

Average Values

When we want to find the average or mean of a finite collection of objects we simply sum the values together, and divide by the number of objects in the collection. Thus, the average of the scores 45, 64, and 90 on a test is

$$\frac{45 + 64 + 90}{3} = \frac{199}{3} \approx 66.33$$

What about a continuous function? Is there an intuitive notion of the average value of a continuous function? Let's think about a physical example. Suppose we know the instantaneous velocity of a person hiking in the mountains over some interval of time. What should the average velocity of the hiker be? It seems intuitive to define the average velocity as the total distance traveled divided by the time it took to travel this distance. In this way, another person who was traveling at the average velocity of the first hiker for the same amount of time would travel the same distance. We can find the distance traveled by calculating the integral of the hiker's velocity over the interval of time the hiker traveled. The time traveled will simply be the difference of the endpoints of the interval of time.

Mathematically, we say that the average value of a function f over an interval $[a, b]$ is given by

$$\frac{1}{b-a} \int_a^b f(x) dx$$

which is in complete analogy with the above thought experiment. The average value of a function over a given interval also has a useful graphical interpretation. If we integrate the average value over the interval $[a, b]$, we will find the same area as under the function f . Thus, the average value gives the height of a rectangular box with width $[a, b]$ that has the same area as the function f .

Example 1 Find the average value of x^3 over $[0, 3]$.

Solution Using the above formula we know the average value is

$$\frac{1}{3-0} \int_0^3 x^3 dx = \frac{1}{3} \frac{x^3}{3} \Big|_0^3 = \frac{3^3}{9} - \frac{0}{9} = 3$$

Example 2 Find the average value of $\sin(x)$ over $[0, 2\pi]$.

Solution We already know that

$$\int_0^{2\pi} \sin(x) dx = -\cos(2\pi) + \cos(0) = 0$$

so the average value is 0.

The above result is a problem in the realm of electrical circuits, where AC currents and voltages are represented by sinusoidal functions. Over the appropriate intervals of 2π these functions have an average value of 0. Even if the average value over an interval is not 0, all segments of length 2π are cancelled out when we find the average value. If we were to think

of the average power moving through one of these systems, it would surely not be 0, simply because the average voltage and current are 0. Instead, we would use the root mean square or RMS value, in which we find the average value of the square of a function, and take the square root of the output.

$$\text{RMS value of } f(x) = \sqrt{\frac{1}{b-a} \int_a^b f^2(x) dx}$$

Unfortunately, we do not currently have the tools to evaluate the required integrals to find the RMS values of sinusoidal functions, given by

$$\sqrt{\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} I^2 \sin^2(t) dt}$$

where I represents the amplitude of the signal (and t_1 and t_2 occur at the end of a period of the waveform). Nevertheless, we can find the RMS value for other functions (although we cannot show it, it turns out that the value of the above expression is $I/\sqrt{2}$).

Example 3 Find the RMS value of $f(x) = x$ over $[-4, 4]$

Solution The first step in the process is to find the average value of $f^2(x) = x^2$.

$$\frac{1}{4 - (-4)} \int_{-4}^4 x^2 dx = \frac{1}{8} \frac{x^3}{3} \Big|_{-4}^4 = \frac{1}{24} (64 - (-64)) = \frac{16}{3}$$

Now we take the square root of this value, to find

$$\text{RMS of } f(x) = \sqrt{\frac{16}{3}} = \frac{4}{\sqrt{3}}$$

Example 4 Suppose you are the coach of a runner in a one-on-one race, who is facing another runner notorious for his running techniques. Your runner's best technique is to run at a constant pace throughout the whole race. If you know the opponent's running pattern is given by

$$v(t) = \begin{cases} 5 & 0 \leq t < 1 \\ 4 + t & 1 \leq t < 3 \\ 15 - t^2 & 3 \leq t \leq \sqrt{15} \end{cases}$$

based on previous races, at what constant pace should you advise your runner to run (velocity is in miles per hour)?

Solution In order to win the race, the constant pace will have to be slightly higher than the average velocity of the erratic runner. In order to find the average of this piecewise defined function we do not find the average of each piece. Instead, we find the total distance traveled, by integrating, and then divide the final result by the time it took to run.

$$\begin{aligned} \int_0^{\sqrt{15}} v(t) dt &= \int_0^1 5 dt + \int_1^3 (4 + t) dt + \int_3^{\sqrt{15}} (15 - t^2) dt = 5t \Big|_0^1 + \left(4t + \frac{t^2}{2}\right) \Big|_1^3 + \left(15t - \frac{t^3}{3}\right) \Big|_3^{\sqrt{15}} \\ &= (5 - 0) + \left(12 + \frac{9}{2} - 4 - \frac{1}{2}\right) + \left(15\sqrt{15} - \frac{\sqrt{15}^3}{3} - 45 + \frac{27}{3}\right) \approx 19.73 \end{aligned}$$

Now we divide by $\sqrt{15}$ to find the average speed, which is

$$\frac{19.73}{\sqrt{15}} \approx 5.094$$

Now in order to be sure our runner wins, we would like a constant pace slightly higher than this, so we might advise 5.25 miles per hour.

If we think about the erratic runner in the above race, it seems that at some point he must have been running at his average velocity. If he was always running lower than the average speed, then the average would be too high, and similarly if he was always running faster the average would be too slow. This means that at some points he was running faster and at some points running slower. Because in the physical world we cannot instantly accelerate, at least at some points he must have been running the average speed (even if only for a very short fraction of time). This result holds more generally, as long as we have continuous functions (so in this way we can think that a physical velocity must be continuous).

The Mean Value Theorem for Definite Integrals

If f is continuous on $[a, b]$, then at some point c in $[a, b]$

$$f(c) = \frac{1}{b-a} \int_a^b f(x) dx$$

This theorem tells us (not too surprisingly) that if the integral of a continuous function over a finite interval is 0, then the value of the function must be 0 at some point in the interval. Compare this with the behavior of the sine function.