) = 0.5\left(-A + \sqrt{A^2 + 4AU}\right)$. Notice that only one of the two roots will result in Y being positive. Provided that $A \gg 4U$, we can use the Taylor series approximation to give $Y(U) \approx 0.5\left(-A + A\left(1 + \frac{2U}{A} - \frac{2U^2}{A^2}\right)\right) = U - \frac{U^2}{A}$. From this, $Y(1) = 1 - \frac{1}{A}$ and $Y(0.5) = 0.5 - \frac{0.25}{A} = 0.5Y(1) + \frac{0.25}{A}$. We would like the error, $\frac{0.25}{A}$ to be 1% of $Y(1)$, i.e. $\frac{0.25}{A} \le 0.01\left(1 - \frac{1}{A}\right)$. Solving this gives $A \ge 26$.

25. (a) The $-$ terminal of the opamp is a virtual earth so the current through the resistor is $\frac{X}{R}$. All the current flows through the diode whose voltage is $-U$. Therefore we have $\frac{X}{R} = k \exp\frac{-U}{V_T}$ from which $U = -V_T \ln\frac{X}{kR}$.

    (b) The second opamp is a non-inverting amplifier with a gain of 3 so $W = 3U = -3V_T \ln\frac{X}{kR}$. For the third opamp, the current thorough the diode is $\frac{Y}{2R} = k \exp\frac{-W}{V_T} = k \exp\left(3 \ln\frac{X}{kR}\right) = k\left(\frac{X}{kR}\right)^3$. From this we find that $Y = \frac{2}{k^2R^2}X^3$. This we have made a circuit that cubes its input voltage (times a scale factor).

26. (a) Despite the current I_B, it is still the case that $V_+ = 0$ and so, because of the negative feedback, $P = 0$ also. KCL @ P gives $\frac{0-U}{10} + I_B + \frac{0-Y}{40} = 0$ from which $-4U + 40I_B - Y = 0$ or $Y = -4U + 40I_B$. Substituting $I_B = 0.0001\,\text{mA}$ gives $Y = -4U + 0.004$ so there is an output error of 4 mV.

    (b) This time, the current I_B flows through the 8 kΩ resistor so $V_+ = -8I_B$. As before, we assume that $Q = V_+$ also. Then KCL @ Q gives $\frac{Q-U}{10} + I_B + \frac{Q-Y}{40} = 0$ from which $-4U + 40I_B - Y + 5Q = 0$. Substituting $Q = -8I_B$ gives $-4U + 40I_B - Y - 40I_B = 0$ which simplifies to $Y = -4U$. Thus in this circuit, the bias currents do not cause any error.

    The moral is that when designing opamp circuits, you should try to make the Thévenin resistance seen by the two input terminals the same. If you achieve this, and if the bias currents are the same at both inputs (usually approximately true) there will be no resultant errors.

E1.1 Circuit Analysis

Problem Sheet 4 (lectures 9 & 10)

Key: [A]= easy ... [E]=hard

Note: A "dimensioned sketch" should show the values on the x and y axes corresonding to significant places on the corresponding graph.

1. [B] For each of the following waveforms, determine the corresponding phasor in both the form $a+jb$ and $r\angle\theta$.
   (a) $8\cos\omega t$.
   (b) $3\cos\omega t + 4\sin\omega t$.
   (c) $2\cos\left(\omega t + \frac{\pi}{4}\right)$.
   (d) $8\sin\omega t$.
   (e) $-2\cos\omega t$.
   (f) $-4\sin\left(\omega t - \frac{\pi}{2}\right)$.
   (g) $8\cos\left(\omega t + \frac{\pi}{4}\right) + 5\sin\left(\omega t - \frac{\pi}{4}\right)$.

2. [B] For each of the following phasors, determine the corresponding waveform in both the form $a\cos\omega t + b\sin\omega t$ and $a\cos(\omega t + \theta)$. (a) 1, (b) -2, (c) $3j$, (d) $-4j$, (e) $j-1$, (f) $3-4j$, (g) $2e^{j\frac{\pi}{2}}$, (h) $4e^{-j\frac{\pi}{6}}$.

3. [B] For each of the following cases say which of the two waveforms or phasors leads the other:
   (a) $\sin\omega t$ and $\cos\omega t$.
   (b) $\sin(\omega t + \pi)$ and $\cos\omega t$.
   (c) $\sin(\omega t - \pi)$ and $\cos\omega t$.
   (d) $(1+j)$ and $(2+j)$.
   (e) $(1+j)$ and $(1-j)$.
   (f) $(-1+j)$ and $(-1-j)$.
   (g) 1 and $1\angle 350°$.

4. [B] Draw a dimensioned sketch of the waveform of i in the circuit of Fig. 4(a) when v has the waveform shown in Fig. 4(b).

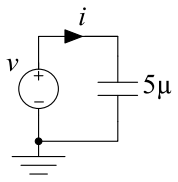


Fig. 4(a)

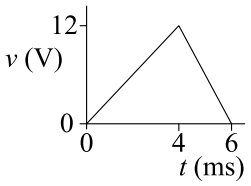


Fig. 4(b)

5. [B] For each of the circuits shown in Fig. 5(a)-(d) determine the average value of $y(t)$ when $x(t) = 4 + 2\cos\omega t$ for some non-zero frequency ω.

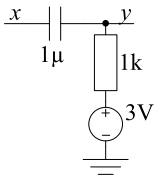


Fig. 5(a)

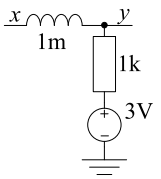


Fig. 5(b)

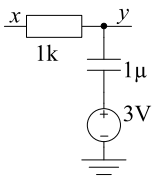


Fig. 5(c)

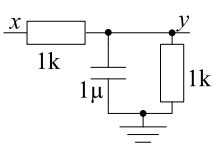


Fig. 5(d)

6. [B] Find the value of a single inductor equivalent to the circuit shown in Fig. 6.

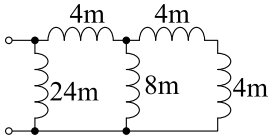


Fig. 6

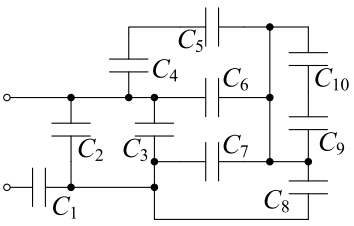


Fig. 7

7. [B] Find the value of a single capacitor equivalent to the circuit shown in Fig. 7 given that each of the capacitors has a value of $1\,\mu$F.

8. [B] Find the average value of v in the circuit of Fig. 8 if $u(t) = 2 + 3\cos\omega t$.

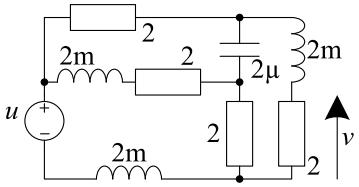


Fig. 8

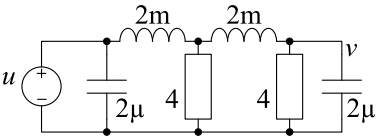


Fig. 9

9. [B] Find the average value of v in the circuit of Fig. 9 if $u(t) = 8 - 2\cos\omega t$.

10. [B] Find the complex impedance of the circuit shown in Fig. 10 for (a) $\omega = 0$, (b) $\omega = 1000$, (c) $\omega = 2000$ and (d) $\omega = \infty$.

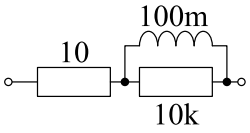


Fig. 10

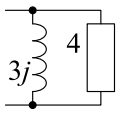


Fig. 11(a)

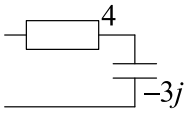


Fig. 11(b)

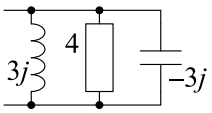


Fig. 11(c)

11. [B] The components in Fig. 11 are labelled with their impedances. Calculate both the complex impedance and the complex admittance for each of the three networks.

12. [B] The components in Fig. 12 are labelled with their impedances. Determine the values of a parallel inductor and resistor that will have the same overall impedance at (a) 1 kHz and (b) 10 kHz. Hint: first calculate the admittance of the original network.

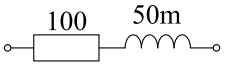


Fig. 12

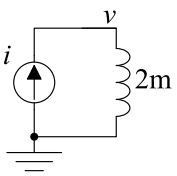


Fig. 13(a)

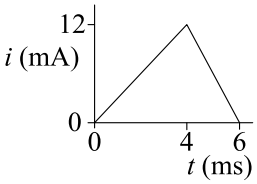


Fig. 13(b)

13. [C] Draw a dimensioned sketch of the waveform of $v(t)$ in the circuit of Fig. 13(a) when i has the waveform shown in Fig. 13(b).

14. [C] The three current i_1, i_2, i_3 in Fig. 14 are equal to $5\cos\left(\omega t + \frac{3\pi}{4}\right)$, $2\cos\left(\omega t + \frac{\pi}{4}\right)$ and $\sqrt{8}\cos\omega t$ but not necessarily in that order. Determine which current is which and find both the phasor I and time-waveform $i(t)$.

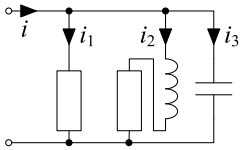


Fig. 14

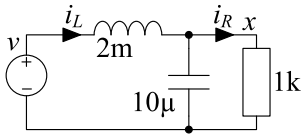


Fig. 17(a)

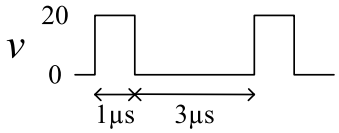


Fig. 17(b)

15. [D] Draw a dimensioned sketch of the waveform of i in the circuit of Fig. 15 when v has the waveform shown in Fig. 4(b) given that at time $t = 0$, (a) $i(0) = 0$ and (b) $i(0) = 2\,\text{A}$.

16. [D] Draw a dimensioned sketch of the waveform of v in the circuit of Fig. 16(a) when i has the waveform shown in Fig. 16(b) given that at time $t = 0$, (a) $v(0) = 0$ and (b) $v(0) = -5\,\text{V}$.

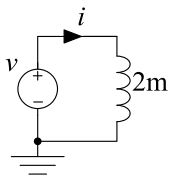


Fig. 15

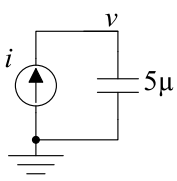


Fig. 16(a)

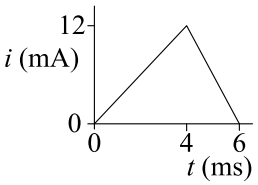


Fig. 16(b)

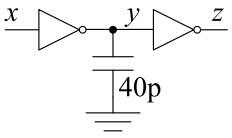


Fig. 18

17. [D] In the circuit of Fig. 17(a), the voltage v has the periodic waveform shown in Fig. 17(b) with a period of $4\,\mu$s and an amplitude of 20 V.
    (a) State the duty cycle of v.
    (b) Determine the average value of x.
    (c) Determine the average value of i_R.
    (d) Determine the average value of i_L.
    (e) Assuming that x is constant (at its average value), draw a dimensioned sketch of the waveform of $i_L(t)$ and determine its maximum and minimum values.
    (f) Assuming that x is constant (at its average value), determine the average, positive peak and negative peak of the powers absorbed by each of R, L and C.

18. [D] In the circuit of Fig. 18, the output logic levels from the inverter are 0 V and 5 V and the inverter has a maximum output current of $\pm 2\,\text{mA}$. The inverter senses a low voltage when its input is $< 1.5\,\text{V}$. If x changes from logic 0 to logic 1, determine the delay until z changes. Ignore the inverter input currents and any delays inside the inverters themselves.

E1.1 Circuit Analysis

Problem Sheet 4 - Solutions

1. (a) 8, (b) $3 - j4 = 5\angle -0.93$ $(-53°)$, (c) $1.4 + j1.4 = 2\angle 0.79$ $(45°)$, (d) $-j8 = 8\angle -1.57$ $(90°)$, (e) $-2 = 2\angle 3.14$ $(180°)$, (f) 4, (g) $2.12 + j2.12 = 3\angle 0.79$ $(45°)$.

2. (a) $\cos \omega t$, (b) $-2\cos \omega t = 2\cos(\omega t + \pi)$, (c) $-3\sin \omega t = 3\cos\left(\omega t + \frac{\pi}{2}\right)$, (d) $4\sin \omega t = 4\cos\left(\omega t - \frac{\pi}{2}\right)$, (e) $-\cos \omega t - \sin \omega t = 1.4\cos\left(\omega t + \frac{3\pi}{4}\right)$, (f) $3\cos \omega t + 4\sin \omega t = 5\cos(\omega t - 0.93)$, (g) $-2\sin \omega t = 2\cos\left(\omega t + \frac{\pi}{2}\right)$, (h) $3.46\cos \omega t + 2\sin \omega t = 4\cos\left(\omega t - \frac{\pi}{6}\right)$.

3. (a) $\cos \omega t$ by $\frac{\pi}{2}$, (b) $\sin(\omega t + \pi)$ by $\frac{\pi}{2}$, (c) $\sin(\omega t - \pi)$ by $\frac{\pi}{2}$: note that $\sin(\omega t + \pi)$ and $\sin(\omega t - \pi)$ are actually the same waveform, (d) $(1 + j)$ by 0.322 rad, (e) $(1 + j)$ by $\frac{\pi}{2}$, (f) $(-1 - j)$ by $\frac{\pi}{2}$, (g) 1 by 10°. Because angles are only defined to within a multiple of 360°, you always need to be careful when comparing them. To find out which is leading, you need to take the difference in phase angles and then add or subtract multiples of 360° to put the answer into the range ±180°. Note that a sine wave is defined for all values of t (not just for $t > 0$) and so there is no such thing as the "first peak" of a sine wave.

4. $i = C\frac{dv}{dt}$. $\frac{dv}{dt}$ is 3000 V/s for the first 4 ms and −6000 V/s for the next 2 ms. So $i = +15$ or -30 mA. (see Fig. 4)

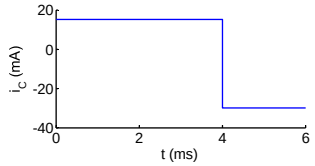


Fig. 4

5. The average value of $x(t)$ is $X = 4$ (note that we use capital letters for quantities that do not vary with time). For averages (or equivalently for DC or $\omega = 0$) capacitors act as open circuit and inductors as short circuits; this gives the simplified circuits shown below. So this gives (a) $Y = 3$, (b) $Y = X = 4$, (c) $Y = X = 4$, (d) $Y = \frac{1}{2}X = 2$.

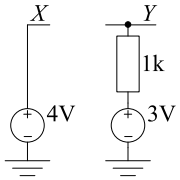


Fig. 5(a)

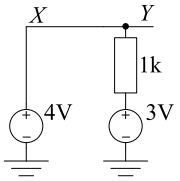


Fig. 5(b)

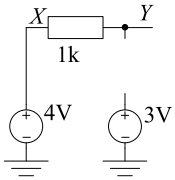


Fig. 5(c)

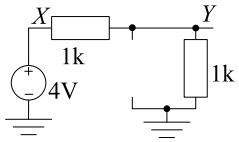


Fig. 5(d)

6. $4 + 4 = 8$. $8||8 = 4$. $4 + 4 = 8$. $24||8 = 6$ mH.

7. C_9 and C_{10} are short-circuted and play no part in the circuit. We can merge series and parallel capacitors as follows: $C_{4,5} = 0.5$, $C_{7,8} = 2$, $C_{2,3} = 2$. Now merge $C_{4,5}$with C_6 to give $C_{4,5,6} = 1.5$ and merge this with $C_{7,8} = 2$ to give $C_{4,5,6,7,8} = \frac{6}{7}$. Now merge this with $C_{2,3} = 2$ to give $C_{2,3,4,5,6,7,8} = \frac{20}{7}$. Finally merge this with $C_1 = 1$ to give $C = \frac{20}{27}\,\mu$F.

8. To determine average values, we can treat C as open circuit and L as short circuit. The original circuit simplifies to that shown in Fig. 8. So we have a simple potential divider and $\bar{v} = \frac{1}{2}\bar{u} = 1$ V where the overbar denotes "average value".

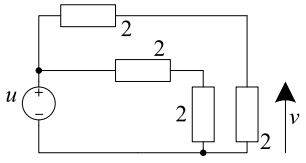


Fig. 8

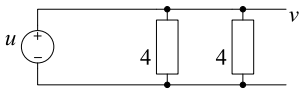


Fig. 9

9. To determine average values, we can treat C as open circuit and L as short circuit. The original circuit simplifies to that shown in Fig. 9 and so $\bar{v} = \bar{u} = 8\,\text{V}$ where the overbar denotes "average value".

10. $Z = R_S + R_P||j\omega L$. (a) $Z = R_S = 10$, (b) $Z = 10 + 10000||100j = 11 + 100j = 100.6\angle 83.7°$, (c) $Z = 10 + 10000||200j = 14 + 200j = 200.4\angle 86°$, (d) $Z = 10010$.

11. We denote the impedance by Z and the admittance by $Y = \frac{1}{Z}$. (a) $Z = 1.44 + 1.92j$ and $Y = 0.25 - 0.33j$, (b) $Z = 4 - 3j$ and $Y = 0.16 + 0.12j$, (c) $Z = 4$ and $Y = 0.25$. Notice that the imaginary part of the impedance (the reactance) is positive for inductive circuits and negative for capacitive circuits, but that the imaginary part of the admittance (the susceptance) has the opposite sign. In part (c), the impedances of the inductor and the capacitor have cancelled out leaving an overall impedance that is purely real; because the impedances are frequency dependent, this cancellation will only happen at one particular frequency which is called the network's "resonant frequency". I strongly advise you to learn how to do these complex arithmetic manipulations using the built-in capabilities of the Casio fx-991.

12. (a) $\omega = 6283$ so the impedance is $Z = 100 + 314j$. Taking the reciprocal gives $Y = \frac{1}{Z} = 0.92 - 2.89j\,\text{mS}$. Since parallel admittances add, the parallel component values must be $R_P = \frac{1000}{0.92} = 1087\Omega$ and $L_P = \frac{1000}{2.89\omega} = 55.1\,\text{mH}$. For case (b), we now have $\omega = 62832$ and, following the same argument, we get $R_P = 98.8\,\text{k}\Omega$ and $L_P = 50.05\,\text{mH}$. As the frequency goes up, the series resistor becomes a less significant part of the total impedance and its effect becomes less. This means that L_P becomes approximately equal to the original inductance and R_P becomes larger.

13. $v = L\frac{di}{dt}$. $\frac{di}{dt}$ is 3 A/s for the first 4 ms and -6 A/s for the next 2 ms. So $v = +6$ or $-12\,\text{mV}$. (see Fig. 13)

14. If we define the voltage phase to be ϕ will be $\angle i_1 = \phi + 0$, $\phi - \frac{\pi}{2} < \angle i_2 < \phi + 0$, $\angle i_3 = \phi + \frac{\pi}{2}$ as the CIVIL mnemonic reminds us. Thus i_3 will have the most positive phase shift. It follows that $i_1 = 2\cos\left(\omega t + \frac{\pi}{4}\right)$, $i_2 = \sqrt{8}\cos\omega t$ and $i_3 = 5\cos\left(\omega t + \frac{3\pi}{4}\right)$ and that $\phi = \frac{\pi}{4}$. As phasors these are $I_1 = 1.4 + j1.4 = 2\angle 45°$, $I_2 = 2.8$, $I_3 = -3.5 + j3.5 = 5\angle 135°$. Adding these together gives $I = 0.71 + j4.95 = 5\angle 1.43\,(82°)$. So $i(t) = 0.71\cos\omega t - 4.95\sin\omega t = 5\cos(\omega t + 1.43)$ A.

15. $i(t) = i(0) + \frac{1}{L}\int_{\tau=0}^{t} v(\tau)d\tau$ where $v(\tau) = 3000\tau$ for $0 \le \tau \le 4\,\text{ms}$ and $v(\tau) = 36 - 6000\tau$ for $4\,\text{ms} \le \tau \le 6\,\text{ms}$ (obtain this formula by finding the straight line equalling 12 at $\tau = 4\,\text{ms}$ and 0 at $\tau = 6\,\text{ms}$). So, for the first segment, $i = \frac{1}{L} \times 1500t^2$ which reaches 12 A at $t = 0.004$. For the second segment, $i = \frac{1}{L} \times \left(36t - 3000t^2\right) + c$. To find c, we force $i = 12$ at $t = 0.004$. This gives $c = -36\,\text{A}$. When $t = 0.006$ we then get $i = 18\,\text{A}$. For part (b), we just add 2 A to the curve. (see Fig. 15)

16. $v = \frac{1}{C}\int i dt$ where $i = 3t$ or $i = 36 \times 10^{-3} - 6t$. So, for the first segment, $v = \frac{1}{C} \times 1.5t^2$ which reaches 4.8 V at $t = 0.004$. For the second segment, $v = \frac{1}{C} \times \left(36 \times 10^{-3}t - 3t^2\right) + a$. To find a, we force $v = 4.8$ at $t = 0.004$. This gives $a = -14.4\,\text{V}$. When $t = 0.006$ we then get $v(t) = 7.2\,\text{V}$. For part (b), we just subtract 5 V from the curve. (see Fig. 16)

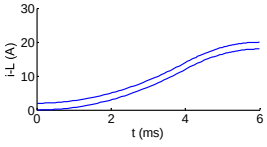


Fig. 15

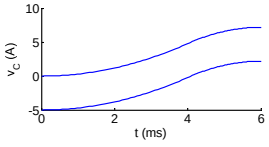


Fig. 16

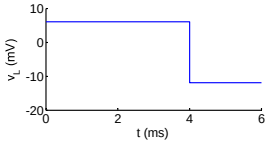


Fig. 13

17. (a) The duty cycle is 0.25 = 25%, (b) $\bar{x} = \bar{v} = \frac{1}{4} \times 20 = 5\,\text{V}$, (c) $\bar{i}_R = \frac{\bar{x}}{R} = 5\,\text{mA}$, (d) Since $\bar{i}_C = 0$, $\bar{i}_L = \bar{i}_R = 5\,\text{mA}$, (e) The voltage across the inductor is $v - x = L\frac{di_L}{dt}$. So when $v = 20$, $\frac{di_L}{dt} = \frac{15}{L} = 7.5\,\text{kA/s}$. So the total change in i_L over the $1\,\mu\text{s}$ interval is 7.5 mA. It follows that $i_L$varies from its average value of 5 mAby ± 3.75 mAand has minimum and maximum values of 1.25 and 8.75 mA (see Fig. 17(a)). (f) Average powers are $P_R = 25\,\text{mW}$, $P_L = P_C = 0$. Max powers are $P_R = 25\,\text{mW}$, $P_L = v_L i_L = 15 \times 8.75 = 131.25\,\text{mW}$, $P_C = v_C i_C = 5 \times 3.75 = 18.75\,\text{mW}$. Min powers are $P_R = 25\,\text{mW}$, $P_L = v_L i_L = -5 \times 8.75 = -43.75\,\text{mW}$ (see Fig. 17(b)), $P_C = v_C i_C = 5 \times -3.75 = -18.75\,\text{mW}$ (see Fig. 17(c)). Note that during the time that it is positive

$(0.5 < t < 2.5\,\text{ms})$, the average value of i_C is $\overline{i_C} = 1.375\,\text{mA}$ and so the total rise in v_C will be $\Delta v_C = \frac{\overline{i_C}\Delta t}{C} = \frac{1.375\times 2}{10} = 275\,\text{mV}$ (i.e. $\pm 138\text{mV}$around its mean) which is small compared to its mean value of 5 V; this justifies the assumption that it is constant.

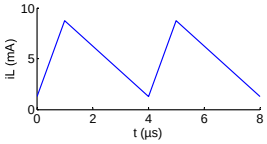


Fig. 17(a)

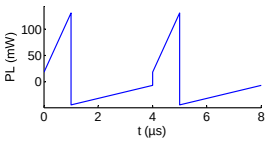


Fig. 17(b)

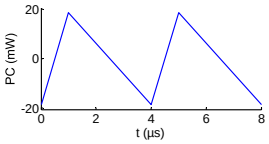


Fig. 17(c)

18. When x changes from low to high, y will change from high to low. The maximum current is 2 mA so $\frac{dy}{dt} = -\frac{i}{C} = -50\,\text{MV/s}$. So the time to fall from 5 V to 1.5 V is $\frac{3.5}{50} \times 10^{-6} = 70\,\text{ns}$.

E1.1 Circuit Analysis

Problem Sheet 5 (Lectures 11, 12 & 13)

Key: [A]= easy ... [E]=hard

Note: A "sketch" should show the values on the x and y axes corresonding to significant places on the corresponding graph.

1. [C] For each of the circuits in Fig. 1(i)-(vi),
   (a) Find the transfer function $\frac{Y}{X}(j\omega)$.
   (b) Find expressions for the low and high frequency asymptotes of $H(j\omega)$.
   (c) Sketch the straight line approximation to the magnitude response, $|H(j\omega)|$, indicating the frequency (in rad/s) and the gain of the approximation (in dB) at each of the corner frequencies.

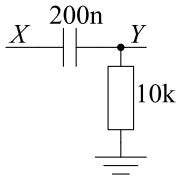


Fig. 1(i)

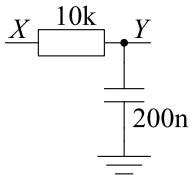


Fig. 1(ii)

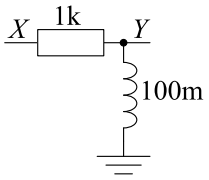


Fig. 1(iii)

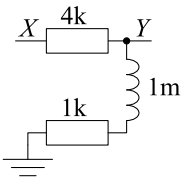


Fig. 1(iv)

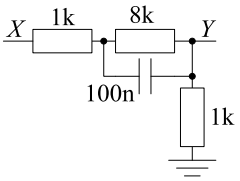


Fig. 1(v)

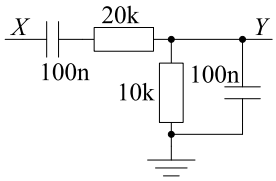


Fig. 1(vi)

2. [C] Sketch a straight line approximation for the phase response, $\angle H(j\omega)$, of the circuit in Fig. 1(v), indicating the frequency (in rad/s) and phase (in rad) at each of the corner frequencies.

3. [B] A "C-weighting filter" in audio engineering has the form $H(j\omega) = \frac{k(j\omega)^2}{(j\omega+a)^2(j\omega+b)^2}$ where $a = 129$ and $b = 76655\,\text{rad/s}$. Calculate k exactly so that $|H(j\omega)| = 1$ at $\omega = 2000\pi\,\text{rad/s}$.

4. [C] Design circuits with each of the magnitude responses given in Fig. 4(i)-(iii) using, in each case, a single capacitor and appropriate resistors.

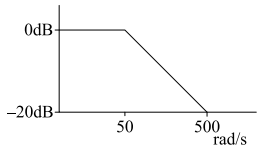


Fig. 4(i)

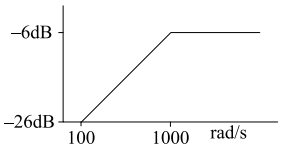


Fig. 4(ii)

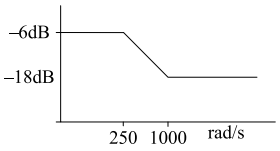


Fig. 4

5. [B] Express each of the following transfer functions in a standard form in which the numerator and denominator are factorized into linear and quadratic factors of the form $(j\omega+p)$ and $\left((j\omega)^2 + qj\omega + r^2\right)$ where p, q and r are real with an additional numerator factor of the form $A(j\omega)^k$ if required. Quadratic terms should be factorized if possible.
   (a) $\frac{-2\omega^2-2j\omega^3}{1-2\omega^2+\omega^4}$
   (b) $\frac{-2\left(1+\omega^2\right)}{(1-\omega^2)+2j\omega}$
   (c) $\frac{10(j\omega)^2+2j\omega+10}{(j\omega)^2+2j\omega+1}$
   (d) $\frac{1}{j\omega+6(j\omega)^{-1}+5}$

6. [B] Without doing any algebra, determine the low and high frequency asymptotes of the following transfer functions:
   (a) $\frac{-2\omega^2-2j\omega^3}{1+\omega^4}$
   (b) $\frac{2(j\omega)^3+3}{4(j\omega)^4+1}$
   (c) $\frac{j\omega\big(2(j\omega)^6+3\big)\big(5(j\omega)^3+4j\omega+3\big)}{2\big((j\omega)^5+1\big)\big((j\omega)^5+5\big)}$
   (d) $\frac{12}{j\omega+6(j\omega)^{-1}}$

7. [D] A circuit has a transfer function whose low and high frequency asymptotes are $A\,(j\omega)^{\alpha}$ and $B\,(j\omega)^{\beta}$ respectively. What constraints can you place on α and β if you know that the magnitude of the transfer function is less than G at all frequencies where G is a fixed real-valued constant.

8. [C] For each of the following transfer functions, sketch the straight line approximation to the magnitude response and determine the gain of this approximation at $\omega = 1000\,\text{rad/s}$: (a) $\frac{5(1+j\omega/500)}{(1+j\omega/100)(1+j\omega/2000)}$, (b) $\frac{2(1+j\omega/5000)}{(1+j\omega/100)}$, (c) $\frac{3j\omega(1+j\omega/500)}{(1+j\omega/100)(1+j\omega/2000)(1+j\omega/5000)}$.

9. [C] The frequency response of the circuit in Fig. 9 is given by $\frac{\left(\frac{j\omega}{p}\right)^2}{\left(\frac{j\omega}{p}\right)^2+2\zeta\left(\frac{j\omega}{p}\right)+1}$ where the corner frquency, $p = \frac{1}{\zeta RC}$ rad/s. Using capacitors of value $C = 10\,\text{nF}$, design a filter with $\zeta = \sqrt{0.5}$ and a corner frequency of $\frac{p}{2\pi} = 1$ kHz. Determine the value of the transfer function at $\frac{\omega}{2\pi} = 100\,\text{Hz}$, 1 kHz and 10 kHz.

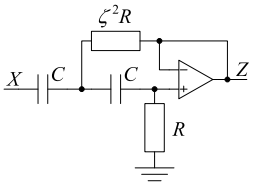


Fig. 9

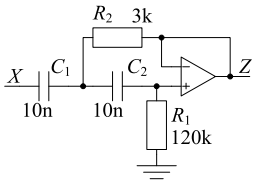


Fig. 11(a)

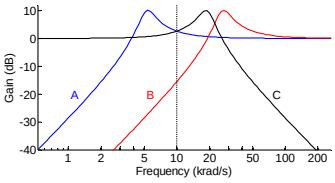


Fig. 11(b)

10. [D] A high-pass "Butterworth" filter of order $2N$ consists of N cascaded copies of Fig. 9 all having the same corner frequency, but with the k^{th} stage having $\zeta_k = \cos\left(\left(\frac{2k-1}{4N}\right)\pi\right)$ where $k = 1, \ldots, N$. ("cascaded" means you connect the output of one stage to the input of the next). Design a 4th order high-pass Butterworth filter with a corner frequency of 1 kHz. Write an expression for its transfer function and sketch its magnitude response using just the high and low asymptotes. Butterworth filters are widely used because they have a very smooth magnitude response without any peaks; the transfer function satisfies $|H(j\omega)|^2 = \frac{\left(\frac{\omega}{p}\right)^{4N}}{\left(\frac{\omega}{p}\right)^{4N}+1}$.

11. [C] The circuit of Fig. 11(a) is a high-pass filter whose magnitude response is marked "A" in Fig. 11(b) . Using the filter transformations described in lectures, design filters with the magnitude responses marked "B" and "C" on the graph. Relative to "A", these are respectively shifted up in frequency by a factor of 5 and reflected in the axis $\omega = 10,000$.

12. [C] (a) Find the resonant frequency, ω_r, at which the impedance of the network in Fig. 12(i) is real. (b) Determine the Q of the circuit at ω_r. (c) Find R_P and L_P in the circuit of Fig. 12(ii) so that the two networks have the same impedance at ω_r.

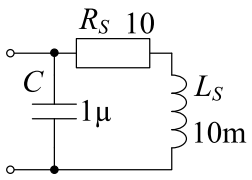


Fig. 12(i)

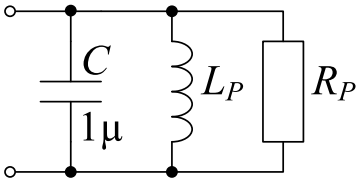


Fig. 12(ii)

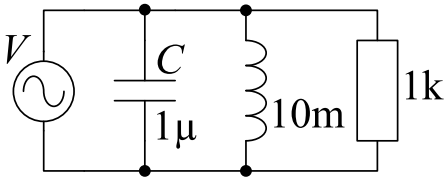


Fig. 13

13. [C] In the circuit of Fig. 13, $\omega = 10\,000$ and the phasor $V = 10$. Find (a) the peak power supplied by V and (b) the peak power absorbed by C.

14. [C] Determine the transfer function for each of the circuits Fig. 14(i)-(iv).

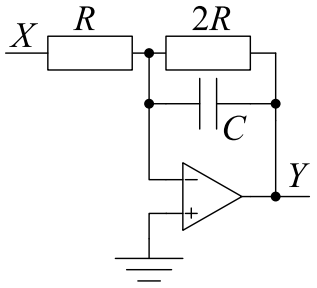


Fig. 14(i)

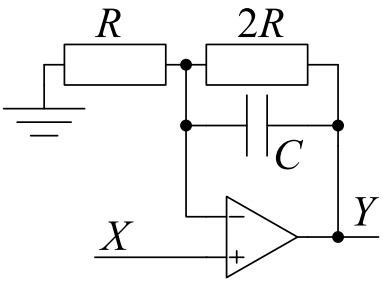


Fig. 14(ii)

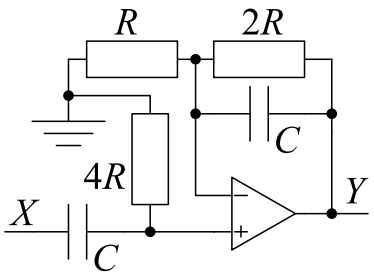


Fig. 14(iii)

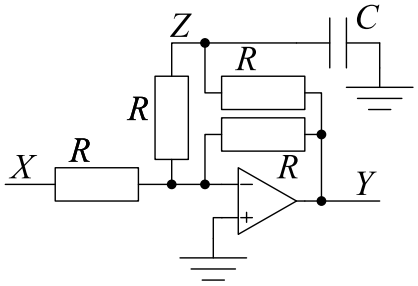


Fig. 14(iv)

15. [D] For the circuit in Fig. 15,
    (a) Find the transfer function $\frac{Z}{W}(j\omega)$ and explain why this is equal to $\frac{Y}{W}(j\omega)$
    (b) Hence, by applying KCL at node W and using part (a) to substitute for W, show that the transfer function $\frac{Y}{X}(j\omega) = \frac{1}{R_1R_2C_1C_2(j\omega)^2+(R_1+R_2)C_1j\omega+1}$.
    (c) From the transfer function expression we can express the corner frequency and damping factor as $p^2 = \frac{1}{R_1R_2C_1C_2}$ and $\zeta = \frac{p(R_1+R_2)C_1}{2}$. By eliminating p between these equations, show that $\left(1+\frac{R_2}{R_1}\right)\left(1+\sqrt{1-\frac{C_1}{\zeta^2C_2}}\right) = 2$. Explain why this means that we must choose $C_1 \leq \zeta^2 C_2$.
    (d) Assuming that you only have available capacitors of values 10 nF, 22 nF and 47 nF, design a filter with $p = 1000 \times 2\pi$ and $\zeta = 0.5$. Choose C_1 and C_2 first. then $\frac{R_2}{R_1}$ and lastly R_1.

16. [D] For the circuit of Fig. 16,
    (a) Find the transfer function $\frac{Y}{X}(j\omega)$.
    (b) Find the frequency, ω_0, at which $\left|\frac{Y}{X}(j\omega)\right|$ is maximum and its value at this maximum.
    (c) Find the 3dB bandwidth of the circuit and the value of $Q = \frac{1}{2\zeta}$.

    [Note: If you find yourself doing loads of algebra, you are using the wrong method.]

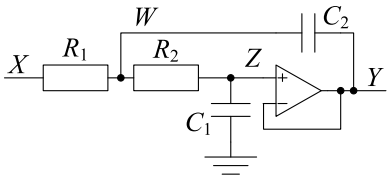


Fig. 15

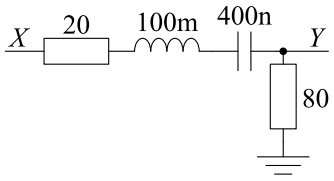


Fig. 16

E1.1 Circuit Analysis

Problem Sheet 5 - Solutions

1. Each of the circuits may be viewed as a potential divider, so we can write down the transfer function without doing any nodal analysis. In two cases, one element of the potential divider consists of a parallel $R||C$ combination. This parallel combination has the impedance $\frac{1}{1/R+j\omega C} = \frac{R}{1+j\omega RC}$. Graphs of the magnitude responses are shown in Fig. 1(i)-(vi) together with their straight-line approximations.

   (i) $\frac{Y}{X} = \frac{R}{R+1/j\omega C} = \frac{j\omega RC}{1+j\omega RC} = \frac{j\omega/500}{1+j\omega/500}$ where $RC = 2\,\text{ms}$. LF asymptote is $0.002j\omega$; HF asymptote is 1. Denominator corner frequency is $\frac{1}{RC} = 500\,\text{rad/s}$. At the corner, the gain is $1 = 0\,\text{dB}$.

   (ii) $\frac{Y}{X} = \frac{1/j\omega C}{R+1/j\omega C} = \frac{1}{1+j\omega RC} = \frac{1}{1+j\omega/500}$ where $RC = 2\,\text{ms}$. LF asymptote is 1; HF asymptote is $500\,(j\omega)^{-1}$. Denominator corner frequency is $\frac{1}{RC} = 500\,\text{rad/s}$. At the corner, the gain is $1 = 0\,\text{dB}$.

   (iii) $\frac{Y}{X} = \frac{j\omega L}{R+j\omega L} = \frac{j\omega L/R}{1+j\omega L/R}$ where $\frac{L}{R} = 100\,\mu\text{s}$. LF asymptote is $10^{-4}j\omega$; HF asymptote is 1. Denominator corner frequency is $\frac{R}{L} = 10^4\,\text{rad/s}$. At the corner, the gain is $1 = 0\,\text{dB}$.

   (iv) For convenience, we define $R = 1\,\text{k}$. Then $\frac{Y}{X} = \frac{R+j\omega L}{5R+j\omega L} = 0.2\frac{1+j\omega\frac{L}{R}}{1+j\omega\frac{L}{5R}}$ where $\frac{L}{R} = 1\,\mu\text{s}$. LF asymptote is 0.2; HF asymptote is 1. Numerator corner frequency is $\frac{R}{L} = 10\,\text{krad/s}$ with a gain at the corner of $0.2 = -14\,\text{dB}$. Denominator corner frequency is $\frac{5R}{L} = 50\,\text{krad/s}$ with a gain at the corner of $1 = 0\,\text{dB}$. As can be seen in Fig. 1(iv), the magnitude response turns up at 10 krad/s and then flattens out again at 50 krad/s . In between these two frequencies the slope $\frac{\log|H|}{\log\omega} = +1$ or, equivalently, +6 dB/octave or +20 dB/decade ; all these are the same as saying that $|H| \propto \omega$; thus from $\omega = 10\,\text{krad/s}$ to 50 krad/s, the frequency increases by a factor of 5 and the gain also increases by a factor of 5.

   (v) For convenience, we define $R = 1\,\text{k}$. Then $\frac{Y}{X} = \frac{R}{2R+\frac{8R}{1+8j\omega RC}} = \frac{1+8j\omega RC}{10+16j\omega RC} = 0.1\frac{1+8j\omega RC}{1+1.6j\omega RC}$ where $RC = 100\,\mu\text{s}$. LF asymptote is 0.1; HF asymptote is 0.5. Numerator corner frequency is $\frac{1}{8RC} = 1250\,\text{rad/s}$ with a gain at the corner of $0.1 = -20\,\text{dB}$. Denominator corner frequency is $\frac{1}{1.6RC} = 6.25\,\text{krad/s}$ with a gain at the corner of $0.5 = -6\,\text{dB}$. As in the previous part, both the frequency and the gain change by a factor of 5 between the corner frequencies.

   (vi) For convenience, we define $R = 10\,\text{k}$. Then $\frac{Y}{X} = \frac{\frac{R}{1+j\omega RC}}{2R+\frac{1}{j\omega C}+\frac{R}{1+j\omega RC}} = \frac{j\omega RC}{2j\omega RC(1+j\omega RC)+(1+j\omega RC)+j\omega RC} = \frac{j\omega RC}{1+4j\omega RC+2(j\omega RC)^2}$ where $RC = 1\,\text{ms}$. LF asymptote is $0.001j\omega$; HF asymptote is $500\,(j\omega)^{-1}$. We can factorize the denominator to give $1 + 4j\omega RC + 2\,(j\omega RC)^2 = \left(1+\frac{j\omega}{a}\right)\left(1+\frac{j\omega}{b}\right)$ where a and b are -1 times the roots of the quadratic equation $2R^2C^2x^2 + 4RCx + 1$ or $\frac{1\pm\sqrt{0.5}}{RC}$. This gives denominator corner frequencies $a = 293$ and $b = 1707\,\text{rad/s}$. The gain in between these two frequencies can be obtained by substituting $\omega = a$ into the LF asymptote expression to give a value of $aRC = 1 - \sqrt{0.5} = 0.293 = -10.7\,\text{dB}$.

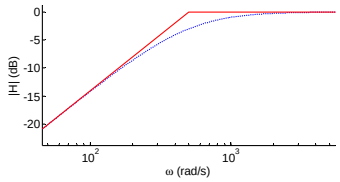


Fig. 1(i)

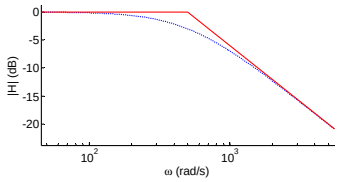


Fig. 1(ii)

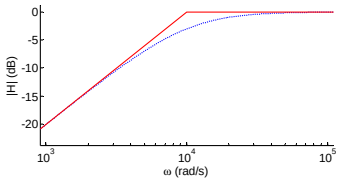


Fig. 1(iii)

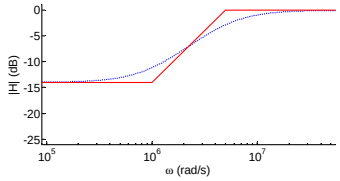


Fig. 1(iv)

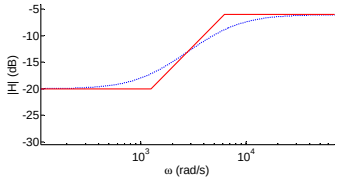


Fig. 1(v)

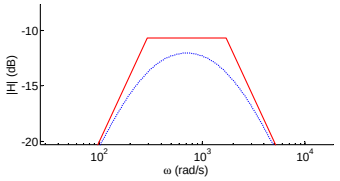


Fig. 1(vi)

2. For convenience, we define $R = 1\,\text{k}$. Then $\frac{Y}{X} = \frac{R}{2R+\frac{8R}{1+8j\omega RC}} = \frac{1+8j\omega RC}{10+16j\omega RC} = 0.1\frac{1+8j\omega RC}{1+1.6j\omega RC}$ where $RC = 100\,\mu\text{s}$. LF asymptote is 0.1; HF asymptote is 0.5; both of these are real and so have zero phase shift. The magnitude plot has a numerator corner frequency of $\frac{1}{8RC} = 1250\,\text{rad/s}$ and denominator corner frequency of $\frac{1}{1.6RC} = 6.25\,\text{krad/s}$. Each of these generates a pair of corner frequencies on the phase plot at $0.1\times$ and $10\times$ the frequency. Thus we have corners at $\omega = 125\,(+)$, $625\,(-)$, $12.5\,\text{k}\,(-)$, $62.5\,\text{k}\,(+)\,\text{rad/s}$ where the sign in parentheses indicates the gradient change $\pm\frac{\pi}{4}\,\text{rad/decade}$. Between 125 and 625 rad/s the gradient is $\frac{\pi}{4}\,\text{rad/decade}$ so the phase will change by $\frac{\pi}{4}\times\log_{10}\frac{625}{125} = \frac{\pi}{4}\times 0.7 = +0.55\,\text{rad}$. This is therefore the phase shift for the flat part of the phase response. (see Fig. 2).

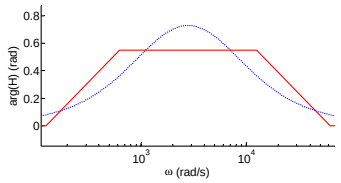


Fig. 2

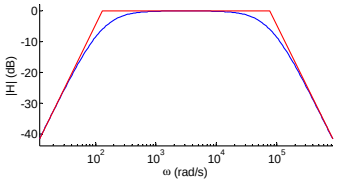


Fig. 3

3. At $\omega = 2000\pi = 6283.2$ we know $|H(j\omega)| = 1$. Hence $k = \frac{|j\omega+a|^2|j\omega+b|^2}{|j\omega|^2} = \frac{(\omega^2+a^2)(\omega^2+b^2)}{\omega^2} = \frac{3.9495\times10^7\times5.9155\times10^9}{3.9478\times10^7} = 5.918\times10^9$.

4. (i) This is the same low-pass filter as Fig. 1(ii) but with a corner frequency of 50 rad/s. So we want $RC = \frac{1}{50} = 20\,\text{ms}$. One possible choice is shown in Fig. 4(i).

   (ii) This is the same high-pass filter as Fig. 1(i) but with a corner frequency of 1000 rad/s and a high frequency gain of $0.5 = -6$ dB. So we want $RC = \frac{1}{1000} = 1\,\text{ms}$. One possible choice is shown in Fig. 4(ii); the two resistors give the correct high frequency gain.

   (iii) We want a circuit whose gain decreases from $\frac{1}{2}$ at low frequencies to $\frac{1}{8}$ at high frequencies. We can do this by using a capacitor to short out part of the vertical limb of the potential divider at high frequencies as shown in Fig. 4(iii). This design has a gain of $\frac{1}{8}$ when the capacitor is a short circuit; with the capacitor open circuit (low frequencies), we add in an additional $6R$ which gives a gain of $\frac{1}{2}$. The impedance of $6R||C$ is $\frac{R+\frac{6R}{1+6j\omega RC}}{8R+\frac{6R}{1+6j\omega RC}} = \frac{7+6j\omega RC}{14+48j\omega RC}$ which, as a check, we see has the correct LF and HF asymptotes. The numerator corner frequency is at $\omega = \frac{7}{6RC}$ which needs to be at 1000 rad/s. From this, $RC = 1.17\,\text{ms}$ so one possible set of value is $C = 100\,\text{nF}$ and $R = 12\,\text{k}\Omega$.

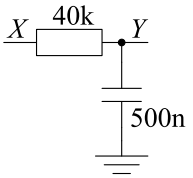


Fig. 4(i)

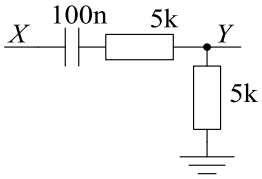


Fig. 4(ii)

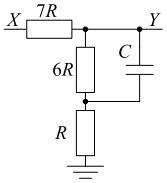


Fig. 4

5. (a) $\frac{-2\omega^2-2j\omega^3}{1-2\omega^2+\omega^4} = \frac{2(j\omega)^2(j\omega+1)}{\big((j\omega)^2+1\big)\big((j\omega)^2+1\big)}$.

   (b) $\frac{-2\big(1+\omega^2\big)}{(1-\omega^2)+2j\omega} = \frac{2(j\omega+1)(j\omega-1)}{(j\omega+1)(j\omega+1)} = \frac{2(j\omega-1)}{(j\omega+1)}$

   (c) $\frac{10(j\omega)^2+2j\omega+10}{(j\omega)^2+2j\omega+1} = \frac{10\big((j\omega)^2+0.2j\omega+1\big)}{(j\omega+1)(j\omega+1)}$

   (d) $\frac{1}{j\omega+6(j\omega)^{-1}+5} = \frac{j\omega}{(j\omega)^2+5j\omega+6} = \frac{j\omega}{(j\omega+2)(j\omega+3)}$

6. To find the low frequency asymptote, you take the lowest power of $j\omega$ in each of the numerator and denominator factors and multiply them together. Likewise, for the high frequency asymptote, you take the highest power of $j\omega$ in each of the factors. There is no need (or indeed advantage) to do any factorization or to multiply out existing factors.

   (a) $H_{\rm LF} = \frac{-2\omega^2}{1} = 2(j\omega)^2$, $H_{\rm HF} = \frac{-2j\omega^3}{\omega^4} = 2(j\omega)^{-1}$

   (b) $H_{\rm LF} = \frac{3}{1} = 3$, $H_{\rm HF} = \frac{2(j\omega)^3}{4(j\omega)^4} = 0.5(j\omega)^{-1}$

   (c) $H_{\rm LF} = \frac{j\omega\times3\times3}{2\times1\times5} = 0.9j\omega$, $H_{\rm HF} = \frac{j\omega\times2(j\omega)^6\times5(j\omega)^3}{2\times(j\omega)^5\times(j\omega)^5} = 5$

   (d) $H_{\rm LF} = \frac{12}{6(j\omega)^{-1}} = 2j\omega$, $H_{\rm HF} = \frac{12}{j\omega} = 12(j\omega)^{-1}$

7. We must have $\alpha \geq 0$ because, if α were negative, $(j\omega)^\alpha$ would increase without limit as $\omega \to 0$ and the transfer function would exceed G at some point. Similarly, we must have $\beta \leq 0$ because otherwise $(j\omega)^\beta$would increase without limit as $\omega \to \infty$. A consequence of this is that the order of the numerator can never exceed that of the denominator in a transfer function whose magnitude is bounded.

8. Graphs of the transfer functions are shown in Fig. 8(a)-(c).

   (a) We have a LF asymptote of $5 = 14\,$dB. We have corner frequencies at $\omega = 100\,(-)$, $500\,(+)$, $2000\,(-)$ where the sign in parentheses indicates the polarity of gradient change. To estimate the gain at $\omega = 1000$, we assume that a factor $\left|1 + \frac{j\omega}{a}\right|$ is equal to 1 if $\omega < a$ or else $\frac{\omega}{a}$ if $\omega > a$. This gives $|H(1000j)| \simeq \left|\frac{5(\omega/500)}{(\omega/100)(1)}\right| = 1 = 0\,$dB.

   (b) We have a LF asymptote of $2 = 6\,$dB. We have corner frequencies at $\omega = 100\,(-)$, $5000\,(+)$. Using the same technique as in part (a), $|H(1000j)| \simeq \left|\frac{2(1)}{(\omega/100)}\right| = \left|\frac{200}{1000}\right| = 0.2 = -14\,$dB.

   (c) We have corner frequencies at $\omega = 100\,(-)$, $500\,(+)$, $2000\,(-)$, $5000\,(-)$. We have a LF asymptote of $3j\omega = 6\,$dB which at the first corner ($\omega = 100$) is $300j = 50\,$dB. Using the same technique as in part (a), $|H(1000j)| \simeq \left|\frac{3\times\omega(\omega/500)}{(\omega/100)(1)(1)}\right| = \left|\frac{3000}{5}\right| = 600 = 55.6\,$dB. To obtain this expression from the transfer function, any term whose corner frequency is $> \omega$ has been replaced by (1).

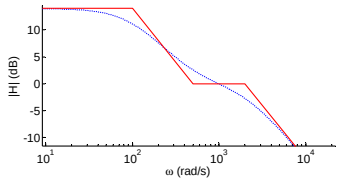


Fig. 8(a)

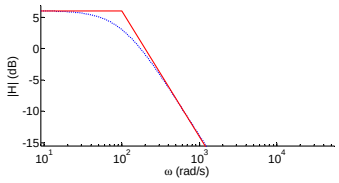


Fig. 8(b)

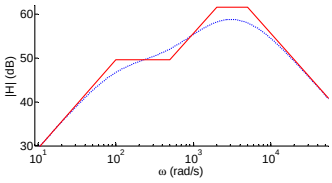


Fig. 8(c)

9. The corner frequency is $p = \frac{1}{\zeta RC} = 2\pi \times 1000$. Rearranging this gives $R = \frac{1}{2000\pi\zeta C} = 22508\,\Omega$. The upper resistor therefore has a value $\zeta^2 R = 0.5R = 11254\,\Omega$. The complete circuit is shown in Fig. 9. At $\omega = 100\,$Hz, 1 kHz and 10 kHz the transfer function is $-0.0099 + 0.0014j = 0.01\angle172°$, $0.707j = 0.707\angle90°$ and $0.9899 + 0.1414j = 1\angle8°$ respectively.

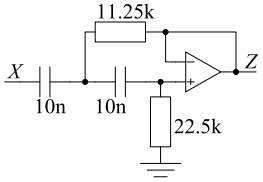


Fig. 9

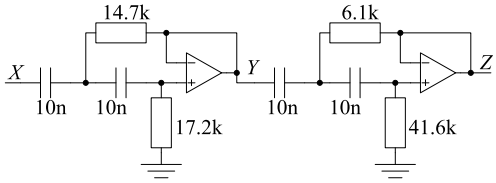


Fig. 10(a)

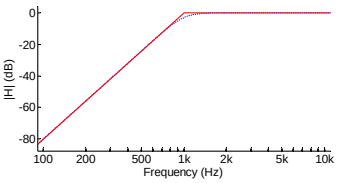


Fig. 10(b)

10. For a 4th order filter, we need $N = 2$ and from the formula given in the question, we use $\zeta_1 = \cos\left(\frac{\pi}{8}\right) = 0.924$ and $\zeta_2 = \cos\left(\frac{3\pi}{8}\right) = 0.383$. If we stick to $C = 10$ nF as in Q9, we obtain $R_1 = \frac{1}{2000\pi\zeta_1 C} = 17.2\,\text{k}\Omega$ and $R_2 = \frac{1}{2000\pi\zeta_2 C} = 41.6\,\text{k}\Omega$ with $\zeta_1^2 R_1 = 14.7\,\text{k}\Omega$ and $\zeta_2^2 R_2 = 6.1\,\text{k}\Omega$. This gives the circuit shown in Fig. 10(a). The transfer function is $H(j\omega) = \frac{\left(\frac{j\omega}{p}\right)^4}{\left(\left(\frac{j\omega}{P}\right)^2+2\zeta_1\left(\frac{j\omega}{p}\right)+1\right)\left(\left(\frac{j\omega}{P}\right)^2+2\zeta_2\left(\frac{j\omega}{p}\right)+1\right)}$; this is plotted in Fig. 10(b).

11. To shift the frequency response up by a factor of 5, we need to divide the value of each C or L component by 5. This gives the circuit of Fig. 11(a). We could also, if we wanted, multiply all the capacitor values by k and divide all the resistor values by k for any scale factor k without changing the transfer function. For this particular circuit, it would be a bad idea to use a value of $k > 1$ because, at $3\,\text{k}\Omega$ the feedback resistor is already a little on the low side for many op-amps (which have a limited current output capability).

    To reflect the magnitude response in the line $\omega_m = 10000$, we need to convert resistors into capacitors and vice-versa. From the notes, the formulae are: $R' = \frac{k}{\omega_m C}$, $C' = \frac{1}{\omega_m k R}$. For the circuit of Fig. 11(b), I have chosen $k = 3.33$ in order to get reasonable component values but other choices are also possible. A full analysis of this low-pass filter circuit is the subject of question 15.

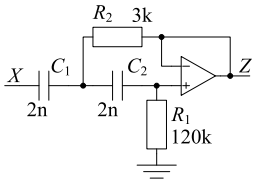


Fig. 11(a)

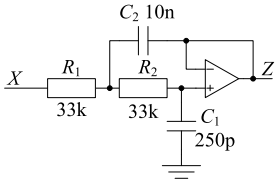


Fig. 11(b)

12. (a) The parallel combination of $C||(R+L)$ has an impedance $Z = \frac{\frac{1}{j\omega C}(R+j\omega L)}{\frac{1}{j\omega C}+R+j\omega L} = \frac{R+j\omega L}{1+j\omega RC+(j\omega)^2 LC}$. We want to find the value of ω that makes this real. The easiest way to do this is to insist that the ratio of imaginary to real part is the same for the numerator and denominator (this implies that they have the same argument). Thus $\frac{\omega_r L}{R} = \frac{\omega_r RC}{1-\omega_r^2 LC}$ from which cross multiplying (after dividing both numerators by ω_r) gives $L - \omega_r^2 L^2 C = R^2 C$ from which $\omega_r = \sqrt{\frac{L-R^2C}{L^2C}} = 9\,950\,\text{rad/s}$. Note that this is close, but not exactly equal to, $\omega_0 = 10\,000$ where the capacitor and inductor impedances have the same magnitude. The value of Z at resonance can now be found as the ratio between the real (or equivalently the imaginary) parts of the numerator and denominator of the previous expression. Thus $Z = \frac{R+j\omega L}{1+j\omega RC+(j\omega)^2 LC} = \frac{R}{1-\omega^2 LC} = \frac{j\omega L}{j\omega RC} = 1000$.

    (b) By definition Q equals ω_r times the average stored energy divided by the average power loss. If the input voltage phasor is V, then the peak energy stored in the capacitor is $\frac{1}{2}C\left|V\right|^2$ and its average stored energy is half this, namely $\frac{1}{4}C\left|V\right|^2$. The current through the resistor is $I_R = \frac{V}{R+j\omega_r L}$. The peak energy stored in the inductor is $\frac{1}{2}L\left|I_R\right|^2 = \frac{1}{2}L\frac{|V|^2}{R^2+\omega_r^2 L^2} = \frac{1}{2}L\frac{|V|^2}{R^2+\frac{L-R^2C}{L^2C}L^2} = \frac{1}{2}L\frac{C|V|^2}{R^2C+L-R^2C} = \frac{1}{2}C\left|V\right|^2$ which is the same as the peak capacitor energy; likewise, the average energy stored in the inductor is $\frac{1}{4}C\left|V\right|^2$. The average power loss in the resistor is $\frac{1}{2}R\left|I_R\right|^2 = \frac{RC}{2L}\left|V\right|^2$. Calculating Q from its definition gives $Q = \omega_r\frac{\frac{1}{4}C|V|^2+\frac{1}{4}C|V|^2}{\frac{RC}{2L}|V|^2} = \frac{\omega_r L}{R} = 9.95$. Since the capacitor and inductor store the same amount of energy on average, the Q can be determined more simply as $Q = \omega_r\frac{\frac{1}{2}L|I_R|^2}{\frac{1}{2}R|I_R|^2} = \frac{\omega_r L}{R} = 9.95$.

    (c) Note that the capacitor is unchanged in the two networks, so we can ignore it when matching their impedances. When choosing components to make two networks have the same impedance, your have a choice: you can either match their impedances or their admittances. You get the same answer in either case, but the algebra can sometimes be much simpler in one case than the other. In this question, it is easiest to use admittances because the components whose values are unknown are in parallel and so their admittances add: the total admittance of R_P and L_P in parallel is $\frac{1}{R_P} - \frac{j}{\omega_r L_P}$ and R_P and L_P remain unentangled in this expression. The admittance of $R_S + L_S$ is

$\frac{1}{R_S+j\omega_r L_S} = \frac{R_S-j\omega_r L_S}{R_S^2+\omega_r^2 L_S^2} = \frac{1}{R_P} - \frac{j}{\omega_r L_P}$. Equating the real and imaginary parts of this equation gives, $R_P = \frac{R_S^2+\omega_r^2 L_S^2}{R_S} = 1\,\text{k}\Omega$ and $L_P = \frac{R_S^2+\omega_r^2 L_S^2}{\omega_r^2 L_S} = L_S + \frac{R^2}{\omega_r^2 L_S} = 10.1\,\text{mH}$.

13. (a) At $\omega = 10\,000$, $Z_L = 100j$ and $Z_C = -100j$. Therefore the currents in L and C are equal and opposite. So the peak power supplied by V is the peak power absorbed by the resistor which equals $\frac{|V|^2}{R} = 100\,\text{mW}$.

    (b) The energy stored in the capacitor at time t is $W_C = \frac{1}{2}Cv(t)^2$. So the power absorbed by the capacitor is $\frac{dW_C}{dt} = Cv\frac{dv}{dt}$. Since you are told that the phasor $V = 10$, you know that the waveform $v(t) = 10\cos(\omega t)$ and, differentiating gives $\frac{dv}{dt} = -10\omega\sin(\omega t)$. Multiplying everything out gives $Cv\frac{dv}{dt} = -100\omega C\cos(\omega t)\sin(\omega t) = -50\omega C\sin(2\omega t)$. This has a peak value of $50\omega C = 500\,\text{mW}$. As is common in resonant circuits, this is 5 times greater than the answer to part (a).

14. (i) This is an inverting amplifier: $\frac{Y}{X} = -\frac{Z_F}{R} = -\frac{1}{R} \times \frac{2R}{1+2j\omega RC} = -\frac{2}{1+2j\omega RC}$.

(ii) This is a non-inverting amplifier: $\frac{Y}{X} = 1 + \frac{Z_F}{R} = 1 + \frac{2}{1+2j\omega RC} = \frac{3+2j\omega RC}{1+2j\omega RC}$.

(iii) This is the same as the previous circuit, but with an additional CR circuit at the input. $\frac{Y}{X} = \frac{4j\omega RC}{1+4j\omega RC} \times \frac{3+2j\omega RC}{1+2j\omega RC}$. This has corner frequencies at $\omega RC = \frac{1}{4}\,(-),\ \frac{1}{2}\,(-),\ \frac{3}{2}\,(+)$.

(iv) The circuit has negative feedback so we can assume $V_+ = V_- = 0$. KCL @ V_- gives: $\frac{0-X}{R} + \frac{0-Y}{R} + \frac{0-Z}{R} = 0$ from which $-Z = X + Y$. Now KCL @ Z gives: $(Z-0)j\omega C + \frac{Z-Y}{R} + \frac{Z}{R} = 0$ from which $Y - Z\,(2 + j\omega RC) = 0$. Substituting $-Z = X + Y$ gives $Y + (X+Y)\,(2 + j\omega RC) = 0$ from which $\frac{Y}{X} = -\frac{2+j\omega RC}{3+j\omega RC}$.

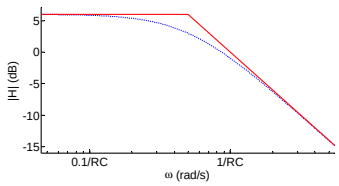


Fig. 14(i)

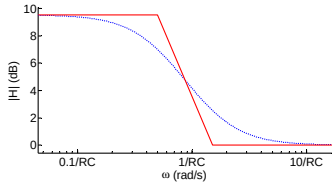


Fig. 14(ii)

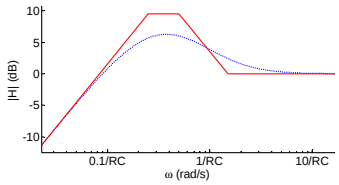


Fig. 14(iii)

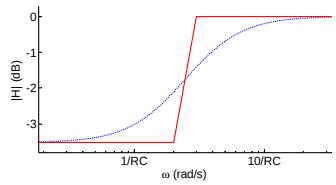


Fig. 14(iv)

15. In this circuit, the output, Y, is fed back to both V_+ and V_- so it is not immediately obvious that the overall feedback is negative. However, we see that $V_- = Y$ whereas $|V_+|$ will be attenuated by the network and will be $< Y$, so all is well. We can therefore assume that $Z = V_+ = V_- = Y$.

(a) $\frac{Z}{W}$ is just a potential divider, so $\frac{Z}{W} = \frac{Y}{W} = \frac{\frac{1}{j\omega C_1}}{R_2 + \frac{1}{j\omega C_1}} = \frac{1}{1+j\omega R_2 C_1}$. $Y = Z$ as noted above. From this we get $W = Y\,(1 + j\omega R_2 C_1)$.

(b) KCL @ W gives: $\frac{W-X}{R_1} + \frac{W-Z}{R_2} + (W-Y)\,j\omega C_2 = 0$ from which (substituting $Z = Y$),

$W\,(R_1 + R_2 + j\omega R_1 R_2 C_2) - Y\,(R_1 + j\omega R_1 R_2 C_2) - XR_2 = 0.$

Substituting the expression for W above gives

$Y\,(1 + j\omega R_2 C_1)\,(R_1 + R_2 + j\omega R_1 R_2 C_2) - Y\,(R_1 + j\omega R_1 R_2 C_2) = XR_2$

from which $Y\left(R_2 + j\omega R_2\,(R_1 + R_2)\,C_1 + (j\omega)^2\,R_1 R_2^2 C_1 C_2\right) = XR_2$.

Hence $\frac{Y}{X}\,(j\omega) = \frac{1}{R_1 R_2 C_1 C_2 (j\omega)^2 + (R_1+R_2) C_1 j\omega + 1}$.

(c) Squaring the expression for ζ gives $\zeta^2 = \frac{p^2 (R_1+R_2)^2 C_1^2}{4} = \frac{(R_1+R_2)^2 C_1^2}{4R_1 R_2 C_1 C_2}$ which gives $\frac{4R_1 R_2}{(R_1+R_2)^2} = \frac{C_1}{\zeta^2 C_2}$.

Using quite a common algebraic trick, we can write the numerator as the difference of two squares:

$$\frac{4R_1 R_2}{(R_1+R_2)^2} = \frac{(R_1+R_2)^2 - (R_1-R_2)^2}{(R_1+R_2)^2} = 1 - \left(\frac{R_1-R_2}{R_1+R_2}\right)^2 = 1 - \left(\frac{2R_1}{R_1+R_2} - 1\right)^2 = 1 - \left(\frac{2}{1+\frac{R_2}{R_1}} - 1\right)^2.$$

Rearranging $\frac{C_1}{\zeta^2 C_2} = 1 - \left(\frac{2}{1+\frac{R_2}{R_1}} - 1\right)^2$ gives $\frac{2}{1+\frac{R_2}{R_1}} = 1 + \sqrt{1 - \frac{C_1}{\zeta^2 C_2}}$ from which

$\left(1 + \frac{R_2}{R_1}\right)\left(1 + \sqrt{1 - \frac{C_1}{\zeta^2 C_2}}\right) = 2.$

The usefulness of this relationship is that it allows you to determine the resistor ratio, $\frac{R_2}{R_1}$, if you know the capacitor ratio $\frac{C_2}{C_1}$. For the square root to be a real number, we must have $1 - \frac{C_1}{\zeta^2 C_2} \geq 0$ which implies $C_1 \leq \zeta^2 C_2$.

(d) We must have $\frac{C_2}{C_1} \geq \frac{1}{\zeta^2} = 4$. Given our restricted choice of capacitor value, we must therefore choose $C_2 = 47$ nF and $C_1 = 10$ nF. So, substituting $\frac{C_1}{\zeta^2 C_2} = 0.851$ into the expression from the

previous part, we find $\left(1+\frac{R_2}{R_1}\right)\times 1.386 = 2$ from which $\frac{R_2}{R_1} = 0.443$. From the expression for p^2, we can write $0.443R_1^2 = R_1R_2 = \frac{1}{p^2C_1C_2} = 53.9\times 10^6$. Hence $R_1 = \sqrt{\frac{53.9\times 10^6}{0.443}} = 11\,\text{k}\Omega$ and $R_2 = 0.443R_1 = 4.9\,\text{k}\Omega$.

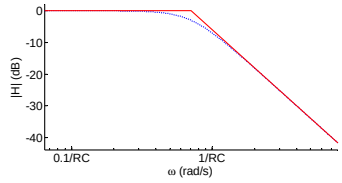


Fig. 15(mag)

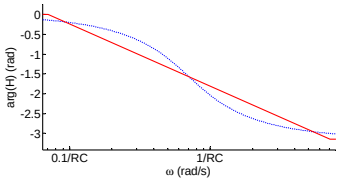


Fig. 15(phase)

16. (a) This circuit is a potential divider, so (setting $R = 20$) we can write down the transfer function: $\frac{Y}{X} = \frac{4R}{5R+j\omega L+\frac{1}{j\omega C}} = \frac{4j\omega RC}{1+5j\omega RC+(j\omega)^2LC} = \frac{2\zeta\left(\frac{j\omega}{a}\right)}{1+2\zeta\left(\frac{j\omega}{a}\right)+\left(\frac{j\omega}{a}\right)^2}$ where $a = \sqrt{\frac{1}{LC}} = 5000$ and $\zeta = 2.5aRC = 0.1$.

(b) To find the maximum of $\left|\frac{Y}{X}\right|$ it is easiest to find instead the maximum of $\left|\frac{Y}{X}\right|^2 = \frac{Y\times Y^*}{X\times X^*}$ where the * denotes the complex conjugate. Note that (i) a number multiplied by its complex conjugate is just the sum of the squares of its real and imaginary parts and that (ii) the magnitude of a complex fraction is the magnitude of the numerator divided by the magnitude of the denominator; very rarely is it necessary to multiply the top and bottom of a fraction by the complex conjugate of the denominator.

The difficult way to find the maximum is to differentiate the expression $\left|\frac{Y}{X}\right|^2 = \left|\frac{4j\omega RC}{1+5j\omega RC+(j\omega)^2LC}\right|^2 = \frac{(4\omega RC)^2}{(1-\omega^2LC)^2+(5\omega RC)^2}$ and set the derivative to zero. Much easier is to take the first expression above: $\left|\frac{Y}{X}\right|^2 = \left|\frac{4R}{5R+j\omega L+\frac{1}{j\omega C}}\right|^2 = \frac{16R^2}{25R^2+\left(\omega L-\frac{1}{\omega C}\right)^2}$. This is clearly maximized by making $\left(\omega L-\frac{1}{\omega C}\right) = 0$ which means $\omega_0 = \sqrt{\frac{1}{LC}}$. At this frequency $\frac{Y}{X} = 0.8 = -1.9\,\text{dB}$.

(c) The 3dB bandwidth is when $\left|\frac{Y}{X}\right|^2$ has fallen by a factor of 2. This will happen when $\left(\omega L-\frac{1}{\omega C}\right)^2 = 25R^2$ or $\omega L-\frac{1}{\omega C} = \pm 5R$. So we need to solve the quadratic equation $LC\omega^2 \pm 5RC\omega - 1 = 0$. The solution is $\omega = \frac{\pm 5RC\pm\sqrt{25R^2C^2+4LC}}{2LC}$ of which the positive solutions are $\omega = \frac{\pm 5RC+\sqrt{25R^2C^2+4LC}}{2LC}$. This gives $\omega_{3\text{dB}} = \{4525,\ 5525\}$. The bandwidth is the difference between these which is $\frac{10RC}{LC} = 1000\,\text{rad/s}$. Notice that ω_0 is the geometric mean of the two 3dB frequencies but is not the arithmetic mean which is 5025 rad/s. The Q (quality factor) of the resonance is $Q = \frac{1}{2\zeta} = 5$. This also equals the ratio of $\omega_0$to the bandwidth and the height of the peak above the intersection of the asymptotes.

The circles in Fig. 16 indicate ω_0 and the two 3dB frequencies.

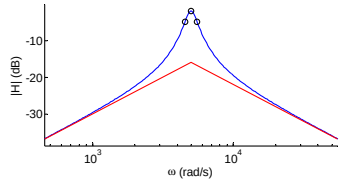


Fig. 16(mag)

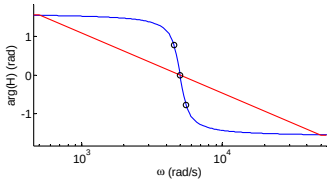


Fig. 16(phase)

E1.1 Circuit Analysis

Problem Sheet 6 (Lectures 14, 15 & 16)

Key: [A]= easy ... [E]=hard

Note: A tilde-superscript on a phasor denotes division by $\sqrt{2}$, i.e. $\widetilde{V} = \frac{1}{\sqrt{2}}V$. This means that $\left|\widetilde{V}\right|$ equals the RMS value of a phasor V.

1. [A] Say which of the following waveforms include negative exponentials and which include positive exponentials: (a) $2-4e^{-3t}$, (b) $2+4e^{3t}$, (c) $2+4e^{-3t}$, (d) $-2-4e^{3t}$, (e) $2+4e^{-t/-3}$.

2. [B] Suppose $v(t) = 5 + 2e^{-100t}$.
   (a) Determine the time constant, τ, of the negative exponential.
   (b) Determine the time at which $v(t) = 5.5\,\mathrm{V}$.
   (c) Give an expression for the time taken for $v(t)$ to fall from A to B where $5 < B < A < 7$.

3. [B] If $V = -200j$ in Fig. 3, find the phasor value of I and the complex power absorbed by each of the components including the voltage source.

4. [B] If $v(t) = \begin{cases} 0 & t < 0 \\ 5 & t \geq 0 \end{cases}$ in Fig. 4 (below),
   (a) find an expression for $x(t)$ for $t \geq 0$.
   (b) Sketch a graph of $x(t)$ for $-RC \leq t \leq 3RC$.
   (c) Determine the time at which $x(t) = 4.5$.

5. [C] For each of the circuits shown in Fig. 5(i)-(vi) determine (a) the time constant (b) the DC gain $\frac{Y}{X}\big|_{\omega=0}$ and (c) the high frequency gain $\frac{Y}{X}\big|_{\omega=\infty}$. In each case, determine these in two ways: directly from the circuit and via the transfer function.

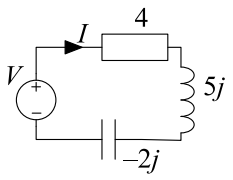


Fig. 3

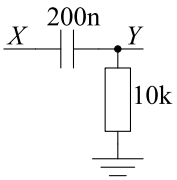


Fig. 5(i)

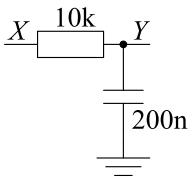


Fig. 5(ii)

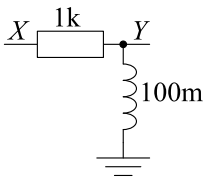


Fig. 5(iii)

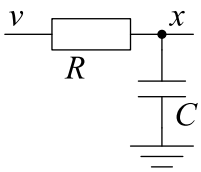


Fig. 4

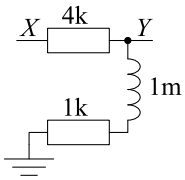


Fig. 5(iv)

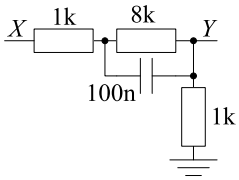


Fig. 5(v)

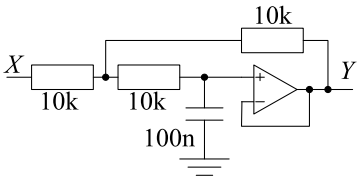


Fig. 5(vi)

6. [C] For each of the periodic waveforms shown in Fig. 6(i)-(iii) determine (a) the mean value and (b) the rms value.

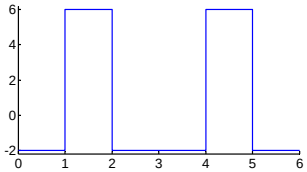


Fig. 6(i)

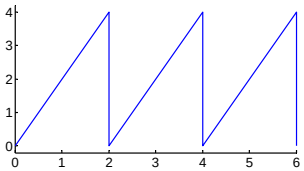


Fig. 6(ii)

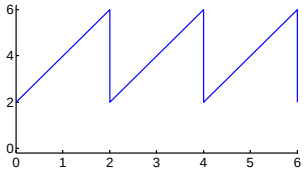


Fig. 6(iii)

7. [C] If $v(t) = \begin{cases} 0 & t < 0 \\ 5 & t \geq 0 \end{cases}$ in Fig. 7, determine an expression for $x(t)$ for $t \geq 0$ and sketch its graph.

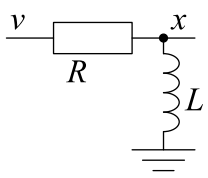


Fig. 7

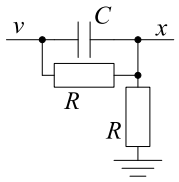


Fig. 8

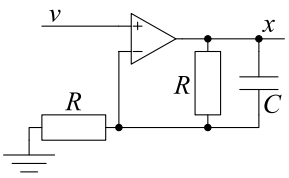


Fig. 9

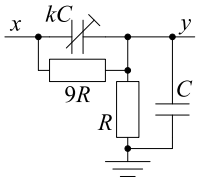


Fig. 10

8. [C] If $v(t) = \begin{cases} 2 & t < 0 \\ 6 & t \geq 0 \end{cases}$ in Fig. 8, determine an expression for $x(t)$ for $t \geq 0$ and sketch its graph.

9. [C] If $v(t) = \begin{cases} 4 & t < 0 \\ 1 & t \geq 0 \end{cases}$ in Fig. 9, find an expression for $x(t)$ for $t \geq 0$ and sketch a graph of $x(t)$ for the time interval $-RC \leq t \leq 3RC$.

10. [C] Fig. 10 shows a simplified circuit diagram for an oscilloscope probe which includes an adjustable capacitor of value kC.
    (a) Determine the transfer function, $\frac{Y}{X}(j\omega)$ and determine its value at $\omega = 0$ and $\omega = \infty$.
    (b) Determine the time constant of the circuit.
    (c) Determine an expression for $y(t)$ if $x(t) = \begin{cases} 0 & t < 0 \\ 10 & t \geq 0 \end{cases}$.
    (d) The variable capacitance, kC, is adjusted to the value that makes the amplitude of the transient equal to zero. Determine the value of k that achieves this.
    (e) Simplify the expression for $\frac{Y}{X}(j\omega)$ when k has the value calculated in the previous part.

11. [C] In the diagram of Fig. 11 power is being transmitted from a source to a load via two transformers having turns ratios of $1 : n$ and $n : 1$ respectively.
    (a) If $\widetilde{V}_L = 240\,\text{V}$ and the average power dissipated in R_L is $10\,\text{kW}$, calculate the value of R_L.
    (b) If $R_S = 0.5\,\Omega$, calculate the power dissipated in R_S when (i) $n = 1$ and (ii) $n = 5$.

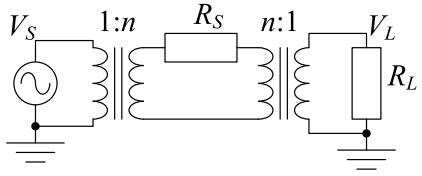


Fig. 11

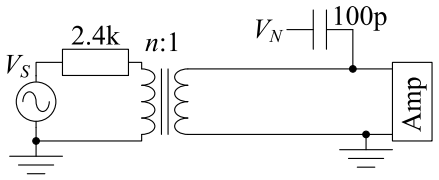


Fig. 12

12. [C] The circuit in Fig. 12 represents a microphone connected to an amplifier via a transformer and a long cable.
    (a) Determine the Thévenin output impedance of the microphone+transformer combination when $n = 4$.
    (b) The cable is subject to 50 Hz interference capacitively coupled from the mains, $\widetilde{V}_N = 230\,\text{V}$, via a capacitor of value 100 pF. If the RMS microphone signal amplitude is $\widetilde{V}_S = 1\,\text{V}$, calculate the ratio of the signal and the noise at the amplifier in dB if (i) $n = 1$ and (ii) $n = 4$.

13. [C] In the circuit of Fig. 13, the transformer may be assumed to be ideal.
    (a) Calculate the average power dissipated in each of R_1 and R_2 if $\widetilde{V}_s = 1$, $n_1 = 2$, $n_2 = 3$, $R_1 = 10$ and $R_2 = 20$.
    (b) Calculate, in terms of n_1, n_2, R_1 and R_2, the effective resistance seen by the voltage source.

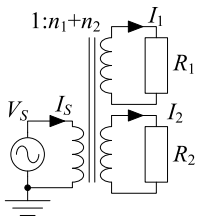


Fig. 13

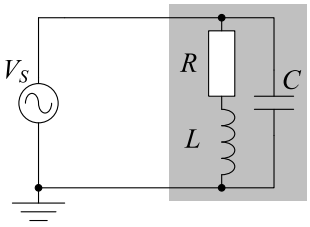


Fig. 14

14. [C] In the circuit of Fig. 14, $\widetilde{V}_S = 230$ at 50 Hz, $L = 8\,\text{mH}$ and $R = 1.6\,\Omega$.
    (a) If $C = 0$ (i.e. the capacitor is omitted), calculate the apparent power, average power and reactive power absorbed by the load (shaded region) and also its power factor.
    (b) Determine the value of C needed to increase the power factor to 0.9. Using this value, recalculate the quantities from part (a).

15. [D] Calculate the waveform $y(t)$ in Fig. 15(i) when,
    (a) $v(t) = \begin{cases} 0 & t \le 0 \\ 5\sin 2000\pi t & t > 0 \end{cases}$ as shown in Fig. 15(ii).
    (b) $v(t) = \begin{cases} 0 & t \le 0 \\ 5\sin 2000\pi t & 0 < t \le 1\,\text{ms} \\ 0 & t > 1\,\text{ms} \end{cases}$ as shown in Fig. 15(iii)

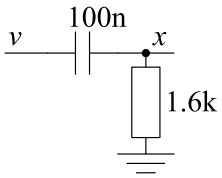


Fig. 15(i)

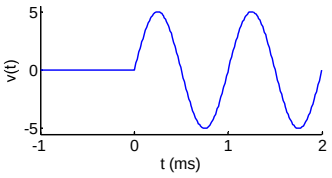


Fig. 15(ii)

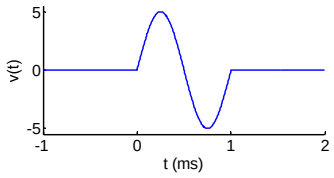


Fig. 15(iii)

16. [D] If the switch in Fig. 16 is $\begin{cases} \text{open} & t < 0 \\ \text{closed} & 0 \le t < 2\,\text{ms} \\ \text{open} & t \ge 2\,\text{ms} \end{cases}$, determine expressions for $i(t)$ for each of these periods and sketch graphs of $i(t)$ and $v(t)$ for $-1\,\text{ms} \le t \le 4\,\text{ms}$.

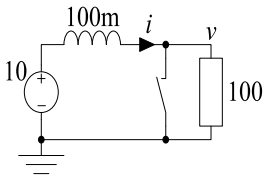


Fig. 16

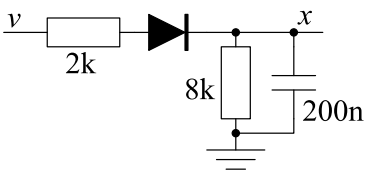


Fig. 17(i)

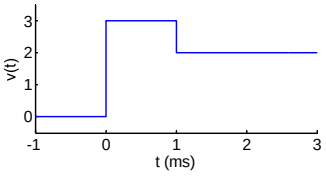


Fig. 17(ii)

17. [E] In Fig. 17(i), $v(t) = \begin{cases} 0 & t < 0 \\ 3 & 0 \le t < 1\,\text{ms} \\ 2 & t \ge 1\,\text{ms} \end{cases}$ as shown in Fig. 17(ii). If the diode has a forward voltage drop of 0.7 V, find expressions for $x(t)$ for $t \ge 0$.

E1.1 Circuit Analysis

Problem Sheet 6 - Solutions

1. (a) Negative, (b) Positive, (c) Negative, (d) Positive, (e) Positive.

2. (a) The time constant is $\frac{1}{100} = 10\,\text{ms}$. (b) We need to solve $5.5 = 5 + 2e^{-100t} \Rightarrow e^{-100t} = 0.25 \Rightarrow -100t = \ln 0.25 = -1.386 \Rightarrow t = 13.86\,\text{ms}$. Alternatively, we can use the standard formula, derived in lectures, $t = \tau \ln\left(\frac{7-5}{5.5-5}\right) = 10 \times \ln 4 = 13.86\,\text{ms}$. (iii) The general formula, derived in lectures, is $T_{A\to B} = \tau \ln\left(\frac{A-5}{B-5}\right)$.

3. The current is $I = \frac{-200j}{4+5j-2j} = \frac{-200j}{4+3j} = -24 - 32j = 40\angle -127°$. So $\left|\widetilde{I}\right|^2 = \frac{40^2}{2} = 800$. The complex power absorbed by each of the passive components is $\left|\widetilde{I}\right|^2 Z$; this gives $\left|\widetilde{I}\right|^2 R = 3.2\,\text{kW}$, $\left|\widetilde{I}\right|^2 Z_L = 4000j = 4\,\text{kVAR}$ and $\left|\widetilde{I}\right|^2 Z_C = -1600j = -1.6\,\text{kVAR}$. The current through the source (following the passive sign convention) is $-I = 24 + 32j$ and the complex power absorbed by it is $\widetilde{V}\left(-\widetilde{I}\right)^* = -141j\,(17 - 22.6j) = (-3.2 - 2.4j)$ kVA. As expected, the total complex power sums to zero.

4. (a) The DC gain of the circuit is 1, so the steady state output is $x_{SS}(t) = \begin{cases} 0 & t < 0 \\ 5 & t \geq 0 \end{cases}$. Because x is the voltage across a capacitor, it must be continuous, so $x(0+) = x(0-) = 0$. So the complete expression is $x(t) = x_{SS}(t) + (x(0+) - x_{SS}(0+))\,e^{\frac{-t}{\tau}} = 5 - 5e^{\frac{-t}{\tau}}$ where $\tau = RC$. (b) $x(t)$ is plotted in Fig. 4.

   (c) Using the standard formula, $t = \tau \ln\left(\frac{0-5}{4.5-5}\right) = \tau \ln(10) = 2.3RC$.

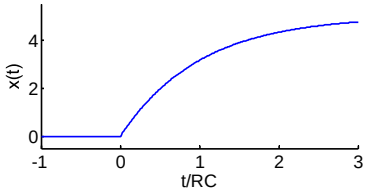


Fig. 4

5. (i) From the circuit, the time constant is $RC = 2\,\text{ms}$. The DC gain may be obtained by treating C as an open circuit and is 0; the HF gain may be obtained by treating C as a short-circuit and is therefore 1. The transfer function (using potential divider formula) is $\frac{Y}{X}(j\omega) = \frac{j\omega RC}{1+j\omega RC}$ which happily gives the same values: $\frac{Y}{X}(0) = 0$, $\frac{Y}{X}(\infty) = 1$, $\tau = \frac{1}{\text{denominator corner frequency}} = RC$. The importance of $\frac{Y}{X}(\infty)$ is that it gives the gain for a step input discontinuity, i.e. $\frac{Y}{X}(\infty) = \frac{\text{output discontinuity}}{\text{input discontinuity}}$.

   (ii) From the circuit: $\tau = RC = 2\,\text{ms}$, DC gain (C open-circuit) = 1, HF gain (C short-circuit) = 0. The transfer function (using potential divider formula) is $\frac{Y}{X}(j\omega) = \frac{1}{1+j\omega RC}$ which gives the same values.

   (iii) From the circuit: $\tau = \frac{L}{R} = 0.1\,\text{ms}$, DC gain ($L$ short-circuit) = 0, HF gain (L open-circuit) = 1. The transfer function (using potential divider formula) is $\frac{Y}{X}(j\omega) = \frac{j\omega L}{R+j\omega L}$ which gives the same values. Note that if the denominator is $(p + j\omega q)$, the time constant is $\frac{q}{p}$ and the corner frequency is $\frac{p}{q}$.

   (iv) To obtain the time constant, we need to determine the Thévenin resistance seen by the inductor. To do this, we set the input voltage, X, to zero (thereby shorting node X to ground) and find the resistance between the inductor terminals (with the inductor removed). The two resistors are in series, so we get $R_{Th} = 1 + 4 = 5\,\text{k}$. From this, $\tau = \frac{L}{R} = 200\,\text{ns}$, DC gain ($L$ short-circuit) = 0.2 (potential divider), HF gain (L open-circuit) = 1. The transfer function (using potential divider formula) is $\frac{Y}{X}(j\omega) = \frac{R+j\omega L}{5R+j\omega L}$ where $R = 1\,\text{k}$. This gives the same values and is, perhaps, a marginally easier way to determine them.

(v) To determine the Thévenin resistance seen by the capacitor, we set $X = 0$ and measure the resistance at the capacitor terminals. Since X is connected to ground, the two 1 k resistors are in series and so we have 8 k in parallel with 2 k which gives 1.6 k. From this, $\tau = RC = 0.16$ ms, DC gain (C open-circuit) $= 0.1$ (potential divider), HF gain (C short-circuit) $= 0.5$. The transfer function (using potential divider formula) is $\frac{Y}{X}(j\omega) = \frac{R}{2R+\frac{8R}{1+8j\omega RC}} = \frac{1+8j\omega RC}{10+16j\omega RC}$ where $R = 1$ k and we used the formula for $Z_{8R||C} = \frac{8R}{1+j\omega 8RC}$.

(vi) To determine the Thévenin resistance at the capacitor terminals is not trivial because of the dependent voltage source that is the opamp. If we set $X = 0$ and replace the capacitor with a voltage source V as in Fig. 5(i), we can use nodal analysis to determine I and then calculate $R_{Th} = \frac{V}{I}$. Since the op-amp is a unit-gain buffer, $Y = V$. KCL at node W gives: $\frac{W}{10} + \frac{W-V}{10} + \frac{W-V}{10} = 0 \Rightarrow W = \frac{2}{3}V \Rightarrow I = \frac{V-W}{10} == \frac{\frac{1}{3}V}{10} = \frac{V}{30} \Rightarrow R_{Th} = 30$ k. So, finally, we get $\tau = R_{Th}C = 3$ ms. For the DC gain, we make C an open-circuit as in Fig. 5(ii). Negative feedback means $V_+ = Y$ and, since there is no current through the resistor connected to V_+, we must also have $W = Y$. KCL at node W then gives $W = Y = X$, so the DC gain is 1. For the HF gain, the capacitor acts a a short circuit so $V_+ = 0$ which in turn means that $Y = 0$ so the gain is 0.

Rather easier is the transfer function approach. We know $V_+ = Y$ and V_+ is determined from W by an RC potential divider giving: $\frac{Y}{W} = \frac{V_+}{W} = \frac{1}{1+j\omega RC}$. KCL at W gives $\frac{W-X}{R} + \frac{W-Y}{R} + \frac{W-Y}{R} = 0 \Rightarrow 3W - 2Y = X$. We now substitute for W using the previous equation $\frac{Y}{W} = \frac{1}{1+j\omega RC}$ to get $3Y(1+j\omega RC) - 2Y = X \Rightarrow \frac{Y}{X} = \frac{1}{1+3j\omega RC}$. From this we can easily get: $\tau = 3RC$, $\frac{Y}{X}(0) = 1$ and $\frac{Y}{X}(\infty) = 0$.

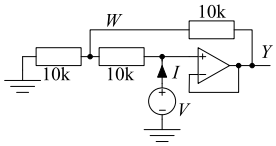


Fig. 5(i)

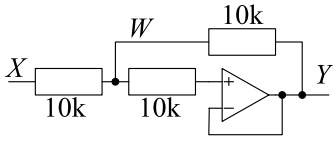


Fig. 5(ii)

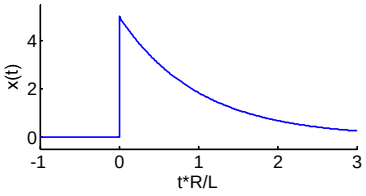


Fig. 7

6. (i) $v(t)$ equals 6 for $\frac{1}{3}$ of the time and -2 for $\frac{2}{3}$ of the time. So its average value is $\overline{v} = \frac{1}{3}\times 6 + \frac{2}{3}\times(-2) = \frac{2}{3}$. Similarly, the average value of v^2 is $\overline{v^2} = \frac{1}{3}\times 36 + \frac{2}{3}\times 4 = 14\frac{2}{3}$. So $V_{rms} = \sqrt{14.67} = 3.83$. This is higher than the average value $\overline{v}$.

   (ii) During the first period ($0 \le t \le 2$), the formula for v can be derived as $v = 2t$. To find the average value, we integrate over one period, and divide by the length of the period. So $\overline{v} = \frac{1}{2}\int_{t=0}^{2} 2t\,dt = \frac{1}{2}\left[t^2\right]_0^2 = 2$. This is also pretty obvious from looking at the waveform. In the same way, $\overline{v^2} = \frac{1}{2}\int_{t=0}^{2} (2t)^2\,dt = \frac{1}{2}\left[\frac{4}{3}t^3\right]_0^2 = 5\frac{1}{3}$ giving $V_{rms} = \sqrt{5.33} = 2.31$.

   (iii) This is the same as the previous waveform but shifted up by +2. You can perform integrations similar to the previous part or, easier, just modify the previous answers. $\overline{v}$ (which previously equalled 2) will be increased by 2 to become $\overline{v} = 4$. Adding a constant onto a random variable does not affect its variance, so $\overline{v^2} - (\overline{v})^2$ will be unchanged at $5\frac{1}{3} - 2^2 = 1\frac{1}{3}$. It follows that $\overline{v^2} = (\overline{v})^2 + 1\frac{1}{3} = 17\frac{1}{3}$. Taking the square root gives $V_{rms} = \sqrt{17.33} = 4.16$.

7. Method 1 (inductor current continuity): For $t < 0$, $x = 0$ and the current through the inductor is $i = \frac{v-x}{R} = 0$. It follows that at time $t = 0+$, the current through the resistor (which equals the current through the inductor) will still be zero and $x(0+) = v(0+) = 5$. From this value is will decay to a steady state value $x_{SS} = 0$ since the inductor is a short circuit for DC. Thus, $x(t) = x_{SS}(t) + (x(0+) - x_{SS}(0+))\,e^{\frac{-t}{\tau}} = 0 + 5e^{\frac{-t}{\tau}}$ where $\tau = \frac{L}{R}$. This is plotted in Fig. 7.

   Method 2 (transfer function): The transfer function of the circuit is (from potential divider equation) $\frac{X}{V} = \frac{j\omega L}{R+j\omega L}$. From this we get the DC gain, $G_{DC} = 0$, the HF gain, $G_{HF} = 1$, and the time constant is $\frac{L}{R}$. The DC gain allows us to calculate the steady state $x_{SS}(t) = G_{DC}v(t) \equiv 0$. The output discontinuity at $t = 0$ is given by $\Delta x = G_{HF}\Delta v = 1\times 5 = 5$. So $x(0+) = x_{SS}(0-) + \Delta x = 0 + 5 = 5$. Finally we put everything together to get: $x(t) = x_{SS}(t) + (x(0+) - x_{SS}(0+))\,e^{\frac{-t}{\tau}} = 0 + (5-0)\,e^{\frac{-t}{\tau}}$.

8. Method 1 (capacitor voltage continuity): The time constant of the circuit is obtained by setting $v = 0$ and finding the Thévenin resistance across the capacitor terminals. Since v is connected to ground, the two resistors are in parallel and $R_{Th}{=}\frac{1}{2}R$ giving $\tau = \frac{1}{2}RC$. The DC gain of the circuit is 0.5, so $x_{SS}(t) = 0.5v(t) = \begin{cases} 1 & t < 0 \\ 3 & t \geq 0 \end{cases}$. For $t < 0$, the capacitor voltage is $v - x = 2 - 1 = 1$. This must remain continuous and so $v(0+) - x(0+) = 1 \Rightarrow x(0+) = v(0+) - 1 = 5$. Putting everything together, we get $x(t) = x_{SS}(t) + (x(0+) - x_{SS}(0+))\, e^{\frac{-t}{\tau}} = 3 + (5-3)\, e^{\frac{-t}{\tau}} = 3 + 2e^{\frac{-t}{\tau}}$. This is plotted in Fig. 8.

   Method 2 (transfer function): The transfer function of the circuit is (from potential divider equation) $\frac{X}{V} = \frac{R}{R+\frac{R}{1+j\omega RC}} = \frac{1+j\omega RC}{2+j\omega RC}$. From this we get the DC gain, $G_{DC} = 0.5$, the HF gain, $G_{HF} = 1$, and the time constant is $0.5RC$. The DC gain allows us to calculate the steady state as above. The output discontinuity at $t = 0$ is given by $\Delta x = G_{HF}\Delta v = 1 \times 4 = 4$. So $x(0+) = x_{SS}(0-) + \Delta x = 1 + 4 = 5$. Finally we put everything together to get: $x(t) = x_{SS}(t) + (x(0+) - x_{SS}(0+))\, e^{\frac{-t}{\tau}} = 3 + (5-3)\, e^{\frac{-t}{\tau}} = 3 + 2e^{\frac{-t}{\tau}}$.

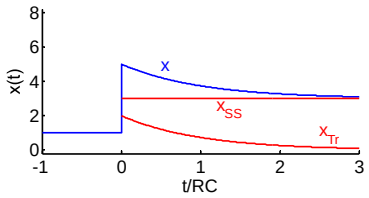


Fig. 8

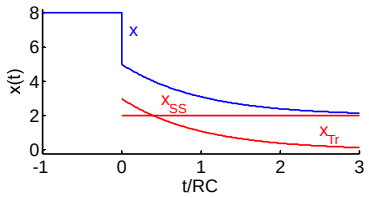


Fig. 9

9. For opamp circuits, it is easiest to use the transfer function to determine the relevant circuit parameters. This is a non-inverting amplifier with a gain of $\frac{X}{V} = 1 + \frac{\frac{R}{1+j\omega RC}}{R} = \frac{2+j\omega RC}{1+j\omega RC}$. Thus we have a DC gain, $G_{DC} = 2$, a high frequency gain $G_{HF} = 1$ and a time constant $\tau = RC$. At $t = 0$, the input discontinuity is $\Delta V = v(0+) - v(0-) = -3$ and so the ouput discontinuity is $\Delta X = x(0+) - x(0-) = G_{HF}\Delta V = 1 \times -3 = -3$. The steady state output is given by $x_{SS}(t) = G_{DC}v(t) = \begin{cases} 8 & t < 0 \\ 2 & t \geq 0 \end{cases}$. So this gives $x(0+) = -3 + x(0-) = -3 + 8 = 5$.

   Putting everything together, we get $x(t) = x_{SS}(t) + (x(0+) - x_{SS}(0+))\, e^{\frac{-t}{\tau}} = 2 + (5-2)\, e^{\frac{-t}{\tau}} = 2 + 3e^{\frac{-t}{RC}}$.

10. (a) The impedance of $R||C$ is $\frac{R}{1+j\omega RC}$. We can regard the circuit as a potential divider and so the gain is $\frac{Y}{X} = H(j\omega) = \frac{\frac{R}{1+j\omega RC}}{\frac{9R}{1+j9k\omega RC} + \frac{R}{1+j\omega RC}}$ which simplifies to $H(j\omega) = \frac{1+j9k\omega RC}{10+j9(k+1)\omega RC}$. The DC gain is $H(0) = 0.1$ and the high frequency gain is $H(\infty) = \frac{k}{k+1}$.

    (b) The time constant is the reciprocal of the denominator corner frequency and is $\tau = 0.9(k+1)RC$.

    (c) The steady state output for an input of $x = 10$ is $y_{SS}(t) = H(0) \times 10 = 1$. The input step is $\Delta x = 10$ and so the output step is $\Delta y = H(\infty)\Delta x = \frac{10k}{k+1}$. Since $y(t) = 0$ for $t < 0$, it follows that $y(0+) = \frac{10k}{k+1}$. So the output for $t > 0$ is $y(t) = 1 + \left(\frac{10k}{k+1} - 1\right) e^{-\frac{t}{\tau}} = 1 + \frac{9k-1}{k+1}e^{-\frac{t}{\tau}}$.

    (d) The value of k that makes the transient amplitude zero is $k = \frac{1}{9}$. Note that the transient amplitude will be positive or negative according to whether k is greater or less than this value.

    (e) If $k = \frac{1}{9}$ then $H(j\omega) = 0.1$ and is independent of ω.

11. (a) Average power is $\frac{\widetilde{V}^2}{R_L}$, so $R_L = \frac{\widetilde{V}^2}{10\,\mathrm{k}} = 5.76\,\Omega$. (b) Current through R_L is $\widetilde{I}_L = \frac{\widetilde{V}}{R} = 41.7\,\mathrm{A}$. The current through R_S is $\frac{\widetilde{I}_L}{n}$ so the power dissipation is $\frac{\widetilde{I}_L^2 R_S}{n^2}$. This gives (i) 868 W and (ii) 34.8 W.

12. (a) Impedances are transformed by the square of the turns ratio because the voltage decreases by n and the current increases by n so that the ratio of voltage over current decreases by n^2. So when $n = 4$, the impedance at the output of the secondary is $\frac{2400}{16} = 150\,\Omega$.

(b) At 50 Hz, the capacitor impedance is $Z_C = \frac{1}{j2\pi 50\times 100\,\mathrm{p}} = -j31.8\,\mathrm{M\Omega}$. With the transformed source impedance from part (a), we get the equivalent circuit shown in Fig. 12. Using superposition, $\widetilde{V}_A = \widetilde{V}_N \times \frac{Rn^{-2}}{Rn^{-2}+Z_C} + \widetilde{V}_S n^{-1} \times \frac{Z_C}{Rn^{-2}+Z_C} = \frac{n^{-1}Z_C\widetilde{V}_S + Rn^{-2}\widetilde{V}_N}{Rn^{-2}+Z_C}$. The ratio of the signal and noise voltages is equal to the ratio of the two terms in the numerator, so the ratio of the signal and noise voltage magnitudes is $\frac{n^{-1}|Z_C|\left|\widetilde{V}_S\right|}{Rn^{-2}\left|\widetilde{V}_N\right|} = \frac{31.8\,\mathrm{M}\times 1}{2400\times 230} \times n = 57.66 \times n$. Converting this to decibels gives $20\log_{10} 57.66 + 20\log_{10} n = 35.2 + 20\log_{10} n$. This gives (i) 35.2 dB for $n = 1$ and (ii) 47.3 dB for $n = 4$. So, slightly surprisingly, using a transfomer to *reduce* the voltage coming from the microphone actually makes the signal-to-noise ratio better.

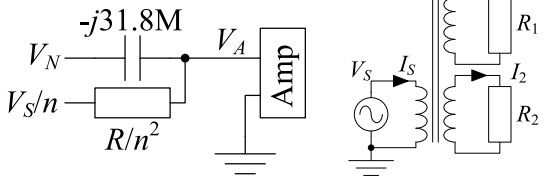


Fig. 12

Fig. 13

13. (a) $\widetilde{V}_1 = n_1\widetilde{V}_S = 2$ so the average power dissipated in R_1 is $\frac{\widetilde{V}_1^2}{R_1} = 400\,\mathrm{mW}$. Similarly, $\widetilde{V}_2 = n_2\widetilde{V}_S = 3$, so the average power dissipated in R_2 is $\frac{\widetilde{V}_2^2}{R_2} = 450\,\mathrm{mW}$.

    (b) From the ideal transformer equations, $1\times I_S + n_1\times(-I_1) + n_2\times(-I_2) = 0$ (the minus signs arise because in Fig. 13 I_1 and I_2 are defined as coming out of the transformer). Rearranging this and using Ohm's law gives $I_S = n_1 I_1 + n_2 I_2 = \frac{n_1 V_1}{R_1} + \frac{n_2 V_2}{R_2} = \frac{n_1^2 V_S}{R_1} + \frac{n_2^2 V_S}{R_2}$. So $R_{eff} = \frac{V_S}{I_S} = \frac{1}{\frac{1}{n_1^{-2}R_1} + \frac{1}{n_2^{-2}R_2}}$.

    This is the same as the parallel combination of $n_1^{-2}R_1$ and $n_2^{-2}R_2$, i.e. the parallel combination of the individual winding resistances transferred from the secondaries to the primary.

14. (a) The inductor impedance is $j\omega L = j\times 100\pi\times 0.008 = 2.51\,\Omega$. So the current is $I = \frac{V_S}{R+j\omega L} = 41.5 - 65.1j$. So $S = VI^* = P + jQ = 9.54 + 14.98j\,\mathrm{kVA}$. So the apparent power is $|S| = 17.8\,\mathrm{kVA}$and the average and reactive powers are P and Q given earlier. The power factor is $\cos\phi = \frac{P}{|S|} = 0.54$.

    (b) Adding the capacitor will not consume any average power and so will not affect P at all. We need to reduce Q to $P\tan(\arccos 0.9) = 4.62\,\mathrm{kVAR}$ since $\tan\phi = \frac{Q}{P}$ and we want $\cos\phi = 0.9$. It follows that $Q_C = 4.62 - 14.98 = -10.36\,\mathrm{kVAR} = \frac{-|V_S|^2}{|Z_C|} = -230^2\omega C$. This gives $C = 623\,\mu\mathrm{F}$. Now $S = P + jQ = 9.54 + 4.62j\,\mathrm{kVA}$. So the apparent power is $|S| = 10.6\,\mathrm{kVA}$ and the average and reactive powers are P and Q given earlier. The power factor is $\cos\phi = \frac{P}{|S|} = 0.9$.

15. Notice first that since there are no input discontinuities, there will be no output discontinuities either. The circuit transfer function is $\frac{X}{V} = \frac{j\omega RC}{1+j\omega RC}$. The gain at $\omega = 2000\pi$ is $G = 0.503 + 0.5j = 0.709\angle 44.8°$. The phasor corresponding to $v(t) = 5\sin\omega t$ is $V = -5j$ and so the steady state output will be $X = GV = 2.5 - 2.51j = 3.54\angle -45.2°$ which corresponds to a waveform $x_{SS}(t) = 2.5\cos\omega t + 2.51\sin\omega t$. The time constant is $RC = 0.16\,\mathrm{ms}$.

    (a) At time $t = 0+$ we have $x_{SS}(0+) = 2.5\,\mathrm{V}$. Since there is no output discontinuity, $x(0+) = x(0-) = 0$. Putting this together gives $x(t) = x_{SS}(t) + (x(0+) - x_{SS}(0+))\,e^{\frac{-t}{\tau}} = 2.5\cos\omega t + 2.51\sin\omega t + (0-2.5)\,e^{\frac{-t}{RC}}$. This is plotted in Fig. 15(i).

    (b) Substituting $t = 1$ into the previous expression gives $x(1-) = x(1+) = 2.495$ (very close to the steady state value since it has had $6\frac{1}{4}$ time constants to converge). For $t > 1$, the steady state is $x_{SS}(t) \equiv 0$. Therefore we get $x(t) = 2.495e^{\frac{-(t-1)}{\tau}}$. This is plotted in Fig. 15(ii).

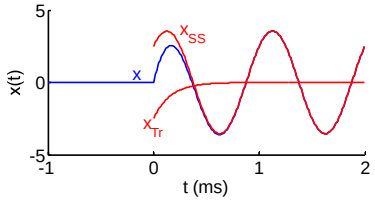


Fig. 15(i)

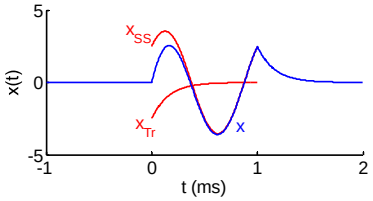


Fig. 15(ii)

16. For the steady state when $t < 0$, we can treat the inductor as a short circuit and so $i = \frac{10}{R} = 100\,\text{mA}$. When the switch is closed, there is a constant 10 V across the inductor and so $\frac{di}{dt} = \frac{V}{L} = 100\,\text{V/s}$. Therefore the current through the inductor will increase linearly at this rate for 2 ms (from an initial value of 100 mA) and will reach a value of 300 mA. Since there is no resistor in series with the inductor, the current increases linearly rather than exponentially; you can, if you wish, regard this as a limiting case of a negative exponential that has an infinite time constant.

    When the switch is opened at $t = 2$ ms, the current will decay from its peak value of 300 mA back down to its steady state value of 100 mA with a time constant of $\frac{L}{R} = 1\,\text{ms}$. Thus for $t > 2\,\text{ms}$, we have $i(t) = 100 + (300 - 100)\,e^{\frac{-(t-2)}{1}}$ (in units of mA and ms). All this is plotted in Fig. 16(i).

    When the switch is open, $v(t) = Ri(t)$. However, when the switch is closed, $v(t) \equiv 0$. We therefore get the voltage waveform plotted in Fig. 16(v).

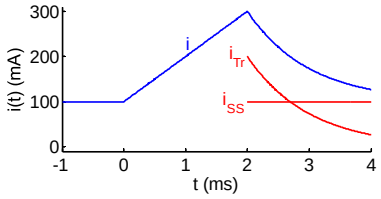


Fig. 16(i)

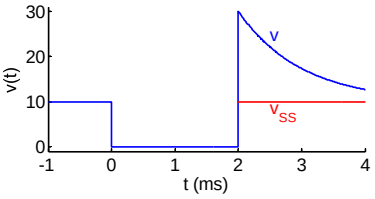


Fig. 16(v)

17. In this question, we have two different circuits according to whether the diode is off or on. These are shown in Fig. 17(off),(on). When the diode is off, we have a DC steady state $x_{SS}(t) = 0$ and a time constant $\tau_{Off} = RC = 1.6\,\text{ms}$. On the other hand, when the diode is on, we can get the DC steady state by doing KCL for the shaded supernode: $\frac{x+0.7-v}{2} + \frac{x}{8} = 0 \Rightarrow x_{SS} = \frac{4}{5}(v - 0.7)$. We obtain the time constant by setting all voltage sources to zero and finding the Thévenin resistance and the capacitor terminals: this is $R_{Th} = 2\,\text{k}||8\,\text{k} = 1.6\,\text{k}$; this gives a time constant $\tau_{On} = 0.32\,\text{ms}$.

    For $t < 0$, $v = x = 0$ and so, since $v - x < 0.7$, the diode will be off. When v changes to 3, the diode will turn on and will charge the capacitor up to a steady state voltage of $\frac{4}{5}(3 - 0.7) = 1.84$. When v now changes to 2 V, the diode will turn off and x will fall towards the "off" steady state of 0 V. However, it will never reach this value, because when x reaches $v - 0.7 = 1.3\,\text{V}$ the diode will turn on again resulting in a new steady state of $\frac{4}{5}(2 - 0.7) = 1.04\,\text{V}$. So this means we actually have four distinct time segments: $t < 0$, $0 \le t < 1$, $1 \le t < T_x$, $t \ge T_x$ wher T_x is the, as yet unknown, time at which the diode turns on for the last time.

    Segment 1 (Diode Off, $t < 0$, $x = v = 0$).

    Segment 2 (Diode On, $0 \le t < 1$, $v = 3$, $x_{SS} = \frac{4}{5}(3 - 0.7) = 1.84$, $\tau_{On} = 0.32\,\text{ms}$): $x(t) = x_{SS}(t) + (x(0+) - x_{SS}(0+))\,e^{\frac{-t}{\tau}} = 1.84 + (0 - 1.84)\,e^{\frac{-t}{\tau}} = 1.84 - 1.84e^{\frac{-t}{\tau_{On}}}$. At $t = 1$ this gives $x(1) = 1.76$.

    Segment 3 (Diode Off, $1 \le t < T_x$, $v = 2$, $x_{SS} = 0$, $\tau_{Off} = 1.6\,\text{ms}$): Capacitor voltage continuity means that $x(1+) = 1.76$. So $x(t) = x_{SS}(t) + (x(1+) - x_{SS}(1+))\,e^{\frac{-t}{\tau}} = 0 + (1.76 - 0)\,e^{\frac{-t}{\tau}} = 1.76e^{\frac{-(t-1)}{\tau_{Off}}}$. We need to know when the voltage x reaches 1.3 V ($t = T_x$) because that is when the diode will turn on again. Solving $1.76e^{\frac{-(T_x-1)}{\tau_{Off}}} = 1.3 \Rightarrow T_x = 1.48\,\text{ms}$.

    Segment 4 (Diode On, $t \ge 1.48\,\text{ms}$, $v = 2$, $x_{SS} = \frac{4}{5}(2 - 0.7) = 1.04$, $\tau_{On} = 0.32\,\text{ms}$): $x(t) = x_{SS}(t) + (x(T_x+) - x_{SS}(T_x+))\,e^{\frac{-t}{\tau}} = 1.04 + (1.3 - 1.04)\,e^{\frac{-t}{\tau}} = 1.04 + 0.26e^{\frac{-(t-T_x)}{\tau_{On}}}$.

    All four segments are plotted in Fig. 17(iii).

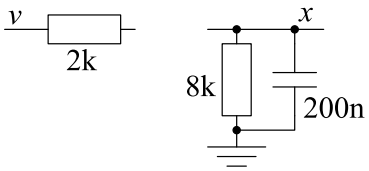


Fig. 17(off)

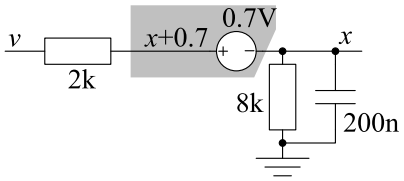


Fig. 17(on)

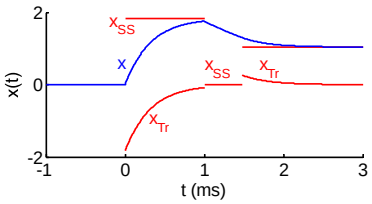


Fig. 17(iii)

E1.1 Circuit Analysis

Problem Sheet 7 (Lectures 17 & 18)

Key: [A]= easy ... [E]=hard

Note: In this problem sheet u and Z_0 are the propagation velocity and characteristic impedance of a transmission line and the forward and backward waves at the point x are $f_x(t) = f(t - \frac{x}{u})$ and $g_x(t) = g(t + \frac{x}{u})$ with, in the case of sinusoidal waves, the corresponding phasors being F_x and G_x.

[A] Find the propagation velocity, u, and characteristic impedance, Z_0, of a transmission line whose capacitance and inductance are 50 pf/m and 500 nH/m respectively. Express the propagation velocity also as a fraction of the speed of light. [B] The line in Fig. has $Z_0 = 100\,\Omega$. For each of the cases below, calculate the reflection coefficients at both ends of the line and describe the waves that would arise from a short positive pulse at V_S.$R_S = 10$ and $R_L = 100$.$R_S = 10$ and $R_L = 1000$.$R_S = 100$ and $R_L = 1000$.

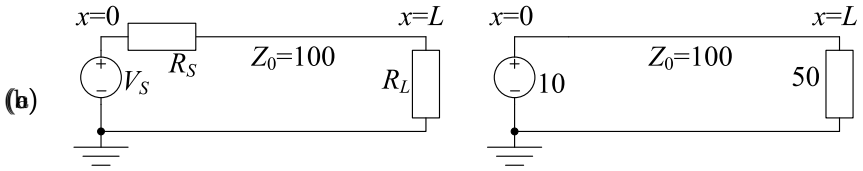


Fig. Fig.

[C] The line in Fig. is driven by a 10 V DC voltage source. Determine the voltage and current in the line and hence the forward and backward waves $f(t - \frac{x}{u})$ and $g(t + \frac{x}{u})$. Determine also the power carried by the two waves and verify that their differnce equals the total power delivered to the load.

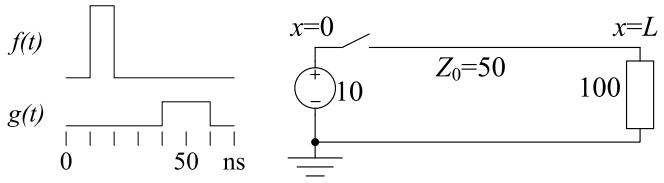


Fig. Fig.

[C] A transmission line has a propagation velocity of 15 cm/ns and a characteristic impedance of $100\,\Omega$. The forward and backward waves are shown in Fig. and have amplitudes of 9 V and 3 V respectively. Draw dimensioned sketches of the voltage and current waveforms at (a) $x = 0$ and (b) $x = 300$ cm. In each case, give the value of the peak voltage and peak current.

[C] The transmission line shown in Fig. has a propagation velocity of 15 cm/ns and a characteristic impedance of $50\,\Omega$. The length of the line is $L = 300$ cm.

(a) Determine the reflection coefficients at both ends of the line when the switch is held closed.

(b) Calculate the steady state DC forward and backward waves when the switch has been closed for a long time.

(c) If the switch is closed at time $t = 0$, determine the forward and backward waves at $x = 0$. Hence determine the voltage waveforms at $x = 0$ and $x = L$.

[C] A length of transmission line with $Z_0 = 100$ and $u = 20$ cm/ns is terminated in a short circuit at $x = L$. Find the shortest lengths of line, L, for which the impedance at 20 MHz at $x = 0$ will equal (a) 50 pF and (b) $1\,\mu$H.

[C] In Fig. , $L = 5$ m, $u = 20$ cm/ns, $Z_0 = 100$, $R_L = 50$ and the frequency of operation is 50 MHz.

(a) If the forward wave phasor at $x = 0$ is $F_0 = 6j$, determine the forward wave phasors, F_x, at $x = 1$, 2, 3, 4 and 5 metres.

(b) Calculate the reflection coefficient at $x = L$.

(c) Determine the backward wave phasors, G_x, at $x = 0$, 1, 2, 3, 4 and 5.

(d) Determine the line voltage phasors, V_x, at $x = 0$, 1, 2, 3, 4 and 5.

(e) Determine the Voltage Standing Wave Ratio: $VSWR = \frac{\max(|V_x|)}{\min(|V_x|)}$.

(f) Determine the line impedance, $\frac{V_0}{I_0}$, at $x = 0$.

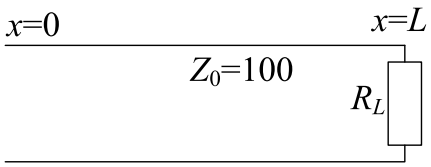


Fig.

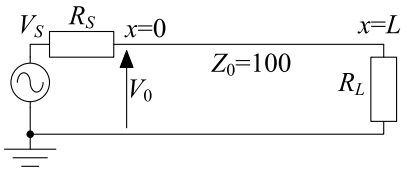


Fig.

[C] Repeat question for $R_L = 100$.

[C] In Fig. , $L = 1$ m, $u = 15$ cm/ns, $Z_0 = 100$, $R_S = 10$, $R_L = 150$ and the frequency of operation is 20 MHz.

(a) Calculate the reflection coefficient, ρ_L at $x = L$. Hence calculate the phasor ratio $\frac{G_0}{F_0}$ at $x = 0$.

(b) Calculate the line impedance $\frac{V_0}{I_0}$ at $x = 0$.

(c) By treating the circuit at the source end as a potential divider, calculate V_0 if $V_S = 10$.

(d) Calculate F_0 and hence calculate F_L, G_L and the load voltage, V_L.

(e) Calculate the complex power supplied by the source.

E1.1 Circuit Analysis

Problem Sheet 7 - Solutions

The propagation velocity (from page 17-4 of the notes) is $u = \sqrt{\frac{1}{L_0 C_0}} = 2 \times 10^8$ m/s which is $\frac{2}{3}$ of the speed of light. The characteristic impedance is $Z_0 = \sqrt{\frac{L_0}{C_0}} = 100\,\Omega$. These are typical characteristics for the twisted pair cable used in computer networks. The reflection coefficients is given by $\rho = \frac{R - Z_0}{R + Z_0}$ where R is the Thévenin impedance at the relevant end of the line. This gives (a) $\rho_S = -0.818$, $\rho_L = 0$ so a pulse at V_S will travel down the line as a forward wave and stop when it reaches the load (no reflections), (b) $\rho_S = -0.818$, $\rho_L = 0.818$ so a positive pulse at V_S will travel down the line as a forward wave, be reflected at $x = L$ and travel back towards the source as a backward wave. When it reaches $x = 0$ it will be reflected and inverted and will result in a negative pulse traveling as a forward wave. This whole process will be repeated for ever and you will get an infinite sequence of pulses each one smaller than than the previous one, (c) $\rho_S = 0$, $\rho_L = 0.818$ so a positive pulse at V_S will travel down the line as a forward wave, be reflected at $x = L$ and then travel back towards $x = 0$ where it will stop. Since the voltage source is DC, $f()$ and $g()$ will be constants and the waves will be independent of x and t. From Ohm's law, we know that the voltage and current in the line are 10 V and 0.2 A so we must have $f + g = 10$ and $f - g = 0.2 \times 100 = 20$. Solving these two equations gives $f = 15$ and $g = -5$. As we would expect, the ratio $\frac{g}{f} = -0.333$ is equal to the reflection coefficient at the load. The power carried by the two waves is $\frac{f^2}{Z_0} = 2.25$ W and $\frac{g^2}{Z_0} = 0.25$ W and the difference between these does indeed equal the power absorbed by the load: $\frac{10^2}{50} = 2$ W. At $x = 0$, $v_0(t) = f(t) + g(t)$ and $i_0(t) = Z_0^{-1}\,(f(t) - g(t))$ which gives the waveforms shown in Fig. (a). The peak voltage and current are 9 V and 90 mA respectively. At $x = 300$ cm, $f\left(t - \frac{x}{u}\right)$ is delayed by 20 ns while $g\left(t + \frac{x}{u}\right)$ is advanced by the same amount. The new voltage and current waveforms are as shown in Fig. (b). The forward and backward waves now overlap and the peak voltage and current are 12 V and 60 mA respectively.

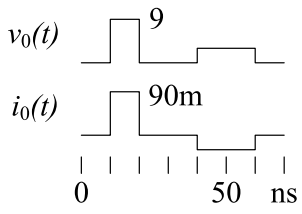


Fig. (a)

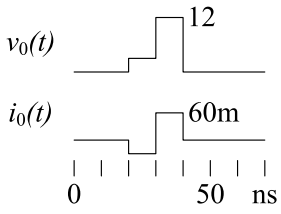


Fig. (b)

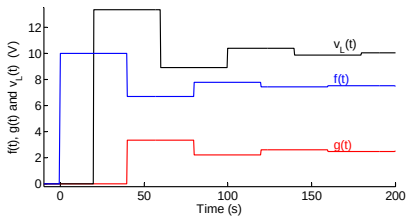


Fig.

(a) $\rho_L = \frac{100-50}{100+50} = 0.33$, $\rho_0 = \frac{0-50}{0+50} = -1$.
(b) In the steady state we must have $V = 10$ V and $I = 0.1$ A which means that $f + g = 10$ and $f - g = 0.1 Z_0 = 5$. We can solve these two equations to give $f = 7.5$ and $g = 2.5$.
(c) From the notes (page 17-11) the forward wave is the sum of an infinite number of copies of the input signal; each extra copy is delayed by an additional round-trip propagation delay ($\frac{600}{15} = 40$ ns) and multiplied by an additional factor $\rho_L \rho_0 = -0.33$. This is shown in Fig. ; the input signal jumps to 10 V when the switch is closed and stays there forever. Onto this is added a copy of the input signal delayed by 40 ns and multiplied by -0.33 which means that $f(t)$ jumps to 6.67 at $t = 40$ ns. With pulse waveforms, it is sometimes easier to keep track of the *changes* in the signals rather than their absolute values. In this case, the input signal only changes once: by +10 V at $t = 0$. It follows that $f(t)$ will have a change of +10 at $t = 0$ followed by a change of $10\rho_L\rho_0 = -3.33$ at $t = 40$ followed by a change of $10\,(\rho_L\rho_0)^2 = 1.11$ at $t = 80$ followed by a change of $10\,(\rho_L\rho_0)^3 = -0.37$ at $t = 120$ and so on.
So, $f(0) = 10$, $f(40) = 6.67$, $f(80) = 7.78$, $f(120) = 7.41$, $f(160) = 7.53$, $f(200) = 7.49$, $f(240) = 7.5)$, $\cdots$. As we can see, $f(t) \to 7.5$ as predicted in part (a).
$g(t) = \rho_L f(t - \frac{2L}{u})$ is just the same as $f(t)$ but delayed by the round-trip propagation delay and multiplied by $\rho_l = 0.33$.
The voltage waveform at $x = 0$ is equal to $f(t) + g(t)$ and has a constant value of 10 V for $t \geq 0$ as is obvious from the circuit.
The voltage waveform at $x = L$ is $v_L(t) = f_L(t) + g_L(t) = f(t - 20) + g(t + 20)$, that is, the sum of $f(t)$ delayed by 20 ns and $g(t)$ advanced by 20 ns. This reaches a peak value of $10\,(1 + \rho_L) = 13.3$ V.

Note that the propagation velocity in SI units is 20×10^7 m/s. At 20 MHz the wavenumber is $k = \frac{\omega}{u} = \frac{2\pi \times 2 \times 10^7}{20 \times 10^7} = 0.628$ rad/m or, equivalently, the wavelength is $\lambda = \frac{u}{f} = \frac{2\pi}{k} = 10$ m. The impedance of the line at $x = 0$, which we will call Z, is, by definition, equal to the phasor $Z = \frac{V_0}{I_0} = \frac{F_0+G_0}{Z_0^{-1}(F_0-G_0)}$. Because the line is short circuited at $x = L$, the reflection coefficient at $x = L$ is $\rho_L = -1$. G_0 and F_0 are related by $G_0 = F_0\rho_L e^{-2jkL}$; the ρ_L factor comes from the reflection at $x = L$ and the e^{-2jkL} factor is the phase shift arising from the time delay for the wave to travel from $x = 0$ to $x = L$ and back again. Substituting for G_0 in the previous expression for Z gives

$$Z = Z_0\frac{F_0 + F_0\rho_L e^{-2jkL}}{F_0 - F_0\rho_L e^{-2jkL}} = Z_0\frac{1 - e^{-2jkL}}{1 + e^{-2jkL}} = Z_0\frac{e^{-jkL}\left(e^{jkL} - e^{-jkL}\right)}{e^{-jkL}\left(e^{jkL} + e^{-jkL}\right)} = jZ_0 \tan kL$$

which leads to $L = \frac{n\lambda}{2} + \frac{1}{k}\tan^{-1}\frac{Z_T}{jZ_0}$ where the $\frac{n\lambda}{2}$ term arises because tan() repeats every π. We want to choose L so that (a) $Z_T = \frac{1}{j\omega C} = -159j$ or (b) $Z_T = j\omega L = 126j$ and in each case choose n to make L as small as possible while still remaining positive. This gives (a) $L = 3.39$ m and (b) $L = 1.43$ m.

(a) The wavelength is $\lambda = \frac{u}{f} = \frac{20\times10^7}{50\times10^6} = 4$ m which means that we get a phase shift of $-\frac{\pi}{2}$ for every metre we move along the line. Equivalently $k = \frac{\omega}{u} = \frac{2\pi}{\lambda} = 1.571$ rad/m. Thus at $x = 0$, 1, 2, 3, 4, 5, $F_x = F_0e^{-jkx} = 6j$, 6, $-6j$, -6, $6j$, 6. Notice that the phase repeats every 4 m since this is one wavelength.
(b) The reflection coefficient at $x = L$ is $\rho_L = \frac{R_L - Z_0}{R_L + Z_0} = -0.333$.
(c) It follows that $G_L = \rho_L F_L = -0.333 \times 6 = -2$. As with F_x in part (a), G_x will get a phase shift of $-\frac{\pi}{2}$ for every metre we move backwards along the line towards $x = 0$, i.e. $G_x = G_L e^{-jk(L-x)}$. If you substitute for G_L you can write this as $G_x = \rho_L F_L e^{-jk(L-x)} = \rho_L F_0 e^{-jkL}e^{-jk(L-x)} = \rho_L F_0 e^{-jk(2L-x)}$. Thus at $x = 0$, 1, 2, 3, 4, 5, $G_x = 2j$, -2, $-2j$, 2, $2j$, -2.
(d) $V_x = F_x + G_x$, so at $x = 0$, 1, 2, 3, 4, 5, $V_x = 8j$, 4, $-8j$, -4, $8j$, 4. We can see that when F_x and G_x are in phase (at $x = 0$, 2, 4) their magnitudes add to give $|V_x| = 6 + 2 = 8$ whereas when they are out of phase (at $x = 1$, 3, 5) the subtract to give $|V_x| = 6 - 2 = 4$.
(e) From part (d), the VSWR is $\frac{8}{4} = 2$. Because $|G_x| = |\rho_L|\,|F_x|$, the VSWR is $\frac{1+|\rho_L|}{1-|\rho_L|}$ which, if you express ρ_L in terms of R_L equals $\max\left(\frac{R_L}{Z_0}, \frac{Z_0}{R_L}\right) = \max(0.5, 2) = 2$.
(f) The line impedance at $x = 0$ is $\frac{V_0}{I_0} = Z_0\frac{F_0+G_0}{F_0-G_0} = Z_0\frac{8j}{4j} = Z_0 \times 2 = 200$.

(a) The forward wave is unaffected so $F_x = F_0e^{-jkx} = 6j$, 6, $-6j$, -6, $6j$, 6 as before.
(b) The line termination is matched and the reflection coefficient is now 0.
(c) It follows that $G_x = 0\,\forall x$.
(d) $V_x = F_x + G_x = F_x + 0 = F_x$, so $V_x = 6j$, 6, $-6j$, -6, $6j$, 6.
(e) $|V_x|$ is always equal to 6 and so the VSWR equals 1, its minimum possible value. Measuring the VSWR gives a way of telling when a line is matched without having to measure either Z_0 or R_L.
(f) The line impedance at $x = 0$ is $\frac{V_0}{I_0} = Z_0\frac{F_0+G_0}{F_0-G_0} = Z_0\frac{F_0+0}{F_0-0} = Z_0 = 100$.

(a) We have $\lambda = 7.5$ m and $k = \frac{2\pi}{\lambda} = 0.838$. The reflection coefficient is $\rho_L = \frac{R_L - Z_0}{R_L + Z_0} = 0.2$. Hence $\frac{G_0}{F_0} = \rho_L e^{-2jkL} = -0.0209 - 0.1989j$.
(b) $Z_T = \frac{V_0}{I_0} = Z_0\frac{F_0+G_0}{F_0-G_0} = Z_0\frac{1+\frac{G_0}{F_0}}{1-\frac{G_0}{F_0}} = 100 \times (0.8874 - 0.3677j) = 88.74 - 36.77j = 96.06\angle -22.5°$.
(c) Hence $V_0 = V_S\frac{Z_T}{R_S+Z_T} = 10 \times (0.911 - 0.033j) = 9.11 - 0.33j$.
(d) Since $V_0 = F_0 + G_0 = F_0\left(1 + \frac{G_0}{F_0}\right)$, we can write $F_0 = \frac{V_0}{1+\frac{G_0}{F_0}} = 9.00 + 1.49j$. From this we get $F_L = F_0e^{-jkL} = 7.13 - 5.69j$. We can now calculate $G_L = \rho_L F_L = 1.43 - 1.14j$ and then work out $V_L = F_L + G_L = 8.56 - 6.83j$. Notice that once you know F and G at one point on the line, you can easily calculate their values at any other point whereas V varies in a rather more complicated manner; because of this, we convert from V to F, then move F to a different place and convert back to V again.
(e) The complex power supplied by the source is

$$\frac{\left|\tilde{V}_S\right|^2}{(R_S + Z_T)^*} = \frac{\frac{1}{2}|V_S|^2}{(R_S + Z_T)^*} = \frac{50}{98.7 + 36.8j} = 0.445 - 0.166j \text{ VA}$$

. You might expect this to be purely real because the circuit appears to contain only resistors; however, there are implicit capacitors and inductors in the transmission line.