Mathematics-XII
Continuity and Differentiability
Worksheet

Differentiation

1. Differentiate each of te following with respect to x :

   a. $\sec^{-1}\left(\frac{\sqrt{x}+1}{\sqrt{x}-1}\right)+\sin^{-1}\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right)$

   b. $\sin^{-1}\left(\frac{\sqrt{1+x}+\sqrt{1-x}}{2}\right); \quad 0<x<1$

   c. $\cos^{-1}\left(\frac{3\sin x+4\cos x}{5}\right)$

   d. $\cos^{-1}\left(\frac{12x+5\sqrt{1-x^2}}{13}\right)$

   e. $\tan^{-1}\left(\frac{a\cos x-b\sin x}{b\cos x+a\sin x}\right); \quad \frac{-\pi}{2}<x<\frac{\pi}{2}, \ \frac{a}{b}\tan x>-1$

   f. $\sin^{-1}\left(\frac{2^{x+1}}{1+4^x}\right)$

   g. $\tan^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right); \quad 0<x<\frac{\pi}{2}$

   h. $\tan^{-1}\left(\frac{5ax}{a^2-6x^2}\right); \quad 6x^2<a^2, a>0$

   i. $\tan^{-1}\left(\frac{2x}{1+15x^2}\right)$

   j. $\tan^{-1}\left(\frac{\sqrt{x}(3-x)}{1-3x}\right)$

   k. $\sin^{-1}\left(x^2\sqrt{1-x^2}-x\sqrt{1-x^4}\right)$

   l. $\sin^{-1}(\cos x)+\cos^{-1}(\sin x)$

   m. $\log_7(\log x)$

2. Prove that :

   a. $\frac{dy}{dx}=-e^{y-x}$, if $e^x+e^y=e^{x+y}$.

   b. $\frac{dy}{dx}=\frac{x-y}{x\log x}$, if $x=e^{\frac{x}{y}}$.

   c. $\frac{dy}{dx}=\frac{(1+\log y)^2}{\log y}$, if $y^x=e^{y-x}$.

   d. $\frac{dy}{dx}=\frac{y}{x}, \frac{d^2y}{dx^2}=0$, if $x^m y^n=(x+y)^{m+n}$.

e. $\frac{dy}{dx}=\frac{\sin^2(a+y)}{\sin a}$, if $x\sin(a+y)+\sin a\cos(a+y)=0$.

f. $\frac{dy}{dx}=\frac{5}{1+25x^2}$, if $y=\tan^{-1}\frac{4x}{1+5x^2}+\tan^{-1}\frac{2+3x}{3-2x}$.

g. $\left(x^2+1\right)y_2+xy_1=0$, if $y=\log\left(x+\sqrt{x^2+1}\right)$.

h. $\frac{dy}{dx}=\frac{ny}{\sqrt{x^2+a^2}}$, if $y=\left(x+\sqrt{x^2+a^2}\right)^n$.

i. $x\frac{dy}{dx}=\frac{y^2}{1-y\log x}$. if $y=x^{x^{x..\infty}}$.

j. $(2y-1)\frac{dy}{dx}=1$, if $y=\sqrt{x+\sqrt{x+\sqrt{x+...}}}$

k. $\frac{dy}{dx}=\frac{y^2\tan x}{y\log\cos x-1}$ if $y=(\cos x)^{(\cos x)^{(\cos x)...\infty}}$.

l. $\frac{dy}{dx}=\sqrt{\frac{1-y^2}{1-x^2}}$, if $\sqrt{1-x^2}+\sqrt{1-y^2}=a(x-y)$.

m. $\frac{dy}{dx}=\frac{x^2}{y^2}\sqrt{\frac{1-y^6}{1-x^6}}$, if $\sqrt{1-x^6}+\sqrt{1-y^6}=a^3\left(x^3-y^3\right)$.

n. $y^2+\left(\frac{dy}{dx}\right)^2=a^2+b^2$, if $y=a\sin x+b\cos x$.

o. $\frac{d^2y}{dx^2}=\frac{\cos x}{(1-\sin x)^2}$, if $y=\tan x+\sec x$.

p. $\frac{dy}{dx}=\frac{-y\log x}{x\log y}$, if $x=e^{\cos 2t}, y=e^{\sin 2t}$.

q. $\left(x^2+4\right)(y_1)^2=n^2\left(y^2+4\right)$, if $x=\sec\theta-\cos\theta, y=\sec^n\theta-\cos^n\theta$.

r. $\left(1-x^2\right)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+p^2y=0$, if $x=\sin t, y=\sin pt$.

s. $y^2\frac{d^2y}{dx^2}-x\frac{dy}{dx}+y=0$, if $x=a\cos\theta+b\sin\theta, y=a\sin\theta-b\cos\theta$.

3. Find $\frac{dy}{dx}$, if $y^x+x^y+x^x=a^b$.

4. If $y=\tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right), x^2\le 1$, find $\frac{dy}{dx}$.

5. Find the derivative of $f\left(e^{\tan x}\right)$ with respect to x at $x=0$, it is given that $f'(1)=5$.

6. If $f(x)=|\cos x-\sin x|$, find $f'\left(\frac{\pi}{6}\right)$.

7. Differentiate $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ with respect to $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$.

8. Differentiate $\tan^{-1}\frac{\sqrt{1-x^2}}{x}$ with respect to $\cos^{-1}\left(2x\sqrt{1-x^2}\right)$, where $x \in \left(\frac{1}{\sqrt{2}}, 1\right)$.

9. Differentiate $\tan^{-1}\left(\frac{x}{1+\sqrt{1-x^2}}\right)$ with respect to $\sin\left(2\cot^{-1}\sqrt{\frac{1+x}{1-x}}\right)$.

10. If $x = a\sin 2t(1+\cos 2t), y = b\cos 2t(1-\cos 2t)$, find $\frac{dy}{dx}$ at $x = \frac{\pi}{4}$.

11. Find $\left.\frac{d^2y}{dx^2}\right|_{\theta=\frac{\pi}{4}}$, $x = a\cos^3\theta, y = a\sin^3\theta$.

Continuity

12. Find the value of *k*, if given functions are continuous at the indicated points:

a. $f(x) = \begin{cases} \frac{x^3+x^2-16x+20}{(x-2)^2} & ; x \neq 2 \\ k & ; x = 2 \end{cases}$ at *x=2,*

b. $f(x) = \begin{cases} \frac{1-\cos kx}{x\sin x} & ; x \neq 0 \\ \frac{1}{2} & ; x = 0 \end{cases}$ at *x=0,*

c. $f(x) = \begin{cases} \frac{2^{x+2}-16}{4^x-16} & ; x \neq 2 \\ k & ; x = 2 \end{cases}$ at *x=2,*

d. $f(x) = \begin{cases} \frac{\sqrt{1+kx}-\sqrt{1-kx}}{x} & ; -1 \leq x < 0 \\ \frac{2x+1}{x-1} & ; 0 \leq x \leq 1 \end{cases}$ at *x=0,*

e. $f(x) = \begin{cases} \frac{\sqrt{3}\sin x - \cos x}{x-\frac{\pi}{6}} & ; x \neq \frac{\pi}{6} \\ k & ; x = \frac{\pi}{6} \end{cases}$ at $x = \frac{\pi}{6}$.

13. Given $f(x) = \frac{1}{x-1}$, find the point of discontinuity of the composite function $y = f(f(x))$.

14. If $f(x) = \frac{\sqrt{2}\cos x - 1}{\cot x - 1}, x \neq \frac{\pi}{4}$, then find the value of $f\left(\frac{\pi}{4}\right)$ so that $f(x)$ becomes continuous at $x = \frac{\pi}{4}$.

15. Let $f(x)=\begin{cases} \dfrac{1-\cos 4x}{x^2} & ; x<0 \\ a & ; x=0 \\ \dfrac{b\sqrt{x}}{\sqrt{16+\sqrt{x}}-4} & ; x>0 \end{cases}$. For what values of a and b, f is continuous at *x=0*?

16. Find the values of *a,b* and *c* for which the given function is continuous at *x=0,*

$$f(x)=\begin{cases} \dfrac{\sin(a+1)x+\sin x}{x} & ; x<0 \\ c & ; x=0 \\ \dfrac{\sqrt{x+bx^2}-\sqrt{x}}{b\sqrt{x^3}} & ; x>0 \end{cases}.$$

17. Find the values of *p* and *q* for which the given function is continuous at $x=\dfrac{\pi}{2}$,

$$f(x)=\begin{cases} \dfrac{1-\sin^3 x}{3\cos^2 x} & ; x<\dfrac{\pi}{2} \\ \text{p} & ; x=\dfrac{\pi}{2} \\ \dfrac{q(1-\sin x)}{(\pi-2x)^2} & ; x>\dfrac{\pi}{2} \end{cases}.$$

18. Show that the function *f* given by, $f(x)=\begin{cases} \dfrac{e^{\frac{1}{x}}+1}{e^{\frac{1}{x}}-1} & ; x\neq 0 \\ 0 & ; x=0 \end{cases}$ is discontinous at *x=0.*

Differentiability

19. Prove that the greatest integer function defined by $f(x)=[x]$ is not differentiable at $x=1$.

20. Show that $f(x)=|x-5|$ is continuous but not differentiable at *x=5.*

21. Show that the function f defined by $f(x)=\begin{cases} x^2-1 , & x\leq 1 \\ 1-x , & x>1 \end{cases}$ is continuous but not differentiable at *x=1.*

22. Let $f(x)=x|x| \;\forall\; x\in R.$ Discuss the derivative of *f(x)* at *x=0.*

23. For what values of a and b is the function $f(x)=\begin{cases} x^2 & , x\leq 1 \\ 2ax+b & , x>1 \end{cases}$ derivable at *x=1?*

24. Find the values of p and q so that $f(x)=\begin{cases} x^2+3x+p & ; x\leq 1 \\ qx+2 & ; x>1 \end{cases}$ is differentiable at *x=1?*

25. If *f* is differentiable at *x=a*, then find

a. $\lim_{x \to a} \frac{xf(a) - af(x)}{x - a}$

b. $\lim_{x \to a} \frac{x^2 f(a) - a^2 f(x)}{x - a}$

26. A function $f : R \to R$ satisfies the equation $f(x+y) = f(x)f(y)$ for all $x, y \in R, f(x) \neq 0$. Suppose that the function is differentiable at $x = 0$ and $f'(0) = 2$. Prove that $f'(x) = 2f(x)$.

Rolle's and Mean Value (Lagrange's) Theorems

27. Verify Rolle's theorem for the following functions:
   a. $f(x) = x^3 - 9x^2 + 26x - 24,\ x \in [2,4]$
   b. $f(x) = \cos x + \sin x,\ x \in [0, 2\pi]$.

28. It is given that for the function $f(x) = x^3 - 6x^2 + px + q,\ x \in [1,3]$, Rolle's theorem holds with $c = 2 + \frac{1}{\sqrt{3}}$, find the values of p and q.

29. If $f : [-5,5] \to R$ is differentiable function and if $f'(x)$ does not vanish anywhere, then prove that $f(-5) \neq f(5)$.

30. Verify Mean Value theorem for the following functions:
   a. $f(x) = (x-3)(x-6)(x-9),\ x \in [3,5]$
   b. $f(x) = \sin x - \sin 2x,\ x \in [0, \pi]$.

31. Using Mean Value theorem prove that there is a point on the curve $y = 2x^2 - 5x + 3$ between the points *A* (1,0) and *B* (2,1), where tangent is parallel to the chord *AB*. Also find that point.

32. Find a point on the curve $y = \sqrt{x-2}$ defined in the interval [2,3] where the tangent is parallel to the joining the end points of the curve.

33. Find c so that $f'(c) = \frac{f(6) - f(4)}{6 - 4}$, where $f(x) = \sqrt{x+2}$ and $c \in (4,6)$.

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