Congruence

Consider real symmetric matrices of size n by n only (these are orthogonal diagonalizable). n is the order of the matrix.

If $B = P^T AP$ where P is invertible then A and B are said to be *congruent*.

Under similarity, the eigenvalues stay the same, since the characteristic equation is same.

Question: **Under congruence, what stays the same?**

For $c_i \neq 0$ we have congruence

$$\begin{pmatrix} c_1 & 0 & \dots \\ 0 & c_2 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & \dots \\ 0 & \lambda_2 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} c_1 & 0 & \dots \\ 0 & c_2 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} c_1^2\lambda_1 & 0 & \dots \\ 0 & c_2^2\lambda_2 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

so we can change the value of each eigenvalue to one of $+1$, -1 or 0. Note permutation matrices are orthogonal, so we can change the order of the eigenvalues. Let p be the number of $+1$'s, m be the number of -1's and z be the number of 0's. Note $p + m + z = n$, the *order*. The integer $n - z$ is the *rank* and $p - m$ is called the *signature*. From the order, rank and signature we can recover p, m and z.

Computational note: we can obtain the diagonal matrix above by using simultaneous row and column operations in a manner like ordinary row reduction — consider what EAE^T means for elementary E. This avoids having to compute eigenvalues.

In the complex case, where Hermitian matrices are used instead, λ_i is real. Use P^* instead of P^T. Then $\overline{c_i}\lambda_i c_i = |c_i|^2\lambda_i$ so again the eigenvalues can be changed to one of $+1$, -1 or 0 and similar results hold.

Theorem (**Sylvester's law of inertia**) Symmetric matrices are congruent iff they have the same order, rank and signature (or equivalently, the same p, m and z).

Proof: Let

$$\Lambda_{p,m,z} = \text{Diagonal}\,(\overbrace{1,\dots,1}^{p},\overbrace{-1,\dots,-1}^{m},\overbrace{0,\dots,0}^{z}).$$

It suffices to show $\Lambda_{p,m,z}$ is not congruent to any other $\Lambda_{p',m',z'}$. For this the case $p < p'$ is like all other cases. Suppose $\Lambda_{p,m,z} = P^T\Lambda_{p',m',z'}P$. Consider subspaces, where the x_i's are arbitrary,

$$V_1 = \{P^{-1}\begin{pmatrix} x_1 \\ \vdots \\ x_{p'} \\ 0 \\ \vdots \\ 0 \end{pmatrix}\}, \qquad V_{-1} = \{\begin{pmatrix} \vdots \\ 0 \\ x_{p+1} \\ \vdots \\ x_{p+m} \\ 0 \\ \vdots \end{pmatrix}\}, \qquad \text{and} \qquad V_0 = \{\begin{pmatrix} 0 \\ \vdots \\ 0 \\ x_{p+m+1} \\ \vdots \\ x_{p+m+z} \end{pmatrix}\}$$

Since † $\mathbf{v}^T\Lambda_{p,m,z}\mathbf{v}$ is $>$, $<$ or $=0$ for all nonzero $\mathbf{v} \in V_j$, as j is 1, -1 or 0, we see that the V_j are *disjoint* subspaces of n-space. Yet their dimensions add to $p' + m + z$ which is greater than n.

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† Let $\mathbf{v} = P^{-1}\begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix}$ be in V_1. Then $\mathbf{v}^T\Lambda_{p,m,z}\mathbf{v} = (\,\mathbf{x} \quad \mathbf{0}\,)\,(P^{-1})^T\Lambda_{p,m,z}P^{-1}\begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} =$
$(\,\mathbf{x} \quad \mathbf{0}\,)\,\Lambda_{p',m',z'}\begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} = x_1^2 + \dots + x_{p'}^2 > 0$ if $\mathbf{v} \neq \mathbf{0}$.