Unit- 2 (B)

Theory of Quadratic Equations

2.1 Definition :

The equation, in which the unknown variable quantity has maximum power two, is called quadratic. Example -

$(x + 5)^2 = 0$

$4x^2 - 5x + 2 = 0$

$9x^2 = 64$

The equation $ax^2 + bx + c = 0$ is a general form of the the quadratic equation. Where $a, b, c, \in R$ and $a \neq 0$

2.2 Roots of an quadratic equation

Those values of x, which statisfy the equation $ax^2 + bx + c = 0, a \neq 0$ are called the roots of the equation.

If α, β be the roots of $ax^2 + bx + c = 0$. Then this equation can be written as $a (x - \alpha) (x - \beta) = 0$ because $x - \alpha = 0$ or $x - \beta = 0$ gives $x = \alpha$, and β (where $a \neq 0$)

2.3 A quadratic equation has two roots only

Let α, β, and γ be the three separate roots of the equation

$ax^2 + bx + c = 0$ .....(1)

Then $a\alpha^2 + b\alpha + c = 0$ ....(2)

$a\beta^2 + b\beta + c = 0$ ....(3)

$a\gamma^2 + b\gamma + c = 0$ ....(4)

Subtracting (2) and (3)

$a (\alpha^2 - \beta^2) + b (\alpha - \beta) = 0$

$[(\alpha + \beta) + b] (\alpha - \beta) = 0$

$\Rightarrow \alpha - \beta = 0 \ \Rightarrow \alpha = \beta$

But α and β are separate roots, therefore

$a (\alpha + \beta) + b = 0$ ....(5)

similarly

$a (\alpha - \gamma) + b = 0$ ....(6)

subtracting (5) and (6) we get

$a (\alpha - \gamma) = 0$

$\Rightarrow \alpha - \gamma = 0 \neq 0$

$\Rightarrow \alpha = \gamma$

Therefore if we suppose three roots of a quadratic equaiton then two of them will be always equal.

Hence there are always two roots of an quadratic equation

2.4 Nature of the roots

The nature of quadratic equation $ax^2 + bx + c = 0$ is decided by the expression $b^2 - 4ac$ which is called Discriminant. It is denoted by Δ. The discriminant $\Delta = b^2 - 4ac$ decides the roots of quadratic equation in the following way.

(i) if $\Delta > 0 \Rightarrow b^2 - 4ac > 0$ that is, when $b^2 - 4ac$ is positive, both roots will be real and unequal.

(ii) If $\Delta < 0$ that is $b^2 - 4ac < 0$ then both roots will be imaginary.

(iii) If $\Delta = 0$ that is values $b^2 - 4ac = 0$ then both roots will be real and equal. The values of Δ will be $\frac{-b}{2a}$

(iv) If Δ is perfect square and a, b, c are rational numbers, then both roots will be rational because $b^2 - 4ac$ will be rational numbers.

(v) If $b^2 - 4ac$ is not perfect square then $b^2 - 4ac$ will be irrational number, therefore roots will be irrational numbers.

2.5 Relations between /roots and Coefficients of Quadratic Equation)

Let α, β be the roots of quadratic equation then we know that

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad ....(1)$$

$$\beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \quad ....(2)$$

adding (1) and (2)

$$\alpha+\beta=\frac{-b+\sqrt{b^2-4ac}}{2a}+\frac{-b-\sqrt{b^2-4ac}}{2a}$$

The sum of roots $\alpha + \beta = -b/a$

Multiplying (1) and (2)

$$\alpha.\beta = \left[\frac{-b+\sqrt{b^2-4ac}}{2a}\right]+\left[\frac{-b-\sqrt{b^2-4ac}}{2a}\right]$$

$$= \frac{-b^2-(b^2-4ac)}{4a^2} = c/a$$

Note - If $\sqrt{b^2-4ac}$ is imaginary then roots will be complex and conjugate if one is a + *ib* then other will be a – ib

2.6 Formation of quadratic Equation -

To construct an quadratic equation whose roots α and β are given

Let the given equation be $ax^2 + bx + c = 0$

We know that $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta=\frac{c}{a}$

also $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$ ....(i)

Putting the value of $\frac{-b}{a}$ and $\frac{c}{a}$ in (1)

$x^2 - (\alpha + \beta)x + \alpha\beta = 0$ ....(ii)

That is

x^2 – (Sum of roots) x+Product of roots = 0

Therefore, to construct the equation we have have the following steps.

(1) Find the sum of roots

(2) Find the product of roots

(3) Put the values in the equation ...(ii)

Eaxmple 1 - Find the nature of the roots

(i) $x^2 - 6x + 9 = 0$

(ii) $x^2 - 5x - 6 = 0$

(iii) $2x^2 + 3x - 4 = 0$

(iv) $x^2 - x + 1 = 0$

Solution (i) - The discriminant of the roots $x^2 - 6x + 9 = 0$ will be

$\Delta = b^2 - 4ac = 36 - 4 \times 1 \times 9$

$= 36 - 36 = 0$

The roots will be real and equal.

(ii) The discriminant of the equation $x^2 - 5x - 6 = 0$ will be

$\Delta = b^2 - 4ac$

$= 25^2 - 4 \times 1 \times (-6)$

$= 25 + 24$

$= 49 > 0$

> 0 and perfect square, the roots will be real and rational.

(iii) The discriminant of the equation $2x^2 + 3x - 4 = 0$ will be

$\Delta = b^2 - 4ac$

$= 9 - 4.\ 2(-4)$

$= 9 + 32$

$= 41$

> 0 and it is not perfect square therefore roots will real and irrational

(iv) The discriminant of the equation $x^2 - x + 1 = 0$ will be

$\Delta = b^2 - 4ac$

$= 1 - 4$

$= -3$

< 0 the root will be complex number which are conjugate to each other.

Example- 2:If α and βbe the roots of the equation $x^2 - px + q = 0$ then prove that $\{\alpha^2/\beta + \beta^2/\alpha\} - p^2(1/\alpha + 1/\beta) + 3(\alpha + \beta) = 0$ (M.P. 95)

Solution : α, β are the roots of the equation.

$x^2 + px + q = 0$

$\therefore\ \alpha + \beta = p$ and $\alpha\beta = q$

$$\text{L.H.S.}=\left(\frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha}\right)-p^2\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)+3(\alpha+\beta)$$

$$=\frac{\alpha^3+\beta^3}{\alpha\beta}-p^2\frac{\alpha+\beta}{\alpha\beta}+3(\alpha+\beta)$$

$$=\frac{(\alpha+\beta)[(\alpha+\beta)^2-3\alpha\beta]}{\alpha\beta}-\frac{p^2(\alpha+\beta)}{\alpha\beta}+3(\alpha+\beta)$$

$$=\frac{p[p^2-3q]}{q}-\frac{p^3}{q}+3q$$

$$=\frac{p^2-3pq-p^3+3pq}{q}$$

= 0

Example-3 : If α,β be the roots of the equation $ax^2 + bx + c = 0$ then find the equation whose roots are 1/α and 1/β (M.P.)1992

Solution :

The given equation is $ax^2 + bx + c = 0$

The roots of this equation are α and β

∴ $\alpha + \beta = -b/a, \alpha\beta = c/a$

The equation whose roots are $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ will be

$$x^2 - \frac{\alpha+\beta}{\alpha\beta}x + \frac{1}{\alpha\beta} = 0$$

Putting the values of $\alpha + \beta$ and $\alpha\beta$ we get

$$x^2 - \left[\frac{-b/a}{c/a}\right] x + 1/(c/a) = 0$$

$$x^2 + \frac{b}{c}x + \frac{a}{c} = 0$$

$cx^2 + bx + a = 0$ will be the required equation.

Example 4 : It the ratio of roots of the equation $ax^2 + bx + c = 0$ be r : 1 then show that $(r^2 + 1)/r = (b^2 - 2ac)/ac$

Solution : Let α and rα be the roots of the equation $ax^2 + bx + c = 0$ then

$\alpha + r\alpha = -b/a$

$\alpha(1 + r) = -b/a$ ....(1)

and $\alpha . r\alpha = c/a$

$\alpha^2 r = c/a$ ....(2)

Squaring (1), we get

$\alpha^2(1 + r)^2 = b^2/a^2$

Dividing by (2) we get

$$\frac{\alpha^2(1+r)^2}{\alpha^2 r} = \frac{b^2/a^2}{c/a}$$

$$\frac{1+2r+r^2}{r} = \frac{b^2}{ac}$$

$$\frac{1+r^2}{r} + 2 = \frac{b^2}{ac}$$

$\Rightarrow (r^2 + 1)/r = (b^2/ac) - 2$

$$\Rightarrow \frac{b^2 - 2ac}{ac}$$

Example 5 : If the difference of the roots of equation $x^2 - px + q = 0$ be 1 then show that $p^2 + 4q^2 = (1 + 2q)^2$

Solution : Let α be the roots of the equation $x^2 - px + q = 0$

Then other roots will be $(\alpha + 1)$

∴ $\alpha + (\alpha + 1) = p$ and $\alpha(\alpha + 1) = q$

$2\alpha + 1 = p$

$2\alpha = p - 1$

$$\Rightarrow \alpha = \frac{p-1}{2}$$

$$\frac{(p-1)}{2}\left(\frac{p-1}{2}+1\right) = q$$

$(p - 1)(p + 1) = 4q$

$p^2 - 1 = 4q$

$p^2 = 4q + 1$

Now L.H.S. = $p^2 + 4q^2$

$= 4q + 1 + 4q^2$

$= (2q)^2 + 2. 2q + 1$

$= (2q + 1)^2$

= R.H.S.

Example 6 : If one root of the equation $x^2 + px + q = 0$ and $x^2 + qx + p = 0$ be common then prove that $p - q = 0$ or $p + q + 1 = 0$

Solution : Let α be the common root of given equ.. Therefore

$x^2 + p\alpha + q = 0$

we get

$\alpha^2 + p\alpha + q = 0$ ....(1)

Similarly from second equation,

$\alpha^2 + q\alpha + p = 0$ ....(2)

Solving (1) and (2),

$$\frac{\alpha^2}{p^2 - q^2} = \frac{\alpha}{q-p} = \frac{1}{q-p}$$

$$\frac{\alpha^2}{p^2 - q^2} = \frac{1}{q-p} \Rightarrow \alpha = 1$$

$$\frac{\alpha^2}{p^2 - q^2} = \frac{1}{q-p}$$

$\alpha^2(q - p) = p^2 - q^2$

$(q - p)\alpha^2 + q^2 - p^2 = 0$

$(q - p)(\alpha^2 + p + q) = 0$
$q - p = 0$ or $\alpha^2 + p + q = 0$
$p = q$ or $p + q + 1 = 0$
because $\alpha = 1$

Excrcise 2.1

1 Find the nature of the following equation

(i) $x^2 - 3x + 7 = 0$

(ii) $2y^2 - 11y + 9 = 0$

(iii) $3y^2 - 2y\sqrt{6} + 2 = 0$

2 If α, β be the roots of the equation $ax^2 + bx + c$

(i) $\alpha^2 + \beta^2$ (ii) $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$

(iii) $\alpha^2\beta + \alpha\beta^2$

(iv) $\alpha^2 - \beta^2$ (v) $\alpha^3 - \beta^3$

(vi) $\frac{\alpha}{\beta^2} + \beta/\alpha^2$

3. If the roots of the equation $4x^2 + 15x + m = 0$ then find the values of m (M.P. 87)

4 If the roots of the equation $(a^2 + b^2)x^2 - 2(ap + bq)x + p^2 + q^2 = 0$ are equal then show that $\frac{a}{b} = \frac{p}{a}$

5 If one of the root of the equation $ax^2 + 10x + 5 = 0$ be three times of other root then find the value of a (M.P. 89)

6. For which values of m the roots of the equation $x^2 - 2x(1 + 3m) + 7(3 + 2m) = 0$ are equal (AICBSE 85)

7. If the difference of the roots of the equation $x^2 - px + q = 0$ and $x^2 - qx + p = 0$ are equal then show that $p + q + 4 = 0$ or $p = q$

8. Find the condition when the roots of the equation $ax^2 + bx + c = 0$ are square the other root (M.P. 88, 91)

9. If the ratio of the roots of the equation $lx^2 + mx + n = o$ be p : q then show that

$$\sqrt{p/q} + \sqrt{q/p} + \sqrt{n/l} = 0$$

10. If the ratio of the roots of the equation $x^2 + px + q = 0$ be the same as the root of equation $x^2 + p_1x + q_1 = 0$ then show that $p^2 q_1 = p_1^2 . q$ (M.P. 94)

11. If α, β be the roots of the equation $ax^2 + bx + c = 0$ and $\alpha + \delta, \beta + \delta$ be the roots of the equation $Ax^2 + Bx + C = 0$. Then show that $\frac{b^2 - 4ac}{B^2 - AC} = (a/A)^2$ (M.P. 93)

12. If the roots of the equation $(1 + m^2)x^2 + 2mcx + c^2 - a^2 = 0$ be equal then show that $c = a\sqrt{1+m^2}$ (M.P. 85)

13. If the roots of equation $ax^2 + bx + q = 0$ be in the ratio m : n show that

$$\sqrt{m/n} + \sqrt{n/m} = \sqrt{\frac{p^2}{q}}$$

14. If α and β be the roots of the equation $px^2 + qx + r = 0$ then find the equation whose roots are $\frac{\beta}{\alpha}$ and $\frac{\beta}{\alpha}$ (M.P. 90, 94)

15. If α,β be the roots of the equation $3x^2 - 4x + 1 = 0$ then show that $9x^2 - 28x + 3 = 0$ will be an equation whose roots are $\frac{\alpha^2}{\beta}$ and $\frac{\beta^2}{\alpha}$

16. Find the equation whose roots are $3 + \sqrt{5}$ and $3 - \sqrt{5}$ (M.P. 87)

17. If α, β be the roots of the equation $2x^2 - 5x - 7 = 0$ then find the equation whose roots are $2\alpha + 3\beta$ and $3\alpha + 2\beta$

18. If α, β be the roots of the equation $x^2 - (1 + n^2)x + \frac{1}{2}(1 + n^2 + n^4) = 0$ then show that $\alpha^2 + \beta^2 = n^2$ (MP 89)

19. If α,β be the roots of the equation $x^2 - px + q = 0$ then find the value of $\alpha^2(\alpha^2/\beta + \beta) + \beta^2(\frac{\beta^2}{\alpha} + \alpha)$

20. If one of the roots of the equation $ax^2 + bx + c = 0$ and $cx^2 + bx + a = 0$ be common then show that $a + b + c = 0$ or $a - b + c = 0$

21. If one of the roots of the equations $x^2 + ax + bc = 0$ and $x^2 + bx + ac = 0$ be common then show that the rest of the roots of these equations satisfy $x^2 + cx + ab = 0$.

Answer

1. (i) Imaginary and Unequal
   (ii) Rational and Unequal
   (iii) Real and equal
2. (i) $(b^2 - 2ac) / a^2$,
   (ii) $(b^2 - 2ac) / c^2$,
   (iii) $-bc/a^2$, (iv) $(-b\sqrt{b^2-4ac}) / a^2$,
   (v) b^2, (vi) $(3abc - b^2) / ac^2$,
3. $\frac{225}{16}$ 5. $\frac{15}{4}$
6. $2, -10/19$,
8. $b^3 + ca^2 + ac^2 = 3abc$
14. $prx^2 + (2rp - q^2)\, x + rp = 0$
16. $x^2 - 6x + 4 = 0$
17. $2x^2 - 25x + 82 = 0$
19. $\frac{p(p^2-2q)\,(b^2-3q)}{q}$

2.8 Quadratic Inequations:

The Quadratic in-equation is one of the following expression.

$ax^2 + bx + c \leq 0$ or $ax^2 + bx + c \geq 0$ where a, b, c are real numbers and $a \neq 0$

The values of x, which satisfy the above given inequation, are called the solution of the inequation.

Example 7 : Find the solution of the following inequation and represent them on the line

(i) $2 - 3x - 2x^2 \geq 0$

(ii) $15x^2 + 4x - 4 \leq 0$

Solution :

(i) $2 - 3x - 2x^2 \geq 0$

$2 - 4x + x - 2x^2 \geq 0$

$2(1 - 2x) + x(1 - 2x) \geq 0$

$(1 - 2x)(2 + x) \geq 0$

$(1 - 2x) \geq 0$ or $(2 + x) \geq 0$

or $1 \leq 2x$ and $x \leq -2$

$x \geq 1/2$ and $x \leq -2$

This type of solution is not possible therefore the required solution will be $\{x : -2 \leq x \leq 1/2\}$

The above solution can be given on the following line

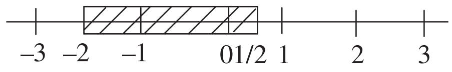

Fig- 2.1

The solution set will the set of all those – 2 and $\frac{1}{2}$. The end points –2 and $\frac{1}{2}$ are also included.

(ii) $15x^2 + 4x - 4 \leq 0$

$15x^2 + 10x - 6x - 4 \leq 0$

$5x(3x + 2) - 2(3x + 2) \leq 0$

$(3x + 2)(5x - 2) \leq 0$

$3x + 2 \leq 0$ or $5x - 2 \leq 0$

$x \leq -2/3$ and $x \geq 2/5$

But such type of solution is not possible. The required solution will be

$\{x : -2/3 \leq x \leq 2/5\}$

The above solution can be given on the following line.

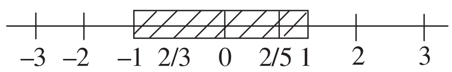

Fig. 2.2

The solution will lie between $\frac{-2}{3}$ and $\frac{2}{5}$, The end points $\frac{-2}{3}$ and $\frac{2}{5}$ are also included in the solution set.

Exercise 2.2

1. Find the solution of the following inequation and represent them on the line :
   (i) $x^2 + 7x + 10 \leq 0$
   (ii) $2y^2 + 5y - 3 \geq 0$
2. Solve the equation $\frac{12x}{4x^2+9} \leq 1$

Answer

1. (i) $\{x : -5 \leq x \leq -2\}$
   (ii) $\{y : -3 \leq y \leq 1/2\}$
2 All real values of x $\{x : \geq 0)\}$

Subjective Questions :

1. If one root of the equation $x^2 + px + q = 0$ is the twice of the second root, then prove that $2p^2 = 9q$.
M.P. 93, 2006B

2. Solve the quadratic equation $2x^2 - 13x + 15 = 0$ by vedic method. **M.P. 2010**

3. If roots of the equation $ax^2 + bx + c = 0$ are in the ratio $r : 1$ then Prove that $\frac{r^2+1}{r} = \frac{b^2-2ac}{ac}$.
MP 2014, M.P. 97, 2012

4. If one root of the equation $ax^2 + 10x + 5 = 0$ is three times of the second root, then find the value of a. **MP 2014, M.P. 1998**

5. if α, β are the roots of the equation $ax^2 - bx + b = 0$ then prove that $\sqrt{\frac{\alpha}{\beta}} + \sqrt{\frac{\beta}{\alpha}} = \sqrt{\frac{b}{a}}$.
M.P. 2006 B,C

6. If α and β are the roots of the equation $3x^2 - 6x + 4 = 0$ then find the value of $2\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) + 3\alpha\beta$.
MP 2013, M.P. 2006 C

7. Find the value of a if the roots of the equation $x^2 - (3a - 1)x + 2a^2 + 2a - 11 = 0$ are equal.
M.P. 2011

8. If the sum of the roots of the equation $ax^2 + bx + c = 0$ is equal to sum of the square of their reciprocals, then show that.
$2ca^2 = ab^2 + bc^2$ **M.P. 2009**

9. If the roots of the equation $ax^2 + cx + c = 0$ are in the ratio $p : q$ then prove that
$\sqrt{\frac{p}{q}} + \sqrt{\frac{q}{p}} + \sqrt{\frac{c}{a}} = 0$ **M.P. 1997**

10. If one root of the equations $ax^2 + bx + c = 0$. *and cx2 + bx + a = 0 is common, then prove that*
$a + b + c = 0$ or $a - b + c = 0$ **M.P. 1997**