Chapter 12

Conic Sections

Smooth, regular curves are common in the world around us. The orbit of the space shuttle around the Earth is one kind of smooth curve; another is the shape of domed buildings such as the United States Capitol. Television receiving dishes are a very common curved shape, as are the curved rearview mirrors on your car. Mathematically, however, an interesting property of all the curves just mentioned is that they are expressed as quadratic equations in two variables.

We discussed quadratic equations in Chapter 5. In the current chapter, we extend that discussion to other forms of equations and their related graphs, called conic sections: parabolas, circles, ellipses, and hyperbolas. We will describe each curve and analyze the equation used to graph it. We will also discuss properties that make these curves so useful in so many different areas, from engineering to architecture to astronomy. We conclude the chapter with a project that further explores the various conic sections we have studied.

12.1 APPLICATION

A satellite dish receiver is in the shape of a parabola. A cross section of the dish shows a diameter of 13 feet at a distance of 2.5 feet from the vertex of the parabola. Write an equation for the parabola.

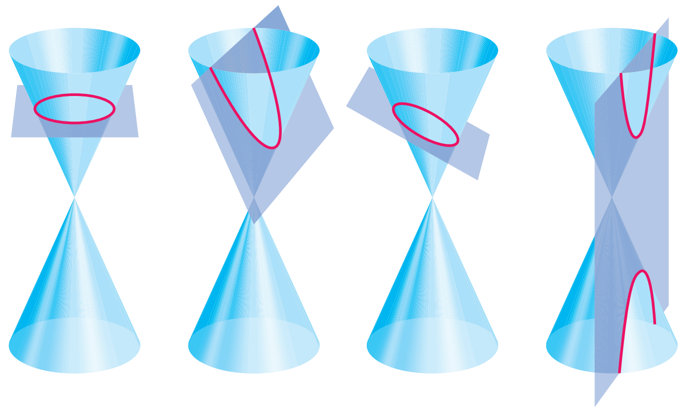
After completing this section, we will discuss this application further. See page 000.

12.1 Parabolas

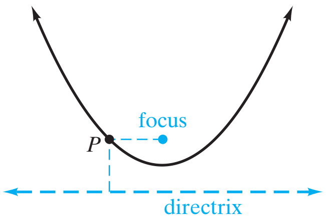
OBJECTIVES

1 Write quadratic equations in the form $y = a(x - h)^2 + k$.
2 Understand the effects of the constants h and k on the graph of a quadratic equation in the form $y = (x - h)^2 + k$.
3 Graph vertical parabolas.
4 Graph horizontal parabolas.
5 Write a quadratic equation for a parabola, given its vertex and a point on the curve.
6 Model real-world situations by using parabolas.

Conic sections are the curves obtained by intersecting a plane and a right circular cone. A plane perpendicular to the cone's axis cuts out a circle; a plane parallel to a side of the cone produces a parabola; a plane at an arbitrary angle to the axis of the cone forms an ellipse; and a plane parallel to the axis cuts out a hyperbola. If we extend the cone through its vertex and form a second cone, you find the second branch of the hyperbola. All these curves can be described as graphs of second-degree equations in two variables.



Circle Parabola Ellipse Hyperbola



We have described the shape of a graph of a quadratic equation as a parabola. In this section, we define that curve geometrically. A **parabola** is defined as the set of all points in a plane that are equidistant from a line and a point not on the line. The line is called the **directrix** and the point is called the **focus** (plural, *foci*).

12.1.1 Writing Quadratic Equations in the Form $y = a(x - h)^2 + k$

The standard form of a quadratic function is $y = ax^2 + bx + c$. In this section, we will write the quadratic equation in the form $y = a(x - h)^2 + k$, where a, h, and k are real numbers. To do that, we will need to complete the square.

To write a quadratic equation in the form $y = a(x - h)^2 + k$,

- Isolate the x-terms to one side of the equation.
- Factor out the leading coefficient.
- Add the value needed to complete the square to both sides of the equation.
- Rewrite the trinomial as a binomial squared.
- Solve the equation for y.

EXAMPLE 1 Write the quadratic equations in the form $y = a(x - h)^2 + k$. Identify a, h, and k.

a. $y = x^2 - 4x + 7$ **b.** $y = -2x^2 - 16x - 35$

Solution

a.

$$y = x^2 - 4x + 7$$

$$y - 7 = x^2 - 4x \quad \text{Isolate the } x\text{-terms.}$$

$$y - 7 + 4 = x^2 - 4x + 4 \quad \text{Add } \left(\tfrac{-4}{2}\right)^2 = 4 \text{ to both sides.}$$

$$y - 3 = (x - 2)^2 \quad \text{Write the trinomial as a binomial squared.}$$

$$y = (x - 2)^2 + 3 \quad \text{Solve for } y.$$

In the equation $y = (x - 2)^2 + 3$, $a = 1$, $h = 2$, and $k = 3$.

b.

$$y = -2x^2 - 16x - 35$$

$$y + 35 = -2x^2 - 16x \quad \text{Isolate the } x\text{-terms.}$$

$$y + 35 = -2(x^2 + 8x) \quad \text{Factor out the leading coefficient, } -2.$$

$$y + 35 + [-2(16)] = -2(x^2 + 8x + 16) \quad \text{Add } -2\left(\tfrac{8}{2}\right)^2\text{, or } -2(16)\text{, to both sides.}$$

$$y + 3 = -2(x + 4)^2 \quad \text{Write the trinomial as a binomial squared.}$$

$$y = -2(x + 4)^2 - 3 \quad \text{Solve for } y.$$

In the equation $y = -2(x + 4)^2 - 3$, or $y = -2[x - (-4)]^2 + (-3)$, $a = -2$, $h = -4$, and $k = -3$.

HELPING HAND Note that the leading coefficient a is also the same value as the real number a in our new equation.

12.1.1 Checkup

In exercises 1 and 2, write the quadratic equations in the form $y = a(x - h)^2 + k$. Identify a, h, and k.

1. $y = x^2 + 6x + 7$ **2.** $y = 3x^2 - 12x + 5$

3. When you rewrite a quadratic equation in the form $y = a(x - h)^2 + k$, is the new equation equivalent to the original equation? Explain.

12.1.2 Understanding the Effects of the Constants

In Chapter 5, we described the graph of a quadratic equation in two variables, $y = f(x) = ax^2 + bx + c$, as a parabola. We discovered that the values of the real numbers a, b, and c affected the shape of the graph.

If the quadratic equation is written in the form $y = a(x - h)^2 + k$, do the values of the real numbers a, h, and k affect the shape of the parabola? To find out, complete the following set of exercises.

Discovery 1

Effect of the Real Numbers *h* and *k* of a Quadratic Equation on a Parabola

1. Sketch the graphs of the following quadratic equations of the form $y = a(x - h)^2 + k$, where $a = 1$ and $h = 0$. Label the vertex of each graph.

$$y = x^2 + 2 \qquad y = x^2 - 2$$

2. Write a rule for determining the y-coordinate of the vertex of a parabola from the equation of the parabola.
3. Sketch the graphs of the following quadratic equations of the form $y = a(x - h)^2 + k$, where $a = 1$ and $k = 0$. Label the vertex of each graph.

$$y = (x + 3)^2 \qquad y = (x - 3)^2$$

4. Write a rule for determining the x-coordinate of the vertex of a parabola from the equation of the parabola.
5. Sketch the graph of the following quadratic equations of the form $y = a(x - h)^2 + k$, where $a = 1$. Label the vertex of each graph.

$$y = (x + 3)^2 + 2 \qquad y = (x - 3)^2 - 2$$

6. Write a rule for determining the coordinates of the vertex of a parabola from the equation of the parabola.

The vertex of the parabola is (h, k). Since the x-coordinate of the vertex is h, the axis of symmetry is the graph of the vertical line $x = h$. The graph of a quadratic equation of the form $y = a(x - h)^2 + k$ is called a **vertical parabola**, because its axis of symmetry is a vertical line.

Remember that the real number a is also the leading coefficient of the equation $y = ax^2 + bx + c$. Therefore, it also affects the shape of the graph.

SUMMARY OF THE EFFECTS OF THE REAL NUMBERS *a*, *h*, AND *k* OF A QUADRATIC EQUATION ON A VERTICAL PARABOLA

The real numbers a, h, and k of a quadratic equation in the form $y = a(x - h)^2 + k$ affect the graph of the equation.

If $a > 0$, then the graph is concave upward (opens upward).
If $a < 0$, then the graph is concave downward (opens downward).
If $|a| > 1$, then the graph is narrower than it would be if $a = 1$.
If $|a| < 1$, then the graph is wider than it would be if $a = 1$.

The vertex of the graph is (h, k).
The axis of symmetry is the line graphed by $x = h$.

EXAMPLE 2 Determine the vertex and axis of symmetry for the graph of each equation. Describe the graph, but do not draw it.

a. $y = 2(x - 4)^2 - 3$ **b.** $y = -x^2 - 4x - 8$

Solution

a. $y = 2(x - 4)^2 - 3$

or $y = 2(x - 4)^2 + (-3)$ — Write the equation in $y = a(x - h)^2 + k$ form.

We see that $a = 2$, $h = 4$, and $k = -3$.

Since $a = 2$ and $2 > 0$, the graph is concave upward.

Since $|2| = 2 > 1$, then the graph is narrower than it would be if $a = 1$.

The vertex is (h, k), or $(4, -3)$.

The axis of symmetry is the graph of $x = 4$.

b. $y = -x^2 - 4x - 8$

First, write the equation in the form $y = a(x - h)^2 + k$.

$y + 8 = -x^2 - 4x$ — Isolate the x-terms.

$y + 8 = -1(x^2 + 4x)$ — Factor out the leading coefficient.

$y + 8 + [-1(4)] = -1(x^2 + 4x + 4)$ — Add $-1(\frac{4}{2})^2$, or $-1(4)$, to both sides.

$y + 4 = -1(x + 2)^2$ — Write the trinomial as a binomial squared.

$y = -1(x + 2)^2 - 4$ — Solve for y.

or $y = -1[x - (-2)]^2 + (-4)$

Therefore, $a = -1$, $h = -2$, and $k = -4$.

Since $a = -1$ and $-1 < 0$, the graph is concave downward.

Since $|-1| = 1$, the graph is the same width as it would be if $a = 1$.

The vertex is (h, k), or $(-2,-4)$.

The axis of symmetry is the graph of $x = -2$.

12.1.2 Checkup

In exercises 1 and 2, determine the vertex and axis of symmetry for the graph of each equation. Describe the graph, but do not draw it.

1. $y = 0.5(x + 3)^2 + 1$

2. $y = -x^2 + 6x - 5$

3. Summarize how you can determine the concavity and narrowness of a parabola from the constants in its equation.

4. Summarize how you can determine the vertex and the axis of symmetry of a parabola from the constants in its equation.

12.1.3 Graphing a Vertical Parabola

To graph a parabola, we use the information that we can determine from its equation and add points to establish a pattern for the curve.

> To graph a vertical parabola,
>
> - Locate and label the vertex, (h, k).
> - Graph the axis of symmetry, $x = h$, with a dashed line.
> - Graph enough points to see a pattern. The x- and y-intercepts are important points to determine. Connect the points with a smooth curve.

EXAMPLE 3 Graph the vertical parabola for $y = 2(x - 4)^2 - 3$.

Solution

$y = 2(x - 4)^2 - 3$

$y = 2(x - 4)^2 + (-3)$ — Write the equation in the form $y = a(x - h)^2 + k$.

Therefore, $a = 2$, $h = 4$, and $k = -3$.

The vertex is (h, k), or $(4, -3)$.

The axis of symmetry is the line $x = 4$.

The graph opens upward, because $a = 2 > 0$.

The graph is narrower than it would be if $a = 1$, because $|a| = |2| > 1$.

The y-intercept is the point on the graph where $x = 0$. Substitute 0 for x and solve for y.

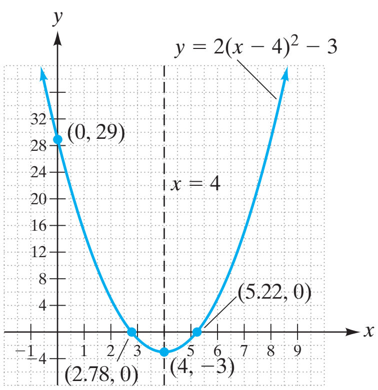
$$y = 2(x - 4)^2 - 3$$
$$y = 2(0 - 4)^2 - 3$$
$$y = 2(16) - 3$$
$$y = 32 - 3$$
$$y = 29$$

The y-intercept is $(0, 29)$.

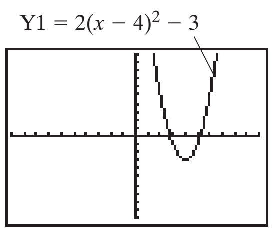
The x-intercept is the point on the graph where $y = 0$. Therefore, substitute 0 for y and solve for x.

$$y = 2(x - 4)^2 - 3$$
$$0 = 2(x - 4)^2 - 3$$
$$2(x - 4)^2 = 3$$
$$(x - 4)^2 = \frac{3}{2}$$
$$x - 4 = \pm\sqrt{\frac{3}{2}}$$
$$x = 4 \pm \sqrt{\frac{3}{2}}$$
$$x = 4 \pm \sqrt{\frac{3}{2} \cdot \frac{2}{2}}$$
$$x = 4 \pm \frac{\sqrt{6}}{2}$$
$$x = \frac{8 \pm \sqrt{6}}{2}$$

The x-intercepts are about $(5.22, 0)$ and $(2.78, 0)$.



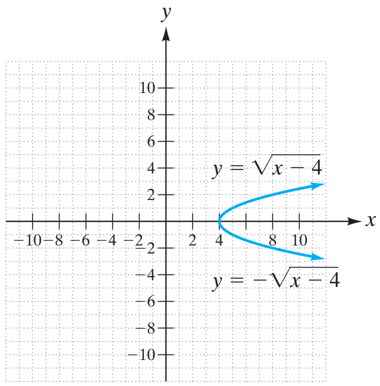
Calculator Check

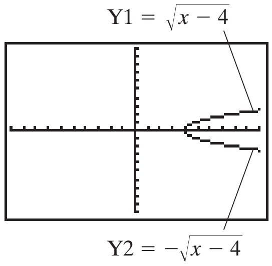


$(-10, 10, -10, 10)$

12.1.3 Checkup

1. Graph the vertical parabola for $y = -\frac{2}{3}(x + 5)^2 + 2$.
2. Name some points that are useful to locate in graphing a vertical parabola, and explain why they are useful.





(−10, 10, −10, 10)

12.1.4 Graphing a Horizontal Parabola

Some parabolas open left or right. In such a case, the parabola has a horizontal axis of symmetry and is called a **horizontal parabola**.

For example, graph the horizontal parabola $x = y^2 + 4$.

In order to graph this parabola, we will solve for y.

$$\begin{aligned} x &= y^2 + 4 \\ x - 4 &= y^2 \\ y^2 &= x - 4 \\ y &= \pm\sqrt{x - 4} \end{aligned}$$

We will graph two curves, $y = \sqrt{x-4}$ and $y = -\sqrt{x-4}$, on the same coordinate plane.

To graph a horizontal parabola on your calculator, graph two curves. Note that a horizontal parabola does not represent a function, because it does not pass the vertical-line test.

The form of a quadratic equation that will graph a horizontal parabola is $x = a(y - k)^2 + h$. To write an equation in this form, we complete the square as we did for a vertical parabola, only this time for the variable y instead of x. In addition, just as the real numbers a, h, and k affect a vertical parabola, they also affect a horizontal parabola.

SUMMARY OF THE EFFECTS OF THE REAL NUMBERS *a*, *h*, AND *k* OF A QUADRATIC EQUATION ON A HORIZONTAL PARABOLA

The real numbers a, h, and k of a quadratic equation in the form $x = a(y - k)^2 + h$ affect its graph.

If $a > 0$, then the graph opens to the right.
If $a < 0$, then the graph opens to the left.
If $|a| > 1$, then the graph is narrower than it would be if $a = 1$.
If $|a| < 1$, then the graph is wider than it would be if $a = 1$.
The vertex of the graph is (h, k).
The axis of symmetry is the line graphed by $y = k$.

EXAMPLE 4 Graph the horizontal parabolas.

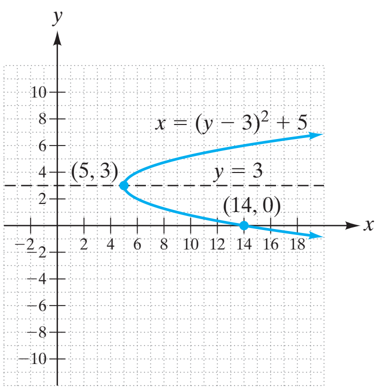
a. $x = (y - 3)^2 + 5$ **b.** $x = -2y^2 - 4y - 5$

Solution

a. $x = (y - 3)^2 + 5$

The equation is written in the form $x = a(y - k)^2 + h$. Therefore, $a = 1$, $h = 5$, and $k = 3$.

The vertex is (h, k), or $(5, 3)$.
The axis of symmetry is the graph of $y = k$, or $y = 3$.
The graph opens to the right, because $a = 1 > 0$.



The y-intercept is the point on the graph where $x = 0$. Substitute 0 for x and solve for y.

$$x = (y - 3)^2 + 5$$
$$0 = (y - 3)^2 + 5$$
$$(y - 3)^2 = -5$$
$$y - 3 = \pm\sqrt{-5}$$
$$y = 3 \pm i\sqrt{5}$$

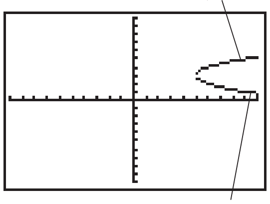
y is an imaginary number. Therefore, the graph has no y-intercept. In fact, we know this is so because the vertex is located at (3, 5) and opens to the right.

The x-intercept is the point on the graph where $y = 0$.

$$x = (y - 3)^2 + 5$$
$$x = (0 - 3)^2 + 5$$
$$x = 14$$

The x-intercept is (14, 0).

Calculator Check



$(-10, 10, -10, 10)$

To graph these equations on your calculator, you will need to solve for y.

$$x = (y - 3)^2 + 5$$
$$x - 5 = (y - 3)^2 \quad \text{Isolate the } y \text{ terms to one side of the equation.}$$
$$\pm\sqrt{x - 5} = y - 3 \quad \text{Principle of square roots}$$
$$3 \pm \sqrt{x - 5} = y \quad \text{Solve for } y.$$
$$y = 3 \pm \sqrt{x - 5}$$

b. $x = -2y^2 - 4y - 5$

First, write the equation in the form $x = (y - k)^2 + h$.

$$x = -2y^2 - 4y - 5$$
$$x + 5 = -2y^2 - 4y \quad \text{Isolate the } y \text{ terms to one side of the equation.}$$
$$x + 5 = -2(y^2 + 2y) \quad \text{Factor out the leading coefficient.}$$
$$x + 5 + [-2(1)] = -2(y^2 + 2y + 1) \quad \text{Add } -2\left(\frac{2}{2}\right)^2 = -2(1) \text{ to complete the square.}$$
$$x + 3 = -2(y + 1)^2 \quad \text{Write the trinomial as a binomial squared.}$$
$$x = -2(y + 1)^2 - 3 \quad \text{Solve for } x.$$
$$x = -2[y - (-1)]^2 + (-3) \quad \text{Write in the form } x = a(y - k)^2 + h.$$

We see that $a = -2$, $h = -3$, and $k = -1$.

The vertex is $(-3, -1)$.

The axis of symmetry is $y = -1$.

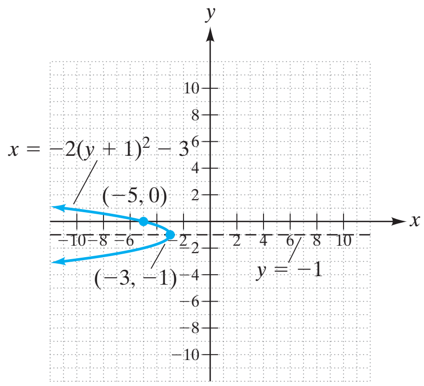
The parabola opens to the left, because $a = -2 < 0$.

The graph has no y-intercept, because the vertex of the graph is located at $(-3, -1)$ and the graph opens to the left.

Determine the x-intercept by substituting 0 for y and solving for x.

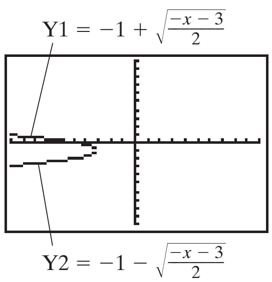
$$x = -2(y + 1)^2 - 3$$
$$x = -2(0 + 1)^2 - 3$$
$$x = -5$$

The x-intercept is $(-5, 0)$.



Calculator Check

Solve the equation for y. First, we complete the square as in the previous solution, obtaining $x = -2(y + 1)^2 - 3$.



(−10, 10, −10, 10)

$$x = -2(y + 1)^2 - 3$$

$$x + 3 = -2(y + 1)^2 \qquad \text{Isolate the } y \text{ terms to one side of the equation.}$$

$$\frac{-x - 3}{2} = (y + 1)^2 \qquad \text{Divide both sides by } -2.$$

$$\pm\sqrt{\frac{-x - 3}{2}} = y + 1 \qquad \text{Principle of square roots}$$

$$-1 \pm \sqrt{\frac{-x - 3}{2}} = y \qquad \text{Solve for } y.$$

$$y = -1 \pm \sqrt{\frac{-x - 3}{2}}$$

12.1.4 Checkup

In exercises 1 and 2, graph the horizontal parabola for each relation.

1. $x = 2(y + 1)^2 - 3$

2. $x = -\frac{1}{2}y^2 + 4y - 7$

3. How does a horizontal parabola differ from a vertical one?

4. How can the constants in the equation for a horizontal parabola help you graph the equation?

12.1.5 Writing Quadratic Equations, Given the Vertex and a Point on the Graph

Earlier, we learned that although two points determine a straight line, you need three points to determine a curve. Thus, an infinite number of parabolas can be drawn through any two given points. However, if we know that one of these points is the vertex and if we know that the parabola is vertical or horizontal, then we can write an equation for the specific parabola that passes through these points.

EXAMPLE 5

a. Write an equation of a vertical parabola with a vertex of (2, 6) and passing through the point (−1, 4).

b. Write an equation of a horizontal parabola with a vertex of (−1, 1) and a y-intercept of (0, 2).

Solution

a. Since the vertex is (2, 6), it follows that $h = 2$ and $k = 6$. Also, we know that the point (−1, 4) is a solution of the equation. We will substitute −1 for x and 4 for y, as well as 2 for h and 6 for k, in the equation $y = a(x - h)^2 + k$ and solve for a.

$$y = a(x - h)^2 + k$$
$$4 = a[(-1) - 2]^2 + 6 \quad \text{Substitute.}$$
$$4 = a(-3)^2 + 6$$
$$4 = 9a + 6$$
$$a = -\frac{2}{9}$$

Now, we write an equation using the known values for a, h, and k.

$$y = a(x - h)^2 + k$$
$$y = -\frac{2}{9}(x - 2)^2 + 6$$

The graph of the equation $y = -\frac{2}{9}(x - 2)^2 + 6$ is a vertical parabola with a vertex of (2, 6) and passing through the point (−1, 4).

b. First, we substitute values for h, k, x, and y. Given the vertex (−1, 1), we know that $h = -1$ and $k = 1$.

We use the coordinates of the y-intercept for x and y, $x = 0$ and $y = 2$. Then we solve for a.

$$x = a(y - k)^2 + h$$
$$0 = a(2 - 1)^2 + (-1) \quad \text{Substitute.}$$
$$0 = a(1)^2 - 1$$
$$0 = a - 1$$
$$a = 1$$

Write an equation using $a = 1$, $h = -1$, and $k = 1$.

$$x = a(y - k)^2 + h$$
$$x = 1(y - 1)^2 + (-1)$$
$$x = (y - 1)^2 - 1$$

The graph of the equation $x = (y - 1)^2 - 1$ is a horizontal parabola with a vertex of (−1, 1) and a y-intercept of (0, 2).

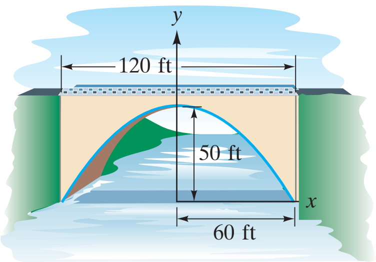
12.1.5 Checkup

1. Write an equation of a vertical parabola with a vertex of (−6, 2) and passing through (−3, 5).
2. Write an equation of a horizontal parabola with a vertex of (1, −2) and passing through (9, −4).
3. When you are asked to write an equation of a parabola and are given points on its graph to do so, is it important to know whether the parabola is a vertical parabola or a horizontal parabola? Explain.

12.1.6 Modeling the Real World

Parabolas occur in many different applications. For example, architects and engineers frequently use the shape of a parabola for support arches in bridges and buildings. The suspension cables used to support bridges like the Golden Gate in San Francisco or the Verrazano Narrows in New York City are parabolic in shape. (Free-hanging cables, such as telephone lines, have a shape called a *catenary*, which we will not discuss in this text.) Parabolic reflectors have the property that all light striking the inside surface of the reflector and coming in parallel to the axis of the parabola is reflected to the same point, the focus of the parabola. Thus, parabolic reflectors are used in television receiving dishes and mirrors in astronomical telescopes.

EXAMPLE 6 A concrete bridge is designed with an arch in the shape of a parabola. The road over the bridge is 120 feet long and the maximum height of the arch is 50 feet. Write an equation for the parabolic arch.



Solution

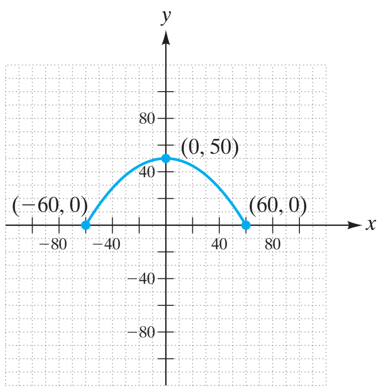
Graph the curve and label the vertex and both x-intercepts.

According to the figure, the vertex of the parabola is located at its maximum height, or (0, 50). The x-intercepts are (60, 0) and (−60, 0). We need only one x-intercept to find an equation, so we will use (60, 0). Substitute 0 for h, 50 for k, 60 for x, and 0 for y in the equation for a vertical parabola, and solve for a.

$$y = a(x - h)^2 + k$$
$$0 = a(60 - 0)^2 + 50$$
$$0 = 3600a + 50$$
$$a = -\frac{1}{72}$$

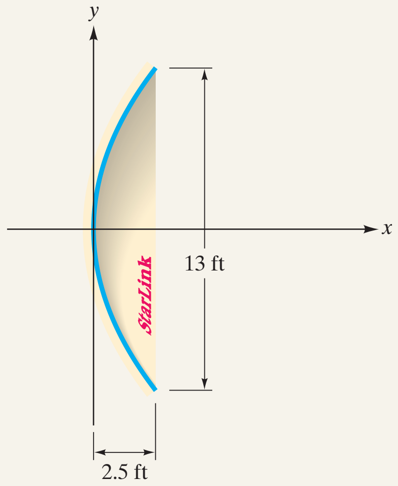
We will limit our graph to a portion of an arch by restricting the domain to $-60 \le x \le 60$.

An equation for the parabola is $y = -\frac{1}{72}(x - 0)^2 + 50$, or $y = -\frac{1}{72}x^2 + 50$, where $-60 \le x \le 60$, measured in feet.



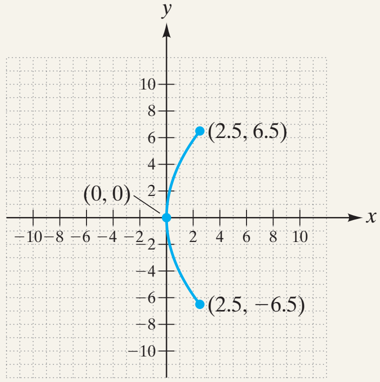
APPLICATION

A satellite dish receiver is in the shape of a parabola. A cross section of the dish shows a diameter of 13 feet at a distance of 2.5 feet from the vertex of the parabola. Write an equation for the parabola.



Discussion

According to the figure, the horizontal parabola has a vertex at the origin. If the parabola is 13 feet in diameter at a distance along the axis of 2.5 feet from the vertex, the radius is 6.5 feet. We can thus label two points on the parabola as (2.5, 6.5) and (2.5, −6.5).



Substitute 0 for h, 0 for k, 2.5 for x, and 6.5 for y in the equation for a horizontal parabola, and solve for a.

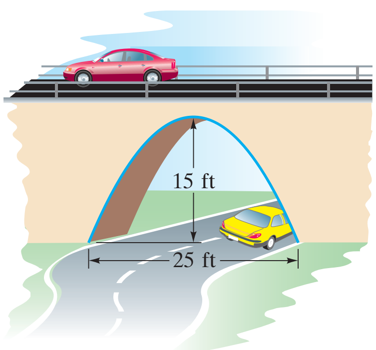
$$x = a(y - k)^2 + h$$
$$2.5 = a(6.5 - 0)^2 + 0$$
$$2.5 = 42.25a$$
$$a = \frac{10}{169}$$

Since our object is not modeled by the complete graph, we need to limit the x-values by restricting the domain.

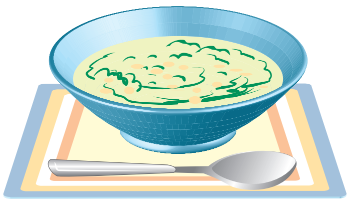
An equation for the parabola is $x = \frac{10}{169}y^2$, where $-6.5 \le y \le 6.5$.

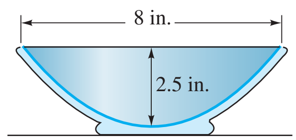
12.1.6 Checkup

1. An arched underpass has the shape of a parabola. A road passing under the arch is 25 feet wide, and the maximum height of the arch is 15 feet. Write an equation for the parabolic arch.



2. A soup bowl has a cross section with a parabolic shape, as shown in the figure. The bowl has a diameter of 8 inches and is 2.5 inches deep. Write an equation for its shape.





12.1 Exercises

Write each equation in the form $y = a(x - h)^2 + k$.

1. $y = 2x^2 - 36x + 165$ **2.** $y = -4x^2 + 24x - 34$ **3.** $y = 3x^2 + 48x + 197$
4. $y = -2x^2 - 28x - 89$ **5.** $y = -3x^2 + 42x - 157$ **6.** $y = 5x^2 - 30x + 32$
7. $y = -x^2 - 20x - 103$ **8.** $y = 2x^2 + 44x + 227$ **9.** $y = \frac{1}{2}x^2 - 4x + 13$
10. $y = \frac{2}{3}x^2 + 8x + 7$ **11.** $y = 1.5x^2 - 6x + 2.4$ **12.** $y = -2.5x^2 + 20x - 34.3$

Determine the vertex and axis of symmetry, and describe the graph, for each equation. Do not graph the equation.

13. $y = 3(x - 12)^2 + 5$ **14.** $y = -5(x + 4)^2 + 9$ **15.** $y = -2(x + 13)^2 - 5$
16. $y = 4(x - 5)^2 - 7$ **17.** $y = \frac{2}{5}(x - 5)^2 + 7$ **18.** $y = -\frac{1}{5}(x + 10)^2 - 11$
19. $y = -\frac{1}{4}(x + 3)^2 - 10$ **20.** $y = \frac{5}{7}(x + 14)^2 - 7$ **21.** $y = -0.8(x + 9)^2 - 1$
22. $y = 3.7(x - 1)^2 + 4.8$ **23.** $y = 4.2(x - 12)^2 - 5$ **24.** $y = -0.75(x + 5)^2 - 6$
25. $y = 4x^2 + 24x + 25$ **26.** $y = -3x^2 + 12x$ **27.** $y = -\frac{1}{2}x^2 + 6x - 7$
28. $y = \frac{2}{3}x^2 - 8x + 35$ **29.** $y = 1.2x^2 + 2.4x - 4.6$ **30.** $y = -0.8x^2 + 8x - 9.3$

Graph each parabola.

31. $y = 3(x + 5)^2 - 1$ **32.** $y = (x + 2)^2 + 3$ **33.** $y = -(x - 2)^2 - 2$
34. $y = -4(x - 1)^2 + 1$ **35.** $y = \frac{1}{2}(x + 2)^2 - 5$ **36.** $y = -\frac{3}{4}(x + 2)^2 + 3$
37. $y = -\frac{2}{3}(x - 3)^2 + 4$ **38.** $y = \frac{4}{5}x^2 - 3$ **39.** $y = 1.6x^2$
40. $y = 0.5(x + 2)^2$ **41.** $y = -0.2(x + 1)^2 + 2$ **42.** $y = -1.4x^2 + 3$
43. $x = (y + 3)^2 - 1$ **44.** $x = -3(y + 1)^2 + 8$ **45.** $x = 2(y - 5)^2$
46. $x = 2y^2 - 3$ **47.** $x = -\frac{1}{2}(y + 4)^2 - 7$ **48.** $x = \frac{1}{4}(y - 2)^2$
49. $x = 0.8(y - 1)^2 + 2$ **50.** $x = -1.5(y + 2)^2 + 2.5$ **51.** $y = 2x^2 - 12x + 23$
52. $x = 3y^2 + 6y - 5$ **53.** $x = -2y^2 - 16y - 22$ **54.** $y = -4x^2 - 16x - 7$
55. $y = \frac{1}{3}x^2 + 4x + 7$ **56.** $x = \frac{3}{4}y^2 + 3y - 2$ **57.** $x = -0.4y^2 - 0.8y + 2.6$
58. $y = -1.5x^2 - 3x - 4.5$

Write an equation of the vertical parabola with the given vertex and passing through the given point.

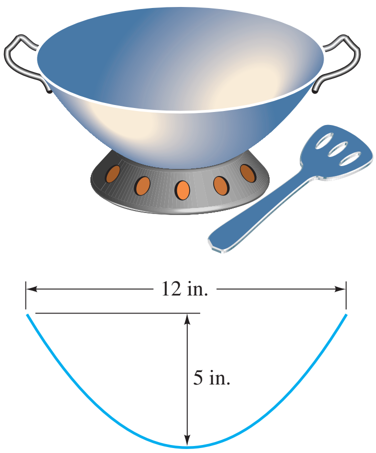
59. Vertex at $(3, -2)$ and passing through $(5, 2)$.
60. Vertex at $(5, 3)$ and passing through $(3, -1)$.
61. Vertex at $(-1, 4)$ and passing through $(-3, 12)$.
62. Vertex at $(-2, -1)$ and passing through $(1, 26)$.
63. Vertex at $(-4, -5)$ and passing through $(0, 3)$.
64. Vertex at $(-3, 11)$ and passing through $(3, -13)$.
65. Vertex at $(0, -2)$ and passing through $(-2, 0)$.
66. Vertex at $(4, -5)$ and passing through $(6, 1)$.

In exercises 67–72, write an equation of the horizontal parabola with the given vertex and passing through the given point.

67. Vertex at $(4, 3)$ and passing through $(6, 2)$.
68. Vertex at $(-3, -5)$ and passing through $(-5, -6)$.
69. Vertex at $(-5, -4)$ and passing through $(-7, -2)$.
70. Vertex at $(4, -3)$ and passing through $(10, -6)$.
71. Vertex at $(-8, 2)$ and passing through $(2, -3)$.
72. Vertex at $(0, -1)$ and passing through $(-6, 1)$.
73. A steel bridge has an arch shaped like a parabola. The maximum height of the arch is 35 feet, and the road on the bridge spans a distance of 90 feet. Write an equation for the parabolic arch.
74. A wooden bridge has an arch shaped like a parabola, with a maximum height of 20 feet. The road on the bridge stretches a distance of 75 feet. Write an equation for the parabolic arch of the bridge.

75. A serving dish has a cross section with a parabolic shape, with a diameter of 10 inches and a depth of 4.5 inches. Write an equation for the cross-sectional shape of the dish.

76. A wok (a pan used for stir-frying) has a cross section with a parabolic shape. The wok has a diameter of 12 inches and a depth of 5 inches. Write an equation for the cross-sectional shape of the wok.



77. An architect wishes to build a parabolic-shaped arch as the entryway to a building she is designing. She wishes to have the maximum height of the arch be 9 feet and the width at the base be 9 feet. Write an equation for the shape of the arch.

78. The cross section of an earthen dam has a parabolic shape with a maximum height of 10 feet. The base of the cross section measures 16 feet. Write an equation for the shape of the cross section of the dam.

12.1 Calculator Exercises

To graph a horizontal parabola with your calculator, you will need to solve the equation $x = a(y - k)^2 + h$ for y.

$$x = a(y - k)^2 + h$$

$$a(y - k)^2 + h = x \qquad \text{Exchange expressions.}$$

$$a(y - k)^2 = x - h \qquad \text{Subtract } h \text{ from both expressions.}$$

$$(y - k)^2 = \frac{x - h}{a} \qquad \text{Divide both expressions by } a.$$

$$y - k = \pm\sqrt{\frac{x - h}{a}} \qquad \text{Apply the square-root principle.}$$

$$y = k \pm \sqrt{\frac{x - h}{a}} \qquad \text{Add } k \text{ to both expressions.}$$

Therefore, to graph the parabola using your calculator, first store $Y1 = K + \sqrt{\frac{(X - H)}{A}}$ and $Y2 = K - \sqrt{\frac{(X - H)}{A}}$. Then store values for a, h, and k in the locations A, H, and K, and graph the curve. In this way, you need only store new values of a, h, and k each time to graph, saving you the trouble of repeatedly entering expressions for Y1 and Y2. For example, if you wish to graph $x = (y - 3)^2 + 5$, note that $h = 5$, $k = 3$, and $a = 1$. Store 1 in A, 5 in H, and 3 in K. Then press [GRAPH], and the calculator will graph the two functions stored in Y1 and Y2.

Use this procedure to graph the following equations:

1. $x = 2(y - 5)^2 + 3$

2. $x + 4 = \frac{1}{2}(y + 3)^2$

3. $x - 2 = 0.3y^2$

12.1 Writing Exercise

In this section, you saw that the equation of a parabola can be used to describe an arch such as one you might see on a bridge or an entranceway. Are all arches parabolic in shape? Try to find a reference that discusses arches and explains the various shapes you might encounter. Summarize what you have found in a short paper, listing the reference you used.

12.2 Circles

OBJECTIVES

1. Graph circles with the center at the origin.
2. Graph circles with the center not at the origin.
3. Write equations of circles.
4. Model real-world situations by using circles.

APPLICATION

Many railroad viaducts are constructed in the shape of a semicircle. A stone-arch railroad viaduct at Rockville, Pennsylvania, over the Susquehanna River is made of 48 semicircular arches, each with a span of 70 feet. Use Figure 12.1 to write equations that model each of the first two arches.

After completing this section, we will discuss this application further. See page 000.

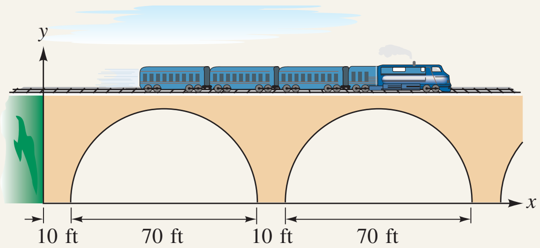
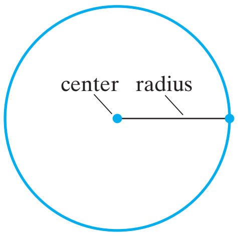


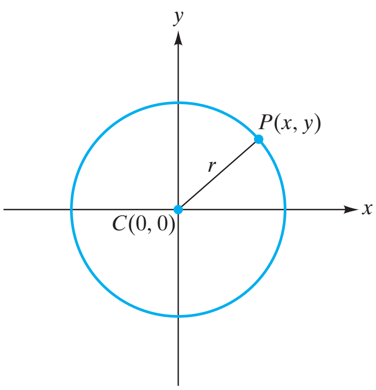
Figure 12.1



A **circle** is the set of points in a plane that are equidistant from a given point, called the **center**. The **radius** (plural, *radii*) of the circle is the distance between each of its points and the center.

12.2.1 Graphing Circles with the Center at the Origin

Since a circle is defined as a set of points equidistant from the center, we can use the distance formula to determine an equation of a circle. We begin with a circle having its center at the origin $C(0, 0)$, passing through a point $P(x, y)$, and having a radius r.



$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \quad \text{Distance formula}$$
$$r = \sqrt{(x - 0)^2 + (y - 0)^2} \quad \text{Substitute.}$$
$$r^2 = x^2 + y^2 \quad \text{Square both sides.}$$
$$\text{or} \quad x^2 + y^2 = r^2$$

CENTER–RADIUS FORM OF THE EQUATION OF A CIRCLE WITH CENTER AT THE ORIGIN

The center–radius form of the equation of a circle with its center at the origin and radius r is

$$x^2 + y^2 = r^2, \text{ where } r > 0$$

We can now graph a circle, using this equation.

To graph a circle with the center at the origin,

- Locate and label the origin as the center of the circle.
- Locate and label the two x-intercepts $(r, 0)$ and $(-r, 0)$.
- Locate and label the two y-intercepts $(0, r)$ and $(0, -r)$.
- Sketch the circle containing the preceding two points.

EXAMPLE 1 Graph.

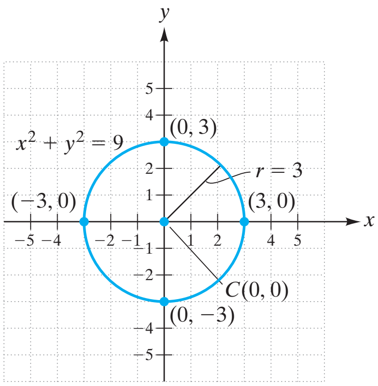
a. $x^2 + y^2 = 9$ **b.** $x^2 + y^2 = 10$

Solution

a. $x^2 + y^2 = 9$

$x^2 + y^2 = 3^2$ Rewrite the constant as a square.

Since $r^2 = 9 = 3^2$, it follows that $r = 3$. The circle has its center at the origin and a radius of 3.



Calculator Check

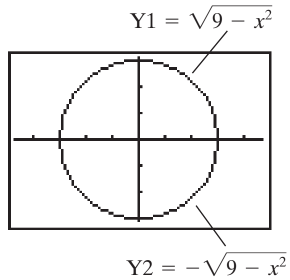
To graph the equation on your calculator, first solve for y.

$x^2 + y^2 = 9$

$y^2 = 9 - x^2$ Subtract x^2 from both sides.

$y = \pm\sqrt{9 - x^2}$ Principle of square roots

The graph does not represent a function and must be entered as two functions. The graph of $y = \sqrt{9 - x^2}$ is a semicircle above the x-axis, and the graph of $y = -\sqrt{9 - x^2}$ is a semicircle below the x-axis.

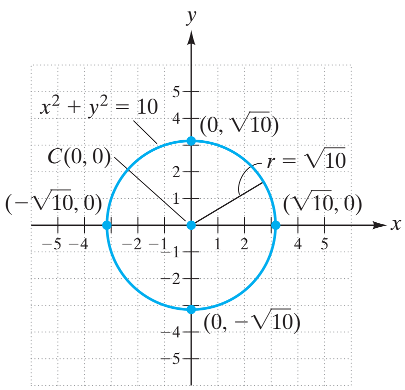


(−4.7, 4.7, −3.1, 3.1)

b. $x^2 + y^2 = 10$

$x^2 + y^2 = (\sqrt{10})^2$ Rewrite the constant as a square.

Since $r^2 = 10 = (\sqrt{10})^2$, it follows that $r = \sqrt{10} \approx 3.16$.



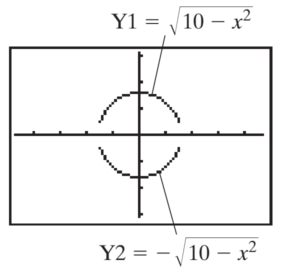
Calculator Check

To graph the equation on your calculator, first solve for y.

$x^2 + y^2 = 10$

$y^2 = 10 - x^2$ Subtract x^2 from both sides.

$y = \pm\sqrt{10 - x^2}$ Principle of square roots



(−9.4, 9.4, 2, −6.2, 6.2, 2)

HELPING HAND We must be careful which window we use to produce the graph. The window must be of square dimensions, or the graph will be distorted and not appear to be a circle. Producing a square window setting is discussed in the Calculator Exercises at the end of this section.

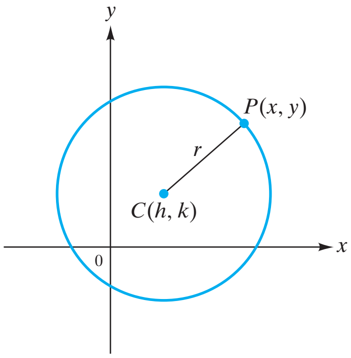
12.2.1 Checkup

In exercises 1 and 2, sketch the graph of the equation. Check the graphs on your calculator.

1. $x^2 + y^2 = 25$ **2.** $x^2 + y^2 = 30$

3. Explain why the equation $x^2 + y^2 = r^2$ is not a function when solved for y.

12.2.2 Graphing Circles with the Center Not at the Origin



We can also use the distance formula to determine an equation of a circle having a radius r and a center $C(h, k)$ that is not at the origin. We begin with a circle having a center $C(h, k)$ and passing through a point $P(x, y)$.

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \qquad \text{Distance formula.}$$

$$r = \sqrt{(x - h)^2 + (y - k)^2} \qquad \text{Substitute.}$$

$$r^2 = (x - h)^2 + (y - k)^2 \qquad \text{Square both sides.}$$

or $(x - h)^2 + (y - k)^2 = r^2$

CENTER–RADIUS FORM OF THE EQUATION OF A CIRCLE

The center–radius form of the equation of a circle with center $C(h, k)$ and radius r is

$$(x - h)^2 + (y - k)^2 = r^2$$

HELPING HAND The center–radius form also applies to a circle with its center at the origin.

$$(x - h)^2 + (y - k)^2 = r^2$$
$$(x - 0)^2 + (y - 0)^2 = r^2$$
$$x^2 + y^2 = r^2$$

To graph a circle with its center not at the origin,

- Locate and label the center of the circle, (h, k).
- Locate and label the two points on the graph that are a horizontal distance r from the center.
- Locate and label the two points on the graph that are a vertical distance r from the center.
- Sketch the circle containing the preceding two points.

EXAMPLE 2 Sketch the graph of each equation.

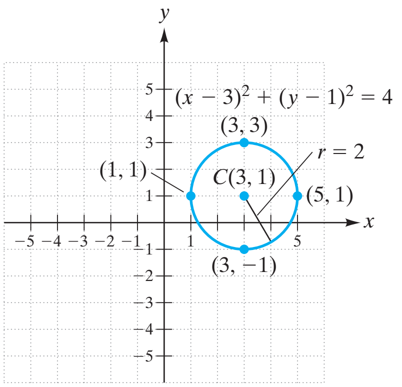
a. $(x - 3)^2 + (y - 1)^2 = 4$ **b.** $(x + 3)^2 + y^2 = 11$

Solution

a. $(x - 3)^2 + (y - 1)^2 = 4$

$(x - 3)^2 + (y - 1)^2 = 2^2$

We have the equation for a circle with $h = 3, k = 1$, and $r = 2$. The center of the circle, (h, k), is $(3, 1)$, and the radius is 2.



Calculator Check

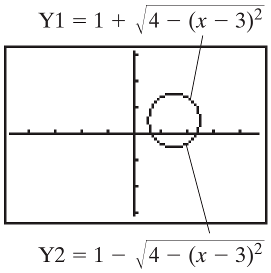
To graph this equation on your calculator, you must solve for y.

$(x - 3)^2 + (y - 1)^2 = 4$

$(y - 1)^2 = 4 - (x - 3)^2$ *Isolate the y-terms.*

$y - 1 = \pm\sqrt{4 - (x - 3)^2}$ *Principle of square roots*

$y = 1 \pm \sqrt{4 - (x - 3)^2}$ *Solve for y.*

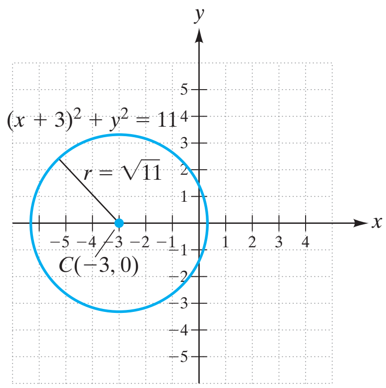


$(-9.4, 9.4, 2, -6.2, 6.2, 2)$

b. $(x + 3)^2 + y^2 = 11$

$[x - (-3)]^2 + (y - 0)^2 = (\sqrt{11})^2$ *Center–radius form for a circle*

Therefore, $h = -3, k = 0$, and $r = \sqrt{11}$. The center is $(-3, 0)$ and the radius is $\sqrt{11} \approx 3.32$.

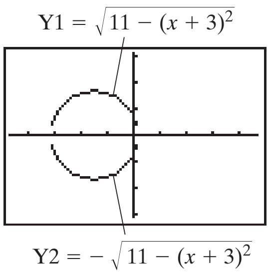


To graph this equation on your calculator, you must solve for y.

$(x + 3)^2 + y^2 = 11$

$y^2 = 11 - (x + 3)^2$ *Isolate the y-terms.*

$y = \pm\sqrt{11 - (x + 3)^2}$ *Solve for y.*



$(-9.4, 9.4, 2, -6.2, 6.2, 2)$

If we can write an equation in the center–radius form, we can sketch the circular graph. In Example 3, we must first write the equation in center–radius form by completing the square twice before we graph.

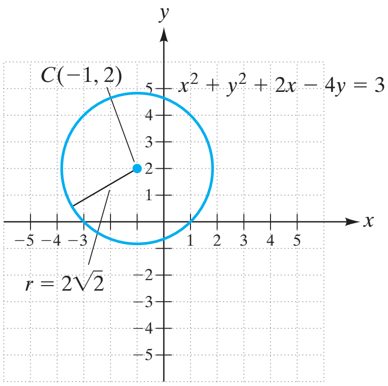
EXAMPLE 3 Sketch the graph of the equation $x^2 + y^2 + 2x - 4y = 3$.

Solution

$$x^2 + y^2 + 2x - 4y = 3$$

Group the x-terms and the y-terms.

$$(x^2 + 2x) + (y^2 - 4y) = 3$$



Complete both squares. For the x-terms, add 1 to both sides because $\left(\frac{2}{2}\right)^2 = 1$. For the y-terms, add 4 to both sides because $\left(\frac{-4}{2}\right)^2 = 4$.

$$(x^2 + 2x + 1) + (y^2 - 4y + 4) = 3 + 1 + 4$$

Write each trinomial as a binomial square, and simplify the right side of the equation.

$$(x + 1)^2 + (y - 2)^2 = 8$$

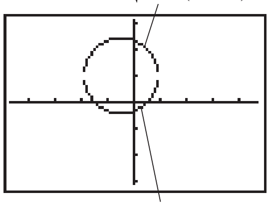
Write the equation in center–radius form.

$$[x - (-1)]^2 + (y - 2)^2 = (2\sqrt{2})^2$$

Remember, $\sqrt{8} = 2\sqrt{2}$.

The center of the circle is $(-1, 2)$ and the radius is $2\sqrt{2} \approx 2.83$.

$\text{Y1} = 2 + \sqrt{8 - (x + 1)^2}$



$\text{Y2} = 2 - \sqrt{8 - (x + 1)^2}$

$(-9.4, 9.4, 2, -6.2, 6.2, 2)$

Calculator Check

Solve the equation for y. Complete the squares as in the previous solution.

$$x^2 + y^2 + 2x - 4y = 3$$
$$(x + 1)^2 + (y - 2)^2 = 8$$
$$(y - 2)^2 = 8 - (x + 1)^2 \quad \text{Isolate the } y\text{-terms.}$$
$$y - 2 = \pm\sqrt{8 - (x + 1)^2} \quad \text{Principle of square roots}$$
$$y = 2 \pm \sqrt{8 - (x + 1)^2}$$

12.2.2 Checkup

In exercises 1–3, sketch the graph of each equation. Check, using your calculator.

1. $(x - 2)^2 + (y - 2)^2 = 4$ **2.** $(x + 2)^2 + (y + 2)^2 = 9$ **3.** $x^2 + y^2 - 6x + 4y = 3$

4. Explain how the constants in the center–radius form of the equation of a circle help you to graph the equation.

12.2.3 Writing Equations of Circles

If we can determine the center and radius of a circle, we can use the center–radius form to write an equation of the circle.

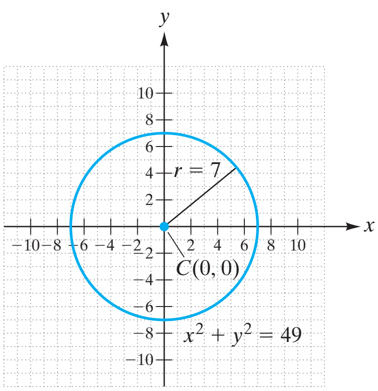
EXAMPLE 4 Write an equation of each circle, given the following information:

a. Center at the origin and radius 7

b. Center at $(-1, 3)$ and radius 4

c. Center at $(3, 1)$ and passing through the point $(5, 4)$

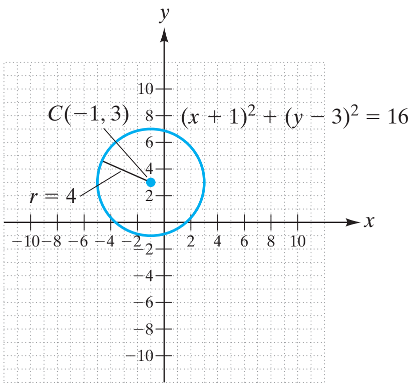
Solution

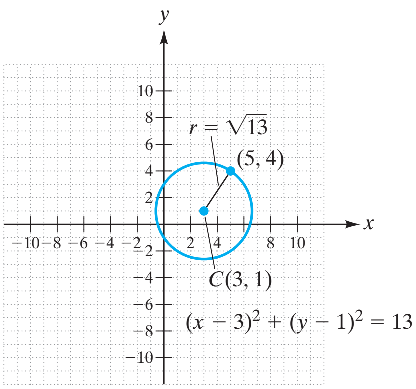


a. Since the circle has its center at the origin, use the equation $x^2 + y^2 = r^2$ and substitute 7 for r.

$$x^2 + y^2 = r^2$$
$$x^2 + y^2 = 7^2$$
$$x^2 + y^2 = 49$$

The graph of the equation $x^2 + y^2 = 49$ is a circle with the center at the origin and radius 7.





b. Substitute values of h, k, and r in the center–radius form of the equation.

$$(x - h)^2 + (y - k)^2 = r^2$$
$$[x - (-1)]^2 + (y - 3)^2 = 4^2 \quad \text{Substitute } -1 \text{ for } h, 3 \text{ for } k, \text{ and } 4 \text{ for } r.$$
$$(x + 1)^2 + (y - 3)^2 = 16$$

c. We need to determine the radius of the circle, or the distance between the points (3, 1) and (5, 4).

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \quad \text{Distance formula}$$
$$r = \sqrt{(5 - 3)^2 + (4 - 1)^2} \quad \text{Substitute.}$$
$$r = \sqrt{2^2 + 3^2}$$
$$r = \sqrt{13}$$

Now use the center–radius form to write the equation.

$$(x - h)^2 + (y - k)^2 = r^2$$
$$(x - 3)^2 + (y - 1)^2 = (\sqrt{13})^2$$
$$(x - 3)^2 + (y - 1)^2 = 13$$

The graph of the equation $(x - 3)^2 + (y - 1)^2 = 13$ is a circle with center (3, 1) and passing through the point (5, 4).

12.2.3 Checkup

1. Write the equation of a circle with its center at the origin and a radius of 2.5 units.
2. Write the equation of a circle with its center located at (1.5, 2.5) and a radius of 3 units.
3. Write the equation of a circle with its center located at (2, −3) and passing through the point (5, 1).
4. Explain why knowing where the center of a circle lies and knowing one point through which the circle passes provide enough information to be able to write the equation of the circle.

12.2.4 Modeling the Real World

We can model many situations by using circles or parts of circles. The design of arches often involves parts of circles. Almost every state in the country has a domed building. Famous buildings, such as St. Peter's and St. Paul's Cathedrals in the United Kingdom, have a cross section of a semicircle. We also may need to model a design that involves more than one circle. One such example involves **concentric circles**: circles with the same center, but different radii. As long as we can determine the centers and radii of these designs, we can write equations for the curves.

EXAMPLE 5 Dolores is building a circular fountain and a concrete walkway around it. The landscape architect gave her a drawing of the design, as shown in Figure 12.2, with distances measured in feet.

Figure 12.2

a. Use the equations given to determine the radius of each circle.
b. Determine the area of the fountain and the concrete walkway combined.
c. Determine the area of the walkway.
d. Determine the circumference of the outer edge of the walkway.

Solution

a. Inner circle

$$x^2 + y^2 - 9 = 0$$
$$x^2 + y^2 = 9$$
$$x^2 + y^2 = 3^2$$

The radius is 3 feet.

Outer circle

$$x^2 + y^2 - 25 = 0$$
$$x^2 + y^2 = 25$$
$$x^2 + y^2 = 5^2$$

The radius is 5 feet.

b. The area of the fountain and the concrete walkway is the area of the outer circle with radius 5 feet.

$$A = \pi r^2$$
$$A = \pi(5)^2$$
$$A = 25\pi$$

The total area covered is 25π square feet, or approximately 78.54 square feet.

c. The area of the walkway is the difference of the area of the outer circle and the area of the inner circle.

The inner circle has a radius of 3 feet.

$$A = \pi r^2$$
$$A = \pi(3)^2$$
$$A = 9\pi$$

The area of the outer circle (found in part b) minus the area of the inner circle is the area of the walkway.

$$25\pi - 9\pi = 16\pi$$

The walkway area is 16π square feet, or approximately 50.27 square feet.

d. The radius of the outer circle is 5 feet.

$$C = 2\pi r$$
$$C = 2\pi(5)$$
$$C = 10\pi$$

The outer circumference for the walkway is 10π feet, or approximately 31.42 feet.

APPLICATION

Many railroad viaducts are constructed in the shape of a semicircle. A stone-arch railroad viaduct at Rockville, Pennsylvania, over the Susquehanna River is made of 48 semicircular arches, each with a span of 70 feet. Use Figure 12.3 to write equations that model each of the first two arches.

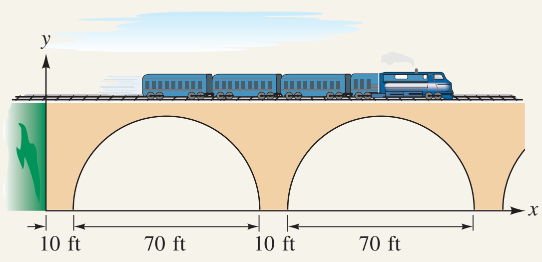


Figure 12.3

Discussion

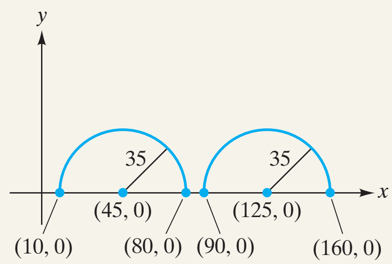
Label the coordinates of the x-intercepts and the center of each semicircle.

First arch	Second arch
center $(h, k) = (45,0)$	center $(h, k) = (125, 0)$
radius $r = 35$	radius $r = 35$
equation	equation

$$(x - h)^2 + (y - k)^2 = r^2 \qquad (x - h)^2 + (y - k)^2 = r^2$$
$$(x - 45)^2 + (y - 0)^2 = 35^2 \qquad (x - 125)^2 + (y - 0)^2 = 35^2$$
$$(x - 45)^2 + y^2 = 1225 \qquad (x - 125)^2 + y^2 = 1225$$

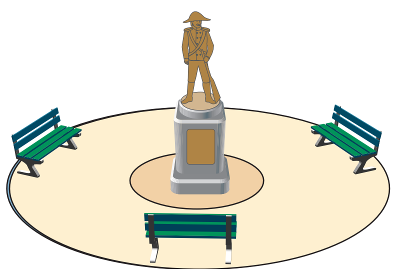
Note that, to limit the equation to a semicircle, we must restrict the y-values to be nonnegative.

The first arch is modeled by the equation $(x - 45)^2 + y^2 = 1225$, where $y \geq 0$. The second arch is modeled by the equation $(x - 125)^2 + y^2 = 1225$, where $y \geq 0$.



12.2.4 Checkup

1. The townsfolk of Pleasantville erected a statue of the town founder, General M. I. Pleasant, in the center of town. To do so, they constructed a circular park in the center of town. The park was covered in grass, except for a concrete circle in the middle upon which the statue stood. The landscape architect's drawing described the large circle covering the park by the equation $x^2 + y^2 = 5625$ and the small circle of concrete by the equation $x^2 + y^2 = 100$. The two circles are concentric, with distances measured in feet.
   a. Determine the radius of each circle.
   b. Determine the area of the park and the area of the concrete circle.
   c. Determine the area of the grass.
   d. Determine the circumference of the park.



2. Jack built a semicircular patio next to the family room of his home. The patio spanned a distance of 12 feet adjacent to his house. Write an equation of the circle that models the patio shape.

12.2 Exercises

Sketch the graph of each equation. Check the graphs on your calculator.

1. $x^2 + y^2 = 16$ **2.** $x^2 + y^2 = 64$ **3.** $x^2 + y^2 = 2.25$ **4.** $x^2 + y^2 = 5.29$

5. $x^2 + y^2 = \dfrac{100}{49}$ **6.** $x^2 + y^2 = \dfrac{64}{25}$ **7.** $x^2 + y^2 = 13$ **8.** $x^2 + y^2 = 11$

9. $(x - 1)^2 + (y - 4)^2 = 16$ **10.** $(x - 7)^2 + (y - 3)^2 = 81$ **11.** $(x + 2)^2 + (y + 5)^2 = 36$

12. $(x + 6)^2 + (y + 2)^2 = 100$ **13.** $(x - 3)^2 + (y + 1)^2 = 33$ **14.** $(x + 4)^2 + (y - 7)^2 = 65$

15. $(x + 1.5)^2 + (y - 2)^2 = 6.25$ **16.** $(x - 3.5)^2 + (y + 4)^2 = 30.25$ **17.** $(x + 1)^2 + (y - 3)^2 = \dfrac{121}{25}$

18. $(x - 2)^2 + (y + 4)^2 = \dfrac{144}{25}$ **19.** $x^2 + y^2 - 10x + 6y = -18$ **20.** $x^2 + y^2 - 14x + 4y = -44$

21. $x^2 + y^2 + 2x - 14y = -25$ **22.** $x^2 + y^2 - 6y = 55$ **23.** $x^2 + y^2 + 4x = 96$

24. $x^2 + y^2 + 8x + 8y = -16$ **25.** $x^2 + y^2 - 6x + 12y = 11.25$ **26.** $x^2 + y^2 + 10x - 4y = 43.25$

27. $x^2 + y^2 - 3.6x + 4.8y = 11.25$ **28.** $x^2 + y^2 + 7.2x + 8.6y = 2.19$ **29.** $x^2 + y^2 - \dfrac{6}{5}x - \dfrac{4}{5}y = \dfrac{131}{25}$

30. $x^2 + y^2 + \dfrac{1}{2}x - \dfrac{3}{2}y = \dfrac{71}{16}$

In exercises 31–40, write the equation of a circle with the given information.

31. Center at the origin and a radius of 4.5.

32. Center at the origin and a radius of 6.8.

33. Center at $(-3, 7)$ and a radius of 11.

34. Center at $(4, -11)$ and a radius of 6.

35. Center at $(4.6, 2.3)$ and a radius of 5.8.

36. Center at $(-0.5, -1.5)$ and a radius of 5.5.

37. Center at $(3, 5)$ and passing through $(9, 13)$.

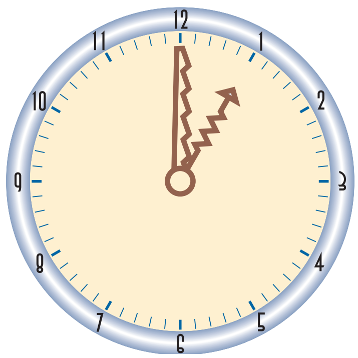
38. Center at $(-2, 4)$ and passing through $(1, 8)$.

39. Center at $(-3, 2)$ and passing through $(-2, 4)$.

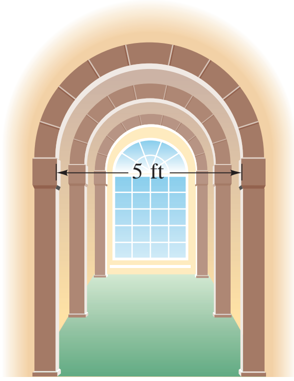
40. Center at $(-4, -3)$ and passing through $(3, 4)$.

41. A fine china dinner plate has a platinum ring around its outer edge and another platinum ring separating the center of the plate from the edge. The inner ring is described by the equation $x^2 + y^2 = 9$, and the outer ring is described by the equation $x^2 + y^2 = 25$, with distances measured in inches. Find the radius of each of the concentric circles. Determine the circumference of each circle. What is the difference in the circumferences of the two circles?

42. A wall clock is circular with a ring containing the numerals. The inner border of the ring is described by the equation $x^2 + y^2 = 22.5625$, and the outer border of the ring is described by the equation $x^2 + y^2 = 30.25$, with distances measured in inches. Find the radius of each of the concentric circles. Determine the circumference of each circle. What is the difference in the circumferences of the two circles?

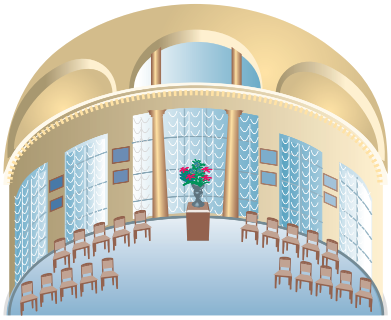


43. An arch over an entrance is semicircular with an opening that is 5 feet wide. Write an equation that models the arch.



44. Trains traveling on the Settle–Carlisle Railway pass over a series of arch viaducts. The Ribblehead viaduct has 24 semi-circular arches that span a length of 440 yards. Given that the first arch is 5 feet from the end of the span, the span of the arch is 40 feet, and the distance between arches is 5 feet, write equations that model each of the first two arches.

45. One of the most famous rooms of the Alexander Palace in Russia is the Semi-Circular Hall. If you were to design such a hall spanning a maximum width of 200 feet, what equation would you use to model the shape of your hall?



46. The heart of the United States Capitol is the Rotunda, an imposing circular room with a diameter of 96 feet. Write an equation that models a semicircle of half of the Rotunda divided by the diameter.

12.2 Calculator Exercises

Part 1. To store the center – radius form of a circle in your calculator, you will need to solve

$(x - h)^2 + (y - k)^2 = r^2$ for y.

$$(x - h)^2 + (y - k)^2 = r^2$$

$$(y - k)^2 = r^2 - (x - h)^2$$

Subtract $(x - h)^2$ from both expressions.

$$y - k = \pm\sqrt{r^2 - (x - h)^2}$$

Apply the square-root principle.

$$y = k \pm \sqrt{r^2 - (x - h)^2}$$

Add k to both expressions.

Therefore, to graph the circle with your calculator, first store $Y1 = K + \sqrt{R^2 - (X - H)^2}$ and $Y2 = K - \sqrt{R^2 - (X - H)^2}$. Then store values for r, h, and k in locations R, H, and K, and graph the curve. In this way, each time you wish to graph the equation of a circle, you need only store the new values of r, h, and k, saving you from having to repeatedly enter the expressions for Y1 and Y2. For example, if you wish to graph $(x + 3)^2 + y^2 = 11$, store $\sqrt{11}$ in R, -3 in H, and 0 in K. Then press GRAPH, and the calculator will graph the two functions stored in Y1 and Y2.

Use this procedure to graph the following equations:

1. $x^2 + (y + 3)^2 = 25$ **2.** $(x - 1)^2 + (y + 1)^2 = 9$

3. $(x - 1.5)^2 + (y + 2.5)^2 = 25$

Part 2. Complete the following example to help you better understand how to use your calculator to graph circles:

When you graph a circle with your calculator, the window you use is very important. The standard window will cause the circle to be distorted. To see this, graph the circle for $x^2 + y^2 = 9$, using the standard window, ZOOM 6. The graph does not appear circular. The horizontal diameter is longer than the vertical diameter. Next, graph the same equation, using the decimal window, ZOOM 4. This graph does appear circular, with both vertical and horizontal diameters the same length. The integer window, ZOOM 8 ENTER, also removes the distortion.

If you graph a circle by using the standard window, you can remove the distortion by using the square window, ZOOM 5. Do this with the same equation, $x^2 + y^2 = 9$, using first the standard window and then the square window. Check the settings after doing so, and you will see that to remove the distortion, the calculator settings were reset in the ratio of 3 to 2. In other words, the minimum and maximum settings for the x-axis must be 1.5 times as large as the minimum and maximum settings for the y-axis in order to avoid distorting the graph.

If you were to choose your own settings for a graph, describe how you would do so to avoid distortion. Then answer the following questions:

1. If you set Ymin equal to -100 and Ymax equal to 100, how should you set Xmin and Xmax? Use this setting to graph $x^2 + y^2 = 8100$.
2. If you set Ymin equal to 0 and Ymax equal to 500, how should you set Xmin and Xmax? Use this setting to graph $(x - 250)^2 + (y - 250)^2 = 40{,}000$.
3. If you set Ymin equal to -60 and Ymax equal to 60, how should you set Xmin and Xmax? Use this setting to graph $x^2 + y^2 - 60x - 60y + 900 = 0$.

12.2 Writing Exercise

You have now seen two forms of equations,

$$y = a(x - h)^2 + k$$

which graphs as a parabola, and

$$(x - h)^2 + (y - k)^2 = r^2$$

which graphs as a circle. Discuss the similarities between the two equations. What roles do the constants h and k play in the graphs of the equations? What roles do the other constants a and r play in the graphs of the equations?

12.3 Ellipses

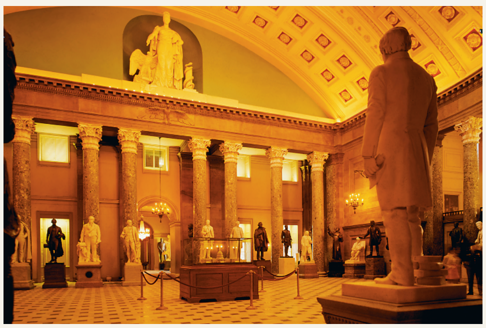
OBJECTIVES

1. Graph ellipses with centers at the origin.
2. Understand the effects of the constants a and b on the graph of an equation of an ellipse.
3. Graph ellipses with centers not at the origin.
4. Write equations of ellipses.
5. Model real-world situations by using ellipses.

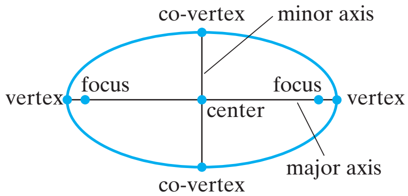
APPLICATION

A whispering gallery is an elliptical-shaped room with a dome-shaped ceiling. If two people stand at the foci of the ellipse and whisper, they can hear each other, but others in the room cannot. Statuary Hall in the U.S. Capitol Building is a whispering gallery. The dimensions of the elliptical-shaped hall are 95 feet by 75 feet. Write an equation that models the shape of the hall.

After completing this section, we will discuss this application further. See page 000.

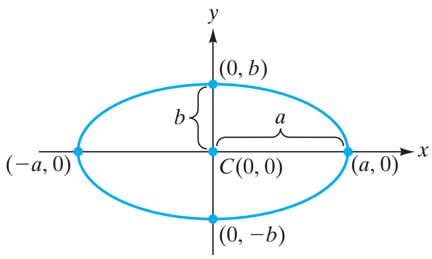


An **ellipse** is the set of points in a plane such that the sum of the distances from each point to two fixed points is constant. Each of the two fixed points is called a **focus** (plural, *foci*). The line containing the foci intersects the ellipse at points called **vertices** (singular, *vertex*). The line segment between the vertices is called the **major axis**, and its midpoint is the *center* of the ellipse. A line perpendicular to the major axis through the center intersects the ellipse at points called the **co-vertices**, and the line segment between the co-vertices is called the **minor axis**.



12.3.1 Graphing Ellipses with the Center at the Origin

Suppose the center of an ellipse is the origin, or $C(0, 0)$, the x-intercepts are $(a, 0)$ and $(-a, 0)$, and the y-intercepts are $(0, b)$ and $(0, -b)$. We can use this information along with the definition of an ellipse to write an equation of an ellipse with its center at the origin.



EQUATION OF AN ELLIPSE WITH CENTER AT THE ORIGIN

The equation of an ellipse with its center at the origin, x-intercepts of $(a, 0)$ and $(-a, 0)$, and y-intercepts of $(0, b)$ and $(0, -b)$ is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ where } a > 0 \text{ and } b > 0$$

We can now graph an ellipse, using this equation.

To graph an ellipse with its center at the origin,

- Locate and label the center of the ellipse at the origin.
- Locate and label the x-intercepts $(a, 0)$ and $(-a, 0)$.
- Locate and label the y-intercepts $(0, b)$ and $(0, -b)$.
- Sketch the elliptical graph containing the points located on the curve.

HELPING HAND If an accurate graph is needed, we will need to determine more points by substituting x-values and solving for the corresponding y-value.

EXAMPLE 1 Sketch the graph.

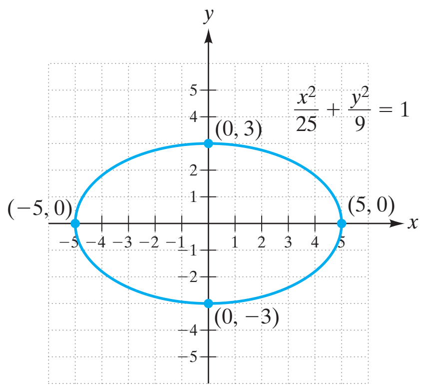
a. $\frac{x^2}{25} + \frac{y^2}{9} = 1$ **b.** $4x^2 + 10y^2 = 100$

Solution

a. Since $a^2 = 25$, it follows that $a = 5$. Because the center is at the origin, the x-intercepts of the ellipse are $(-5, 0)$ and $(5, 0)$.

Since $b^2 = 9$, it follows that $b = 3$. Because the center is at the origin, the y-intercepts of the ellipse are $(0, -3)$ and $(0, 3)$.

Sketch the graph.



Calculator Check

An ellipse is not a function, because it does not pass the vertical-line test. To graph an ellipse on your calculator, solve for y and graph two functions.

$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$

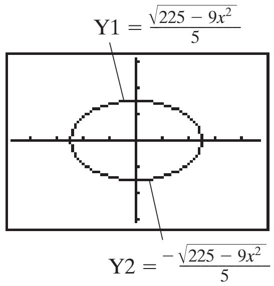
$$9x^2 + 25y^2 = 225 \quad \text{Multiply both sides by 225, the LCD of the fractions.}$$

$$25y^2 = 225 - 9x^2 \quad \text{Subtract } 9x^2 \text{ from both sides.}$$

$$y^2 = \frac{225 - 9x^2}{25} \quad \text{Solve for } y^2.$$

$$y = \pm\sqrt{\frac{225 - 9x^2}{25}} \quad \text{Principle of square roots.}$$

$$y = \pm\frac{\sqrt{225 - 9x^2}}{5}$$



(−9.4, 9.4, 2, −6.2, 6.2, 2)

If an equation is not in the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we may need to rewrite the equation before we attempt to graph it.

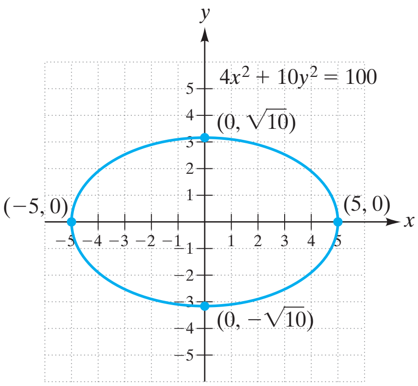
b. To obtain an equation in the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we need the left side to be a sum of two ratios and the right side to be the constant 1. We will divide both sides by the number on the right side to obtain a 1.

$$4x^2 + 10y^2 = 100$$

$$\frac{4x^2}{100} + \frac{10y^2}{100} = \frac{100}{100}$$

$$\frac{x^2}{25} + \frac{y^2}{10} = 1$$

Since $a^2 = 25$, it follows that $a = 5$. The x-intercepts are $(-5, 0)$ and $(5, 0)$. We also know that if $b^2 = 10$, $b = \sqrt{10}$. The y-intercepts are $(0, \sqrt{10})$ and $(0, -\sqrt{10})$.



Calculator Check

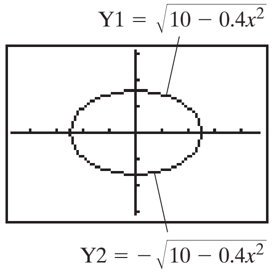
To graph the equation on your calculator, first solve for y.

$$4x^2 + 10y^2 = 100$$

$$10y^2 = 100 - 4x^2 \quad \text{Subtract } 4x^2 \text{ from both sides.}$$

$$y^2 = 10 - 0.4x^2 \quad \text{Divide both sides by 10.}$$

$$y = \pm\sqrt{10 - 0.4x^2} \quad \text{Principle of square roots}$$



$(-9.4, 9.4, 2, -6.2, 6.2, 2)$

12.3.1 Checkup

1. Sketch the graph of the equation $\frac{x^2}{4} + \frac{y^2}{16} = 1$.
2. Sketch the graph of the equation $x^2 + 9y^2 = 36$.
3. Explain why you should rewrite the equation of an ellipse in the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to sketch its graph more easily.

12.3.2 Understanding the Effects of the Constants *a* and *b*

If an equation is written in the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, do the values of the real numbers a and b affect the shape of the ellipse? To find out, complete the following set of exercises.

Discovery 2

Effect of the Real Numbers *a* and *b* on the Shape of an Ellipse

1. Sketch the graphs of the following equations in the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > b$:

$$\frac{x^2}{25} + \frac{y^2}{9} = 1 \qquad \frac{x^2}{36} + \frac{y^2}{16} = 1$$

Label the x- and y-intercepts.

2. Sketch the graphs of the following equations in the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a < b$:

$$\frac{x^2}{9} + \frac{y^2}{25} = 1 \qquad \frac{x^2}{16} + \frac{y^2}{36} = 1$$

Label the x- and y-intercepts.

3. Sketch the graphs of the following equations in the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a = b$:

$$\frac{x^2}{25} + \frac{y^2}{25} = 1 \qquad \frac{x^2}{16} + \frac{y^2}{16} = 1$$

Label the x- and y-intercepts.
Write a rule for determining the shape of an ellipse from its equation.

If $a > b$ in the equation, the major axis is the x-axis. If $a < b$ in the equation, the major axis is the y-axis. If $a = b$, the lengths of the major axis and the minor axis are equal, and the graph is a circle. We can see why by simplifying the equation. For example, simplify $\frac{x^2}{25} + \frac{y^2}{25} = 1$.

$$\frac{x^2}{25} + \frac{y^2}{25} = 1$$

$$x^2 + y^2 = 25 \qquad \text{Multiply both sides of the equation by 25.}$$

The result is an equation of a circle with its center at the origin and a radius of 5.

EXAMPLE 2 Describe the relationship between the major and minor axes of the graph of each equation.

a. $\frac{x^2}{100} + \frac{y^2}{49} = 1$ **b.** $\frac{x^2}{5} + \frac{y^2}{7} = 1$

Solution

a. $\frac{x^2}{100} + \frac{y^2}{49} = 1$

Since $a^2 = 100$, $a = 10$.

Since $b^2 = 49$, $b = 7$.

Also, since $10 > 7$, $a > b$. The major axis is the x-axis. The minor axis is the y-axis.

b. $\frac{x^2}{5} + \frac{y^2}{7} = 1$

Since $a^2 = 5$, $a = \sqrt{5}$.

Since $b^2 = 7$, $b = \sqrt{7}$.

Also, since $\sqrt{5} < \sqrt{7}$, $a < b$. The major axis is the y-axis. The minor axis is the x-axis.

12.3.2 Checkup

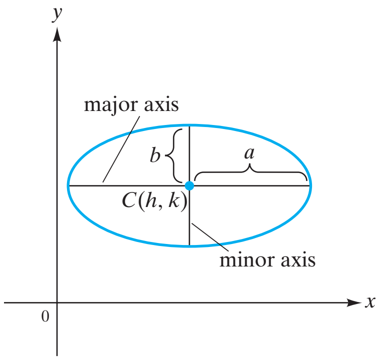
In exercises 1–3, describe the relationship between the major and minor axes of the graph of each equation.

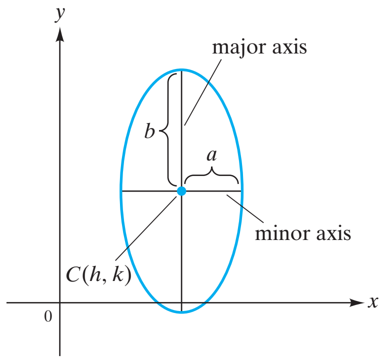
1. $\frac{x^2}{36} + \frac{y^2}{49} = 1$ **2.** $\frac{x^2}{49} + \frac{y^2}{36} = 1$ **3.** $\frac{x^2}{20} + \frac{y^2}{20} = 1$

4. All circles are ellipses, but not all ellipses are circles. Explain.

12.3.3 Graphing Ellipses with Their Centers Not at the Origin

An equation of an ellipse with its center $C(h, k)$ not at the origin can also be derived.





EQUATION OF AN ELLIPSE WITH CENTER NOT AT THE ORIGIN

The equation of an ellipse with center $C(h, k)$, a horizontal distance a between the center and a point on the graph, and a vertical distance b between the center and a point on the graph is

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1, \text{ where } a > 0 \text{ and } b > 0$$

HELPING HAND The real numbers a and b also affect the graph of an ellipse with its center not at the origin in the same manner as they do an ellipse with its center at the origin. If $a > b$, the major axis is parallel to the x-axis. If $a < b$, the major axis is parallel to the y-axis. If $a = b$, the graph is a circle.

If we can write an equation in the form $\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$, we can sketch an ellipse.

To graph an ellipse with its center not at the origin,

- Locate and label the center of the ellipse, (h, k).
- Locate and label the two points on the graph located a horizontal distance a from the center.
- Locate and label the two points on the graph located a vertical distance b from the center.
- Sketch the graph containing the points located on the ellipse.

EXAMPLE 3 Sketch the graph of the given equation.

a. $\frac{(x + 3)^2}{16} + \frac{(y - 2)^2}{9} = 1$

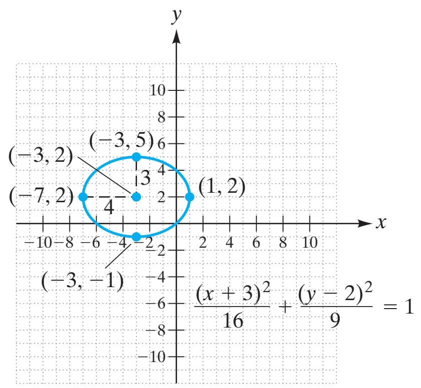
b. $25(x - 3)^2 + 10(y + 2)^2 = 100$

Solution

a.
$$\frac{(x+3)^2}{16}+\frac{(y-2)^2}{9}=1$$

$$\frac{[x-(-3)]^2}{4^2}+\frac{(y-2)^2}{3^2}=1 \qquad \text{Rewrite the equation.}$$

We see that $h=-3, k=2, a=4,$ and $b=3$. Since $a>b$, the major axis is parallel to the x-axis. Graph the center of the ellipse at $(-3, 2)$. Since $a=4$, the distance between the center and each of the vertices is 4. The vertices are $(-7, 2)$ and $(1, 2)$. Since $b=3$, the distance between the center and each of the co-vertices is 3. The co-vertices are $(-3, 5)$ and $(-3, -1)$. Sketch the graph.



Calculator Check

To graph this equation on your calculator, you must solve for y.

$$\frac{(x+3)^2}{16}+\frac{(y-2)^2}{9}=1$$

$$9(x+3)^2+16(y-2)^2=144 \qquad \text{Multiply both sides by 144.}$$

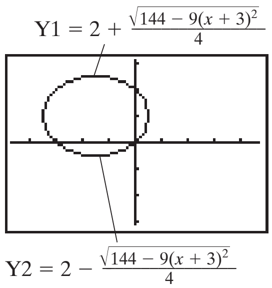
$$16(y-2)^2=144-9(x+3)^2 \qquad \text{Isolate the } y\text{-terms to one side.}$$

$$(y-2)^2=\frac{144-9(x+3)^2}{16} \qquad \text{Divide both sides by 16.}$$

$$y-2=\pm\sqrt{\frac{144-9(x+3)^2}{16}} \qquad \text{Principle of square roots}$$

$$y-2=\pm\frac{\sqrt{144-9(x+3)^2}}{4}$$

$$y=2\pm\frac{\sqrt{144-9(x+3)^2}}{4}$$



$(-9.4, 9.4, 2, -6.2, 6.2, 2)$

b.
$$25(x-3)^2+10(y+2)^2=100$$

$$\frac{25(x-3)^2}{100}+\frac{10(y+2)^2}{100}=\frac{100}{100} \qquad \text{Divide both sides by 100.}$$

$$\frac{(x-3)^2}{4}+\frac{(y+2)^2}{10}=1 \qquad \text{Simplify.}$$

$$\frac{(x-3)^2}{2^2}+\frac{[y-(-2)]^2}{(\sqrt{10})^2}=1 \qquad \text{Write the equation in the desired form.}$$

(continued on next page)

Calculator Check

Solve for y and enter the equation into your calculator.

$$25(x-3)^2+10(y+2)^2=100$$

$$10(y+2)^2=100-25(x-3)^2 \qquad \text{Isolate the } y\text{-terms.}$$

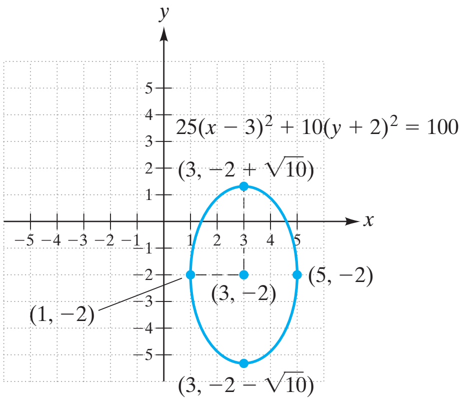
$$(y+2)^2=\frac{100-25(x-3)^2}{10} \qquad \text{Divide both sides by 10.}$$

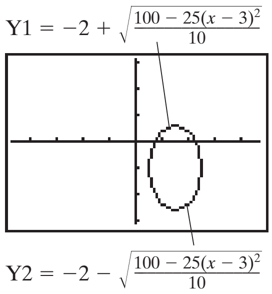
$$y+2=\pm\sqrt{\frac{100-25(x-3)^2}{10}} \qquad \text{Principle of square roots}$$

$$y=-2\pm\sqrt{\frac{100-25(x-3)^2}{10}}$$

(continued on next page)

According to the equation, $h = 3$, $k = -2$, $a = 2$, and $b = \sqrt{10}$. Since $a < b$, the major axis is parallel to the y-axis. Graph the center first. The distance between the center and one of the co-vertices is 2. The distance between the center and one of the vertices is $\sqrt{10}$, or about 3.16. Sketch the graph.





(−9.4, 9.4, 2, −6.2, 6.2, 2)

12.3.3 Checkup

1. Sketch the ellipse for the equation $\frac{(x-2)^2}{9} + \frac{y^2}{16} = 1$.
2. Sketch the ellipse for the equation $49(x + 3)^2 + 4(y - 4)^2 = 196$.
3. In graphing an ellipse, will the major axis always be the x-axis? Explain.

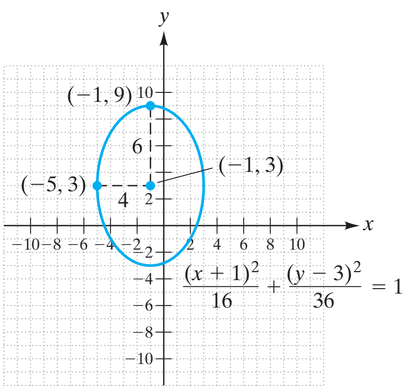
12.3.4 Writing Equations of Ellipses

If we can determine the center, the direction of the major axis, the distance between the center and one of the vertices, and the distance between the center and one of the co-vertices, we can write an equation of a given ellipse.

EXAMPLE 4

a. Write an equation of an ellipse whose major axis is vertical, with the center located at $(-1, 3)$, the distance between the center and one of the co-vertices equal to 4, and the distance between the center and one of the vertices equal to 6.

b. Write an equation of an ellipse with its center at $(-2, 1)$, one vertex at $(1, 1)$, and one co-vertex at $(-2, -1)$.



Solution

Because the distance between the center and the co-vertices is 4, the minor axis is horizontal, with $a = 4$. Because the major axis is vertical and the distance between the center and the vertices is 6, $b = 6$.

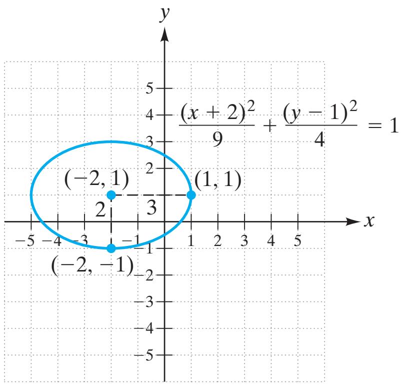
a. Substitute values of h, k, a, and b into the equation.

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

$$\frac{[x-(-1)]^2}{4^2} + \frac{(y-3)^2}{6^2} = 1 \qquad \text{Substitute } -1 \text{ for } h, 3 \text{ for } k, 4 \text{ for } a, \text{ and } 6 \text{ for } b.$$

$$\frac{(x+1)^2}{16} + \frac{(y-3)^2}{36} = 1$$

The graph of the equation $\frac{(x+1)^2}{16} + \frac{(y-3)^2}{36} = 1$ is an ellipse with center located at $(-1, 3)$, the distance between the center and one of the co-vertices equal to 4, and the distance between the center and one of the vertices equal to 6.



b. First, graph the ellipse.

We see that the center $C(h, k)$ is $(-2, 1)$. Also, the distance a between the center and the vertex $(1, 1)$ is $a = |-2 - 1| = 3$. The distance b between the center and the co-vertex $(-2, -1)$ is $b = |1 - (-1)| = 2$.

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

$$\frac{[x-(-2)]^2}{3^2} + \frac{(y-1)^2}{2^2} = 1 \qquad \text{Substitute } -2 \text{ for } h, -1 \text{ for } k, 3 \text{ for } a, \text{ and } 2 \text{ for } b.$$

$$\frac{(x+2)^2}{9} + \frac{(y-1)^2}{4} = 1$$

The graph of the equation $\frac{(x+2)^2}{9} + \frac{(y-1)^2}{4} = 1$ is an ellipse with its center at $(-2, 1)$, one vertex at $(1, 1)$, and one co-vertex at $(-2, -1)$.

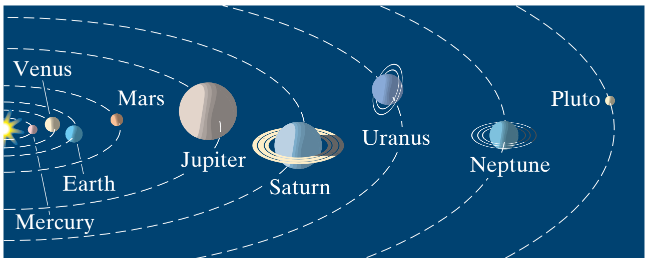
12.3.4 Checkup

1. Write an equation of an ellipse whose major axis is horizontal, with the center at $(1.5, -2)$, the distance between the center and one of the vertices equal to 5, and the distance between the center and one of the co-vertices equal to 2.
2. Write an equation of an ellipse with its center at $(3, 2)$, one vertex at $(3, 6)$, and one co-vertex at $(6, 2)$.
3. If you knew only where the center of an ellipse lay and you knew the distance from the center to one of the vertices, would you have enough information to write the equation of the ellipse? Explain.

12.3.5 Modeling the Real World

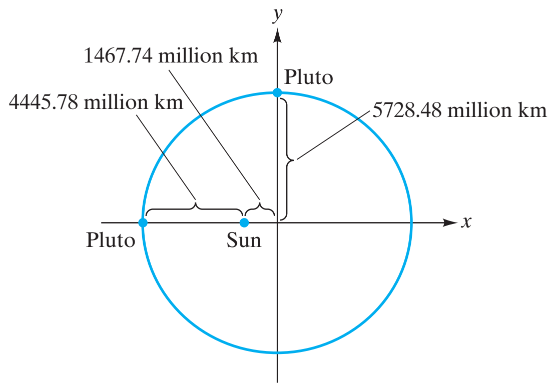
A number of different kinds of situations involve elliptical shapes. One very important concept is the orbital paths of planets about the Sun, as well as the orbits of satellites about planets and the paths of comets. If equations can be derived to model the paths of these objects, we can determine their movement in space and predict when certain planets or comets can be sighted. Remember that in the real world situations change, and sometimes these models will need to be changed to reflect new data.

EXAMPLE 5 The orbit of the planet Pluto around the Sun can be modeled by an ellipse, with the Sun at one focus. The Sun is approximately 1467.74 million kilometers from the center of the ellipse. At its closest point, Pluto is approximately 4445.78 million kilometers from the Sun. The minimum distance from Pluto to the center of the ellipse is approximately 5728.48 million kilometers. Write an equation for the model and sketch the graph of the equation.



Solution

A sketch using the information given will help us determine the model.



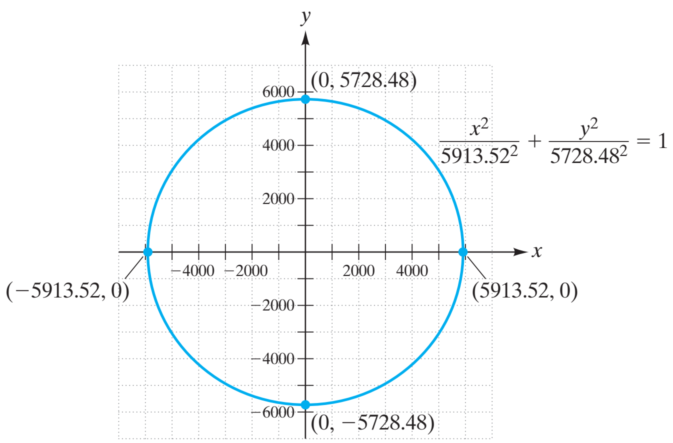
Therefore, we know the following:

$$a = 4445.78 + 1467.74 = 5913.52 \text{ million kilometers}$$
$$b = 5728.48 \text{ million kilometers.}$$

An equation that models the orbital path of Pluto is

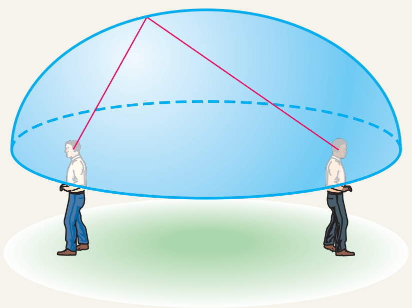
$$\frac{x^2}{5913.52^2} + \frac{y^2}{5728.48^2} = 1.$$

Sketch the graph.



APPLICATION

A whispering gallery is an elliptical-shaped room with a dome-shaped ceiling. If two people stand at the foci of the ellipse and whisper, they can hear each other, but others in the room cannot. Statuary Hall in the U.S. Capitol Building is a whispering gallery. The dimensions of the elliptical-shaped hall are 95 feet by 75 feet. Write an equation that models the shape of the hall.



Discussion

A sketch will help us visualize the shape. We will set the center of the hall at the origin.

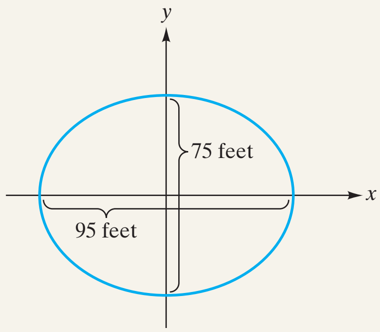
According to the sketch, $a = \frac{95}{2} = 47.5$ feet and $b = \frac{75}{2} = 37.5$ feet. An equation for an ellipse to model the shape of the hall is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{x^2}{47.5^2} + \frac{y^2}{37.5^2} = 1$$

$$\frac{x^2}{2256.25} + \frac{y^2}{1406.25} = 1$$

A model for the shape of the room is $\frac{x^2}{2256.25} + \frac{y^2}{1406.25} = 1$.



12.3.5 Checkup

1. The orbit of Mars around the Sun can be modeled by an ellipse, with the Sun at one of the foci. The Sun is approximately 21.24 million kilometers from the center of the ellipse. At its closest, Mars is approximately 206.75 million kilometers from the Sun. The minimum distance of Mars to the center of the ellipse is approximately 226.94 million kilometers. Write an equation for the model and sketch the graph of the equation.

2. The Colosseum in Rome is elliptical in shape. The dimensions are approximately 190 meters by 155 meters. Assuming that the center of the Colosseum is the origin, and that the length is the major axis, write an equation for the ellipse that can be used to represent the shape of the Colosseum.

12.3 Exercises

Sketch the graph of each equation. Check, using your calculator.

1. $\frac{x^2}{81} + \frac{y^2}{49} = 1$ **2.** $\frac{x^2}{64} + \frac{y^2}{36} = 1$ **3.** $\frac{x^2}{64} + \frac{y^2}{10} = 1$ **4.** $\frac{x^2}{121} + \frac{y^2}{75} = 1$

5. $\frac{x^2}{36} + \frac{y^2}{81} = 1$ **6.** $\frac{x^2}{49} + \frac{y^2}{144} = 1$ **7.** $\frac{x^2}{5} + \frac{y^2}{25} = 1$ **8.** $\frac{x^2}{50} + \frac{y^2}{81} = 1$

9. $\frac{x^2}{15} + \frac{y^2}{30} = 1$ **10.** $\frac{x^2}{55} + \frac{y^2}{27} = 1$ **11.** $\frac{x^2}{42.25} + \frac{y^2}{12.25} = 1$ **12.** $\frac{x^2}{27.04} + \frac{y^2}{53.29} = 1$

13. $49x^2 + 4y^2 = 196$ **14.** $64x^2 + 25y^2 = 1600$ **15.** $9x^2 + 16y^2 = 144$ **16.** $4x^2 + 36y^2 = 144$

Describe the relationship between the major and minor axes of the graph of each equation.

17. $\frac{x^2}{121} + \frac{y^2}{169} = 1$ **18.** $\frac{x^2}{225} + \frac{y^2}{400} = 1$ **19.** $\frac{x^2}{144} + \frac{y^2}{100} = 1$ **20.** $\frac{x^2}{196} + \frac{y^2}{121} = 1$

21. $\frac{x^2}{400} + \frac{y^2}{400} = 1$ **22.** $\frac{x^2}{256} + \frac{y^2}{256} = 1$

For each equation, sketch the ellipse. Check, using your calculator.

23. $\frac{(x-5)^2}{36} + \frac{(y-3)^2}{25} = 1$ **24.** $\frac{(x-6)^2}{25} + \frac{(y-4)^2}{4} = 1$ **25.** $\frac{x^2}{9} + \frac{(y-4)^2}{16} = 1$

26. $\frac{(x+6)^2}{36} + y^2 = 1$ **27.** $\frac{(x+4)^2}{49} + \frac{(y+6)^2}{81} = 1$ **28.** $\frac{(x+3)^2}{64} + \frac{(y+5)^2}{100} = 1$

29. $16(x+1)^2 + 49(y+3)^2 = 784$ **30.** $4(x-6)^2 + 36(y-2)^2 = 144$ **31.** $100(x+2)^2 + 25(y-5)^2 = 2500$

32. $4(x-5)^2 + 25(y+5)^2 = 100$ **33.** $x^2 + 9(y+2)^2 = 9$ **34.** $16(x-6)^2 + y^2 = 16$

In exercises 35–42, write an equation of an ellipse from the given information.

35. The ellipse has a major axis that is horizontal, with the center located at $(-2, -2)$. The distance between the center and one of the vertices of the ellipse is 5, and the distance between the center and one of the co-vertices is 3.

36. The ellipse has a major axis that is horizontal, with the center located at $(4, 2)$. The distance between the center and one of the vertices of the ellipse is 4, and the distance between the center and one of the co-vertices is 2.

37. The ellipse has a major axis that is vertical, with the center located at $(-5, 3)$. The distance between the center and one of the vertices of the ellipse is 6, and the distance between the center and one of the co-vertices is 3.

38. The ellipse has a major axis that is vertical, with the center located at $(3, -3)$. The distance between the center and one of the vertices of the ellipse is 6, and the distance between the center and one of the co-vertices is 4.

39. The ellipse has its center at $(2, 3)$, one vertex at $(9, 3)$, and one co-vertex at $(2, 1)$.

40. The ellipse has its center at $(-5, -2)$, one vertex at $(-9, -2)$, and one co-vertex at $(-5, 1)$.

41. The ellipse has its center at $(-3, 2)$, one vertex at $(-3, -3)$, and one co-vertex at $(0, 2)$.

42. The ellipse has its center at $(6, -1)$, one vertex at $(6, 6)$, and one co-vertex at $(2, -1)$.

43. Your town decides to build an arena for rodeos. The arena is to be elliptical, with external dimensions of 600 feet by 450 feet. Assume that the center of the arena is the origin. Write an equation that models the shape of the arena.

44. A receptionist's table is designed to be elliptical, with dimensions of 10 feet by 4 feet. Assume that the center of the table is the origin. Write an equation that models the shape of the table.

45. The orbit of Mercury around the Sun can be modeled by an ellipse, with the Sun at one of the foci. The Sun is approximately 11.93 million kilometers from the center of the ellipse. At its closest, Mercury is approximately 45.97 million kilometers from the Sun. The minimum distance from Mercury to the center of the ellipse is approximately 56.65 million kilometers. Write an equation for the model and sketch the graph of the equation.

46. The orbit of Saturn around the Sun can be modeled by an ellipse, with the Sun at one of the foci. The Sun is approximately 79.92 million kilometers from the center of the ellipse. At its closest, Saturn is approximately 1347.26 million kilometers from the Sun. The minimum distance of Saturn to the center of the ellipse is approximately 1424.94 million kilometers. Write an equation for the model and sketch the graph of the equation.

47. An architectural design for the bank president's office calls for it to be an elliptical-shaped oval office with a major axis of 40 feet and a minor axis of 30 feet. Write an equation that models the shape of the office.

48. The design for the princess palace will include a ballroom with an elliptical shape. The major axis of the ellipse will be 120 feet long and the minor axis will be 80 feet wide. Write an equation that models the shape of the ballroom.

49. The Oval Office in the White House is an ellipse with the long axis measuring 35.83 feet and the short axis measuring 29 feet. Assuming the center of the office is the origin, write an equation for the ellipse that can be used to represent the shape of the Oval Office.

50. The Oval Office at the Truman Library was expected to be an exact replica of the Oval Office in the White House. However, because of an error on the blueprint, the replica was not built to the exact dimensions of the original. The Truman Oval Office has a long axis measuring 32.75 feet and a short axis measuring 27.25 feet. Assuming that the center of the office is the origin, write an equation for the ellipse that can be used to represent the shape of the Oval Office at the Truman Library.

51. The Museum of Science and Industry in Chicago has a whispering gallery. When a visitor stands at one of the foci of the ellipse, the sound from this focus reflects directly to a person standing at the other focus. The ellipse has dimensions of 13.5 feet by 40.6 feet. Write an equation that models the shape of the gallery.

52. St. Paul's Cathedral was constructed with a whispering gallery. The floor of the gallery is a circular ellipse with a diameter of 34 meters. Write an equation that models the shape of the gallery.

12.3 Calculator Exercises

To store the equation of an ellipse in your calculator, you will need to solve $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ for y.

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

$$\frac{(y-k)^2}{b^2} = 1 - \frac{(x-h)^2}{a^2} \qquad \text{Subtract } \frac{(x-h)^2}{a^2} \text{ from both expressions.}$$

$$(y-k)^2 = b^2 - \frac{b^2(x-h)^2}{a^2} \qquad \text{Multiply both expressions by } b^2.$$

$$y - k = \pm\sqrt{b^2 - \frac{b^2(x-h)^2}{a^2}} \qquad \text{Apply the square-root principle.}$$

$$y = k \pm \sqrt{b^2 - \frac{b^2(x-h)^2}{a^2}} \qquad \text{Add } k \text{ to both expressions.}$$

Therefore, to graph the ellipse with your calculator, first store $Y1 = K + \sqrt{B^2 - \frac{B^2(X-H)^2}{A^2}}$ and $Y2 = K - \sqrt{B^2 - \frac{B^2(X-H)^2}{A^2}}$. Then store values for a, b, h, and k in locations A, B, H, and K, and graph the curve. In this way, each time you wish to graph the equation of an ellipse, you need only store the new values of a, b, h, and k, saving you from having to repeatedly enter the expressions for Y1 and Y2. For example, if you wish to graph $\frac{(x+3)^2}{16} + \frac{(y-2)^2}{9} = 1$, store 4 in A, 3 in B, −3 in H, and 2 in K. Then press GRAPH, and the calculator will graph the two functions stored in Y1 and Y2.

Use this procedure to graph the following ellipses:

1. $\frac{x^2}{9} + \frac{(y-2)^2}{25} = 1$

2. $\frac{(x+3)^2}{16} + \frac{(y-1)^2}{4} = 1$

3. $3(x-2)^2 + 4(y+2)^2 = 36$

12.3 Writing Exercise

In this section, it was stated that all circles are ellipses, but not all ellipses are circles. Explain, using examples.

12.4 Hyperbolas

OBJECTIVES

1 Graph hyperbolas with their centers at the origin.
2 Graph hyperbolas with their centers not at the origin.
3 Write equations of hyperbolas.
4 Model real-world situations by using hyperbolas.

APPLICATION

Long-range navigation (LORAN) is a radio navigation system developed during World War II. The system enables a pilot to guide aircraft by maintaining a constant difference between the aircraft's distances from two fixed points: the master station and the slave station. Write an equation for the hyperbola depicted in Figure 12.4.

After completing this section, we will discuss this application further. See page 000.

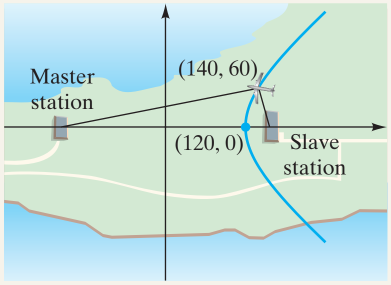
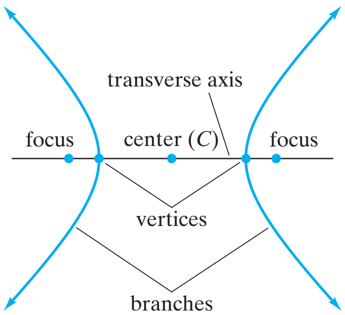


Figure 12.4



A **hyperbola** is the set of points in a plane such that the absolute value of the difference of the distance of each point from two fixed points is constant. Each fixed point is called a *focus*, and the point midway between the foci is called the *center*. The line containing the foci is the **transverse axis**. The graph is made up of two parts called **branches**. Each branch intersects the transverse axis at a point called the *vertex*.

12.4.1 Graphing Hyperbolas with Their Centers at the Origin

We can use the distance formula to determine an equation of a hyperbola with its center at the origin, or $C(0, 0)$, and its vertices located equidistant from the origin on the x- or y-axis.

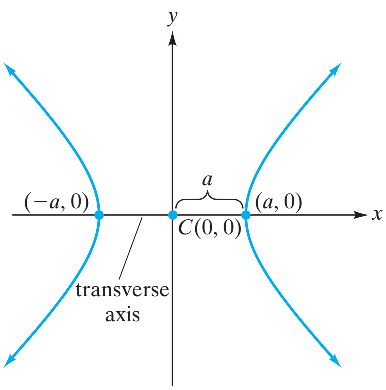


Figure 12.5a

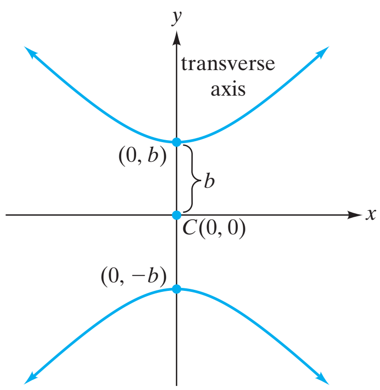


Figure 12.5b

In Figure 12.5a, the hyperbola opens to the left and to the right, and the transverse axis is horizontal. We call such a hyperbola a **horizontal hyperbola**. In Figure 12.5b, the hyperbola opens upward and downward, and the transverse axis is vertical. We call such a hyperbola a **vertical hyperbola**.

Can we determine the type of hyperbola from its equation? To find out, complete the following set of exercises.

Discovery 3

Effect of the Form of the Equation on the Orientation of the Hyperbola

1. Sketch the graph of the following equation in the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$:

$$\frac{x^2}{25} - \frac{y^2}{9} = 1$$

2. Sketch the graph of the following equation in the form $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$:

$$\frac{y^2}{9} - \frac{x^2}{25} = 1$$

Write a rule for explaining the orientation of a hyperbola from its equation.

If the equation of a hyperbola is in the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then the transverse axis is the x-axis. If the equation is in the form $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$, then the transverse axis is the y-axis.

THE EQUATION OF A HYPERBOLA WITH CENTER AT THE ORIGIN

The equation of a horizontal hyperbola with center at the origin and x-intercepts $(-a, 0)$ and $(a, 0)$ is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \text{ where } a > 0 \text{ and } b > 0$$

The equation of a vertical hyperbola with center at the origin and y-intercepts $(0, -b)$ and $(0, b)$ is

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1, \text{ where } a > 0 \text{ and } b > 0$$

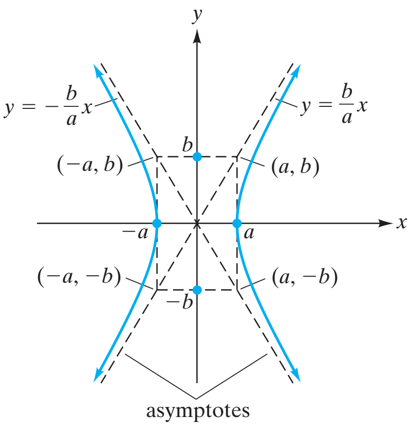


Figure 12.6

To graph a hyperbola, we need to see that, as the branches extend indefinitely, they get closer and closer to, but never touch, two lines such as shown in Figure 12.6. These lines are called **asymptotes**.

To graph a hyperbola's asymptotes, we need to draw a central rectangle with corners at (a, b), $(-a, b)$, $(a, -b)$, and $(-a, -b)$. The asymptotes are the extended diagonals of this rectangle. Note that the center is the intersection of the diagonals. We can write equations for the asymptotes because we know at least two points on each line. The equations of the asymptotes are $y = \pm\frac{b}{a}x$.

HELPING HAND The central rectangle and the two asymptotes are not part of the graph of the hyperbola and are drawn with dashed lines.

We can now graph a hyperbola by using its equation and this aid:

To graph a hyperbola with the center at the origin,

- Locate and label the center of the hyperbola at the origin.
- Locate and label the x-intercepts or y-intercepts.
- With a dashed line, draw a central rectangle with corners at (a, b), $(-a, b)$, $(a, -b)$, and $(-a, -b)$.
- With a dashed line, draw the asymptotes (the extended diagonals of the central rectangle) and label them.
- Sketch the hyperbola containing the vertices that were located and approaching the asymptotes.

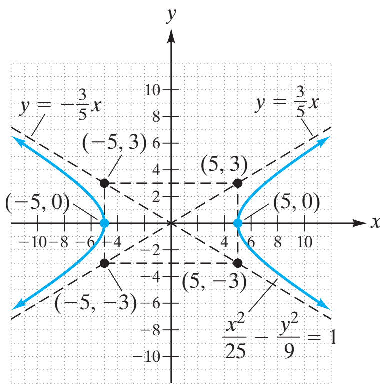
EXAMPLE 1 Sketch the graphs of the equations.

a. $\dfrac{x^2}{25} - \dfrac{y^2}{9} = 1$

b. $-4x^2 + 10y^2 = 100$

Solution

a. Since $a^2 = 25$ and $b^2 = 9$, it follows that $a = 5$ and $b = 3$. The equation given is for a horizontal hyperbola with its center at the origin. The x-intercepts of the hyperbola are $(-5, 0)$ and $(5, 0)$. To draw the asymptotes, we need to draw a central rectangle with corners at $(5, 3)$, $(-5, 3)$, $(5, -3)$, and $(-5, -3)$ and extend the diagonals of the rectangle. The equations of the diagonals are $y = \pm\frac{b}{a}x$, or $y = \pm\frac{3}{5}x$. Sketch the hyperbola by including the x-intercepts and allowing the graph to approach the asymptotes.



Calculator Check

A hyperbola is not a function, because it does not pass the vertical-line test. To graph a hyperbola on your calculator, solve for y and graph two functions.

$$\frac{x^2}{25} - \frac{y^2}{9} = 1$$

$$9x^2 - 25y^2 = 225 \qquad \text{Multiply both sides by 225, the LCD of the fractions.}$$

$$-25y^2 = 225 - 9x^2 \qquad \text{Solve for } y.$$

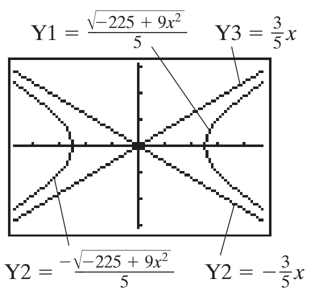
$$y^2 = \frac{225 - 9x^2}{-25}$$

$$y = \pm\sqrt{\frac{225 - 9x^2}{-25}} \qquad \text{Principle of square roots}$$

$$y = \pm\sqrt{\frac{-225 + 9x^2}{25}} \qquad \text{Multiply the radicand by } \tfrac{-1}{-1}.$$

$$y = \pm\frac{\sqrt{-225 + 9x^2}}{5}$$

It is desirable to graph the asymptotes. Graph the two equations $y = \pm\frac{3}{5}x$.



(−9.4, 9.4, 2, −6.2, 6.2, 2)

b. To obtain an equation in the form for a hyperbola, we need the left side to be a difference of two ratios and the right side to be the constant 1. First, we will divide both sides by the number on the right side to obtain a 1.

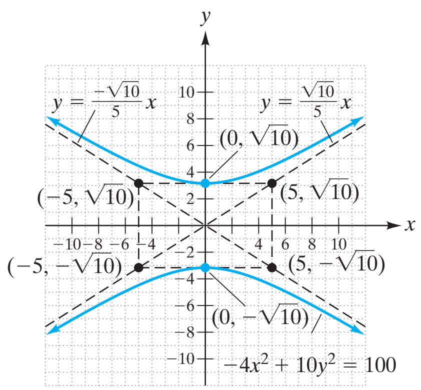
$$-4x^2 + 10y^2 = 100$$

$$\frac{-4x^2}{100} + \frac{10y^2}{100} = \frac{100}{100} \quad \text{Divide both sides by 100.}$$

$$-\frac{x^2}{25} + \frac{y^2}{10} = 1$$

$$\frac{y^2}{10} - \frac{x^2}{25} = 1$$

Since $a^2 = 25$ and $b^2 = 10, a = 5$ and $b = \sqrt{10}$. The equation is in the form for a vertical hyperbola with its center at the origin. The y-intercepts are $(0, \sqrt{10})$ and $(0, -\sqrt{10})$. To draw the asymptotes, we need to draw a central rectangle with corners at $(5, \sqrt{10}), (5, -\sqrt{10}), (-5, \sqrt{10})$, and $(-5, -\sqrt{10})$ and extend the diagonals of the rectangle. The equations for the diagonals are $y = \pm\frac{\sqrt{10}}{5}x$. Sketch the hyperbola by including the y-intercepts and allowing the graph to approach the asymptotes.



Calculator Check

To graph the equation on your calculator, first solve for y.

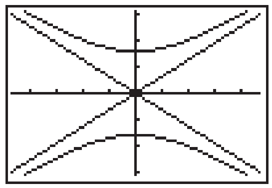
$$-4x^2 + 10y^2 = 100$$

$$10y^2 = 100 + 4x^2 \quad \text{Add } 4x^2 \text{ to both sides.}$$

$$y^2 = 10 + 0.4x^2 \quad \text{Divide both sides by 10.}$$

$$y = \pm\sqrt{10 + 0.4x^2} \quad \text{Principle of square roots}$$

Also, graph the equations for the asymptotes, $y = \pm\frac{\sqrt{10}}{5}x$.



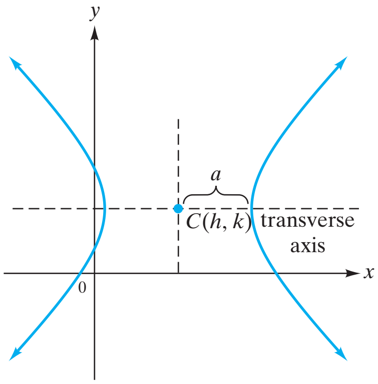
(−9.4, 9.4, 2, −6.2, 6.2, 2)

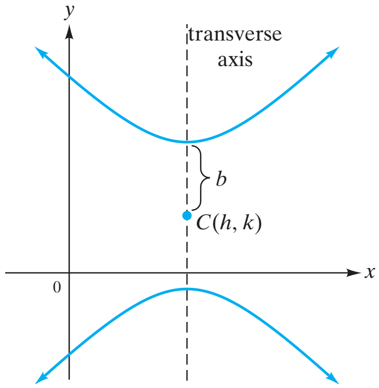
12.4.1 Checkup

1. Sketch the graph of the equation $\frac{x^2}{9} - \frac{y^2}{4} = 1$, along with its asymptotes.
2. Sketch the graph of the equation $4y^2 = 36 + 9x^2$, along with its asymptotes.
3. A parabola may appear to be one branch of a hyperbola. Do you think it really is? Explain. (*Hint:* Consider the general form of the equation for each.)

12.4.2 Graphing Hyperbolas with Their Centers Not at the Origin

An equation of a hyperbola with center $C(h, k)$ not at the origin can also be derived.





EQUATION OF A HYPERBOLA WITH CENTER NOT AT THE ORIGIN

The equation of a horizontal hyperbola with center (h, k) and distance a between the center and one of the vertices is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1, \text{ where } a > 0 \text{ and } b > 0$$

The equation of a vertical hyperbola with center (h, k) and distance b between the center and one of the vertices is

$$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1, \text{ where } a > 0 \text{ and } b > 0$$

If we can write an equation in one of these forms, we can sketch a hyperbola.

To graph a hyperbola with center not at the origin,

- Locate and label the center of the hyperbola.
- Locate and label the two points on the graph that have a horizontal distance a or a vertical distance b from the center.
- With a dashed line, draw a central rectangle with corners at $(h-a, k-b)$, $(h-a, k+b)$, $(h+a, k+b)$, and $(h+a, k-b)$.
- With a dashed line, draw the asymptotes (the extended diagonals of the central rectangle) and label them.
- Sketch the hyperbolic graph containing the vertices that were located and approaching the asymptotes.

EXAMPLE 2 Sketch the hyperbola of the equation $\frac{(x+3)^2}{16} - \frac{(y-2)^2}{9} = 1$.

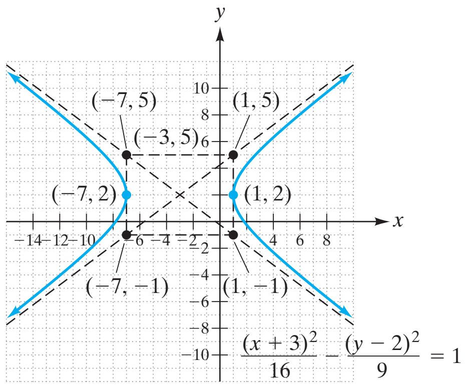
Solution

$$\frac{(x+3)^2}{16} - \frac{(y-2)^2}{9} = 1$$

$$\frac{[x-(-3)]^2}{4^2} - \frac{(y-2)^2}{3^2} = 1 \qquad \text{Rewrite the equation.}$$

We see that $h = -3, k = 2, a = 4$, and $b = 3$. The equation represents a horizontal hyperbola. The center of the hyperbola is $(-3, 2)$. Since $a = 4$, the distance between the center and each of the vertices is 4. The vertices are $(-7, 2)$ and $(1, 2)$.

To draw the asymptotes, we need to draw a central rectangle and extend its diagonals. Since $b = 3$, the distance between the center and the side of the central rectangle is 3. This will enable us to locate the two points $(-3, 5)$ and $(-3, -1)$. Use the four points $(-7, 5)$, $(1, 5)$, $(1, -1)$, and $(-7, -1)$ to draw the central rectangle and its diagonals. Finally, sketch the hyperbola.



Calculator Check

You will need to solve the equation for y to graph the hyperbola on your calculator.

$$\frac{(x+3)^2}{16} - \frac{(y-2)^2}{9} = 1$$

$$9(x+3)^2 - 16(y-2)^2 = 144$$

Multiply both sides by 144.

$$-16(y-2)^2 = 144 - 9(x+3)^2$$

Isolate the y-terms to one side.

$$(y-2)^2 = \frac{144 - 9(x+3)^2}{-16}$$

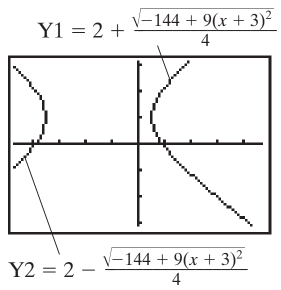
Divide both sides by -16.

$$y - 2 = \pm\sqrt{\frac{144 - 9(x+3)^2}{-16}}$$

Principle of square roots

$$y - 2 = \pm\frac{\sqrt{-144 + 9(x+3)^2}}{4}$$

$$y = 2 \pm \frac{\sqrt{-144 + 9(x+3)^2}}{4}$$



$(-9.4, 9.4, 2, -6.2, 6.2, 2)$

12.4.2 Checkup

1. Sketch the hyperbola of the equation $\frac{(x-1)^2}{9} - \frac{(y+3)^2}{4} = 1$, along with its asymptotes.

2. Explain how the asymptotes of a hyperbola relate to the branches of the hyperbola.

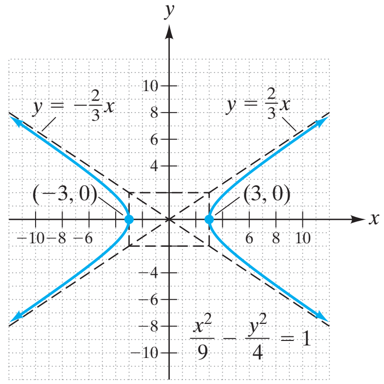
12.4.3 Writing Equations of Hyperbolas

We can write equations of hyperbolas, given various kinds of information, but some of the algebra involved can get complicated. In this text, we will limit our discussions to hyperbolas with a center at the origin, and we will always start with knowing the coordinates of at least one vertex and the equations for the asymptotes.

EXAMPLE 3 Write an equation of a hyperbola with the center at the origin, a vertex at $(3, 0)$, and equations of the asymptotes $y = \pm\frac{2}{3}x$.

Solution

First graph the hyperbola. The hyperbola is a horizontal hyperbola, because the given vertex is an x-intercept. Since we know the hyperbola has its center



at the origin and a vertex at (3, 0), the other vertex is (−3, 0). We can draw the central rectangle, using the slopes of the equations of the asymptotes to determine the vertices of the rectangle. That is, from the origin, count a slope of $\frac{2}{3}$. One corner of the central rectangle is at (3, 2). The opposite corner of the rectangle will be at (−3, −2).

Because the given vertex is the x-intercept, we will write an equation of a horizontal hyperbola. The value of a is the x-coordinate of the vertex, or $a = 3$. The y-coordinates of the central rectangle are the positive and negative values of b. In our equation, we use the positive value of b, or $b = 2$. We can also find b algebraically. Since the slope of the asymptote is $\frac{2}{3} = \frac{b}{a}$ and $a = 3$, we can solve for b. $\frac{2}{3} = \frac{b}{3}$, or $b = 2$.

The equation of the hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\frac{x^2}{3^2} - \frac{y^2}{2^2} = 1 \quad \text{Substitute 3 for } a \text{ and 2 for } b.$$

$$\frac{x^2}{9} - \frac{y^2}{4} = 1.$$

12.4.3 Checkup

1. Write an equation of a hyperbola with its center at the origin, a vertex at (0, 4), and equations of the asymptotes $y = \pm\frac{1}{2}x$.

2. How does the equation of a hyperbola differ from the equation of an ellipse?

12.4.4 Modeling the Real World

Hyperbolas are frequently used as models of situations that occur in the fields of optics and accoustics, because light and sound waves striking a hyperbolic surface at a certain angle (toward one focus) are reflected in a specific direction (toward the other focus). We can write equations for situations that involve hyperbolic shapes, as long as we have enough information to determine values for a and b in the given equations for the hyperbolas.

EXAMPLE 4 A hyperbolic mirror is used in some telescopes. Such a mirror has the property that a light ray directed at one focus will be reflected to the other focus. Using Figure 12.7, write an equation to model the hyperbolic mirror's surface.

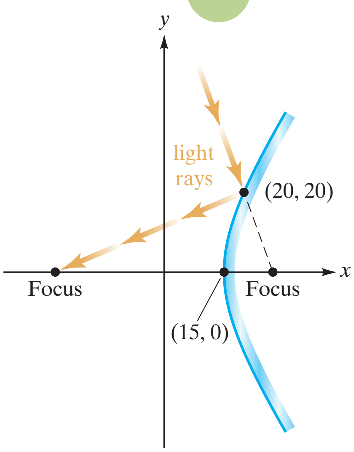


Figure 12.7

Solution

The vertex of the hyperbola is an x-intercept and the center is at the origin. Therefore, the x-coordinate of the x-intercept is $a = 15$. We can determine b by substituting 20 for x and 20 for y in the equation for a horizontal hyperbola and solving for b.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\frac{20^2}{15^2} - \frac{20^2}{b^2} = 1 \quad \text{Substitute.}$$

$$20^2b^2 - (15^2)(20^2) = 15^2b^2 \quad \text{Clear the fractions.}$$

$$20^2b^2 - 15^2b^2 = (15^2)(20^2) \quad \text{Isolate the } b \text{ terms to one side.}$$

$$b^2(20^2 - 15^2) = (15^2)(20^2) \quad \text{Factor out } b^2.$$

$$b^2 = \frac{(15^2)(20^2)}{20^2 - 15^2} \quad \text{Solve for } b.$$

$$b^2 = \frac{3600}{7}$$

Substitute 15 for a and $\frac{3600}{7}$ for b^2 in the equation for a horizontal hyperbola with center at the origin.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\frac{x^2}{15^2} - \frac{y^2}{\frac{3600}{7}} = 1$$

$$\frac{x^2}{225} - \frac{7y^2}{3600} = 1$$

We must restrict our model to the right branch of the hyperbola, or when $x \geq 15$.

The hyperbolic mirror surface can be modeled with the equation

$$\frac{x^2}{225} - \frac{7y^2}{3600} = 1, \text{ where } x \geq 15$$

APPLICATION

Long-range navigation (LORAN) is a radio navigation system developed during World War II. The system enables a pilot to guide aircraft by maintaining a constant difference between the aircraft's distances from two fixed points: the master station and the slave station. Write an equation for the hyperbola depicted in Figure 12.8.

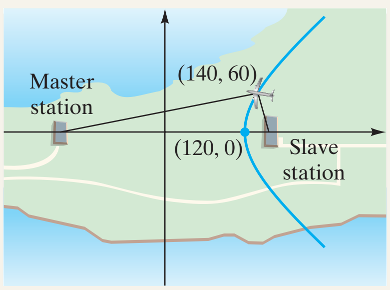


Figure 12.8

Discussion

The vertex of the hyperbola is an x-intercept and the center is at the origin. Therefore, the x-coordinate of the x-intercept is $a = 120$. We can determine b by substituting the coordinates of the given point on the graph—140 for x and 60 for y—in the equation of a horizontal hyperbola and solving for b.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\frac{140^2}{120^2} - \frac{60^2}{b^2} = 1$$

Substitute.

$$140^2b^2 - (120^2)(60^2) = 120^2b^2$$

Clear the fractions.

$$140^2b^2 - 120^2b^2 = (120^2)(60^2)$$

Isolate the b terms to one side.

(Continued on page 992)

$$b^2(140^2 - 120^2) = (120^2)(60^2)$$

Factor out b^2.

$$b^2 = \frac{(120^2)(60^2)}{140^2 - 120^2}$$

Solve for b^2.

$$b^2 = \frac{129{,}600}{13}$$

Substitute 120 for a and $\frac{129{,}600}{13}$ for b^2 in the equation of a horizontal hyperbola with center at the origin.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\frac{x^2}{120^2} - \frac{y^2}{\frac{129{,}600}{13}} = 1$$

$$\frac{x^2}{14{,}400} - \frac{13y^2}{129{,}600} = 1$$

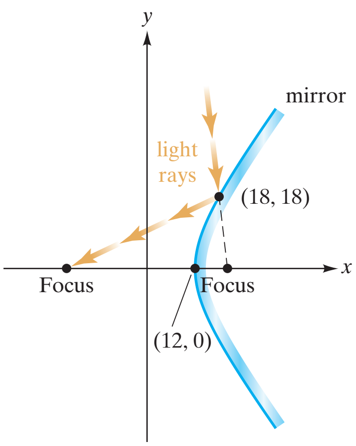
We must restrict our model to the right branch of the hyperbola ($x > 0$).

The figure can be modeled with the equation

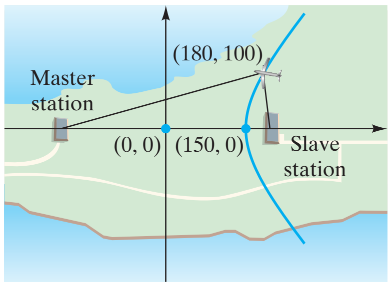
$$\frac{x^2}{14{,}400} - \frac{13y^2}{129{,}600} = 1, \text{ where } x \geq 120$$

12.4.4 Checkup

1. In the figure, a hyperbolic mirror is depicted. Write an equation that models the hyperbolic mirror's surface.



2. Write an equation that models the hyperbola depicted for an aircraft guided by a navigation system that employs hyperbolic tracking.



12.4 Exercises

Sketch the graph of each equation. Also, sketch the asymptotes for each graph.

1. $\frac{x^2}{16} - \frac{y^2}{9} = 1$
2. $\frac{x^2}{49} - \frac{y^2}{25} = 1$
3. $\frac{x^2}{4} - \frac{y^2}{49} = 1$
4. $\frac{x^2}{16} - \frac{y^2}{64} = 1$
5. $\frac{x^2}{64} - \frac{y^2}{25} = 1$
6. $\frac{x^2}{100} - \frac{y^2}{49} = 1$
7. $\frac{x^2}{25} - \frac{y^2}{81} = 1$
8. $\frac{x^2}{81} - \frac{y^2}{144} = 1$
9. $16x^2 = 25y^2 + 400$
10. $81x^2 = 121y^2 + 9801$
11. $16x^2 = 25y^2 - 400$
12. $81x^2 = 121y^2 - 9801$
13. $49x^2 = 16y^2 + 784$
14. $100x^2 = 36y^2 + 3600$
15. $64y^2 = 4x^2 + 256$
16. $100y^2 = 49x^2 + 4900$
17. $\frac{x^2}{36} - \frac{y^2}{20} = 1$
18. $\frac{x^2}{30} - \frac{y^2}{64} = 1$
19. $10y^2 - 49x^2 = 490$
20. $100y^2 = 50x^2 + 5000$
21. $\frac{(x-2)^2}{25} - \frac{(y-1)^2}{16} = 1$

22. $\frac{(x-3)^2}{64} - \frac{(y-6)^2}{25} = 1$

23. $\frac{(x-3)^2}{64} - \frac{(y+4)^2}{36} = 1$

24. $\frac{(x+2)^2}{49} - \frac{(y-5)^2}{100} = 1$

25. $\frac{(y-5)^2}{121} - \frac{(x-3)^2}{81} = 1$

26. $\frac{(y-7)^2}{4} - \frac{(x+4)^2}{9} = 1$

27. $\frac{(y+6)^2}{36} - \frac{(x+2)^2}{49} = 1$

28. $\frac{(y+1)^2}{100} - \frac{(x+3)^2}{16} = 1$

29. $\frac{(x-5)^2}{10} - \frac{(y+1)^2}{25} = 1$

30. $\frac{(x+4)^2}{81} - \frac{(y-5)^2}{55} = 1$

Write an equation of a hyperbola with its center at the origin from the given information.

31. The hyperbola has a vertex at $(-4, 0)$ and the equations of the asymptotes are $y = \pm x$.

32. The hyperbola has a vertex at $(1, 0)$ and the equations of the asymptotes are $y = \pm 2x$.

33. The hyperbola has a vertex at $(5.6, 0)$ and the equations of the asymptotes are $y = \pm\frac{1}{4}x$.

34. The hyperbola has a vertex at $(-2.4, 0)$ and the equations of the asymptotes are $y = \pm\frac{1}{3}x$.

35. The hyperbola has a vertex at $(0, 5)$ and the equations of the asymptotes are $y = \pm\frac{5}{3}x$.

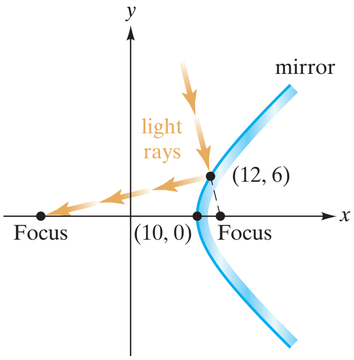
36. The hyperbola has a vertex at $(0, -2)$ and the equations of the asymptotes are $y = \pm\frac{4}{3}x$.

37. The hyperbola has a vertex at $(0, -3.5)$ and the equations of the asymptotes are $y = \pm\frac{1}{2}x$.

38. The hyperbola has a vertex at $(0, 4.5)$ and the equations of the asymptotes are $y = \pm\frac{3}{5}x$.

In each figure, a hyperbolic mirror is shown. Write an equation that models the mirror's surface.

39.



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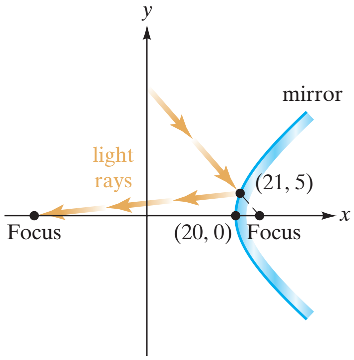
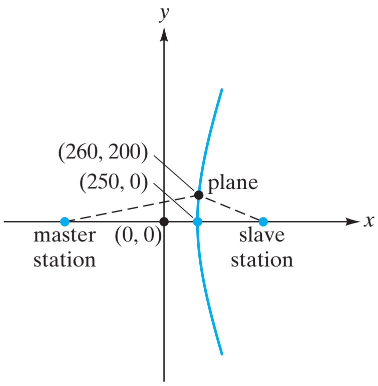


In each figure, an aircraft is guided by a navigation system that employs hyperbolic tracking. Write an equation that models the hyperbola depicted.

41.



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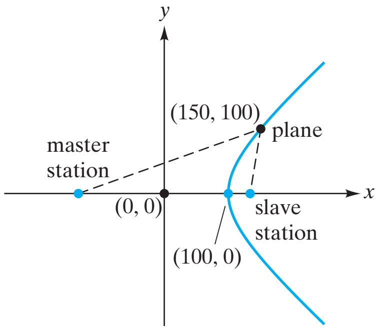


12.4 Calculator Exercises

To store the equation of a hyperbola in your calculator, you will need to solve the equation of a horizontal hyperbola, $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$, or the equation of a vertical hyperbola, $\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$, for y. First we will solve the equation of a horizontal hyperbola for y.

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

$$-\frac{(y-k)^2}{b^2} = -\frac{(x-h)^2}{a^2} + 1$$ Subtract $\frac{(x-h)^2}{a^2}$ from both expressions.

$$\frac{(y-k)^2}{b^2} = \frac{(x-h)^2}{a^2} - 1$$ Multiply both expressions by -1.

$$(y-k)^2 = \frac{b^2(x-h)^2}{a^2} - b^2$$ Multiply both expressions by b^2.

$$y - k = \pm\sqrt{\frac{b^2(x-h)^2}{a^2} - b^2}$$ Apply the square-root principle.

$$y = k \pm \sqrt{\frac{b^2(x-h)^2}{a^2} - b^2}$$ Add k to both expressions.

Therefore, to graph the horizontal hyperbola with your calculator, first store $Y1 = K + \sqrt{\frac{B^2(X-H)^2}{A^2} - B^2}$ and $Y2 = K - \sqrt{\frac{B^2(X-H)^2}{A^2} - B^2}$. Then store values for a, b, h, and k in locations A, B, H, and K, and graph the curves. In this way, each time you wish to graph the equation of a horizontal hyperbola, you need only store the new values of a, b, h, and k, and graph, saving you from having to repeatedly enter the expressions for Y1 and Y2. For example, if you wish to graph $\frac{(x+3)^2}{16} - \frac{(y-2)^2}{9} = 1$, store 4 in A, 3 in B, -3 in H, and 2 in K. Then press GRAPH, and the calculator will graph the two functions stored in Y1 and Y2.

If you also wish to graph the asymptotes of the hyperbola, remember that the slopes of the asymptotes are given by $\pm\frac{b}{a}$, and the asymptotes will cross at (h, k). Using the point–slope form of a linear equation, you should be able to show that the equations of the two asymptotes are $y = k \pm \frac{b}{a}(x - h)$. (This derivation is left for you.) Thus, if you also store

$$Y3 = K + \frac{B}{A}(X - H) \quad \text{and} \quad Y4 = K - \frac{B}{A}(X - H)$$

the calculator will graph the asymptotes as well.

You should be able to derive the equation of a vertical hyperbola in a similar fashion, with the result $y = k \pm \sqrt{\frac{b^2(x-h)^2}{a^2} + b^2}$. (This derivation is also left for you.) With appropriate changes in the forms for Y1 and Y2, you can now graph a vertical hyperbola. (Note that the change is only in the operation sign within the radicand. This makes it easy for you to quickly change the form stored in the calculator.)

Use these forms to graph the following hyperbolas.

1. $\frac{(x-3)^2}{4} - \frac{(y-5)^2}{9} = 1$

2. $\frac{(y+2)^2}{16} - \frac{(x+1)^2}{25} = 1$

3. $\frac{(x-1)^2}{3} - y^2 = 1$

4. $\frac{y^2}{9} - (x+2)^2 = 1$

12.4 Writing Exercises

You have now seen four different forms for conic sections: parabolas, circles, ellipses, and hyperbolas. To be sure that you can recognize each form, state the form for each of the following equations. After you have identified the type of conic section each form represents, check your results by graphing the equation.

1. $(x-2)^2 + (y+3)^2 = 9$

2. $\frac{x^2}{36} - \frac{y^2}{16} = 1$

3. $y = 5(x-3)^2 + 7$

4. $\frac{x^2}{25} + \frac{y^2}{49} = 1$

5. $y = -2x^2 - 5$

6. $\frac{(y-4)^2}{25} - \frac{(x+2)^2}{9} = 1$

7. $\frac{(x-5)^2}{16} + \frac{(y+2)^2}{4} = 1$

8. $x^2 + y^2 = 15$

9. $x^2 + y^2 - 10x - 4y + 4 = 0$

10. $y = 3x^2 - 12x + 7$

11. $4x^2 - y^2 - 8x - 4y = 4$

12. $4x^2 + y^2 - 8x + 4y + 4 = 0$

CHAPTER 12 SUMMARY

After completing this chapter, you should be able to define the following key terms in your own words.

12.1
conic sections
parabola
directrix
focus
vertical parabola
horizontal parabola

12.3
ellipse
focus
vertices
major axis
co-vertices
minor axis

12.4
hyperbola
transverse axis
branches
horizontal hyperbola
vertical hyperbola
asymptotes

12.2
circle
center
radius
concentric circles

Reflections

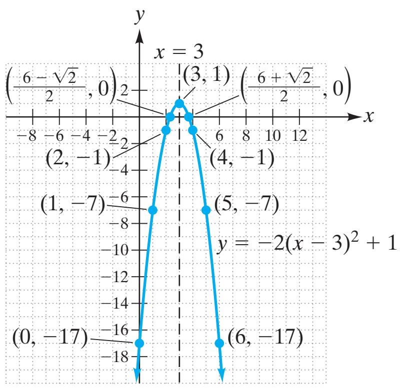
1. What is a conic section?
2. State the definition of a parabola.
3. Given the equation of a parabola, $y = a(x - h)^2 + k$, explain how each constant influences the graph of the parabola.
4. Explain the difference between a vertical parabola and a horizontal parabola.
5. State the definition of a circle.
6. Given the equation of a circle, $(x - h)^2 + (y - k)^2 = r^2$, explain how each constant influences the graph of the circle.
7. State the definition of an ellipse.
8. Given the equation of an ellipse, $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, explain how each constant influences the graph of the ellipse.
9. State the definition of a hyperbola.
10. Given the equation of a hyperbola, $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$, explain how each constant influences the graph of the hyperbola.

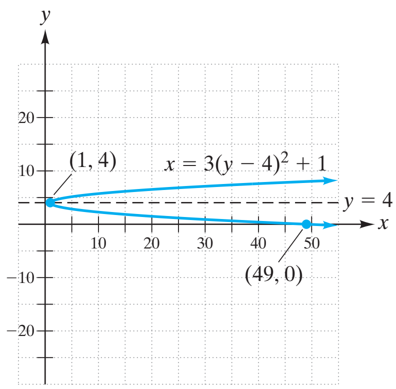
CHAPTER 12 SECTION-BY-SECTION REVIEW

12.1

Recall	Examples
Write a quadratic equation in the form $y = a(x - h)^2 + k$. • Isolate the x-terms. • Complete the square. • Solve for y.	Write $y = 2x^2 + 4x - 3$ in the form $y = a(x - h)^2 + k$ $y = 2x^2 + 4x - 3$ $y + 3 = 2x^2 + 4x$ $y + 3 = 2(x^2 + 2x)$ $y + 3 + 2(1) = 2(x^2 + 2x + 1)$ $y + 5 = 2(x + 1)^2$ $y = 2(x + 1)^2 - 5$
Graph a vertical parabola, $y = a(x - h)^2 + k$. • Locate and label the vertex, (h, k). • Graph the axis of symmetry, $x = h$, with a dashed line. • Determine the x- and y-intercepts. • Determine other points in order to draw the curve.	Graph $y = -2(x - 3)^2 + 1$. $a = -2, h = 3$, and $k = 1$ The vertex is $(3, 1)$. The line of symmetry is $x = 3$. The x-intercept occurs when $y = 0$. $0 = -2(x - 3)^2 + 1$ $2(x - 3)^2 = 1$ $(x - 3)^2 = \frac{1}{2}$

(Continued on page 996)

Recall	**Examples**
	$x - 3 = \pm\sqrt{\frac{1}{2}}$ $x = 3 \pm \sqrt{\frac{1}{2}}$ $x = 3 \pm \sqrt{\frac{1}{2} \cdot \frac{2}{2}}$ $x = 3 \pm \frac{\sqrt{2}}{2}$ $x = \frac{6 \pm \sqrt{2}}{2}$ $x \approx 2.29, 3.71$ The x-intercepts are about (2.29, 0) and (3.71, 0). The y-intercept occurs when $x = 0$. $y = -2(0 - 3)^2 + 1$ $y = -2(-3)^2 + 1$ $y = -17$ The y-intercept is (0, −17). 
Graph a horizontal parabola, $x = a(y - k)^2 + h$. • Locate and label the vertex, (h, k). • Graph the axis of symmetry, $y = k$, with a dashed line. • Determine the x- and y-intercepts. • Determine other points in order to draw the curve.	Graph $x = 3(y - 4)^2 + 1$. $a = 3, h = 1, k = 4$ The vertex is (1, 4). The axis of symmetry is $y = 4$. The x-intercept occurs when $y = 0$. $x = 3(0 - 4)^2 + 1$ $x = 3(-4)^2 + 1$ $x = 49$ The x-intercept is (49, 0). The y-intercept occurs when $x = 0$. $0 = 3(y - 4)^2 + 1$ $3(y - 4)^2 = -1$ $(y - 4)^2 = \frac{-1}{3}$ $y - 4 = \pm\sqrt{\frac{-1}{3}}$ $y = 4 \pm \sqrt{\frac{-1}{3}}$ $y = 4 \pm \sqrt{\frac{-1}{3} \cdot \frac{3}{3}}$ $y = 4 \pm \frac{\sqrt{-3}}{3}$

Recall	Examples
	$y = \frac{4 \pm i\sqrt{3}}{3}$ $y = \frac{12 \pm i\sqrt{3}}{3}$ Since y is an imaginary number, the parabola has no y-intercept. 
Write an equation of a parabola. • Determine values for a, h, and k. • Substitute the values for a, h, and k into the general form of a parabola.	Write an equation for a horizontal parabola with a vertex of (1, 4) and passing through the point (−2, −1). Substitute −2 for x, −1 for y, 1 for h, and 4 for k. Then solve for a. $x = a(y - k)^2 + h$ $-2 = a(-1 - 4)^2 + 1$ $-2 = 25a + 1$ $a = \frac{-3}{25}$ $x = a(y - k)^2 + h$ $x = \frac{-3}{25}(y - 4)^2 + 1$

Write each equation in the form $y = a(x - h)^2 + k$.

1. $y = 4x^2 + 40x + 97$ **2.** $y = \frac{1}{4}x^2 + 4x + 28$ **3.** $y = 1.5x^2 - 6x + 6$

Determine the vertex and axis of symmetry and describe the graph for each equation. Do not graph the equation.

4. $y = -2(x + 3)^2 + 4$ **5.** $y = 0.5(x - 5)^2 - 2$ **6.** $y = x^2 - 12x + 43$ **7.** $y = 3x^2 - 8$

Graph each parabola.

8. $y = \frac{1}{2}(x + 3)^2 - 2$ **9.** $y = -2x^2 + 4x + 2$

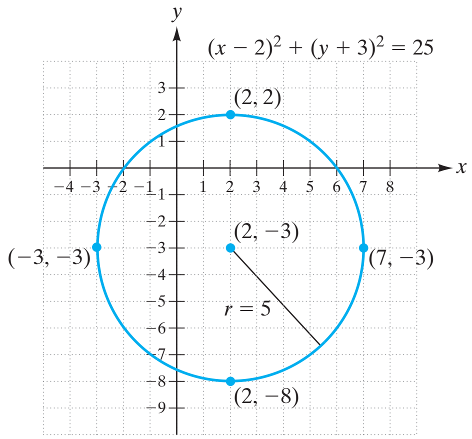
In exercises 10 and 11, write an equation of a parabola from the given information.

10. The parabola is a vertical parabola with the vertex at (2, 2), and it passes through (0, 3).

11. The parabola is a horizontal parabola with the vertex at (−5, −1), and it passes through (3, 1).

12. Find an equation of the parabola that defines the arch in a bridge over a river where the road over the arch is 120 meters long and the maximum height of the arch is 45 meters.

12.2

Recall	Examples
Graph a circle, $(x - h)^2 + (y - k)^2 = r^2$. • Locate and label the center of the circle, (h, k). • Locate and label two points that are a horizontal distance r from the center. • Locate and label two points that are a vertical distance r from the center. • Sketch the graph containing the points located on the circle.	Graph the circle $(x - 2)^2 + (y + 3)^2 = 25$. $(x - 2)^2 + (y + 3)^2 = 25$ $(x - 2)^2 + [y - (-3)]^2 = 5^2$ $h = 2, k = -3$, and $r = 5$ The center of the circle is $(2, -3)$. Two points a horizontal distance of 5 from the center are $(-3, -3)$ and $(7, -3)$. Two points a vertical distance of 5 from the center are $(2, 2)$ and $(2, -8)$. 
Write an equation of a circle. • Determine values for h, k, and r. • Substitute the values for h, k, and r into the general form of a circle.	Write an equation of a circle with its center at $(3, -1)$ and passing through the point $(5, 2)$. Substitute 3 for h, -1 for k, 5 for x, and 2 for y. Then solve for r. $(x - h)^2 + (y - k)^2 = r^2$ $(5 - 3)^2 + [2 - (-1)]^2 = r^2$ $2^2 + 3^2 = r^2$ $13 = r^2$ $r = \pm\sqrt{13}$ $(x - h)^2 + (y - k)^2 = r^2$ $(x - 3)^2 + [y - (-1)]^2 = (\sqrt{13})^2$ $(x - 3)^2 + (y + 1)^2 = 13$

Sketch the graph of each equation.

13. $x^2 + y^2 = 144$ **14.** $x^2 + y^2 = 60$ **15.** $(x - 4)^2 + (y + 3)^2 = 25$

16. $(x + 1.5)^2 + (y + 2.5)^2 = 12.25$

Write each equation in center–radius form and sketch the graph.

17. $x^2 + y^2 - 14x + 8y + 29 = 0$ **18.** $x^2 + y^2 - 10x - 12y = 29$

In exercises 19–21, write the equation of a circle with the given information.

19. Center at the origin and a radius of 5.6.

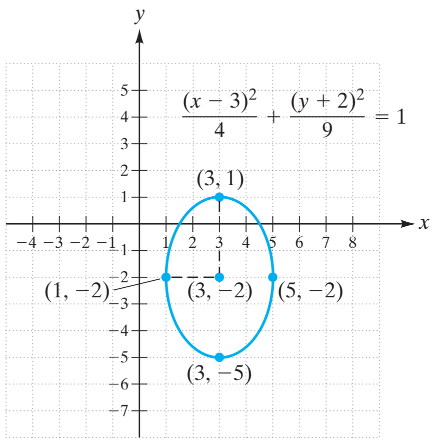
20. Center at $(8, -3)$ and a radius of 9.

21. Center at $(2, 5)$ and passing through $(6, 8)$.

22. A circular pool has a fountain resting on a circular pedestal in the center. The designer of the fountain specified the shape of the fountain's pool by the equation $x^2 + y^2 = 144$ and the shape of the pedestal by the equation $x^2 + y^2 = 25$, with measurements in feet. Use these equations to find the radius of each of the concentric circles. Determine the circumference of each circle. What is the difference in the circumferences of the two circles?

23. An arch over a doorway is semicircular with an opening that is 6 feet wide. Write an equation that models the arch.

12.3

Recall	Examples
Graph an ellipse, $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$. • Locate and label the center of the ellipse, (h, k). • Locate and label two points that are a horizontal distance a from the center of the ellipse. • Locate and label two points that are a vertical distance b from the center of the ellipse. • Sketch the graph containing the points located on the ellipse.	Graph $\frac{(x-3)^2}{4} + \frac{(y+2)^2}{9} = 1$. $\frac{(x-3)^2}{4} + \frac{(y+2)^2}{9} = 1$ $\frac{(x-3)^2}{2^2} + \frac{[y-(-2)]^2}{3^2} = 1$ $h = 3, k = -2, a = 2$, and $b = 3$ The center of the ellipse is $(3, -2)$. Two points a horizontal distance of 2 from the center are $(1, -2)$ and $(5, -2)$. Two points a vertical distance of 3 from the center are $(3, 1)$ and $(3, -5)$. 
Write an equation of an ellipse. • Determine values for h, k, a, and b. • Substitute the values for h, k, a, and b into the general form of an ellipse.	Write an equation of an ellipse with the center located at $(2, 5)$, one co-vertex at $(4, 5)$, and one vertex at $(2, 0)$. Since the co-vertex is $(4, 5)$ and the center is $(2, 5)$, the horizontal distance a is $\|2 - 4\| = 2$. Since the vertex is $(2, 0)$ and the center is $(2, 5)$, the vertical distance b is $\|5 - 0\| = 5$. Substitute 2 for a, 5 for b, 2 for h, and 5 for k. $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ $\frac{(x-2)^2}{2^2} + \frac{(y-5)^2}{5^2} = 1$ $\frac{(x-2)^2}{4} + \frac{(y-5)^2}{25} = 1$

Describe the relationship between the major and minor axes of the graphs of each equation. Sketch the graphs.

24. $\frac{x^2}{121} + \frac{y^2}{64} = 1$ **25.** $\frac{x^2}{25} + \frac{y^2}{49} = 1$ **26.** $\frac{x^2}{10} + \frac{y^2}{40} = 1$ **27.** $25x^2 + 9y^2 = 225$

Sketch the ellipse for each equation.

28. $\frac{(x-5)^2}{49} + \frac{(y-2)^2}{16} = 1$ **29.** $\frac{(x+3)^2}{25} + \frac{(y+1)^2}{36} = 1$ **30.** $4(x-4)^2 + 9(y+3)^2 = 36$

In exercises 31–33, write an equation of an ellipse from the given information.

31. The ellipse has a major axis that is horizontal, with the center located at $(4, 2)$. The distance between the center and one of the vertices of the ellipse is 3, and the distance between the center and one of the co-vertices is 1.

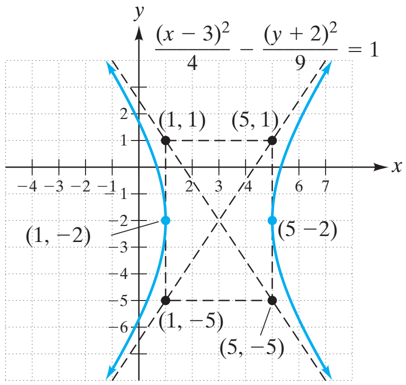
32. The ellipse has a major axis that is vertical, with the center located at $(-3.5, 2)$. The distance between the center and one of the vertices of the ellipse is 4, and the distance between the center and one of the co-vertices is 3.

33. The ellipse has its center at $(2, -3)$, one vertex at $(2, 3)$, and one co-vertex at $(5, -3)$.

34. The orbit of Jupiter around the Sun can be modeled by an ellipse, with the Sun at one of the foci. The Sun is approximately 37.34 million kilometers from the center of the ellipse. At its closest, Jupiter is approximately 740.58 million kilometers from the Sun. The minimum distance of Jupiter to the center of the ellipse is approximately 777.02 million kilometers. Write an equation for the model and sketch the graph of the equation.

35. The governor of your state has asked you to design an oval office for him. The office is to be shaped like an ellipse with a major axis of 50 feet and a minor axis of 40 feet. Write an equation that models the shape of the office.

12.4

<table>
<tr><th>Recall</th><th>Examples</th></tr>
<tr><td>
Graph a horizontal hyperbola,
$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$
or a vertical hyperbola, $\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$.
• Locate and label the center of the hyperbola, (h, k).
• Locate and label two points, the vertices, that are a horizontal distance a from the center of the hyperbola or a vertical distance b from the center.
• With a dashed line, draw the central rectangle with corners at $(h-a, k-b)$, $(h-a, k+b)$, $(h+a, k+b)$, and $(h+a, k-b)$.
• With a dashed line, draw the asymptotes that are extensions of the central rectangle's diagonals, and label the line.
• Sketch the hyperbola containing the vertices that were located and approaching the asymptotes.
</td><td>
Graph $\frac{(x-3)^2}{4} - \frac{(y+2)^2}{9} = 1$.
$\frac{(x-3)^2}{4} - \frac{(y+2)^2}{9} = 1$
$\frac{(x-3)^2}{2^2} - \frac{[y-(-2)]^2}{3^2} = 1$
$h = 3, k = -2, a = 2$, and $b = 3$.
The center of the hyperbola is $(3, -2)$. The vertices at a horizontal distance of 2 from the center are $(1, -2)$ and $(5, -2)$.
The corners of the central rectangle are at $(1, -5)$, $(1, 1)$, $(5, 1)$, and $(5, -5)$.


</td></tr>
<tr><td>
Write an equation of a hyperbola with its center at the origin.
• Determine the two vertices.
• Determine values for a and b.
• Substitute the values of a and b into the form of a hyperbola. (Note that $h = 0$ and $k = 0$.)
</td><td>
Write an equation of a hyperbola with its center at the origin, a vertex at $(2, 0)$, and equations of the asymptotes are $y = \pm\frac{3}{4}x$.
The hyperbola is a horizontal hyperbola, because the vertex is an x-intercept. The other vertex is $(-2, 0)$.
The x-coordinate of the vertex, 2, is a.
Solve $\frac{3}{4} = \frac{b}{a}$ for b.
$\frac{3}{4} = \frac{b}{2}$
$4b = 6$
$b = 1.5$
$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$
$\frac{(x-0)^2}{2^2} - \frac{(y-0)^2}{1.5^2} = 1$
$\frac{x^2}{4} - \frac{y^2}{2.25} = 1$
</td></tr>
</table>

Sketch the graph of each equation. Also, sketch the asymptotes for each graph.

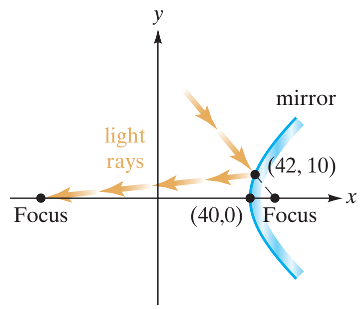
36. $\frac{x^2}{81} - \frac{y^2}{49} = 1$ **37.** $\frac{y^2}{40} - \frac{x^2}{64} = 1$ **38.** $25x^2 - 36y^2 = 900$ **39.** $\frac{(x-3)^2}{64} - \frac{(y-6)^2}{49} = 1$

In exercises 40–41, write an equation of a hyperbola with its center at the origin from the given information.

40. The hyperbola has a vertex at (5, 0) and the equations of the asymptotes are $y = \pm\frac{4}{5}x$.

41. The hyperbola has a vertex at (−3, 0) and the equations of the asymptotes are $y = \pm 4x$.

42. Write an equation that models the surface of the hyperbolic mirror shown in the figure.



CHAPTER 12 CHAPTER REVIEW

Write the equation of a parabola from the given information.

1. The parabola is a vertical parabola with a vertex at (3, 5) and passing through (2, 9).

2. The parabola is a horizontal parabola with a vertex at (−2, −3) and passing through (2, −4).

Write the equation of a circle from the given information.

3. Center at the origin and a radius of 4.5.

4. Center at (−3, 5) and a radius of 4.

5. Center at (3, 1) and passing through (−1, 4).

Write the equation of an ellipse from the given information.

6. The ellipse has a major axis that is horizontal, with the center located at (−4, −2). The distance between the center and one of the vertices of the ellipse is 6, and the distance between the center and one of the co-vertices is 3.

7. The ellipse has a major axis that is vertical, with the center located at (5, 2.5). The distance between the center and one of the vertices of the ellipse is 4, and the distance between the center and one of the co-vertices is 2.

8. The ellipse has its center at (−2, 2), one vertex at (4, 2), and one co-vertex at (−2, 0).

Write the equation of a hyperbola with its center at the origin from the given information.

9. The hyperbola has a vertex at (−3, 0) and the equations of the asymptotes are $y = \pm\frac{1}{2}x$.

Graph exercises 10–23.

10. $y = -\frac{1}{2}(x+5)^2 - 6$ **11.** $y = 2x^2 - 28x + 94$ **12.** $x^2 + y^2 - 10x + 14y + 25 = 0$

13. $x^2 + y^2 = 121$ **14.** $x^2 + y^2 = 40$ **15.** $(x+2)^2 + (y-5)^2 = 36$

16. $\frac{(x+2)^2}{49} + \frac{(y-3)^2}{36} = 1$ **17.** $20x^2 + 5y^2 = 100$ **18.** $25(x-1)^2 + (y+5)^2 = 25$

19. $\frac{x^2}{81} + \frac{y^2}{49} = 1$ **20.** $\frac{y^2}{25} - \frac{x^2}{4} = 1$ **21.** $\frac{x^2}{16} - \frac{y^2}{55} = 1$

22. $64x^2 - 4y^2 = 256$ **23.** $\frac{(x+5)^2}{81} - \frac{(y-7)^2}{9} = 1$

24. The orbit of the largest asteroid, Ceres, around the Sun can be modeled by an ellipse, with the Sun at one of the foci. The Sun is approximately 40.20 million kilometers from the center of the ellipse. At its closest, Ceres is approximately 374.20 million kilometers from the Sun. The minimum distance of Ceres to the center of the ellipse is approximately 412.44 million kilometers. Write an equation for the model and sketch the graph of the equation.

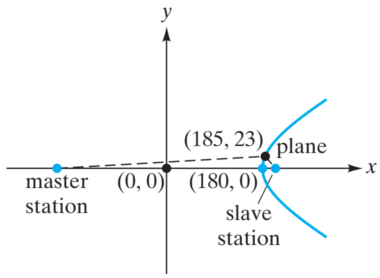
25. An oval room has the shape of an ellipse. The major axis of the room measures 100 feet and the minor axis of the room measures 75 feet. Write an equation that models the shape of the room.

26. The Gateway Arch in St. Louis is a stainless-steel catenary that is 630 feet tall and 630 feet wide at its base. If you were to design an arch that had the shape of a parabola, but with the same height and width as the Gateway Arch, what equation would you write to model the shape of your arch?

27. A funnel has a circular opening at the top given by the equation $x^2 + y^2 = 4$ and a circular opening at the bottom given by the equation $x^2 + y^2 = 0.25$. Use these equations to find the radius of each of the circular openings. Determine the circumference of each circle. What is the difference in the circumferences of the two openings?

28. An arch over a storefront is semicircular with an opening that is 10 feet wide. Write an equation that models the arch.

29. In the figure, an aircraft is guided using a navigation system that employs hyperbolic tracking. Write an equation that models the hyperbola depicted.



CHAPTER 12 TEST

TEST-TAKING TIPS

When studying for a test, you should try to improve your notes. Read your notes and textbook with pencil in hand. Identify and label text or figures according to categories. This chapter had many different forms for conic sections. If you organize your notes according to the various forms and work examples for each form, you will recall the forms better during a test. Try to think of exercises that involve each form. Compare the forms to see what their differences are. List key points that help identify the forms. If something confuses you, place a big question mark next to it, and try to get help with the question before taking the test. Then go back and read the confusing material again to see if the help reduced your confusion.

1. Write the equation $y = -2x^2 + 28x - 90$ in the form $y = a(x - h)^2 + k$.

Graph each parabola. Show the axis of symmetry and label the vertex.

2. $y = 0.5x^2 + 4x + 1$

3. $x = 2(y - 3)^2 + 2$

Graph each equation. Label any vertices.

4. $x^2 + y^2 = 81$

5. $(x - 3)^2 + (y + 5)^2 = 16$

6. $x^2 + y^2 + 2x - 4y = 4$

7. $\dfrac{(x - 5)^2}{49} + \dfrac{(y + 4)^2}{36} = 1$

8. $81x^2 + 25y^2 = 2025$

9. $4x^2 - 25y^2 = 100$

10. $\dfrac{(y + 6)^2}{9} - \dfrac{(x - 3)^2}{4} = 1$

In exercises 11–18, write an equation from the given information.

11. A vertical parabola has a vertex at $(-3, -2)$ and passes through $(-2, -4)$.

12. A horizontal parabola has a vertex at $(-3, 4)$ and passes through $(9, 6)$.

13. A circle has a center at $(2, -1)$ and a radius of 4.

14. A circle has a center at $(6, 2)$ and passes through $(9, 6)$.

15. An ellipse has a major axis that is horizontal, with center at $(4, 1)$. The distance between the center and one of the vertices of the ellipse is 3, and the distance between the center and one of the co-vertices is 2.

16. An ellipse has its center at $(-3, 2)$, one vertex at $(-3, 6)$, and one co-vertex at $(-5, 2)$.

17. A hyperbola has its center at the origin and a vertex at $(2, 0)$. The equations of its asymptotes are $y = \pm\frac{2}{3}x$.

18. A hyperbola has its center at the origin and a vertex at $(0, -2)$. The equations of its asymptotes are $y = \pm x$.

19. The orbit of Uranus around the Sun can be modeled by an ellipse, with the Sun at one of the foci. The Sun is approximately 135.00 million kilometers from the center of the ellipse. At its closest, Uranus is approximately 2737.32 million kilometers from the Sun. The minimum distance of Uranus to the center of the ellipse is approximately 2869.15 million kilometers. Write an equation for the model and sketch the graph of the equation.

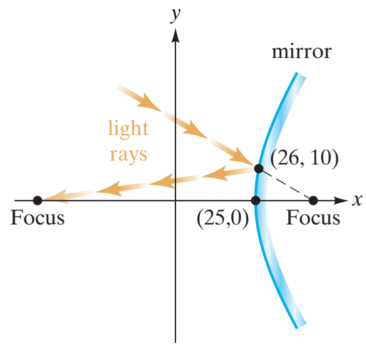
20. An elliptical conference table has a major axis measuring 8 feet and a minor axis measuring 5 feet. Write an equation that models the shape of the table.

21. An underpass is semicircular with a width of 26 feet. Write an equation that models the underpass.

22. A footbridge over a creek has an arch in the shape of a parabola. The maximum height of the arch is 5 feet, and the horizontal length of the bridge over the arch is 12 feet. Write an equation that models the shape of the arch.

23. A planetarium has a circular room whose outer edge is given by the equation $x^2 + y^2 = 400$, measured in feet. Use this equation to find the radius of the circular room. Find the circumference of the room.

24. Write an equation that models the surface of the hyperbolic mirror shown in the figure.



25. Given the equation of a circle, $(x - h)^2 + (y - k)^2 = r^2$, explain how the constants h, k, and r influence the graph of the circle.

Chapter 12 Project

Part 1. General Equation for a Conic Section

The general equation for a conic section is

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

where x and y are the variables and the letters a through f are coefficients. It is an interesting fact that the type of conic section which results can be determined by finding the sign of $b^2 - 4ac$. The following conditions apply:

- If $b^2 - 4ac < 0$, the conic section is an ellipse, a circle, a point, or no curve.
- If $b^2 - 4ac = 0$, the conic section is a parabola, one line, or no curve.
- If $b^2 - 4ac > 0$, the conic section is a hyperbola or two intersecting lines.

We will check some of these conditions with the following table, which lists cases with values for the coefficients:

$$ax^2 + bxy + cy^2 + dx + ey + f = 0.$$

Case	a	b	c	d	e	f	$b^2 - 4ac$	Equation	Equation Solved for y	Type of Conic Section and Sketch
1.	9	0	4	0	0	−36				
2.	1	0	1	−6	−8	0				
3.	4	0	0	0	−1	0				
4.	25	0	−1	0	0	−100				
5.	0	0	0	4	−1	0				
6.	1	0	1	0	0	0				
7.	25	0	−1	0	0	0				

- First, use the values of the coefficients to determine the value of $b^2 - 4ac$.
- Next, substitute the values of the coefficients into the general equation to write the equation for each case.
- Solve the equation for y. You may have to use the principle of square roots or the quadratic formula to do so. Remember that when you are solving for y by means of the quadratic formula, carry along the x variable as part of the coefficients for the quadratic equation of y.
- Use your calculator to graph the equations you have solved for y. Remember that the graph may not be a function, and you may have to graph the equations in two parts.
- After graphing the equation(s) for y, describe the type of conic section the case represents, and sketch the graph in the table.
- Finally, check the value for $b^2 - 4ac$ against the conditions listed above to see if, in fact, $b^2 - 4ac$ did predict what type of conic section resulted.

Part 2. Parabolas

You have learned how to graph a parabola by first rewriting the equation as $y = a(x - h)^2 + k$ in order to locate the vertex and then plotting some additional coordinate pairs. However, earlier you learned that you can graph $y = ax^2 + bx + c$ by using a table of values to locate and then graph coordinate pairs. Try both methods on the following exercises:

1. $y = 2x^2 - 20x + 57$ **2.** $y = 9 - 6x - 3x^2$ **3.** $2y = x^2 + 12x + 6$

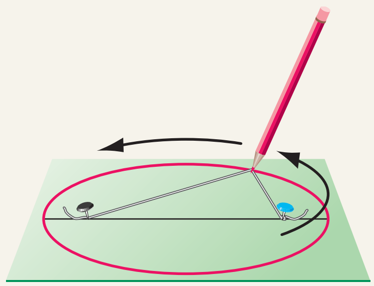
Discuss the advantages and disadvantages of each method. Does one method give you additional information about the graph? Explain.

Part 3. Circles

In this chapter, you saw examples of concentric circles that occur in real-world applications. Some other places where concentric circles occur are in the cross section of a piece of pipe and in buildings with a dome, such as the Capitol Building of the United States. Search the literature or Web sites until you find an example that involves concentric circles with their dimensions given. Then use the example to write equations that model the concentric circles you found.

Part 4. Ellipses

An ellipse can be sketched by using two tacks and some thread. Tie the two tacks together with the piece of thread. Then stick the tacks into a piece of paper, keeping some slack in the thread. Use a pencil to hold the thread taut, and draw a curve. The curve will be an ellipse. See the sketch.



Explain why this method works. (*Hint*: Review the definition of an ellipse in this chapter.)

Now use the tacks and thread to sketch an ellipse on a piece of graph paper. After drawing the ellipse, draw a set of axes (an x-axis and a y-axis) on your sketch, and determine the vertices and co-vertices of your ellipse. Use this information to write the equation of your ellipse.

In this chapter, we discussed whispering galleries. Search the library or Web sites for an example of a whispering gallery. Find the dimensions and write an equation that models the shape of the floor of the gallery.

Part 5. Hyperbolas

As you have learned in this chapter, the equation of a horizontal hyperbola with its center at the origin is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Discover what happens to the shape of the hyperbola when $a < b$, when $a > b$, and when $a = b$. To do so, graph exercises 1 and 2 on the same coordinate plane. Then graph exercises 1 and 3 on the same coordinate plane.

1. $\frac{x^2}{9} - \frac{y^2}{9} = 1$ **2.** $\frac{x^2}{9} - \frac{y^2}{4} = 1$ **3.** $\frac{x^2}{9} - \frac{y^2}{16} = 1$

4. Is the hyperbola from exercise 2 wider or narrower than that from exercise 1?

5. Is the hyperbola from exercise 3 wider or narrower than that from exercise 1?

6. Describe how the hyperbola in which $a > b$ compares with the hyperbola in which $a = b$.

7. Describe how the hyperbola in which $a < b$ compares with the hyperbola in which $a = b$.

8. Do you think the preceding results will still apply in graphing a vertical hyperbola?

CHAPTERS 1–12 CUMULATIVE REVIEW

Simplify and write with positive exponents.

1. $(-2x^{-3}y^2)^2$ **2.** $\left(\dfrac{4x^{3/4}}{x^{-1/4}}\right)^2$

Simplify.

3. $(3.2a^2 - 2.6ab + 1.7b^2) - (4a^2 + 2.6ab - b^2)$

4. $(3x + 2y)(3x - 2y)$

5. $(2x + 6)(2x - 3)$

6. $(x - 4)^2$

7. $\dfrac{25x^2y^4z^2}{-5xyz}$

8. $\dfrac{2r^2s + 4rs - 8s^2}{2rs}$

9. $\dfrac{x^2 - x - 6}{x^2 - 9}$

10. $\dfrac{x}{x + 2} + \dfrac{3}{x - 2} - \dfrac{x - 1}{x^2 - 4}$

11. $\dfrac{2m - 3}{m + 1} \cdot \dfrac{2m^2 + 5m + 3}{4m^2 - 9}$

12. $\dfrac{16x}{y^3 + y^2 - 12y} \div \dfrac{4x^2y}{y^2 - y - 20}$

13. $\sqrt{-18} - \sqrt{-50} + 4\sqrt{-8}$

14. $\sqrt[3]{16a^2b^2} \cdot \sqrt[3]{8ab^2}$

15. $(\sqrt{2} - \sqrt{3})(\sqrt{2} + \sqrt{3})$

16. $\dfrac{\sqrt{32xy^2z}}{\sqrt{8x^3y}}$

17. $\dfrac{\sqrt{x} + 4}{\sqrt{x} - 3}$

18. $x^{2/3}(x^{3/4} - x^{1/3})$

Factor completely if possible.

19. $25m^2 - 36n^2$

20. $2x^2 - 4x - 16$

21. $x^2 + 6x - 6$

Graph and label as indicated. Determine the domain and range of each function.

22. $f(x) = \dfrac{2}{3}x + 1$; three points on the graph

23. $y = x^2 + 5x + 4$; vertex, y-intercept, x-intercept, enough points to determine the curve, and the axis of symmetry

24. $y = \dfrac{x^2 + 4}{x + 2}$; enough points to determine the curve

25. $h(x) = \sqrt{x - 5}$; enough points to determine the curve

26. $y = 3^x$; enough points to determine the curve

27. $f(x) = \ln(x - 3)$; enough points to determine the curve

Graph the given conic section.

28. $x = y^2 + 2y - 5$

29. $\dfrac{x^2}{25} + \dfrac{y^2}{100} = 1$

30. $\dfrac{x^2}{25} - \dfrac{y^2}{100} = 1$

31. $x^2 + y^2 - 10 = 0$

Solve.

32. $2(x + 3.1) = (x - 4.2) + (x - 6)$

33. $x^2 - 2x - 12 = 12$

34. $x^2 - 5x + 9 = 0$

35. $\dfrac{x + 1}{x + 2} = \dfrac{x - 1}{x + 3}$

36. $4\sqrt{x + 1} - 14 = -2$

37. $e^{2x - 4} = e^{x(x - 2)}$

38. $2 \log_3 x = \log_3 5$

Solve. Write the solution set in interval notion.

39. $2x - 3 < 3(x - 4)$

40. $x^2 + 2x \geq 2$

Solve.

41.
$$\begin{aligned} x + 2y - 2z &= 3 \\ -x + 3y + 3z &= -3 \\ 2x - 2y + z &= 5 \end{aligned}$$

In exercises 42 and 43, write an equation of a line that satisfies the given conditions.

42. Passes through the points $(3, -1)$ and $(-2, 2)$.

43. Is perpendicular to $x + 4y = 2.3$ and passes through $(-2, 1)$.

44. Write a quadratic equation for a curve that passes through the points $(0, -6)$, $(-4, 0)$, and $(1, 0)$.

45. Given $f(x) = \dfrac{2}{3}x + 6$, find $f^{-1}(x)$.

46. Happy Recipe Company bought a rebuilt copier for \$525 to reproduce its latest recipe book. If it costs \$5 per book for materials to print the books and each book sells for \$15, determine the break-even point (the point at which the revenue equals the cost).

47. Nathan has 15 feet of landscaping timbers to place diagonally across his rectangular flower garden. If he wants to use all of the timbers and have the length of the garden be 3 feet more than the width, determine the dimensions of the garden.

48. A tank is drained by two separate drains. When the drain on the left is used alone, it takes 4 hours to drain the tank. When the drain on the right is used alone, it takes 6.5 hours to drain the tank. How long will it take to drain the tank if both drains are used? (Round your answer to the nearest tenth of an hour.)

49. If money in an account is continuously compounded, the amount in the account will grow to k times the original investment in $t = \frac{\ln k}{r}$ years when the annual rate of interest is r. How long will it take the amount invested to triple if the annual interest rate is 4.5%?

50. The orbit of the Moon around the Earth can be modeled by an ellipse, with the Earth at one of the foci. The Earth is approximately 21,120 kilometers from the center of the ellipse. The minimum distance of the Moon to the center of the ellipse is approximately 383,419 kilometers. At its closest, the Moon is approximately 362,880 kilometers from the Earth. Write an equation for the model and sketch the graph of the equation.