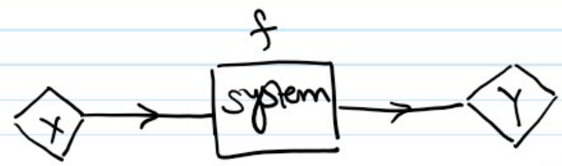
# L-4

21 April 2021 12:56

## Range of a Function:




input (domain) output (Range)

$f(x) = x^2$

$x = 2$ ④

$x = -1$ 1

For Range of a function first of all we shall determine domain of the function.

Domain of the function may be

① Separate elements

(2) In interval $[a, b]$

(3) Real number or set of real number.

[1] For Separate elements:

Let function $y = f(x)$ is defined for sepate elements $x = \{x_1, x_2, x_3, \dots x_n\}$

$$\text{Range} = \{ f(x_1), f(x_2), f(x_3), \dots, f(x_n) \}$$

Ex. 1. Find range of the function.

$$f(x) = {}^{7-x}P_{x-3}$$

Solution:

$7 - x > 0$ $x < 7$ $3 \leq x \leq 5$

$x - 3 \geq 0$ $x \geq 3$ $x \in I$

$7 - x \geq x - 3$ $\boxed{x = \{3, 4, 5\}}$

$10 \geq 2x \Rightarrow \boxed{x \leq 5}$

$7 - x \geq x - 3$

$10 \geq 2x \Rightarrow \boxed{x \leq 5}$ $\quad \boxed{x = \{3, 4, 5\}}$

$$\text{Range} = \{f(3), f(4), f(5)\}$$

$$= \{{}^4P_0, {}^3P_1, {}^2P_2\}$$

$$\text{Range} = \{24, 3, 1\}$$

Ex: find range of the function.

$$f(x) = \sec^{-1}x + \sin^{-1}x$$

Sol$^n$: given $f(x) = \sec^{-1}x + \sin^{-1}x$


$$\boxed{\begin{matrix} x \geq 1 \\ \text{or } x \leq -1 \end{matrix}} \cap \boxed{-1 \leq x \leq 1}$$

domain: $x = \{-1, 1\}$

$$\text{Range} = \{f(-1), f(1)\}$$

$$f(-1) = \sec^{-1}(-1) + \sin^{-1}(-1)$$

$$= \pi + (-\pi/2) = \pi/2$$

$$f(1) = \sec^{-1}(1) + \sin^{-1}(1)$$

$$= 0 + \pi/2 = \pi/2$$

$$\text{Range} = \{\pi/2\}$$

[2] An interval $[a, b]$: If domain of the function $y = f(x)$ is an interval $[a, b]$ in this case we find out maximum and minimum value of $f(x)$ in $[a, b]$ by using concept of maxima & minima or by using

minimum value of f(x) in [a,b] by using concept of maxima & minima or by using A.M ≥ G.M ≥ H.M or by any other method.

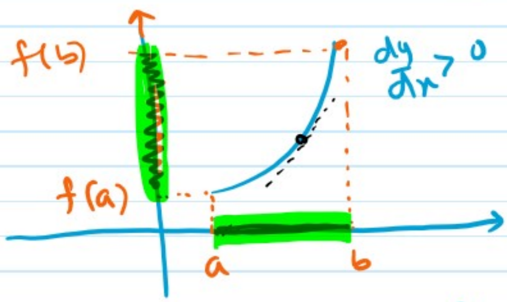
Range = [fmin, fmax]

Concept of maxima & minima:

step I: find $\frac{dy}{dx} = f'(x)$

step II: If $\frac{dy}{dx} > 0$ or $\frac{dy}{dx} < 0$ $\forall\ x \in (a,b)$

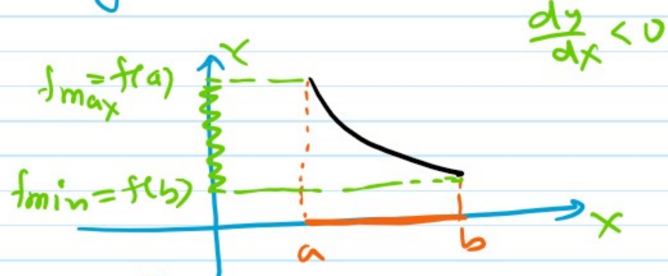
(A)




Range = [f(a), f(b)]

OR

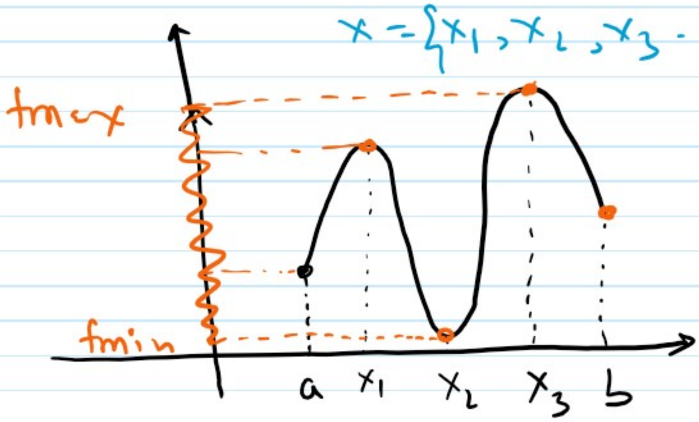
(B)




Range = [fmin, fmax]

step III If $\frac{dy}{dx} = 0$ for some real values of x in (a,b)

$x = \{x_1, x_2, x_3 \ldots\}$ (say)



working method: Then find out

$$\left.\begin{aligned} f(a) &= \text{- - - -} \\ f(x_1) &= \text{- - -} \\ f(x_2) &= \text{- - -} \\ &\vdots \\ f(x_n) &= \dots \\ f(b) &= \text{- - - -} \end{aligned}\right\}$$ Then find out fmin & fmax

Range = [fmin, fmax]

Ex1. find range of the function

$$f(x) = \sqrt{x-2} + \sqrt{6-x}$$

sol$^n$

$$f(x) = \sqrt{x-2} + \sqrt{6-x}$$

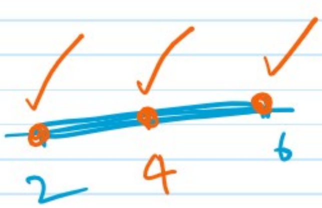
$$x - 2 \geq 0 \quad \cap \quad 6 - x \geq 0$$

$$x \geq 2 \quad \cap \quad x \leq 6$$

Domain: $x \in [2, 6]$

$$f'(x) = \frac{1}{2}(x-2)^{1/2-1}(1) + \frac{1}{2}(6-x)^{1/2-1}(-1)$$

$$= \frac{1}{2}\frac{1}{\sqrt{x-2}} - \frac{1}{2\sqrt{6-x}}$$




$f'(x) = 0$

$$\frac{1}{2\sqrt{x-2}} = \frac{1}{2\sqrt{6-x}}$$

$$x - 2 = 6 - x$$

$\boxed{x = 4}$ lies in $[2, 6]$

$$f(2) = 0 + 2$$

$$f(4) = \sqrt{2} + \sqrt{2} = 2\sqrt{2}$$

$f(4) = \sqrt{2} + \sqrt{2} = 2\sqrt{2}$

$f(6) = 2$

$\text{Range} = [2, 2\sqrt{2}]$

Ex. find range of the function.

$$f(x) = \sin^{-1}x + \cos^{-1}x + \tan^{-1}x$$

sol$^n$

$$f(x) = \sin^{-1}x + \cos^{-1}x + \tan^{-1}x$$

$$-1 \le x \le 1 \;\cap\; -1 \le x \le 1 \;\cap\; (-\infty, \infty)$$

$$x \in [-1, 1]$$

$$f'(x) = \cancel{\frac{1}{\sqrt{1-x^2}}} - \cancel{\frac{1}{\sqrt{1-x^2}}} + \frac{1}{1+x^2}$$

$$f'(x) = \frac{1}{1+x^2} \qquad \boxed{f'(x) > 0}$$

$$f(-1) = -\pi/2 + \pi + (-\pi/4) = \pi/4$$

$$f(1) = \pi/2 + 0 + \pi/4 = \frac{3\pi}{4}$$

$$\text{Range} = \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$$

Ex. find range of the function.

$$f(x) = \sin^2 x - \sin x + 1$$

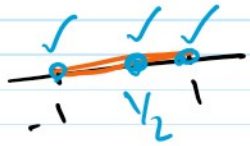
sol$^n$

$$y = \sin^2 x - \sin x + 1$$

let $\sin x = t$

$$y = t^2 - t + 1 \;; \qquad t \in [-1, 1]$$

$\frac{dy}{dt} = 2t - 1 \qquad \frac{dy}{dt} = 0 \qquad 2t - 1 = 0$




$\frac{dy}{dt} = 2t - 1$ $\quad\quad$ $\frac{dy}{dt} = 0, \quad 2t - 1 = 0$

$t = 1/2$

$Y(-1) = 1 + 1 + 1 = 3$

$Y(1/2) = \frac{1}{4} - \frac{1}{2} + 1 = 3/4$

$Y(1) = 1 - 1 + 1 = 1$

Range $= [3/4, 3]$

Ex: find range of the function.

$f(x) = \sqrt{25 - x^2}$

sol$^n$

$25 - x^2 \geq 0$

$x^2 - 25 \leq 0$

$(x-5)(x+5) \leq 0$

$x \in [-5, 5]$

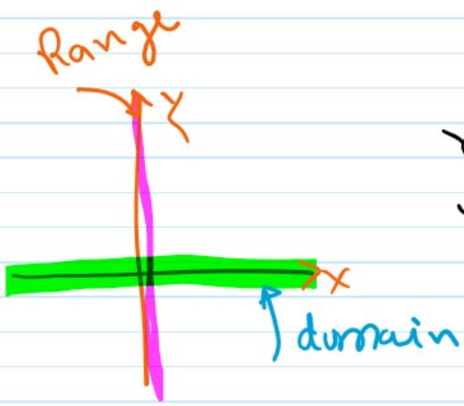
$f'(x) = \frac{1}{2\sqrt{25-x^2}}(-2x)$

$f'(x) = 0$ $\quad\quad$ $\boxed{x = 0}$

$f(-5) = 0$ $\quad\quad$ $f(0) = 5$

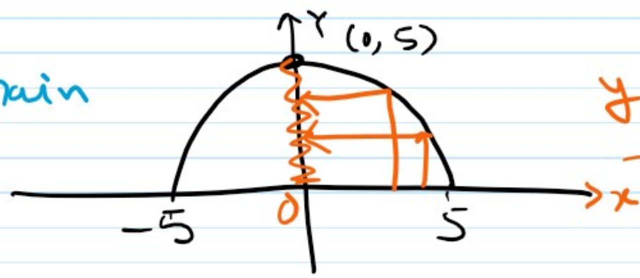
$f(5) = 0$

Range $= [0, 5]$




$Y = \sqrt{25 - x^2}$ $\quad\quad$ $Y \geq 0$

$y^2 = 25 - x^2$ ; $\quad x^2 + y^2 = 25$




$y \in [0, 5]$

[3] Real number or set of real numbers:

In this case we find out a condition in $y$ by using condition of real $x$ in different cases.

[A] If $\dfrac{y}{1} = \dfrac{\text{Linear}}{\text{quadratic}}$ or $\dfrac{\text{quadratic}}{\text{Quadratic}}$

Then transform above relation as:

$$y(\text{quadratic}) = 1\cdot(\text{Linear})$$

$$Ax^2 + Bx + C = 0$$

For real $x$ $\boxed{B^2 - 4AC \geq 0}$ Condition in $y$

Ex. find Range of the function.

$$f(x) = \frac{x}{x^2+1}$$

sol$^n$. Domain $x \in R$.

$$\frac{y}{1} = \frac{x}{1+x^2}$$

$$y + y\cdot x^2 = x$$

$$y\cdot x^2 - x + y = 0$$

$$Ax^2 + Bx + C = 0$$

$A = y$

$B = -1$

$C = y$

$$(-1)^2 - 4\cdot y\cdot y \geq 0$$

$$1 - 4y^2 \geq 0$$

$$4y^2 - 1 \leq 0$$

$$y^2 - \frac{1}{4} \leq 0$$

$$\left(y-\frac{1}{2}\right)\left(y+\frac{1}{2}\right) \leq 0$$


$$y \in \left[-\frac{1}{2}, \frac{1}{2}\right]$$

Ex. 2. $Y = \dfrac{x^2}{\phantom{x}}$

Ex. 2.

$$Y = \frac{x^2}{x^2+x+1}$$

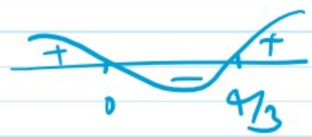
$$y \cdot x^2 + y \cdot x + y = x^2$$

$$(y-1)x^2 + y \cdot x + y = 0$$

$$y^2 - 4 \cdot (y-1) \cdot y \geq 0$$

$$y\,[y - 4y + 4] \geq 0$$

$$y\,(4-3y) \geq 0$$




$$y\,(3y-4) \leq 0$$

$$y \in [0, 4/3]$$

[B] $y = \dfrac{\text{Linear}}{\text{Linear}}$ ;

$$Y = \frac{ax+b}{cx+d}$$

domain $cx+d \neq 0$

$x \neq -d/c$

$x \in R - \{-d/c\}$

$$Y = \lim_{x \to \infty} \frac{ax+b}{cx+d} = \lim_{x \to \infty} \frac{a+b/x}{c+d/x} = a/c$$

$$\boxed{y \in R - \{a/c\}}$$

Ex:

$$y = \frac{x-1}{2x-1}$$

$$\boxed{y \in R - \{1/2\}}$$

$$2yx - y = x - 1$$

$$2yx - x = y - 1$$

$$x = \left(\frac{y-1}{2y-1}\right)$$

$2y - 1 \neq 0$

$$x = \left(\frac{y-1}{2y-1}\right)$$ $\quad 2y-1 \neq 0$

$y \neq 1/2$

[C] $\quad y = f(x^2)$ $\quad$ ex. $\quad y = \dfrac{1}{x^2-1}$, $\quad y = \dfrac{x^2}{x^2+1}$

$y = x^2 - 1$, $\quad y = \dfrac{x^2-1}{x^2+1}$

working method : $\quad y = \dfrac{1}{2-x^2}$ $\quad$ etc.

Step1: find out value of $x^2$ as a function of $y$.

$x^2 = g(y)$ (say)

as $x^2 \geq 0$

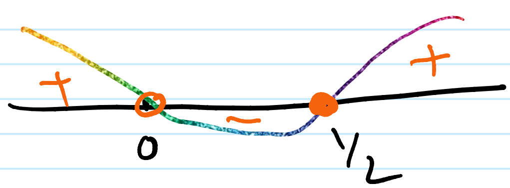
$\Rightarrow \boxed{g(y) \geq 0}$

Ex. $\quad y = \dfrac{1}{2-x^2}$ $\qquad x \in R - \{\pm\sqrt{2}\}$

$2 - x^2 = \dfrac{1}{y}$ $\qquad x^2 \geq 0$

$2 - \dfrac{1}{y} = x^2$ $\qquad 2 - \dfrac{1}{y} \geq 0$

$\dfrac{2y-1}{y} \geq 0$ $\qquad y \neq 0$




$\boxed{y \in (-\infty, 0) \cup [\tfrac{1}{2}, \infty)}$

[D] $\quad y = f(\{x\})$ $\quad$ e.g. $\quad \boxed{y = \dfrac{\{x\}}{1+\{x\}}}$, $\quad y = \{x\}^2 + 1$

Then find out

$\{x\} = g(y)$ $\qquad$ we know that $\quad 0 \leq \{x\} < 1$

$\boxed{0 \leq g(y) < 1}$

Ex. find range of the function

$$y = \frac{\{x\}}{1+\{x\}}$$

sol$^n$.

$$\frac{y}{1} = \frac{\{x\}}{1+\{x\}} \Rightarrow y + y\cdot\{x\} = \{x\}$$

$$y = (1-y)\{x\}$$

$$\{x\} = \frac{y}{1-y}$$

$$\boxed{0 \le \frac{y}{1-y} < 1}$$

I  II

I:

$$\frac{y}{1-y} \ge 0$$

$$\frac{y}{y-1} \le 0$$


$0 \le y < 1$

II:

$$\frac{y}{1-y} - 1 < 0$$

$$\frac{y-1+y}{1-y} < 0$$

$$\frac{2y-1}{y-1} > 0$$


$y \in (-\infty, \frac{1}{2}) \cup (1, \infty)$

$0 \le y < 1/2$

Range $\in [0, 1/2)$

Method: let $\{x\} = t$

$$y = \frac{t}{1+t}; \quad t \in [0,1)$$

$$\frac{dy}{dt} = \frac{(1+t)\cdot 1 - t\cdot 1}{(1+t)^2} = \frac{1}{(1+t)^2}$$

$$y(0) = \frac{0}{1+0} = 0$$

$$y(1) = \frac{1}{1+1} = \frac{1}{2}$$

$$y(1) = \frac{1}{1+1} = \frac{1}{2}$$

Range $\in [0, \frac{1}{2})$