Motion in Two & Three Dimension

MOTION IN A PLANE

The motion in which the movement of a body is restricted to a plane is called motion in a plane.

The general approach to solve problem on this topic is to resolve the motion into two mutually perpendicular. One along X-axis and other along Y-axis. These two motion are independent of each other and can be treated as two separate rectilinear motions.

The velocity v and acceleration a can be resolved into its x and y components say

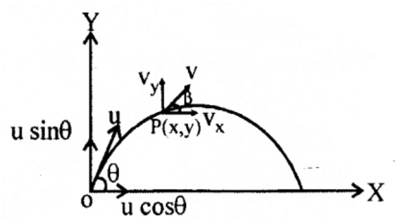
$$v = v_x\hat{i} + a_y\hat{j}$$
$$a = a_x i + a_y j$$

x-component of motion

$$v_x = u_x + a_x t$$
$$x = u_x t + \frac{1}{2}a_x t^2$$
$$v_x^2 - u_x^2 = 2a_x x$$
$$x\left(\frac{u_x + v_x}{2}\right)t$$

y- component of motion

$$v_y = u_y + a_y t$$
$$y = u_y t + \frac{1}{2}a_y t^2$$
$$v_y^2 - u_y^2 = 2a_y y$$
$$y = \left(\frac{u_y + v_y}{2}\right)t$$



PROJECTILE

Projectile is the name given to a body thrown with some initial velocity in any arbitrary direction and then allowed to move under the influence of gravity alone.

Example: A football kicked by the player, a stone thrown from the top of building, a bomb released from a plane. The path followed by the projectile is called a trajectory. The projectile moves under the action of two velocities:

(1) A uniform velocity in the horizontal direction, which does not change (if there is no air resistance)

(2) A uniformly changing velocity in the vertical direction due to gravity.
The horizontal and vertical motions are independent of each other.

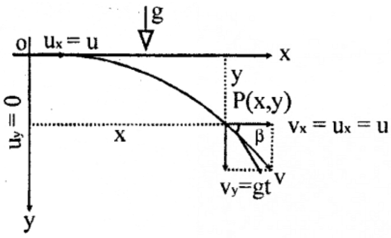
1. **Oblique projectile:** In this, the body is given an initial velocity making an angle θ with the horizontal and it moves under the influence of gravity along a parabolic path.

2. **Horizontal projectile**: In this, the body is given an initial velocity directed along the horizontal and then it moves under the influence of gravity along a parabolic path.

Motion along x – axis

$$u_x = u, a_x = 0$$

$$x = u_x t + \frac{1}{2} a_x t^2$$



$$x = ut + 0$$

$$\therefore t = \frac{x}{u} \qquad ...(1)$$

Motion along y – axis

$$u_y = 0, a_y = g$$

$$y = u_y t + \frac{1}{2} a_y t^2$$

$$y = 0 + \frac{1}{2} g t^2$$

$$y = \frac{1}{2} g t^2 \qquad ...(2)$$

From equation (1) & (2) we get $y = \left(\frac{g}{2u^2}\right) x^2$ which is the equation of a parabola.

Velocity at any instant

$$\vec{v} = v_x \hat{i} + v_y y$$

$$v = \sqrt{u^2 + g^2 t^2}$$

If β is the angle made by $\vec{v}$ with the horizontal, then

$$\tan\beta = \frac{v_y}{v_x} = \frac{gt}{u}$$

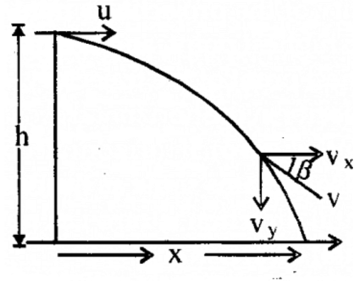
If h is the distance of the ground from the point of projection, T is the rime taken to strike the ground and R is the horizontal range of the projectile then.

$$T = \sqrt{\frac{2h}{g}}$$

$$R = u\sqrt{\frac{2h}{g}}$$

$$\tan\beta = \frac{v_y}{v_x 1} = \frac{u\sin\theta - gt}{u\cos\theta} = \tan\theta - \frac{gt}{u\cos\theta}$$

Case 1: If the projectile is projected from the top of the tower of height 'h', in horizontal direction, then the height of tower, range & time of flight are related as:



$$h = \frac{1}{2}gt^2 \text{ and } x = vt$$

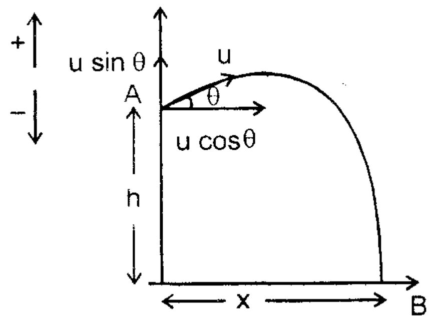
Case 2: If a particle is projected at an angle (θ) in upward direction from the top of tower of height h with velocity u, then

$$u_y = u\sin\theta$$

$$a_y = -g$$

$$u_x = u\cos\theta$$

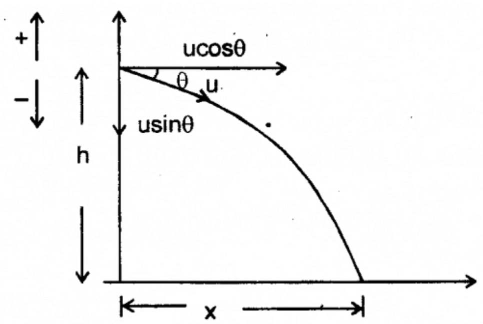
$$h = +u\sin\theta.t - \frac{1}{2}gt^2 \text{ and } x = u\cos\theta.t$$



Case 3: If a body is projected at an angle (θ) from the top of tower in downward direction then

$$u_y = -u\sin\theta, u_x = u\cos\theta, a_x = 0$$

$$a_y = +g - h = -u\sin\theta.t - \frac{1}{2}gt^2 \text{ and } x = u\cos\theta.t$$



PROJECTILE MOTION Equation of Trajectory:

Let the point from which the projectile is thrown into space is taken as the origin, horizontal direction in the plane of motion is taken as the X-axis-, the vertical direction is taken as the Y-axis, Let the projectile be thrown with a velocity u making an angle θ with the X-axis.

The components of the initial velocity in the X-direction and Y-direction are u cos θ and u sin θ respectively. Then at any instant of time t, Motion along x - axis

$$u_x = u\cos\theta, a_x = 0$$

$$x = u_x t + \frac{1}{2}a_y t^2$$

$$x = (u\cos\theta)t \qquad ...(1)$$

Motion along y-axis

$$u_y = u\sin\theta, a_y = -g$$

$$y = u_y t + \frac{1}{2}a_y t^2$$

$$y = u\sin\theta + \frac{1}{2}gt^2 \qquad ...(2)$$

from equation (1) and (2) we get

$$y = x\tan\theta = \frac{9}{2u^2\cos^2\theta}x^2$$

which is the equation of a parabola. Hence the path followed by the projection is parabolic.

Velocity at any Point:

Let v_y be the vertical velocity of projectile at time t. (at P). And v_x be the horizontal component of velocity at time t.

$$\therefore v_y = u\sin\theta - gt \qquad ...(1)$$

$$v_x = u\cos\theta \qquad ...(2)$$

$$v = \sqrt{v_x{}^2 + v_y{}^2}$$

$$= \sqrt{u^2\cos^2\theta + u^2\sin^2\theta - 2gt\sin\theta + g^2t^2}$$

$$\boxed{v = \sqrt{u^2 + g^2t^2 - 2gt\sin\theta}}$$

and the instantaneous angle (β) with horizontal is given by

$$\tan\beta = \frac{v_y}{v_x 1} = \frac{u\sin\theta - gt}{u\cos\theta} = \tan\theta - \frac{gt}{u\cos\theta}$$

Time of Flight:

The time of flight of the projectile is given by

$\boxed{T = 2t = \frac{2u\sin\theta}{g}}$, where 't' is the time of ascent or descent.

Maximum Height:

Maximum height attained by the projectile is given by:

$$\boxed{H = \frac{u^2}{2g}\sin^2\theta}$$

Range:

The horizontal range of the projectile is given by

$$\boxed{R = \frac{u^2\sin\theta}{g}}$$

$\boxed{R_{max} = \frac{u^2}{g}}$ at 8 = 45° ($\because$ maximum value of sin2θ = 1)

In case of vertical motion, 8 = 90° so maximum height u² attained $= H = \frac{u^2}{2g}$

Keep in Memory

1. The horizontal range of the projectile is same at two angles of projection for 8 and (90 - θ).
2. The height attained by the projectile above the ground is the largest when the angle of projection with the horizontal is 90° (Vertically upward projection). In such a case time of flight is largest but the range is the smallest (zero).
3. If the velocity of projection is doubled. The maximum height attained and the range become 4 times, but the time of flight is doubled.
4. When the horizontal range of the projectile is maximum, (θ = 45°), then the maximum height attained is 1/4th of the range.
5. For a projectile fired from the ground, the maximum height is attained after covering a horizontal distance equal to half of the range.

   The velocity of the projectile is minimum but not zero at the highest point, and is equal to *u* cos θ i.e. at the highest point of the trajectory, the projectile has net velocity in the horizontal direction (vertical component is zero). Horizontal component of velocity also remains same as the component of g in horizontal direction is zero i.e., no acceleration in horizontal direction.

PROJECTILE ON AN INCLINED PLANE

The body is thrown from a plane OA inclined at an angle α with the horizontal, with a constant velocity u in a direction making an angle θ with the horizontal.'

The body returns back on the same plane OA. Hence the net displacement of the particle in a direction normal to the plane OA is zero.

u_x = u cos (θ - α) along the incline, + x axis)

u_y = u sin (θ - α) along the incline, + y axis)

a_x = g sin β along - x axis, as retardation

a_y = g cos β along - y axis, as retardation

$$s = ut + \frac{1}{2}at^2$$

or $$0 = u\sin(\theta - \alpha)T - \frac{1}{2}g\cos\alpha T^2$$

The time of flight of the projectile is given by

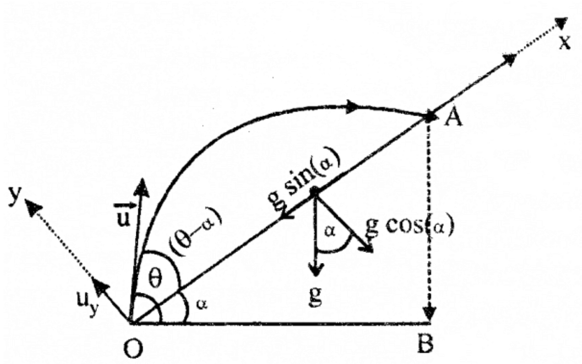
$$\boxed{T = \frac{2\sin(\theta - \alpha)}{g\cos(\alpha)}}$$

If maximum height above the inclined plane is H,

$$\boxed{H = \frac{u^2\sin^2(\theta - \alpha)}{2g}}$$

The horizontal range of the projectile is given by

$$OB = u\cos\theta\, t = \boxed{\frac{2u^2\sin(\theta-\alpha)\cos\theta}{g\cos\alpha} = R}$$



The range of the projectile at the inclined plane is given by

$$OA = \boxed{\frac{OB}{\cos\alpha} = \frac{2u^2\sin(\theta-\alpha)\cos\theta}{g\cos^2\alpha}} = R$$

Condition for R to be maximum:

Since $R = \dfrac{u^2}{g\cos^2\alpha}\quad[2\sin(\theta-\alpha)\cos\theta]$

$= \dfrac{u^2}{g\cos^2\alpha}\quad[\sin(2\theta-\alpha)-\sin\alpha]$

{2 sin A cos B = sin(A + B) + sin (A – B)]

R is maximum when sin (29 -a) is maximum i.e.,

$$\sin(2\theta-\alpha)=1 \text{ or } \boxed{\theta=\frac{\pi}{4}+\frac{\alpha}{2}} \Rightarrow \boxed{R_{max} = \frac{u^2}{g\cos^2\alpha}[1-\sin\alpha]}$$

or R_{max}(on inclined plane) $= \dfrac{R_{max}(\text{on horizontal plane})}{1+\sin\alpha}$

Where R_{mav} (on horizontal plane) $= \dfrac{u^2}{2g}$

Condition for T to be maximum

$T = \dfrac{2u\sin(\theta-\alpha)}{g\cos\alpha}$ so T is max when sin $(\theta-\alpha)$ is maximum

ii.e., $\sin(\theta-\alpha)=1$ or $\boxed{\theta=\frac{\pi}{2}+\alpha \Rightarrow T=\frac{2u}{g\cos\alpha}}$

It means that if θ_1 is the angle for projectile for which T' is maximum & θ_2 is the angle for which R is maximum, then $\theta_1 = 2\theta_2$.

Keep in Memory

1. Angular displacement behaves like vector, when its magnitude is very very small. It follows laws of vector addition.

2. Angular velocity and angular acceleration are axial vectors.

3. Centripetal acceleration always directed towards the centre of the circular path and is always perpendicular to the instantaneous velocity of the particle.

4. Equation of trajectory of an oblique projectile in terms of range (R) is

$$y = x\tan\theta - \left(1 - \frac{x}{R}\right)$$

5. There are two unique times at which the projectile is at the same height h(< H) and the sum of these two times equals the time of flight T, Since, $h = (u\sin\theta)t - \frac{1}{2}gt^2$ is a quadratic in time, so it has two unique roots t_1 and t_2 (say) such that sum of roots $(t_1 + t_2)$ is $\frac{2u\sin\theta}{g}$ and product $(t_x t_2)$ is $\frac{2h}{g}$. The time lapse $(t_1 - 1_2)$ is $(t_1 - t_2)^2 - 4t_1 + t_2$

$$\Rightarrow t_1 - t_2\sqrt{\frac{4u^2\sin^2\theta}{g^2} - \frac{8h}{g}}.$$

CIRCULAR MOTION

Circular motion maybe divided into two types

(1) Motion in a horizontal Circle: It is an accelerated motion even if the speed is uniform.

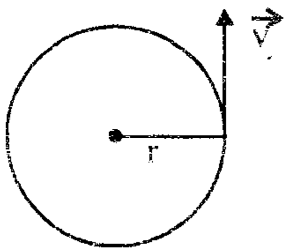
(a) Uniform Circular Motion: An object moving in a circle with a constant speed is said to be in uniform circular motion.

Angular displacement: Change in angular position is called angular displacement

Angular Velocity: Rate of change of angular displacement is called angular velocity

$$\omega = \frac{d\theta}{dt}$$

Relation between linear velocity and angular velocity



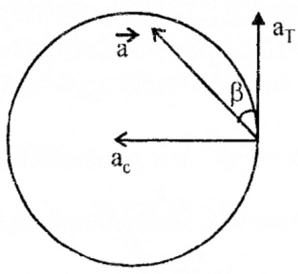
$$\vec{v} = \vec{\omega} \times \vec{r}$$

in magnitude, $v = r\omega$

Angular acceleration: Rata of change of angular velocity is called angular acceleration.

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

Relation between linear acceleration and angular acceleration



$$\vec{a}_T = \vec{\alpha} \times \vec{r}$$

in magnitude, $a_T = r\,\alpha$

Centripetal acceleration: Acceleration acting on a body moving in uniform circular motion is called centripetal acceleration. It arises due to the change in the direction of the velocity vector.

Magnitude of certipetal acceleration is

$$a_c = \frac{v^2}{r} = r\omega^2$$

$$\therefore \qquad \omega = \frac{2\pi}{R} = 2\pi\upsilon \quad \left(\upsilon = \frac{1}{T} = \text{frequency}\right)$$

$$\therefore \qquad a_c = 4\pi^2\upsilon^2 r$$

This acceleration is always directed radially towards the centre of the circle.

Centripetal Force: The force required to keep a body moving in uniform circular motion is called centripetal force.

$$F_c = \frac{mv^2}{r} = mr\omega^2$$

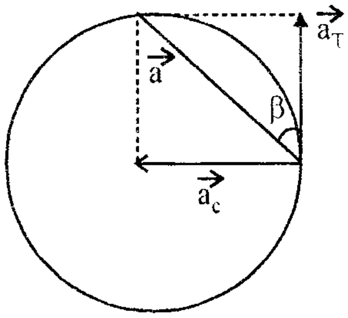
It always directed radially inwards.

Centrifugal Force: Centrifugal force is a fictitious force which acts on a body in rotating (non-inertial frames) frame of reference.

Magnitude of the centrifugal is $F = \frac{mv^2}{r}$.

This force is always directed radially outwards and is also called corolious force.

(b) Non-uniform circular motion: An object moving in a circle with variable speed is said to be in non uniform circular motion. If the angular velocity varies with time, the object has two accelerations possessed by it, centripetal acceleration (a_c) and angential accelaration (a_T) and both perpendicular to each other



Net acceleration

$$a = \sqrt{a_c^2 + a_T^2}$$

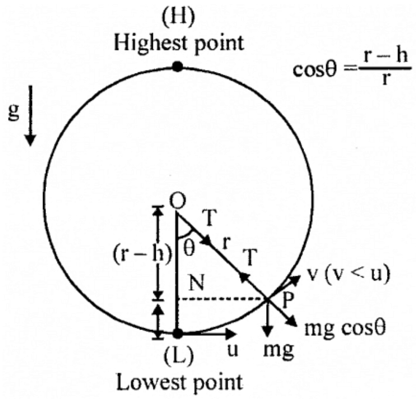
$$a = \sqrt{(r^2\omega^2 + r^2\alpha^2)}$$

$$a = r\sqrt{\omega^4 + \alpha^2}$$

$$\tan\beta = \frac{a_c}{a_T}$$

(2) Motion in a vertical Circle: When an object moves in a vertical circle, acceleration due to gravity acts at every point and changes the speed at every point.

Consider an object of mass m is attached to a string of length 1 to be whirled in a vertical circle.



Velocity at point P at vertical height h

Loss in K.E = gain in P.E.

$$\frac{1}{2}mu^2 - \frac{1}{2}mv^2 = mgh - 0$$

$$\therefore \quad v = \sqrt{u^2 - 2gh}$$

Tension at point P

$$T - mg\cos\theta = \frac{mv^2}{r}$$

$$T = mg\cos\theta + \frac{mv^2}{r}$$

$$T = \frac{m}{r}(u^2 + gr - 3gh)$$

Tension at the lowest point

$$T_L = \frac{m}{r}(u^2 + gr)$$

Tension at the highest point

$$T_H = \frac{m}{r}(u^2 + 5gr)$$

The minimum velocity required at the bottom to complete the circle is given by

$$v_{min} \geq \sqrt{5gr}$$

The velocity at the highest point is $v = \sqrt{gr}$

Keep in Memory

1. Circular motion is uniform if $a_T = r\alpha = 0$, that is angular velocity remains constant and radial acceleration $a_c = \frac{v^2}{r} = rw^2$ is constant.

2. When a_E or a is present angular velocity varies with time and net acceleration is

$$a = \sqrt{a_c^2 + a_E^2}$$

3. If $a_T = 0$ or $\alpha = 0$, no work is done in circular motion.

4. In vertical circle $T_L - T_H = 6g$

5. In vertical circle if $u = \sqrt{2gr}$, the body continuous to oscillate about the lowest point L. The arc of oscillation is a semicircle.

6. In vertical circle, if the speed of body at lowest point lies between $\sqrt{2gr}$ and $\sqrt{5gr}$, the body leaves the circular path somewhere between L and H.