

## Solutions to the problems in Circuit Theory

1. We have the circuit on the right, with a driving voltage  $U_S = 5\text{ V}$ , and we want to know  $U$  and  $I$ .

a.  $R = 1000\ \Omega$ ;

the total resistance in the circuit is then

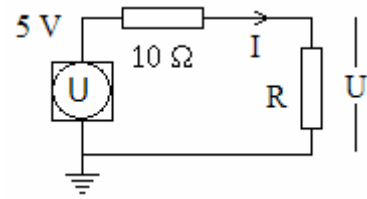
$R_{\text{tot}} = 1010\ \Omega$ , and we can use Ohm's law to find

$$I = U_S/R_{\text{tot}} = 5/1010\text{ A} = 4.95\text{ mA}$$

and  $U = RI = 4.95\text{ V}$ .

b.  $R = 15\ \Omega$ ; repeating the calculation we see that  $R_{\text{tot}} = 25\ \Omega$ ,  $I = U_S/R_{\text{tot}} = 5/25\text{ A} = 200\text{ mA}$  and  $U = RI = 3.0\text{ V}$ .

c.  $R = 0.1\ \Omega$ ; this time  $R_{\text{tot}} = 10.1\ \Omega$ ,  $I = U_S/R_{\text{tot}} = 5/10.1\text{ A} = 495\text{ mA}$  and  $U = RI = 49.5\text{ mV}$ .



2. In the circuit above,  $R_{\text{tot}} = (10 + R)\ \Omega$ . The current  $I = U_S/R_{\text{tot}}$  and  $U = RI = R U_S/R_{\text{tot}}$ . The power dissipated in the resistor is then

$$P = UI = R(U_S/R_{\text{tot}})^2 = RU_S^2/(10+R)^2.$$

To find the maximum in  $P$  we take the derivative with respect to  $R$ , which is

$$dP/dR = U_S^2[(R+10)^2 - 2R(10+R)]/(10+R)^4 = U_S^2(R+10-2R)/(10+R)^3 = U_S^2(10 - R)/(10+R)^3;$$

This function has an extreme value when  $10 - R = 0$ , or  $R = 10$ ,

and since  $P = 0$  at  $R = 0$  and  $P \rightarrow 0$  also as  $R \rightarrow \infty$ , the power must be a maximum when  $R = 10\ \Omega$ .

This is a completely general result:

***A two-pole circuit such as this one (a Thevenin equivalent circuit) delivers maximum output power when the load resistor connected equals the internal resistance.***

3. We have a circuit as in the figure right, with a source current  $I = 0.5\text{ A}$ . From Kirchhoff's second law,

$$I = I_x + I_y.$$

a.  $R = 1000\ \Omega$ . The parallel combination of resistances has an effective value  $R_x$ , which is obtained as

$$1/R_x = 1/10 + 1/R \text{ or } R_x = 10R/(10+R);$$

in this case,  $R_x = 10000/1010\ \Omega = 9.901\ \Omega$ .

In general,  $U = IR_x = 10 IR/(10+R)$ , and we see that

$I_x = U/R = 10 I/(10+R)$  and  $I_y = U/10 = IR/(10+R)$  ("current division formulas"). We thus find

$$U = 10 IR/(10+R) = 4.95\text{ V},$$

$$I_x = 10 I/(10+R) = 4.95\text{ mA}, \text{ and}$$

$$I_y = IR/(10+R) = 495\text{ mA}.$$

b.  $R = 15\ \Omega$ . We repeat the calculation using the same formulae to find

$$U = 10 IR/(10+R) = 3\text{ V},$$

$$I_x = 10 I/(10+R) = 200\text{ mA}, \text{ and}$$

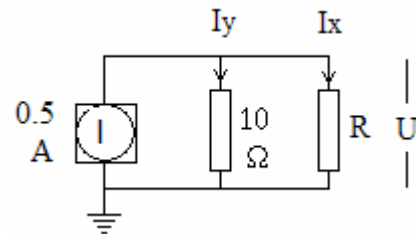
$$I_y = IR/(10+R) = 300\text{ mA}.$$

c. Finally,  $R = 0.1\ \Omega$ . The same calculation gives

$$U = 10 IR/(10+R) = 49.5\text{ mV},$$

$$I_x = 10 I/(10+R) = 495\text{ mA}, \text{ and}$$

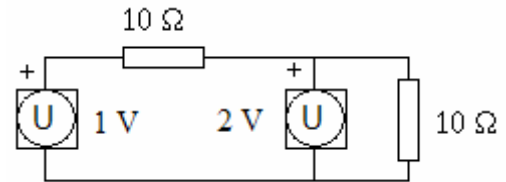
$$I_y = IR/(10+R) = 4.95\text{ mA}.$$



4.

We want to know the currents through the two resistors in the figure on the right.

Of course, we use the simplest possible solution!



We see immediately that the voltage over the resistor on the right must *always* be 2 V (parallel to an ideal voltage source), so by **Ohm's law** the current through this resistor must be  $I_r = U/R = 2/10 \text{ A} = 0.2 \text{ A}$ .

We can use the same method for the resistor on the left, but let us be a little more sophisticated and assume that the current through this resistor is  $I_L$ .

The voltage on the right voltage source is higher, so we assume  $I_L$  flows through the resistor from right to left.

We then use **Kirchhoff's voltage law**. If we circulate from the bottom left corner clockwise through the left loop, we sum up the voltages as

$$+1 + 10 I_L - 2 = 0, \quad \text{or} \quad 10 I_L - 1 = 0; \quad I_L = 0.1 \text{ A}.$$

**Answer:** The currents are  $I_L = 0.1 \text{ A}$  and  $I_R = 0.2 \text{ A}$ .

5. We want to know the voltage at point A (relative to ground). Let us use the superposition principle, starting with the *left* voltage source!

We can re-draw the circuit as in the figure below, right, where we have short-circuited the right voltage source and moved the right resistor down a bit.

From point A to ground we now have a total resistance  $R'$  equal to  $10 \Omega$  in parallel to  $10 \Omega$ , i.e.  $1/R' = (1/10) + (1/10) = (2/10)$ ,  $R' = 5 \Omega$ .

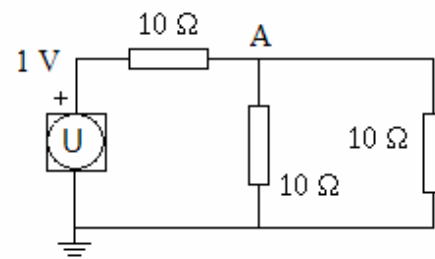
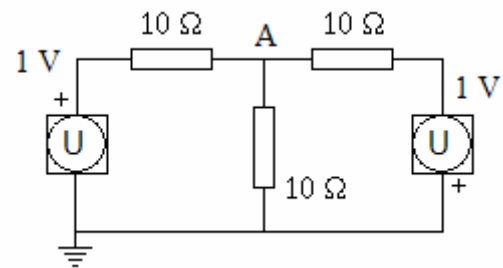
From the voltage divider theorem we get  $U_A(1) = U [5/(5+10)] \text{ V} = U/3 = (1/3) \text{ V}$ .

We then repeat the calculation for the *right* source,

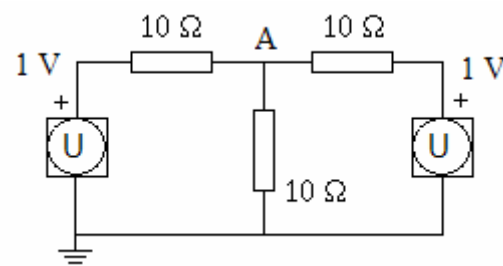
deleting instead the left one. From the symmetry of the circuit we see immediately that we would get the same magnitude for the second contribution, i.e.  $|U_A(2)| = (1/3) \text{ V}$ , but since the polarity of the two sources are opposite  $U_A(2) = - (1/3) \text{ V}$ .

Finally, we add up the two contributions to find  $U_A = 0 \text{ V}$ .

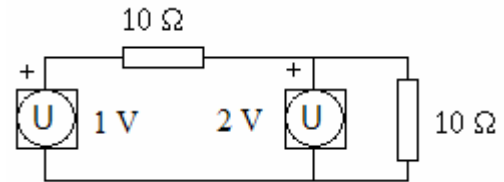
(We could use instead Kirchhoff's laws in the two loops to find the same result.)



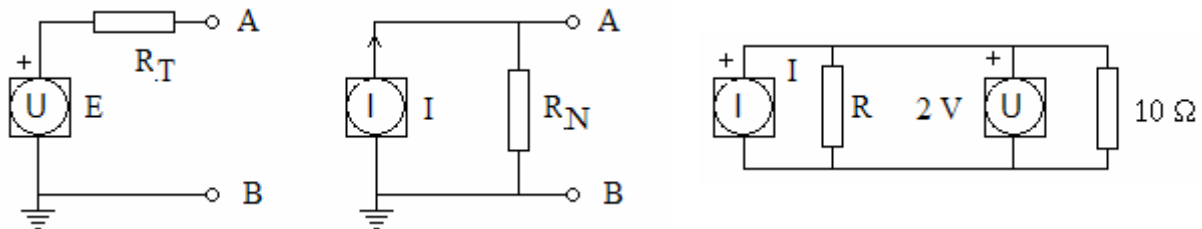
6. We have here essentially the same problem as in 5), except that the polarity of the right-hand source has been reversed. The solution must also be identical, except that now  $U_A(1) = U_A(2) = 1/3 \text{ V}$ , such that the total  $U_A = 2/3 \text{ V}$ .



7. We have again the circuit used in problem 4):  
 Now we want to replace the left voltage source plus the left-hand resistor with a current source in parallel with a resistor, in such a way that nothing is changed in the properties of the circuit. This problem is equivalent to replacing a Thévenin equivalent circuit with a Norton one!



The left figure below shows the two equivalent (Thévenin and Norton) circuits, and we draw the new desired circuit on the right:

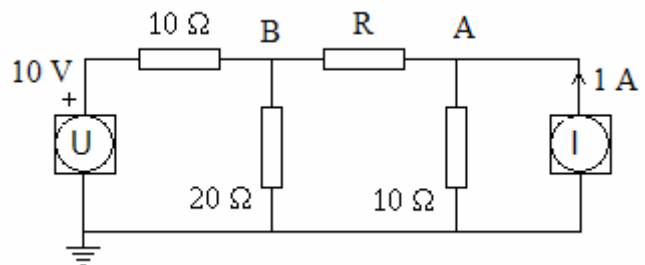


Referring to the left figure above, the measured resistances between A and B, when the voltage source is short-circuited and the current source is open, are simply  $R_T$  and  $R_N$ , and equivalence implies  $R_T = R_N$ . Also, the output voltages between A and B are  $U_{AB} = E$  and  $U_{AB} = IR_N$ , respectively, and thus  $E = IR_N$ .

The solution for the new circuit in the right-hand figure above is thus  
 $R = 10 \Omega$  and  
 $I = (E/R_T) = 1/10 \text{ A} = 0.1 \text{ A}$ .

8. We want to know the voltages  $U_A$  and  $U_B$  at points A and B, respectively, in the figure below; the resistor  $R = 10 \Omega$ .

We use the superposition theorem, since we have both a current source and a voltage source. **Removing the current source** leaves us with four resistors in an array. The total resistance seen by the voltage source is  $10 \Omega$  in series with a parallel combination of  $20 \Omega$  and  $(10+10) \Omega$ .



The total resistance is easily seen to be  $20 \Omega$ . There is thus a current of  $0.5 \text{ A}$  driven by the voltage source; this divides equally between the two  $20\text{-}\Omega$  branches, the current from B to A must be  $0.25 \text{ A}$ , and the voltages  $U_A(1) = 0.25 \cdot 10 \text{ V} = 2.5 \text{ V}$ ,  $U_{BA}(1) = U_B - U_A = 0.25 \cdot 10 \text{ V} = 2.5 \text{ V}$ , and thus  $U_B(1) = 5 \text{ V}$ .

**Removing (shorting!) the voltage source**, we have a more complicated situation. The total resistance seen by the current source is  $10 \Omega$  in parallel with a three-resistor network. Working from the left, we have  $10 \Omega$  in parallel with  $20 \Omega$ , giving  $R_{\text{eff}}(1) = 10 \cdot 20 / (10 + 20) \Omega = 20/3 \Omega$ . This acts in series with  $R$ , giving a total of  $R_{\text{eff}}(2) = (10 + 20/3) \Omega = 50/3 \Omega$ .

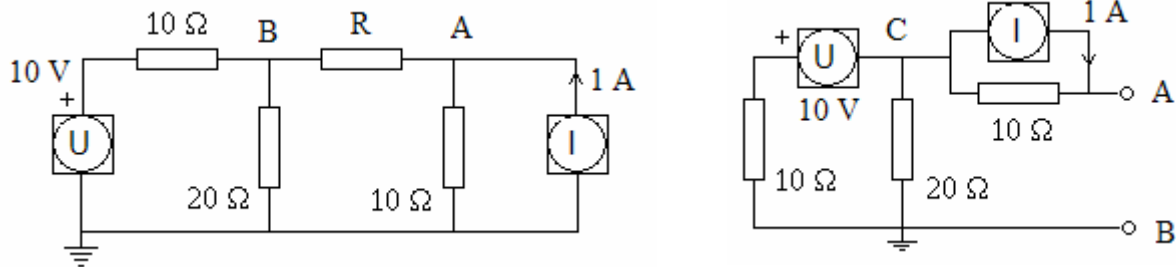
The current from the current source divides between the  $10 \Omega$  resistor and the series combination according to the current divider rule, and thus the current through  $R$  must be  $I_R = 1 \cdot 10 / (10 + R_{\text{eff}}(2)) \text{ A} = 10 / (10 + 50/3) \text{ A} = 10 \cdot 3 / 80 \text{ A} = 0.375 \text{ A}$ , resulting in a voltage  $U_{BA} = -10 \cdot 0.375 \text{ V} = -3.75 \text{ V}$  (the current is opposite to the first one!). The current through the  $10 \Omega$  resistor is then  $0.625 \text{ A}$ , giving  $U_A(2) = 6.25 \text{ V}$ , and we find  $U_B(2) = (6.25 - 3.75) \text{ V} = 2.5 \text{ V}$ .

**The total voltages are then  $U_A = 8.75 \text{ V}$  and  $U_B = 7.5 \text{ V}$ .**

9. We have the same circuit as in problem 8), except that we move the ground point to point B.  
 a. We want to analyse the effect of changing R, so for simplicity we want to replace the rest of the circuit with a Thévenin equivalent.

We can start by re-drawing the circuit as specified above. (This is actually not necessary!) The left figure below is the original circuit, while the right-hand side shows the similarly complicated result after re-drawing, *omitting R*.

*(Trace the circuits and convince yourself that they are actually equivalent!)*



The Thévenin equivalent internal resistance  $R_T$  is the total resistance between A and B after shorting the voltage source and opening the current source. This must be the sum of the  $10\ \Omega$  resistor originally in parallel with the current source and the parallel combination of 10 and  $20\ \Omega$  on the left, i.e.

$$R_T = (10 + 10 \cdot 20 / (10 + 20))\ \Omega = (10 + 20/3)\ \Omega = 16.67\ \Omega.$$

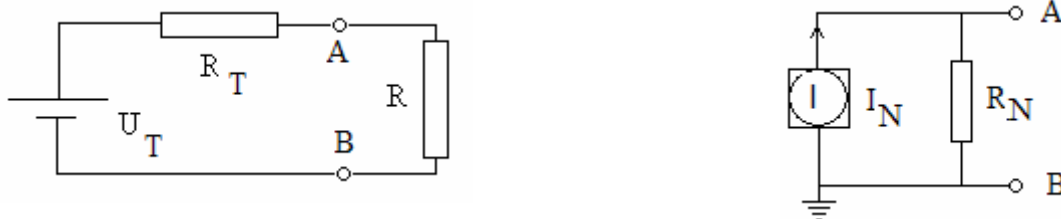
The Thevenin voltage  $U_T$  can also easily be calculated, basically by a superposition method:

When no resistance R is connected, the current  $I = 1\ \text{A}$  flows only through the  $10\ \Omega$  resistor in parallel with the source, giving rise to a voltage drop of 10 V with point A positive relative to the point C defined in the figure.

The voltage source is also part of a local circuit where it gives rise to a voltage drop over the  $20\ \Omega$  resistor. We can calculate this voltage using the voltage divider theorem as  $U_{BC} = 10 \cdot 20 / (10 + 20)\ \text{V} = 20/3\ \text{V} \approx 6.67\ \text{V}$ , with point B being positive relative to point C.

We have defined point B as ground; using the information above we find that  $U_C = -6.67\ \text{V}$  and thus the open circuit voltage  $U_A = (10 - 6.67) = 3.33\ \text{V} = U_T$ .

We can thus draw an equivalent circuit (left below), with  $U_T = 3.33\ \text{V}$  and  $R_T = 16.67\ \Omega$ .



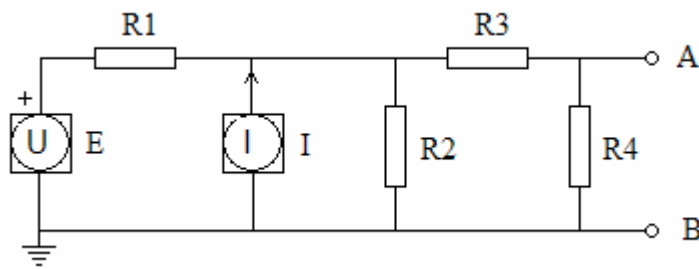
- b. We want to replace the circuit with a Norton model instead.

The necessary conversion equations were derived in Problem 7 (see above), so we give immediately the results:

$I_N = U_T / R_T = (10/3) / (50/3)\ \text{A} = 0.2\ \text{A}$ ,  $R_N = R_T = 16.67\ \Omega$ . The model is shown on the right above.

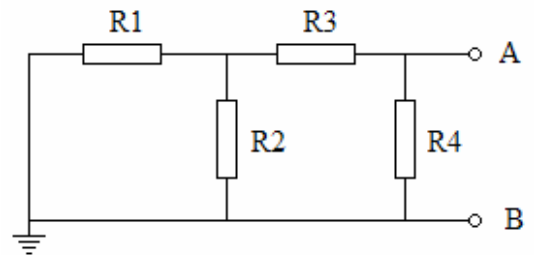
- c. What value for the external resistor R would dissipate the largest power? This was solved in Problem 2), where we found that this occurred for  $R = R_T = R_N = 16.67\ \Omega$ .

10. a. We want to replace the circuit below with an equivalent Thevenin two-pole circuit, calculated between A and B:



$$\begin{aligned} E &= 5 \text{ V} \\ I &= 2 \text{ A} \\ R_1 &= 40 \ \Omega \\ R_2 &= 60 \ \Omega \\ R_3 &= 6 \ \Omega \\ R_4 &= 20 \ \Omega \end{aligned}$$

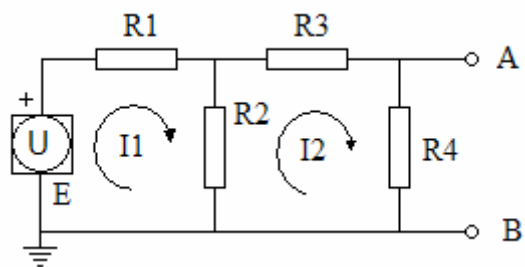
We start by short-circuiting all voltage sources and opening all current sources, to find the equivalent resistance, seen between A and B. We then get the circuit on the right-hand side.



$R_1 // R_2$  is then  $R'$ ,  $1/R' = 1/R_1 + 1/R_2$ ; numerically  $1/R' = 1/40 + 1/60 = 5/120 = 1/24$ , and  $R' = 24 \ \Omega$ .

The total Thévenin equivalent resistance  $R_T$  as seen between A and B is then  $R_4$  in parallel with the series combination of  $R_3$  and  $R'$ , such that  $1/R_T = (1/R_4) + [1/(R_3+R')]$ , or numerically  $1/R_T = (1/20) + [1/(6 + 24)] = 1/20 + 1/30 = 5/60 = 1/12$ , such that  $R_T = 12 \ \Omega$ .

Because we have both a voltage source and a current source, we use the superposition principle to find the Thévenin equivalent voltage  $U_T$  between A and B. We start by removing the current source (fig  $\rightarrow$ ) and introducing two circulating currents,  $I_1$  and  $I_2$ , fulfilling Kirchhoff's current law.



Summing the voltages in the left-hand loop,

$$U - I_1 (R_1+R_2) + I_2 R_2 = 0, \quad (1)$$

and in the right-hand loop

$$I_1 R_2 - I_2 (R_2+R_3+R_4) = 0. \quad (2)$$

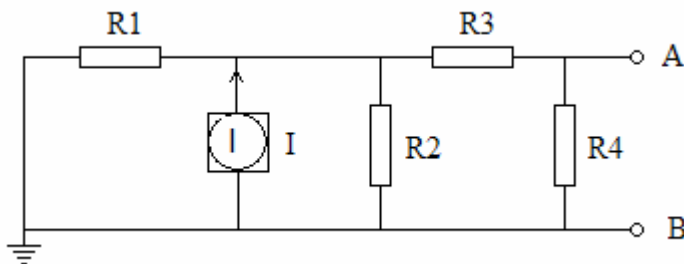
(2) gives  $I_1 = I_2 (R_2+R_3+R_4)/R_2$ , numerically  $I_1 = I_2 (60+6+20)/60 = (86/60) I_2$ ;

Inserting into (1) gives

$$U = (86/60)I_2(R_1+R_2) - I_2 R_2, \text{ or } U = I_2 [(86/60)(R_1+R_2) - R_2]; \text{ numerically, we get}$$

$$I_2 = 5 / [(86/60)(100) - 60] = 5 / (500/6) \text{ A} = 3/50 \text{ A}.$$

The contribution from the voltage source to the output voltage is then  $U_T(1) = I_2 R_4 \text{ V} = 1.2 \text{ V}$ .



We now remove (short) the voltage source. We see that the current source "sees" three parallel resistors:

$R_1$ ,  $R_2$ , and the series combination  $R' = R_3+R_4 = 26 \ \Omega$ .

The total parallel resistance is then

$$\begin{aligned} 1/R &= 1/R_1 + 1/R_2 + 1/R' = \\ &= 1/40 + 1/60 + 1/26 = 25/312, \end{aligned}$$

and the voltage over the current source and over each parallel resistor "leg" is  $U = IR = 624/25 \text{ V}$ .

In particular, this is the voltage over  $(R_3+R_4)$ ; the voltage over  $R_4$  is  $U_T(2) = U R_4 / (R_3+R_4) = (20/26)(624/25) \text{ V} = (4/5)24 \text{ V} = 19.2 \text{ V}$ .

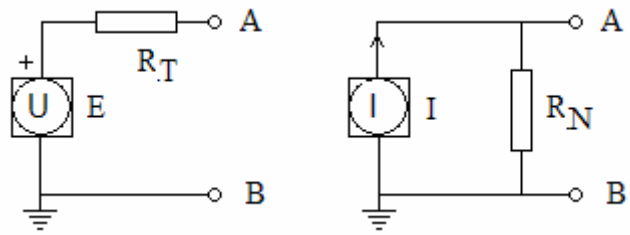
We find the total Thévenin output voltage as  $U_T = U_T(1) + U_T(2) = 20.4 \text{ V}$ .

b. In this problem we want to replace the same circuit with an equivalent Norton two-pole circuit. We can either make the whole calculation again, or use the equivalence of the Norton and Thévenin circuits:

The output voltages from the two circuits must be equal, that is

$$U_{AB} = E = U_T = I_N R_N,$$

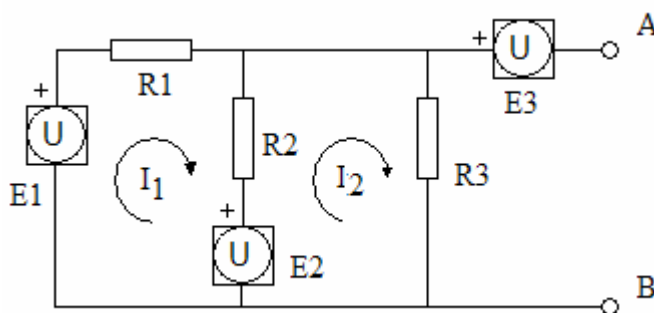
and, obviously,  $R_N = R_T$ .



Using these equations we see immediately that  $R_N = R_T = 12 \Omega$ ,

and that the equivalent Norton current  $I_N = U_T/R_T = 1.7 \text{ A}$

11. a. We want to replace the circuit below with an equivalent Thévenin two-pole circuit.



$$\begin{aligned} E1 &= 12 \text{ V} \\ E2 &= 10 \text{ V} \\ E3 &= 6 \text{ V} \\ R1 &= 80 \Omega \\ R2 &= 40 \Omega \\ R3 &= 80 \Omega \end{aligned}$$

Let us introduce circulating currents  $I_1$  and  $I_2$  as defined in the figure! (These currents flow in closed loops and thus automatically satisfy Kirchhoff's second law.)

Walking around the left-hand loop we find

$$E_1 - I_1 R_1 - I_1 R_2 + I_2 R_2 - E_2 = 0 \quad (1)$$

and from the other loop we get

$$E_2 + I_1 R_2 - I_2 R_2 - I_2 R_3 = 0. \quad (2)$$

Eq. (2) can be re-written to give  $I_2 = (E_2 + I_1 R_2)/(R_2 + R_3)$ , which inserted in (1) gives

$$E_1 - I_1(R_1 + R_2) + I_2 R_2 - E_2 = E_1 - I_1(R_1 + R_2) + (E_2 + I_1 R_2) R_2 / (R_2 + R_3) - E_2;$$

$$E_1 - E_2 + E_2 R_2 / (R_2 + R_3) = E_1 - E_2 R_3 / (R_2 + R_3) = I_1 [R_1 + R_2 - R_2^2 / (R_2 + R_3)];$$

Numerically,

$$12 - 10 \cdot 80 / 120 = I_1 (80 + 40 - 40^2 / 120); \quad 12 - 20 / 3 = I_1 (120 - 40 / 3), \text{ or } 16 / 3 = I_1 \cdot 320 / 3;$$

we find  $I_1 = 16 / 320 \text{ A} = 0.050 \text{ A}$

which inserted into the equation for  $I_2$  gives  $I_2 = (10 + 0.05 \cdot 40) / (40 + 80) = 0.10 \text{ A}$ .

The output voltage  $U_{AB}$ , which equals the Thévenin voltage  $U_T$ , is then

$$U_{AB} = U_T = I_2 R_3 - E_3 = (0.1 \cdot 80 - 6) \text{ V} = 2 \text{ V}.$$

The total resistance  $R_T$  is simply all three resistances in parallel;

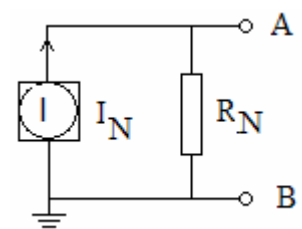
$$R_T = R_1 R_2 R_3 / (R_1 R_2 + R_2 R_3 + R_1 R_3) = 20 \Omega.$$

b. To replace this by a Norton circuit, we use again the simple conversion formulas to find

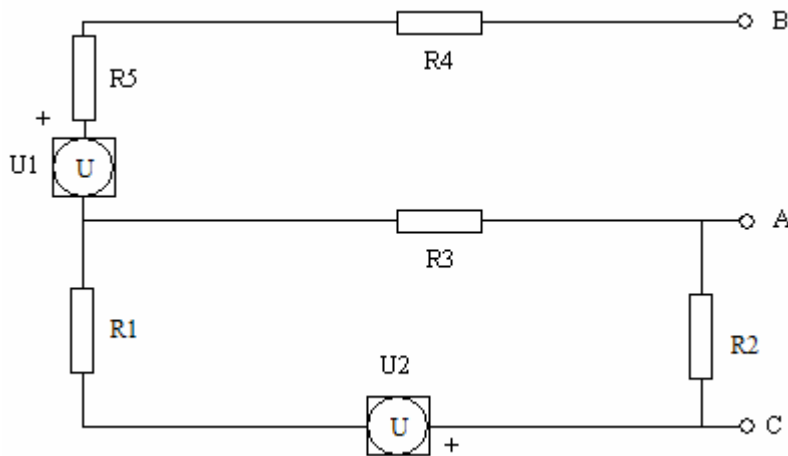
$$I_N = U_T / R_T = 2 / 20 \text{ A} = 0.1 \text{ A}, \quad \text{and } R_N = R_T = 20 \Omega.$$

c. From the Norton equivalent circuit shown on the right we immediately realise that short-circuiting the output connections will give an output current equal to the output of the current source, i.e.

$$I_{\text{short}} = I_N = 0.1 \text{ A}.$$



12.



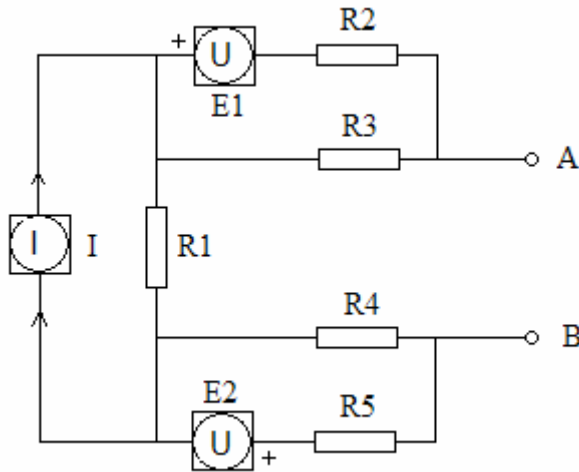
- $U_1 = 2 \text{ V}$
- $U_2 = 21 \text{ V}$
- $R_1 = 100 \ \Omega$
- $R_2 = 100 \ \Omega$
- $R_3 = 10 \ \Omega$
- $R_4 = 10 \ \Omega$
- $R_5 = 100 \ \Omega$

We seek the potential differences between points A, B and C in the figure above.

We see immediately that current can only circulate in the closed loop in the lower part of the figure. Let us assume that this current is  $I$ , and that it circulates in the counter-clockwise direction, i.e. flowing from point C to point A (right-hand side). Using Ohm's law we find that  $U_2 = I (R_1 + R_2 + R_3)$ , or  $I = U_2 / (R_1 + R_2 + R_3) = 21 / 210 \text{ A} = 0.1 \text{ A}$ .

- a. The voltage between points A and B is then  $U_{AB} = -I R_3 + U_1 = (-0.1 \cdot 10 + 2) \text{ V} = \mathbf{1 \text{ V}}$ .  
(Point B more *positive*.)
- b. The voltage between points A and C is  $U_{AC} = -I R_2 = -0.1 \cdot 100 \text{ V} = \mathbf{10 \text{ V}}$ .  
(Point C more *positive*.)

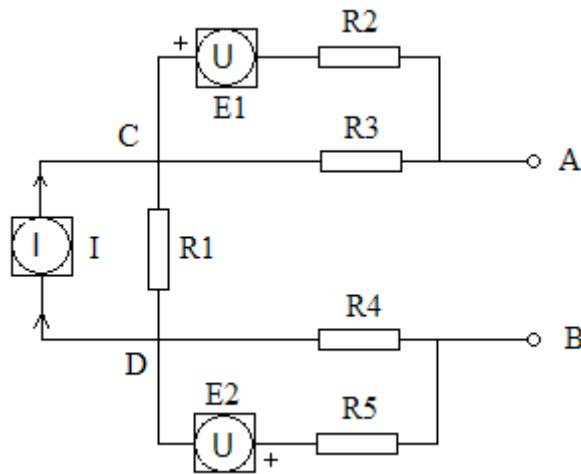
13. a. We want to replace the circuit below with a simpler Thévenin equivalent circuit.



- $I = 1 \text{ A}$
- $E_1 = 6 \text{ V}$
- $E_2 = 12 \text{ V}$
- $R_1 = 8 \ \Omega$
- $R_2 = 6 \ \Omega$
- $R_3 = 12 \ \Omega$
- $R_4 = 8 \ \Omega$
- $R_5 = 8 \ \Omega$

Although this circuit looks complicated, it is actually quite simple as is obvious when it is re-drawn as shown below. In that figure it is obvious that the circuit consists of three completely independent circuits, and as long as the output connections are "open" there is no interaction between the three parts.

1. Voltage source  $E_1$  creates a voltage drop over  $R_3$  given by  $U_1 = E_1 R_3 / (R_2 + R_3) = 6 \cdot 12 / 18 \text{ V} = 4 \text{ V}$ . Point C is positive relative to point A.
2. The current source creates a voltage drop over  $R_1$  given by  $U_2 = I R_1 = 1 \cdot 8 \text{ V} = 8 \text{ V}$ . Point C is positive relative to point D.
3. Voltage source  $E_2$  creates a voltage drop over  $R_4$  given by  $U_3 = E_2 R_4 / (R_4 + R_5) = 12 \cdot 8 / 16 \text{ V} = 6 \text{ V}$ . Point B is positive relative to point D.



$$\begin{aligned}
 I &= 1 \text{ A} \\
 E1 &= 6 \text{ V} \\
 E2 &= 12 \text{ V} \\
 R1 &= 8 \ \Omega \\
 R2 &= 6 \ \Omega \\
 R3 &= 12 \ \Omega \\
 R4 &= 8 \ \Omega \\
 R5 &= 8 \ \Omega
 \end{aligned}$$

We sum up the individual contributions; assuming  $U_B = 0$  we find  $U_A = (0 - 6 + 8 - 4) \text{ V} = -2 \text{ V}$ , which we can define as the equivalent Thévenin voltage  $U_T = -2 \text{ V}$ . (B is positive relative to A.)

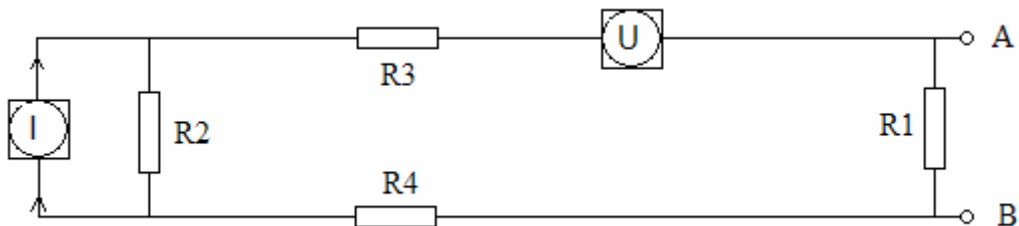
The total equivalent source resistance  $R_T$  is easily seen to be  $(R2//R3) + R1 + (R4//R5)$ , so  $R_T = R_2 R_3 / (R_2 + R_3) + R_1 + R_4 R_5 / (R_4 + R_5) = (6 \cdot 12 / 18 + 8 + 8 \cdot 8 / 16) \ \Omega = (4 + 8 + 4) \ \Omega = 16 \ \Omega$ .

b. We use again the voltage-to-current transformation;

$$R_N = R_T = 16 \ \Omega, \text{ and}$$

$$I_N = U_T / R_T = -2 / 16 \text{ A} = -125 \text{ mA}. \text{ (Current flows from A toward B.)}$$

14. The figure shows an industrial installation. A signal is sent by a current source through a double wire to an instrument on the right. An interference voltage  $U$  is coupled to the signal wires.



$$\begin{aligned}
 I &= 10 \text{ mA} & R1 &= 100 \ \Omega & R3 &= R4 = 10 \ \Omega \\
 U &= 10 \text{ V} & R2 &= 1 \text{ M} \ \Omega
 \end{aligned}$$

a. We want to know the output voltage  $U_{AB}$  between points A and B due to the current source:

The total resistance in the loop,  $R_{loop} = R1 + R3 + R4 = 120 \ \Omega$  is very much smaller than the internal resistance  $R2$  of the current source, and the current through the loop is thus

$$I_{loop} = I R2 / (R2 + R_{loop}) \approx I.$$

The desired voltage  $U_{AB} = R1 I_{loop}$ ;

The exact value is then  $U_{AB} = R1 I R2 / (R2 + R_{loop}) = 0.99988 \text{ V} \approx 1 \text{ V}$ .



b. We also want to know the output voltage  $U_{AB}$  due to the interfering voltage source  $U$ . Ohm's law tells us that the current due to this source is  $I_i = U/(R_1 + R_2 + R_3 + R_4)$ , and thus

$$U_{AB} = I_i R_1 = U R_1 / (R_1 + R_2 + R_3 + R_4) = 10 \cdot 100 / 1000120 \text{ V} = 9.9988 \cdot 10^{-4} \text{ V} \approx \mathbf{1.0 \text{ mV}}.$$

15. Let us assume that  $\sqrt{j} = a + jb$ ! We know then that  $(a + jb)^2 = j$ .

But if  $(a + jb)^2 = a^2 - b^2 + 2jab = j$ , we can identify

$$a^2 = b^2 \text{ and}$$

$$2ab = 1, \text{ or } ab = 1/2, \text{ or } b = 1/2a.$$

Then,  $a^2 = (1/2a)^2 = 1/4a^2$  and  $a = \pm 1/\sqrt{2} = b$ .

We find that  $\sqrt{j} = \pm(1/\sqrt{2})(1 + j)$ .

16. We have  $Z = (1 + 2j)/(1 - j)$ .

We first simplify  $Z$  to

$$\begin{aligned} Z &= (1 + 2j)/(1 - j) = (1 + 2j)(1 + j)/[(1 - j)(1 + j)] = (1 + 2j + j + 2j^2)/(1 - j^2) = \\ &= (1 + 3j - 2)/(1 - j^2) = (3j - 1)/(1 + 1) = (3j - 1)/2. \end{aligned}$$

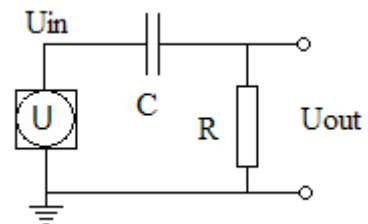
$$|Z| \text{ is then } |Z| = (1/2)(3^2 + 1)^{1/2} = 10^{1/2}/2 \approx 1.581.$$

$$\arg(Z) = \arctan(\text{Im } Z)/\text{Re}(Z) = \arctan(3/(-1)) = \arctan(-3) \approx 1.893 \text{ rad } (\approx 108.4^\circ).$$

17. a. We want to know the transfer function

$$G(j\omega) = U_{out}/U_{in}.$$

In this case we can easily apply the  $j\omega$  method, using the voltage divider formula:



$$U_{out} = U_{in} \cdot R / (R + X_C) = U_{in} \cdot R / (R + (1/j\omega C)) = U_{in} \cdot j\omega CR / (j\omega CR + 1);$$

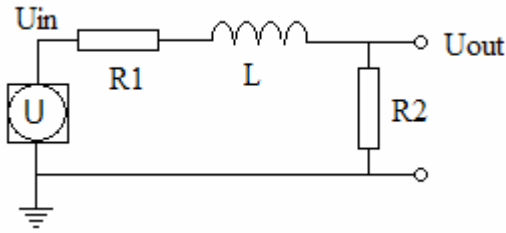
$$G(j\omega) = U_{out}/U_{in} = j\omega RC / (j\omega RC + 1).$$

b. If we exchange  $R$  and  $C$  we get instead

$$U_{out} = U_{in} \cdot X_C / (R + X_C) = U_{in} \cdot (1/j\omega C) / (R + (1/j\omega C)) = U_{in} / (j\omega CR + 1);$$

$$G(j\omega) = U_{out}/U_{in} = 1 / (j\omega RC + 1).$$

18. We want to know  $G = U_{out}/U_{in}$  as a function of frequency for the circuit below.

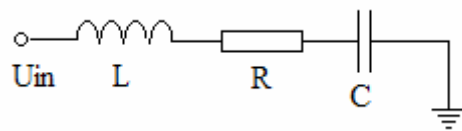


We use again the  $j\omega$  method. The output voltage can be written

$$U_{out} = U_{in} \cdot R_2 / (R_1 + R_2 + j\omega L), \text{ and thus}$$

$$G(j\omega) = U_{out}/U_{in} = R_2 / (R_1 + R_2 + j\omega L).$$

19. We want to know the properties of the L-C-R series resonance circuit shown on the left.



a. The total impedance  $Z$  can be calculated using the  $j\omega$  method:

$$Z = j\omega L + R + 1/(j\omega C) = R + j(\omega L - (1/\omega C))$$

b.  $Z$  is real at the resonance frequency  $\omega_0$ , i.e. when

$$(\omega_0 L - (1/\omega_0 C)) = 0, \text{ or } \omega_0^2 = 1/LC, \text{ which can be written } \omega_0 = (LC)^{-1/2}.$$

c. We assume that  $U_{in} = 1$  V rms, and that  $\omega = \omega_0$  (i.e.  $Z = R$ ). We can then calculate the current as  $i = U_{in}/Z = U_{in}/R$ .

The voltage drop over the inductance is  $U_L = j i \omega_0 L = j U_{in} \omega_0 L/R;$

the voltage drop over the resistance is  $U_R = i R = U_{in} R/R = U_{in};$

the voltage drop over the capacitance is  $U_C = i /j\omega_0 C = i /j\omega_0 C = -j U_{in}/\omega_0 C R.$

d. The quality factor  $Q$  is defined as  $Q = \omega_0 L/R$ . The voltage over the inductance can thus be written  $U_L = j U_{in} \omega_0 L/R = j U_{in} Q$ , and its magnitude is  $|U_L| = Q U_{in}$ .

e. At  $\omega = \omega_0$ ,  $\omega_0 L - (1/\omega_0 C) = 0$  and  $|Z| = R$ .

We now increase  $\omega$  from  $\omega_0$  to  $(\omega_0 + \Delta\omega)$ , where  $\Delta\omega = \omega_0/2Q$  and  $Q \gg 1$ . We saw in a) that the general expression for  $Z$  is  $Z = R + j(\omega L - (1/\omega C))$  and thus  $|Z|^2 = R^2 + [\omega L - (1/\omega C)]^2$ .

Here, we have  $\omega = \omega_0 + \Delta\omega = \omega_0 + (\omega_0/2Q) = \omega_0(1 + 1/2Q);$

we then have

$$\omega L - (1/\omega C) = \omega_0(1 + 1/2Q)L - 1/[\omega_0(1 + 1/2Q)C] = \omega_0 L + \omega_0 L/2Q - 1/[\omega_0 C(1 + 1/2Q)];$$

but  $Q = \omega_0 L/R$ , so  $\omega_0 L/2Q = R/2$ , and  $[1 + 1/2Q]^{-1} \approx (1 - 1/2Q) = (1 - R/2\omega_0 L)$ .

Then,

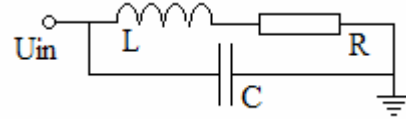
$$\omega L - (1/\omega C) = \omega_0 L + R/2 - 1/[\omega_0 C(1 + 1/2Q)] \approx \omega_0 L + R/2 - (1 - R/2\omega_0 L)/\omega_0 C = \omega_0 L + R/2 - 1/\omega_0 C + R/(2LC\omega_0^2).$$

However, we know already that  $(\omega_0 L - 1/\omega_0 C) = 0$  and  $\omega_0^2 = 1/LC$ , and thus

$$\omega L - (1/\omega C) = \omega_0 L - 1/\omega_0 C + R/2 + R/(2LC\omega_0^2) = 0 + R/2 + R/2 = R.$$

We then find  $|Z|^2 = R^2 + [\omega L - (1/\omega C)]^2 = R^2 + R^2 = 2R^2$ , and thus  $|Z| = R\sqrt{2}$ , implying that the impedance has increased by a factor  $\sqrt{2}$  (i.e. by 3 dB!) from its minimum at  $\omega_0$ .

20. a. We have a **parallel resonance circuit** as shown on the right. The impedance  $Z$  of this circuit can be calculated as



$$\begin{aligned} (1/Z) &= (1/X_C) + [1/(R + j\omega L)] = \\ &= j\omega C + [1/(R + j\omega L)] = \\ &= [j\omega C(R + j\omega L) + 1]/(R + j\omega L) = [j\omega RC - \omega^2 LC + 1]/(R + j\omega L), \\ &\text{and thus} \\ Z &= (R + j\omega L)/[(1 - \omega^2 LC) + j\omega RC]. \end{aligned}$$

b. We define the resonance frequency  $\omega_0$  as the frequency at which  $Z$  is real. We re-write  $Z$  as

$$\begin{aligned} Z &= (R + j\omega L)/[(1 - \omega^2 LC) + j\omega RC] = \\ &= (R + j\omega L) [(1 - \omega^2 LC) - j\omega RC]/[(1 - \omega^2 LC) + j\omega RC][(1 - \omega^2 LC) - j\omega RC] = \\ &= [R(1 - \omega^2 LC) + j\omega L(1 - \omega^2 LC) - j\omega R^2 C + \omega^2 RLC]/[(1 - \omega^2 LC)^2 + (\omega RC)^2]; \end{aligned}$$

This is real when  $j\omega L(1 - \omega^2 LC) - j\omega R^2 C = 0$ , i.e.  $1 - \omega^2 LC - R^2 C/L = 0$ . We find that

$$\omega_0^2 = (1 - R^2 C/L)/(LC), \text{ or}$$

$$\omega_0 = [(1 - R^2 C/L)/(LC)]^{1/2} = [(L - R^2 C)/(L^2 C)]^{1/2} = [(1/(LC)) - (R^2/L^2)]^{1/2}$$

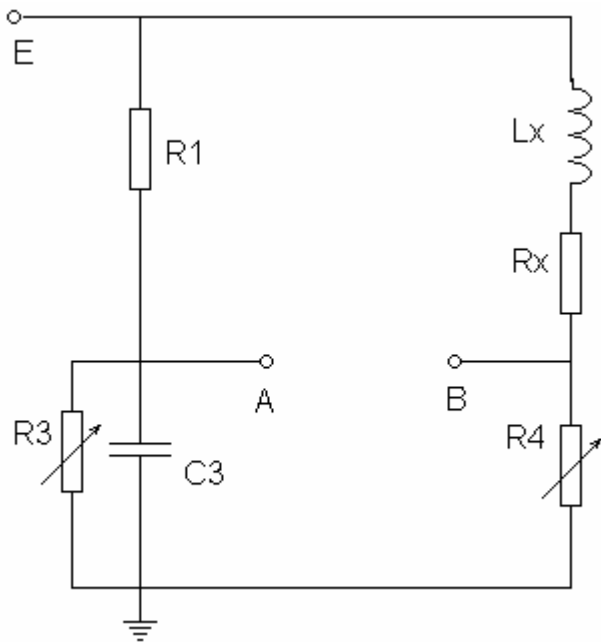
If  $Q = \omega_0 L/R \gg 1$ , then  $R/L \ll \omega_0$ ;

but then  $R^2/L^2 \ll \omega_0^2$  and the second term on the right in the expression

$$\omega_0^2 = [(1/(LC)) - (R^2/L^2)]$$

must be negligible such that  $\omega_0^2 \approx 1/(LC)$ , the same expression as for the series resonance circuit.

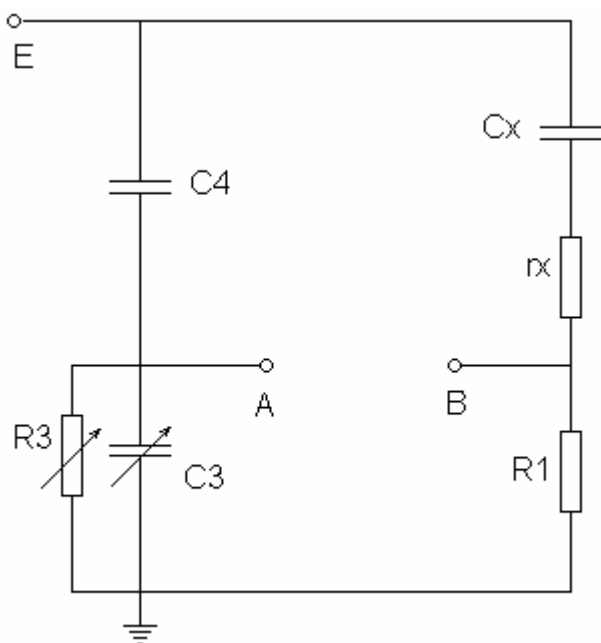
21. The figure below shows a Maxwell bridge for measuring inductances. The unknown coil, with inductance  $L_x$  and resistance  $R_x$ , is connected as shown; arrows denote variable resistances.



a. We want an expression for  $L_x$  at balance. The bridge is balanced when the voltages at A and B are identical. The voltage at B can be written  $U_B = E R_4 / (R_4 + R_x + j\omega L_x)$ ; to find the voltage at A, we first calculate the parallel impedance  $Z_p$  between A and earth as  $1/Z_p = 1/R_3 + j\omega C_3 = (1 + j\omega R_3 C_3) / R_3$ , after which we write  $U_A = E Z_p / (R_1 + Z_p) = E R_3 / (R_1 + R_3 + j\omega R_1 R_3 C_3)$ . To get the simplest possible solution we now put  $1/U_A = 1/U_B$ , to get  $(R_4 + R_x + j\omega L_x) / R_4 = (R_1 + R_3 + j\omega R_1 R_3 C_3) / R_3$ ; we see that  $L_x$  appears only in an imaginary term. Putting the *imaginary* terms equal we find  $j\omega R_3 L_x = j\omega R_1 R_3 R_4 C_3$ , or  $L_x = R_1 R_4 C_3$ .

b. We can get further information by putting the *real* terms equal, i.e.  $R_3 (R_4 + R_x) = R_4 (R_1 + R_3)$ , giving  $R_x = R_1 R_4 / R_3$ .

22. The figure below shows a Schering bridge, used for measuring capacitances. The unknown capacitance  $C_x$ , with resistance  $r_x$ , is connected as shown; arrows denote variable impedances. We want to know the balance conditions for  $C_x$  and  $r_x$ .



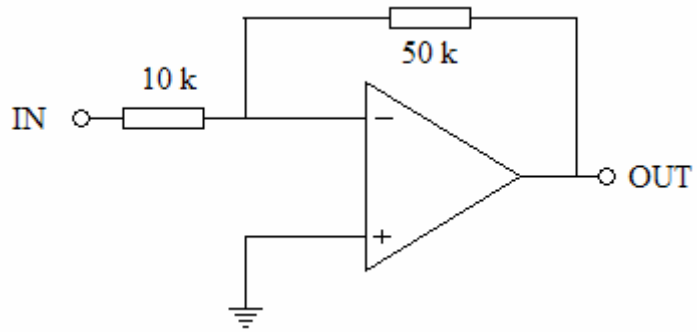
The parallel impedance  $Z_p$  between A and earth is found from  $1/Z_p = 1/R_3 + j\omega C_3 = (1 + j\omega R_3 C_3) / R_3$ , and the series impedance connected between point B and the power source is  $Z_s = r_x + 1/j\omega C_x = (1 + j\omega r_x C_x) / j\omega C_x$ . To find the balance condition, we can either use  $U_A = U_B$ , or the "diagonal impedance match" method. Let us try the latter this time!

We have the bridge in balance when  $R_1 / j\omega C_4 (= R_1 X_{C4}) = Z_p Z_s = [R_3(1 + j\omega r_x C_x)] / [j\omega C_x (1 + j\omega R_3 C_3)]$ ; we find  $R_1 C_x + j\omega R_1 R_3 C_x C_3 = R_3 C_4 + j\omega r_x R_3 C_x C_4$ .

From the real parts we find  $R_1 C_x = R_3 C_4$ , or  $C_x = R_3 C_4 / R_1$ ; from the imaginary parts we see  $R_1 R_3 C_x C_3 = r_x R_3 C_x C_4$ , or  $r_x = R_1 C_4 / C_3$ .

23. We want to know the transfer function ("gain") for the circuit below.

Let us assume there is a current  $I$  flowing from the input terminal, through the two resistors, and to the output. (No current may enter the *input* of an amplifier, but the *output* may both supply and sink currents.)



From Ohm's law, and using the fact that the input terminals must have the same potential ("infinite gain"), we can write

$$I = (U_{in} - 0)/R_{in} = (0 - U_{out})/R_F, \text{ where } R_{in} = 10 \text{ k}\Omega \text{ and the feedback resistor } R_F = 50 \text{ k}\Omega.$$

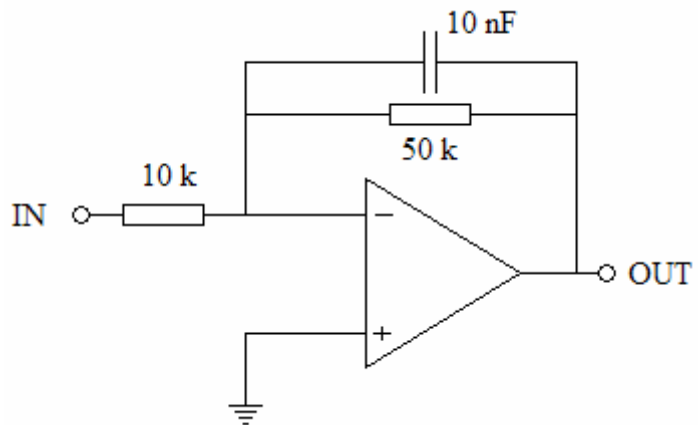
We find immediately  $U_{in}/R_{in} = -U_{out}/R_F$ , and thus

$$G = U_{out}/U_{in} = -R_F/R_{in} = -5.$$

24. We want to know the gain  $G = U_{out}/U_{in}$  as a function of frequency for this circuit.

Considering that no current can enter the amplifier input, the gain  $G$  can be found to be

$G = -Z/R_1$ , where  $R_1 = 10 \text{ k}\Omega$  and  $Z$  is the parallel combination of  $C = 10 \text{ nF}$  and  $R_2 = 50 \text{ k}\Omega$ .



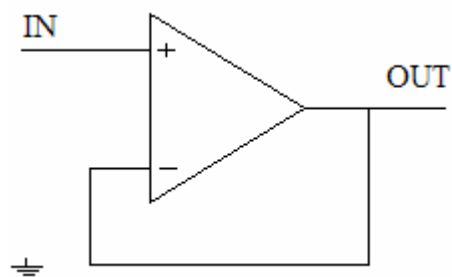
$$1/Z = 1/R_2 + j\omega C = (1 + j\omega R_2 C)/R_2,$$

$$\text{and } Z = R_2/(1 + j\omega R_2 C);$$

$$G = -(R_2/R_1)/(1 + j\omega R_2 C), \text{ or numerically } G = -5/(1 + 5 \cdot 10^{-4} j\omega).$$

25. In the circuit on the right, the voltage on the inverting input is equal to the output voltage.

Since the voltages at the two input terminals must be the same,  $U_{in} = U_{out}$ , and thus  $G = U_{out}/U_{in} = 1$ .



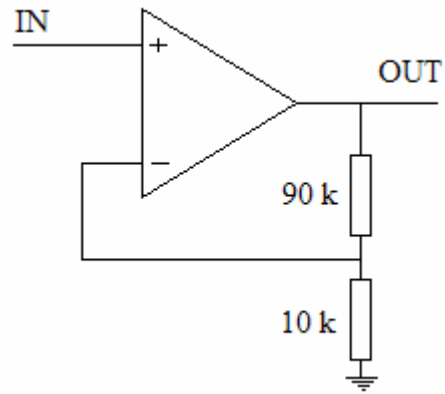
(The name, "voltage follower", derives from the fact that the output voltage tracks or follows the input voltage. The advantage of this circuit is that there will be only a very small input current, because of the very high ("infinite") input impedance, and an instrument attached to the output will thus measure the correct value with no risk of "loading" effects.)

26.

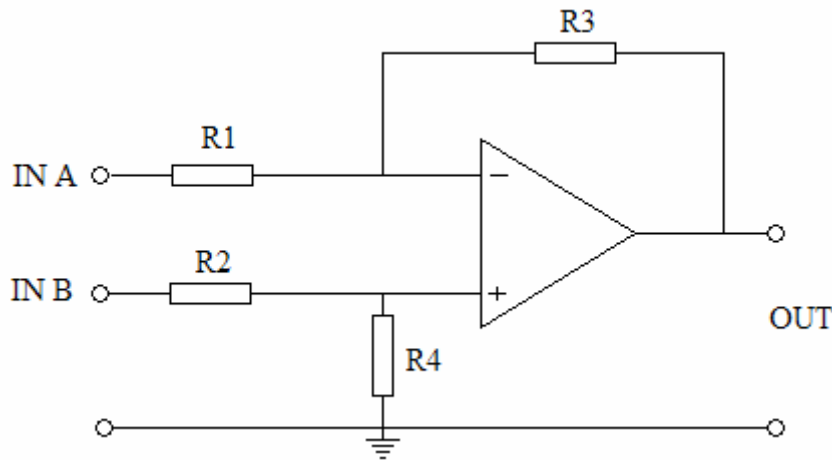
In this circuit, we can assume a current is flowing through the resistor chain, from amplifier output to ground. No current enters the amplifier inputs.

The voltage on the two inputs must be the same, and from the voltage divider formula  $U_{in} = U_{out} \cdot 10 / (10 + 90) = U_{out} / 10$ ;

$$G = U_{out} / U_{in} = 10.$$



27. We want to analyze the amplification of the differential amplifier shown below.



We know that no current can enter an amplifier input. The current from input A must thus go from the input terminal, through  $R_1$  and  $R_3$ , to the amplifier output. If we call this current  $I_A$  and the voltage at the amplifier inputs  $U_V$ , Ohm's law gives

$$I_A = (U_A - U_V) / R_1 = (U_V - U_{out}) / R_3. \quad (1)$$

Because the amplification is "infinite", we must have the same voltage  $U_V$  at **both** inputs. The current  $I_B$  from input B must go to the non-inverting input and then down to ground, and thus

$$I_B = (U_B - U_V) / R_2 = (U_V - 0) / R_4 = U_V / R_4. \quad (2)$$

Eq. (2) gives  $(U_B - U_V) / R_2 - U_V / R_4 = 0$ ;  $U_B / R_2 = U_V [(1/R_2) + (1/R_4)] = U_V (R_2 + R_4) / R_2 R_4$  and thus  $U_V = U_B R_4 / (R_2 + R_4)$ .

(This result can be obtained immediately by the voltage division theorem!)

Eq. (1) can be re-written as

$$U_{out} = U_V + U_V (R_3 / R_1) - U_A (R_3 / R_1) \quad (1')$$

Inserting (2') into (1), we find

$$U_{out} = U_B R_4 / (R_2 + R_4) + U_B (R_3 / R_1) [R_4 / (R_2 + R_4)] - U_A (R_3 / R_1).$$

a. We assume all resistors are equal, i.e.  $R_1 = R_2 = R_3 = R_4$ :

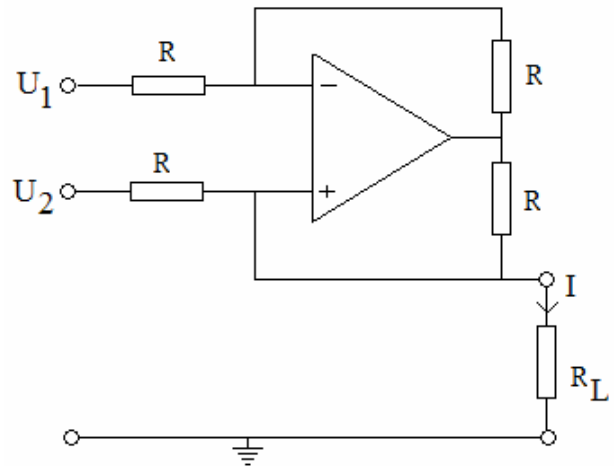
$$U_{out} = U_B (1/2) + U_B (1/2) - U_A = U_B - U_A.$$

b. We assume  $R_1 = R_2 = 10 \text{ k}\Omega$  and  $R_3 = R_4 = 30 \text{ k}\Omega$  :

$$U_{out} = U_B (3/4) + U_B 3(3/4) - 3 U_A = 3(U_B - U_A).$$

28. Let us call the output voltage of the operational amplifier  $U_{out}$ , as usual. We know that  
 a) the amplifier input impedance is “infinite”, so no current enters the inputs,  
 b) the amplification is “infinite”, so the two input terminals of the amplifier must have the same potential relative to ground. We can call this potential  $U_{in}$ .

If we have a voltage  $U_1$ , this voltage must give rise to a current  $I_1$  which flows from terminal  $U_1$  to the inverting input, and from there to the amplifier output. We then know that  $I_1 = (U_1 - U_{in})/R = (U_{in} - U_{out})/R$ , which we can rewrite as  $U_{out} = 2U_{in} - U_1$ .



The situation in the lower branch is more complicated, but we see that the current  $I$  must be the sum of the current coming from input terminal  $U_2$  and the current coming from the output terminal, so

$$I = (U_2 - U_{in})/R + (U_{out} - U_{in})/R = (U_2 - 2U_{in} + U_{out})/R;$$

if we insert the expression for  $U_{out}$  derived above we find

$$I = (U_2 - 2U_{in} + U_{out})/R = (U_2 - 2U_{in} + 2U_{in} - U_1)/R = (U_2 - U_1)/R.$$

The output current thus does not depend on  $R_L$ , only on  $R$  and the difference between the input voltages, and ***the circuit is a current source.***