# Solving ODEs Euler Method & RK2/4

Major: All Engineering Majors

Authors: Autar Kaw, Charlie Barker

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# Euler Method

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# Euler's Method

$$\frac{dy}{dx} = f(x,y), y(0) = y_0$$

$$\text{Slope} = \frac{Rise}{Run}$$

$$= \frac{y_1 - y_0}{x_1 - x_0}$$

$$= f(x_0, y_0)$$

$$y_1 = y_0 + f(x_0, y_0)(x_1 - x_0)$$

$$= y_0 + f(x_0, y_0)h$$

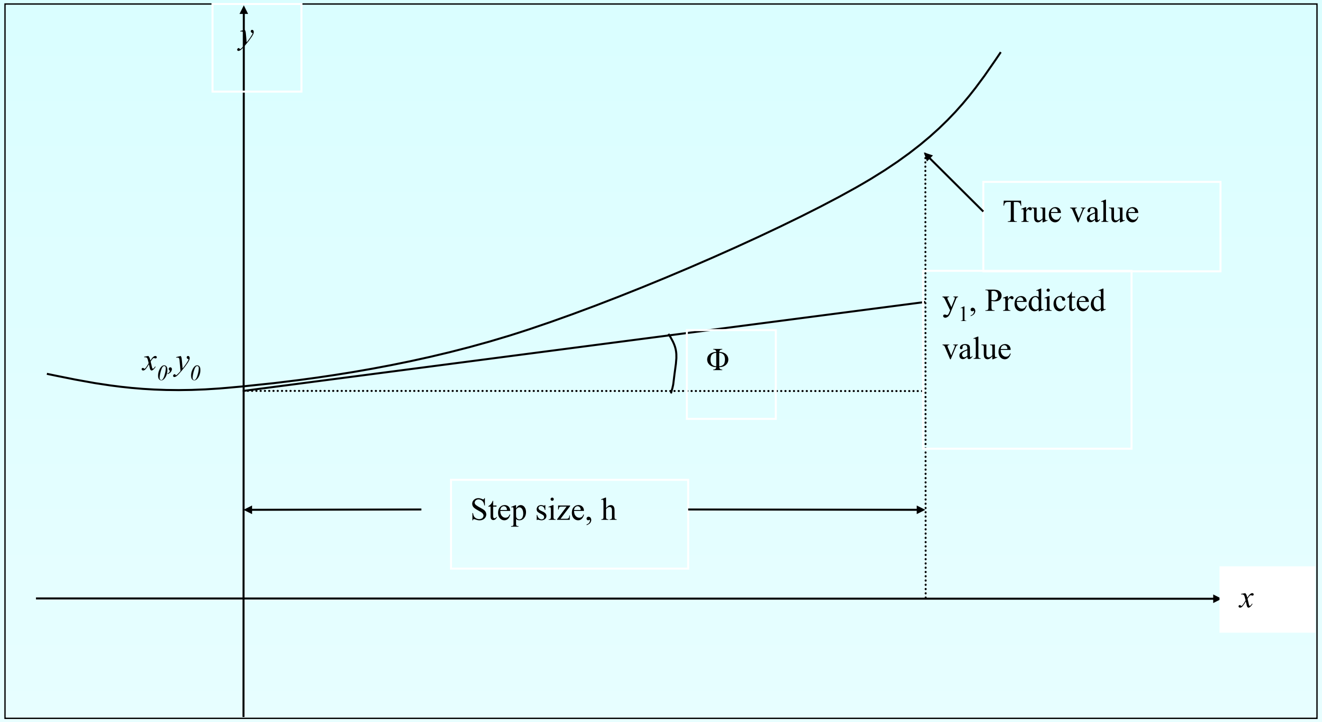


**Figure 1** Graphical interpretation of the first step of Euler's method

# Euler's Method

$$y_{i+1} = y_i + f(x_i, y_i)h$$

$$h = x_{i+1} - x_i$$

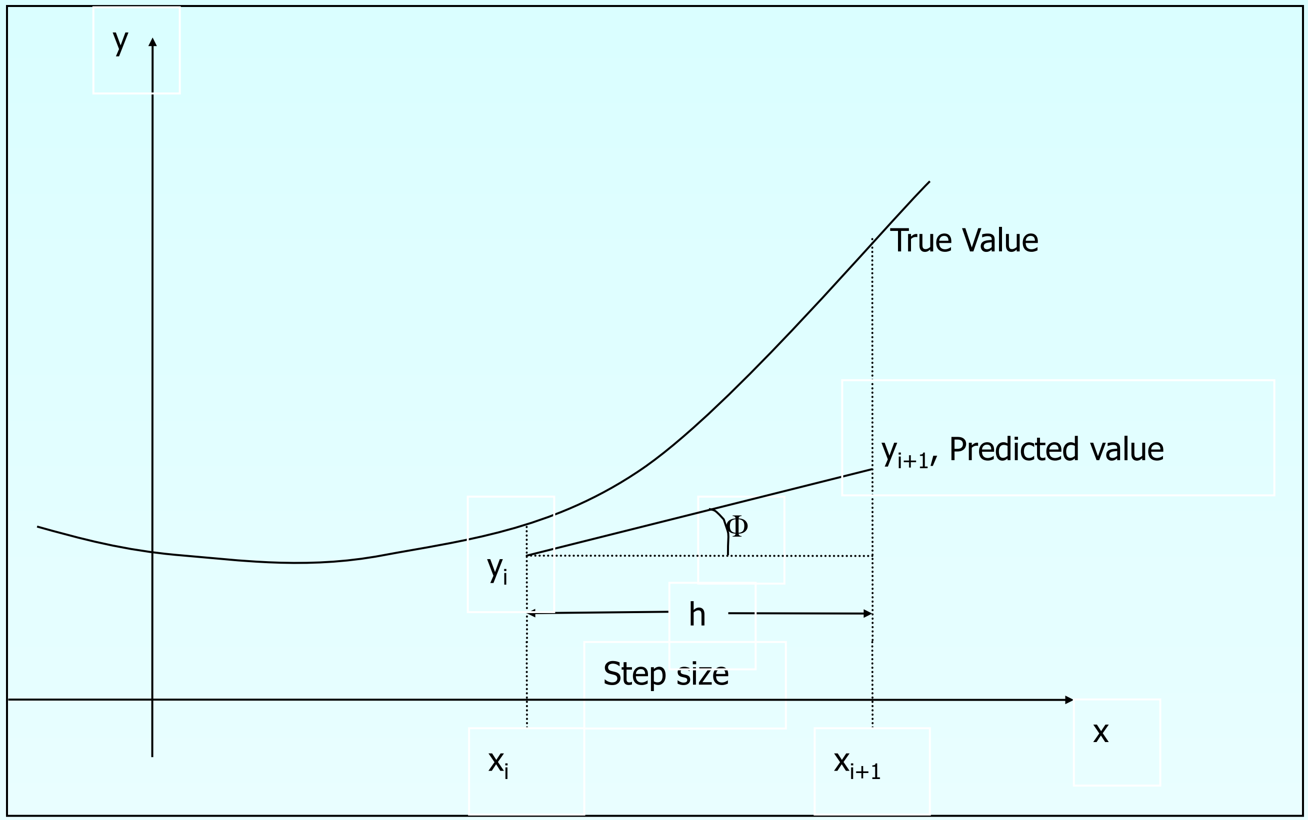


**Figure 2.** General graphical interpretation of Euler's method

# How to write Ordinary Differential Equation

How does one write a first order differential equation in the form of

$$\frac{dy}{dx} = f(x, y)$$

**Example**

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$$

is rewritten as

$$\frac{dy}{dx} = 1.3e^{-x} - 2y, y(0) = 5$$

In this case

$$f(x, y) = 1.3e^{-x} - 2y$$

# Example

A ball at 1200K is allowed to cool down in air at an ambient temperature of 300K. Assuming heat is lost only due to radiation, the differential equation for the temperature of the ball is given by

$$\frac{d\theta}{dt} = -2.2067\times10^{-12}\left(\theta^4 - 81\times10^8\right), \theta(0) = 1200K$$

Find the temperature at $t = 480$ seconds using Euler's method. Assume a step size of

$h = 240$ seconds.

# Solution

Step 1:

$$\frac{d\theta}{dt} = -2.2067\times10^{-12}\left(\theta^4 - 81\times10^8\right)$$

$$f(t,\theta) = -2.2067\times10^{-12}\left(\theta^4 - 81\times10^8\right)$$

$$\theta_{i+1} = \theta_i + f(t_i,\theta_i)h$$

$$\begin{aligned}\theta_1 &= \theta_0 + f(t_0,\theta_0)h\\ &= 1200 + f(0,1200)240\\ &= 1200 + \left(-2.2067\times10^{-12}\left(1200^4 - 81\times10^8\right)\right)240\\ &= 1200 + (-4.5579)240\\ &= 106.09K\end{aligned}$$

$\theta_1$ is the approximate temperature at $t = t_1 = t_0 + h = 0 + 240 = 240$

$$\theta(240) \approx \theta_1 = 106.09K$$

# Solution Cont

**Step 2:** For $i=1, \quad t_1 = 240, \quad \theta_1 = 106.09$

$$\begin{aligned}
\theta_2 &= \theta_1 + f(t_1, \theta_1)h \\
&= 106.09 + f(240, 106.09)240 \\
&= 106.09 + \left(-2.2067\times10^{-12}\left(106.09^4 - 81\times10^8\right)\right)240 \\
&= 106.09 + (0.017595)240 \\
&= 110.32K
\end{aligned}$$

$\theta_2$ is the approximate temperature at $t = t_2 = t_1 + h = 240 + 240 = 480$

$$\theta(480) \approx \theta_2 = 110.32K$$

# Solution Cont

The exact solution of the ordinary differential equation is given by the solution of a non-linear equation as

$$0.92593 \ln \frac{\theta - 300}{\theta + 300} - 1.8519 \tan^{-1}\left(0.00333\theta\right) = -0.22067 \times 10^{-3} t - 2.9282$$

The solution to this nonlinear equation at t=480 seconds is

$$\theta(480) = 647.57K$$

# Comparison of Exact and Numerical Solutions

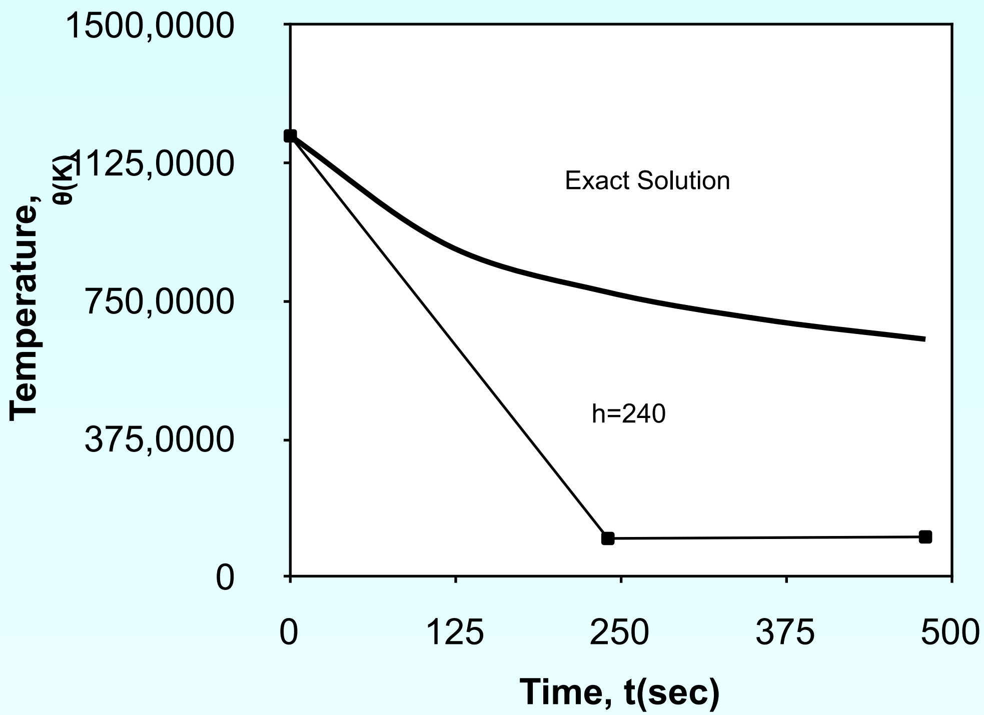


**Figure 3.** Comparing exact and Euler's method

# Effect of step size

**Table 1. Temperature at 480 seconds as a function of step size, h**

| Step, h | θ(480) | $E_t$ | $\lvert\epsilon_t\rvert$% |
|---|---|---|---|
| 480 | −987.81 | 1635.4 | 252.54 |
| 240 | 110.32 | 537.26 | 82.964 |
| 120 | 546.77 | 100.80 | 15.566 |
| 60 | | 32.607 | 5.0352 |

$\theta(480) = 647.57K$ (exact)

# Comparison with exact results

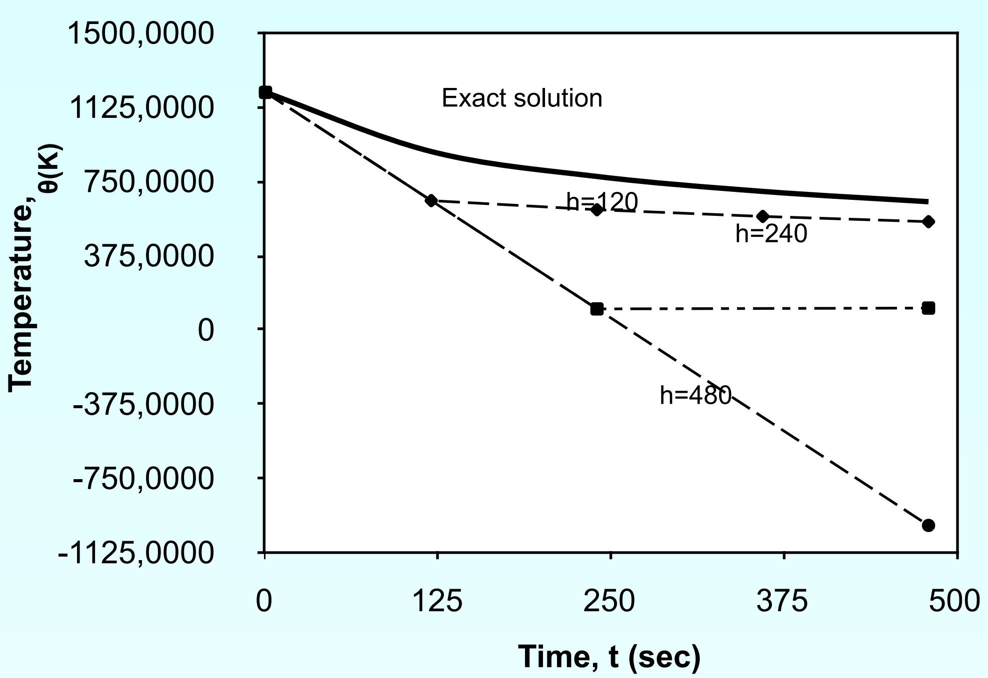


**Figure 4.** Comparison of Euler's method with exact solution for different step sizes

# Effects of step size on Euler's Method

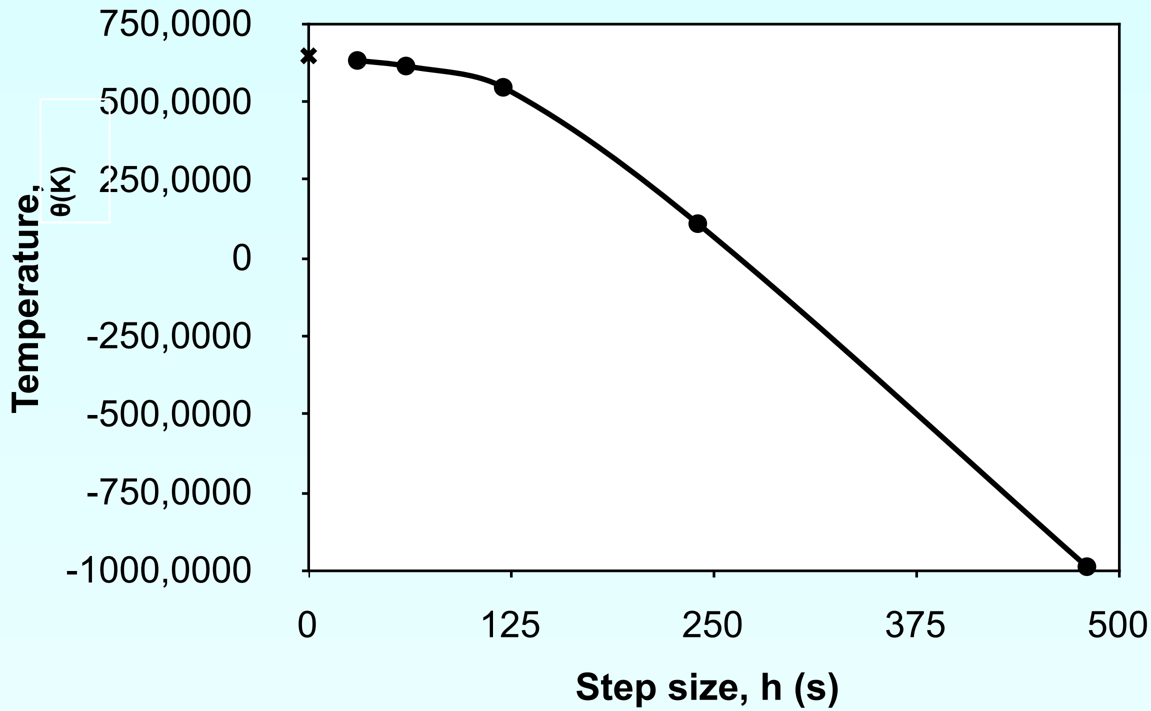


**Figure 5.** Effect of step size in Euler's method.

# Errors in Euler's Method

It can be seen that Euler's method has large errors. This can be illustrated using Taylor series.

$$y_{i+1} = y_i + \left.\frac{dy}{dx}\right|_{x_i, y_i} (x_{i+1} - x_i) + \frac{1}{2!} \left.\frac{d^2 y}{dx^2}\right|_{x_i, y_i} (x_{i+1} - x_i)^2 + \frac{1}{3!} \left.\frac{d^3 y}{dx^3}\right|_{x_i, y_i} (x_{i+1} - x_i)^3 + \ldots$$

$$y_{i+1} = y_i + f(x_i, y_i)(x_{i+1} - x_i) + \frac{1}{2!} f'(x_i, y_i)(x_{i+1} - x_i)^2 + \frac{1}{3!} f''(x_i, y_i)(x_{i+1} - x_i)^3 + \ldots$$

As you can see the first two terms of the Taylor series

$y_{i+1} = y_i + f(x_i, y_i)h$ are the Euler's method.

The true error in the approximation is given by

$$E_t = \frac{f'(x_i, y_i)}{2!} h^2 + \frac{f''(x_i, y_i)}{3!} h^3 + \ldots \qquad E_t \propto h^2$$

# Runge 2$^{nd}$ Order Method

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# Runge-Kutta $2^{nd}$ Order Method

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# Runge-Kutta 2nd Order Method

For $\frac{dy}{dx} = f(x, y), y(0) = y_0$

Runge Kutta 2nd order method is given by

$$y_{i+1} = y_i + \left(a_1 k_1 + a_2 k_2\right)h$$

where

$$k_1 = f\left(x_i, y_i\right)$$

$$k_2 = f\left(x_i + p_1 h, y_i + q_{11} k_1 h\right)$$

# Heun’s Method

<u>Heun’s method</u>

Here $a_2$=1/2 is chosen

$$a_1 = \frac{1}{2}$$

$$p_1 = 1$$

$$q_{11} = 1$$

resulting in

$$y_{i+1} = y_i + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$$

where

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + h, y_i + k_1 h)$$

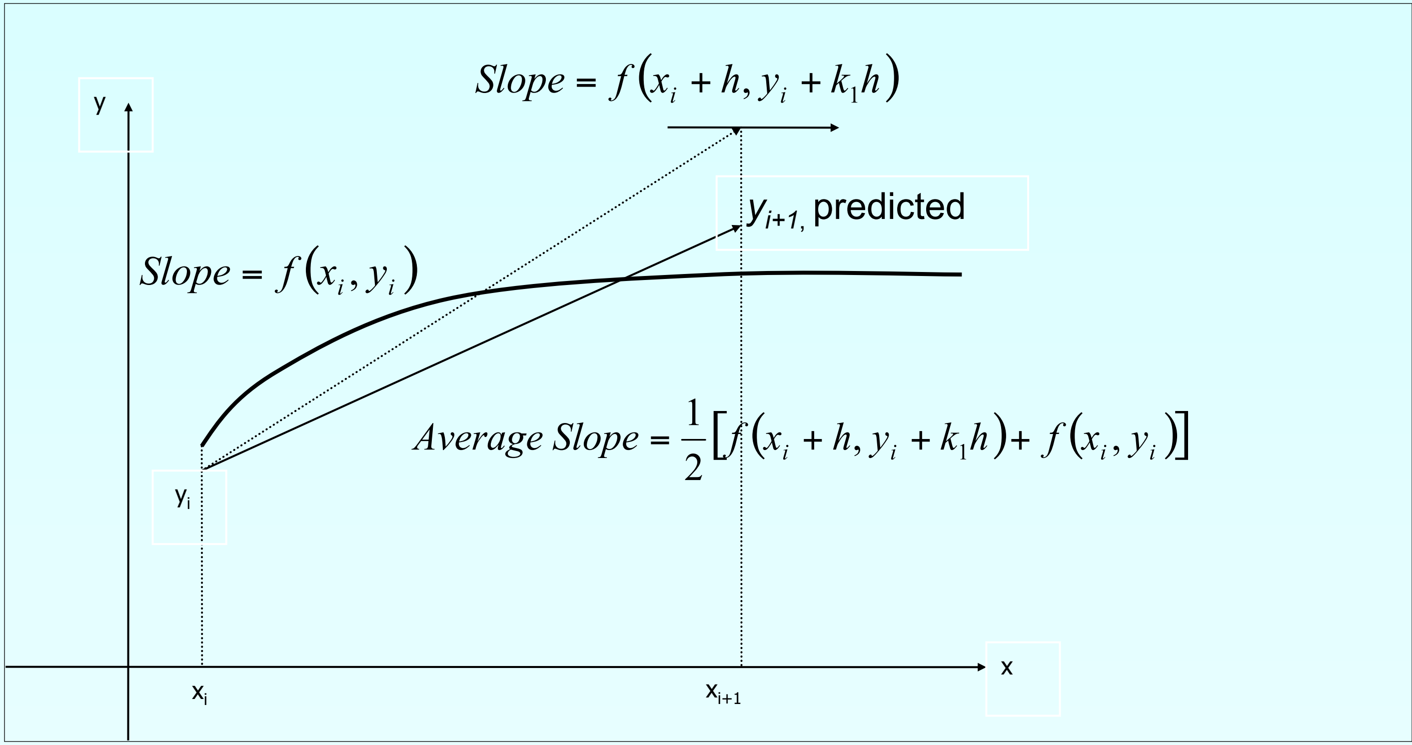


**Figure 1** Runge-Kutta 2nd order method (Heun’s method)

# Midpoint Method

**Here** $a_2 = 1$ **is chosen, giving**

$$a_1 = 0$$

$$p_1 = \frac{1}{2}$$

$$q_{11} = \frac{1}{2}$$

resulting in

$$y_{i+1} = y_i + k_2 h$$

where

$$k_1 = f\left(x_i, y_i\right)$$

$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right)$$

# Ralston's Method

Here $a_2 = \frac{2}{3}$ is chosen, giving

$$a_1 = \frac{1}{3}$$

$$p_1 = \frac{3}{4}$$

$$q_{11} = \frac{3}{4}$$

resulting in

$$y_{i+1} = y_i + \left(\frac{1}{3}k_1 + \frac{2}{3}k_2\right)h$$

where

$$k_1 = f\left(x_i, y_i\right)$$

$$k_2 = f\left(x_i + \frac{3}{4}h, y_i + \frac{3}{4}k_1 h\right)$$

# How to write Ordinary Differential Equation

How does one write a first order differential equation in the form of

$$\frac{dy}{dx} = f(x, y)$$

**Example**

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$$

is rewritten as

$$\frac{dy}{dx} = 1.3e^{-x} - 2y, y(0) = 5$$

In this case

$$f(x, y) = 1.3e^{-x} - 2y$$

# Example

A ball at 1200K is allowed to cool down in air at an ambient temperature of 300K. Assuming heat is lost only due to radiation, the differential equation for the temperature of the ball is given by

$$\frac{d\theta}{dt} = -2.2067\times 10^{-12}\left(\theta^4 - 81\times 10^8\right), \theta(0) = 1200K$$

Find the temperature at $t = 480$ seconds using Heun's method. Assume a step size of $h = 240$ seconds.

$$\frac{d\theta}{dt} = -2.2067\times 10^{-12}\left(\theta^4 - 81\times 10^8\right)$$

$$f(t,\theta) = -2.2067\times 10^{-12}\left(\theta^4 - 81\times 10^8\right)$$

$$\theta_{i+1} = \theta_i + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$$

# Solution

Step 1: $i = 0, t_0 = 0, \theta_0 = \theta(0) = 1200K$

$$
\begin{aligned}
k_1 &= f(t_0, \theta_o) \\
&= f(0, 1200) \\
&= -2.2067\times10^{-12}\left(1200^4 - 81\times10^8\right) \\
&= -4.5579
\end{aligned}
$$

$$
\begin{aligned}
k_2 &= f(t_0 + h, \theta_0 + k_1 h) \\
&= f(0 + 240, 1200 + (-4.5579)240) \\
&= f(240, 106.09) \\
&= -2.2067\times10^{-12}\left(106.09^4 - 81\times10^8\right) \\
&= 0.017595
\end{aligned}
$$

$$
\begin{aligned}
\theta_1 &= \theta_0 + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h \\
&= 1200 + \left(\frac{1}{2}(-4.5579) + \frac{1}{2}(0.017595)\right)240 \\
&= 1200 + (-2.2702)240 \\
&= 655.16K
\end{aligned}
$$

# Solution Cont

**Step 2:** $i = 1, t_1 = t_0 + h = 0 + 240 = 240, \theta_1 = 655.16K$

$$\begin{aligned} k_1 &= f(t_1, \theta_1) \\ &= f(240, 655.16) \\ &= -2.2067\times10^{-12}\left(655.16^4 - 81\times10^8\right) \\ &= -0.38869 \end{aligned}$$

$$\begin{aligned} k_2 &= f(t_1 + h, \theta_1 + k_1 h) \\ &= f(240 + 240, 655.16 + (-0.38869)240) \\ &= f(480, 561.87) \\ &= -2.2067\times10^{-12}\left(561.87^4 - 81\times10^8\right) \\ &= -0.20206 \end{aligned}$$

$$\begin{aligned} \theta_2 &= \theta_1 + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h \\ &= 655.16 + \left(\frac{1}{2}(-0.38869) + \frac{1}{2}(-0.20206)\right)240 \\ &= 655.16 + (-0.29538)240 \\ &= 584.27K \end{aligned}$$

# Solution Cont

The exact solution of the ordinary differential equation is given by the solution of a non-linear equation as

$$0.92593\ln\frac{\theta - 300}{\theta + 300} - 1.8519\tan^{-1}(0.00333333\theta) = -0.22067\times 10^{-3}t - 2.9282$$

The solution to this nonlinear equation at t=480 seconds is

$$\theta(480) = 647.57K$$

# Comparison with exact results

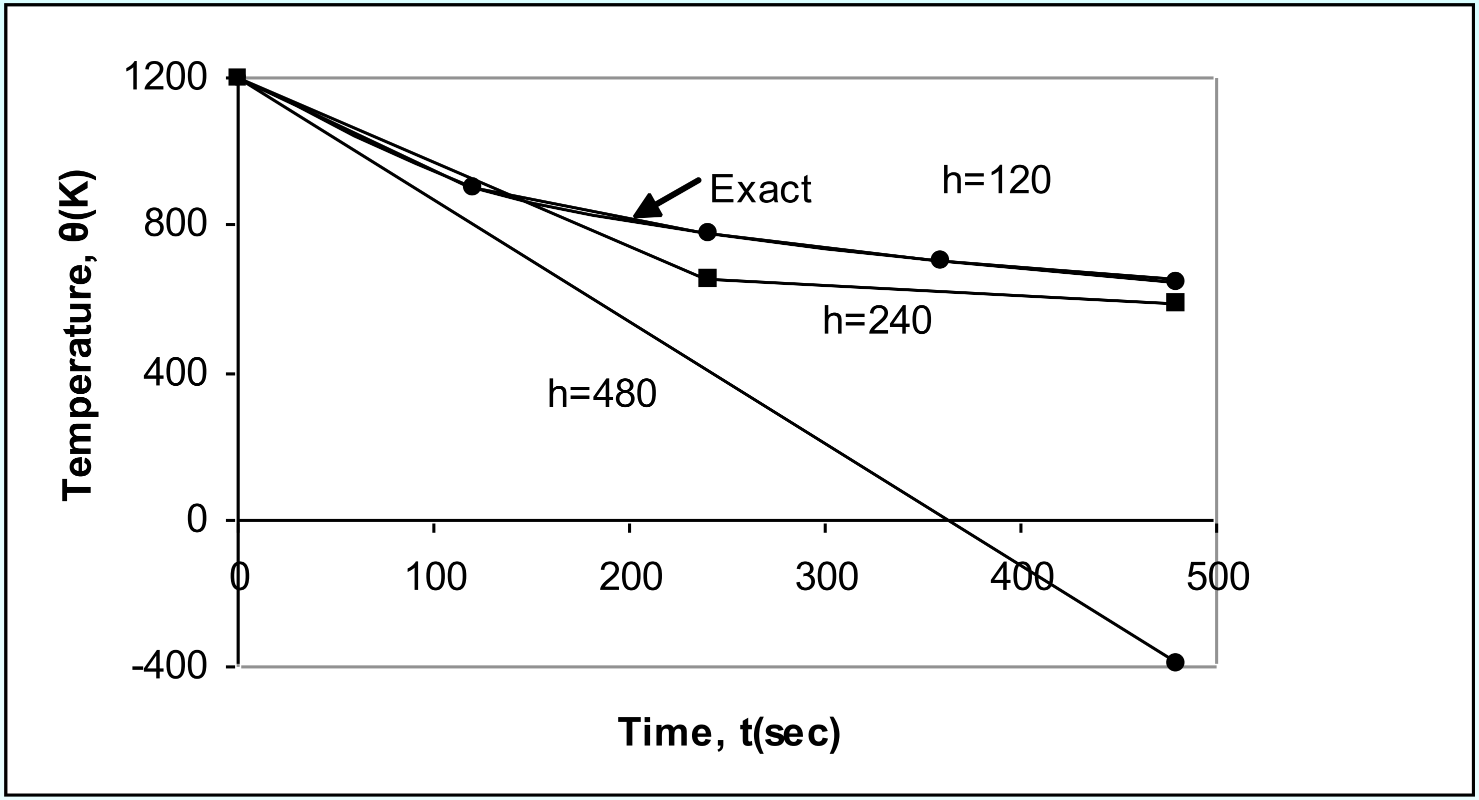


**Figure 2.** Heun's method results for different step sizes

# Effect of step size

**Table 1. Temperature at 480 seconds as a function of step size, h**

| Step size, $h$ | $\theta(480)$ | $E_t$ | $\lvert\epsilon_t\rvert\%$ |
|---|---|---|---|
| 480 | −393.87 | 1041.4 | 160.82 |
| 240 | 584.27 | 63.304 | 9.7756 |
| 120 | 651.35 | −3.7762 | 0.58313 |
| 60 | 649.91 | −2.3406 | 0.36145 |

$$\theta(480) = 647.57K$$ (exact)

# Effects of step size on Heun's Method

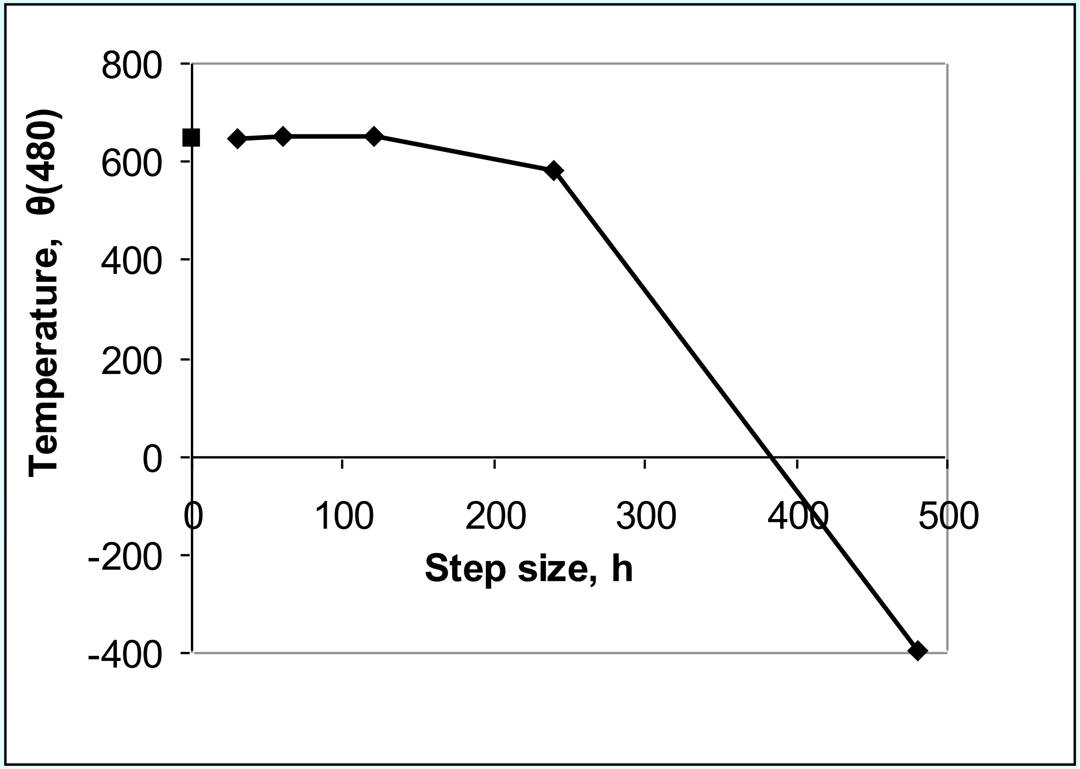


**Figure 3.** Effect of step size in Heun's method

# Comparison of Euler and Runge-Kutta 2nd Order Methods

**Table 2.** Comparison of Euler and the Runge-Kutta methods

| Step size, $h$ | $\theta(480)$ | | | |
|---|---|---|---|---|
| | Euler | Heun | Midpoint | Ralston |
| 480 | −987.84 | −393.87 | 1208.4 | 449.78 |
| 240 | 110.32 | 584.27 | 976.87 | 690.01 |
| 120 | 546.77 | 651.35 | 690.20 | 667.71 |
| 60 | 614.97 | 649.91 | 654.85 | 652.25 |

$$\theta(480) = 647.57K \quad \text{(exact)}$$

# Comparison of Euler and Runge-Kutta 2nd Order Methods

**Table 2**. Comparison of Euler and the Runge-Kutta methods

| Step size, h | $\left\|\in_t\right\|\%$ | | | |
|---|---|---|---|---|
| | Euler | Heun | Midpoin | Ralston |
| 480 | 252.54 | 160.82 | 86.612 | 30.544 |
| 240 | 82.964 | 9.7756 | 50.851 | 6.5537 |
| 120 | 15.566 | 0.58313 | 6.5823 | 3.1092 |
| 60 | 5.0352 | 0.36145 | 1.1239 | 0.7229 9 |
| 30 | 2.2864 | 0.097625 | 0.22353 | |

$$\theta(480) = 647.57K \quad \text{(exact)}$$

# Comparison of Euler and Runge-Kutta 2nd Order Methods

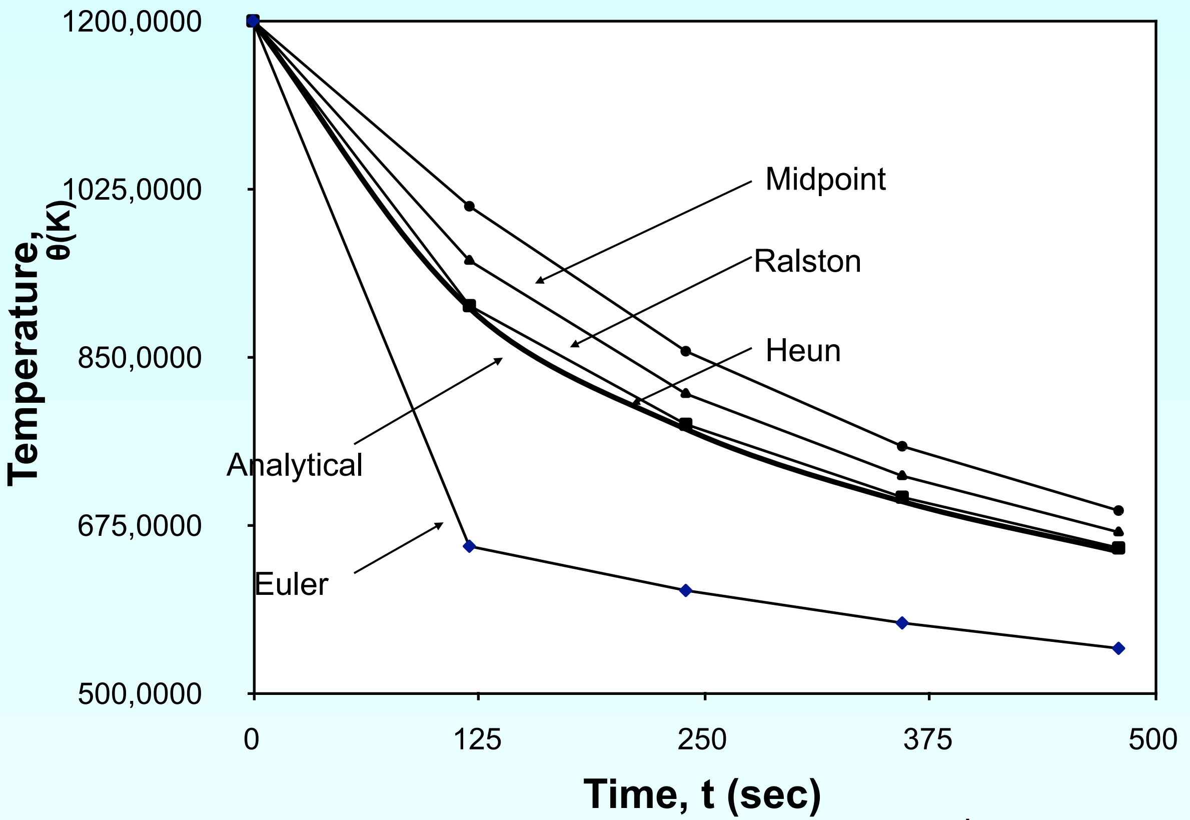


**Figure 4.** Comparison of Euler and Runge Kutta 2nd order methods with exact results.

# Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/runge_kutta_2nd_method.html

# Runge 4$^{th}$ Order Method

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# Runge-Kutta $4^{th}$ Order Method

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# Runge-Kutta 4th Order Method

For $\frac{dy}{dx} = f(x,y), y(0) = y_0$

Runge Kutta 4th order method is given by

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$

where

$$k_1 = f(x_i, y_i)$$

$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right)$$

$$k_3 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2 h\right)$$

$$k_4 = f(x_i + h, y_i + k_3 h)$$

# How to write Ordinary Differential Equation

How does one write a first order differential equation in the form of

$$\frac{dy}{dx} = f(x, y)$$

**Example**

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$$

is rewritten as

$$\frac{dy}{dx} = 1.3e^{-x} - 2y, y(0) = 5$$

In this case

$$f(x, y) = 1.3e^{-x} - 2y$$

# Example

A ball at 1200K is allowed to cool down in air at an ambient temperature of 300K. Assuming heat is lost only due to radiation, the differential equation for the temperature of the ball is given by

$$\frac{d\theta}{dt} = -2.2067\times10^{-12}\left(\theta^4 - 81\times10^8\right),\ \theta(0) = 1200K$$

Find the temperature at $t = 480$ seconds using Runge-Kutta 4th order method.

Assume a step size of $h = 240$ seconds.

$$\frac{d\theta}{dt} = -2.2067\times10^{-12}\left(\theta^4 - 81\times10^8\right)$$

$$f(t,\theta) = -2.2067\times10^{-12}\left(\theta^4 - 81\times10^8\right)$$

$$\theta_{i+1} = \theta_i + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)h$$

# Solution

Step 1: $i = 0, t_0 = 0, \theta_0 = \theta(0) = 1200$

$$k_1 = f(t_0, \theta_o) = f(0,1200) = -2.2067\times10^{-12}\left(1200^4 - 81\times10^8\right) = -4.5579$$

$$k_2 = f\left(t_0 + \frac{1}{2}h, \theta_0 + \frac{1}{2}k_1 h\right) = f\left(0 + \frac{1}{2}(240), 1200 + \frac{1}{2}(-4.5579)240\right)$$
$$= f(120, 653.05) = -2.2067\times10^{-12}\left(653.05^4 - 81\times10^8\right) = -0.38347$$

$$k_3 = f\left(t_0 + \frac{1}{2}h, \theta_0 + \frac{1}{2}k_2 h\right) = f\left(0 + \frac{1}{2}(240), 1200 + \frac{1}{2}(-0.38347)240\right)$$
$$= f(120, 1154.0) = 2.2067\times10^{-12}\left(1154.0^4 - 81\times10^8\right) = -3.8954$$

$$k_4 = f(t_0 + h, \theta_0 + k_3 h) = f(0 + (240), 1200 + (-3.984)240)$$
$$= f(240, 265.10) = 2.2067\times10^{-12}\left(265.10^4 - 81\times10^8\right) = 0.0069750$$

# Solution Cont

$$\theta_1 = \theta_0 + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)h$$

$$= 1200 + \frac{1}{6}\left(-4.5579 + 2\left(-0.38347\right) + 2\left(-3.8954\right) + \left(0.069750\right)\right)240$$

$$= 1200 + \frac{1}{6}\left(-2.1848\right)240$$

$$= 675.65K$$

$\theta_1$ is the approximate temperature at

$$t = t_1 = t_0 + h = 0 + 240 = 240$$

$$\theta\left(240\right) \approx \theta_1 = 675.65K$$

# Solution Cont

**Step 2:** $i = 1, t_1 = 240, \theta_1 = 675.65K$

$$k_1 = f(t_1, \theta_1) = f(240, 675.65) = -2.2067 \times 10^{-12}(675.65^4 - 81 \times 10^8) = -0.44199$$

$$k_2 = f\left(t_1 + \frac{1}{2}h, \theta_1 + \frac{1}{2}k_1 h\right) = f\left(240 + \frac{1}{2}(240), 675.65 + \frac{1}{2}(-0.44199)240\right)$$

$$= f(360, 622.61) = -2.2067 \times 10^{-12}(622.61^4 - 81 \times 10^8) = -0.31372$$

$$k_3 = f\left(t_1 + \frac{1}{2}h, \theta_1 + \frac{1}{2}k_2 h\right) = f\left(240 + \frac{1}{2}(240), 675.65 + \frac{1}{2}(-0.31372)240\right)$$

$$= f(360, 638.00) = 2.2067 \times 10^{-12}(638.00^4 - 81 \times 10^8) = -0.34775$$

$$k_4 = f(t_1 + h, \theta_1 + k_3 h) = f(240 + (240), 675.65 + (-0.34775)240)$$

$$= f(480, 592.19) = 2.2067 \times 10^{-12}(592.19^4 - 81 \times 10^8) = -0.25351$$

# Solution Cont

$$\theta_2 = \theta_1 + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)h$$

$$= 675.65 + \frac{1}{6}\left(-0.44199 + 2\left(-0.31372\right) + 2\left(-0.34775\right) + \left(-0.25351\right)\right)240$$

$$= 675.65 + \frac{1}{6}\left(-2.0184\right)240$$

$$= 594.91K$$

$\theta_2$ is the approximate temperature at

$$t_2 = t_1 + h = 240 + 240 = 480$$

$$\theta\left(480\right) \approx \theta_2 = 594.91K$$

# Solution Cont

The exact solution of the ordinary differential equation is given by the solution of a non-linear equation as

$$0.92593\ln\frac{\theta-300}{\theta+300}-1.8519\tan^{-1}\left(0.00333\theta\right)=-0.22067\times10^{-3}t-2.9282$$

The solution to this nonlinear equation at t=480 seconds is

$$\theta(480)=647.57K$$

# Comparison with exact results

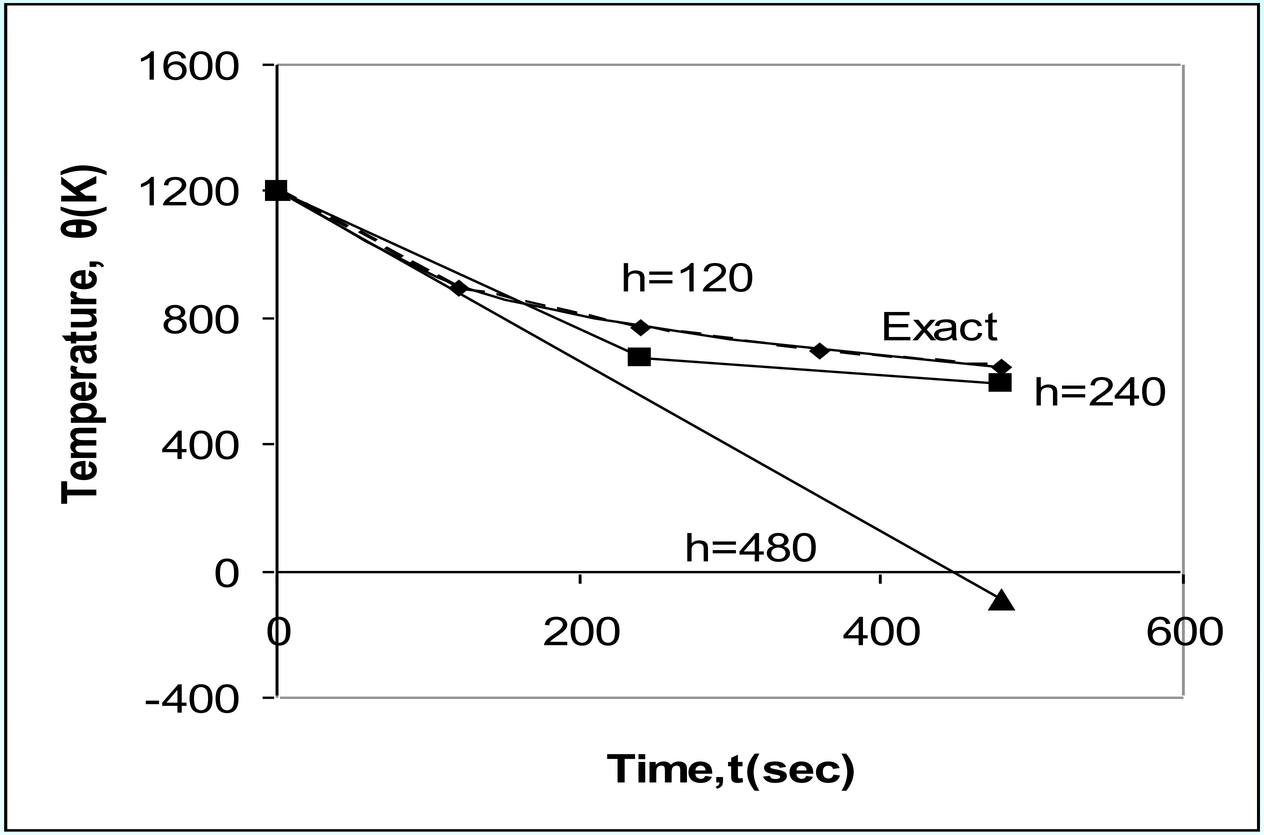


**Figure 1.** Comparison of Runge-Kutta 4th order method with exact solution

# Effect of step size

**Table 1. Temperature at 480 seconds as a function of step size, h**

| Step size, h | θ (480) | $E_t$ | $\vert\epsilon_t\vert\%$ |
|---|---|---|---|
| 480 | −90.278 | 737.85 | 113.94 |
| 240 | 594.91 | 52.660 | 8.1319 |
| 120 | 646.16 | 1.4122 | 0.21807 |
| 60 | 647.54 | 0.033626 | 0.0051926 |
| 30 | 647.57 | 0.00086900 | 0.00013419 |

$\theta(480) = 647.57K$ (exact)

# Effects of step size on Runge-Kutta 4$^{th}$ Order Method

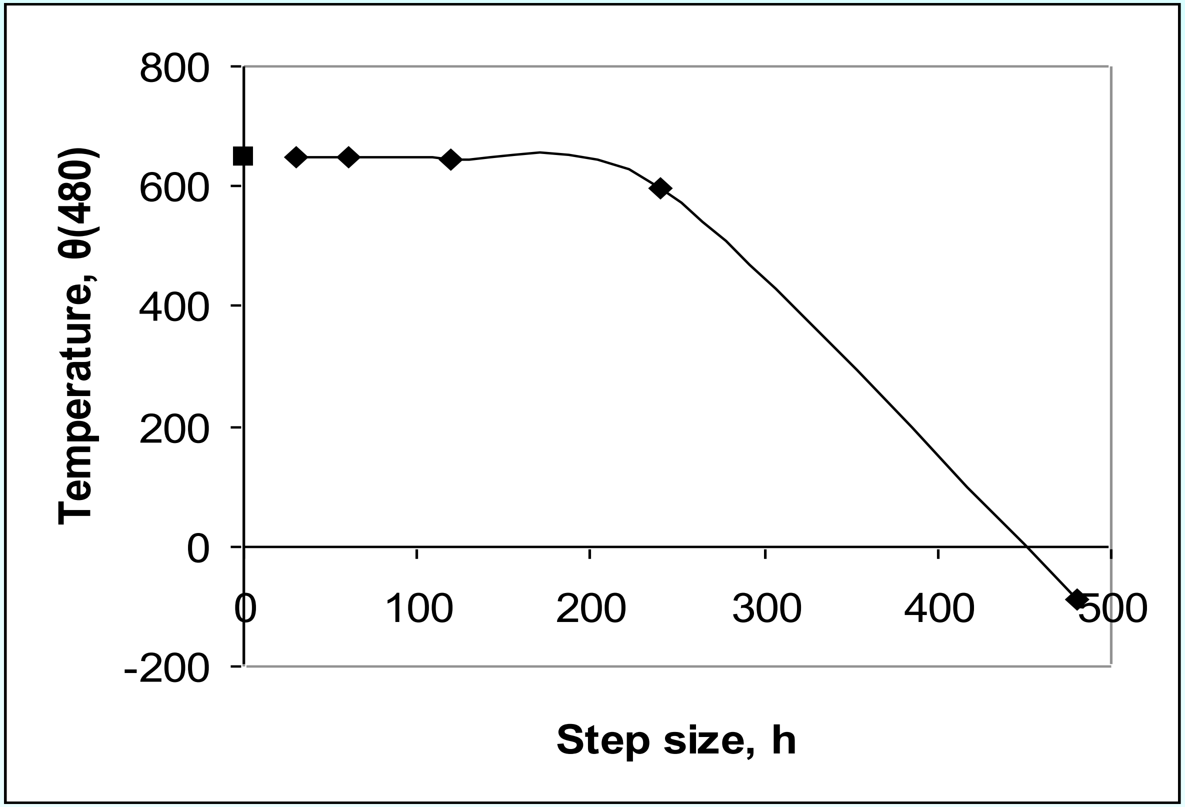


**Figure 2.** Effect of step size in Runge-Kutta 4th order method

# Comparison of Euler and Runge-Kutta Methods

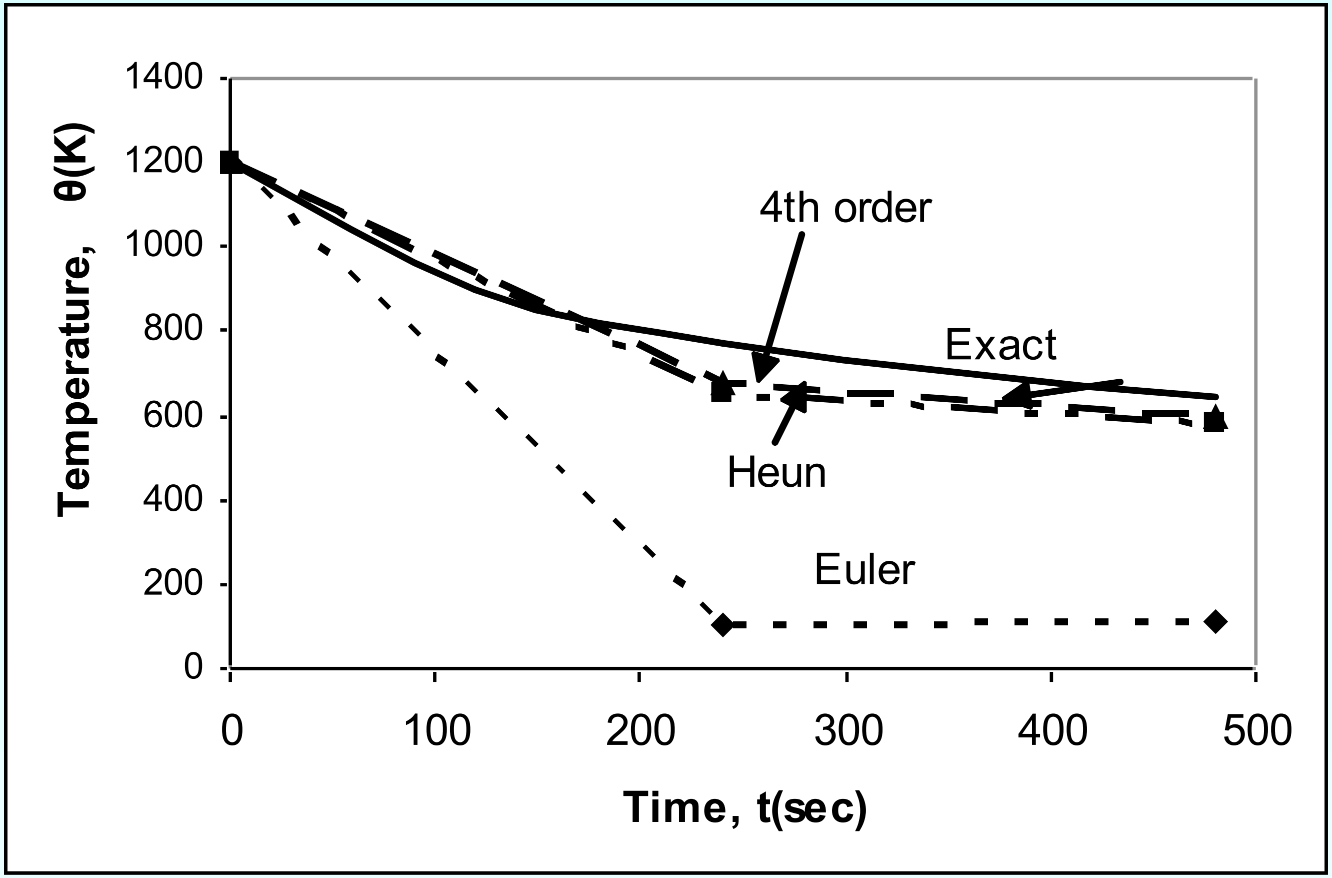


**Figure 3.** Comparison of Runge-Kutta methods of 1st, 2nd, and 4th order.

# THE END