JEE ADVANCED

Class XI

Chapter 1

Gravitation

The interaction between the masses of two bodies is known as the gravitation.

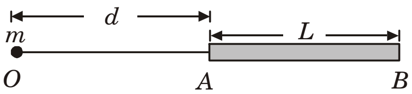
Newton's law of gravitation

- Any two particles in the universe attract each other with a force which is directly proportional to the product of their masses and inversely proportional to the square of the distance between them and it acts along the line joining the masses, i.e., if two particles of masses m_1 and m_2 be separated by a distance r, the force of attraction F is given by,

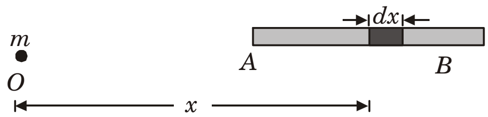
$$F = G\frac{m_1 m_2}{r^2}$$

- G is the universal gravitational constant. It is basically a conversion factor to adjust the number and units so that they come out to the correct value. $G = 6.7 \times 10^{-11}$ Nm^2/kg^2
  The dimension of G is given by $[M^{-1}L^3T^{-2}]$

Illustration 1 : Find the gravitational force in micronewton of attraction on the point mass m placed at O by a thin rod of mass M and length L as shown in figure.
[Take m = 10 kg, $G = 6.6 \times 10^{-11}$ Nm^2/kg^2, M = 1000 kg, L = 1 m, d = 10 cm



Soln.:



First we need to find the force due to an element of length dx. The mass of the element is $dm = \left(\frac{M}{L}\right)dx$

so, $dF = G\frac{Mm}{L}\frac{dx}{x^2}$

$\therefore$ The net gravitational force is $F = \frac{GMm}{L}\int_d^{d+L}\frac{dx}{x^2} = \frac{GMm}{L}\left[\frac{1}{d} - \frac{1}{L+d}\right] = \frac{GMm}{d(L+d)} = 6\ \mu N$

Notice that when $d >> L$, we find $F = \frac{GMm}{d^2}$, the result for two point masses.

Gravitational attraction between heavenly bodies

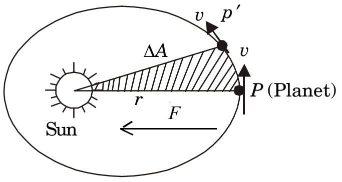
- Newton's law of gravitation is a universal law. It can explain the motion of heavenly bodies like the planets, the moon and the sun. The gravitational attraction between the sun and its planets provides the necessary centripetal force for the planets to describe the circular motion round the sun. Gravitational force of attraction between the earth and the sun is given by,

$$F = 3.53 \times 10^{22} \text{ newton.}$$

- Gravitational attraction between earth and the moon $= 1.976 \times 10^{20}$ newton.

Kepler's law of planetary motion

- **First law.** The orbit of an object moving around another in space is elliptical with the stationary objects located at one of the focal points of the ellipse.



In other words the earth travels around the sun in an ellipse, and the sun is at a focal point of that ellipse.

- **Second law.** The position vector from the sun to the planet sweeps out equal area in equal time, i.e., areal velocity (the area covered by the position vector of planet per unit time) of a planet around the sun always remains constant.

$$\frac{\Delta A}{\Delta t} = \text{Constant}$$

- **Third law.** The square of the time period of a planet around the sun is proportional to the cube of the semi-major axis of the ellipse or mean distance of the planet from the sun. i.e.

$$T^2 \propto a^3$$

Illustration 2 : A saturn year is 29.5 times the earth year. How far is saturn from the sun (M) if the earth is 1.5×10^8 km away from the sun?

Soln.: It is given that $T_s = 29.5\ T_e$; $R_e = 1.5 \times 10^{11}$ m

Now according to Kepler's third law $\frac{T_s^2}{T_e^2} = \frac{R_s^3}{R_e^3}$

$$R_s = R_e\left(\frac{T_s}{T_e}\right)^{2/3} = 1.5\times10^{11}\left(\frac{29.5\ T_e}{T_e}\right)^{2/3} = 1.43\times10^{12}\ \text{m} = 1.43\times10^{9}\ \text{km}$$

Gravity and acceleration due to gravity

- Gravity is the force of attraction exerted by the earth towards its centre on a body lying on or near the surface of earth. Gravity is basically a special case of gravitation and is also known as earth's gravitation.
- When a body is falling under the action of gravity the body would possess an acceleration which is called acceleration due to gravity, usually denoted by the letter 'g'.

For a body of mass 'm'

Force due to gravity, $F = mg$ ...(1)

From Newton's law of gravitation,

$$F = G\frac{M_e m}{R_e^2} \quad ...(2)$$

M_e = Mass of the earth

R_e = Radius of the earth

From (1) and (2), $mg = G\frac{M_e m}{R_e^2}$

$$g = \frac{GM_e}{R_e^2}$$

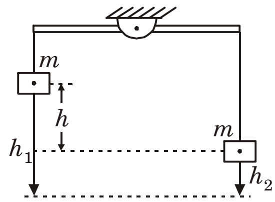
Variation of acceleration due to gravity

- **Variation with altitude:** Let us consider an object of mass m at a height h above the surface of the earth. If 'g' be the acceleration due to gravity at this place, then,

$$g' = \left(1 - \frac{2h}{R_e}\right)g$$

Thus, the decrease in the acceleration due to gravity $= g - g' = \frac{2h}{R_e}g$

Illustration 3 : Two equal masses m and m are hung from a balance whose scale pans differ in vertical height by h. Calculate the error in weighing, if any in terms of density of earth ρ.



Soln.: As with height g varies as $g' = \dfrac{g}{[1+h/R]^2} = g\left[1 - \dfrac{2h}{R}\right]$

and in accordance with figure, $h_1 > h_2$,

so W_1 will be lesser than W_2 and $W_2 - W_1 = mg_2 - mg_1 = 2mg\left[\dfrac{h_1}{R} - \dfrac{h_2}{R}\right]$

or $\quad W_2 - W_1 = 2m\dfrac{GM}{R^2}\dfrac{h}{R} \quad \left[\text{as } g = \dfrac{GM}{R^2} \text{ and } (h_1 - h_2) = h\right]$

or $\quad W_2 - W_1 = \dfrac{2mhG}{R^3} \times \left(\dfrac{4}{3}\pi R^3 \rho\right) = \dfrac{8}{3}\pi\rho Gmh \quad \left[\text{as } M = \dfrac{4}{3}\pi R^3 \rho\right]$

- **Variation with depth:** For an object of mass m at a depth h below the surface of the earth,

$$\boxed{g' = \left(1 - \frac{h}{R_e}\right)g}$$

Decrease in acceleration due to gravity at this place $= g - g' = \dfrac{h}{R_e}g$

- **Variation of 'g' due to shape of the earth:** The earth is not a perfect sphere. The diameter in the equator plane is about 21 km more than the diameter along the poles. Due to this, the acceleration due to gravity is more at the poles and less at the equator.
- **Variation of 'g' due to rotation of the earth:** The earth rotates about its own axis from west to east. All the bodies on the surface of the earth, except those lying on the poles, execute circular motion about the axis of the earth. If λ be the latitude at any place, then the value of acceleration due to gravity at the place is given by,

$$\boxed{g' = g - \omega^2 R_e \cos^2\lambda}$$

where ω is the angular velocity of motion of the earth around its polar axis.

At the equator, $\quad \lambda = 0$

$\qquad g' = g - \omega^2 R_e$

At the poles, $\quad \lambda = 90°$

$\qquad g' = g$

Illustration 4 : A body is suspended on a spring balance in ship sailing along the equator with a speed v. Show that scale reading will be very close to $W_0(1 \pm 2\omega v/g)$ where ω is the angular speed of the earth and W_0 is the scale reading when the ship is at rest. Explain also the significance of plus or minus sign.

Soln.: We know that at equator due to rotation of earth $g' = (g - R\omega^2)$, so if m is the mass of body

$$W_0 = m(g - R\omega^2) = m\left[g - \frac{V^2}{R}\right] \quad [\text{as } V = R\omega] \qquad .....(i)$$

Now when the ship is moving along the equator with a speed v, the speed of ship relative to the centre of earth will be $V \pm v$. The plus sign holds if the ship is moving in the direction of motion of earth, *i.e.*, west to east and minus if the ship is moving in opposite direction. So the weight recorded by the spring balance in the moving ship will be

$$W = m\left[g - \frac{(V \pm v)^2}{R}\right] \qquad .....(ii)$$

Dividing equation (ii) by (i), $\frac{W}{W_0} = \left[1 - \frac{(V \pm v)^2}{gR}\right]\left[1 - \frac{V^2}{gR}\right]^{-1}$

or $\frac{W}{W_0} = \left[1 - \frac{(V \pm v)^2}{gR}\right]\left[1 + \frac{V^2}{gR}\right]$ [as $(1 - x)^{-1} = 1 + x$ if $x << 1$]

or $\frac{W}{W_0} = \left[1 - \frac{(V \pm v)^2}{gR} + \frac{V^2}{gR} + ...\right] \simeq \left[1 \mp \frac{2Vv}{gR}\right]$

or $W \simeq W_0\left[1 \mp 2\frac{\omega v}{g}\right]$ [as $V = R\omega$]

Here negative sign holds for the ship moving in the direction of motion of earth, i.e., from west to east, *i.e.,* the body in the moving ship will weigh less if the ship is moving from west to east, in the direction of motion of earth and more if the ship is moving from east to west, opposite to the motion of earth.

- **Variation of *g* due to some other features:** The surface of the earth is uneven. The presence of mountains, plateaus and valleys would cause a variation in g.
  Further, the earth's density is not uniform, the inner core is heavier than the mantle. The density of the crust of earth varies from region to region over earth's surface. Thus g varies from region to region.

Gravitational field

- The space surrounding a material body in which another body experiences force by virtue of its mass is called gravitational field.
- The intensity of gravitational field at any point is the force on the unit mass placed at that point i.e.

$$\vec{E} = \frac{\vec{F}}{m} = G\frac{M_e}{R_e^3}\hat{R}_e$$

Magnitude of $E = g$

Gravitational potential

- The gravitational potential at a point in the gravitational field of a body is defined as the work done in bringing a unit mass from infinity to that point.
- The gravitational force on a unit mass placed at a distance r from the object of mass $M = \frac{GM}{r^2}$
- Work done by this force in moving the unit mass through a distance $dr = \frac{GM}{r^2}\mathrm{d}r$.
- Thus, the gravitational potential at a point r from the body is given by,

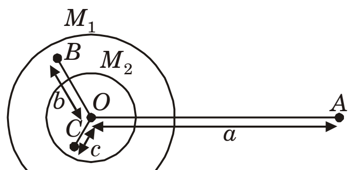
$$V = \int_{\infty}^{r} \frac{GM}{r^2}\mathrm{d}r$$

$$V = -\frac{GM}{r}$$

- **Gravitational potential energy.** The gravitational potential energy of a body at a point is defined as the work done in bringing the body from infinity to that point.

$$U = -\frac{GMm}{r}$$

Illustration 5 : Two concentric shells of masses M_1 and M_2 are situated as shown in figure. Find the force on a particle of mass m when the particle is located at (a) $r = a$ (b) $r = b$ (c) $r \le c$. The distance r is measured from the centre of the shell.



[M_1 = 500 kg, M_2 = 1500 kg, $a = \sqrt{6.6}$ m, $b = \sqrt{3.3}$,
$G = 6.6 \times 10^{-11}$ Nm²/kg²]

Take 10^{-8} N as unit.

Soln.: We know that attraction at an external point due to spherical shell of mass M is $\frac{GM}{r^2}$ while at an internal point is zero. So

(a) For $r = a$, the point is external for both the shell ; so $E_A = \frac{G(M_1 + M_2)}{a^2}$

$\therefore \quad F_A = mE_A = \frac{G[M_1 + M_2]m}{a^2} = 2m$ units

(b) For $r = b$, the point is external to the shell of mass M_2 and internal to the shell of mass M_1 ; so

$E_B = \frac{GM_2}{b^2} + 0$

$\therefore \quad F_B = mE_B = \frac{GM_2 m}{b^2} = 3m$ units

(c) For $r = c$, the point is internal to both the shells, so $E_C = 0 + 0 = 0 \quad \therefore \quad F_C = mE_C = 0$

Motion of the satellites-orbital velocity

- The celestial bodies which revolve around the planets in close and stable orbits are called as satellites.
- Satellites revolve round the earth in definite orbits and are kept in these orbits by the gravitational attraction of the earth. If a satellite of mass m circles round the earth at a height of h above the surface of the earth, then its velocity v in the orbit is given by the following equation,

$$\frac{mv^2}{R+h} = G\frac{M_e m}{(R_e+h)^2} \qquad \text{or,} \quad v = \sqrt{\frac{GM_e}{(R_e+h)}}$$

- If g is the acceleration due to gravity on the earth, then,

$$g = \frac{GM_e}{R_e^2} \Rightarrow GM_e = gR_e^2 \qquad \therefore v = \sqrt{\frac{gR_e^2}{(R_e+h)}}$$

$$\boxed{v = R_e\sqrt{\frac{g}{R_e+h}}}$$

- The time period of revolution of the satellite about the earth is given by,

$$T = \frac{2\pi r}{v} = \frac{2\pi(R_e+h)}{R_e\sqrt{\frac{g}{R_e+h}}}$$

$$\boxed{T = \frac{2\pi}{R_e}\sqrt{\frac{(R_e+h)^3}{g}}}$$

- For a satellite revolving very near to earth's surface, $R_e >> h$, then

$$v = \sqrt{gR_e} = 8 \text{ km/sec}$$

and $\quad T = 2\pi\sqrt{\frac{R_e}{g}} = 84$ minutes

Illustration 6 : A small satellite revolves round a planet of mean density 10 gm/cm^3, the radius of the orbit being slightly greater than the radius of the planet. Calculate the time of revolution of the satellite. Take $G = 6.6 \times 10^{-8}$ C.G.S. unit.

Soln.:
- Mean density of planet, $\rho = 10$ gm/cm^3
- Radius of satellite $\simeq$ Radius of planet

To calculate the time of revolution of the satellite.

Let M is the mass of the planet and r is the radius of the orbit of the satellite which is nearly equal to the radius of the planet.

The orbital velocity of the satellite round the planet, $v = \sqrt{\dfrac{GM}{r}}$ ...(1)

Mass of the planet, $M = \dfrac{4}{3}\pi r^3 \rho$

Putting the value of M in equation (1), we get,

$$v = \sqrt{\frac{4}{3}\pi r^3 \rho . \frac{G}{r}} = r\sqrt{\frac{4}{3}\pi\rho G}$$

Time of revolution, $T = \dfrac{2\pi r}{v} = \dfrac{2\pi}{\sqrt{\frac{4}{3}\pi\rho G}} = \sqrt{\dfrac{3\pi}{\rho G}} = \sqrt{\dfrac{3 \times 3.14}{10 \times 6.6 \times 10^{-8}}}$ = 3780 seconds = 1 hr 3 min

Geostationary satellite and parking orbit:	If the time period of the satellite be exactly equal to the period of revolution of the earth, then it will appear to be stationary at the same place on the earth. This is known as *parking orbit*. And such a satellite is called a geo-static or geo-stationary or geo-synchronous satellite and is used as a communication satellite.
Energy of an orbiting satellite:	Kinetic energy of the satellite = $\frac{1}{2}mv^2$ $= \frac{1}{2}m\left(\sqrt{\frac{GM}{r}}\right)^2 = \frac{GMm}{2r}$ Gravitational potential energy of the satellite = $-\frac{GMm}{r}$ $\therefore$ Total energy of the satellite = $\frac{GMm}{2r} - \frac{GMm}{r}$ $\boxed{E = -\frac{GMm}{2r}}$
Escape velocity	• The escape velocity on earth is defined as the minimum velocity with which a body has to be projected vertically upward from the surface of the earth, so that it just goes beyond the gravitational attraction of the earth and never returns on it. • It can be proved using the concept of gravitational potential energy that the escape velocity on earth's surface is given by, $v_e = \sqrt{\frac{2GM_e}{R_e}} = \sqrt{2gR_e}$ • We know that the orbital velocity of a satellite close to the earth surface is given by, $v_0 = \sqrt{gR_e}$ $\therefore$ $v_e = \sqrt{2}v_0$ $v_e = 11.2$ km/sec

Illustration 6 : An artificial satellite is moving in a circular orbit around the earth with a speed equal to half the magnitude of escape velocity from the earth.
(a) Determine the height of the satellite above the earth's surface.
(b) If the satellite is stopped suddenly in its orbit and allowed to fall freely on to the earth, find the speed with which it hits the surface of the earth.
Take, radius of the earth = 6400 km and $g = 9.8\ m/s^2$.

Soln.: If R be the radius of the earth and h be the height of the satellite above the earth's surface, the orbital velocity v_0 and the escape velocity v_e of the satellite are given by,

$$v_0 = \sqrt{\frac{gR^2}{R+h}} \quad \text{and} \quad v_e = \sqrt{2gR}$$

(a): It is given that, $v_0 = \frac{1}{2}v_e$

$$i.e.\ \sqrt{\frac{gR^2}{R+h}} = \frac{1}{2}\sqrt{2gR} \quad \text{or} \quad \frac{gR^2}{R+h} = \frac{1}{2}gR$$

$\therefore \quad h = R = 6400$ km

(b): Potential energy of the satellite at a distance $R + h$ (= $2R$) from the centre of the earth = $-\frac{GMm}{2R}$

Where M and m are the masses of the earth and the satellite respectively.

Potential energy of the satellite at a distance R from the centre of the earth i.e., on the surface of the earth = $-\frac{GMm}{R}$

$\therefore$ Change in potential energy of the satellite = $-\frac{GMm}{R} + \frac{GMm}{2R} = -\frac{GMm}{2R}$

This decrease in potential energy will be converted into kinetic energy and thus,

$$\frac{1}{2}mv^2 = \frac{GMm}{2R} \quad \text{or,} \quad v^2 = \frac{GM}{R} = gR \quad \left[\because g = \frac{Gm}{R^2}\right]$$

$\therefore \quad v = \sqrt{gR} = \sqrt{9.8 \times 6400 \times 10^3} = 7.92 \times 10^3$ m/s = 7.92 km/s

Thus the satellite would hit the surface of the earth with a speed of 7.92 km/s.

Competitions Window

- Gravitational force is attractive force between two masses M_1 and M_2 separated by a distance r. It is given by $F = -G(M_1M_2/r^2)$, where G is universal gravitational constant, $G = 6.67 \times 10^{-11}$ Nm^{2}kg^{-2}. The negative sign shows that force is attractive.
- Dimensional formula of G is $[M^{-1}L^3T^{-2}]$.
- The gravitation is the central force. It acts along the line joining the particles. It is a conservative force. The work done by it is independent of the path followed. This force is attractive in nature.
- It is the weakest force in nature. It is 10^{38} times smaller than nuclear force and 10^{36} times smaller than electric force.
- Gravitation is independent of the presence of other bodies around it.
- The gravitational pull of the earth is called gravity.
- The gravitation forces between two bodies are action-reaction pairs. The law of gravitation is universal as it is applicable to all bodies, irrespective of their size, shape or position. This force does not depend upon the state of the bodies, nature of the intervening medium, temperature and other physical condition of the bodies.
- In motion of the planets and satellites, force of gravitation provides the necessary centripetal force due to which earth revolves around the sun and moon around the earth.
- The value of acceleration due to gravity increases as we go from equator to the pole.
- The value of the acceleration due to gravity on the moon is about one sixth of that on the earth and on the sun is about 27 times that on the earth.
- Among the planets, the g is minimum on the mercury.
- Acceleration due to gravity on the surface of the earth is given by, $g = GM/R^2$, where M is the mass of the earth and R is the radius of the earth.
- Acceleration due to gravity at a height h above the surface of the earth is given by

  $$g_h = GM/(R+h)^2 \cong g\,(1 - 2h/R).$$

  The approximation is true when $h << R$.
- Value of g at depth d from earth's surface

  (a) $g' = g\left[1 - \frac{d}{R}\right]$ (b) $g' = \frac{GM}{R^3}(R - d)$.

  Again, the approximation is true for $d << R$.
- The value of g at a height h from the surface and $h << R$ is $g' = g\left(1 - \frac{2h}{R}\right)$.
- The value of g at latitude λ is

  (a) $g' = g - \omega^2 R_e \cos^2\lambda$

  (b) At the equator $\lambda = 0$. $\therefore$ $g' = g - \omega^2 R_e$

  (c) At the poles $\lambda = \pi/2$. $\therefore$ $g' = g$.
- The decrease in g with latitude is caused by the rotation of the earth about its own axis. A part of g is used to provide the centripetal acceleration for rotation about the axis.
- The rotation of the earth about the sun has no effect on the value of g.

- Decrease in g in going from poles to the equator is about 0.35%.
- The weight of the body varies in the same manner as the g does. ($W = mg$).
- The gravitational force of attraction acting on a body of unit mass at any point in the gravitational field is defined as the intensity of gravitational field (E_g) at that point. $E_g = \frac{F}{m} = \frac{GM}{r^2}$.
- The gravitational potential energy of a mass m at a point above the surface of the earth at a height h is given by $\frac{-GMm}{R+h}$. The negative sign implies that as R increases, the gravitational potential energy decreases and becomes zero at infinity.
- The body is moved from the surface of earth to a point at a height h above the surface of earth then change in potential energy will be mgh.
- Gravitational potential at a point above the surface of the earth at a height h is $-GM/(R + h)$. Its unit is joule/kilogram.
- Gravitational mass, M_g is defined by Newton's law of gravitation.

$$M_g = \frac{F_g}{g} = \frac{W}{g} = \frac{\text{Weight of body}}{\text{Acceleration due to gravity}} \qquad \frac{(M_1)_g}{(M_2)_g} = \frac{F_{g_1} g_2}{g_1 F_{g_2}}$$

- Let ω_0 = angular speed of the satellite, v_0 = orbital speed of the satellite, then $v_0 = (R + h)\omega_0$, where R = radius of the earth and h = height of the satellite above the surface of the earth.
  Let g = acceleration due to gravity on the surface of the earth, T = time period of the satellite, M = mass of the earth. Then different quantities connected with satellite at height h are as follows:

  (a) $\omega_0 = \left[\frac{GM}{(R+h)^3}\right]^{1/2} = R\left[\frac{g}{(R+h)^3}\right]^{1/2}$.

  (b) $T = \frac{2\pi}{\omega_0}$ and frequency of revolution, $\nu = \frac{\omega_0}{2\pi}$.

  (c) $v_0 = \left[\frac{GM}{R+h}\right]^{1/2} = R\left[\frac{g}{R+h}\right]^{1/2}$

  Very near the surface of the earth, we get the values by putting $h = 0$. That is:

  (i) $\omega_0 = \left[\frac{GM}{R^3}\right]^{1/2} = \left[\frac{g}{R}\right]^{1/2}$ (ii) $v_0 = \left[\frac{GM}{R}\right]^{1/2} = [g\,R]^{1/2}$

  (iii) $T \cong 2\pi\left[\frac{R}{g}\right]^{1/2}$ = 5078 sec = 1 hour 24.6 minute.

- Altitude or height of satellite above the earth's surface, $h = \left(\frac{T^2R^2g}{4\pi^2}\right)^{1/3} - R$.
- Angular momentum, $L = mv(R+h) = [m^2GM(R+h)]^{1/2}$.
- Above the surface of the earth, the acceleration due to gravity varies inversely as the square of the distance from the centre of the earth. $g' = \frac{gR^2}{(R+h)^2}$.
- The gravitational potential energy of a satellite of mass m is $U = \frac{-GMm}{r}$, where r is the radius of the orbit. It is negative.
- Kinetic energy of the satellite is $K = \frac{1}{2}mv_0^2 = \frac{GMm}{2r}$.
- Total energy of the satellite $E = U + K = -\frac{GMm}{2r}$. –ve sign indicates that it is the binding energy of the satellite.

- Total energy of a satellite at a height equal to the radius of the earth is given by

$$-\frac{GMm}{2(R+R)} = -\frac{GMm}{4R} = \frac{1}{4}mgR .$$

where $g = GM/R^2$ is the acceleration due to gravity on the surface of the earth.
When the total energy of the satellite becomes zero or greater than zero, the satellite escapes from the gravitational pull of the earth.
- If the radius of planet decreases by $n\%$, keeping the mass unchanged, the acceleration due to gravity on its surface increases by $2n\%$ *i.e.* $\frac{\Delta g}{g} = -\frac{2\Delta R}{R}$
- If the mass of the planet increases by $m\%$ keeping the radius constant, the acceleration due to gravity on its surface increases by $m\%$ $\left[\frac{\Delta g}{g} = \frac{\Delta M}{M}\right]$ where R = constant.
- If the density of planet decreases by $p\%$ keeping the radius constant, the acceleration due to gravity decreases by $p\%$.
- If the radius of the planet decreases by $q\%$ keeping the density constant, the acceleration due to gravity decreases by $q\%$.
- For the planets orbiting around the sun, the angular speed, the linear speed and kinetic energy change with time but the angular momentum remains constant.
- The minimum velocity with which a body must be projected in the atmosphere so as to enable it to just overcome the gravitational attraction of the earth is called escape velocity. *i.e.* $v_e = \sqrt{2gR}$, where R = radius of earth.
- The relation between orbital velocity of satellite and escape velocity is $v_e = \sqrt{2}\, v_o$. *i.e.* if the orbital velocity of a satellite revolving close to the earth happens to increase to $\sqrt{2}$ times, the satellite would escape.
- There is no atmosphere on the moon because the escape velocity on the moon is low.
- If the orbital radius of the earth around the sun be one fourth of the present value, then the duration of the year will be one eighth of the present value.
- Weightlessness is a situation in which the effective weight of the body becomes zero. Circumstances when this condition arises.
  (i) When the body is taken at the centre of the earth.
  (ii) When a body is lying in a freely falling lift, ($a = g$).
  (iii) When the body is inside a space craft or satellite which is orbiting around the earth.
- **Kepler's first law (law of orbit)** - Every planet revolves around the sun in an elliptical orbit. The sun is situated at one focus of the ellipse.
- **Kepler's second law (law of area)** - The radius vector drawn from the sun to a planet sweeps out equal areas in equal intervals of time. *i.e.* the areal velocity of planet around the sun is constant.
- **Kepler's third law (law of period)** - The square of the time period of revolution of a planet around the sun is directly proportional to the cube of semi-major axis of its elliptical orbit, *i.e* $T^2 \propto a^3$, where a = semi-major axis of the elliptical orbit of the planet around the sun.
- **Shape of orbit of a satellite** - The shape of orbit of a satellite depends upon its velocity of projection v from earth.
  (a) If $v < v_o$, the satellite falls back to earth following a spiral path.
  (b) If $v = v_o$, the satellite revolves in circular path/orbit around earth.
  (c) If $v > v_o$, but less than escape velocity v_e, *i.e.* $v_o < v < v_e$, the satellite shall revolve around earth in elliptical orbit.
  (d) If $v = v_e$, the satellite will escape away following a parabolic path.

(e) If $v > v_e$, the satellite will escape away following a hyperbolic path.

- **Geo-stationary satellite/Parking orbit**
  (a) Time period = 24 hour
  It is synchronous with earth.
  (b) The angular velocity of satellite must be in the same direction as that of the earth. It thus revolves around earth from west to east. Its relative angular velocity with respect to earth is zero.
  (c) The orbit of satellite should be circular.
  (d) The orbit should be in equatorial plane of earth. It contains centre of earth as well as equator.
  (e) It should be at 36000 km from the surface of earth. It is thus (36000 + 6400) km or 42400 km from the centre of earth.
  Radius of parking orbit = 42400 km.
  (f) A satellite revolving in stable orbit does not require any energy from an external source. The work done by the satellite in completing its orbit is zero.
  (g) Acceleration due to gravity is used up in providing centripetal acceleration to the satellite. Hence effective g inside the satellite is zero.
  (h) Its orbital velocity = 3.1 km/sec
  Its angular velocity = $\frac{2\pi}{24}$ radian/hour.

- **For a satellite orbiting near earth's surface**
  (a) Time period = 84 minute approximately
  (b) Orbital velocity = 8 km/sec
  (c) Angular speed $\omega = \frac{2\pi}{84}\frac{\text{radian}}{\text{min}} = 0.00125\ \frac{\text{radian}}{\text{sec}}$.

- **Inertial mass and gravitational mass**
  (a) Inertial mass = $\frac{\text{force}}{\text{acceleration}}$
  (b) Gravitational mass = $\frac{\text{weight of body}}{\text{acceleration due to gravity}}$
  (c) They are equal to each other in magnitude.
  (d) Gravitational mass of a body is affected by the presence of other bodies near it. Inertial mass of a body remains unaffected by the presence of other bodies near it.

Miscellaneous Examples

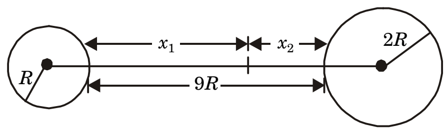
1. Two spherical bodies of mass M and $5M$ and radii R and $2R$ respectively are released in free space with initial separation between their centres equal to $12R$. If they attract each other due to gravitational force only, then find the distance covered by the smaller body just before collision.

Soln.: Mass of smaller body = M, Mass of larger body = $5M$, Radius of smaller body = R

Radius of larger body = $2R$, Initial separation between their centers = $12R$

To find the distance covered by the smaller body just before collision.

It is given that only gravitational force is acting between two bodies, therefore the force between them will increase as the distance decreases.



Let the spheres collide after time t, when the smaller sphere covered distance x_1 and bigger sphere covered distance x_2. The gravitational force acting between two spheres depends on the distance which is a variable quantity.

The gravitational force, $F(x) = \dfrac{GM \times 5M}{(12R - x)^2}$

Acceleration of smaller body, $a_1(x) = \dfrac{G \times 5M}{(12R - x)^2}$

Acceleration of bigger body, $a_2(x) = \dfrac{GM}{(12R - x)^2}$

From equation of motion, $x_1 = \dfrac{1}{2} a_1(x)\, t^2$ and $x_2 = \dfrac{1}{2} a_2(x)\, t^2 \Rightarrow \dfrac{x_1}{x_2} = \dfrac{a_1(x)}{a_2(x)} \Rightarrow x_1 = 5x_2$

It is given that $x_1 + x_2 = 12R - (R + 2R) = 9R \Rightarrow x_1 + \dfrac{x_1}{5} = 9R \Rightarrow x_1 = \dfrac{45R}{6} = 7.5R$

$\therefore$ Required distance = $7.5R$.

---

2. How far from earth must a body be along a line towards the sun so that the sun's gravitational pull on it balances that of the earth? The sun is 9.3×10^7 miles away and its mass is $3.24 \times 10^5\, M_e$, where M_e is the mass of the earth.

Soln.: Distance between sun and earth, $d = 9.3 \times 10^7$ miles

Mass of the sun, $M_s = 3.24 \times 10^5\, M_e$

Where M_e is the mass of the earth.

To find a point at which gravitational force by earth is balanced by gravitational force by sun.

Let the body be placed at a distance of x miles from the earth. So, if mass of the sun be M_s and if it be at a distance of d miles from the earth, then the sun is at a distance $(d - x)$ miles from the body. Thus,

or $\dfrac{GmM_s}{(d-x)^2} = \dfrac{GmM_e}{x^2}$ or $\dfrac{M_s}{M_e} = \left(\dfrac{d-x}{x}\right)^2$ or $\dfrac{d-x}{x} = \sqrt{\dfrac{M_s}{M_e}}$

or $\dfrac{d}{x} - 1 = \sqrt{3.24 \times 10^5}$ or $\dfrac{d}{x} = 1 + \sqrt{3.24 \times 10^5}$

$\therefore$ $x = \dfrac{d}{1 + \sqrt{3.24 \times 10^5}} = \dfrac{9.3 \times 10^7}{1 + \sqrt{3.24 \times 10^5}} = 1.63 \times 10^5$ miles $\therefore$ Required distance = 1.63×10^5 miles.

---

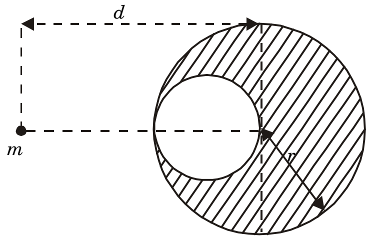
3. In a lead sphere of radius r, a spherical cavity of diameter r is made. The surface of the cavity touches the outside of the lead sphere and passes through the centre of the sphere. The mass of the solid sphere was M. With what force, according to the universal law of gravitation, will the sphere attract a small sphere of mass m placed at a distance d from the centre of the lead sphere on the straight line joining the centres of the spheres and the side of the hollow?

Soln.: Radius of lead sphere = r, Radius of cavity = $r/2$, Mass of solid sphere = M

Distance between small sphere of mass m and $M = d$

To find gravitational force at m

If the lead sphere be a solid one then the gravitational force of attraction on the mass m would be,



$$F = G\frac{Mm}{d^2}$$

Now, we consider this force as the resultant of the forces (1) due to the hollow sphere and (2) due to the sphere of the size of the hollow.

Let these forces be F_1 and F_2 respectively. Thus, $F = F_1 + F_2$

Now, the volume of the cavity $= \frac{4}{3}\pi\left(\frac{r}{2}\right)^3$

$\therefore$ mass of the lead covering the cavity $= M' = \frac{4}{3}\pi\left(\frac{r}{2}\right)^3 \rho$ where ρ is the density of the lead.

$$\therefore \quad M' = \frac{1}{8}\times\frac{4}{3}\pi r^3\rho = \frac{M}{8}$$

Now, the centre of the cavity is at a distance of $\left(d - \frac{r}{2}\right)$ from the mass m. So, $F_2 = \dfrac{G\frac{M}{8}\times m}{\left(d-\frac{r}{2}\right)^2}$

$$\therefore \text{ Required force } F_1 = F - F_2 = \frac{GMm}{d^2} - \frac{GMm}{8\left(d-\frac{r}{2}\right)^2} = \frac{GMm}{8d^2}\times\frac{7d^2 - 8dr + 2r^2}{\left(d-\frac{r}{2}\right)^2}$$

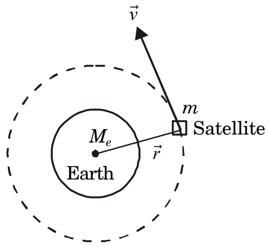
---

4. A satellite of mass m is moving in a circular orbit of radius r. Calculate its angular momentum with respect to the centre of the orbit in terms of the mass of the earth.

Soln.: Mass of satellite $= m$, Radius of orbit $= r$

To calculate the angular momentum of the satellite.

The angular momentum of the satellite with respect to the centre of orbit is given by, $\vec{J} = \vec{r}\times m\vec{v}$



where $\vec{r}$ is the radius vector of the satellite with respect to the centre of the orbit and $\vec{v}$ is its velocity.

The angle between $\vec{r}$ and $\vec{v}$ is 90° in the case of circular orbit.

$\therefore \quad \vec{J} = mvr \sin 90° = mvr$ ...(1)

Now the gravitational force of attraction between the earth and the satellite $= \dfrac{GM_e m}{r^2}$

This force provides the necessary centripetal force to the satellite for its circular motion.

$\therefore \quad \dfrac{GM_e m}{r^2} = \dfrac{mv^2}{r}$ or $mv^2r = G\,M_e\,m$

or $m^2v^2r^2 = G\,M_e\,m^2\,r$ or $mvr = (G\,M_e\,m^2\,r)^{1/2}$ ...(2)

From (1) & (2) we get, $J = (G\,M_e\,m^2\,r)^{1/2}$

---

5. Two particles of mass m_1 and m_2 initially at rest are infinite distance apart. Show that at any instant the relative velocity of approach of the particles due to gravitational attraction only is $\sqrt{2G(m_1 + m_2)/r}$, where r is their separation at the instant.

Soln.: We know the relative motion of two particles subjected only to their mutual interaction is equivalent to the motion of a particle of mass equal to the reduced mass under a force equal to their interaction. Thus, if m' be the reduced mass of the particles of masses m_1 and m_2 then,

$$\frac{1}{m'} = \frac{1}{m_1} + \frac{1}{m_2} \qquad \text{or,} \quad m' = \frac{m_1 m_2}{m_1 + m_2}$$

Now, when the particles are at a distance x, then gravitational force between them is given by,

$$F = G.\frac{m_1 m_2}{x^2}$$

The work done when they are displaced by dx, we have,

$$dW = F \cdot dx = G.\frac{m_1 m_2}{x^2}dx$$

Thus, total work done to bring the particles from infinite separation to one when they are separated by a distance r is obtained by

$$W = \int dW = Gm_1 m_2 \int_r^{\infty} \frac{1}{x^2}dx = G.\frac{m_1 m_2}{r} \qquad ...(1)$$

If the velocity of approach of the particles be v, then the gain in kinetic energy of the particles are

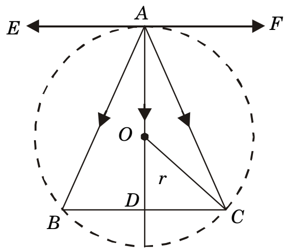
$$K.E. = \frac{1}{2}m'v^2 = \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}v^2 \qquad ...(2)$$

From (1) and (2), $\frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}v^2 = G.\frac{m_1 m_2}{r}$ $\quad \therefore \boxed{v = \sqrt{\frac{2G(m_1 + m_2)}{r}}}$

---

6. Three identical bodies, each of mass m are located at the vertices of an equilateral triangle with side l. At what speed should they move if they all revolve under the influence of one another's gravity in a circular orbit, circumscribing the triangle while still preserving the equilateral triangle?

Soln.: Let ABC be the equilateral triangle.



The radius of the orbit $= OC = r = \frac{DC}{\cos 30°} = \frac{l/2}{\sqrt{3}/2} = \frac{l}{\sqrt{3}}$

Since the particles A, B and C describe the common circle, they have a common angular velocity.

The centripetal force required for each particle for describing a circular path is provided by the gravitational force of attraction on it by the other two particles.

Now, the gravitational force on the mass at A due to mass at $B = \frac{Gm^2}{l^2}$ along AB

The component of force along AO (towards the centre) $= \frac{Gm^2}{l^2}\cos 30°$ $\quad (\because \angle BAD = \frac{1}{2}\angle BAC = 30°)$

$$= \frac{Gm^2}{l^2}.\frac{\sqrt{3}}{2}$$

Similarly, the component of the force exerted by the mass at C on the mass at A along $AD = \frac{\sqrt{3}}{2}.\frac{Gm^2}{l^2}$

The components of the forces along AE and AF being equal and opposite, will be cancelled out.

Therefore, net force on $A = 2.\frac{\sqrt{3}}{2}.\frac{Gm^2}{l^2} = \sqrt{3}\frac{Gm^2}{l^2}$ directed towards the centre O of the circle.

Now, if v be the velocity of each particle, $\frac{mv^2}{r} = \sqrt{3}.\frac{Gm^2}{l^2}$

or, $v^2 = \frac{\sqrt{3}Gm^2 r}{ml^2} = \frac{\sqrt{3}\,Gm^2\,l}{m\,l^2\sqrt{3}}$ $\quad \left[\because r = \frac{l}{\sqrt{3}}\right]$

$= \frac{Gm}{l}$ $\quad \therefore \boxed{v = \sqrt{\frac{Gm}{l}}}$

---

7. A rocket is fired vertically from the ground with a resultant vertical acceleration of 10 m/s^2. The fuel is finished in one minute and it continues to move up. What is the maximum height reached?

Soln.: Acceleration = 10 m/s^2

Time = 1 minute = 60 sec

To find maximum height reached.

We have f = 10 m/s^2, t = 1 min = 60 s

Thus, the rocket rises a height of h_1 metre in 1 minute, given by,

$$h_1 = \frac{1}{2} \times 10 \times (60)^2 = 18000 \text{ m}$$

Again, if after 1 min the velocity of the rocket becomes v m/s, then $v = 10 \times 60 = 600$ m/s

After this the rocket has no acceleration of its own and it is under the retardation due to gravity. The rocket would rise up till its velocity becomes zero. So, if it rises h_2 metre further, then,

$0 = (600)^2 - 2 \times 9.8 \times h_2$ $\qquad \therefore \quad h_2 = 18367.35$ m

Thus, total height of the rocket = $h_1 + h_2$ = 18000 + 18367.35 = 36367.35 m

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8. A juggler is maintaining four balls in motion, making each in turn rise to a height of 90 cm from his hand. With what velocity does he project them? And where will the other three balls be at the instant when one is just leaving his hand?

Soln.: Let u be the velocity of projection of each ball. The maximum height reached by the ball is given by

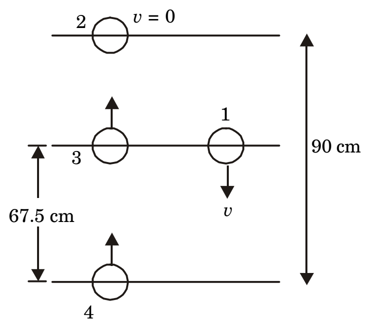
$$H = \frac{u^2}{2g} \quad \therefore \quad u = \sqrt{2gH} \quad \text{or,} \quad u = \sqrt{2 \times 980 \times 90} = 420 \text{ cm/s}$$

Total time of flight of each ball, $t = \dfrac{2u}{g} = \dfrac{2 \times 420}{980} = \dfrac{6}{7}$ sec

Now, the balls are thrown at equal interval of time which is given by $t' = \dfrac{6/7}{4} = \dfrac{3}{14}$ sec.

when one ball is in his hand, the previous ball had been thrown just $\dfrac{3}{14}$ sec earlier. This will be at a height of h_1, given by,

$$h_1 = ut - \frac{1}{2}gt^2 = 420 \times \frac{3}{14} - \frac{1}{2} \times 980 \times \left(\frac{3}{14}\right)^2 = 67.5 \text{ cm}$$



The ball previous to that was thrown $\dfrac{3}{7}$ sec earlier.

Hence its height is given by

$$h_2 = 420 \times \frac{3}{7} - \frac{1}{2} \times 980 \times \left(\frac{3}{7}\right)^2 = 90 \text{ cm}$$

Similarly, the height of the third ball is obtained as,

$$h_3 = 420 \times \frac{9}{14} - \frac{1}{2} \times 980 \times \left(\frac{9}{14}\right)^2 = 67.5 \text{ cm.}$$

Thus the three balls are at the height of 67.5 cm, 90 cm and 67.5 cm respectively when one is just leaving his hand. This situation is shown in figure with their direction of velocities also.

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9. A cosmic body A moves to the sun of mass m_s with a velocity v_0 from a very large distance, such that the perpendicular distance of the direction of v_0 from the sun is l (called impact parameter). Find the minimum distance of the body from the sun on its subsequent motion.

Soln.: Velocity of the cosmic body = v_0, Mass the sun = m_s

Perpendicular distance of the direction of v_0 from the sun = l

To find the minimum distance of the body from the sun on its subsequent motion.

The cosmic body A is under the only central force of the gravitational field of sun. Therefore both the energy and angular momentum conservation laws hold.

v = velocity of body A at minimum distance $r_{\min}$ from the sun. Using law of conservation of energy,

$$\frac{1}{2} m_A v_0^2 + 0 = \frac{1}{2} m_A v^2 - \frac{G m_A m_S}{r_{\min}} \quad \Rightarrow \quad v^2 = v_0^2 + \frac{2Gm_s}{r_{\min}} \qquad ...(1)$$

Using law of conservation of angular momentum, $m_A v_0 l = m_A v r_{\min} \Rightarrow v = \dfrac{v_0 l}{r_{\min}}$ ...(2)

Using (1) & (2), $\dfrac{v_0^2 l^2}{r_{\min}^2} = v_0^2 + \dfrac{2Gm_s}{r_{\min}}$ $\Rightarrow r^2_{\min} + r_{\min}\left(\dfrac{2Gm_s}{v_0^2}\right) - l^2 = 0$

$\therefore\ r_{\min} = \dfrac{1}{2}\left[-\dfrac{2Gm_s}{v_0^2} \pm \sqrt{\left(\dfrac{2Gm_s}{v_0^2}\right)^2 + 4l^2}\right]$ $\Rightarrow r_{\min} = \dfrac{Gm_s}{v_0^2}\left[\sqrt{1+\left(\dfrac{lv_0^2}{Gm_s}\right)^2} - 1\right]$

---

10. Two earth's satellites move in a common plane along circular orbit. The orbital radius of one satellite is r while that of other satellite is Δr less. What time interval separates the periodic approaches of the satellites to each other over the minimum distance?

Soln.: Radius of Ist satellite = r, Radius of IInd satellite = $r - \Delta r$

Difference in time of revolution.

If m_1 be the mass of Ist satellite revolving in a circular orbit of radius r and v_1 be its orbital speed, then for the circular motion of this satellite

$\dfrac{m_1 v_1^2}{r} = \dfrac{Gm_1 M_e}{r^2}$ or $v_1 = \sqrt{\dfrac{GM_e}{r}}$

If T_1 is the time period of Ist satellite then, $T_1 = \dfrac{2\pi r}{v_1} = \dfrac{2\pi}{\sqrt{GM_e}} r^{3/2}$

Similarly, for the second satellite whose orbital radius is Δr less as compared to that of Ist satellite, time period is: $T_2 = \dfrac{2\pi(r - \Delta r)^{3/2}}{\sqrt{GM_e}}$

$\therefore$ periodic time interval = $T_1 - T_2 = \dfrac{2\pi}{\sqrt{GM_e}}\,[r^{3/2} - (r - \Delta r)^{3/2}]$

$$= \frac{2\pi}{\sqrt{GM_e}}\left[r^{3/2} - r^{3/2}\left(1 - \frac{\Delta r}{r}\right)^{3/2}\right] = \frac{2\pi}{\sqrt{GM_e}} r^{3/2}\left[1 - \left(1 - \frac{3}{2}.\frac{\Delta r}{r}\right)\right] = \frac{2\pi}{\sqrt{GM_e}} r^{3/2}.\frac{3\Delta r}{2r} = \frac{3\pi}{\sqrt{GM_e}} r^{1/2}.\Delta r$$

---

11. The density inside a solid sphere of radius R is expressed as $\rho = \rho_0(R/r)$ where ρ_0 is the density at the surface and r shows the distance from the centre. Calculate the gravitational field due to this sphere at a distance $2R$ from its center.

Soln.: Radius = R, Density of surface = ρ_0, Inside density, $\rho = \rho_0\ (R/r)$

To calculate gravitational field at a distance $2r$ from the centre.

We have to calculate the gravitational field at a point outside the sphere. For this purpose the sphere can be supposed to be divided into various thin concentric shells and each shell can be replaced by a point mass at its centre having mass equal to the mass of the shell. Thus, the whole sphere can be replaced by a point mass situated at its centre having mass equal to the mass of the given sphere. If M be the mass of the given sphere, then the gravitational field at the given point is

$E = \dfrac{GM}{(2R)^2} = \dfrac{GM}{4R^2}$ ...(1)

The mass M can be calculated as under—

Consider a concentric spherical shell of radius r and thickness dr. Its volume is: $dV = (4\pi r^2)\ dr$ and its mass is: $dM = \rho dV = \left[\rho_0 \dfrac{R}{r}\right](4\pi r^2)dr = 4\pi\rho_0 R r dr$

Hence, the mass of the whole sphere is $M = \int_0^R 4\pi\rho_0 R r dr = 2\pi\rho_0 R^3$

Thus from (1) and (2), the required gravitational field is $E = \dfrac{2\pi G\rho_0 R^3}{4R^2} = \dfrac{1}{2}\pi G\rho_0 R$

---

12. A satellite of mass M_s is moving around the earth in a circular orbit of radius R_s. It starts losing energy slowly at a constant rate α due to resistive force of air. If mass and radius of the earth are M_e and R_e respectively, then calculate the time taken by satellite to fall on earth's surface.

Soln.: If v_s is the orbital velocity of satellite in its orbit of radius R_s, then for circular motion of the satellite

$$\frac{M_s v_s^2}{R_s} = \frac{M_e M_s}{R_s^2} G \text{ or } v_s^2 = \frac{GM_e}{R_s} \qquad ...(1)$$

If v_s' be the orbital velocity of satellite when its orbit touches the earth's surface, then

$$\frac{M_s (v_s')^2}{R_e} = \frac{M_e M_s G}{R_e^2} \quad \text{or} \quad (v_s')^2 = \frac{GM_e}{R_e} \qquad ...(2)$$

Now *K.E.* of the satellite in an orbit of radius R_s is given by

$$K = (1/2)\, M_s v_s^2 = \frac{1}{2}\frac{GM_s M_e}{R_s} \qquad ...(3)$$

Similarly, *K.E.* of the satellite in an orbit of radius R_e is given by

$$K' = (1/2)\, M_s\, (v_s')^2 = \frac{1}{2}\frac{M_s M_e G}{R_e} \qquad ...(4)$$

Now, potential energy of the satellite when it is at a distance R_s from earth

$$= U = -\frac{GM_s M_e}{R_s} \qquad ...(5)$$

Similarly, potential energy of satellite on the surface of earth

$$= U' = -\frac{GM_s M_e}{R_e} \qquad ...(6)$$

Hence, total energy of satellite when it is in orbit of radius R_s,

$$E = K + U = \frac{GM_s M_e}{2R_s} - \frac{GM_s M_e}{R_s} = -\frac{GM_s M_e}{2R_s} \qquad ...(7)$$

Similarly, total energy of satellite on the surface of earth

$$E' = K' + U' = \frac{GM_s M_e}{2R_e} - \frac{GM_s M_e}{R_e} = -\frac{GM_s M_e}{2R_e}$$

Thus when satellite falls on the earth, the loss in the energy is given by

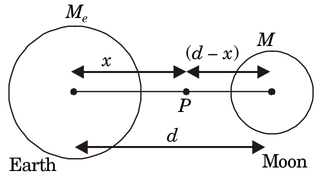
$$\Delta E = -\frac{GM_s M_e}{2R_s} - \left[-\frac{GM_s M_e}{2R_e}\right] = \frac{GM_s M_e}{2}\left[\frac{1}{R_e} - \frac{1}{R_s}\right]$$

If t is the time taken by satellite to fall to the earth, then $\alpha t = \Delta E = \frac{GM_s M_e}{2}\left[\frac{1}{R_e} - \frac{1}{R_s}\right]$

$$t = \frac{GM_s M_e}{2\alpha}\left[\frac{1}{R_e} - \frac{1}{R_s}\right]$$

---

13. At what distance from the centre of the moon is the point at which the strength of the resultant of the Earth's and Moon's gravitational fields is equal to zero? The earth's mass is assumed to be η times that of the moon, and the distance between the centres of these planets is η' times greater than the radius of the earth.



Soln.: Let us assume that

M = Mass of the moon and R_e = Radius of the earth

Hence, Mass of earth, $M_e = \eta M$

And distance between centres of earth and moon $d = \eta' R_e$

Suppose the resultant of the earth's and moon's gravitational fields is zero at some point P at a distance x from earth's centre.

Hence gravitational field due to earth at point $P = \frac{GM_e}{x^2} = \frac{G\eta M}{x^2}$ ...(1)

and gravitational field due to moon at point $P = \dfrac{GM}{(d-x)^2}$

Hence, net field at P

$\dfrac{G\eta M}{x^2} - \dfrac{GM}{(d-x)^2} = 0$ or $\dfrac{G\eta M}{x^2} = \dfrac{GM}{(d-x)^2}$ or $\dfrac{\sqrt{\eta}}{x} = \dfrac{1}{(d-x)}$

or $x = \sqrt{\eta}(d-x)$ or $x(\sqrt{\eta}+1) = \sqrt{\eta}d$ or $x = \dfrac{\sqrt{\eta}d}{\sqrt{\eta}+1} = \dfrac{\sqrt{\eta}\eta' R_e}{\sqrt{\eta}+1}$

Hence, the distance from the centre of the moon $= d - x = \eta' R_e - \dfrac{\sqrt{\eta}\eta' R_e}{\sqrt{\eta}+1} = \dfrac{\eta' R_e}{\sqrt{\eta}+1}$

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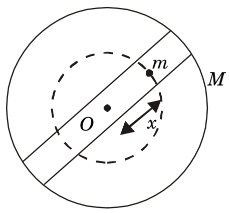
14. A smooth tunnel is made along diameter of the earth. A body of mass 10 kg is dropped from one end of the tunnel. Find its kinetic energy and potential energy when it passes through the centre of the earth. Given the gravitational acceleration of the earth at its surface is 10 ms^{-2} and radius of the earth is 6.4×10^6 m.

Soln.: Potential energy of the body at the surface



$= -\dfrac{GMm}{R} = -\dfrac{gR^2 m}{R}$ $\left[\because g = \dfrac{GM}{R^2}\right]$

$= -gRm = -10 \times 6.4 \times 10^6 \times 10$

$= -6.4 \times 10^8$ J = Total energy at the surface

When the body is dropped into the tunnel, the force on it at a distance of x from the centre is given by,

$F = -G\dfrac{4\pi x^3 \rho m}{3x^2} = -\dfrac{4}{3}\pi G\rho m x = -\dfrac{gm}{R}x$

Thus, the body executes a simple harmonic motion with angular frequency ω given by, $\omega = \sqrt{\dfrac{g}{R}}$

$\therefore$ Kinetic energy at the centre O of the earth $= \dfrac{1}{2}m\omega^2 R^2 = \dfrac{1}{2}m \times \dfrac{g}{R} \times R^2 = \dfrac{1}{2}mgR$

$= \dfrac{1}{2} \times 10 \times 10 \times 6.4 \times 10^6 = 3.2 \times 10^8$ J

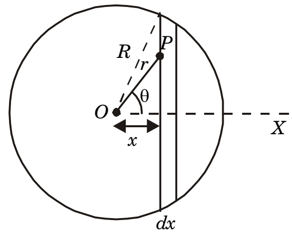
Thus, potential energy at $O = 3.2 \times 10^8 - (-6.4 \times 10^8) = 9.6 \times 10^8$ J

---

15. A uniform solid sphere of mass M and radius R is cut into two by a diametrical plane. Show that the resultant attraction between the two halves is $3GM^2/16 R^2$.

Soln.: Consider a strip of thickness dx at a distance x from the centre O of the sphere.

The area of the strip $= \pi (R^2 - x^2)$.



$\therefore$ Mass of the strip $= \pi (R^2 - x^2)\, dx\rho$ where ρ is the density of the solid.

At a point P, r cm from O, the force per unit mass is

$G \cdot \dfrac{M}{r^2} = G \cdot \dfrac{4\pi r^3 \rho}{3r^2} = \dfrac{4}{3}\pi G r\rho = F$ (say)

The component of this force along OX is $F \cos\theta = \dfrac{4}{3}\pi G r\rho . \dfrac{x}{r} = \dfrac{4}{3}\pi G\rho x$

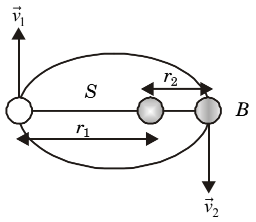
The sine components will cancel out by taking corresponding elements above and below OX. Same is the case with all point masses on the strip.

$\therefore$ The attraction on the strip along $OX = \dfrac{4}{3}\pi\, G\rho x \times \pi (R^2 - x^2)\, dx\, .\, \rho = \dfrac{4}{3}\pi^2 G\rho^2 (R^2 x - x^3)\, dx$

$\therefore$ Total attraction between the hemispheres $= \dfrac{4}{3}\pi^2 G\rho^2 \int_0^R (R^2 x - x^3)dx = \dfrac{4}{3}\pi^2 G\rho^2 \left[\dfrac{R^4}{2} - \dfrac{R^4}{4}\right]$

$= \dfrac{1}{3}\pi^2 G\rho^2 R^4 = \dfrac{3}{16} G\dfrac{M^2}{R^2}$

16. A planet of mass m moves along an ellipse around the Sun so that its maximum and minimum distances from the Sun are equal to r_1 and r_2 respectively. Calculate the angular momentum of this planet relative to the centre of the sun.



Soln.: According to law of conservation of angular momentum

$mv_1r_1 = mv_2r_2$ ...(1)

and from the conservation of energy $-\frac{GM_sm}{r_1} + \left(\frac{1}{2}\right)mv_1^2 = -\frac{GM_sm}{r_2} + \left(\frac{1}{2}\right)mv_2^2$

where M_s is the mass of the sun.

$\therefore \quad GM_s\left[\frac{1}{r_2} - \frac{1}{r_1}\right] = \frac{v_2^2}{2} - \frac{v_1^2}{2} = \frac{v_1^2r_1^2}{2r_2^2} - \frac{v_1^2}{2}$ or $GM_s\left(\frac{r_1 - r_2}{r_1r_2}\right) = \frac{v_1^2}{2}\left(\frac{r_1^2}{r_2^2} - 1\right) = \frac{v_1^2}{2}\left(\frac{r_1^2 - r_2^2}{r_2^2}\right)$

$\therefore \quad v_1^2 = \frac{2GM_s(r_1 - r_2)r_2^2}{r_1r_2(r_1^2 - r_2^2)} = \frac{2GM_sr_2}{r_1(r_1 + r_2)}$ or $v_1 = \sqrt{\left[\frac{2GM_sr_2}{r_1(r_1 + r_2)}\right]}$

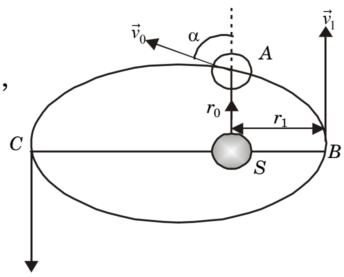
$\therefore \quad L = mv_1r_1 = m\sqrt{\left[\frac{2GM_sr_1r_2}{r_1 + r_2}\right]}$

---

17. A planet A moves along an elliptical orbit around the Sun. At the moment when it was at the distance r_0 from the sun its velocity was equal to v_0 and angle between the radius vector $\vec{r}_0$ and the velocity vector $\vec{v}_0$ was equal to α. Calculate the maximum and minimum distance that will separate this planet from the Sun during its orbital motion.

Soln.: The situation is shown in figure. According to law of conservation of angular momentum

$|\vec{L}_A| = |\vec{r}_0 \times m\vec{v}_0| = mv_0r_0 \sin\alpha = |\vec{L}_B| = mv_1r_1$



Hence, $v_1 = \frac{v_0r_0\sin\alpha}{r_1}$ ...(1)

Now, from the conservation of total energy between the points A and B,

$(T.E.)_A = (T.E.)_B$

$\frac{1}{2}mv_0^2 - \frac{GM_sm}{r_0} = \frac{1}{2}m_1v_1^2 - \frac{GM_sm}{r_1}$ or $v_0^2 - \frac{2GM_s}{r_0} = v_1^2 - \frac{2GM_s}{r_1}$

or $v_0^2r_1^2 - \frac{2GM_s}{r_0}r_1^2 = v_1^2r_1^2 - 2GM_sr_1$ ...(2)

Substituting the value of v_1r_1 from (1) in equation (2), we get

$\left(v_0^2 - \frac{2GM_s}{r_0}\right)r_1^2 = v_0^2r_0^2 \times \sin^2\alpha - 2GM_sr_1$ or $\left(v_0^2 - \frac{2GM_s}{r_0}\right)r_1^2 + (2GM_s)r_1 - v_0^2r_0^2\sin^2\alpha = 0$

Solving the equation for r_1

$$r_1 = \frac{-2GM_s \pm \sqrt{[4G^2M_s^2 + 4v_0^2r_0^2\sin^2\alpha\{v_0^2 - (2GM_s/r_0)\}]}}{2\{v_0^2 - (2GM_s/r_0)\}}$$

$$= \frac{GM_s \mp \sqrt{\left[G^2M_s^2 - (v_0^2r_0^2\sin^2\alpha)\{(2GM_s/r_0) - v_0^2\}\right]}}{\{(2GM_s/r_0) - v_0^2\}} = \frac{1 \mp \sqrt{1 - \frac{v_0^2r_0^2\sin^2\alpha}{GM_s}\left\{\frac{2}{r_0} - \frac{v_0^2}{GM_s}\right\}}}{\left(\frac{2}{r_0} - \frac{v_0^2}{GM_s}\right)}$$

$$r_1 = \frac{r_0\left[1 \mp \sqrt{\{1 - \eta\sin^2\alpha(2 - \eta)\}}\right]}{(2 - \eta)} \qquad \left(\text{Where } \eta = \frac{v_0^2r_0}{GM_s}\right)$$

---

18. Calculate the radius of the circular orbit of a stationary Earth satellite, which remains motionless with respect to its surface. What are its velocity and acceleration in the inertial frame fixed at a given moment to the centre of the earth?

Soln.: Because the satellite is stationary relative to the earth. This implies that the time period for both earth and satellite is same.

Hence time period for satellite = T = 24 hours.

Suppose v be the velocity of a satellite of mass m revolving around the earth at a height h above its surface. Hence $\frac{mv^2}{R_e+h} = \frac{GmM_e}{(R_e+h)^2}$ or $v = \sqrt{\frac{GM_e}{R_e+h}}$

Now time period $T = \frac{2\pi(R_e+h)}{v} = 2\pi\frac{(R_e+h)^{3/2}}{\sqrt{(GM_e)}}$ or $(R_e+h)^3 = \left(\frac{T}{2\pi}\right)^2 GM_e$

or $r = (R_e + h) = \left[\left(\frac{T}{2\pi}\right)^2 GM_e\right]^{1/3}$

Thus radius of circular orbit of stationary satellite $r = (R_e + h) = \left[\left(\frac{T}{2\pi}\right)^2 GM_e\right]^{1/3}$

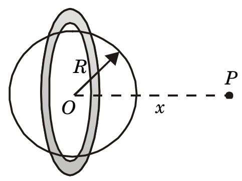
$= \left[\left(\frac{24\times 3600}{2\times 3.14}\right)^2 \times 6.67\times 10^{-11}\times 5.96\times 10^{24}\right]^{1/3}$ = 4.2×10^4 km

Now, velocity = $\sqrt{\frac{GM_e}{R_e+h}} = \sqrt{\frac{6.67\times 10^{-11}\times 5.96\times 10^{24}}{4.2\times 10^7}}$ = 3100 m/sec

and acceleration = $\frac{v^2}{r} = \frac{GM_e}{(R_e+h)^2} = \frac{6.67\times 10^{-11}\times 5.96\times 10^{24}}{(4.2)^2\times 10^{14}}$ = 0.22 m/sec^2

---

19. A planet of mass M and radius R is spherical in shape. The mass is uniformly distributed over the volume of the sphere. The planet is encircled by a concentric ring shaped body of mass $M/2$. The plane of the ring is in equatorial plane of the planet and it can be considered to be a thin ring of radius $(3/2)R$. Calculate the magnitude of gravitational field strength at a point P at a distance x from the common centre along a line perpendicular to the plane of the ring. (Assume $x > R$).



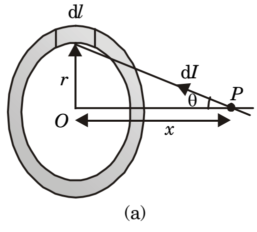
Soln.: Let us first calculate the field at P due to the ring shaped body.

Let the mass of the ring be $M/2$ (say m) and its radius $3R/2$ (say r).

Consider a small element of length dl on the ring.

Mass of this element is $dm = (m/2\pi r)\, dl$

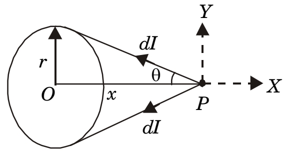
The field strength (dI) at P due to this element $dI = G\frac{m\,dl}{2\pi r}\cdot\frac{1}{(r^2+x^2)}$



(a)

in the direction shown in figure (a).

If a similar element is considered diametrically opposite to the previous element, field due to both will be numerically equal in the directions shown in figure (b).



(b)

Obviously the components along Y-axis cancel each other, and components along the X-axis get added up. Hence, the direction of resultant field is towards O. [For all elements, such pair can be made]

$\therefore\ dI_x = dI\cos\theta = G\frac{mdl}{2\pi r}\frac{\cos\theta}{(r^2+x^2)} = G\frac{mxdl}{2\pi r(r^2+x^2)^{3/2}}$ $\left[\because \cos\theta = \frac{x}{\sqrt{r^2+x^2}}\right]$

$\therefore$ Net field at P is $I = G\frac{mx}{2\pi r(r^2+x^2)^{3/2}}\int_0^{2\pi r} dl = G\frac{mx}{(r^2+x^2)^{3/2}}$

Here, $m = M/2$ and $r = 3R/2$

$$\therefore \quad I_{\text{ring}} = \frac{GMx}{2\left[\frac{9R^2}{4} + x^2\right]^{3/2}} = \frac{4GMx}{(9R^2 + 4x^2)^{3/2}}$$

We know that field due to a sphere of mass M and radius R, for an outside point is $I_{sp} = GM/x^2$

Hence the resultant field at P is $I = \frac{4GMx}{(9R^2 + 4x^2)^{3/2}} + \frac{GM}{x^2}$ towards the common centre.

---

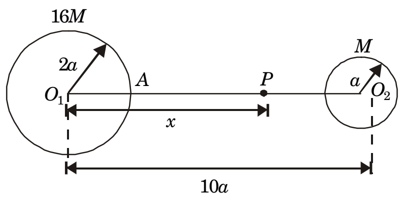
20. Distance between the centres of two stars is $10a$. The masses of these stars are M and $16M$ and their radii a and $2a$ respectively. A body of mass m is fired straight from the surface of the large star towards the smaller star. What would be its minimum initial speed to reach the surface of the smaller star? Obtain an expression in terms of G, M and a.

Soln.: The two stars are separated by distance of $10a$. The projected body experiences a large attractive force due to the larger star and a smaller force in opposite direction due to the other star. There is a point on the line joining the centre of the two stars where the force due to both the stars become equal. Beyond this point, the attractive force of the smaller star becomes greater and hence the body collides with the surface of the smaller star. Let P be the point at a distance x from the centre of the larger star where the gravitational field intensity due to two stars becomes equal and opposite. If our body is imparted sufficient enough kinetic energy to reach point P, it will reach the smaller star.

$$G\frac{16M}{x^2} = \frac{GM}{(10a - x)^2}$$

or $16(10a - x)^2 - x^2 = 0$ or $(40a - 5x)(40a - 3x) = 0$

$\therefore \quad x = 8a$ or $x = 40\,a/3$



Latter is > $10a$ and hence discarded.

Now, let us apply the principle of energy conservation.

The body is provided enough energy so that it reaches P and loses all its kinetic energy.

Potential energy of the body on the surface of the larger star is contribution of both the stars. Hence

$$U_A = -\frac{Gm(16M)}{2a} - \frac{GmM}{8a} = -\frac{65Gm}{8a}$$ [Take note of distances $2a$ and $8a$]

Potential energy at P is $U_P = -\frac{GMm}{2a} - \frac{G(16M)m}{8a} = -\frac{5GMm}{2a}$

From the law of conservation of energy, $U_A + \frac{1}{2}mv_{\min}^2 = U_P$

or $-\frac{65GMm}{8a} + \frac{1}{2}mv_{\min}^2 = -\frac{5GMm}{2a}$ or $\frac{1}{2}mv_{\min}^2 = \frac{65GMm}{8a} - \frac{5GMm}{2a} = \frac{45}{8}\frac{GMm}{a}$

or $v^2_{\min} = \frac{45GM}{4a}$ or $v_{\min} = \frac{3}{2}\sqrt{\frac{5GM}{a}}$.

The above example has been deliberately provided to emphasize the usefulness of conservation principles in solving the problems related to gravitation. Gravitation force is conservative in nature. This increases the usefulness of laws of conservation in solving problems related to gravitation.

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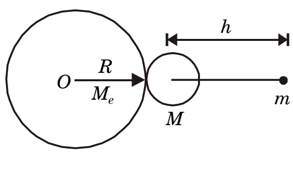
21. A cord of length 64 m is used to connect a 100 kg astronaut to spaceship whose mass is much larger than that of the astronaut. Assume that the spaceship is orbiting close to earth's surface and the astronaut and the spaceship fall on the same straight line from the earth centre. The radius of earth is $R = 6400$ km. Estimate the tension in the string. Neglect the gravitational pull of satellite on the astronaut.



Soln.: The situation has been shown in the figure. Centre of earth is O. The astronaut is made to revolve around the earth with the angular velocity of the satellite. The centripetal force of the astronaut is provided by the gravitational attraction of the earth and the tension in the string.

Let the mass of the satellite be M.

For the satellite, $\frac{GM_eM}{R^2} = M\omega^2R$ [$\because$ satellite is very close to the earth]

or $\omega^2 = \frac{GM_e}{R^3}$ or $\omega = \sqrt{\frac{GM_e}{R^3}}$ ...(i)

Both tension and gravitational force of earth, pull the astronaut towards the centre.

$\therefore$ Centripetal force $= T + \frac{GM_em}{(R+h)^2}$ or $m\omega^2 (R + h) = T + \frac{GM_em}{(R+h)^2}$

Since, $GM_e/R^2 = g$, $\therefore$ $T = m\omega^2(R+h) - \frac{gR^2m}{(R+h)^2} = m.\frac{GM_e(R+h)}{R^3} - \frac{gR^2m}{(R+h)^2}$ [Using (i)]

$$= mg\left(\frac{R+h}{R}\right) - \frac{gm}{\left(1+\frac{h}{R}\right)^2} = mg\left(1+\frac{h}{R}\right) - mg\left[1+\frac{h}{R}\right]^{-2}$$

Expanding the second term and neglecting higher order terms as $h << R$

$$= mg + mg\frac{h}{R} - mg\left[1 - \frac{2h}{R}\right] = \frac{3mgh}{R} = \frac{3\times 100\times 9.8\times 64}{6400\times 1000} = 2.94 \times 10^{-2}\text{ N.}$$

---

22. Two satellites S_1 and S_2 revolve around a planet in coplanar circular orbits in the same sense. The radii of orbits of S_1 and S_2 are r_1 and r_2 respectively. At certain moment their position vectors with respect to the planet are inclined at an angle θ.

(a) What is the angular velocity (ω) of S_2 as actually observed by an astronaut in S_1?

(b) Find ω when S_1 and S_2 are nearest and farthest to each other.

(c) If the period of revolution of S_1 and S_2 are 1 hour and 8 hour respectively and radius of orbit of S_1 is 10^4 km, find the angular speed of S_2 observed by an astronaut in S_1, when they are closest to each other.

Soln.: (a) Figure (a) shows the situation. θ = angle between their position vectors.

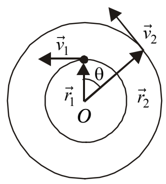
Angle between their velocity vectors $\vec{v}_1$ and $\vec{v}_2$ will also be θ, as the angle between perpendiculars to two straight lines is same as angle between the two lines.

$\vec{v}_2 - \vec{v}_1$ = relative velocity of S_2 with respect to S_1, $\vec{r}_2 - \vec{r}_1$ = position vector of S_2 with respect to S_1.

Let $\vec{\omega}$ = angular velocity of S_2 with respect to S_1. Then, $\vec{v}_2 - \vec{v}_1 = \vec{\omega}\times(\vec{r}_2 - \vec{r}_1)$

Taking cross product on both sides with $(\vec{r}_2 - \vec{r}_1)$

$(\vec{r}_2 - \vec{r}_1)\times(\vec{v}_2 - \vec{v}_1) = (\vec{r}_2 - \vec{r}_1)\times[\vec{\omega}\times(\vec{r}_2 - \vec{r}_1)]$ ...(i)



(a)

We know that $\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}.\vec{C}) - \vec{C}(\vec{A}.\vec{B})$ [This rule is known as "bac(k)-cab" rule]

$\therefore$ $(\vec{r}_2 - \vec{r}_1)\times[\vec{\omega}\times(\vec{r}_2 - \vec{r}_1)] = \vec{\omega}[(\vec{r}_2 - \vec{r}_1).(\vec{r}_2 - \vec{r}_1)] - (\vec{r}_2 - \vec{r}_1)[(\vec{r}_2 - \vec{r}_1).\vec{\omega}] = \vec{\omega}|\vec{r}_2 - \vec{r}_1|^2 - 0$

Because $(\vec{r}_2 - \vec{r}_1).\vec{\omega} = 0$ as $(\vec{r}_2 - \vec{r}_1)$ and $\vec{\omega}$ are perpendicular vectors.

Hence, from (i)

$\therefore$ $\vec{\omega}|\vec{r}_2 - \vec{r}_1|^2 = (\vec{r}_2 - \vec{r}_1)\times(\vec{v}_2 - \vec{v}_1)$, $\vec{\omega} = \frac{\vec{r}_2\times\vec{v}_2 + \vec{r}_1\times\vec{v}_1 - \vec{r}_2\times\vec{v}_1 - \vec{r}_1\times\vec{v}_2}{|\vec{r}_2 - \vec{r}_1|^2}$

$$\vec{\omega} = \frac{r_2v_2\hat{n} + r_1v_1\hat{n} - r_2v_1\sin(90-\theta)\hat{n} - r_1v_2(\sin 90 - \theta)\hat{n}}{|\vec{r}_2 - \vec{r}_1|^2}$$

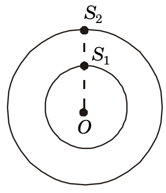
where $\hat{n}$ = a unit vector perpendicular to the plane of the figure.

$\sin(90 + \theta) = \cos\theta$, where $(90 + \theta)$ is angle between $\vec{r}_2$ and $\vec{v}_1$

and $\sin(90 - \theta) = \cos\theta$, where $(90 - \theta)$ is the angle between $\vec{r}_2$ and $\vec{v}_1$ and $\vec{r}_1$ and $\vec{v}_2$.

$\therefore$ $\omega = \frac{r_2v_2 + r_1v_1 - (r_2v_1 + r_1v_2)\cos\theta}{r_2^2 + r_1^2 - 2r_1r_2\cos\theta}$ ...(ii)

(b) When S_1 and S_2 are nearest to each other, $\theta = 0°$

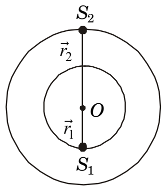


(b)

$$\therefore \quad \omega = \frac{r_2v_2 + r_1v_1 - (r_2v_1 + r_1v_2)}{r_2^2 + r_1^2 - 2r_1r_2} \quad \text{(from (ii)}$$

$$= \frac{v_2(r_2 - r_1) - v_1(r_2 - r_1)}{(r_2 - r_1)^2} = \frac{(v_2 - v_1)}{(r_2 - r_1)}$$

When S_1 and S_2 are farthest to each other, $\theta = 180°$



(c)

$$\therefore \quad \omega = \frac{r_2v_2 + r_1v_1 + r_2v_1 + r_1v_2}{(r_2 + r_1)^2} = \frac{(r_2 + r_1)(v_2 + v_1)}{(r_2 + r_1)^2} = \frac{v_2 + v_1}{r_2 + r_1}$$

(c) We know that for a satellite, $T^2 \propto r^3$ where T = time period, r = radius of path

$$\therefore \quad T_1^2 \propto r_1^3 \text{ and } T_2^2 \propto r_2^3 \Rightarrow \frac{T_1^2}{T_2^2} = \frac{r_1^3}{r_2^3} \qquad \text{or} \quad \frac{1}{8^2} = \left(\frac{10^4}{r_2}\right)^3 \Rightarrow \frac{10^4}{r_2} = \left(\frac{1}{64}\right)^{1/3} = \frac{1}{4}$$

or $r_2 = 4 \times 10^4$ km

Now $v_1 = \dfrac{2\pi r_1}{T_1} = \dfrac{2\pi \times 10^4}{1} = 2\pi \times 10^4$ km/hr. and $v_2 = \dfrac{2\pi r_2}{T_2} = \dfrac{2\pi \times 4 \times 10^4}{8} = \pi \times 10^4$ km/hr

When they are closest to each other, $\omega = \dfrac{v_2 - v_1}{r_2 - r_1} = -\dfrac{\pi \times 10^4}{4 \times 10^4 - 10^4} = -\dfrac{\pi}{3}$ radian/hr

The (–ve) sign shows that relative to S_1, S_2 is rotating in clockwise sense (with respect to figure a).

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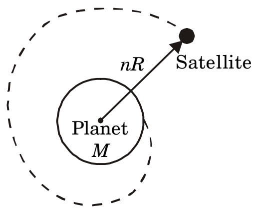
23. A satellite of a planet revolves in a circular orbit whose radius exceeds the radius of the planet n times. The radius of the planet is R. Suddenly, the satellite starts experiencing slight resistance due to cosmic dust. Assuming the resistance force to depend on the velocity of satellite as $F = \alpha v^3$, where α is a constant. Find how long the satellite will stay in orbit until it falls into the planet's surface? Assume the mass of satellite as m, and that of planet M.



Soln.: The satellite slows down, reducing its radius of revolution and eventually falls down on the planet (see figure).

When the radius of revolution is nR, the velocity of satellite is given as

$$\frac{mv_1^2}{nR} = \frac{GMm}{(nR)^2} \qquad \text{or,} \quad v_1 = \sqrt{\frac{GM}{nR}}$$

The satellite falls on the planet's surface when the radius of the orbit reduces to R. Let the velocity be v_2 just before touching the planet.

$$\frac{mv_2^2}{R} = \frac{GMm}{R^2} \qquad \text{or} \quad v_2 = \sqrt{\frac{GM}{R}}.$$

$$\text{Now,} \quad F = m\frac{dv}{dt} \qquad \text{or,} \quad m\frac{dv}{dt} = \alpha v^3 \qquad \text{or,} \quad \frac{dv}{v^3} = \frac{\alpha}{m}dt$$

$$\text{Integrating} \int_{v_1}^{v_2} \frac{dv}{v^3} = \frac{\alpha}{m}\int_0^t dt \qquad \text{or,} \quad -\frac{1}{2}\left[\frac{1}{v^2}\right]_{v_1}^{v_2} = \frac{\alpha}{m}t \quad \text{or} \quad \frac{1}{v_1^2} - \frac{1}{v_2^2} = \frac{2\alpha}{m}t$$

$$\text{or,} \quad \frac{nR}{GM} - \frac{R}{GM} = \frac{2\alpha}{m}t \qquad \text{or} \quad t = \frac{m(n-1)R}{2\alpha GM}$$

Alternatively, at any time t let the orbital radius of the satellite be r. Then $\dfrac{GMm}{r^2} = \dfrac{mv^2}{r}$ or, $v^2 = \dfrac{GM}{r}$

The total energy of the satellite at this instant is E = kinetic energy + potential energy

$$= \frac{1}{2}mv^2 - \frac{GMm}{r} = \frac{GMm}{2r} - \frac{GMm}{r} = -\frac{GMm}{2r}$$

The rate of change of energy with changing radius is $\frac{dE}{dr} = \frac{GMm}{2r^2}$ ...(i)

But the rate of work done by the force must also be equal to the rate of change of energy. Hence

$$\frac{dE}{dt} = -F.v \text{ or, } \frac{dE}{dt} = -\alpha v^4 = -\alpha\left(\frac{GM}{r}\right)^2$$

$$\Rightarrow \frac{dE}{dr} \cdot \frac{dr}{dt} = -a\left(\frac{GM}{r}\right)^2 \quad \text{...(ii)}$$

From (i) and (ii) we can relate r and t

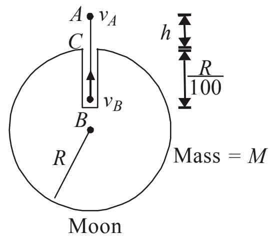
$$\frac{GMm}{2r^2}\mathrm{d}r = -\alpha\left(\frac{GM}{r}\right)^2 \mathrm{d}t \quad \text{or,} \quad \mathrm{d}t = -\frac{m}{2GM.\alpha}\mathrm{d}r$$

Integrating $\int_0^t \mathrm{d}t = -\frac{m}{2GM.\alpha}\int_{nR}^{R} \mathrm{d}r$ or $\boxed{t = \frac{m}{2\alpha GM}(n-1)R.}$

---

24. A body is projected vertically upwards from the bottom of a crater of moon of depth $\frac{R}{100}$ where R is the radius of moon with a velocity equal to the escape velocity on the surface of moon. Calculate maximum height attained by the body from the surface of the moon.

Soln.: BC is the crater of moon. B denotes bottom of crater.



Depth of crater = $\frac{R}{100}$, Radius of moon = R, Speed of particle at $B = v_B$

Escape velocity on surface of moon = v_e $\quad\therefore\quad v_e = \sqrt{\frac{2GM}{R}}$

At highest point $v_A = 0$

Mechanical energy is conserved in the process.

Decrease in kinetic energy = $\frac{1}{2}mv_B^2$

Increase in $PE = U_A - U_B$

$$V_A = \text{Potential at } A = -\frac{GM}{(R+h)}, \quad V_B = \text{Potential at } B = -\frac{GM}{R^3}\left[1.5R^2 - 0.5\left(R - \frac{R}{100}\right)^2\right]$$

$$= -\frac{GM}{R^3}\left[\frac{3R^2}{2} - \left(\frac{1}{2}\right)\left(\frac{99R}{100}\right)^2\right] = -\frac{GM}{R^3} \times \frac{R^2}{2}[3 - 0.98] = -\frac{GM}{R} \times \frac{2.02}{2}$$

$$\therefore \quad \text{Increase in } PE = -\frac{GMm}{R+h} + \frac{GMm}{R} \times (1.01)$$

Equate KE and PE.

$$\frac{1}{2}mv_B^2 = -\frac{GMm}{(R+h)} + \frac{GMm \times 1.01}{R}$$

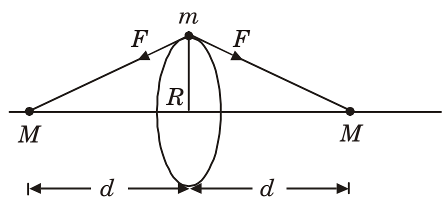
$$\frac{GM}{R} = GM\left[-\frac{1}{R+h} + \frac{1.01}{R}\right] \quad \text{or} \quad \frac{1}{R} = -\frac{1}{R+h} + \frac{1.01}{R} \quad \text{or} \quad \frac{1}{R+h} = \frac{0.01}{R} = \frac{1}{100R}$$

or $100R = R + h$ or $h = 99R$

---

25. There are two fixed heavy masses of magnitude M of high density on x-axis at $(d, 0, 0)$ and $(-d, 0, 0)$. A small mass m moves in a circle of radius R in the y-z plane between the heavy masses. Find the speed of the small particle.

Soln.: Force of attraction between M and m is $F = \frac{GMm}{R^2 + d^2}$



By symmetry F_x components will cancel.

$\therefore$ The net force, which provides the centripetal force, is given by

$$2F_y = 2 \cdot \frac{GMm}{(R^2 + d^2)} \times \frac{R}{(R^2 + d^2)^{1/2}} = 2R\frac{GMm}{(R^2 + d^2)^{3/2}}$$

$\Rightarrow \quad \dfrac{mv^2}{R} = 2R\dfrac{GMm}{(R^2+d^2)^{3/2}} \qquad \Rightarrow \quad v = \left\{\dfrac{2GMR^2}{(R^2+d^2)^{3/2}}\right\}^{1/2}$

---

26. Two satellite of same mass are launched in the same orbit around the earth so that they rotate opposite to each other. If they collide elastically obtain the total energy of the system before and just after the collision. Describe the subsequent motion of the wreckage.

Soln.: Potential energy of the satellite in its orbit $= -\dfrac{GMm}{r} \quad \Rightarrow \quad \text{K.E.} = \dfrac{|U|}{2} = \dfrac{GMm}{2r}$

where m is mass of satellite, M the mass of the earth and r the orbital radius.

Total energy of one satellite = kinetic energy + potential energy $= \dfrac{GMm}{2r} - \dfrac{GMm}{r} = -\dfrac{GMm}{2r}$

For two satellites, total energy $E = -\dfrac{GMm}{r}$

Let v' be the velocity after collision. By conservation of mementum $m\vec{v}_1 + m\vec{v}_2 = 0 = (m+m)v' \quad \Rightarrow \quad v' = 0$

The wreckage of mass $(2m)$ has no kinetic energy, but it has only potential energy. So, energy after collision $= \dfrac{-GM(2m)}{r}$

The wreckage falls down under gravity.

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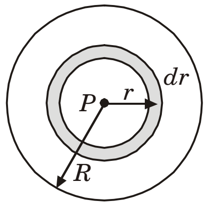
27. A uniform sphere has a mass M and radius R. Find the gravitational pressure P inside the sphere, as a function of the distance r from its centre.

Soln.: Consider a layer of thickness dr at a distance r from the centre of the sphere.



Now force due to the layer $dF = 4\pi r^2 dr\rho \qquad \Rightarrow \quad (dP)4\pi r^2 = \dfrac{G\left(\frac{4}{3}\pi r^3\rho\right)\left(4\pi r^2 dr\rho\right)}{r^2}$

(where r is the mean density of sphere)

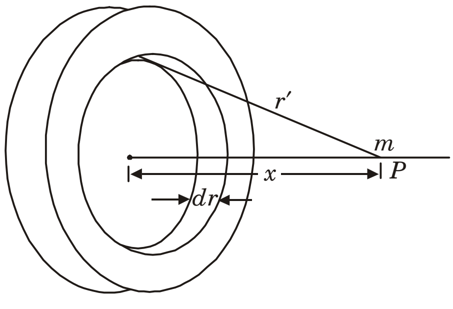
or, $\quad dP = G\dfrac{4}{3}\pi\rho^2 r dr \qquad \therefore \quad P = G\displaystyle\int_t^R \dfrac{4}{3}\pi\rho^2 r dr \qquad$ or $\quad P = G\dfrac{2\pi}{3}\rho^2(R^2 - r^2)$

$= \dfrac{3}{8}G\dfrac{\{1 - r^2/R^2\}M^2}{\pi R^4} \qquad [\because \ \rho = M/(4/3)\pi R^3]$

---

28. What is the gravitational potential energy of a particle of mass m kept at a distance x from the centre of a disc of mass M on its axis? The radius of the disc is R.

Soln.: The gravitational potential of the differential ring is given as $dV = -G\dfrac{dM}{r'}$



where $dM = \dfrac{M}{\pi R^2} \times 2\pi r dr = \dfrac{2Mrdr}{R^2} \quad$ and $\quad r' = \sqrt{r^2 + x^2}$

$\Rightarrow \quad dV = -G\left(\dfrac{2Mrdr}{R^2}\right)\Big/\sqrt{r^2+x^2}$

$\Rightarrow$ The potential due to the entire disc at the point P is given as

$V = \displaystyle\int dV = -\dfrac{2Gm}{R^2}\int_0^R \dfrac{rdr}{\sqrt{r^2+x^2}} = -\dfrac{GM}{R^2}\int_0^R -\dfrac{2rdr}{\sqrt{r^2+x^2}}$

Let, $r^2 + x^2 = t^2 \qquad \Rightarrow \quad 2rdr = 2tdt$

$\therefore \quad V = -\dfrac{GM}{R^2}\displaystyle\int \dfrac{2tdt}{t} = -\dfrac{2GM}{R^2}t \qquad \Rightarrow \quad V = -\dfrac{2GM}{R^2}\left|\sqrt{r^2+x^2}\right|_0^R = -\dfrac{2GM}{R^2}\left[\sqrt{R^2+x^2} - x\right]$

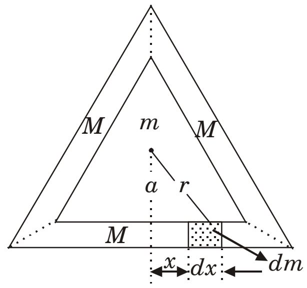
The gravitational potential energy of the system $U = mV = -\dfrac{2GMm}{R^2}\left(\sqrt{R^2+x^2} - x\right)$

---

29. Three uniform rods, each of mass M and length l are connected to form an equilateral triangle in a gravity free space. Another small body of mass m is kept at the centroid. Find the minimum velocity v to be given to mass m so that it escapes the gravitational pull of the triangle.

Soln.: The gravitational potential energy between the point mass m and the elementary segment is given as $dU = -\frac{Gm(dm)}{r} = -Gm\left\{\left(\frac{M}{l}\right)dx\right\}\Big/\sqrt{a^2+x^2} = -\frac{GMm}{l}\frac{dx}{\sqrt{a^2+x^2}}$

$\therefore$ The total gravitational potential energy associated with the system



$$U = 3\int dU = -\left(\frac{3GMm}{l}\right)2\int_0^{l/2}\frac{dx}{\sqrt{a^2+x^2}} = -\frac{6GMm}{l}\left[\ln\left|\frac{x}{a}+\frac{\sqrt{a^2+x^2}}{a}\right|\right]_0^{l/2}$$

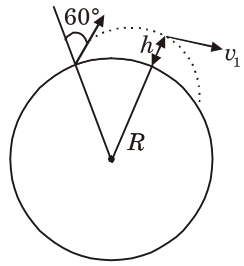
$$\Rightarrow \quad U = -\frac{6GMm}{l}\ln\left|\frac{l/2+\sqrt{a^2+(l/2)^2}}{a}\right| U = -\frac{6GMm}{l}\cdot\ln\left|\frac{l+\sqrt{l^2+4a^2}}{2a}\right|$$

For minimum velocity $|U| = \frac{1}{2}mv^2 \quad \therefore \quad v = \left[\frac{12GM}{l}\cdot\ln\left|\frac{l+\sqrt{l^2+4a^2}}{2a}\right|\right]^{1/2}$

Putting $a = \frac{l}{2\sqrt{3}}, \quad v = \left[\frac{12GM}{l}\ln(2+\sqrt{3})\right]^{1/2}$

---

30. A missile is launched at an angle of 60° to the vertical with a velocity $\sqrt{0.75gR}$ from the surface of the earth (R is the radius of the earth). Find its maximum height from the surface of earth. (Neglect air resistance and rotation of earth.)



Soln.: From conservation of mechanical energy

$$\frac{1}{2}mv_0^2 - \frac{GMm}{R} = \frac{1}{2}mv_1^2 - \frac{GMm}{R+h} \quad \text{......(i)}$$

Also from conservation of angular momentum

$$mv_0R\sin 60° = mv_1(R+h) \quad \text{......(ii)}$$

Solving (i) and (ii) and putting $v_0 = \sqrt{\frac{3GM}{4R}}$, we get $h \approx 0.25\,R$ (Approx.)

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31. A satellite of mass 2×10^3 kg has to be shifted from an orbit of radius $2R$ to another orbit of radius $3R$, where R is the radius of earth, Calculate the minimum energy required. [R = 6400 km and g = 10 m/s^2.]

Soln.: Total mechanical energy of a satellite in a circular orbit of radius $r = -\frac{1}{2}\frac{GMm}{r}$

E_1 = energy in first orbit $= -\frac{1}{2}\frac{GMm}{2R}$, $\quad E_2$ = energy in second orbit $= -\frac{1}{2}\frac{GMm}{3R}$

ΔE, energy (minimum) required $= E_2 - E_1 = \frac{1}{2}\frac{GMm}{R}\left(\frac{1}{2}-\frac{1}{3}\right) \Rightarrow \Delta E = \frac{GMm}{12R} = \frac{mgR^2}{12R} \quad (GM = gR^2)$

$$\Rightarrow \quad \Delta E = \frac{mgR}{12} = 1.067\times 10^{10}\ \text{J}$$

---

32. An artificial satellite (mass m) of a planet (mass M) revolves in a circular orbit whose radius is n times the radius R of the planet. In the process of motion, the satellite experiences a slight resistance due to cosmic dust. Assuming the force of resistance on satellite to depend on velocity as $F = -av^2$ where a is a constant, calculate how long the satellite will stay iln the orbit before it falls onto the planet's surface.

Soln.: Air resistance $F = -av^2$, where orbital velocity $v = \sqrt{\frac{GM}{r}}$; r = the distance of the satellite from planet's center. $\Rightarrow F = -\frac{GMa}{r}$

The work done by the resistance force $dW = F\cdot dx = [Fvdt]$

$$= \left[\frac{GMa}{r}\sqrt{\frac{GM}{r}}dt\right] = \left[\frac{(GM)^{3/2}a}{r^{3/2}}dt\right] \quad \text{.......(i)}$$

$\Rightarrow$ The loss of energy of the satellite $= dE \quad \therefore \quad \frac{dE}{dr} = \frac{d}{dr}\left[-\frac{GMm}{2r}\right] = \frac{GMm}{2r^2}$

$$\Rightarrow \quad dE = \frac{GMm}{2r^2}dr \qquad \text{.......(ii)}$$

Since $dE = -dW$ (work energy theorem)

$$-\frac{GMm}{2r^2}dr = \frac{(GM)^{3/2}a}{r^{3/2}}dt \qquad \Rightarrow \quad t = -\frac{m}{2a\sqrt{GM}}\int_{nR}^{P}\frac{dr}{\sqrt{r}} \qquad \Rightarrow \quad t = -\frac{m\sqrt{R}(\sqrt{n}-1)}{a\sqrt{GM}} = (\sqrt{n}-1)\frac{m}{a\sqrt{gR}}.$$

---

33. Two satellites A and B, of equal mass, move in the equatorial plane of earth close to the earth's surface. Satellite A moves in the same direction as that of the rotation of the earth while satellite B moves in the opposite direction. Determine the ratio of the kinetic energy of B to that of A in the reference frame fixed to earth.

Soln.: The orbital speed of a satellite very close to earth $= v_0 = \sqrt{\frac{GM}{R}} = \sqrt{g_0 R}$

The peripheral speed of earth $= v_e = R\omega_e = \frac{2\pi}{T_e}R$

$\Rightarrow$ The velocities of the satellites relative to earth are given by

$= v_{r_1} = \frac{2\pi}{T_e}R + \sqrt{g_0 R}$ and $v_{r_2} = \sqrt{g_0 R} - \frac{2\pi}{T_e}R$

Positive and negative sign are for the satellites orbiting form east to west to east repectively because earth rotates from west to east

$$\Rightarrow \quad \frac{\text{KE}_1}{\text{KE}_2} = \frac{\frac{1}{2}mv_{r_1}^2}{\frac{1}{2}mv_{r_2}^2} = \left(\frac{v_{r_1}}{v_{r^2}}\right)^2 = \frac{\left(\frac{2\pi}{T_e}R + \sqrt{g_0 R}\right)^2}{\left(-\frac{2\pi}{T_e}R + \sqrt{g_0 R}\right)^2}$$

Putting $R = 6.4 \times 10^6$ m, $g_0 = 9.8$ m/sec^2 and $T_e = 86400$ sec, $\quad \Rightarrow \frac{\text{KE}_1}{\text{KE}_2} = 1.265$

EXERCISE

MCQs — One Correct Option

1. A body is taken from the surface of the earth to the moon. What will be the change in the weight of the body?
(a) increase (b) decrease
(c) remains same
(d) first decrease and then increase.

2. Newton's law of gravitation is valid for
(a) small bodies only
(b) planets only
(c) both small and big bodies
(d) only valid for solar system

3. When astronauts returned from the moon they had to live for some days in a caravan with the air pressure inside lower than that outside. The reason for this is that
(a) it is easier to move about
(b) the men have to get used to earth
(c) they need to get used to normal air pressure gradually
(d) none of the above

4. The orbital velocity of a body in an orbit of radius R from the center of the earth is
(a) 0 (b) $\sqrt{\frac{GM}{R}}$
(c) $\sqrt{\frac{2GM}{R}}$ (d) none of these

5. A planet has $2^{1/3}$ times the radius of the earth. The escape velocity of a body on the planet will be (mass of planet is equal to that of earth.)
(a) same as earth (b) higher than earth
(c) lower than earth (d) cannot be determined

6. The acceleration due to gravity on a satellite is 1.96 m/s^2, while on the earth it is 9.80 m/s^2. If the jumping from a height of 5 m is safe at the earth, then jumping from what height will be safe at that satellite?
(a) 2.5 m (b) 25 m
(c) 5 m (d) 50 m

7. Four particles of mass m are kept at the corners of a square of side a. What will be the total force on any particle?
(a) $\frac{\sqrt{2}Gm^2}{a^2}$ (b) $(1+\sqrt{2})\frac{Gm^2}{a^2}$
(c) $\frac{(1+2\sqrt{2})Gm^2}{2a^2}$ (d) none of these

8. What is the maximum height reached by a rocket fired with speed equal to 80% of escape velocity from earth's surface?
(a) $\frac{16}{25}R$ (b) $\frac{16}{9}R$
(c) $\frac{9}{16}R$ (d) $\frac{15}{7}R$

9. A planet of mass m moves along an ellipse around the sun so that its maximum and minimum distance from the sun are equal to r_1 and r_2 respectively. The angular momentum M of this planet relative to the centre of the sun is
(a) $\sqrt{\frac{GM_s r_1^3}{r_1 r_2}}$ (b) $\sqrt{\frac{2GM_s r_1 r_2}{r_1+r_2}} \times m$
(c) $\sqrt{\frac{2Gm r_1 r_2}{r_1+r_2}}$ (d) none of these

10. A satellite S is moving in an elliptical orbit around the earth. The mass of the satellite is very small compared to the mass of the earth.
(a) The acceleration of S is always directed towards the centre of the earth.
(b) The angular momentum of S about the centre of the earth changes in direction but its magnitude remains constant.
(c) The total mechanical energy of S varies periodically with time.
(d) The linear momentum of S remains constant in magnitude.

11. A man is in a satellite which is at a distance of 2000 km from the surface of the earth. The man has to be thrown out so that he can escape the earth completely. With what velocity should he be thrown out?
(a) $\sqrt{\frac{2GM}{R}}$ (b) $\sqrt{\frac{2GM}{R+2000}}$
(c) $\sqrt{\frac{GM}{R+2000}}$ (d) $\frac{1}{20}\sqrt{\frac{GM}{5}}$

12. A body has to be placed in an orbit of height H

from the surface of the earth. The body should be projected with a velocity of

(a) $\sqrt{\frac{GM}{R+H}}$ (b) $\sqrt{\frac{GM}{H}}$

(c) greater than $\sqrt{\frac{GM}{R+H}}$

(d) lesser than $\sqrt{\frac{GM}{R+H}}$

13. What amount of energy is required to dislodge a part of the earth with mass 25% mass of the earth, so that it can form a satellite of earth orbiting at a radius of 10000 km from the earth's centre?

(a) 1.8302×10^{32} J (b) 2.232×10^{32} J
(c) 1.532×10^{32} J (d) 0.4067×10^{32} J

14. The total gravitational potential energy of a spherical shell of inner radius r and outer radius R and mass m is

(a) $-\frac{3Gm^2}{(R^3-r^3)^2}\left[\frac{R^5-r^5}{5}-r^3\frac{(R^2-r^2)}{2}\right]$

(b) $\frac{3Gm^2}{(R^3-r^3)^2}\left[\frac{R^5-r^5}{5}+r^3\frac{(R^2-r^2)}{2}\right]$

(c) zero (d) none of these

15. A solid sphere has a mass of 1 kg and radius of 1 m. What is the gravitational potential at the center of the sphere?

(a) $4G$ (b) $-4G$
(c) $-3G/2$ (d) $3G$

16. Two massive particles of masses M and m $(M > m)$ are separated by a distance l. They rotate with equal angular velocity under their gravitational attraction. The linear speed of the particle of mass m is

(a) $\sqrt{\frac{GMm}{(M+m)l}}$ (b) $\sqrt{\frac{GM^2}{(M+m)l}}$

(c) $\sqrt{\frac{Gm}{l}}$ (d) $\sqrt{\frac{Gm^2}{(M+m)l}}$

17. A particle is projected from the mid-point of the line joining two fixed particles each of mass m. If the separation between the fixed particles is l, the minimum velocity of projection of the particle so as to escape is equal to

(a) $\sqrt{\frac{Gm}{l}}$ (b) $\sqrt{\frac{Gm}{2l}}$

(c) $\sqrt{\frac{2Gm}{l}}$ (d) $2\sqrt{\frac{2Gm}{l}}$

18. A projectile is launched from the surface of the earth with a very high speed v at an angle θ with vertical. What is its velocity when it is at the farthest distance from the earth surface? Given that the maximum height reached by the projectile is equal to the height reached when it is launched perpendicular to earth with a velocity $\sqrt{\frac{Gm}{R}}$.

(a) $\frac{v\cos\theta}{2}$ (b) $\frac{v\sin\theta}{2}$

(c) $\sqrt{\frac{GM}{2R}}$ (d) $\sqrt{\frac{GM}{3R}}$

19. The fractional change in the value of free-fall acceleration g for a particle when it is lifted from the surface to an elevation $h (h << R)$ is

(a) h/R (b) $-(2h/R)$
(c) $2h/R$ (d) none of these

20. If the change in the value of g at a height h above the surface of the earth is the same as at a depth x below it, then (both x and h being much smaller than the radius of the earth) :

(a) $x = h$ (b) $x = 2h$
(c) $x = (1/2)h$ (d) $x = h^2$

21. The mean radius of the earth is R, its angular speed about its own axis is ω and the acceleration due to gravity at the earth surface is g. The cube of radius of orbit of geostationary satellite will be

(a) (R^2g/ω) (b) $(R^2\omega/g)$
(c) (Rg/ω^2) (d) (R^2g/ω^2)

22. A particle hanging from a spring stretches it by 1 cm at earth's surface. Radius of earth is 6400 km. At a place 800 km above the earth's surface, the same particle will stretch the spring by

(a) 1 cm (b) 8 cm
(c) 0.1 cm (d) 0. 79 cm

23. A satellite goes along an elliptical path around earth. The rate of change of arc length a swept by the satellite is proportional to

(a) r (b) r^2
(c) $r^{1/2}$ (d) r^{-1}

24. A planet has twice the density of earth, but the acceleration due to gravity on its surface is exactly the same as on the surface of earth, its radius in terms of radius of earth R will be

(a) $R/4$ (b) $R/2$
(c) $R/3$ (d) $R/8$

25. Two particles of mass M and m are initially at rest and infinitely separated. When they move towards each other due to gravitational attraction, their relative velocity at any instant in terms of d distance between them at that instant is

(a) $\left(\frac{2Gd}{M+m}\right)^{1/2}$ (b) $\left[\frac{2G(M+m)}{d}\right]^{1/2}$
(c) $\frac{2G(M+m)}{d}$ (d) $\frac{2Gd}{M+m}$

26. The weight of a body on earth is denoted by W and the acceleration due to gravity is g. Newton's second law can be written as $F = \frac{W}{g}a$. The acceleration due to gravity on the moon is g_1. What is the expression for Newton's law on the moon?

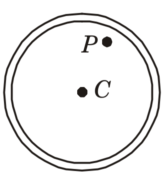
(a) $F = \frac{W}{g_1}a$ (b) $F = \frac{W}{g}g_1 a$
(c) $F = \frac{W}{g}a$ (d) $F = \frac{W}{g_1}ga$

27. The orbital velocity of an artificial satellite in a circular orbit just above earth's surface is v_0. For a satellite orbiting in acircular orbit at an altitude of half of earth's radius is

(a) $\sqrt{\frac{3}{2}}\,v_0$ (b) $\sqrt{\frac{2}{3}}\,v_0$
(c) $\frac{3}{2}v_0$ (d) $\frac{2}{3}v_0$

28. The dimensional formula for gravitational constant is
(a) $[M^{-1}L^3T^{-2}]$ (b) $[M^3L^{-1}T^{-2}]$
(c) $[M^{-1}L^2T^3]$ (d) $[M^2L^3T^{-1}]$

29. The force between a hollow sphere and a point mass at P inside it as shown in the figure



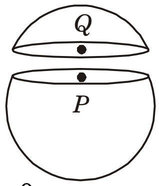
(a) is attractive and constant
(b) is attractive and depends on the position of the point with respect to centre C.
(c) is zero
(d) is repulsive and constant.

30. A body of mass m is moved to a height equal to the radius of the earth R. The increase in its potential energy is
(a) mgR (b) $2mgR$
(c) $\frac{1}{2}mgR$ (d) $\frac{1}{4}mgR$

31. A binary star system consists of two stars. One star has twice the mass of the other. The star rotates about their common centre of mass
(a) Star having the smaller mass has twice angular momentum compared to heavier star
(b) Both stars have same angular momentum about centre of mass
(c) Both stars have same linear speed
(d) Both the stars have same kinetic energy

32. The escape velocity of a particle of mass m varies as
(a) m^2 (b) m
(c) m^0 (d) m^{-1}

33. A spherical shell is cut into two pieces along a chord as shown in figure. For points P and Q



(a) $E_P > E_Q$
(b) $E_P > E_Q$
(c) $E_P = E_Q = 0$ (d) $E_P = E_Q \neq 0$

34. The period of rotation of the earth so as to make any object weight-less on its equator is ($r = 6.4 \times 10^6$ m, $g = 9.8$ m/s^2)
(a) 84 min (b) 74 minutes
(c) 64 minutes (d) 54 minutes

35. The gravitational field due to a mass distribution is given by $l = (A/x^3)$ in X-direction. The gravitational potential at a distance x is equal to
(a) $-A/x^3$ (b) $-A/2x^2$
(c) $-A/x^4$ (d) $A/2x^2$

36. Three particles, each of mass m, are placed at the corners of an equilateral triangle of side d. The potential energy of the system is
(a) $\frac{3Gm^2}{d}$ (b) $\frac{Gm^2}{d}$
(c) $\frac{-3Gm^2}{d}$ (d) none of these

37. The energy required to remove a body of mass m from earth's surface to far away is equal to
(a) $2mgR$ (b) mgR
(c) $-mgR$ (d) zero

38. The energy required to shift a satellite from orbital radius r to orbital radius $2r$ is E. What energy will be required to shift the satellite from orbital radius $2r$ to orbital radius $3r$?
(a) E (b) $E/2$

(c) $E/3$ (d) $E/4$

39. A person brings a mass of 1 kg from infinity to a point A. Initially the mass was at rest but it moves at a speed of 2 m/s as it reaches A. The work done by the person on th mass is –3 J. The potential at infinity is –10 J. The potential at A is

(a) 8 J (b) 10 J
(c) 2 J (d) 5 J

40. The density inside an isolated large solid sphere of radius a = 4 km is given by $\rho = \rho_0\, a/r$ where ρ_0 is the density at the surface and equals to 10^9 kg/m^3 and r denotes the distance from the centre. The gravitational field in m/s^2 due to this sphere at a distance $2a$ from its centre is (Take $G = 6.65 \times 10^{-11}$ Nm^2/kg^2)

(a) 418 m/s^2 (b) 382 m/s^2
(c) 258 m/s^2 (d) 948 m/s^2

MCQs — More than One Correct Option

1. A planet is revolving round the sun in an elliptical orbit. The work done on the planet by the gravitational force of sun is zero

(a) in some parts of the orbit
(b) in any part of the orbit
(c) in no part of the orbit
(d) in one complete revolution

2. Two objects of masses m and $4m$ are at rest at an infinite separation. They move towards each other under mutual gravitational attraction. If G is the universal gravitational constant. Then at separation r

(a) the total energy of the two objects is zero
(b) their relative velocity of approach is $\left(\frac{10Gm}{r}\right)^{1/2}$ in magnitude
(c) the total kinetic energy of the objects is $\frac{4Gm^2}{r}$
(d) net angular momentum of both the particles is zero about any point

3. Let V and E be the gravitational potential and gravitational field. Then select the correct alternative(s)

(a) The plot of E against r (distance from centre) is discontinuous for a spherical shell
(b) The plot of V against r is continuous for a spherical shell
(c) The plot of E against r is discontinuous for a solid sphere
(d) The plot of V against r is continuous for a solid sphere

4. In elliptical orbit of a planet

(a) angular momentum about centre of sun is constant
(b) potential energy is constant
(c) kinetic energy is constant
(d) total mechanical energy is constant

5. For a satellite revolving in circular orbit suppose V_0 is the orbital speed, T its time period, U its potential energy and K the kinetic energy. Now value of G is decreased. Then

(a) V_0 will decrease (b) T will decrease
(c) U will decrease (d) K will decrease

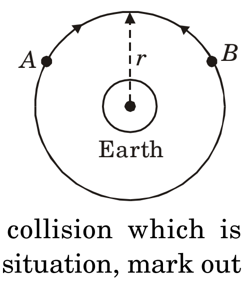
6. Due to a solid sphere magnitude of :

(a) gravitational potential is maximum at centre
(b) gravitational potential is constant
(c) kinetic energy is constant
(d) total mechanical energy is constant

7. A spring of spring constant k is fixed to the ceiling of a lift. The other end of the spring is attached to block hangs in equilibrium. Now the lift starts accelerating downwards with an acceleration $2g$. [Assume the length of the spring to be large and its mass negligible]. Now,

(a) the block would not perform simple harmonic motion and would stick to the ceiling.
(b) the block performs SHM with time period $T = 2\pi\sqrt{\frac{m}{k}}$.
(c) the minimim energy of the spring block system during the motion of the block would be zero
(d) the block performs simple harmonic motion with amplitude $\frac{2mg}{k}$.

8. Consider two satellites A and B of equal mass m, moving in same circular orbit about earth, but in opposite sense as shown in figure.



The orbital radius is r.
The satellites undergo a collision which is perfectly inelastic. For this situation, mark out the correct statement(s). [Take mass of earth as M].

(a) The total energy of the two satellites plus earth system just before collision is $-\frac{GMm}{r}$.

(b) The total energy of the two satellites plus earth system just before collision is $\frac{-2GMm}{r}$.

(c) The total energy of the two satellites plus earth system just before collision is $\frac{-GMm}{2r}$.

(d) The combined mass (two satellites) will fall towards the earth just after collision.

9. For two satellites at distance R and $7R$ above the earth's surface, the ratio of their

(a) total energies is 4 and potential and kinetic energies is 2

(b) potential energies is 4

(c) kinetic energies is 4

(d) total energies is 4

10. If both the mass and radius of the earth decrease by 1%, the value of

(a) acceleration due to gravity would decrease by nearly 1%

(b) acceleration due to gravity would increase by 1%

(c) escape velocity from the earth's surface would decrease by 1%

(d) the gravitational potential energy of a body on earth's surface remain unchanged

Assertion Reason

Each question contains Statement-1 (Assertion) and Statement-2 (Reason). Of these Statements, mark correct choice if

(a) Statements-1 and 2 are true and Statement-2 is a correct explanation for Statement-1

(b) Statements-1 and 2 are true and Statement-2 is not a correct explanation for Statement-1

(c) Statement-1 is true, Statement-2 is false

(d) Statement-1 is false, Statement-2 is true.

1. **Statement-1 :** Smaller the orbit of the planet around the sun, shorter is the time it takes to complete one revolution.
Statement-2 : According to Kepler's third law of planetary motion, square of time period is proportional to cube of mean distance from sun.

2. **Statement-1 :** The value of acceleration due to gravity does not depend upon mass of the body.
Statement-2 : Acceleration due to gravity is a constant quantity.

3. **Statement-1 :** The difference in the value of acceleration due to gravity at pole and equator is proportional to square of angular velocity of earth.
Statement-2 : The value of acceleration due to gravity is minimum at the equator and maximum at the pole.

4. **Statement-1 :** There is no effect of rotation of earth on acceleration due to gravity at poles.
Statement-2 : Rotation of earth is about polar axis.

5. **Statement-1 :** Gravitational potential of earth at every place on it is negative.
Statement-2 : Every body on earth is bound by the attraction of earth.

6. **Statement-1 :** A planet moves faster, when it is closer to the sun in its orbit and vice versa.
Statement-2 : Orbit velocity in orbit of planet is constant.

7. **Statement-1 :** The time period of geostationary satellite is 24 hours.
Statement-2 : Geostationary satellite must have the same time period as the time taken by the earth to complete one revolution about its axis.

8. **Statement-1 :** When distance between two bodies is doubled and also mass of each body is doubled, gravitational force between them remains the same.
Statement-2 : According to Newton's law of gravitation, force is directly proportional to masses of bodies and inversely proportional to square of distance between them.

9. **Statement-1 :** A body becomes weightless at the centre of earth.
Statement-2 : As the distance from centre of earth decreases, acceleration due to gravity increases.

10. **Statement-1 :** Space rockets are usually launched in the equatorial line from west to east.
Statement-2 : The acceleration due to gravity is minimum at the equator.

11. **Statement-1 :** The binding energy of a satellite depends upon the mass of the satellite.
Statement-2 : Binding energy is the negative value of total energy of satellite.

12. **Statement-1 :** We can not move even a finger without disturbing all the stars.
Statement-2 : Every body in this universe attracts every other body with a force which is inversely proportional to the square of distance between them.

13. **Statement-1 :** The speed of satellite always remains constant in an orbit.

Statement-2 : The speed of a satellite depends on its path.

14. **Statement-1 :** Gravitational field intensity is zero both at centre and infinity.

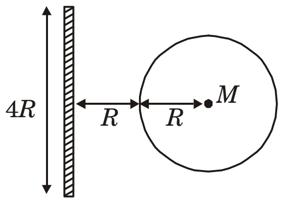
Statement-2 : The dimensions of gravitational field intensity is $[LT^{-2}]$.

15. **Statement-1 :** The square of the period of revolution of a planet is proportional to the cube of its distance from the sun.

Statement-2 : Sun's gravitational field is inversely proportional to the square of its distance from the planet.

Passage Comprehension

PASSAGE 1 : A thin rigid rod of length $4R$ and a sphere of equal mass M and radius R are located in a relative position as shown in the diagram. The centre of the sphere being at a distance of $2R$ from the centre of the rod on its perpendicular bisector. Assume that the system of rod and sphere is in gravity free space.



1. Which of the following best explains the mass density of rod?

(a) Linear (b) Surface
(c) Volume (d) all of the above

2. Let O represents the centre of sphere and A be the centre of the rod. A small element of rod at P makes an angle $\angle AOP = \theta$. The mass of this element is

(a) $\frac{M}{2}\sec^2\theta\, d\theta$ (b) $M\cos^2\theta \cdot d\theta$
(c) $2M\sin\theta \cdot d\theta$ (d) $3M\tan\theta \cdot d\theta$

3. The force on the sphere by the rod is

(a) $\frac{GM^2}{4\sqrt{2R^2}}$ (b) $\frac{GM^2}{4R^2}$
(c) $\frac{GM^2}{R^2}$ (d) none of these.

4. The acceleration of the sphere relative to the rod is

(a) $\frac{GM}{4\sqrt{2}R^2}$ (b) $\frac{GM}{2\sqrt{2}R^2}$
(c) $\frac{GM}{\sqrt{2}R^2}$ (d) $\frac{GM}{R^2}$

5. The rod and sphere collide at a point, which is

(a) at left of rod (b) at right of sphere
(c) at centre of mass of rod + sphere
(d) none of these

PASSAGE 2 : *Earth satellite* : A body moving in an orbit around the earth is called earth's satellite. The moon is the natural satellite of the earth. The first artificial (or man-made) satellite was put into earth's orbit in 1956. Artificial satellites are put into orbit at an altitude of a few hundred kilometres. The satellite is carried in a rocket which is launched from the earth with a velocity greater than the escape velocity. The escape velocity is the velocity with which a body must be projected in order that it may escape the gravitational pull of the earth. When the rocket has achieved the desired height, the satellite is released horizontally by imparting to it a very high speed so that it remains moving in a nearly circular orbit around the earth. This velocity is called the orbital velocity which is about 8 km s^{-1} for a satellite a few hundred kilometers above the earth.

1. The escape velocity of a rocket fired from the earth depends upon

(a) the mass of the rocket
(b) the volume of the fuel in the rocket
(c) the acceleration due to the gravity of the earth
(d) the direction in which the rocket is fired

2. The centripetal force necessary to keep a satellite in a circular orbit around the earth is provide by

(a) a continuous ejection of hot gases by the satellite.
(b) the gravitational attraction between the earth and satellite
(c) the gravitational pull of the sun exerted on the satellite
(d) the weightlessness of the satellite

3. A satellite of mass m is in a stable circular orbit around the earth at an altitude of about 100 kilometers. If M is the mass of the earth, R its radius and g the acceleration due to gravity, the time period T of the revolution of the satellite is given by

(a) $T = 2\pi\sqrt{\frac{R}{g}}$ (b) $T = 2\pi\sqrt{\frac{g}{R}}$
(c) $T = 2\pi\sqrt{\frac{MR}{mg}}$ (d) $T = 2\pi\sqrt{\frac{mR}{Mg}}$

4. An artificial satellite is orbiting the earth at an altitude of 500 km. A bomb is released from the

satellite. This bomb will
(a) explode due to the heat generated by the friction of air
(b) fall freely on the earth
(c) escape into outer space
(d) orbit the earth along with the satellite

PASSAGE 3 : Two stars bound together by gravity orbit each other because of their mutual attraction. Such a pair of stars is referred to as a binary star system. One type of binary system is that of a black hole and a companion star.The black hole is a star that has collapsed on itself and is so massive that not even light rays can escape its gravitational pull. Therefore, when describing the relative motion of a black hole can be assumed negligible compared to that of the companion.

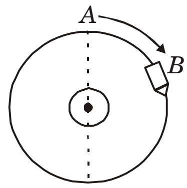
The orbit of the companion star is either elliptical with the black hole at one of the foci or circular with the black hole at the centre. The gravitational potential energy is given by $U = -GmM/r$, where G is the universal gravitational constant, m is the mass of the companion star, M is the mass of the block hole, and r is the distance between the center of the companion star and the center of the black hole. Since the gravitational force is conservative the companion stars total mechanical energy is a constant. Because of the periodic nature of the orbit, there is a simple relation between the average kinetic energy $< K >$ of the companion star and its average potential energy $< U >$. In particular, $< K > = - < U/2 >$.

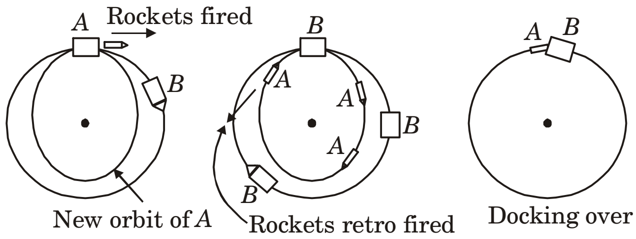
Two special points along the orbit are singled out by astronomers. Perigee is the point at which the companion star is closest to the black hole, and apogee is the point at which it is farthest from the black hole.

1. For circular orbits the potential energy of the companion star is constant throughout the orbit. If the radius of the orbit doubles, what is the new value of the velocity of the companion star?
(a) It is 1/2 of the old value
(b) It is $1/\sqrt{2}$ of the old value
(c) It is the same as old value
(d) It is double the old value

2. Which of the following prevents the companion star from leaving its orbit and falling into the black hole?
(a) The centripetal force
(b) The gravitational force
(c) The companion star's potential energy
(d) The companion star's kinetic energy

PASSAGE 4 : An unmanned satellite A and a space-craft B are orbiting around the earth in same circular orbit as shown. The space-craft is ahead of satellite by some time. Let us consider that some technical problem has come in the satellite and astronaut from B has to make it correct. For this to be done docking of two (A and B) is required (in layman terms connecting A and B). To achieve this, rockets of A has been fired in forward direction and docking takes place as shown in figure below.





Take Mass of earth = 5.98×10^{24} kg
Radius of earth = 6400 km
Orbital radius = 9600 km
Mass of satellite, A = 320 kg
Mass of space-craft = 3200 kg

Assume that initially space-craft B leads satellite A by 100 s i.e., A arrives at any particular position after 100 s of B's arrival. Based on above information answer the following questions :

1. To dock A and B in the above described situation, one can use the rocket system of either one i.e., either of A or of B. To accomplish docking in minimum possible time which is the best way?
(a) To use rocket system of A
(b) To use rocket system of B
(c) Either (a) or (b)
(d) Information insufficient

2. The initial total energy and time period of satellite are respectively,
(a) -6.65×10^{10} J, 9358 s
(b) -6.65×10^{9} J, 9358 s
(c) -6.65×10^{10} J, 9140 s
(d) -6.65×10^{9} J, 9140 s

3. After once returning to the original point *i.e.*, the place from where rockets have been fired, in which direction and with what extent the rockets have to be fired from satellite to again comeback in the original orbit?

(a) forward direction with same extent
(b) backward direction with same extent
(c) forward direction with higher extent
(d) backward direction with higher extent

4. Is the docking done by this much power is proper? If not, then which of the following can cause the proper docking?

(a) no, rocket has to be fired with greater extent
(b) yes, no measure has to be adopted
(c) no, rocket has to be fired in backward direction
(d) no, rocket has to be fired with lesser extent

PASSAGE 5 : The satellites when launched from earth are not given the orbital velocity initially, in practice, a multi-stage rocket propeller carries the space-craft upto its orbit and during each stage rocket has been fired to increase the velocity to acquire the desired velocity for a particular orbit. The last stage of the rocket brings the satellite in circular/elliptical (desired) orbit.

Consider a satellite of mass 150 kg in low circular orbit, in this orbit, we can't neglect the effect of air drag. This air drag opposes the motion of satellite and hence total mechanical energy of earth-satellite system decreases means total energy becomes more negative and hence orbital radius decreases which causes the increase in kinetic energy. When the satellite comes in enough low orbit, the excessive thermal energy generation due to air friction may cause the satellite to burn up. Based on above information, answer the following questions.

1. What is the reason that during launching of satellite, while crossing the atmosphere it doesn't get burnt, but while falling down towards earth or if orbiting in lower orbit, it gets burnt up?

(a) while going up air friction force doesn't come into existence
(b) while going up satellite is with launching vehicle whose speed is controllable
(c) while going up space-craft is protecting the satellite from air friction by itself getting burned
(d) none of these

2. What would be the motion of satellite if air drag has to be considered

(a) moves with uniform speed in the launching orbit
(b) orbital radius decreases continuously as a result moves with non-uniform velocity in elliptical orbit
(c) orbital radius decreases continuously and hence collapses with earth after some time in random manner and there is equal chance of burning up the satellite due to air friction also
(d) moves with non-uniform speed in the launching orbit

3. It has been mentioned in passage that as r decreases, E decreases but K increases. The increases in K is [E = Total mechanical energy, r = orbital radius, K = kinetic energy] is

(a) due to increase in gravitational potential energy
(b) due to decrease in gravitational potential energy
(c) due to work done by air friction force
(d) both (b) and (c)

4. If due to air drag, the orbital radius of earth decreases from R to $R - \Delta R$, $\Delta R << R$, then the expression for increase in orbital velocity Δv is

(a) $\frac{\Delta R}{2}\sqrt{\frac{GM}{R^3}}$ (b) $-\frac{\Delta R}{2}\sqrt{\frac{GM}{R^3}}$

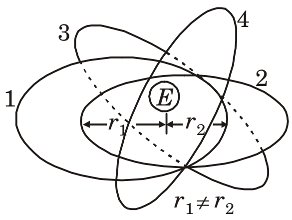
(c) $\Delta R\sqrt{\frac{GM}{R^3}}$ (d) $-\Delta R\sqrt{\frac{GM}{R^3}}$

Matrix Match Type

1.

Column I	Column II
(A) Potential energy of the satellite	(P) $\frac{GmM}{2(R+h)}$
(B) Kinetic energy of the satellite	(Q) $\sqrt{\frac{GM}{(R+h)}}$
(C) Orbital velocity of the satellite	(R) $-\frac{GmM}{(R+h)}$
(D) Escape velocity of the satellite	(S) $\sqrt{\frac{2MG}{R}}$

2. Four identical satellites are orbiting in four elliptical orbits having same semi-major axis but different eccentricities. In Column I some quantities associated with four orbits are given and in Column II the words which can give the information about physical quantities mentioned in Column I. Match the entries of Column I with the entries of Column II.



	Column I		Column II
(A)	Total energy of all four orbits	(P)	Same
(B)	Speed of satellite in all four orbits	(Q)	Different
(C)	Velocity of satellite in all four orbits	(R)	Constant
(D)	Angular momentum of satellites about centre of earth in all four orbits	(S)	Varying

3. An artificial satellite is in circular orbit around the earth. One of the rockets of the satellite is momentarily fired, the direction of firing of rocket is mentioned in Column I and corresponding change(s) are given in Column II. Match the entries of Column I with the entries of Column II

	Column I		Column II
(A)	Towards the earth's centre	(P)	Orbit changes and becomes elliptical
(B)	Away from the earth's centre	(Q)	Orbit plane changes
(C)	At right angle to the plane of orbit	(R)	Semi-major axis of orbit increases
(D)	In forward direction	(S)	Energy of earth-satellite system increases

4.

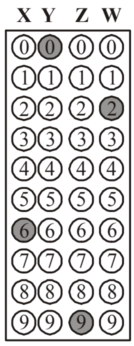
	Column I		Column II
(A)	Elliptical orbit of planet	(P)	Kinetic energy conservation
(B)	Circular orbit of satellite	(Q)	Angular momentum conservation
(C)	Escape velocity	(R)	Independent of mass of particle/satellite
(D)	Orbital velocity	(S)	$\sqrt{\frac{GM}{R}}$

5. On the surface of earth acceleration due to gravity is g and gravitational potential is V. Match the following :

	Column I		Column II
(A)	At height $h = R$, value of g	(P)	decreases by a factor 1/4
(B)	At depth $h = R/2$, value of g	(Q)	decreases by a factor 1/2
(C)	At height $h = R$, value of V	(R)	increases by a factor 11/8
		(S)	increases by a factor 2
		(T)	none

Integer Answer Type

In this section the answer to each of the questions is a single-digit integer, ranging from 0 to 9. If the correct answers to question numbers X, Y, Z and W (say) are 6, 0, 9 and 2, respectively, then the correct darkening of bubbles will look like the following.



1. Three particles, each of mass m, are placed at the vertices of an equilateral triangle of side a. Find the gravitational field intensity at the centroid of the triangle.

2. Find the ratio of the kinetic energy required to be given to the satellite to escape earth's gravitational field to the kinetic energy required to be given so that the satellite moves in circular orbit just above earth's atmosphere.

3. The potential energy of a satellite having mass m and rotating at a height of 6.4×10^6 m from the earth's surface is $-\frac{mgR}{n}$. Find the value of n. (Given $R = 6.4 \times 10^6$ m):

4. A body is projected vertically upwards with a velocity equal to one-third of the escape velocity. The maximum height attained by the body is $h = R/x$. Find the value of x.

5. If a particle is fired vertically upwards from the surface of earth and reaches a height of 6400 km, calculate the initial velocity of the particle. (Assume R = 6400 km and $g = 10\ \text{ms}^{-2}$)

6. The ratio of the energy required to raise a satellite up to a height h above the earth to the kinetic energy of the satellite into the orbit there is xh/R. Find the value of x. (R = radius of earth)

7. The point of suspension of a simple pendulum, with normal time period T_1, is moving vertically upwards according to equation: $y = kt^2$, where $k = 1\ m/s^2$. If new time period is T_2, then find the value of T_2^2.

8. If a satellite is stopped suddenly in its orbit and allowed to fall freely onto the earth. Find the speed with which it hits the surface of the earth.

9. The gravitational field in a region is given by $\vec{E} = (2\hat{i} + 3\hat{j})$ N/kg. No work is done by the gravitational field when a particle is moved on the line $3y + kx = 5$. Find the value of k.

10. A star suddenly shrinks and its density becomes 10^9 times the original value. The value of acceleration due to gravity on its surface will increase by a factor of 10^n g. What is the value of n?

11. Two satellites are orbiting around the earth in circular orbits of the same radius. The mass of satellite A is five times greater than the mass of satellite B. Find the ratio of periods of revolution.

12. Two particles of equal mass m go round a circle of radius R under the action of their mutual gravitational attraction. The speed of each particle is $\frac{1}{2}\sqrt{\frac{GM}{nR}}$. Find the value of n.

13. Two balls A and B are thrown vertically upwards from the same location on the surface of the earth with velocities $2\sqrt{\frac{gR}{3}}$ and $\sqrt{\frac{2gR}{3}}$ respectively, where R is the radius of the earth and g is the acceleration due to gravity on the surface of the earth. Determine the ratio of the maximum height attained by A to that attained by B.

14. Two bodies of masses $M_1 = m$ and $M_2 = 4m$ are placed at a distance r. The gravitational potential at a point on the line joining them where the gravitational field is zero is $\frac{-nGm}{r}$. Find the value of n.

15. The distance between the centres of the earth and the moon is d. The mass of the earth is 81 times that of the moon. $\frac{d}{5n}$ is the distance from the centre of the moon on the line joining the centres of the earth and the moon where the weight of a body zero. Find the value of n.

16. Find the height of the point vertically above the earth's surface at which the acceleration due to gravity becomes 1% of its value at the surface (R is the radius of the earth).

QUESTIONS FROM PREVIOUS YEARS IIT-JEE/JEE Advanced

MCQs : One Correct Option

1. A simple pendulum has a time period T_1 when on the earth's surface, and T_2 when taken to a height R above the earth's surface, where R is the radius of the earth. The value of T_2 / T_1 is

(a) 1 (b) $\sqrt{2}$

(c) 4 (d) 2. **(2001)**

2. A geo-stationary satellite orbits around the earth in a circular orbit of radius 36,000 km. Then, the time period of a spy satellite orbiting a few hundred km above the earth's surface (R_{earth} = 6,400 km) will approximately be

(a) (1/2) hr (b) 1 hr

(c) 2 hr (d) 4 hr **(2002)**

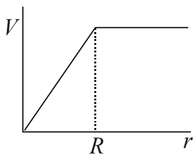
3. A binary star system consists of two stars A and B which have time periods T_A and T_B, radii R_A and R_B and masses M_A and M_B. Then

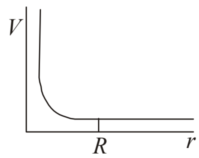
(a) if $T_A > T_B$ then $R_A > R_B$

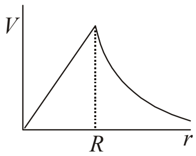
(b) if $T_A > T_B$ then $M_A > M_B$

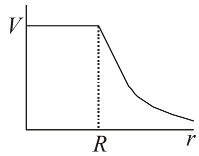
(c) $\left(\frac{T_A}{T_B}\right)^2 = \left(\frac{R_A}{R_B}\right)^3$ (d) $T_A = T_B$. **(2006)**

4. A spherically symmetric gravitational system of particles has a mass density $\rho = \begin{cases} \rho_0 & \text{for } r \le R \\ 0 & \text{for } r > R \end{cases}$ where ρ_0 is a constant. A test mass can undergo circular motion under the influence of the gravitational field of particles. Its speed V as a function of distance r $(0 < r < \infty)$ from the centre of the system is represented by

(a) 

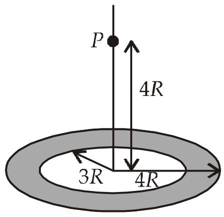
(b) 

(c) 

(d) 

(2008)

5 A thin uniform annular disc (see figure) of mass M has outer radius $4R$ and inner radius $3R$. The work required to take a unit mass from point P on its axis to infinity is



(a) $\frac{2GM}{7R}(4\sqrt{2}-5)$

(b) $-\frac{2GM}{7R}(4\sqrt{2}-5)$

(c) $\frac{GM}{4R}$

(d) $\frac{2GM}{5R}(\sqrt{2}-1)$ **(2010)**

6. A satellite is moving with a constant speed V in a circular orbit about the earth. An object of mass m is ejected from the satellite such that it just escapes from the gravitational pull of the earth. At the time of its ejection, the kinetic energy of the object is

(a) $\frac{1}{2}mV^2$ (b) mV^2

(c) $\frac{3}{2}mV^2$ (d) $2mV^2$ **(2011)**

7. A planet of radius $R = \frac{1}{10} \times (\text{radius of Earth})$ has the same mass density as Earth. Scientists dig a well of depth $\frac{R}{5}$ on it and lower a wire of the same length and of linear mass density 10^{-3} kg m^{-1} into it. If the wire is not touching anywhere, the force applied at the top of the wire by a person holding it in place is (take the radius of Earth = 6×10^6 m and the acceleration due to gravity on Earth is 10 m s^{-2}).

(a) 96 N (b) 108 N

(c) 120 N (d) 150 N

(JEE Advanced 2014)

MCQs : More Than One Correct Option

8. Two spherical planets P and Q have the same uniform density ρ, masses M_P and M_Q, and surface areas A and $4A$, respectively. A spherical planet R also has uniform density ρ and its mass is $(M_P + M_Q)$. The escape velocities from the planets P, Q and R are V_P, V_Q and V_R, respectively. Then

(a) $V_Q > V_R > V_P$ (b) $V_R > V_Q > V_P$

(c) $\frac{V_R}{V_P} = 3$ (d) $\frac{V_P}{V_Q} = \frac{1}{2}$ **(2012)**

9. Two bodies, each of mass M, are kept fixed with a separation $2L$. A particle of mass m is projected from the midpoint of the line joining their centres, perpendicular to the line. The gravitational constant is G. The correct statement(s) is (are)

(a) The minimum initial velocity of the mass m to escape the gravitational field of the two bodies is $4\sqrt{\frac{GM}{L}}$.

(b) The minimum initial velocity of the mass m to escape the gravitational field of the two bodies is $2\sqrt{\frac{GM}{L}}$.

(c) The minimum initial velocity of the mass m to escape the gravitational field of the two bodies is $\sqrt{\frac{2GM}{L}}$.

(d) The energy of the mass m remains constant.

(JEE Advanced 2013)

Assertion and Reason

Each question contains Statement-1 (Assertion) and Statement-2 (Reason). Of these Statements, mark correct choice if

(a) Statements-1 and 2 are true and Statement-2 is a correct explanation for Statement-1

(b) Statements-1 and 2 are true and Statement-2 is not a correct explanation for Statement-1

(c) Statement-1 is true, Statement-2 is false

(d) Statement-1 is false, Statement-2 is true.

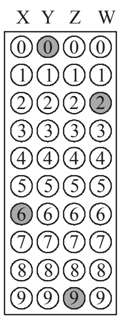
10. Statement-1 : An astronaut in an orbiting space station above the Earth experiences weightlessness.

Statement-2: An object moving around the Earth under the influence of Earth's gravitational force is in a state of free-fall.

(2008)

Integer Type Questions

In this section answer to each of the questions is a single digit integer, ranging from 0 to 9. If the correct answers to question numbers X, Y, Z and W (say) are 6, 0, 9 and 2 respectively, then the correct darkening of bubbles will look like the following



11. Gravitational acceleration on the surface of a planet is $\frac{\sqrt{6}}{11}g$ where g is the gravitational acceleration on the surface of the earth. The average mass density of the planet is $\frac{2}{3}$ times that of the earth. If the escape speed on the surface of the earth is taken to be 11 km s^{-1}, the escape speed on the surface of the planet in km s^{-1} will be **(2010)**

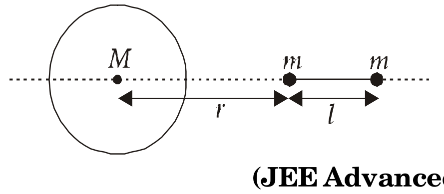
12. A binary star consists of two stars A(mass $2.2M_s$) and B(mass $11M_s$), where M_s is the mass of the sun. They are separated by distance d and are rotating about their centre of mass, which is stationary. The ratio of the total angular momentum of the binary star to the angular momentum of star B about the centre of mass is

(2010)

13. A bullet is fired vertically upwards with velocity v from the surface of a spherical planet. When it reaches its maximum height, its acceleration due to the planet's gravity is $1/4^{th}$ of its value at the surface of the planet. If the escape velocity from the planet is $v_{esc} = v\sqrt{N}$, then the value of N is (ignore energy loss due to atmosphere)

(JEE Advanced 2015)

14. A large spherical mass M is fixed at one position and two identical point masses m are kept on a line passing through the centre of M (see figure). The point masses are connected by a rigid massless rod of length l and this assembly is free to move along the line connecting them. All three masses interact only through their mutual gravitational interaction. When the point mass nearer to M is at a distance $r = 3l$ from M, the tension in the rod is zero for $m = k\left(\frac{M}{288}\right)$. The value of k is



(JEE Advanced 2015)

HINTS & SOLUTIONS

MCQs : One Correct Option

1. **(d) :** At the surface of the earth the weight will be maximum. It will decrease (due to decrease in g) as the body is moved away from the surface of the earth. The weight will be zero at some place between earth and moon, where the gravitational force of attraction on the body by the earth and the moon will be equal. Beyond this upto the moon the weight will increase due to the gravitational force of the moon.

2. **(c) :** Many students have misconception that Newton's law of gravitation is valid only for big (heavenly) bodies. But this is wrong.
Actually gravitational force is the attractive force that one body exerts on another body because of its mass.
Therefore, gravitation exist because of masses of bodies and does not depend on size of the bodies.

3. **(c):** On the surface of the earth, the atmosphere pressure is quite high. The astronauts will feel great discomfort if they move on earth immediately after coming back from moon. To avoid it they need to get used to normal air pressure gradually. That is why they have to live for some days in a caravan with the air pressure lower than outside.

4. **(b):** The orbital velocity of the body does not depend on the mass of the body.
Let the radius of the orbit be R. Let the mass of the body be m and mass of the earth is M. Since the body has to move in a circle along the orbit, the gravitational force should provide the centripetal force.

$$\frac{GmM}{R^2} = \frac{mv^2}{R} \Rightarrow v = \sqrt{\frac{GM}{R}}$$

5. **(c):** *lower than earth*
Escape velocity can be determined by considering the total energy of the planet + body system; When the energy = 0, the body escape from the planet.

Hence $\frac{-GMm}{R} + \frac{1}{2}mv^2 = 0$

The radius of the planet is $2^{1/3} R$

Therefore, $\frac{1}{2}mv^2 = \frac{GMm}{2^{1/3}R} \Rightarrow v = \sqrt{2^{2/3}\frac{GM}{R}}$ ...(1)

Comparing (1) with the escape velocity of earth $\left(\sqrt{\frac{2GM}{R}}\right)$ escape velocity of planet is less than that of earth.

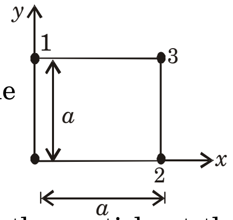
6. **(b):** The safety of jump depends upon the amount of kinetic energy $\left(\frac{1}{2}mv^2\right)$ with which a man hits the ground. The amount of kinetic energy depends on the potential energy (mgh) at the height from which the man is jumping. So in both cases, the potential energy at the respective height should be same.

i.e. $mg_sh_s = mg_eh_e$

or $h_s = \frac{g_eh_e}{g_s} = \frac{9.8 \times 5}{1.96} = 25$ m

Therefore, safe height on the satellite is 25 m.

7. **(c):** The total force acting on any particle is equal to the vectorial sum of individual force acting on the particle. Force of interaction on the particle m placed at a distance L from a mass M is $\frac{GMm}{L^2}$



Force due to particle 1 on the particle at the origin

$= \vec{F}_1 = \frac{Gm \times m}{a^2}\hat{j}$

Force due to particle 2 on the particle at the origin $= \vec{F}_2 = \frac{Gm \times m}{a^2}\hat{i}$

Force due to particle 3 on the particle at the origin $= \vec{F}_3 = \frac{Gm \times m}{(\sqrt{2}a)^2}(\cos 45°\hat{i} + \cos 45°\hat{j})$

Total force $\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3$

$$\Rightarrow \vec{F} = \left(\frac{Gm^2}{a^2} + \frac{Gm^2}{2\sqrt{2}a^2}\right)\hat{i} + \left(\frac{Gm^2}{a^2} + \frac{Gm^2}{2\sqrt{2}a^2}\right)\hat{j}$$

$$= \frac{(1 + 2\sqrt{2})Gm^2}{2\sqrt{2}a^2} \times (\hat{i} + \hat{j})$$

$$|\vec{F}| = \frac{(1 + 2\sqrt{2})Gm^2}{2\sqrt{2}a^2}\sqrt{2}$$

$$|\vec{F}| = \frac{(1 + 2\sqrt{2})Gm^2}{2a^2}$$

8. **(b):** Velocity of rocket, $v = 0.8\ v_e$
Let the rocket reach at a height h. From the law of conservation of energy we have
Energy on surface of earth = energy at height h

or $-\dfrac{GMm}{R}+\dfrac{1}{2}mv^2 = -\dfrac{GMm}{R+h}$

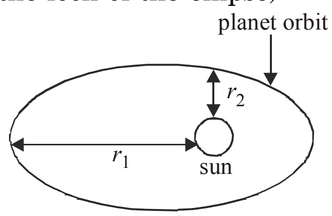
or $v^2 = \dfrac{2GM}{R}-\dfrac{2GM}{R+h}$

or $0.64v_e^2 = 2GM\cdot\dfrac{h}{R(R+h)}$

or $0.32 \times (2gR) = \dfrac{ghR}{R+h}\left[\because \dfrac{GM}{R^2}=g\right]$

or $0.64 = \dfrac{h}{R+h} \Rightarrow h = \dfrac{16}{9}R$

9. **(b) :** Mass of planet = m
Maximum distance of planet from sun = r_1
Minimum distance of planet from sun = r_2
As the planet is under central gravitational interaction. It's angular momentum is conserved about the sun (which is situated at one of the focii of the ellipse)



From conservation of momentum

$mv_1r_1 = mv_2r_2$ or $v_1^2 = \dfrac{v_2^2r_2^2}{r_1^2}$ ...(1)

From the conservation of mechanical energy of the system (sun + planet),

$$-\frac{GM_sm}{r_1}+\frac{1}{2}mv_1^2 = -\frac{GM_sm}{r_2}+\frac{1}{2}mv_2^2$$

or $-\dfrac{GM_s}{r_1}+\dfrac{1}{2}v_2^2\dfrac{r_2^2}{r_1^2} = -\left(\dfrac{GM_s}{r_2}\right)+\dfrac{1}{2}v_2^2$

so $\dfrac{1}{2}v_2^2\left(\dfrac{r_2^2-r_1^2}{r_1^2}\right) = GM_s\left(\dfrac{1}{r_1}-\dfrac{1}{r_2}\right)$

or $\dfrac{1}{2}v_2^2\dfrac{(r_2+r_1)}{r_1^2} = \dfrac{GM_s}{r_1r_2}$

so $v_2 = \sqrt{2GM_sr_1/r_2(r_1+r_2)}$

Hence angular momentum of the planet relative to the centre of the sun,

$$M = mv_2r_2 = \sqrt{\frac{2GM_sr_1r_2}{r_1+r_2}} \times m$$

10. **(a):** when a satellite is moving in an elliptical orbit, its angular momentum $(=\vec{r}\times\vec{p})$ about the centre of earth does not change its direction. The linear momentum $(=m\vec{v})$ does not remain constant as velocity of satellite is constant. The total mechanical energy of S is constant at all locations. The acceleration of S (= centripetal acceleration) is always directed towards the centre of earth.

11. **(b):** This problem can be solved by using the energy principles. For a body to escape from the earth, the total energy ($PE + KE$) should be equal to zero. We calculate the KE and PE and equate it to zero.
Calculating the gravitational potential energy.
Find the change in the gravitational potential energy and the kinetic energy to be needed for the man to escape the earth.

Initial potential energy = $-\dfrac{GMm}{R+2000}$
Where R is the radius of earth.
Calculating the kinetic energy

Initial kinetic energy = $\dfrac{1}{2}mv^2$
Where v is the velocity required to throw the man out.
Applying the energy condition
For a body to escape from earth, the total energy ($PE + KE$) should be = 0

Accordingly, $\dfrac{1}{2}mv^2+\left(-\dfrac{GMm}{R+2000}\right) = 0$

$\Rightarrow -\dfrac{GMm}{R+2000}+\dfrac{1}{2}mv^2 = 0$

$\Rightarrow v = \sqrt{\dfrac{2GM}{R+2000}}$

12. **(c) :** The energy supplied is to overcome the gravitational force and equals potential energy change.
Calculating the gravitational potential.
Find the orbital velocity of the body at a height H from the surface of the earth.
Initial gravitational potential (on the surface of earth) = $-\dfrac{GMm}{R}$
Calculating energy (final) possessed by the mass at height H.
The energy possessed by the body = sum of PE and KE.
Velocity of the mass in an orbit at height $H = \sqrt{\dfrac{GM}{R+H}}$

Therefore KE of the mass = $\dfrac{1}{2}mv^2 = \dfrac{GMm}{2(R+H)}$
PE of the body = gravitational potential energy
$= -\dfrac{GMm}{(R+H)}$
Total energy of the body = KE + PE
$= \dfrac{GMm}{2(R+H)}-\dfrac{GMm}{(R+H)} = -\dfrac{GMm}{2(R+H)}$

Calculating the velocity of projection.

By energy considerations,

Final energy – initial energy = energy to be imparted on the body

$$\Rightarrow GMm\left\{\frac{1}{R}-\frac{1}{2(R+H)}\right\} = \frac{1}{2}mv^2$$

$$\Rightarrow v = \sqrt{\left\{\frac{GM(R+2H)}{R(R+H)}\right\}} > \sqrt{\frac{GM}{R+H}}$$

Therefore correct answer is (c).

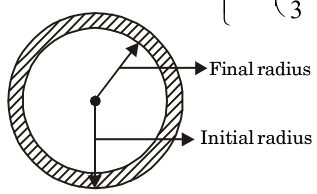
13. **(d) :** 25% of the mass is removed as shown in diagram.

Energy required is equal to change in energy of the system.

The mass removed from the earth will be along the circumference. As a result, the mass and the radius of earth get reduced. The figure shown will explain the situation.

Calculating the radius:

Initial radius of earth $R = \left\{\frac{M}{d\times\left(\frac{4}{3}\pi\right)}\right\}^{1/3}$



The shaded portion indicates the removed mass (= 25% of earth's mass) which is equal to the mass of the satellite)

Mass removed from earth = $0.25M$

Final radius of earth $= \left\{\frac{0.75M}{d\times\left(\frac{4}{3}\pi\right)}\right\}^{1/3} = 0.9085R$

Radius of the satellite $= \left\{\frac{0.25M}{d\times\left(\frac{4}{3}\pi\right)}\right\}^{1/3} = 0.6299R$

Calculating the initial energy of the system

Initial energy = self energy of earth

$$= \frac{-3GM^2}{5R}$$

Substituting, $G = 6.67\times10^{-11}$
$M = 5.98\times10^{24}$
$R = 6.4\times10^{6}$

We get initial energy

$$= -\frac{[3\times6.67\times10^{-11}\times(5.98\times10^{24})^2]}{(5\times6400\times1000)}$$

$= -2.237\times10^{32}$ J

Final energy of the system

Final energy = self energy of earth + self energy of satellite + interaction energy + kinetic energy

Final self energy of earth

$$= -\frac{3GM^2}{5}\times(0.9085R) \qquad ...(1)$$

Final self energy of satellite

$$= -\frac{3GM^2}{5}\times(0.6299R) \qquad ...(2)$$

Interaction energy

$$= -\frac{G(0.75M)(0.25M)}{10^7} \qquad ...(3)$$

Kinetic energy $= \frac{1}{2}(0.25M)\,v^2$

But $v^2 = \left[\frac{G(0.75M)}{10^7}\right]$

Therefore kinetic energy

$$= \frac{\frac{1}{2}G(0.75M)(0.25M)}{10^7} \qquad ...(4)$$

Adding 1 + 2 + 3 + 4 and substituting the appropriate values,

Final energy = -1.8302×10^{32} J

Finding the energy to be added.

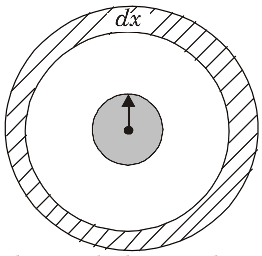
Energy that is to be added = Final energy of the system – initial energy of the system

$= -1.8302\times10^{32} - (-2.237\times10^{32})$

$= 0.4067\times10^{32}$ J

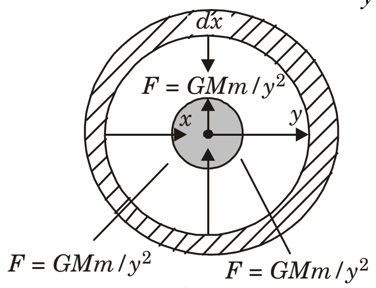
14. **(a) :** Consider that an arbitrarily shape (same as shape of original body) is already filled. Now calculate the work done in bringing a thin shell (same shape as original body) from infinity on to the top of the body. We will consider a spherical shell of inner radius r and outer radius R. Assume that a sphere of radius x is already assembled. We will now find the work done in bringing a thin spherical shell of thickness dx from infinity onto the top of the sphere.

Calculating the force acting on the shell. The force acting on the shell at all points must be the same. If they are same only on parts of the surface, break up the shell in to surfaces such that the force acting on any point in the surface is the same. The forces acting are the same along the surface of the spherical shell. They point inwards. Let us assume that the mass of the shell is m and that the mass of the existing solid sphere is M. This is shown in the figure. We need not break the spherical shell in to small surfaces, since that the value of the force is the same along the surface.



Calculate the work done in bringing the sphere from infinity to top of the arbitrary sized body. It the shell consists of multiple surfaces (with varying forces on each surface), break the shell in to individual surface and calculate the total work done in bringing from infinity to top of the body. Since the force varies with distance, assume that the shell is at a distance y and then calculate the force on the surface. Now integrate this expression with limits of y being from infinity to the surface of the body.

The work done in bringing the shell from infinity to the surface of the sphere is $\int \frac{GMm}{y^2}\mathrm{d}y$.



The limits of y are from ∞ to x.

The value of this is $\frac{GMm}{x}$

Here $M = \frac{4}{3}\pi(x^3 - r^3) \times d$ (d is the density)

$m = 4\pi x^2 dx \times d$

Substituting we have,

$$\mathrm{d}W = G \times 4\pi x^2 d^2 \times \frac{\frac{4}{3}\pi(x^3 - r^3)}{x}\mathrm{d}x$$

Integrate the work done over the entire dimensions of the body. Consider a negative value of this. This will be the gravitational potential of the body.

We need to integrate the value of $\mathrm{d}W$ from r to R.

$$W = \frac{3Gm^2}{(R^3 - r^3)^2}\left[\frac{R^5 - r^5}{5} - r^3\frac{(R^2 - r^2)}{2}\right]$$

The potential energy of the sphere is therefore

$$= -\frac{3Gm^2}{(R^3 - r^3)^2}\left[\frac{R^5 - r^5}{5} - r^3\frac{(R^2 - r^2)}{2}\right]$$

15. **(c) :** The gravitational potential at the centre of the solid sphere is the amount of work do be done in bringing a unit mass body from infinity to the centre of the sphere. The force acting on unit mass body from infinity to the centre of the sphere varying constantly.

Therefore, the potential too varies.

To calculate the net potential, the potential needs to be integrated along the path taken by. the unit mass. (m = 1kg).

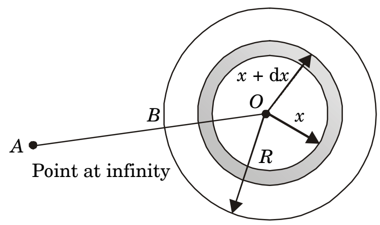
The given figure will explain the situation.

Calculating the potential.

To calculate the potential the work done is calculated. The component of the work done, will give the gravitational potential difference Gravitational force acting on the body as it moves from A (when A is infinity) to B

$= F_1 = \frac{GM}{x^2}$ (where x is the distance of the unit mass from the outer surfaces)

Force on the unit mass inside the sphere from



The shaded portion indicates a thin shell of thickness dx. The work done to be calculated is for displacing the unit mass by distance dx.

B to center O,

$= F_2 = GM\left(\frac{x}{R^3}\right)$

work done in integral form, $\int dW = \int F.dx$

work done from A to B (W_1)

$$\int_\infty^R \mathrm{d}W_1 = \int_\infty^R\left(\frac{GM}{x^2}\right)dx$$

Integrating, $W_1 = \left[-\frac{GM}{x}\right]_\infty^R = \frac{GM}{R}$

Work done from B to centre (W_2)

$$\int \mathrm{d}W_2 = -\int F_2.\mathrm{d}x \Rightarrow \int_R^0 \mathrm{d}W_2 = \int_R^0 GM\left(\frac{x}{R^3}\right)\mathrm{d}x$$

Integrating, $W_2 = \left[\frac{GMx^2}{2R^3}\right]_R^0 = \frac{GM}{2R}$

Therefore net work done = $W_1 + W_2 = \frac{3GM}{2R}$

Potential at the center of the solid sphere = – (work done in bringing an unit mass from infinity to the center of the sphere).

Potential = $-\frac{3GM}{2R}$

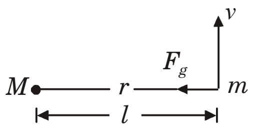
Substituting, M = 1 kg and R = 1 m

Potential = $-\frac{3G}{2}$

16. **(b) :** The system rotates about the centre of mass. The gravitational force acting on the particle m accelerates it towards the centre of the circular path, which has the radius $r = \frac{Ml}{M+m}$

$\Rightarrow F = \frac{mv^2}{r}$

$\Rightarrow \frac{GMm}{l^2} = \frac{mv^2}{\frac{Ml}{M+m}}$



$\Rightarrow v = \sqrt{\frac{GM^2}{(M+m)l}}$

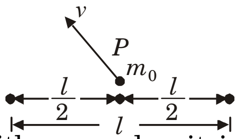
17. **(d) :** The gravitational potential at the mid-point P,

$V = V_1 + V_2 = \frac{-Gm}{(l/2)} - \frac{Gm}{(l/2)} = \frac{4Gm}{l}$

$\Rightarrow$ The gravitational potential energy

$U = -\frac{4Gmm_0}{l}$,

m_0 = mass of particle



When it is projected with a speed v, it just escapes to infinity, and the potential and kinetic energy will become zero.

$\Rightarrow \Delta KE + \Delta PE = 0$

$\Rightarrow \left(0 - \frac{1}{2}m_0v^2\right) + -\left\{-\frac{4Gmm_0}{l}\right\} = 0$

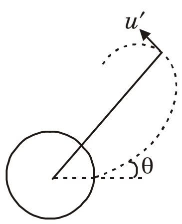
$\Rightarrow v = 2\sqrt{\frac{2Gm}{l}}$.

18. **(b) :** The maximum height reached by the projectile is given by

$\frac{1}{2}mv^2 - \frac{GMm}{R} = -\frac{GMm}{R+h}$

or, $\frac{1}{2}m\frac{GM}{R} - \frac{GMm}{R}$

$= -\frac{GMm}{R+h} = -\frac{GMm}{2R}$



$\therefore\ h = R$

Applying conservation of momentum

$mu'(R+h) = mv\sin\theta R$

$\therefore\ u'(2R) = v\sin\theta R \therefore\ u' = \frac{v\sin\theta}{2}$

19. **(b) :** $g = \frac{GM}{R^2}$ ....(i)

$\frac{dg}{dR} = \frac{-2GM}{R^3}$ putting $dR = h$ we obtain

$\Rightarrow \frac{dg}{h} = \frac{-2GM}{R^2}\cdot\frac{1}{R}$ ....(ii)

From (i) and (ii) $\Rightarrow \frac{dg}{g} = 2\left(\frac{h}{R}\right)$

$\Rightarrow$ Change is –ve. That means g decreases.

20. **(b) :** $g\left(1 - \frac{x}{R}\right) = g\left(1 - \frac{2h}{R}\right)$

$\Rightarrow \frac{x}{R} = \frac{2h}{R} \quad \Rightarrow \frac{x}{h} = 2$

21. **(d) :** $mr\omega^2 = \frac{GMm}{r^2} \quad \Rightarrow r\omega^2 = \frac{GM}{r^2}$

$\Rightarrow r^3 = \frac{GM}{\omega^2} = \frac{GM}{R^2}\cdot\frac{R^2}{\omega^2} \Rightarrow r^3 = g\frac{R^2}{\omega^2}$.

22. **(d) :** $\frac{kx_1}{kx_2} = \frac{F_{gr_1}}{F_{gr_2}} = \left(\frac{r_2}{r_1}\right)^2$

$\Rightarrow \frac{x_1}{x_2} = \left(\frac{800+6400}{6400}\right)^2 = \left(\frac{72}{64}\right)^2 = \left(\frac{9}{8}\right)^2$

$\Rightarrow \frac{x_1}{x_2} = \frac{81}{64}$

$\Rightarrow x_2 = \frac{64}{81}x_1 = \frac{64}{81}$ cm = 0.79 cm.

23. **(d) :** $\frac{dA}{dt} = \frac{1}{2}r^2\omega$ = constant = k

$\Rightarrow \frac{1}{2}r(r\omega) = k \Rightarrow vr = 2k \Rightarrow v \propto (1/r)$.

24. **(b) :** $g = \frac{GM}{R^2} = \frac{G\left(\frac{4}{3}\pi R^3\rho\right)}{R^2} = \frac{4}{3}\pi G\rho R$

$\therefore\ \frac{g_p}{g_e} = \left(\frac{\rho_p}{\rho_e}\right)\left(\frac{R_p}{R_e}\right) \Rightarrow 1 = (2)\frac{R_p}{R_e}$

$\Rightarrow \frac{R_p}{R_e} = 1/2 \Rightarrow R_p = \frac{R}{2}$.

25. **(b) :** The problem may be treated in the centre of mass system coordinates. In this system the displacement and velocities are measured relative to the centre of mass. The reduced mass

$\mu = \frac{mM}{m+M}$

The kinetic energy $= \frac{1}{2}\mu v^2 = \frac{1}{2}\left(\frac{mM}{m+M}\right)v^2$

Work done in moving the masses from infinte distance to a separation distance d is

$GMm\int_{\infty}^{d}\frac{dr}{r^2} = \frac{GMm}{d}$

$\frac{1}{2}\frac{mM}{(M+m)}v^2 - \frac{GMm}{d} = 0$

$\therefore\ v = \sqrt{\frac{2G(M+m)}{d}}$.

26. **(c) :** The mass of the body on earth is W/g. The mass of the body on the moon remain the same namely W/g. Hence Newton's law on the moon is $F = \frac{W}{g}a$.

27. **(b) :** Orbital velocity $= \sqrt{\frac{g_0 R^2}{R+h}}$ where R is radius of earth.

If $h = 0$, $v_0 = \sqrt{\frac{g_0 R^2}{R}} = \sqrt{g_0 R}$

If $h = \frac{R}{2}$, $v = \sqrt{\frac{g_0 R^2}{R + \frac{R}{2}}} = \sqrt{\frac{2g_0 R}{3}} = \sqrt{\frac{2}{3}}\, v_0$.

28. **(a) :** $G = \frac{Fr^2}{m_1 m_2} = \frac{[\text{MLT}^{-2}\text{L}^2]}{[\text{M}^2]} = [\text{M}^{-1}\text{L}^3\text{T}^{-2}]$

29. **(c) :** Since gravitational field inside hollow sphere is zero. Therefore force acting on the particle P and C are zero.

30. **(c) :** $\Delta U = \frac{-GMm}{2R} + \frac{GMm}{R} = \frac{GMm}{2R} = \frac{mgR}{2}$

31. **(a) :** If distance between the two stars is $3l$ then angular momentum of mass m about common centre of mass $= m(2l)^2\omega$ and angular momentum of mass $2m$ about common centre of mass $= (2m)l^2\omega$.

32. **(c) :** $v_{\text{escape}} = \sqrt{\frac{2GM}{R}}$ where M is mass of planet (it is independent of mass of particle m).

33. **(d) :** $E_P + E_Q = 0$ but $E_P = E_Q \neq 0$

34. **(a) :** Put $g_e = 0$, in the expression

$g_e = g_0 - r\omega^2 \Rightarrow \omega = \sqrt{\frac{g_0}{r}}$

or $T = 2\pi\sqrt{\frac{r}{g_0}}$

Putting $r = 6.4 \times 10^6$ m and $g_0 = 9.8$ m/sec^2, we obtain, $T = 84$ min

35. **(d) :** The potential at a distance x is

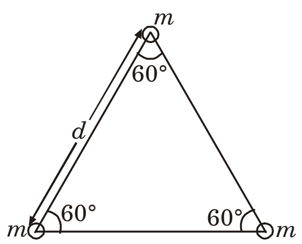
$$V(x) = -\int_\infty^x l\, dx = -\int_\infty^x \frac{A}{x^3}dx$$

$$V(x) = \left[\frac{A}{2x^2}\right]_\infty^x = \frac{A}{2x^2}$$

36. **(c) :** For the system of two particles, gravitational energy is given as $U = -Gm_1m_2/r$

$U_A = U_{12} + U_{23} + U_{31}$

or $U_A = -\frac{3Gm^2}{d}$



[–ve sign indicates that the particles are bounded by their mutual gravitational field]

37. **(b) :** The potential energy of the body on the surface of earth, $U_1 = -\frac{GMm}{R}$.

The potential energy of the body at infinity, $U_2 = 0$

$\Rightarrow \Delta U = U_2 - U_1 = \frac{GMm}{R} = mgR$

$\left(\because g = \frac{GMm}{R^2}\right)$

38. **(c) :** Energy required to shift the satellite from orbital radius r to orbital radius $2r$,

$$E = \left[-\frac{GMm}{4r}\right] - \left[-\frac{GMm}{2r}\right]$$

or $E = \frac{GMm}{r}\left[\frac{1}{2} - \frac{1}{4}\right]$ or $E = \frac{GMm}{4r}$

Energy required to shift the satellite from orbital radius 2r to orbital radius 3r,

$$E' = \left[-\frac{GMm}{6r}\right] - \left[-\frac{GmM}{4r}\right]$$

or $E' = \frac{GmM}{r}\left[\frac{1}{4} - \frac{1}{6}\right]$

or $E' = \frac{GmM}{12r} = \frac{1}{3}\left[\frac{GmM}{4r}\right] \Rightarrow E' = \frac{E}{3}$

39. **(d) :** Work done by external agent

W_{ext}, $= \Delta U + \Delta K = U_A - U_\infty + K_A - K_\infty$

$-3 = U_A - 10 + \frac{1}{2} \times mv^2 - 0$

$-3 = U_A + \frac{1}{2} \times 1 \times 4 - 10 \Rightarrow U_A = 5$ J

40. **(a) :** The gravitational field at the given point is

$E = \frac{GM}{(2a)^2} = \frac{GM}{4a^2}$ .....(i)

The mass M may be calculated as follows. Consider a concentric shell of radius r and thickness dr.

Its volume is $dV = (4\pi r^2)dr$

and its mass is $dM = \rho dV = \left(\rho_0 \frac{a}{r}\right)(4\pi r^2 dr)$

$= 4\pi\rho_0 ar\, dr$.

The mass of the whose sphere is

$$M = \int_0^a 4\pi\rho_0 ar\, dr = 2\pi\rho_0 a^3$$

Thus, by (i) the gravitational field is

$$E = \frac{2\pi G\rho_0 a^3}{4a^2} = \frac{1}{2}\pi G\rho_0 a$$

$\therefore E = \frac{1}{2} \times \frac{22}{7} \times 6.65 \times 10^{-11} \times 10^9 \times 4 \times 10^3$ m/s^2

$\Rightarrow E = 418$ m/s^2

MCQs : More Than One Correct Option

1. **(a, d) :** At two positions, when the planet is closest to the sun (perigee) and when it is farthest from the sun (apogee), velocity vector is perpendicular to force vector *i.e.*, work done is zero. In one complete revolution work done is zero.

2. **(a, b, c, d) :** Initially potential and kinetic both energies are zero and from conservation of mechanical energy total energy of the two objects is zero. Further, decrease in gravitational potential energy = increase in kinetic energy

or $\frac{G(m)(4m)}{r} = \frac{1}{2}\mu v_r^2$ ....(i)

Here, μ = reduced mass $= \frac{(m)(4m)}{m+4m} = \frac{4m}{5}$

Substituting in equation(i), we get

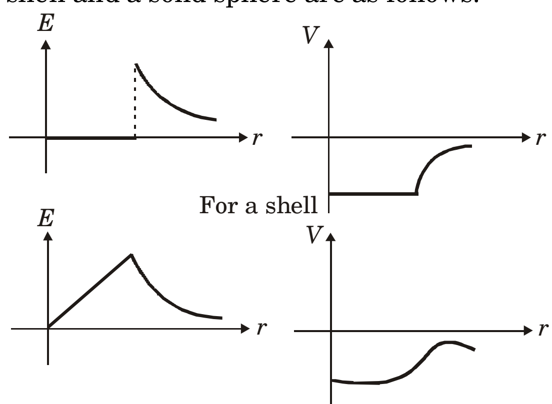
v_r = relative velocity of approach $= \sqrt{\frac{10Gm}{r}}$

From equation (i) total kinetic energy

$= \frac{G(m)(4m)}{r} = \frac{4Gm^2}{r}$

Net torque of two equal and opposite forces acting on two objects is zero. Therefore, angular momentum will remain conserved. Initially both the objects were stationary *i.e.*, angular momentum about any point was zero. Hence, angular momentum of both the particles about any point will be zero at all instants.

3. **(a, b, d) :** *E*-*r* and *V*-*r* graphs for a spherical shell and a solid sphere are as follows.



For a shell

For a solid sphere

4. **(a, d) :** Distance from centre of sun and hence kinetic energy and potential energy go on changing.

5. **(a, d) :** $V_0 = \sqrt{\frac{GM}{r}}$, $T = \frac{2\pi r^{3/2}}{\sqrt{GM}}$

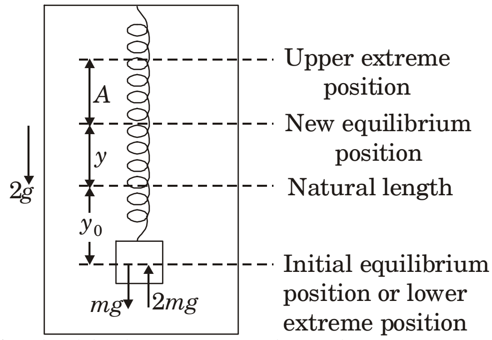
$U = \frac{-GMm}{r}$ $K = \frac{GMm}{2r}$

6. **(a, d) :** At centre, $V = \frac{-1.5\,GM}{R}$

7. **(d) :** Initially the spring is elongated by $y_0 = \frac{mg}{k}$.

When lift starts accelerating down with acceleration 2*g*, the spring will compress and compression in spring in the new equilibrium position, is

$y = \frac{mg}{k}$, and the block performs SHM about this equilibrium position.



As the block is at rest when elongation in spring is $\frac{mg}{k}$, so this is the extreme position and hence amplitude will be $y + y_0 = \frac{2mg}{k}$

Let us consider an instant when elongation in spring is *x* and at this instant velocity of block is *v*, then applying work-energy theorem for this instant at the lower extreme position,

$\frac{mv^2}{2} - 0 - mg(y_0 - x) - \left[\frac{kx^2}{2} - \frac{ky_0^2}{2}\right]$

$\Rightarrow$ Total energy of spring-block system at this instant is, $\frac{mv^2}{2} + \frac{kx^2}{2} = E = mg(y_0 - x) + \frac{ky_0^2}{2}$

$\Rightarrow \quad E = mgy_0 + \frac{mgy_0}{2} - mgx = \frac{3mg}{2}y_0 - mgx$

$= mg\left(\frac{3}{2}y_0 - x\right)$

So, *E* would be minimum when *x* is maximum *i.e.*, $x = y_0$.

8. **(a,d) :** Just before collision, the total energy of two satellites is, $E = -\frac{GMm}{2r} - \frac{GMm}{2r} = -\frac{GMm}{r}$

Let orbital velocity is *v*, then from momentum conservation, $mv - mv = 2m \times v_1$

$\Rightarrow v_1 = 0$

As velocity of combined mass just after collision is zero, the combined mass will fall towards earth. At this instant, the total energy of the system only consists of the gravitational potential energy given by $U = -\frac{GM \times 2m}{2r}$

9. **(b, c, d) :** Distance of the two satellites from the centre of the earth are $r_1 = 2R$ and $r_2 = 8R$ respectively. R = earth's radius. Their potential energies are $V_1 = -\frac{GmM}{r_1}$

and $V_2 = -\frac{GmM}{r_2}$

Their ratio is $\frac{V_1}{V_2} = \frac{r_2}{r_1} = \frac{8R}{2R} = 4.$

The kinetic energy of a satellite can be obtained from relation $\frac{mv^2}{r} = \frac{GmM}{r^2}$

or $K = \frac{1}{2}mv^2 = \frac{GmM}{2r}$

Thus $K_1 = \frac{GmM}{2r_1}$ and $K_2 = \frac{GmM}{2r_2}$

The ratio of their kinetic energies is

$\frac{K_1}{K_2} = \frac{r_2}{r_1} = \frac{8R}{2R} = 4.$

Their total energies are

$E_1 = -\frac{GmM}{r_1} + \frac{GmM}{2r_1} = -\frac{GmM}{2r_1}$

and $E_2 = -\frac{GmM}{r_2} + \frac{GmM}{2r_2} = -\frac{GmM}{2r_2}$

Their ratio is $\frac{E_1}{E_2} = \frac{r_2}{r_1} = \frac{8R}{2R} = 4.$

10. **(b, d) :** The acceleration due to gravity is $g = \frac{GM}{R^2}$

The new value of g would be

$g' = \frac{G(0.99\ M)}{(0.99\ R^2)} \simeq 1.01\ g$

Thus g would increase by about 1%. The new escape velocity would be

$v_e' = \sqrt{\frac{2 \times 0.99M \times G}{0.99\ R}} = \sqrt{\frac{2MG}{R}} = v_e$

Thus the escape velocity will remain unchanged. The potential energy of a body of mass m on earth's surface would be

$-\frac{GM(0.99M)}{(0.99R)} = -\frac{GmM}{R}$

Thus the potential energy will also remain unchanged.

Assertion & Reason

1. **(a) :** According to Kepler's third law of motion, the square of the time period of a planet about the sun is proportional to the cube of the semi major axis of the ellipse or mean distance of the planet from the sun. *i.e.* $T^2 \propto a^3$, when a is smaller, shorter is the time period.

2. **(c) :** Acceleration due to gravity is given by $g = \frac{GM}{R^2}$

Thus it does not depend on mass of body on which it is acting. Also it is not a constant quantity and changes with change in value of both M and R (distance between two bodies). Even for earth it is a constant only near the earth's surface.

3. **(b) :** Acceleration due to gravity,

$g' = g - R_e\omega^2 \cos^2\lambda$

At equator, $\lambda = 0° \therefore \cos 0° = 1$

$\therefore\ g_e = g - R_e\omega^2$

At poles, $\lambda = 90° \therefore \cos 90° = 0$

$\therefore\ g_p = g$

Thus, $g_p - g_e = g - g + R_e\omega^2 = R_e\omega^2$

Also, the value of g is maximum at poles and minimum at equator.

4. **(a) :** At poles, radius of horizontal circle is zero. $\therefore$ Centripetal force $F = mr\omega^2 = 0$. Hence g at poles is not affected by rotation of earth.

5. **(a) :** The gravitational potential at a point in the gravitational field of earth is defined as the work done in bringing a unit mass from infinity to that point. It is attracted by the earth gravitational field, so work is done on the body, so the gravitational potential is negative.

6. **(c) :** According to Kepler's law of planetary motion, a planet revolves around the sun in such a way that its areal velocity is constant. *i.e.,* it move faster, when it is closer the sun and vice-versa.

7. **(a) :** As the geostationay satellite is established in an orbit in the plane of the equator at a particular place, so it move in the same sense as the earth and hence its time period of revolution is equal to 24 hours, which is equal to time period of revolution of earth about its axis.

8. **(a) :** According to Newton's law of gravitation,

$F = \frac{Gm_1m_2}{r^2}$. When m_1, m_2 and r all are doubled,

$F = \frac{G(2m_1)(2m_2)}{(2r)^2} = \frac{Gm_1m_2}{r^2}$,

i.e. F remains the same.

9. **(c) :** Variation of g with depth from surface of earth is given by $g' = g\left(1 - \frac{d}{R}\right)$.

At the centre of earth, $d = R$

$\therefore\ g' = g\left(1-\frac{R}{R}\right) = 0$

$\therefore$ Apparent weight of body $= mg' = 0$

10. **(b) :** We know that earth revolves from west to east about its polar axis. Therefore, all the particles on the earth have velocity from west to east. This velocity is maximum in the equatorial line, as $v = R_e\omega$, where R_e is the radius of earth and ω is the angular velocity of revolution of earth about its polar axis. When a rocket is launched from west to east in equatorial plane, the maximum linear velocity is added to the launching velocity of the rocket, due to which launching becomes easier.

11. **(c) :** Binding energy is the minimum energy required to free a satellite from the gravitational attraction. It is the negative value of total energy of satellite. Let a satellite of mass m be revolving around earth of mass M_e and radius R_e.

Total energy of satellite = $P.E. + K.E.$

$$= -\frac{GM_em}{R_e} + \frac{1}{2}mv^2$$

$$= -\frac{GM_em}{R_e} + \frac{m}{2}\frac{GM_e}{R_e} = -\frac{GMm}{2R_e}.$$

$\therefore$ Binding energy of satellite = – [total energy of satellite] which depend on mass of the satellite.

12. **(d) :** According to Newton's law of gravitation, every body in this universe attracts every other body with a force which is inversely proportional to the square of the distance between them. The distance of the finger from the stars is almost infinity for measuring gravitational fields.

13. **(d) :** If the orbital path of a satellite is circular, then its speed is constant and if the orbital path of a satellite is elliptical, then its speed in its orbit is not constant. In that case its areal velocity is constant.

14. **(b) :** Gravitational field intensity at a point distance r from centre of earth is $E = \frac{GM}{r^2}$.

When $r = \infty$, $E = 0$.

When point is inside the earth, then

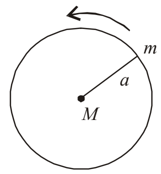
$E = \frac{G}{r^2}\times\frac{4}{3}\pi r^3\rho = \frac{4\pi G\rho r}{3}$ when $r = 0$, $E = 0$.

15. **(a) :** To make our calculations easy, let's take the semi-major axis of the ellipse be equal to the average distance of the Sun from the planet. By applying Newton's law,

$$\frac{GMm}{a^2} = m\omega^2 a$$

ω = angular velocity of the planet

$= \frac{2\pi}{T}$ (T = time period of the planet)



$\therefore\ \frac{GMm}{a^2} = m\frac{(2\pi)^2}{T^2}a$

or, $T^2 = \left(\frac{4\pi^2}{GM}\right)a^3$

or, $T^2 \propto a^3$.

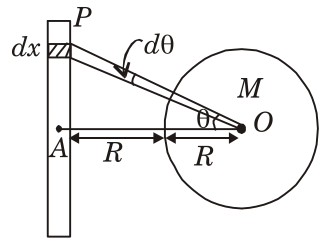
Which is also known as Kepler's third law or the law of period, according to which the square of period of any planet is proportional to the cube of the semi major axis of its orbit.

Passage Comprehension

PASSAGE 1 :

1. **(a) :** As rod is one-dimensional, so linear mass density best explains mass distribution.

2. **(a) :** $dm = \lambda\cdot dx$



$$= \frac{M}{4R}\cdot 2R\sec^2\theta\cdot d\theta$$

$$= \frac{M}{2}\sec^2\theta\cdot d\theta$$

3. **(a) :** $F_{\text{net}} = \int\frac{GM\cdot dm}{(2R\sec\theta)^2}\cos\theta$

$$= \frac{GM^2}{8R^2}\int\cos\theta d\theta$$

$$F_{\text{net}} = \frac{GM^2}{8R^2}\times\frac{2}{\sqrt{2}} = \frac{GM^2}{4\sqrt{2}R^2},$$

which is also equal to the force on the sphere.

4. **(b) :** Magnitude acceleration of each w.r.t ground

$$= \frac{GM}{4\sqrt{2}R^2}$$

$\therefore$ Relative acceleration

$= 2\left(\frac{GM}{4\sqrt{2}R^2}\right) = \frac{GM}{2\sqrt{2}R^2}$, towards the rod.

5. **(c) :** Shift in centre of mass of the system is zero, as no external force acts on it.

PASSAGE 2 :

1. **(c)** **2.** **(b)**

3. **(a)** **4.** **(d)**

PASSAGE 3 :

1. **(b) :** For circular orbits $v = \sqrt{\frac{Gm}{r}}$ or $v \propto \frac{1}{\sqrt{r}}$

If r is doubled, v will become $\frac{1}{\sqrt{2}}$ times.

2. **(b) :** The gravitational force is utilised as the required centripetal force.

PASSAGE 4 :

1. (c) : As it is not known that by how much velocity has changed, which will cause the change in orbital radius and hence time period, so we can use either rocket system to carry out the docking in minimum possible time, through the extent of firing rockets may differ.

2. **(b) :** Time period of A and B in given circular orbit is, $T = \frac{2\pi r^{3/2}}{\sqrt{GM}}$
where r = 9600 km, G = 6.67×10^{-11} N-m^2/kg^2, $M = 5.98 \times 1024$ kg,
which gives T = 9357.79 s $\simeq$ 9358 s
Initial total energy of A,
$E = -\frac{GMm}{2r} = -6.65 \times 10^9$ J

3. **(b) :** To bring back the satellite in same orbit, we have to make the speed of satellite as initial one for which we have to increase v_f by such amount so that it become s equal to v_i, for which we have to fire the rocket in backward direction with same extent.

4. **(d) :** As there is time gap between two after coming back to same position so docking is not proper. To make the docking proper, T must be equal to (9358 – 100) s *i.e.*, greater than what we get. So, for proper docking a must increase which further shows total energy becomes less negative *i.e.*, increases which is possible when rocket has been fired with less extent.

PASSAGE 5 :

1. **(b) :** As satellite is launched using multistage propellor, its speed can be controlled, but when it falls down its kinetic energy increases continously and air drag forces are generally depend on velocity square, so thermal energy dissipation increases to a larger extent and more chances are there for the satellite to burn up.

2. **(c) :** As work is done by air drag force, the total mechanical energy decreases which decreases orbital radius and hence increases the kinetic energy, so r decreases and finally the satellite falls on to the earth or may burn up.

3. **(b) :** From work-energy theorem,
$dK = -dU + W_{\text{air friction}}$
$W_{\text{air friction}}$ is –ve, so
$dK = -dU$ + (a negative quantity)
As K increases, it means U decreases by an amount greater than magnitude of $W_{\text{airfriction}}$.

4. **(a) :** From $v = \sqrt{\frac{GM}{R}}$
Take log on both sides and then differentiate
$\frac{dv}{v} = \frac{-1 \times dR}{2R} \Rightarrow \frac{\Delta v}{v} = \frac{\Delta R}{2R}$ [as $\Delta R = -dR$]
$\Delta v = \frac{\Delta R}{2}\sqrt{\frac{GM}{R^3}}$

Matrix Match Type

1. **(A) → (R), (B) → (P), (C) → (Q), (D) → (S)**

2. **(A) → (P, R), (B) → (Q, S), (C) → (Q, S), (D) → (Q, R) :**
(A) : Energy of earth – satellite system is given by $E = -\frac{GMm}{2a}$ as a (semi-major axis) is same for all four orbits, so energy is same for all. As no dissipative forces are acting, energy remains constant.
(B, C) : As direction differs from time to time, the velocity varies and as distance of satellite from earth is changing, so speed also varies.
(D) : As force is central in nature, angular momentum is constant but different for different orbits.

3. **(A) → (P, R, S), (B) → (P, R, S), (C) → (P, Q, R, S), (D) → (P) :**
(A, B, C) : As rocket has been fired the speed of satellite increases in all these three cases which leads to the increase in kinetic energy and hence total energy becomes less negative or we can say semi-major axis increases.
In C due to thrust exerted by firing of rocket in a direction perpendicular to plane of orbit, the plane of orbit changes.
(D) : Here speed decreases and hence kinetic energy, total energy and the semi-major axis.

4. **(A) → (Q), (B) → (P,Q), (C) → (R), (D) → (R, S)**

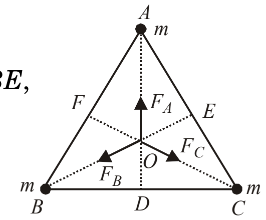
5. **(A) → (Q), (B) → (Q), (C) → (S)**
$g = \frac{GM}{R^2}$, $V = -\frac{GM}{R}$
At height $h = R$: $g' = \frac{g}{1+\frac{h}{R}} = \frac{g}{2}$
$V' = \frac{GM}{2R}$
i.e., g' decrease by a factor 1/2 and V' increases by a factor 2
At depth $h = R/2$:
$g' = g\left(1-\frac{h}{R}\right) = g\left(1-\frac{1}{2}\right) = \frac{g}{2}$

Integer Answer Type

1. **(0)** Given $AB = BC = AC = a$ (see figure). The perpendiculars from A, B and C on opposite sides meet at the centroid O, which bisect the sides AB, BC and AC. Let $r = AO = BO = CO$. Centroid also divides the lines AD, BE and CF in the ratio 2 : 1, *i.e*,

$$AO = \frac{2}{3} AD,\ BO = \frac{2}{3} BE,$$

$$CO = \frac{2}{3} CF.$$



In triangle ABD, $AD = a \sin 60° = \frac{\sqrt{3}\, a}{2}$.

Similarly, $BE = CF = \frac{\sqrt{3}\, a}{2}$.

$\therefore\ r = AO = OB = OC = \frac{2}{3} \times \frac{\sqrt{3}\, a}{2} = \frac{a}{\sqrt{3}}$

The gravitational field intensity at point O is the net force exerted on a unit mass placed at O due to three equal masses m at vertices A, B and C. Since the three masses are equal and their distances from O are also equal, they exert forces F_A, F_B and F_C of equal magnitude. Their directions are shown in the figure. It follows from symmetry of forces that their resultant at point O is zero.

2. **(2)** $\frac{(\text{KE})_{\text{escape}}}{(\text{KE})_{\text{orbit}}} = \frac{\frac{1}{2} m v_e^2}{\frac{1}{2} m v_o^2} = \frac{v_e^2}{v_o^2} = \frac{\frac{2GM}{R}}{\frac{GM}{R}} = 2$

3. **(2)** The potential energy of a satellite is given by

$$U = -\frac{GM_e m}{r}$$

Given $r = R + 6.4 \times 10^6 = R + R = 2R$.

Further $GM_e = gR^2$

$$\therefore\ U = -\frac{gR^2 \cdot m}{2R} = -\frac{mgR}{2}$$

$\therefore\ n = 2$.

4. **(8)** $v_{es.} = \sqrt{\frac{2GM}{R}}$

Initial KE $= \frac{1}{2} m \left(\frac{v_{es.}}{3}\right)^2$

$$= \frac{1}{2} n \left[\frac{1}{3}\sqrt{\frac{2GM}{R}}\right]^2 = \frac{1}{9}\frac{GMm}{R}$$

Let the maximum height attained = h.

The gain in PE $= \frac{GMm}{R} - \frac{GMm}{R+h}$ = loss in KE

i.e., $\frac{GMm}{R} - \frac{GMm}{R+h} = \frac{1}{9}\frac{GMm}{R}$

On solving we get: $\frac{9R}{8} = R + h$ or $h = \frac{R}{8}$

$\therefore\ n = 8$.

5. **(8)** According to law of conservation of energy,

$$\frac{1}{2} mv^2 = \frac{mgh}{1 + \frac{h}{R}}$$

$$\therefore\ v^2 = \frac{2gh}{1 + \frac{h}{R}} = \frac{2 \times 10 \times 6.4 \times 10^6}{1 + \frac{R}{R}}$$

$$= \frac{2 \times 10 \times 6.4 \times 10^6}{2}$$

$\therefore\ v = \sqrt{64 \times 10^6} = 8$ km/sec

6. **(2)** Energy required to raise a satellite upto a height h,

$$E_1 = \Delta U = \frac{mgh}{1 + \frac{h}{R}} \quad \text{...(i)}$$

$$E_2 = \frac{1}{2} mv_0^2$$

$= \frac{1}{2} m \left(\frac{GM}{r}\right)$ [$\because v_0$ is the orbital speed]

$$= \frac{1}{2} m \left(\frac{GM}{R+h}\right) = \frac{1}{2} m \left(\frac{GM}{R^2}\right) \frac{R}{1 + \frac{h}{R}}$$

$$= \frac{mgR}{2\left(1 + \frac{h}{R}\right)} \quad \text{...(ii)}$$

From equations (i) and (ii),

$$\frac{E_1}{E_2} = \frac{2h}{R}$$

$\therefore\ x = 2$.

7. **(5)** $y = kt^2$

The point of suspension of the pendulum is moving upwards. The velocity and acceleration are given by

$$\frac{dy}{dt} = k.\,2t \ \text{ and } \ \frac{d^2y}{dt^2} = 2k$$

where $k = 1$ m/s^2

Thus, the point of suspension of the pendulum is moving upwards with an acceleration of 2 m/s^2. This is the case where the pseudo acceleration is acting downwards.

Hence, effective acceleration due to gravity

$$g' = g + 2 = 12 \text{ m/s}^2$$

$$T_1 = 2\pi\sqrt{\frac{l}{g}},\ T_2 = 2\pi\sqrt{\frac{l}{g'}} \quad \therefore\ \frac{T_1^2}{T_2^2} = \frac{g'}{g} = \frac{12}{10} = \frac{6}{5}$$

Therefore, $T_1^2 = 6$, $T_2^2 = 5$.

8. **(8)** Applying law of conservation of energy.

$$0 - \frac{GMm}{R+h} = \frac{1}{2}mv^2 - \frac{GMm}{R}$$

$$\therefore \quad \frac{1}{2}mv^2 = \frac{GMm}{R} - \frac{GMm}{R+h}$$

$$\frac{1}{2}mv^2 = \frac{GMm}{2R} \qquad (\because h = R)$$

$\therefore$ v = velocity with which it strikes the surface of the earth

$$v = \sqrt{\frac{GM}{R}} = \sqrt{gR} = \sqrt{10 \times 6.4 \times 10^6}$$

$= 8$ km/s.

9. **(2)** The electric field $\vec{E} = 2\hat{i} + 3\hat{j}$ is inclined to x-axis through an angle

$\theta = \tan^{-1}\frac{3}{2}$ or $\tan\theta = 3/2$.

Therefore, the electric field is parallel to a straight line given by $y = \frac{3}{2}x + c$.

The equation of a straight line which is perpendicular to the above line is given by

$$y = -\frac{2}{3}x + c_1 \qquad \text{...(i)}$$

Since no work is done when the particle is moved on the line $3y + kx = 5$ or $y = -\frac{k}{3}x + 5$ ...(ii)

so this line must be perpendicular to electric field as workdone by gravitational field on the particle while moving on the line is zero.

Comparing (i) and (ii), we have $k = 2$.

10. (6) $g = \frac{GM}{R^2}$, $M = \left(\frac{4}{3}\pi R^3\right)\rho \Rightarrow R^3 = \left(\frac{3}{4\pi}\right)\cdot\frac{M}{\rho}$

$$R = \left(\frac{3}{4\pi}\right)^{\frac{1}{3}}\cdot\frac{M^{\frac{1}{3}}}{\rho^{\frac{1}{3}}}$$

$$R = \frac{K}{\rho^{\frac{1}{3}}} \Rightarrow g = GM\cdot\left(\frac{\rho^{\frac{1}{3}}}{K}\right)^2 = \frac{GM\rho^{\frac{2}{3}}}{K^2}$$

Now, $g' = \frac{GM}{K^2}\left(10^9\rho\right)^{\frac{2}{3}}$

$$= \left(\frac{GM\rho^{\frac{2}{3}}}{K}\right)\cdot 10^6 \Rightarrow g' = 10^6 g$$

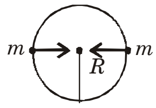
$\therefore \quad n = 6$

11. **(1)** $T^2 = \left(\frac{4\pi^2}{GM}\right)R^3$

T is independent of the satellite's mass m.

$\Rightarrow T_A : T_B = 1 : 1$

12. **(1)** $F = \frac{G\cdot m^2}{(2R)^2}$ and



$$F = \frac{mv^2}{R} \Rightarrow \frac{Gm^2}{4R^2} = \frac{mv^2}{R}$$

or $v^2 = \frac{G\cdot m}{4R}$; $v = \frac{1}{2}\sqrt{\frac{G\cdot m}{R}}$

$\therefore \quad n = 1$

13. **(4)** If h is the maximum height attained, then we have

$$\frac{1}{2}mv^2 - \frac{GMm}{R} = -\frac{GMm}{(R+h)}$$

which gives $\qquad v^2 = \frac{2ghR}{(R+h)}$

$$\left(\because \quad g = \frac{GM}{R^2}\right)$$

For ball A, we have $\frac{4gR}{3} = \frac{2gh_A R}{(R+h_A)} \Rightarrow h_A = 2R$

For ball B, we have $\frac{2gR}{3} = \frac{2gh_B R}{(R+h_B)} \Rightarrow h_B = \frac{R}{2}$

$$\therefore \quad \frac{h_A}{h_B} = 4$$

14. **(9)** If the gravitational field is zero at a point at a distance x from M_1, then

$$\frac{GM_1}{x^2} = \frac{GM_2}{(r-x)^2}$$

or $\frac{x}{(r-x)} = \sqrt{\frac{M_1}{M_2}} = \sqrt{\frac{M_1}{4m}} = \frac{1}{2}$

which gives $x = \frac{r}{3}$. Therefore, $r - x = \frac{2r}{3}$.

The gravitational potential at $x = \frac{r}{3}$ is

$$U = -\frac{GM_1}{x} - \frac{GM_2}{(r-x)}$$

$$= -\frac{GM}{r/3} - \frac{G(4m)}{2r/3} = -\frac{9GM}{r}. \quad \therefore \quad n = 9.$$

15. **(2)** Let x be the required distance from the centre of the moon. The weight of a body of mass m will be zero at this distance if the force due to earth = force due to moon, *i.e.* if

$$\frac{GM_e m}{(d-x)^2} = \frac{GM_m m}{x^2} \quad \text{or} \quad \frac{M_e}{M_m} = \frac{(d-x)^2}{x^2}$$

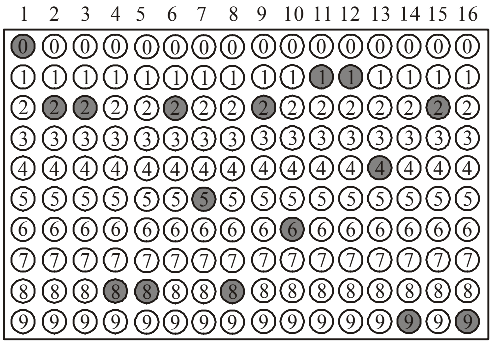
or $81 = \frac{(d-x)^2}{x^2}$ or $9 = \frac{d-x}{x}$

or $x = \frac{d}{10} = \frac{d}{5\times n}$. $\qquad \therefore n = 2$

16. **(9)** $gh = \dfrac{gR^2}{(R+h)^2}$. Given $gh = \dfrac{g}{100}$. Therefore, we have

$$\frac{gR^2}{(R+h)^2} = \frac{g}{100}$$

or $R + h = 10\,R$ or $h = 9\,R$



PREVIOUS YEARS IIT-JEE/JEE Advanced

1. **(d) :** For a simple pendulum, $T = 2\pi\sqrt{\dfrac{l}{g}}$

$\therefore \quad \dfrac{T_2}{T_1} = \sqrt{\dfrac{g_1}{g_2}}$

Now, $g_1 = \dfrac{GM}{R^2}$, $\quad g_2 = \dfrac{GM}{(2R)^2} = \dfrac{GM}{4R^2}$

$\therefore \quad \dfrac{T_2}{T_1} = \sqrt{\dfrac{GM}{R^2} \times \dfrac{4R^2}{GM}} = \sqrt{\dfrac{4}{1}} = \dfrac{2}{1} \quad \therefore \quad \dfrac{T_2}{T_1} = \dfrac{2}{1}$

2. **(c) :** According to Kepler's law, $T^2 \propto R^3$

$\therefore \quad \left(\dfrac{T_2}{T_1}\right)^2 = \left(\dfrac{R_2}{R_1}\right)^3$

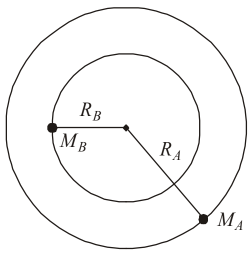
Let the time period at earth's surface = T_1
For geo-stationary satellite, T_2 = 24 hour, R_2 = 36,000 km
For spy-satellite, T_1 = ?, $R_1 \approx$ 6400 km

$\therefore \quad \left(\dfrac{24}{T_1}\right)^2 = \left(\dfrac{36000}{6400}\right)^3 = \left(\dfrac{45}{8}\right)^3 = (5.625)^3 = 178$

or $\dfrac{24}{T_1} = 13.34$ or $T_1 = \dfrac{24}{13.34}$

T_1 = 1.8 hour = Time period at earth's surface.
When spy-satellite is slightly above earth's surface,
T_1 = 2 hour.

3. **(d) :** In case of a binary star system, angular velocity is same.

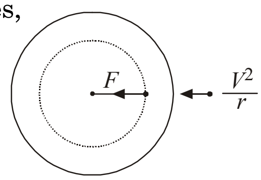


$\omega = \dfrac{2\pi}{T_A} = \dfrac{2\pi}{T_B}$

$\therefore \quad T_A = T_B$.

4. (c) : Case I : $r \le R$

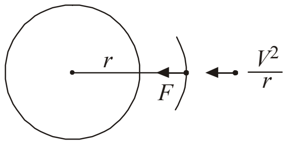
Force on the test mass can be given by $F = mg\dfrac{r}{R}$, where mg is the force on the mass at the surface of the sphere.
Newton's second law gives,



$mg\dfrac{r}{R} = \dfrac{mV^2}{r}$;

$V = \sqrt{\dfrac{g}{R}}\, r \Rightarrow V \propto r$ for $r \le R$

Case II : $r > R$
When the test mass is outside the sphere, application of Newton's law of gravitation and Newton's second law gives



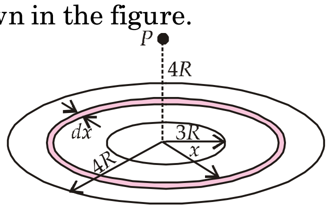
$\dfrac{GMm}{r^2} = \dfrac{mV^2}{r}$, $\quad V = \sqrt{\dfrac{GM}{r}} = \sqrt{\dfrac{g}{r}}\,R \Rightarrow V \propto \dfrac{1}{\sqrt{r}}$.

Therefore, the graph is as shown in alternative (c).

5. **(a) :** Mass per unit area of the disc,

$$\sigma = \frac{\text{Mass}}{\text{Area}} = \frac{M}{\pi\left((4R)^2 - (3R)^2\right)} = \frac{M}{7\pi R^2}$$

Consider a ring of radius x and thickness dx as shown in the figure.



Mass of the ring, $dM = \sigma 2\pi x dx$

$$= \frac{2\pi M x dx}{7\pi R^2}$$

Potential at point P due to annular disc is

$$V_P = \int_{3R}^{4R} -\frac{GdM}{\sqrt{(4R)^2 + (x)^2}} = -\frac{GM2\pi}{7\pi R^2}\int_{3R}^{4R}\frac{xdx}{\sqrt{16R^2 + x^2}}$$

Solving, we get

$$V_P = -\frac{GM2\pi}{7\pi R^2}\left[\sqrt{16R^2 + x^2}\right]_{3R}^{4R}$$

$$= -\frac{2GM}{7R}\left(4\sqrt{2} - 5\right)$$

Workdone in moving a unit mass from P to ∞

$= V_\infty - V_P$

$$= 0 - \left(\frac{-2GM}{7R}\left(4\sqrt{2} - 5\right)\right) = \frac{2GM}{7R}(4\sqrt{2} - 5)$$

6. **(b) :** Escape speed, $v_e = \sqrt{2} \times \text{orbital speed} = \sqrt{2}V$

$\therefore$ Kinetic energy of the object

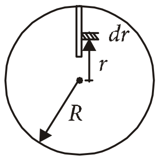
$= \frac{1}{2}mv_e^2 = \frac{1}{2}m(\sqrt{2}V)^2 = mV^2$

7. **(b):** Gravitational field inside the planet at a distance r from centre,

$$g_i = \frac{GM}{R^3}r = \frac{G\left(\frac{4}{3}\pi R^3\right)\rho}{R^3}r = \frac{4}{3}G\pi r\rho$$

Force applied by a person on the wire at rest is the weight of the wire. Since, gravitational field is variable so force on each part of the wire is different.

Consider a small portion dr of the wire at a distance r from centre of the planet, so its weight will be



$$dW = (dm)g_i$$

$$= (\lambda dr)\left(\frac{4}{3}G\pi r\rho\right)$$

$$= \left(\frac{4}{3}G\pi\rho\right)(\lambda r dr)$$

Net weight, $W = \int dW = \left(\frac{4}{3}G\pi\rho\lambda\right)\int_{\left(\frac{4R}{5}\right)}^{R} r dr$

$$W = \left(\frac{4}{3}G\pi\rho\lambda\right)\left(\frac{9}{50}\right)R^2 \quad \text{... (i)}$$

Density of the Earth, $\rho = \dfrac{M_E}{\frac{4}{3}\pi R_E^3}$

Also, $R = \dfrac{R_E}{10}$

Putting the values of ρ and R in equation (i), we get

$$W = \left(\frac{9}{5\times 10^3}\right)g_E\lambda R_E$$

$$= \frac{9}{5\times 10^3}\times 10\times 10^{-3}\times 6\times 10^6 = 108\text{ N}$$

So, net force applied by the person to hold the wire = 108 N.

8. **(b, d) :** The escape velocity for the surface of earth is

$$V_e = \sqrt{\frac{2GM}{R}} = \sqrt{\frac{2\cdot G\rho\frac{4}{3}\pi R^3}{R}}$$

(Since ρ is same for all planet)

$$V_e \propto R \quad \text{...(i)}$$

Surface area of P, $4\pi R_P^2 = A$

Surface area of Q, $4\pi R_Q^2 = 4A$

$$\therefore \frac{R_P}{R_Q} = \frac{1}{2} \Rightarrow R_Q = 2R_P \quad \text{...(ii)}$$

The spherical planet R has mass

$$M_R = M_P + M_Q$$

$$\frac{4}{3}\rho\pi R_R^3 = \frac{4}{3}\rho\pi R_P^3 + \frac{4}{3}\rho\pi R_Q^3 \Rightarrow R_R^3 = R_P^3 + R_Q^3$$

or $R_R^3 = R_P^3 + (2R_P)^3$ (Using (ii))

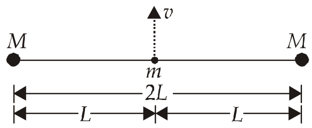
So, $R_R = (9)^{1/3}R_P$

Therefore, $R_R > R_Q > R_P$

From equation (i), $V_R > V_Q > V_P$

and from equation (ii), $\dfrac{V_P}{V_Q} = \dfrac{1}{2}$

9. **(b, d) :** The situation is as shown in the figure



Applying the conservation of mechanical energy, we get

$$-\frac{GMm}{L} - \frac{GMm}{L} + \frac{1}{2}mv^2 = 0 + 0$$

$$\frac{1}{2}mv^2 = \frac{2GMm}{L} \Rightarrow v = \sqrt{\frac{4GM}{L}} = 2\sqrt{\frac{GM}{L}}$$

10. **(a) :** If only the gravitational force of the Earth acts on the astronaut, (that, he is in a state of free fall), he will feel weightless. Remember that the weight you feel is the normal force on your legs by the support.

11. **(3)** On the planet, $g_p = \dfrac{GM_p}{R_p^2} = \dfrac{G}{R_p^2}\left(\dfrac{4}{3}\pi R_p^3\rho_p\right)$

$$= \frac{4}{3}G\pi R_p\rho_p$$

On the earth, $g_e = \dfrac{GM_e}{R_e^2} = \dfrac{G}{R_e^2}\left(\dfrac{4}{3}\pi R_e^3\rho_e\right)$

$$= \frac{4}{3}G\pi R_e\rho_e$$

$$\therefore \frac{g_p}{g_e} = \frac{R_p\rho_p}{R_e\rho_e} \text{ or } \frac{R_p}{R_e} = \frac{g_p\rho_e}{g_e\rho_p} \quad \text{...(i)}$$

On the planet, $v_p = \sqrt{2g_pR_p}$

On the earth, $v_e = \sqrt{2g_eR_e}$

$$\therefore \frac{v_p}{v_e} = \sqrt{\frac{g_pR_p}{g_eR_e}} = \frac{g_p}{g_e}\sqrt{\frac{\rho_e}{\rho_p}} \quad \text{(Using(i))}$$

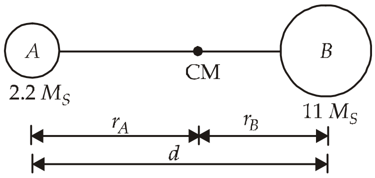
Here, $\rho_p = \frac{2}{3}\rho_e$, $g_p = \frac{\sqrt{6}}{11}g_e$

$\therefore \quad \frac{v_p}{v_e} = \frac{\sqrt{6}}{11}\sqrt{\frac{3}{2}}$

or $v_p = 11 \times \frac{\sqrt{6}}{11} \times \sqrt{\frac{3}{2}}$ ($\because$ $v_e = 11$ km s^{-1} (Given))

$= 3$ km s^{-1}

12. (6) :



Let stars A and B are rotating about their centre mass with angular velocity ω.

Let distance of stars A and B from the centre of mass be r_A and r_B respectively as shown in the figure.

Total angular momentum of the binary stars about the centre of mass is

$L = M_A r_A^2 \omega + M_B r_B^2 \omega$

Angular momentum of the star B about centre of mass is

$L_B = M_B r_B{}^2 \omega$

$$\therefore \quad \frac{L}{L_B} = \frac{(M_A r_A^2 + M_B r_B^2)\omega}{M_B r_B^2 \omega} = \left(\frac{M_A}{M_B}\right)\left(\frac{r_A}{r_B}\right)^2 + 1$$

Since $M_A r_A = M_B r_B$ or $\frac{r_A}{r_B} = \frac{M_B}{M_A}$

$$\therefore \quad \frac{L}{L_B} = \frac{M_B}{M_A} + 1 = \frac{11 M_S}{2.2 M_S} + 1 = \frac{11 + 2.2}{2.2} = 6$$

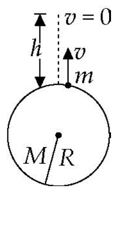
13. (2) : Given situation is shown in the figure.

Let acceleration due to gravity at the surface of the planet be g. At height h above planet's surface $v = 0$.

According to question,

acceleration due to gravity of the planet at height h above its surface becomes $g/4$.

$$g_h = \frac{g}{4} = \frac{g}{\left(1 + \frac{h}{R}\right)^2}$$



$$4 = \left(1 + \frac{h}{R}\right)^2 \Rightarrow 1 + \frac{h}{R} = 2$$

$$\frac{h}{R} = 1 \Rightarrow h = R.$$

So, velocity of the bullet becomes zero at $h = R$.

Also $v_{\text{esc}} = v\sqrt{N} \Rightarrow \sqrt{\frac{2GM}{R}} = v\sqrt{N}$ ... (i)

Applying energy conservation principle,

Energy of bullet at surface of earth

= Energy of bullet at highest point

$$\frac{-GMm}{R} + \frac{1}{2}mv^2 = \frac{-GMm}{2R}$$

$$\frac{1}{2}mv^2 = \frac{GMm}{2R} \quad \therefore v = \sqrt{\frac{GM}{R}}$$

Putting this value in eqn. (i), we get

$$\sqrt{\frac{2\,GM}{R}} = \sqrt{\frac{NGM}{R}} \quad \therefore N = 2$$

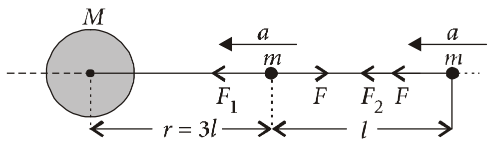
14. (7) : Both the point masses are connected by a light rod so they have same acceleration.

Suppose each point mass is moving with acceleration a towards larger mass M.

Using Newton's 2nd law of motion for point mass nearer to larger mass,

$F_1 - F = ma$

$$\frac{GMm}{(3l)^2} - \frac{Gm^2}{l^2} = ma \quad \text{...(i)}$$



Again using 2nd law of motion for another mass

$F_2 + F = ma$

$$\frac{GMm}{(4l)^2} + \frac{Gm^2}{l^2} = ma \quad \text{...(ii)}$$

From eqn. (i) and (ii), we get

$$\frac{GM}{9l^2} - \frac{Gm}{l^2} = \frac{GM}{16l^2} + \frac{Gm}{l^2}$$

$$\frac{M}{9} - \frac{M}{16} = m + m \Rightarrow \frac{7M}{144} = 2m$$

$$m = \frac{7M}{288} = k\left(\frac{M}{288}\right) \quad \therefore k = 7$$

Notes: