Determinants

Wednesday, April 5, 2023   8:04 PM

Determinant: Determinant is the numerical value of the square matrix. So, to every square matrix A = [aij] of order n, we can associate a number (real or complex) called determinant of the square matrix A. It is denoted by det A or |A|.

Note

(i) Read |A| as determinant A not absolute value of A.

(ii) Determinant gives numerical value but matrix do not give numerical value.

(iii) A determinant always has an equal number of rows and columns, i.e. only square matrix have determinants.

square matrix of order 2x2

$A = \begin{bmatrix} 2 & 4 \\ -3 & 6 \end{bmatrix}$ → value / scalar → Determinant

det(A) = |A| → detA

$2\times6 - (-3)\times4$

$12 + 12 = 24$ → det(A)

det of order 2x2

$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = a\times d - b\times c$ (− fixed)

Ex. If $B = \begin{bmatrix} 3 & -1 \\ -2 & -4 \end{bmatrix}$ find |B| = ?

$|B| = -12 - 2 = -14$

Ex: If $\begin{vmatrix} -3 & x \\ 4 & 8 \end{vmatrix} = 12$ find x.

det. of Order 3x3

$\begin{vmatrix} 1 & 4 & 2 \\ 3 & -1 & 4 \\ 1 & 0 & 2 \end{vmatrix}$

Sign chart

$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$

Expand R_1

$+1\begin{vmatrix} -1 & 4 \\ 0 & 2 \end{vmatrix} - 4\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} + 2\begin{vmatrix} 3 & -1 \\ 1 & 0 \end{vmatrix}$

$1(-2-0) - 4(6-4) + 2(0+1)$

$-2 - 4(2) + 2$

$-2 - 8 + 2$

-8

Expand C_3

$+2\begin{vmatrix} 3 & -1 \\ 1 & 0 \end{vmatrix} - 4\begin{vmatrix} 1 & 4 \\ 1 & 0 \end{vmatrix} + 2\begin{vmatrix} 1 & 4 \\ 3 & -1 \end{vmatrix}$

$2(0+1) - 4(0-4) + 2(-1-12)$

$2 + 16 - 26$

$18 - 26$

-8

Evaluate the determinants in Exercises 1 and 2.

1. $\begin{vmatrix} 2 & 4 \\ -5 & -1 \end{vmatrix}$

2. (i) $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$ (ii) $\begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix}$

3. If $A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$, then show that | 2A | = 4 | A |

4. If $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$, then show that | 3 A | = 27 | A |

Sol. 3 $A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$

$2A = \begin{bmatrix} 2 & 4 \\ 8 & 4 \end{bmatrix}$

L.H.S $|2A| = 8 - 32 = -24$

R.H.S $4\begin{vmatrix} 1 & 2 \\ 4 & 2 \end{vmatrix} = 4(2-8) = 4(-6) = -24$

4. If $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$, then show that $|3A| = 27|A|$

R.H.S $4\begin{vmatrix} 1 & 2 \\ 4 & 2 \end{vmatrix} = 4(2-8) = 4(-6) = -24$

5. Evaluate the determinants

(i) $\begin{vmatrix} 3 & -1 & -2 \\ 0 & 0 & -1 \\ 3 & -5 & 0 \end{vmatrix}$ (ii) $\begin{vmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix}$

Expanding R_2

$-0|\ \ | + 0|\ \ | + 1|\ \ |$

$1(-15+3)$

-12

$3\begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} + 4\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} + 5\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}$

$3(1+6) + 4(1+4) + 5(3-2)$

$21 + 20 + 5$

$41 + 5 = 46$

4-1

(iii) $\begin{vmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{vmatrix}$ (iv) $\begin{vmatrix} 2 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{vmatrix}$

HW

6. If $A = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 1 & -3 \\ 5 & 4 & -9 \end{bmatrix}$, find $|A|$

7. Find values of x, if

(i) $\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix}$ (ii) $\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = \begin{vmatrix} x & 3 \\ 2x & 5 \end{vmatrix}$

Sol:

$10 - 12 = 5x - 6x$

$-2 = -x$

$x = 2$

$2 - 20 = 2x^2 - 24$

$-18 = 2x^2 - 24$

$2x^2 = -18 + 24$

$2x^2 = 6$

$x^2 = 3$

$x = \pm\sqrt{3}$

8. If $\begin{vmatrix} x & 2 \\ 18 & x \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 18 & 6 \end{vmatrix}$, then x is equal to

(A) 6 (B) ±6 (C) – 6 (D) 0

$x^2 - 36 = 36 - 36$

$x^2 = 36$

$x = \pm\sqrt{36}$

$x = \pm 6$

Singular Matrix

If any Matrix whose det is 0 then that Matrix will be a Singular Matrix

Ex: $A = \begin{bmatrix} 4 & 1 \\ 12 & 3 \end{bmatrix}$

$|A| = 12 - 12 = 0$

A → Singular Matrix

Ex- If A is Singular Matrix and $A = \begin{bmatrix} 3 & x \\ 4 & 5 \end{bmatrix}$

and $A = \begin{bmatrix} 3 & x \\ 4 & 5 \end{bmatrix}$

find $x = ?$

Ex Let $B = \begin{bmatrix} y & 4 & y \\ 3 & -2 & 6 \\ 4 & 1 & 2 \end{bmatrix}$

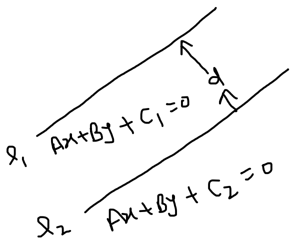
find y is B is singular Matrix.

$y(-4-6) - 4(6-24) + y(3+8) = 0$

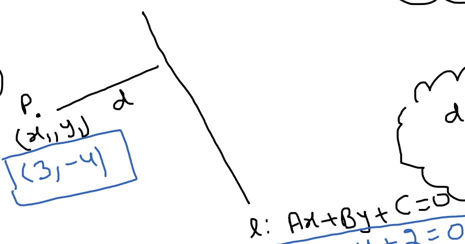
$-10y - 72 + 3y + 8y = 0$

$-10y + 11y = 72$

$\boxed{y = 72}$

① 

$$d = \left| \frac{C_2 - C_1}{\sqrt{A^2+B^2}} \right|$$

② 

$$d = \left| \frac{Ax_1 + By_1 + C}{\sqrt{A^2+B^2}} \right|$$

$P(x_1, y_1)$: $(3, -4)$

$l: Ax + By + C = 0$: $2x - y + 2 = 0$

$d = \left| \frac{6+4+2}{\sqrt{4+1}} \right| = \frac{12}{\sqrt{5}}$ units

Minors vs Cofactors

a_{ij} — elements

M_{ij} → Minors

A_{ij} → Cofactors

$$A_{ij} = (\text{sign}) M_{ij}$$

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

$$\begin{vmatrix} \boxed{2} & -1 & 3 \\ 4 & 2 & 0 \\ 1 & 2 & 1 \end{vmatrix}$$

Minors Minor of element $2 = M_{11} = \begin{vmatrix} 2 & 0 \\ 2 & 1 \end{vmatrix} = (2-0) = 2$

" " " $-1 = M_{12} = \begin{vmatrix} 4 & 0 \\ 1 & 1 \end{vmatrix} = (4-0) = 4$

$M_{13} = 6$

$M_{21} = -6$

$M_{22} = -1$

$M_{23} = 5$

$M_{31} = -6$

$M_{32} = -12$

$M_{33} = 8$

$A_{11} = +2 = 2$ $\quad A_{12} = -4 = -4$ $\quad A_{13} = +6 = 6$

$A_{21} = -(-6)$ $\quad A_{22} = +(-1)$ $\quad A_{23} = -5$

$A_{31} = +(-6)$ $\quad A_{32} = -(-12)$ $\quad A_{33} = +8$

Cofactor Matrix $= \begin{bmatrix} 2 & -4 & 6 \\ 6 & -1 & -5 \\ -6 & 12 & 8 \end{bmatrix}$ ✓

Q. If $A = \begin{bmatrix} 4 & -2 & 3 \\ -1 & 2 & 6 \\ 1 & 2 & -3 \end{bmatrix}$

find cofactor Matrix ?

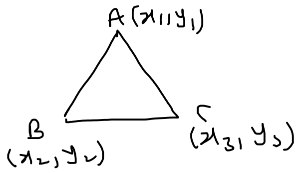
$A_{11} = +(-6-12) = -18$ $A_{12} = -(3-6) = 3$ $A_{13} = +(-2-2) = -4$

$A_{21} = -(6-6) = 0$ $A_{22} = +(-12-3) = -15$ $A_{23} = -(8+2) = -10$

$A_{31} = +(-12-6) = -18$ $A_{32} = -(24+3) = -27$ $A_{33} = +(8-2) = 6$

vb

Area of Δ



$\frac{1}{2}[x_1(y_2-y_3) \ldots$

$$\text{Area of } \Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$ → Area always positive.

1. Find area of the triangle with vertices at the point given in each of the following :
(i) (1, 0), (6, 0), (4, 3)
(ii) (2, 7), (1, 1), (10, 8)
(iii) (−2, −3), (3, 2), (−1, −8)

$$\frac{1}{2}\begin{vmatrix} 1 & 0 & 1 \\ 6 & 0 & 1 \\ 4 & 3 & 1 \end{vmatrix}$$

$\frac{1}{2}\left|1(0-3) + 0 + 1(12-0)\right|$

$\frac{1}{2}(-3+12) = \frac{9}{2}$ Sq. units

$$\frac{1}{2}\begin{bmatrix} -2 & -3 & 1 \\ 3 & 2 & 1 \\ -1 & -8 & 1 \end{bmatrix}$$

$\frac{1}{2}[-2(2+8) + 3(3+1) + 1(-24+2)]$

$\frac{1}{2}[-20 + 12 - 22] = \frac{1}{2}|(-30)| = 15$

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 4 & 5 \\ -1 & 6 & 8 \end{bmatrix}$$

det.

Minors & Cofactors

$$\begin{vmatrix} 4 & 3 & -1 \\ 2 & 1 & 9 \\ 3 & 1 & 4 \end{vmatrix}$$

Cofactors = A_{ij} = Sign M_{ij}

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

M_{11}

[illegible]

$\begin{vmatrix} 3 & 1 & 4 \end{vmatrix}$

$M_{11} = \begin{vmatrix} 1 & 9 \\ 1 & 4 \end{vmatrix} = 4-9 = -5$ $\quad M_{12} = 8-27 = -19$ $\quad M_{13} = 2-3 = -1$

M_{11}: $A_{11} = +(-5) = -5$ $\quad A_{12} = -\begin{vmatrix} 2 & 9 \\ 3 & 4 \end{vmatrix}$

$M_{21} = 12+1 = 13$ $\quad M_{22} = 16+3 = 19$ $\quad M_{23} = 4-9 = -5$

$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$

$M_{31} = 27+1 = 28$ $\quad M_{32} = 36+2 = 38$ $\quad M_{33} = 4-6 = -2$

$\begin{vmatrix} + & - \\ - & + \end{vmatrix}$

3. Using Cofactors of elements of second row, evaluate $\Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$.

R_2 ; determinant

R_2

$a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23}$

element ; Cofactor.

4. Using Cofactors of elements of third column, evaluate $\Delta = \begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix}$.

Adjoint of a Matrix = [Cofactor Matrix]$^{T/'}$ = adj(A)

Q If $A = \begin{bmatrix} 4 & 2 & -1 \\ 3 & 1 & 2 \\ 1 & 0 & 3 \end{bmatrix}$

$|A| = 4(3) - 2(9-2) - 1(0-1)$
$= 12 - 14 + 1 = \cancel{+3}$ ✓ -1

Cofactors

$A_{11} = 3$ $\quad A_{12} = -7$ $\quad A_{13} = -1$

$A_{21} = -6$ $\quad A_{22} = 13$ $\quad A_{23} = 1$

$A_{31} = 3$ $\quad A_{32} = -11$ $\quad A_{33} = -2$

Cofactor Matrix = $\begin{bmatrix} 3 & -7 & -1 \\ -6 & 13 & 1 \\ 3 & -11 & -2 \end{bmatrix}$

Adj(A) = $\begin{bmatrix} 3 & -7 & -1 \\ -6 & 13 & 1 \\ 3 & -11 & -2 \end{bmatrix}'$

Adj(A) = $\begin{bmatrix} 3 & -6 & 3 \\ -7 & 13 & -11 \\ -1 & 1 & -2 \end{bmatrix}$

Inverse of Matrix = A^{-1}

$$A^{-1} = \frac{1}{|A|}\,\text{adj}(A)$$

If $|A| = 0$ then A^{-1} doesn't exist and if $|A| \neq 0$ then A^{-1} does exist

$|A| = -1$ $\quad A^{-1}$ does exist ✓

$|A| = -1 \quad A^{-1}$ does exist ✓

$$A^{-1} = \frac{1}{|A|} \cdot adj(A) = \frac{1}{-1}\begin{bmatrix} 3 & -6 & 3 \\ -7 & 13 & -11 \\ -1 & 1 & -2 \end{bmatrix} = \begin{bmatrix} -3 & 6 & -3 \\ 7 & -13 & 11 \\ 1 & -1 & 2 \end{bmatrix}$$ ✓

Find adjoint of each of the matrices in Exercises 1 and 2.

1. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ **2.** $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix}$

8. $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{bmatrix}$ **9.** $\begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 0 \\ -7 & 2 & 1 \end{bmatrix}$ **10.** $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}$

Let $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 0 \\ -7 & 2 & 1 \end{bmatrix}$ $\quad A^{-1} = \frac{1}{|A|} \cdot adj(A)$

$|A| = 2(-1) - 1(4) + 3(8-7)$

$= -2 - 4 + 3$

$= -6 + 3 = -3$

14. For the matrix $A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$, find the numbers a and b such that $A^2 + aA + bI = O$.

$A^2 + aA + bI = O$

$A^2 = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 8 \\ 4 & 3 \end{bmatrix}$

$\begin{bmatrix} 11 & 8 \\ 4 & 3 \end{bmatrix} + \begin{bmatrix} 3a & 2a \\ a & a \end{bmatrix} + \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$11 + 3a + b = 0$

$3a + b = -11$ $\quad 8 + 2a + 0 = 0$

$3 \times (-4) + b = -11$ $\quad 8 + 2a = 0$

$-12 + b = -11$ $\quad 2a = -8$

$b = -11 + 12$ $\quad \boxed{a = -4}$

$\boxed{b = 1}$

Q. Let $A = \begin{bmatrix} 2 & -2 \\ 4 & 3 \end{bmatrix}$

$A^{-1} = \frac{1}{|A|} \cdot adj(A)$

$\begin{bmatrix} & \\ 4 & 3 \end{bmatrix}$

$A^{-1} = \frac{1}{|A|} \cdot adj(A)$

$|A| = \begin{vmatrix} 2 & -2 \\ 4 & 3 \end{vmatrix} = 6+8 = 14$

A^{-1} does exist

Cofactors

$A_{11} = +3 = 3$ $\quad A_{12} = -4 = -4$

$A_{21} = -(-2) = 2$ $\quad A_{22} = +2 = 2$

$Adj(A) = \begin{bmatrix} 3 & -4 \\ 2 & 2 \end{bmatrix}' = \begin{bmatrix} 3 & 2 \\ -4 & 2 \end{bmatrix}$

$A^{-1} = \frac{1}{|A|} \cdot adj(A)$

$A^{-1} = \frac{1}{14} \begin{bmatrix} 3 & 2 \\ -4 & 2 \end{bmatrix}$

$\begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 0 \\ -7 & 2 & 1 \end{bmatrix}$

$|A| = 2\begin{vmatrix} -1 & 0 \\ 2 & 1 \end{vmatrix} - 1(4+0) + 3(8-7)$

$= 2(-1-0) - 4 + 3$ (fixed.)

$= -2 - 4 + 3$

$= -3$

Cofactors

$A_{11} = +\begin{vmatrix} -1 & 0 \\ 2 & 1 \end{vmatrix} = +(-1-0) = -1$

Applications of Determinants and Matrices

A. Solution of system of linear equations using inverse of a matrix

$a_1x + b_1y + c_1z = d_1$

$a_2x + b_2y + c_2z = d_2$

$a_3x + b_3y + c_3z = d_3$

Coefficient Matrix $\quad$ Variable Matrix $\quad$ Constant Matrix

$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ $\quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $\quad B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

$X = A^{-1} \cdot B$

Solve the following system of equations by matrix method.

$$3x - 2y + 3z = 8$$
$$2x + y - z = 1$$
$$4x - 3y + 2z = 4$$

$3\times1 - 2\times2 + 3\times3$
$3 - 4 + 9$
$-1 + 9 = 8$

$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$A = \begin{bmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $B = \begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$

$X = A^{-1} \cdot B$ — (1)

$A^{-1} = \frac{1}{|A|} \cdot adj(A)$

$|A| = 3(2-3) + 2(4+4) + 3(-6-4)$
$= -3 + 16 - 30$
$= -17$

Cofactors

$A_{11} = +(2-3) = -1$ $A_{12} = -(4+4) = -8$ $A_{13} = +(-6-4) = -10$

$A_{21} = -(-4+9) = -5$ $A_{22} = +(6-12) = -6$ $A_{23} = -(-9+8) = 1$

$A_{31} = +(2-3) = -1$ $A_{32} = -(-3-6) = 9$ $A_{33} = +(3+4) = 7$

$Adj(A) = \begin{bmatrix} -1 & -8 & -10 \\ -5 & -6 & 1 \\ -1 & 9 & 7 \end{bmatrix}' = \begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}$

$A^{-1} = \frac{1}{-17} \begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}$

from (1)

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{17} \begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}_{3\times3} \begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}_{3\times1}$

$= -\frac{1}{17} \begin{bmatrix} -8-5-4 \\ -64-6+36 \\ -80+1+28 \end{bmatrix}_{3\times1}$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{-17} \begin{bmatrix} -17 \\ -34 \\ -51 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$x = 1$
$y = 2$, $z = 3$