SOLUTION OF SYSTEM OF LINEAR EQUATIONS

Lecture 1: **(a) Gauss Elimination method (general).**
(b) Gauss Elimination method (particular case).

In this section, we shall discuss about the numerical computation of the solution of a system of n linear equations of the form

$$a_{11}^{(1)}x_1 + a_{12}^{(1)}x_2 + ... + a_{1n}^{(1)}x_n = b_1^{(1)}$$

$$a_{21}^{(1)}x_1 + a_{22}^{(1)}x_2 + ... + a_{2n}^{(1)}x_n = b_2^{(1)}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_{n1}^{(1)}x_1 + a_{n2}^{(1)}x_2 + ... + a_{nn}^{(1)}x_n = b_n^{(1)}$$

where a_{ij}'s $(i, j=1,2,.....,n)$ are the coefficients of the unknowns $x_1, x_2, ... x_n$ and $b_i\text{'}s\left(i=1, 2, ..., n\right)$ are constants.

In matrix notation, this can be written in the form

$$A\underset{\sim}{x} = \underset{\sim}{b}$$

where $A = \begin{pmatrix} a_{11}^{(1)} \; a_{12}^{(1)} .... a_{1n}^{(1)} \\ a_{21}^{(1)} \; a_{22}^{(1)} .... a_{2n}^{(1)} \\ .... \\ a_{n1}^{(1)} \; a_{n2}^{(1)} .... a_{nn}^{(1)} \end{pmatrix}$ is an (n × n) matrix, $\underset{\sim}{x} = \begin{pmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{pmatrix}$ is a (n ×1) matrix of unknowns and $\underset{\sim}{b} = \begin{pmatrix} b_1^{(1)} \\ b_2^{(1)} \\ \ldots \\ b_n^{(1)} \end{pmatrix}$ is a (n ×1) matrix of prescribed constants.

We assume that det $A \neq 0$, so that the system of n linear equations in n-unknowns has a unique solution. Our aim is to compute n-unknown components $x_1, x_2, ... x_n$, up to desired degree of accuracy.

The methods for solving the system of linear equation can be categorized into two groups:

1. **Direct Method,** (or exact method), where we obtain the solution through a finite number of arithmetic operations, for example, Gauss Elimination method, Crout's method.
2. **Interactive Method,** where a sequence of successive approximations, obtained, which converges to the required solution, up to some desired degree of accuracy, for example, Jacobi's method, Gauss Seidal method.

Gauss-Elimination Method

It is a direct method for finding the solution or the values of unknown of a system of linear equations and is based on the principle of elimination of unknown in successive steps. We first discuss the method considering *n*-equations and then we shall consider in particular, 3-equations with 3-unknowns.

We consider a system of *n*-linear equations with *n*-unknowns as:

$$\left.\begin{array}{l}
a_{11}^{(1)}x_1 + a_{12}^{(1)}x_2 + a_{13}^{(1)}x_3 + a_{14}^{(1)}x_4 + \dots\dots\dots\dots + a_{1,n-1}^{(1)}x_{n-1} + a_{1,n}^{(1)}x_n = b_1^{(1)} \\
a_{21}^{(1)}x_1 + a_{22}^{(1)}x_2 + a_{23}^{(1)}x_3 + a_{24}^{(1)}x_4 + \dots\dots\dots\dots + a_{2,n-1}^{(1)}x_{n-1} + a_{n,n}^{(1)}x_n = b_2^{(1)} \\
a_{31}^{(1)}x_1 + a_{32}^{(1)}x_2 + a_{33}^{(1)}x_3 + a_{34}^{(1)}x_4 + \dots\dots\dots\dots + a_{3,n-1}^{(1)}x_{n-1} + a_{3,n}^{(1)}x_n = b_3^{(1)} \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
a_{n-1,1}^{(1)}x_1 + a_{n-1,2}^{(1)}x_2 + a_{n-1,3}^{(1)}x_3 + a_{n-1,4}^{(1)}x_4 + \dots\dots\dots\dots + a_{n-1,n-1}^{(1)}x_{n-1} + a_{n-1,n}^{(1)}x_n = b_{n-1}^{(1)} \\
a_{n,1}^{(1)}x_1 + a_{n,2}^{(1)}x_2 + a_{n,3}^{(1)}x_3 + a_{n,4}^{(1)}x_4 + \dots\dots\dots\dots + a_{n,n-1}^{(1)}x_{n-1} + a_{n,n}^{(1)}x_n = b_n^{(1)}
\end{array}\right\} \quad (1)$$

where $a_{ij}^{(1)}$'s$\left(i, j = 1, 2, 3\dots\dots\dots, n\right)$ are the coefficient of unknowns and $b_i^{(1)}$'s$\left(i, = 1, 2, 3\dots\dots\dots, n\right)$ are prescribed constants.

Let $a_{11}^{(1)} \neq 0$. Now multiplying the first equation successively by

$$-\frac{a_{21}^{(1)}}{a_{11}^{(1)}}(=m_{21}), -\frac{a_{31}^{(1)}}{a_{11}^{(1)}}(=m_{31}), -\frac{a_{41}^{(1)}}{a_{11}^{(1)}}(=m_{41}), \ldots\ldots\ldots\ldots,$$
$$-\frac{a_{n-1,1}^{(1)}}{a_{11}^{(1)}}(=m_{n-1,1}), -\frac{a_{n1}^{(1)}}{a_{11}^{(1)}}(=m_{n1})$$

and adding respectively with 2nd, 3rd, 4th , ..., (n-1)th and nth equations of the system we get,

$$\left.\begin{array}{l}
a_{11}^{(1)}x_1 + a_{12}^{(1)}x_2 + a_{13}^{(1)}x_3 + a_{14}^{(1)}x_4 + \ldots\ldots\ldots\ldots + a_{1,n-1}^{(1)}x_{n-1} + a_{1,n}^{(1)}x_n = b_1^{(1)} \\
\qquad a_{22}^{(2)}x_2 + a_{23}^{(2)}x_3 + a_{24}^{(2)}x_4 + \ldots\ldots\ldots\ldots + a_{2,n-1}^{(2)}x_{n-1} + a_{2,n}^{(2)}x_n = b_2^{(2)} \\
\qquad a_{32}^{(2)}x_2 + a_{33}^{(2)}x_3 + a_{34}^{(2)}x_4 + \ldots\ldots\ldots\ldots + a_{3,n-1}^{(2)}x_{n-1} + a_{3,n}^{(2)}x_n = b_3^{(2)} \\
\qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\qquad a_{n-1,2}^{(2)}x_2 + a_{n-1,3}^{(2)}x_3 + a_{n-1,4}^{(2)}x_4 + \ldots\ldots\ldots + a_{n-1,n-1}^{(2)}x_{n-1} + a_{n-1,n}^{(2)}x_n = b_{n-1}^{(2)} \\
\qquad a_{n,2}^{(2)}x_2 + a_{n,3}^{(2)}x_3 + a_{n,4}^{(2)}x_4 + \ldots\ldots\ldots\ldots + a_{n,n-1}^{(2)}x_{n-1} + a_{n,n}^{(2)}x_n = b_n^{(2)}
\end{array}\right\} (2)$$

where

$$a_{22}^{(2)} = a_{22}^{(1)} - \frac{a_{21}^{(1)}.a_{12}^{(1)}}{a_{11}^{(1)}},\ a_{23}^{(2)} = a_{23}^{(1)} - \frac{a_{21}^{(1)}.a_{13}^{(1)}}{a_{11}^{(1)}}, \ldots\ldots\ldots\ ;$$

$$a_{32}^{(2)} = a_{32}^{(1)} - \frac{a_{31}^{(1)}.a_{12}^{(1)}}{a_{11}^{(1)}},\ a_{33}^{(2)} = a_{33}^{(1)} - \frac{a_{31}^{(1)}.a_{13}^{(1)}}{a_{11}^{(1)}}, \ldots\ldots\ldots\ ;$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_{n2}^{(2)} = a_{n2}^{(1)} - \frac{a_{n1}^{(1)}.a_{12}^{(1)}}{a_{11}^{(1)}},\ a_{n3}^{(2)} = a_{n3}^{(1)} - \frac{a_{n1}^{(1)}.a_{13}^{(1)}}{a_{11}^{(1)}}, \ldots\ldots\ldots\ ;$$

$$b_2^{(2)} = b_2^{(1)} - \frac{b_1^{(1)}.a_{21}^{(1)}}{a_{11}^{(1)}},\ b_3^{(2)} = b_3^{(1)} - \frac{b_1^{(1)}.a_{31}^{(1)}}{a_{11}^{(1)}}, \ldots\ldots\ldots\ldots\ .$$

It is clear from the system (2) that except the first equation, the rest (n – 1) equations are free from the unknown x_1.

Again assuming $a_{22}^{(2)} \neq 0$, multiplying second equation of the system (2) successively by

$$-\frac{a_{32}^{(2)}}{a_{22}^{(2)}}\left(=m_{32}\right), -\frac{a_{42}^{(2)}}{a_{22}^{(2)}}\left(=m_{42}\right), \ldots\ldots\ldots, -\frac{a_{n-1,2}^{(2)}}{a_{22}^{(2)}}\left(=m_{n-1,2}\right), -\frac{a_{n,2}^{(2)}}{a_{22}^{(2)}}\left(=m_{n,2}\right), \ldots\ldots\ldots$$

and adding respectively to 3rd, 4th,……….,(n–1)th and nth equation of the system (2) we get,

$$\left.\begin{array}{l}
a_{11}^{(1)}x_1 + a_{12}^{(1)}x_2 + a_{13}^{(1)}x_3 + a_{14}^{(1)}x_4 + \ldots\ldots\ldots + a_{1,n-1}^{(1)}x_{n-1} + a_{1,n}^{(1)}x_n = b_1^{(1)} \\
\qquad a_{22}^{(2)}x_2 + a_{23}^{(2)}x_3 + a_{24}^{(2)}x_4 + \ldots\ldots\ldots + a_{2,n-1}^{(2)}x_{n-1} + a_{2,n}^{(2)}x_n = b_2^{(2)} \\
\qquad\qquad a_{33}^{(3)}x_3 + a_{34}^{(3)}x_4 + \ldots\ldots\ldots + a_{3,n-1}^{(3)}x_{n-1} + a_{3,n}^{(3)}x_n = b_3^{(3)} \\
\qquad\qquad a_{43}^{(3)}x_3 + a_{44}^{(3)}x_4 + \ldots\ldots\ldots + a_{4,n-1}^{(3)}x_{n-1} + a_{4,n}^{(3)}x_n = b_4^{(3)} \\
\qquad\qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\qquad\qquad a_{n-1,3}^{(3)}x_3 + a_{n-1,4}^{(3)}x_4 + \ldots\ldots\ldots + a_{n-1,n-1}^{(3)}x_{n-1} + a_{n-1,n}^{(3)}x_n = b_{n-1}^{(3)} \\
\qquad\qquad a_{n,3}^{(3)}x_3 + a_{n,4}^{(3)}x_4 + \ldots\ldots\ldots + a_{n,n-1}^{(3)}x_{n-1} + a_{n,n}^{(3)}x_n = b_n^{(3)}
\end{array}\right\} \quad (3)$$

Here also, we observe that 3rd, 4th up to nth equations of the system (3) are free the unknowns x_1, x_2.

Repeating the same procedure of elimination of the unknowns, lastly we get a system of equation which is equivalent to the system (1) as:

$$\left.\begin{array}{l}
a_{11}^{(1)}x_1 + a_{12}^{(1)}x_2 + a_{13}^{(1)}x_3 + a_{14}^{(1)}x_4 + \ldots\ldots\ldots + a_{1,n-1}^{(1)}x_{n-1} + a_{1,n}^{(1)}x_n = b_1^{(1)} \\
\qquad a_{22}^{(2)}x_2 + a_{23}^{(2)}x_3 + a_{24}^{(2)}x_4 + \ldots\ldots\ldots + a_{2,n-1}^{(2)}x_{n-1} + a_{2,n}^{(2)}x_n = b_2^{(2)} \\
\qquad\qquad a_{33}^{(3)}x_3 + a_{34}^{(3)}x_4 + \ldots\ldots\ldots + a_{3,n-1}^{(3)}x_{n-1} + a_{3,n}^{(3)}x_n = b_3^{(3)} \\
\qquad\qquad\qquad a_{44}^{(4)}x_4 + \ldots\ldots\ldots + a_{4,n-1}^{(4)}x_{n-1} + a_{4,n}^{(4)}x_n = b_4^{(4)} \\
\qquad\qquad\qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\qquad\qquad\qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\qquad\qquad\qquad\qquad a_{n-1,n-1}^{(n-1)}x_{n-1} + a_{n-1,n}^{(n-1)}x_n = b_{n-1}^{(n-1)} \\
\qquad\qquad\qquad\qquad\qquad a_{n,n}^{(n)}x_n = b_n^{(n)}
\end{array}\right\} \quad (4)$$

The non-zero (by assumption) coefficients $a_{11}^{(1)}, a_{22}^{(2)}, a_{33}^{(3)}, ...a_{n,n}^{(n)}$ of the system of equations are known as ***pivots*** and the corresponding equations are known as ***pivotal equations.*** Please note if any of the coefficients $a_{11}^{(1)}, a_{22}^{(2)}, a_{33}^{(3)}, ...a_{n,n}^{(n)}$ are zeros, then the system has to be reshuffled so that they are non-zeros.

Now we can get easily calculate the solution of the system of equations (4) as follows: First we find x_n from *n*th equation, then x_{n-1} from $(n-1)^{\text{th}}$ equation after substituting x_n and then successively we shall get all the unknowns $x_1, x_2, x_3, ..., x_n$ (by the method of back substitution).

Gauss Elimination Method (Particular Case)

In this article we now consider a system of 3-equations with 3-unknowns for better illustration to the readers.

A system of 3-equations with 3-unknowns is given by

$$\left.\begin{aligned} a_{11}^{(1)}x_1 + a_{12}^{(1)}x_2 + a_{13}^{(1)}x_3 &= b_1^{(1)} \\ a_{21}^{(1)}x_1 + a_{22}^{(1)}x_2 + a_{23}^{(1)}x_3 &= b_2^{(1)} \\ a_{31}^{(1)}x_1 + a_{32}^{(1)}x_2 + a_{33}^{(1)}x_3 &= b_3^{(1)} \end{aligned}\right\} \tag{5}$$

where $a_{ij}^{(1)}\text{'}s\left(i,\ j=1,2,3\right)$ and $b_i^{(1)}\text{'}s\left(i=1,2,3\right)$ are known constants.

Let $a_{11}^{(1)} \neq 0$. Multiplying the first equation of (5) successively by $-\dfrac{a_{21}^{(1)}}{a_{11}^{(1)}}$ and $-\dfrac{a_{31}^{(1)}}{a_{11}^{(1)}}$ adding respectively with 2nd and 3rd equation we get, the system as:

$$\left.\begin{aligned} a_{11}^{(1)}x_1 + a_{12}^{(1)}x_2 + a_{13}^{(1)}x_3 &= b_1^{(1)} \\ a_{22}^{(2)}x_2 + a_{23}^{(2)}x_3 &= b_2^{(2)} \\ a_{32}^{(2)}x_2 + a_{33}^{(2)}x_3 &= b_3^{(2)} \end{aligned}\right\} \tag{6}$$

where

$$a_{22}^{(2)} = a_{22}^{(1)} - \frac{a_{12}^{(1)}.a_{21}^{(1)}}{a_{11}^{(1)}},\ a_{23}^{(2)} = a_{23}^{(1)} - \frac{a_{13}^{(1)}.a_{21}^{(1)}}{a_{11}^{(1)}},\ b_{2}^{(2)} = b_{2}^{(1)} - \frac{b_{1}^{(1)}.a_{21}^{(1)}}{a_{11}^{(1)}}$$

$$a_{32}^{(2)} = a_{32}^{(1)} - \frac{a_{12}^{(1)}.a_{31}^{(1)}}{a_{11}^{(1)}},\ a_{33}^{(2)} = a_{33}^{(1)} - \frac{a_{13}^{(1)}.a_{31}^{(1)}}{a_{11}^{(1)}},\ b_{3}^{(2)} = b_{3}^{(1)} - \frac{b_{1}^{(1)}.a_{31}^{(1)}}{a_{11}^{(1)}}$$

Let $a_{22}^{(2)} \neq 0$, Multiplying the second equation by $-\frac{a_{32}^{(2)}}{a_{22}^{(2)}}$ and adding with the 3rd equation of the system (6) we get,

$$\left.\begin{aligned} a_{11}^{(1)}x_1 + a_{12}^{(1)}x_2 + a_{13}^{(1)}x_3 &= b_1^{(1)} \\ a_{22}^{(2)}x_2 + a_{23}^{(2)}x_3 &= b_2^{(2)} \\ a_{33}^{(3)}x_3 &= b_3^{(3)} \end{aligned}\right\} \qquad (7)$$

where $a_{33}^{(3)} = a_{33}^{(2)} - \frac{a_{23}^{(2)}.a_{32}^{(2)}}{a_{22}^{(2)}}$.

The non-zero constants (by assumption) $a_{11}^{(1)}, a_{22}^{(2)}$ and $a_{33}^{(3)}$ are called *pivots* and the corresponding equations are called the *pivotal equations.* Now the value of the unknown x_3 can be obtain easily from the third equation, which can be substituted in the second equation to obtain x_2. Substituting x_3, x_2 in the first equation x_1 also be determined. Thus, all the unknown are completely known by the method of back substitution.