DISTRIBUTION PROBLEMS

When distributing n objects to k containers, the objects can be distinct (e.g. different cards) or identical (e.g. ping-pong balls). Likewise, the containers can be distinct (e.g. labeled boxes) or identical (e.g. urns). If objects are distinct and there can be more than one object in each container, the order may matter (e.g. arranging books on a shelf) or it may not (e.g. putting balls in a bag).

Distributing		**How many objects can each container get?**					
***n* objects**	**to *k* containers**	**no restrictions**	**order matters**	**at most 1**	**at least 1**	**at least 1, order matters**	**exactly 1**
distinct	**distinct**	k^n	$\dfrac{(n+k-1)!}{(k-1)!}$	${}^{k}\mathrm{P}_{n}$	$\mathrm{S}(n, k)\, k!$	${}^{n}\mathrm{P}_{k}\,\dfrac{(n-1)!}{(k-1)!}$	$\begin{cases} k! & \text{if } k = n \\ 0 & \text{otherwise} \end{cases}$
identical	**distinct**	$\dbinom{n+k-1}{n}$		$\dbinom{k}{n}$	$\dbinom{n-1}{k-1}$		$\begin{cases} 1 & \text{if } k = n \\ 0 & \text{otherwise} \end{cases}$
distinct	**identical**	$\sum_{i=1}^{k} \mathrm{S}(n, i)$	$\sum_{i=1}^{k} \mathrm{L}(n, i)$	$\begin{cases} 1 & \text{if } k \geq n \\ 0 & \text{otherwise} \end{cases}$	$\mathrm{S}(n, k)$	$\mathrm{L}(n, k) = \dfrac{n!}{k!}\dbinom{n-1}{k-1}$	$\begin{cases} 1 & \text{if } k = n \\ 0 & \text{otherwise} \end{cases}$
identical	**identical**	$\sum_{i=1}^{k} \mathrm{P}(n, i)$		$\begin{cases} 1 & \text{if } k \geq n \\ 0 & \text{otherwise} \end{cases}$	$\mathrm{P}(n, k)$		$\begin{cases} 1 & \text{if } k = n \\ 0 & \text{otherwise} \end{cases}$

Distributing *n* Distinct Objects to *k* Distinct Containers a b ... n   box 1 box 2 ... box k

Each object can go to any one of the k containers.
So number of distributions = $k \times k \times \ldots \times k = k^n$.
If each container can get at most 1 object, we first choose n containers out of k containers and then put the n objects into these containers in $n!$ ways.
So number of distributions = $\dbinom{k}{n} n! = {}^{k}\mathrm{P}_{n}$.

Distributing *n* Distinct Objects to *k* Distinct Containers if Order Matters

2 objects { a, b } can be arranged in 3 distinct containers in $\dfrac{(2+3-1)!}{(3-1)!} = \dfrac{4!}{2!} = 12$ ways.

The easiest way to understand this formula is to see that each distribution corresponds to a permutation of { a, b, |, | } where the 2 |'s divide the objects among different containers:

Container 1	Container 2	Container 3	
a b			a b | |
b a			b a | |
	a b		| a b |
	b a		| b a |
		a b	| | a b
		b a	| | b a
a	b		a | b |
a		b	a | | b
	a	b	| a | b
b	a		b | a |
b		a	b | | a
	b	a	| b | a

Since $(k-1)$ |'s are needed, the number of distributions = $\dfrac{(n+k-1)!}{(k-1)!}$.

If each container must receive at least 1 object, then this is the same as putting 1 object into each container first. This can be done in ${}^{n}\mathrm{P}_{k}$ ways. So total number of distributions

$$= {}^{n}\mathrm{P}_{k} \times \text{number of ways of arranging } n-k \text{ objects among } k \text{ containers}$$

$$= {}^{n}\mathrm{P}_{k}\ \frac{(n-k+k-1)!}{(k-1)!}$$

$$= {}^{n}\mathrm{P}_{k}\ \frac{(n-1)!}{(k-1)!}\ .$$

Distributing *n* Identical Objects to *k* Distinct Containers○○ ... ○ $\boxed{\textbf{box 1}}$ $\boxed{\textbf{box 2}}$...$\boxed{\textbf{box k}}$

2 identical objects can be put into 3 distinct containers in $\dbinom{2+3-1}{2} = \dbinom{4}{2} = 6$ ways.

The easiest way to understand this formula is to see that each distribution corresponds to a permutation of { o, o, |, | } where the 2 |'s divide the o's among different containers:

Container 1	Container 2	Container 3	
oo			oo | |
o	o		o | o |
o		o	o | | o
	oo		| oo |
	o	o	| o | o
		oo	| | oo

Since $(k-1)$ |'s are needed, the number of distributions = $\dfrac{(n+k-1)!}{n!\,(k-1)!} = \dbinom{n+k-1}{n}$.

If each container can get at most 1 object, then this is the same as choosing n containers out of k containers to receive 1 object. So number of distributions = $\dbinom{k}{n}$.

If each container must receive at least 1 object, then this is the same as putting 1 object into each container first.

So number of distributions = number of ways of distributing $n-k$ objects to k containers

$$= \frac{(n+k-k-1)!}{(n-k)!\,(k-1)!}$$

$$= \frac{(n-1)!}{(n-k)!\,(k-1)!}$$

$$= \binom{n-1}{k-1}\ .$$

Distributing *n* Distinct Objects to *k* Identical Containers a b ... n ...

The number of ways of distributing n distinct objects to k non–empty identical containers is given by $S(n, k)$, the Stirling numbers of the second kind.

For example, the 4 letters {a, b, c, d} can be put into 2 identical containers in $S(4, 2) = 7$ ways:

a, b, c	d
a, b, d	c
a, c, d	b

b, c, d	a
a, b	c, d
a, c	b, d
a, d	b, c

n \ k	1	2	3	4	5	6	7	8	Sum
1	1								1
2	1	1							2
3	1	3	1						5
4	1	7	6	1					15
5	1	15	25	10	1				52
6	1	31	90	65	15	1			203
7	1	63	301	350	140	21	1		877
8	1	127	966	1701	1050	266	28	1	4140

This table can be generated using the recurrence relation $S(n, k) = S(n-1, k-1) + k\,S(n-1, k)$. Each number is equal to the sum of its 'northwestern' neighbour and k times its 'northern' neighbour, where k is the column number. This recurrence relation can be explained as follows:

Suppose that we know the number of ways to distribute $n-1$ distinct objects to k identical containers, and the number of ways to distribute $n-1$ distinct objects to $k-1$ identical containers, and we want to know the number of ways of distributing n objects to k containers. We can either:

(i) start with any of the $S(n-1, k-1)$ combinations and put the nth object into a new container (there is 1 way to do this), or

(ii) start with any of the $S(n-1, k)$ combinations and put the nth object into one of the non–empty container (there are k ways to do this).

Thus the total number of distributions is given by $S(n, k) = S(n-1, k-1) + k\,S(n-1, k)$.

If some containers can be left empty, then the number of distributions $= \sum_{i=1}^{k} S(n, i)$, the Bell numbers.

Distributing *n* Distinct Objects to *k* Identical Containers if Order Matters

If the order is taken into account, first arrange the n objects in $n!$ ways. Then choose $k-1$ out of the $n-1$ possible cut points in each permutation to create k nonempty sets. This can be done in $\binom{n-1}{k-1}$ ways. Since the k containers are identical, every $k!$ permutations create the same distribution. Thus the number of ways of distributing n distinct objects into k nonempty ordered sets is the Lah number $L(n, k) = \frac{n!}{k!}\binom{n-1}{k-1}$, named after Ivo Lah.

For example, the 3 letters { a, b, c } can be arranged in 2 nonempty identical containers in 6 ways:

a b	c
b a	c
a c	b
c a	b
b c	a

c b	a

If some containers can be left empty, then the number of distributions = $\sum_{i=1}^{k} \mathrm{L}(n, i)$.

Distributing *n* Identical Objects to *k* Identical Containers ○ ○...○ ⊔ ⊔ ... ⊔

The number of ways of distributing *n* identical objects to *k* non–empty identical containers equals the number of ways of partitioning the integer *n* into *k* parts, P(*n*, *k*).
For example, 8 identical objects can be placed into 3 identical containers in 5 ways:

oooooo	o	o	8 = 6 + 1 + 1
ooooo	oo	o	8 = 5 + 2 + 1
oooo	ooo	o	8 = 4 + 3 + 1
oooo	oo	oo	8 = 4 + 2 + 2
ooo	ooo	oo	8 = 3 + 3 + 2

n \ *k*	1	2	3	4	5	6	7	8	9	10	Sum
1	1										1
2	1	1									2
3	1	1	1								3
4	1	2	1	1							5
5	1	2	2	1	1						7
6	1	3	3	2	1	1					11
7	1	3	4	3	2	1	1				15
8	1	4	5	5	3	2	1	1			22
9	1	4	7	6	5	3	2	1	1		30
10	1	5	8	9	7	5	3	2	1	1	42

This table can be generated using the recurrence relation P(*n*, *k*) = P(*n* – 1, *k* – 1) + P(*n* – *k*, *k*). Each number is equal to the sum of its 'northwestern' neighbour and its neighbour *k* spaces above, where *k* is the column number. This recurrence relation can be explained as follows:

Each *k*–element partition of the integer *n* either contains '1' or does not contain '1'.
Each *k*–element partition containing '1' corresponds to a *k* – 1 element partition of the integer *n* – 1. For example, the partitions 8 = 6 + 1 + 1 = 5 + 2 + 1 = 4 + 3 + 1 correspond to the 2–element partitions 7 = 6 + 1 = 5 + 2 = 4 + 3.
For each *k*–element partition not containing '1', we can subtract 1 from each part of the partition. For example, one of the 3–element partitions of 8 not containing '1'is 4 + 2 + 2. We can subtract 1 from each part to get 3 + 1 + 1, which is one of the 3–element partitions of 5 (which is 8 – 3). So every *k*–element partition of *n* not containing '1' corresponds to a *k*–element partition of the integer *n* – *k*.
Thus the total number of distributions is given by P(*n*, *k*) = P(*n* – 1, *k* – 1) + P(*n* – *k*, *k*).

If some containers can be left empty, then the number of distributions = $\sum_{i=1}^{k} \mathrm{P}(n, i)$.

Ref: Counting Objects In Boxes
http://2000clicks.com/mathhelp/CountingObjectsInBoxes.aspx