Linear Differential Equations with Variable coefficients

There are two special linear equations with variable coefficients. By using suitable substitution we can transform these equations into linear differential equations with constant coefficients.

Cauchy's Linear Equation (or Euler-Cauchy Equation)

The Differential Equation of the form

$$(a_n x^n D^n + a_{n-1} x^{n-1} D^{n-1} + \cdots + a_2 x^2 D^2 + a_1 x D + a_0) y = f(x)$$

where $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ are constants and $f(x)$ is function of x is called Cauchy's linear Differential Equation.

To solve this, put $z = \log x$ or $x = e^z$

then $$\frac{dz}{dx} = \frac{1}{x}$$

$$Dy = \frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz} = \frac{1}{x} D_1 y$$

$$Dy = \frac{1}{x} D_1 y$$

where $D_1 = \frac{d}{dz}$

$$\Rightarrow xDy = D_1 y$$

$$D^2 y = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right)$$

$$= \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right) + \left(-\frac{1}{x^2}\right)\frac{dy}{dz}$$

$$= \frac{1}{x}\frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx} - \frac{1}{x^2}\frac{dy}{dz}$$

$$= \frac{1}{x^2}\frac{d^2y}{dz^2} - \frac{1}{x^2}\frac{dy}{dz}$$

$$= \frac{1}{x^2}(D_1^2 - D_1)y$$

$$\Rightarrow x^2D^2y = D_1(D_1-1)y$$

$$\Rightarrow x^2D^2 = D_1(D_1-1)$$

III^{ly} $$x^3D^3 = D_1(D_1-1)(D_1-2)$$

using (1), (2), (3) and similar expression in the given equation we get the linear Differential equation with constant coefficients, which can be solved as explained earlier.

Legendre's Linear Differential Equation

Equation of the form

$$\left(a_n(ax+b)^n D^n + a_{n-1}(ax+b)^{n-1}D^{n-1} + \cdots + a_2(ax+b)^2D^2 + a_1(ax+b)D + a_0\right)y = f(x)$$

is called Legendre's linear differential equation, this can be reduced to linear differential equation with constant coefficients by using the transformation

$$z = \log(ax+b) \quad \text{or} \quad ax+b = e^z$$

As above we can prove that

$$(ax+b)D = aD_1$$
$$(ax+b)^2D^2 = a^2D_1(D_1-1)$$
$$(ax+b)^3D^3 = a^3(D_1)(D_1-1)(D_1-2) \quad \text{and so on.}$$

Problems on Cauchy's linear Differential Equations

① Solve $x^2 \frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = x^4$

$$(x^2D^2 - 2xD - 4)y = x^4 \rightarrow ①$$

Soln:- The above differential Equation is Cauchy's linear D.Eqn

It can be converted to linear Differential Equation with constant coefficients by substituting

$$\log x = z \quad \text{or} \quad e^z = x$$

$$xD = D_1$$

$$x^2D^2 = D_1(D_1 - 1)$$

where $D_1 = \frac{d}{dz}$

$$(D_1(D_1-1) - 2D_1 - 4)y = e^{4z}$$

$$(D_1^2 - D_1 - 2D_1 - 4)y = e^{4z}$$

$$(D_1^2 - 3D_1 - 4)y = e^{4z}$$

A.Eqn is $m^2 - 3m - 4 = 0$

$$m = 4, -1$$

$\therefore$ C.F $= c_1e^{4z} + c_2e^{-z}$ → Note this step

(since the independent variable is z, C.F should be in terms of z)

$\therefore$ $\boxed{C.F = c_1x^4 + c_2x^{-1}}$

$$P.I = \frac{1}{D_1^2 - 3D_1 - 4} e^{4z}$$

Replace $D_1 \rightarrow 4$

Denominator becomes zero after replacing D_1 by 4

$$\therefore \quad P.I = z \cdot \frac{1}{2D_1 - 3} e^{4z}$$

$D_1 \to 4$

$$= z \frac{1}{2(4)-3} e^{4z}$$

$$= \frac{ze^{4z}}{5}$$

$$\therefore P.I = \frac{(\log x)\, x}{5}$$

$$y = c_1 x^4 + c_2 x^{-1} + \frac{x \log x}{5}$$

(2) solve $x^2 y'' + 2xy' - 2y = (x+1)^2$

Soln:- The above Differential Eqn can be written as

$$(x^2D^2 + 2xD - 2)\, y = (x+1)^2$$

It can be converted to a linear Differential Equation with constant coefficients by taking the substitution

$$\log x = z \quad \text{or} \quad e^z = x$$

then $\quad xD = D_1$

$x^2 D^2 = D_1(D_1 - 1) \qquad$ where $D_1 = \frac{d}{dz}$

$$\therefore \quad [D_1(D_1-1) + 2D_1 - 2]\, y = (e^z+1)^2$$

$$[D_1^2 - D_1 + 2D_1 - 2]\, y = (e^{2z} + 1 + 2e^z)$$

$$(D_1^2 + D_1 - 2)\, y = e^{2z} + 2e^z + 1$$

$$\therefore \quad m^2 + m - 2 = 0$$

$$m = -2, 1$$

$\therefore$ C.F $= c_1 e^{-2z} + c_2 e^{z}$

$= c_1 x^{-2} + c_2 x$ $\quad (\because e^{z} = x)$

$$P.I = \frac{1}{D_1^2 + D_1 - 2}[e^{2z} + 2e^{z} + 1]$$

$$= \frac{1}{D_1^2+D_1-2} e^{2z} + 2\frac{1}{D_1^2+D_1-2} e^{z} + \frac{1}{D_1^2+D_1-2} e^{0z}$$

Replace $D_1 \rightarrow 2$ $\qquad$ Denominator becomes 0

$$= \frac{1}{2^2+2-2} e^{2z} + 2z\frac{1}{2D_1+1} e^{z} + \frac{1}{0+0-2} e^{0z}$$

$$= \frac{e^{2z}}{4} + \frac{2ze^{z}}{3} - \frac{1}{2}$$

$$P.I = \frac{x^2}{4} + \frac{2}{3} x \log x - \frac{1}{2}$$

$y =$ C.F + P.I

(3) Solve $x^2 y'' - xy' + 2y = x \sin \log x$;

Soln: the above D.Eqn can be written as

$$(x^2D^2 - xD + 2)y = x \sin \log x$$

Put $\log x = z$ or $e^{z} = x$

Then $xD = D_1$

$x^2D^2 = D_1(D_1 - 1)$

where $D_1 = \frac{d}{dz}$

$\therefore$ The given differential Equation becomes

$$(D_1^2 - D_1 - D_1 + 2)\,y = e^z \sin z$$

$$(D_1^2 - 2D_1 + 2)\,y = e^z \sin z$$

$\therefore$ A. Eqn $\quad m^2 - 2m + 2 = 0$

$$m = 1 \pm i$$

$$C.F = e^z (c_1 \cos z + c_2 \sin z)$$

$$\Rightarrow C.F = x(c_1 \cos\log x + c_2 \sin\log x)$$

$$P.I = \frac{1}{D_1^2 - 2D_1 + 2} e^z \sin z \qquad (D_1 \to D_1 + 1)$$

$$= e^z \frac{1}{(D_1+1)^2 - 2(D_1+1) + 2} \sin z$$

$$= e^z \frac{1}{D_1^2 + 2D_1 + 1 - 2D_1 - 2 + 2} \sin z$$

$$= e^z \frac{1}{D_1^2 + 1} \sin z$$

$$D_1^2 \to -1$$

Replacing D_1^2 by -1, we get zero in the denominator

$$\therefore P.I = e^z \cdot z \frac{1}{2D_1} \sin z$$

$$= e^z \cdot z \frac{1}{2} \int \sin z \, dz$$

$$= \frac{e^z \cdot z}{2} (-\cos z)$$

$$P.I = -\frac{x}{2} \log x \cos(\log x), \qquad y = C.F + P.I$$

(4) Solve $x^2y'' - 2xy' + 2y = x^3 + \sin\log x$

Soln :- Given D. Eqn can be written as

$$(x^2D^2 - 2xD + 2)y = x^3 + \sin\log x$$

Put $\log x = z$ or $e^z = x$

then $xD = D_1$

$x^2D^2 = D_1(D_1 - 1)$ where $D_1 = \frac{d}{dz}$

$$\Rightarrow (D_1^2 - D_1 - 2D_1 + 2)y = e^{3z} + \sin z$$

$$\Rightarrow (D_1^2 - 3D_1 + 2)y = e^{3z} + \sin z$$

A. Eqn $m^2 - 3m + 2 = 0$

$m = 1, 2$

$C.F = c_1e^z + c_2e^{2z}$

$C.F = c_1x + c_2x^2$

$$P.I = \frac{1}{D_1^2 - 3D_1 + 2}e^{3z} + \frac{1}{D_1^2 - 3D_1 + 2}\sin z$$

$D_1 \to 3$ $\quad$ $D_1^2 \to -1$

$$= \frac{1}{3^2 - 3(3) + 2}e^{3z} + \frac{1}{-1 - 3D_1 + 2}\sin z$$

$$= \frac{1}{9 - 9 + 2}e^{3z} + \frac{1}{-3D_1 + 1}\sin z$$

Rationalize to get D_1^2

$$= \frac{1}{2}e^{3z} - \frac{1}{(3D_1 - 1)} \times \frac{3D_1 + 1}{(3D_1 + 1)}\sin z$$

$$= \frac{1}{2}e^{3z} - \frac{1}{9D_1^2 - 1}(3D_1 + 1)\sin z$$

Replace $D_1^2 \to -1$

$$= \frac{e^{3z}}{2} - \frac{1}{-10} \, 3D_1(\sin z) + \sin z$$

$$= \frac{e^{3z}}{2} + \frac{1}{10} [3\cos z + \sin z]$$

$$P.I = \frac{x^3}{2} + \frac{1}{10} [3\cos\log x + \sin\log x]$$

$$y = C.F + P.I$$

(5) $x^4 y''' + 2x^3 y'' - x^2 y' + xy = \sin\log x$

Sol:. Observe that given D.Eqn is not in the Standard form of Cauchy's linear D.Eqn.

Divide the given D.Eqn by x

$$\Rightarrow \quad x^3 y''' + 2x^2 y'' - xy' + y = \frac{\sin\log x}{x}$$

$$\Rightarrow \quad (x^3D^3 + 2x^2D^2 - xD + 1)y = \frac{\sin\log x}{x}$$

Put $\log x = z$ or $e^z = x$

then $\quad xD = D_1$

$x^2D^2 = D_1(D_1 - 1)$

$x^3D^3 = D_1(D_1-1)(D_1-2)$

$\therefore$ we have $\left((D_1^2 - D_1)(D_1 - 2) + 2(D_1^2 - D_1) - D_1 + 1\right)y = \frac{\sin z}{e^z}$

$$(D_1^3 - 3D_1^2 + 2D_1 + 2D_1^2 - 2D_1 - D_1 + 1)y = e^{-z}\sin z$$

$$(D_1^3 - D_1^2 - D_1 + 1)y = e^{-z}\sin z$$

$\therefore$ A.Eq^n $\quad m^3 - m^2 - m + 1 = 0$

$m^2(m-1) - (m-1) = 0$

$(m^2-1)(m-1) = 0$

$\Rightarrow m = 1, 1, -1$

Two roots are repeated

$C.F = (C_1 + C_2 z)e^z + C_3 e^{-z}$

$C.F = (C_1 + C_2 \log x)x + C_3 x^{-1}$

$$P.I = \frac{1}{D_1^3 - D_1^2 - D_1 + 1} e^{-z} \sin z$$

Replace

$D_1 \rightarrow D_1 - 1$

$$= e^{-z} \frac{1}{(D_1-1)^3 - (D_1-1)^2 - (D_1 \cancel{+} 1) + 1} \sin z$$

$$= e^{-z} \frac{1}{D_1^3 - 1 - 3D_1^2 + 3D_1 - (D_1^2 + 1 - 2D_1) - D_1 + 1 + 1} \sin z$$

$$= e^{-z} \frac{1}{D_1^3 - 1 - 3D_1^2 + 3D_1 - D_1^2 - 1 + 2D_1 - D_1 + 2} \sin z$$

$$= e^{-z} \frac{1}{D_1^3 - 4D_1^2 + 4D_1} \sin z$$

$D_1^2 \rightarrow -1$

$$= e^{-z} \frac{1}{-D_1 - 4(-1) + 4D_1} \sin z$$

$$= e^{-z} \frac{1}{3D_1 + 4} \sin z$$

$$= e^{-z} \frac{1}{(3D_1+4)} \times \frac{(3D_1 - 4)}{(3D_1 - 4)} \sin z$$

Rationalize to get D_1^2

$$= e^{-z} \frac{1}{9D_1^2-16}(3D_1-4)\sin z$$

$D_1^2 \to -1$

$$= e^{-z} \frac{1}{-9-16}(3D_1(\sin z) - 4\sin z)$$

$D_1 \to \frac{d}{dz}$

$D_1(\sin z) \to \frac{d}{dz}(\sin z)$

$$= -\frac{e^{-z}}{25}[3\cos z - 4\sin z]$$

$$= \frac{e^{-z}}{25}[4\sin z - 3\cos z]$$

$$P.I = \frac{1}{25x}[4\sin(\log x) - 3\cos(\log x)]$$

$y = C.F. \; P.I$

(6) Solve $x^2y'' - (2m-1)xy' + (m^2+n^2)y = n^2x^m \log x$

Soln:- $(x^2D^2 - (2m-1)xD + (m^2+n^2))y = n^2x^m \log x$

Put $\log x = z$ or $e^z = x$

then $xD = D_1$

$x^2D^2 = D_1(D_1-1)$ where $D_1 = \frac{d}{dz}$

$$[D_1^2 - D_1 - (2m-1)D_1 + (m^2+n^2)]y = n^2 \cancel{x} e^{mz} z$$

Consider A. Eqn $K^2 - K - (2m-1)K + m^2+n^2 = 0$

$K^2 - K - 2mK + K + m^2 + n^2 = 0$

$K^2 - 2mK + m^2 + n^2 = 0$

$a = 1$, $b = -2m$, $c = m^2+n^2$

$$K = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{2m \pm \sqrt{4m^2 - 4m^2 - 4n^2}}{2}$$

$$k = \frac{2m \pm \sqrt{-4n^2}}{2}$$

$$k = \frac{2m \pm 2ni}{2}$$

$$k = m + ni$$

$$C.F = e^{mz}[c_1 \cos nz + c_2 \sin nz]$$

$$C.F = x^m [c_1 \cos n \log x + c_2 \sin n \log x]$$

$$P.I = \frac{1}{D_1^2 - D_1 - (2m-1)D_1 + m^2 + n^2} n^2 e^{mz} z$$

n^2 is a constant

$$= n^2 \frac{1}{D_1^2 - \cancel{D_1} - 2mD_1 + \cancel{D_1} + m^2 + n^2} e^{mz} z$$

$$= n^2 \frac{1}{D_1^2 - 2mD_1 + m^2 + n^2} e^{mz} z$$

Replace $D_1 \to D_1 + m$

$$= n^2 e^{mz} \frac{1}{(D_1+m)^2 - 2m(D_1+m) + m^2 + n^2} \cancel{e^{m}} z$$

$$= n^2 e^{mz} \frac{1}{D_1^2 + \cancel{m^2} + \cancel{2mD_1} - \cancel{2mD_1} - \cancel{2m^2} + \cancel{m^2} + n^2} z$$

$$= n^2 e^{mz} \frac{1}{D_1^2 + n^2} z$$

$$= \cancel{n^2} e^{mz} \frac{1}{\cancel{n^2}} \frac{1}{(1 + \frac{D_1^2}{n^2})} z$$

$$= e^{mz} \left(1 + \frac{D_1^2}{n^2}\right)^{-1} z$$

$$= e^{mz} \left(1 - \frac{D_1^2}{n^2} + \left(\frac{D_1^2}{n^2}\right)^2 - \cdots \right) z$$

(8) Solve $(2x+3)^2y'' - 2(2x+3)y' - 12y = 6x$

The above differential Equation is Legendre's linear D.Eqn

It can be written as

$$[(2x+3)^2D^2 - 2(2x+3)D - 12]y = 6x$$

It can be converted to a linear D.Eqn with constant coefficients by the substitution

$$\log(2x+3) = z \quad \text{or} \quad e^z = 2x+3 \Rightarrow x = \frac{e^z - 3}{2}$$

then $(2x+3)D = 2D_1$ "coefficient of x"

$\therefore$ $(2x+2)^2D^2 = 2^2D_1(D_1-1)$

$$\therefore [4(D_1^2 - D_1) - 2(2D_1) - 12]y = 6\left[\frac{e^z-3}{2}\right]$$

$$[4D_1^2 - 4D_1 - 4D_1 - 12]y = 3[e^z - 3]$$

$$[4D_1^2 - 8D_1 - 12]y = 3[e^z - 3]$$

$$[D_1^2 - 2D_1 - 3]y = \frac{3}{4}[e^z - 3]$$

$\therefore$ A.E qn $\quad m^2 - 2m - 3 = 0$

$$m = 3, -1$$

$$C.F = c_1e^{3z} + c_2e^{-z}$$

$$C.F = c_1(2x+3)^3 + c_2(2x+3)^{-1} \qquad [\because e^z = 2x+3]$$

$$P.I = \frac{1}{D_1^2 - 2D_1 - 3}\,\frac{3}{4}[e^z - 3]$$

$$= \frac{3}{4}\left[\frac{1}{D_1^2 - 2D_1 - 3}e^z - \frac{1}{D_1^2 - 2D_1 - 3}3e^{0z}\right]$$

$$P.I = \frac{3}{4}\left[\frac{1}{1-2-3}e^{z} - \frac{1}{-3}3e^{0z}\right]$$

$$= \frac{3}{4}\left[-\frac{e^{z}}{4} + 1\right]$$

$$= \frac{3}{4} - \frac{3e^{z}}{16}$$

$$\therefore P.I = \frac{3}{4} - \frac{3}{16}(2x+3)$$

$$y = C.F + P.I.$$

⑨ Solve $(3x+2)^2 y'' + 3(3x+2)y' - 36y = 3x^2+4x+1$

Soln: The given D.Eqn can be written as

$$[(3x+2)^2 D^2 + 3(3x+2)D - 36]y = 3x^2+4x+1$$

Put $\log(3x+2) = z$ or $e^{z} = 3x+2 \Rightarrow x = \frac{e^{z}-2}{3}$

$$(3x+2)D = 3D_1$$

$$(3x+2)^2 D^2 = 3^2 D_1(D_1-1)$$

$$\therefore [9(D_1^2-D_1) + 3(3D_1) - 36]y = 3\left(\frac{e^{z}-2}{3}\right)^2 + 4\left(\frac{e^{z}-2}{3}\right) + 1$$

$$[9D_1^2 - 9D_1 + 9D_1 - 36]y = \frac{3}{9}[e^{2z}+4-4e^{z}] + \frac{4}{3}[e^{z}-2] + 1$$

Taking LCM.

$$[9D_1^2 - 36]y = \frac{1}{3}[e^{2z}+4-4e^{z}+4e^{z}-8+3]$$

$\div 9$

$$[D_1^2 - 4]y = \frac{1}{27}[e^{2z}-1]$$

$\therefore$ A Eqn: $m^2 - 4 = 0$

$$m = \pm 2$$

$$C.F = c_1 e^{2z} + c_2 e^{-2z}$$

$$C.F = c_1 (3x+2)^2 + c_2 (3x+2)^{-2}$$

$$P.I = \frac{1}{D_1^2-4}\left[\frac{1}{27}(e^{2z}-1)\right]$$

$$= \frac{1}{27}\left[\frac{1}{D_1^2-4}e^{2z} - \frac{1}{D_1^2-4}e^{0z}\right]$$

$D_1 \to 2$ $\qquad$ $D_1 \to 0$

Denominator becomes Zero

$$= \frac{1}{27}\left[z\frac{1}{2D_1}e^{2z} - \frac{1}{-4}\right] \qquad \frac{1}{D_1} \to \int$$

$$= \frac{1}{27}\left[\frac{z}{2}\frac{e^{2z}}{2} + \frac{1}{4}\right]$$

$$= \frac{1}{27}\left[\frac{ze^{2z}}{4} + \frac{1}{4}\right]$$

$$= \frac{ze^{2z}}{108} + \frac{1}{108}$$

$$P.I = \frac{1}{108}\left[(3x+2)^2 \log(3x+2) + 1\right]$$

(10) Solve $(1+x)^2 y'' + (1+x)y' + y = 4\cos\log(x+1)$

Soln: The given D.Eqn can be written as

$$\left[(1+x)^2 D^2 + (1+x)D + 1\right]y = 4\cos\log(x+1)$$

Put $\log(1+x) = z$ or $e^z = 1+x \Rightarrow x = e^z - 1$

$$(1+x)D = 1D_1$$
$$(1+x)^2 D^2 = 1^2 D_1(D_1-1) \qquad \text{Coefficient of } x$$

$$[D_1^2 - D_1 + D_1 + 1]y = 4\cos z$$

$$[D_1^2 + 1]y = 4\cos z$$

A.Eqn $\quad m^2 + 1 = 0$

$$m = \pm i$$

$$C.F = C_1 \cos z + C_2 \sin z$$

$$C.F = C_1 \cos\log(1+x) + C_2 \sin\log(1+x)$$

$$P.I = \frac{1}{D_1^2 + 1} 4\cos z$$

$D_1^2 \to -1$

Denominator becomes zero

$$= 4z \frac{1}{2D_1} \cos z$$

$$= 2z\sin z$$

$\left(\because \frac{1}{D_1}\cos z = \sin z\right)$

$$= 2\log(1+x)\sin(\log(1+x))$$

$$y = C.F + P.I.$$