MASS POINT GEOMETRY

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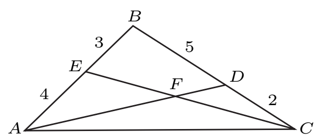
1. Introduction

1.1. **The power of the mass point technique.** This session will introduce a technique that simplifies calculations of ratios in geometric figures in an intuitive way by merging algebra, geometry and basic physics. When the method can be applied, it is far faster than the standard techniques of vectors and area addition. The method is as simple as balancing a see-saw.

Let us begin with a problem involving an old geometry concept. Given a triangle, a *cevian* is a line segment from a vertex to an interior point of the opposite side. (The 'c' is pronounced as 'ch'). Figure 1a illustrates two cevians AD and CE in $\triangle ABC$. Cevians are named in honor of the Italian mathematician Giovanni Ceva who used them to prove his famous theorem in 1678 (cf. Theorem 4). Problem 1 below is not that famous, but it certainly presents a situation that you may stumble upon in everyday life.



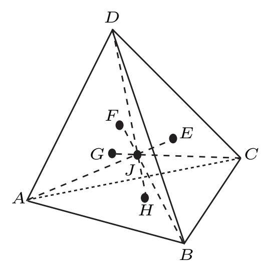


Figure 1. A Triangle with Two Cevians and the Centroid of a Tetrahedron

Problem 1. In $\triangle ABC$, side BC is divided by D in a ratio of 5 to 2 and BA is divided by E in a ratio of 3 to 4 as shown in Figure 1a. Find the ratios in which F divides the cevians AD and CE, i.e. find $EF : FC$ and $DF : FA$.

It is true that Problem 1 can be successfully attacked by vectors or area addition, thereby reducing it to elementary algebra. Yet, we shall see in this session that a much easier and more intuitive solution would result from assigning the *masses* to the vertices of $\triangle ABC$ in such a way that F becomes the *balancing point*, and the problem is reduced to elementary arithmetic.

Examples illustrating the power of our mass-point technique do *not* need to be constrained to the plane. The next problem for instance is set up in 3-dimensional space. Recall that a *tetrahedron* is the mathematical word for what we know as a pyramid: a polyhedron with four triangular faces. Recall that the *centroid* of a triangle is the point of concurrence of the three medians. (cf. Exercise 3).

Problem 2. Consider tetrahedron $ABCD$, and mark the centroids of its four faces by E, F, G and H (cf. Fig. 1b). Prove that the four segments connecting vertices to the centroids of the opposite faces are concurrent, i.e. that AE, BF, CG and DH intersect at a point J, called the centroid of the tetrahedron. In what ratio does this centroid J divide the four segments. For example, what is $AJ : JE$?

1.2. **Archimedes' lever.** The underlying idea of the mass point technique is the principle of the lever, which Archimedes used to discover many of his results. You may have heard the boast of Archimedes upon discovering the lever, *"Give me a place to stand on, and I will move the earth."* Although Archimedes knew his results were correct, based on reasoning with the lever, such justification was unacceptable as proof in Greek mathematics, so he was forced to think of very clever proofs using Euclidean geometry to convince the mathematical community at the time that his results were correct. These proofs are masterpieces of reasoning and the reader is recommended to read some of them to appreciate the elegance of the mathematics.

However, in this session we are going to use Archimedes' *lever*. The basic idea is that of a see-saw with masses at each end. The see-saw will balance if the product of the mass and its distance to the fulcrum is

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the same for each mass. For example, if a baby elephant of mass 100 kg is 0.5 m from the fulcrum, then an ant of mass 1 gram must be located 50 km on the other side of the fulcrum for the see-saw to balance (cf. Fig. 2, not drawn to scale): distance $\times$ mass $= 100\,\text{kg} \times 0.5\,\text{m} = 100,000\,\text{g} \times 0.0005\,\text{km} = 1\,\text{g} \times 50\,\text{km}$.

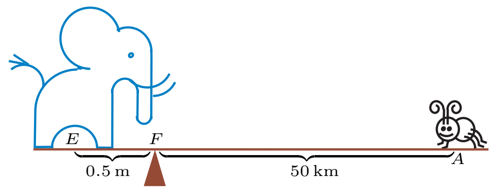


FIGURE 2. Balancing on a See-saw: Elephant and Ant (artwork by Zvezda)

As we shall see, the uses of the lever are more far-reaching than one might imagine. For instance, here is another problem for you to consider. It extends our mass point technique to *transversals*: a transversal of lines l and m is a line that joins a point on l and a point on m. Note that a transversal of two sides in a triangle is a generalization of what we defined earlier as a cevian.

Problem 3. In Figure 3, ED joins points E and D on the sides of $\triangle ABC$ forming a transversal. Cevian BG divides AC in a ratio of 3 to 7 and intersects the transversal ED at point F. Find the ratios $EF : FD$ and $BF : FG$.

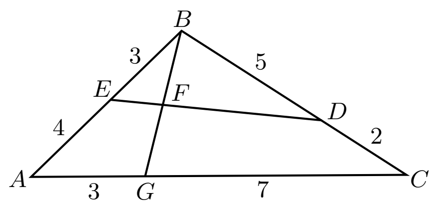


FIGURE 3. A Transversal Problem

2. Definitions and Properties: in the Familiar Setting of Euclidean Geometry

To begin, let me say that I misunderstood mathematics for a long time and it was not until I realized that the definitions, postulates and theorems were the key to everything, that I finally began making some progress. In the end, it still depends on how clever you can be in using the definitions, postulates and theorems to arrive at conjectures and prove more theorems. But if you don't understand these fundamentals completely, you will not go very far in mathematics. This session follows the axiomatic approach to mass points found in Hausner's 1962 paper [5].

2.1. **Objects of mass point geometry.** When developing a new theory, the objects in it must be clearly defined, so there are no ambiguities later on. For example, in ordinary high school geometry, you defined what triangles are and explained what it means for two of them to be congruent. In the slightly more advanced setting of, say, coordinate geometry, it becomes necessary to define even more basic objects, such as a "point" – as a pair of numbers (x, y) called the coordinates of the points, and a "line" – as the set of points which are solutions to linear equations $Ax + By = C$; one would also define what it means for two points P and Q to be the same – $P = Q$ if their corresponding coordinates are equal. In this vein, we define below the main objects of our new *Mass Point* theory.

Definition 1. A *mass point* is a pair (n, P), also written as nP, consisting of a positive number n, the *mass*, and a point P in the plane or in space.

Definition 2. We say that two mass points coincide, $nP = mQ$, if and only if $n = m$ and $P = Q$, i.e. they correspond to the same ordinary point with the same assigned mass.

2.2. **Operations in mass point geometry.** What makes our theory so interesting and powerful is that it combines objects and ideas from both geometry and algebra, and hence makes it necessary for us to define from scratch *operations* on mass points. How can we add two mass points?

Definition 3 (Addition). $nE + mA = (n + m)F$ where F is on EA and $EF : FA = m : n$.

This is the **crucial idea**: adding two mass points nE and mA results in a mass point $(n + m)F$ so that

(a) F is located at the balancing point of the masses on the line segment EA, and
(b) the mass at this location F is the sum $n + m$ of the two original masses.

Definition 4 (Scalar Multiplication)**.** Given a mass point (n, P) and a scalar $m > 0$, we define multiplication of a mass point by a positive real number as $m(n, P) = (mn, P)$.

2.3. Basic properties in mass point geometry.

Property 1 (Closure)**.** *Addition produces a unique sum.*

Property 2 (Commutativity)**.** $nP + mQ = mQ + nP$.

Property 3 (Associativity)**.** $nP + (mQ + kR) = (nP + mQ) + kR = nP + mQ + kR$.

Property 4 (Idempotent)**.** $nP + mP = (n + m)P$.

Property 5 (Distributivity)**.** $k(nP + mQ) = knP + kmQ$.

2.4. More operations on mass points?

Property 6 (Subtraction)**.** *If $n > m$ then $nP = mQ + xX$ may be solved for the unknown mass point xX. Namely, $xX = (n - m)R$ where P is on $\overline{RQ}$ and $RP : PQ = m : (n - m)$.*

Example 1. Given mass points $3Q$ and $5P$, find the location and mass of their difference $5P - 3Q$.

3. Fundamental Examples and Exercises

Let's take a look at the first problem in the introduction. In order to have D as the balancing point of BC we assign a mass of 2 to B and a mass of 5 to C. Now on side BA to have E as the balancing point we assign $2 \cdot 3/4 = 3/2$ to A. Then at the balancing points on the sides of the triangle, we have $2B + 5C = 7D$ or $2B + \frac{3}{2}A = \frac{7}{2}E$. (cf. Fig. 4a)

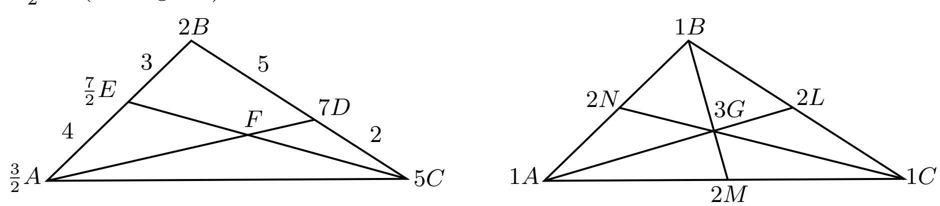


Figure 4. Two Cevians Solved and Medians Concurrent

The center of mass $8.5X$ of the system $\{\frac{3}{2}A, 5C, 2B\}$ is located at the sum $\frac{3}{2}A + 2B + 5C$, which can be calculated in two ways according to our associativity property:

$$\tfrac{5}{2}E + 5C = (\tfrac{3}{2}A + 2B) + 5C = 8.5X = \tfrac{3}{2}A + (2B + 5C) = \tfrac{3}{2}A + 7D.$$

Thus, by definition of addition, X is located on the one hand on segment EC, and on the other hand on segment AD, i.e. at their intersection point F. Hence F is the fulcrum of the see-saw balancing $\frac{3}{2}A$ and $7D$, and of the see-saw balancing $5C$ and $\frac{7}{2}E$. This means that $DF : FA = 3/2 : 7 = 3 : 14$ and $EF : FC = 5 : 7/2 = 10 : 7$. All of this can be written down immediately on the figure in a matter of seconds. □

Many of the following exercises and examples in this section are based on those in an article by Sitomer and Conrad [17]. Since the article is no longer in print and not available in most libraries, I want to make available to you some of the examples from their presentation which expanded my understanding of this technique.

Exercise 1 (Warm-up)**.** If G is on BY, find x and $BG : GY$ provided that

(a) $3B + 4Y = xG$;
(b) $7B + xY = 9G$.

Example 2. In $\triangle ABC$, D is the midpoint of BC and E is the trisection point of AC nearer A (i.e. $AE : EC = 1 : 2$). Let $G = BE \cap AD$. Find $AG : GD$ and $BG : GE$.

Exercise 2 (East Bay Mathletes, April 1999)**.** In $\triangle ABC$, D is on AB and E is the on BC. Let $F = AE \cap CD$, $AD = 3$, $DB = 2$, $BE = 3$ and $EC = 4$. Find $EF : FA$ in lowest terms.

Exercise 3. Show that the medians of a triangle are concurrent and the point of concurrency divides each median in a ratio of 2:1.

(Hint: Assign a mass of 1 to each vertex, cf. Fig. 4b.)

PST 1. (Problem Solving Technique) Given two triangles with the same altitude, their areas are in the same ratio as their bases. In addition, if the triangles have have equal bases, then they have equal areas.

Example 3. Show that all six regions obtained by the slicing a triangle via its three medians have the same area.

Exercise 4 (Varignon's Theorem)**.** If the midpoints of consecutive sides of a quadrilateral are connected, the resulting quadrilateral is a parallelogram.

(Hint: Assign mass 1 to each vertex of the original quadrilateral and find the center of mass in two ways: why does this center lie on each of the line segments joining midpoints of opposite sides?)

Exercise 5. In quadrilateral $ABCD$, E, F, G, and H are the trisection points of AB, BC, CD, and DA nearer A, C, C, A, respectively. Show that $EFGH$ is a parallelogram.

(Hint: Use the point $K = EG \cap FH$.)

Exercise 6. Generalize Exercise 5 to points E, F, G, and H which divide the quadrilateral sides in corresponding ratios of $m : n$.

4. Angle Bisectors, Combining Mass Points and Area, Mass Points in Space

The following problems extend the fundamental idea of mass points in several directions.

4.1. **Using angle bisectors.** To start with, you will need the following famous theorem, which you may have heard in a high school geometry class:

Theorem 1 (Angle Bisector Theorem)**.** *An angle bisector in a triangle divides the opposite side in the same ratio as the other two sides. More precisely, in $\triangle ABC$, if ray $\overrightarrow{BD}$ bisects $\angle ABC$ then $AD : DC = AB : BC$.*

Exercise 7. In $\triangle ABC$, let $AB = c$, $BC = a$ and $CA = b$. Assign a mass to each vertex equal to the length of the opposite side, resulting in mass points aA, bB and cC. Show that the center of mass of this system is located on each angle bisector at a point corresponding to the mass point $(a + b + c)I$.

Exercise 8. Use Exercise 7 to prove that the angle bisectors of the angles of a triangle are concurrent.

Those who know the definition of $\sin A$ may recall the following well-known theorem.

Theorem 2 (Law of Sines)**.** *In $\triangle ABC$ where the opposite sides of $\angle A$, $\angle B$, and $\angle C$ are a, b, and c, respectively, and R is the circumradius of $\triangle ABC$:*

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

Exercise 9. In $\triangle ABC$ with the bisector of $\angle B$ intersecting AC at D:

(a) Show that $AD : DC = \sin C : \sin A$, or equivalently, $AD \sin A = DC \sin C$.
(b) Let $\sin A = 4/5$ and the $\sin C = 24/25$. The bisector BD intersects median AM at point E. Find $AE : EM$ and $BE : ED$.

4.2. **Combining mass points and areas.** As you attempt to solve the following problem keep in mind that it can be solved via a combination of mass points and area addition. This leads to a generalization known as Routh's Theorem (to be discussed later on). Have fun!

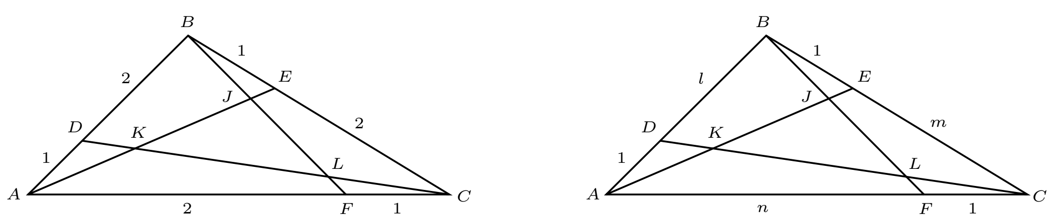


Figure 5. Combining Mass Points with Area Addition and Routh's Theorem

Problem 4. In $\triangle ABC$, D, E, and F are the trisection points of AB, BC, and CA nearer A,B,C, respectively. (cf. Fig. 5a)

(a) Let $BF \cap AE = J$. Show that $BJ : JF = 3 : 4$ and $AJ : JE = 6 : 1$.
(b) Let $CD \cap AE = K$ and $CD \cap BF = L$. Extend part (a) of this problem to show that $DK : KL : LC = 1 : 3 : 3 = EJ : JK : KA = FL : LJ : JB$.
(c) Use parts (a) and (b) and to show that the area of triangle $\triangle JKL$ is one-seventh the area of $\triangle ABC$.

(d) Generalize this problem using points which divide the sides in a ratio of $1 : n$ in place of $1 : 2$ to show the ratio of the areas is $(1-n)^3 : (1-n^3)$.

Part (d) can be generalized even further using different ratios on each side. It is known as Routh's Theorem.(cf. Fig. 5b)

Theorem 3. (Routh) *If the sides AB, BC, CA of $\triangle ABC$ are divided at D, E, F in the respective ratios of $1 : l$, $1 : m$, $1 : n$, then the cevians CD, AE, and BF form a triangle whose area is*

$$\frac{(lmn-1)^2}{(lm+l+1)(mn+m+1)(nl+n+1)}$$

For example, check that when the ratios are all equal, $l = m = n$, Routh's formula yields the answer in part(d). The proof of this theorem is beyond the scope of the present session. See Coxeter [4], Niven [12], and Klamkin [9] for various proofs of this theorem. The proof by Coxeter uses a generalization of mass points called areal coordinates, or normalized barycentric coordinates. It is only four lines long.

4.3. **Mass points in space.** Going in another direction, an extension of the mass point technique can be used to solve problems in space: in 3 dimensions. This is illustrated in Example 4, and Exercises 10 and 11. Thus, for the rest of this subsection, we shall assume the same definitions and properties of addition of mass points in space as those in the plane.

Example 4. Let $ABCD$ be a tetrahedron (cf. Fig. 1b). Assign masses of 1 to each of the vertices. Let H be the point in $\triangle ABC$ such that $1A+1B+1C = 3H$. Let J be the point on DH such that $1D+3H = 4J$. What is the ratio of DJ to JH?

Now let us apply mass points in space to a couple of exercises.

Exercise 10. Fill in the details of the solution above for Problem 2. In particular, show that the four segments from the vertices to centroids of the opposite faces are concurrent at the point J.

In a tetrahedron, *opposite edges* are those pairs of edges that have no vertex in common.

Exercise 11. Show that the three segments joining the midpoints of opposite edges of a tetrahedron bisect each other. (cf. Fig. 6b)

5. Splitting Masses, Altitudes, Ceva and Menelaus

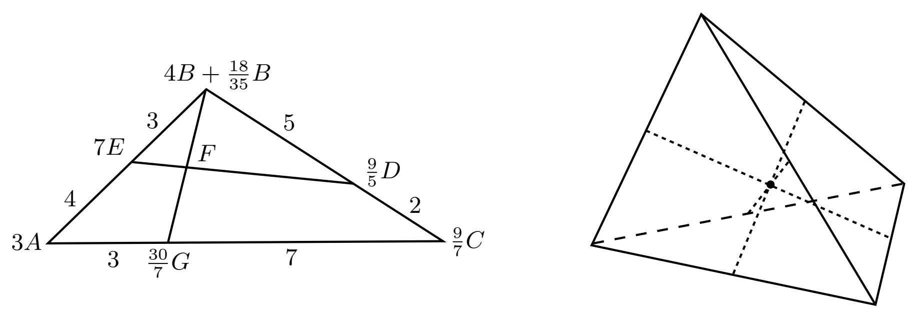


Figure 6. Transversal Problem 3 Solution and Tetrahedron in Exercise 11

5.1. **Splitting masses.** Let's now take a look at Problem 3 stated in the Introduction. This is not as intuitive as the two cevian Problem 1. But once it is shown to work, we can then solve a whole new class of problems with mass points. So let's do it!

Splitting mass points as in $mP + nP = (m+n)P$ is the technique to use when dealing with transversals. The actual assignment of masses is as follows. As a first approximation, assign 4 to B and 3 to A to balance AB at E. Then to balance AC at G assign $\frac{9}{7}$ to C. To balance $\frac{9}{7}C$ at point D, $\frac{18}{35}B$ is needed. So we now have $(4 + \frac{18}{35})B$. This gives $\frac{44}{5}F$ as the center of mass for the masses at A, B and C. Indeed, using associativity of addition:

$$\tfrac{30}{7}G + (4B + \tfrac{18}{35}B) = (3A + \tfrac{9}{7}C) + 4B + \tfrac{18}{35}B = (3A+4B) + (\tfrac{18}{35}B + \tfrac{9}{7}C) = 7E + \tfrac{9}{5}D,$$

from where the center of mass lies on both ED and BG, i.e. it is located at point F.

The sought after ratios can now be read directly from the diagram:

$$EF : FD = 9/5 : 7 = \mathbf{9} : \mathbf{35} \text{ and } BF : FG = 30/7 : 158/35 = 150 : 158 = \mathbf{75} : \mathbf{79}. \quad \square$$

Here is another example.

Example 5. In $\triangle ABC$, let E be on AB such that $AE : EB = 1 : 3$, let D be on BC such that $BD : DC = 2 : 5$, and let F be on ED such that $EF : FD = 3 : 4$. Finally, let ray $\overrightarrow{BF}$ intersect AC at point G. Find $AG : GC$ and $BF : FG$.

Try to use this technique in the following exercises.

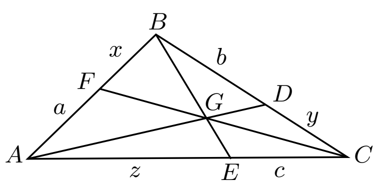
Exercise 12. In Example 5, $AE : EB = 1 : 3$, $BD : DC = 4 : 1$, $EF : FD = 5 : 1$. Show that $AG : GC = 4 : 1$ and $BF : FG = 17 : 7$.

5.2. **Problems which involve altitudes.** Let BD be an altitude of acute $\triangle ABC$. Note that $AD \cdot DC/BD = DC \cdot AD/BD$. So the appropriate masses to assign to A and C, respectively, are DC/BD and AD/BD in order to have the balancing point on AC be at D. Those of you who know some trigonometry will recognize $DC/BD = 1/\tan C = \cot C$ and $AD/BD = 1/\tan A = \cot A$. Therefore, assigning masses *proportional* to $\cot A$ and $\cot C$ to the points C and A, respectively, will balance the side at the foot of the altitude.

Exercise 13. Let $\triangle ABC$ be a right triangle with $AB = 17$, $BC = 15$, and $CA = 8$. Let CD be the altitude to the hypotenuse and let the angle bisector at B intersect AC at F and CD at E. Show that $BE : EF = 15 : 2$ and $CE : ED = 17 : 15$.

Problem 5. The sides of $\triangle ABC$ are $AB = 13$, $BC = 15$ and $AC = 14$. Let BD be an altitude of the triangle. The angle bisector of $\angle C$ intersects the altitude at E and AB at F. Find $CE : EF$ and $BE : ED$.

Exercise 14. Prove that the altitudes of an acute triangle are concurrent using mass points.



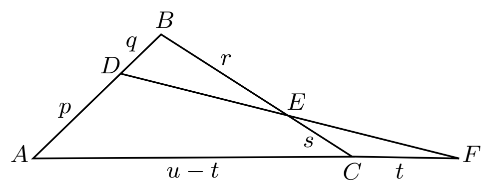


FIGURE 7. The Theorem of Ceva and the Theorem of Menelaus

5.3. **Ceva, Menelaus and Property 3.**

Theorem 4 (Ceva's Theorem). *Three cevians of a triangle are concurrent if and only if the products of the lengths of the non-adjacent parts of the three sides are equal. For example, for $\triangle ABC$ in* Figure 7a*, this means that the three cevians are concurrent iff $abc = xyz$.*

Theorem 5 (Menelaus' Theorem). *If a transversal is drawn across three sides of a triangle (extended if necessary), the products of the non-adjacent segments are equal. For example, for $\triangle ABC$ with transversal intersecting AB in D, BC in E, and AC, externally, in F, the conclusion is $\frac{AD}{DB} \cdot \frac{BE}{EC} \cdot \frac{CF}{FA} = 1$, or equivalently, $AD \cdot BE \cdot CF = DB \cdot EC \cdot FA$.* (cf. Fig. 7b)

Just as Ceva's Theorem is an *if and only if* statement, the converse of Menelaus' Theorem is also true. Use mass point geometry to prove this. Then prove Menelaus' Theorem is true using similar triangles.

6. EXAMPLES OF CONTEST PROBLEMS

6.1. **Math contests versus research Mathematics.** You may still be thinking that the type of problem that yields to a mass point solution is rare, and that it is more like a parlor trick than an important mathematical technique. In this collection of problems that I have assembled, you will see problems that can often be solved by mass point geometry more readily than with the official solution. They come from a wide variety of contests and were often problems the contestants found difficult. Part of the fun of such contests is knowing that a solution which could take you between 5 and 15 minutes exists and trying to find it.

6.2. **The contests surveyed for this collection.** The problems in this section are from are city-wide, regional and national contests(cf. [1, 2, 15, 20]): the New York City Mathematics League (NYCML), the American Regional Mathematics League (ARML), the American High School Mathematics Examination (AHSME), and the American Invitational Mathematics Examination (AIME).

6.3. **Using the fundamental mass point technique.**

Contest Problem 1 (AHSME 1965 #37). Point E is selected on side AB of $\triangle ABC$ in such a way that $AE : EB = 1 : 3$ and point D is selected on side BC so that $CD : DB = 1 : 2$. The point of intersection of AD and CE is F. Find $\frac{EF}{FC} + \frac{AF}{FD}$.

Contest Problem 2 (NYCML S75 #27). In $\triangle ABC$, C' is on side AB such that $AC' : C'B = 1 : 2$, and B' is on AC such that $AB' : B'C = 3 : 4$. If BB' and CC' intersect at P, and if A' is the intersection of ray AP and BC then find $AP : PA'$.

6.4. **Using the angle bisector theorem and the transversal method.**

Contest Problem 3 (ARML 1989 T4). In $\triangle ABC$, angle bisectors AD and BE intersect at P. If the sides of the triangle are $a = 3$, $b = 5$, $c = 7$, with $BP = x$, and $PE = y$, compute the ratio $x : y$, where x and y are relatively prime integers.

Contest Problem 4 (AHSME 1975 #28). In $\triangle ABC$, M is the midpoint of side BC, $AB = 12$ and $AC = 16$. Points E and F are taken on AC and AB, respectively, and lines EF and AM intersect at G. If $AE = 2AF$ then find EG/GF.

Contest Problem 5 (ARML 1992 I8). In $\triangle ABC$, points D and E are on AB and AC, respectively. The angle bisector of $\angle A$ intersects DE at F and BC at T. If $AD = 1, DB = 3, AE = 2$, and $EC = 4$, compute the ratio $AF : AT$.

6.5. **Using ratios of areas** *via* **PST 1.**

Contest Problem 6 (AHSME 1980 #21). In $\triangle ABC$, $\angle CBA = 72°$, E is the midpoint of side AC and D is a point on side BC such that $2BD = DC$; AD and BE intersect at F. Find the ratio of the area of $\triangle BDF$ to the area of quadrilateral $FDCE$.

Contest Problem 7 (AIME 1985 #6). In $\triangle ABC$, cevians AD, BE and CF intersect at point P. The areas of $\triangle$'s PAF, PFB, PBD and PCE are 40, 30, 35 and 84, respectively. Find the area of triangle ABC.

6.6. **Change your point of view.**

Contest Problem 8 (NYCML F76 #13). In $\triangle ABC$, D is on AB such that $AD : DB = 3 : 2$ and E is on BC such that $BE : EC = 3 : 2$. If ray DE and ray AC intersect at F, then find $DE : EF$.

Contest Problem 9 (NYCML S77 #1). In a triangle, segments are drawn from one vertex to the trisection points of the opposite side. A median drawn from a second vertex is divided, by these segments, in the continued ratio $x : y : z$. If $x \geq y \geq z$ then find $x : y : z$.

6.7. **Using special triangles and topics from geometry.**

Contest Problem 10 (NYCML S78 #25). In $\triangle ABC$, $\angle A = 45°$ and $\angle C = 30°$. If altitude BH intersects median AM at P, then $AP : PM = 1 : k$. Find k.

Contest Problem 11 (AHSME 1964 #35). The sides of a triangle are of lengths 13, 14, and 15. The altitudes of the triangle meet at point H. If AD is the altitude to the side of length 14, what is the ratio $HD : HA$?

6.8. **Some especially challenging problems.**

Contest Problem 12 (AIME 1992 #14). In $\triangle ABC$, A', B', and C' are on sides BC, AC, AB, respectively. Given that AA', BB', and CC' are concurrent at the point O, and that $\frac{AO}{OA'} + \frac{BO}{OB'} + \frac{CO}{OC'} = 92$, find the value of $\frac{AO}{OA'} \cdot \frac{BO}{OB'} \cdot \frac{CO}{OC'}$.

Theorem 6. *In $\triangle ABC$, if cevians AD, BE, and CF are concurrent at P then*

$$\frac{PD}{AD} + \frac{PE}{BE} + \frac{PF}{CF} = 1.$$

Contest Problem 13 (AIME 1988 #12). Let P be an interior point of $\triangle ABC$ and extend lines from the vertices through P to the opposite sides. Let $AP = a$, $BP = b$, $CP = c$ and the extensions from P to the opposite sides all have length d. If $a + b + c = 43$ and $d = 3$ then find abc.

Contest Problem 14 (AIME 1989 #15). Point P is inside $\triangle ABC$. Line segments APD, BPE, and CPF are drawn with D on BC, E on CA, and F on AB. Given that $AP = 6$, $BP = 9$, $PD = 6$, $PE = 3$, and $CF = 20$, find the area of triangle ABC.

Contest Problem 15 (Larson [10] problem 8.3.4). In $\triangle ABC$, let D and E be the trisection points of BC with D between B and E. Let F be the midpoint of AC, and let G be the midpoint of AB. Let H be the intersection of EG and DF. Find the ratio $EH : HG$.

7. History and Sources

Mass points were first used by Augustus Ferdinand Möbius in 1827. They didn't catch on right away. Cauchy was quite critical of his methods and even Gauss in 1843 confessed that he found the new ideas of Möbius difficult. I discovered this historical information recently in a little mathematical note by Dan Pedoe in Mathematics Magazine [14]. This was long after I learned about the method.

I first encountered the idea about 30 years ago in a math workshop session entitled "Teeter-totter Geometry" given by Brother Raphael from Saint Mary's College of California in Moraga. Apparently he taught one of the courses each year using only original sources, and that year he was reading Archimedes with his students. It was Archimedes' "principle of the lever" that he used on the day of my visit to show how mass points could be used to make deductions about triangles.

For a very readable account of the assumptions Archimedes makes about balancing masses and locating the center of gravity, I recommend the new book *Archimedes: What Did He Do Besides Cry Eureka?*[19] written by Sherman Stein of U.C.Davis. About twenty-five years ago Bill Medigovich, who was then teaching at Redwood High School in Marin County, California, sent me a 30-sheet packet [11] that he used for a presentation he gave to high school students. I also found the topic discussed in the appendix of *The New York City Contest Problem Book 1975-1984* [16] with a further reference to an article *The Center of Mass and Affine Geometry* [5] written by Melvin Hausner in 1962. Recently, Dover Publications reissued a book published by Hausner [6] in 1965 that was written for a one year course for high school teachers of mathematics at New York University.

As I was preparing for this talk in 2002, I was going through old issues of *Eureka* which are no longer available and found a key paper on the this subject, *Mass Points* [17], that was originally written for the NYC Senior 'A' Mathletes. The authors are Harry Sitomer and Steven R. Conrad. Their paper provided me with what I considered to be the most attractive way to present these ideas. This wonderful journal for problem solvers published by the Canadian Mathematical Society is presently published as *Crux Mathematicorum and Mathematical Mayhem*. There are a few other articles that I used in preparing for the talk. They are listed in the references as Boyd [3], Honsberger [7], and Pedoe(1970) [13].

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