Complex Analysis

Complex Number—A complex number z is an order pair (x, y) of real numbers x and y.

$$z = (x, y) = x + iy$$

$$\text{Re } z = x \text{ (Real part)},$$

$$\text{Im } z = y \text{ (imaginary part)}$$

and $i^2 = -1$, *i.e.* $i = \sqrt{-1}$ (imaginary unit)

(a) $x + iy = a + ib \Leftrightarrow x = a$ and $y = b$.

(b) For $z = x + iy$.

If $x = 0$, then $z = iy$ (pure imaginary)

If $y = 0$, then $z = x$ (pure real)

Addition of complex numbers—

Let $z_1 = x_1 + iy_1$

and $z_2 = x_2 + iy_2$, then

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$

Multiplication of complex numbers—

$$z_1z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$$

Subtraction of complex numbers—

$$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$$

Division of complex numbers—

$$z = \frac{z_1}{z_2} = x + iy$$

where $x = \dfrac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2}$,

$$y = \frac{x_2y_1 + x_1y_2}{x_2^2 + y_2^2}, z_2 \neq 0$$

Practical rule—

$$z = \frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2}$$

$$= \left(\frac{x_1 + iy_1}{x_2 - iy_2}\right) \times \left(\frac{x_2 - iy_2}{x_2 + iy_2}\right)$$

$$= x + iy$$

Complex Conjugate Number—

The complex conjugate of the number $z = x + iy$ is $\bar{z} = x - iy$

(a) $\text{Re } z = x = \dfrac{1}{2}(z + \bar{z})$

and $\text{Im } z = y = \dfrac{1}{2i}(z - \bar{z})$

(b) When z is real, $z = x$, then $z = \bar{z}$

(c) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$,

$\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$

(d) $\overline{z_1 . z_2} = \bar{z}_1 . \bar{z}_2$,

$$\left(\frac{\bar{z}_1}{\bar{z}_2}\right) = \frac{\bar{z}_1}{\bar{z}_2}$$

Polar Form of Complex Numbers

Let (x, y) be the Co-ordinate in castesian co-ordinate system and (r, θ) be the co-ordinates in polar co-ordinates, then

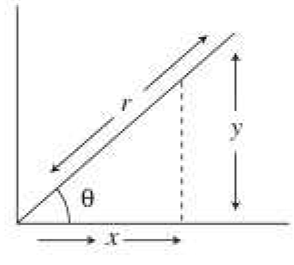
$$x = r\cos\theta, y = r\sin\theta$$

$$\therefore \quad z = x + iy = r(\cos\theta + i\sin\theta)$$

Here $r = |z| = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}$

(absolute value/modulus of z)

$$\theta = arg\ z = \tan^{-1}\left(\frac{y}{z}\right)$$



Triangle inequality—

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

Generalized triangle inequality—

$$|z_1 + z_2 + \ldots\ldots + z_n| \leq |z_1| + |z_2| + \ldots + |z_n|$$

Let $z_1 = r_1 (\cos\theta_1 + i\sin\theta_1)$

and $z_2 = r_2 (\cos\theta_2 + i\sin\theta_2)$

then, $z_1 z_2 = r_1 r_2 [\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$

(a) $|z_1 z_2| = |z_1|\,|z_2|$

(b) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$

(c) $\arg\dfrac{z_1}{z_2} = \arg z_1 - \arg z_2.$

De Moivre's formula—$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.

n^{th} root of $z - \sqrt[n]{z} = \sqrt[n]{r}$

$$\left(\cos\frac{\theta + 2K\pi}{n} + i\sin\frac{\theta + 2K\pi}{n}\right)$$

$$K = 0, 1, 2, \ldots\ldots n - 1$$

Some Elementary Functions

Single valued functions

1. $z^n = (x + iy)^n, n \in N\ (z \neq 0 \text{ if } n < 0)$
2. $e^z = e^x e^{iy} = e^x (\cos y + i\sin y)$, period $= 2\pi i$
3. $\cosh z = \frac{1}{2}(e^z + e^{-z})$,

$\sinh z = \frac{1}{2}(e^z - e^{-z})$

$\tanh z = \dfrac{\sin hz}{\cos hz}, z \neq \left(K + \dfrac{1}{2}\right)\pi i$

$\coth z = \dfrac{\cosh z}{\sinh z}, z \neq K\pi i$

4. $\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$

$\sin z = \dfrac{1}{2i}(e^{iz} - e^{-iz})$

$\tan z = \dfrac{\sin z}{\cos z}, z \neq \left(K + \dfrac{1}{2}\right)\pi i$

$\cot z = \dfrac{\cos z}{\sin z}, z \neq K\pi i$

$\cos iz = \cosh z, \sin iz = i\sinh z$

$\cosh iz = \cos z, \sinh iz = i\sin z$

5. Multiple – Valued Functions—

$\log z = \log|z| + i\arg z$

$= \text{In}\, r + i(\theta + 2n\pi)$

Principal Branch :

$\log z = \log r + i\theta, -\pi < \theta < \pi$

6. $a^z = e^{a\log z}$, a non-integer

If $a = \dfrac{p}{q} \in \theta$ a rational number,

then z^a is q-valued.

If $a \notin \theta$ then z^a is as valued

e.g., $\log 2i = \log|2i| + i\arg 2i$

$= \log 2 + i\left(\dfrac{\pi}{2} + 2n\pi\right)$

$(2i)^i = e^{i\log 2i} = e^{-(\pi/2 + 2n\pi) + i\log 2}$

$= e^{-\pi/2 - 2n\pi}$

$[\cos(\log 2) + i\sin(\log 2)]$

Analytic functions

Differentiability—If G is an open set in $\mathbb{C}$ and $f : G \to \mathbb{C}$.

Then f is differentiable at a point $a \in G$, if

$$f'(a) = \lim_{h\to 0}\frac{f(a+h) - f(a)}{h} \text{ exist}$$

If f is differentiable at each point of G, then f is differentiable on G and $f : G \to \mathbb{C}$.

Continuously differentiable—If $f' : G \to \mathbb{C}$ is continuous analytic function (Holomorphic function)—

A function $f : G \to \mathbb{C}$ is analytic if f is continuously differentiable on G.

Branch of logarithm—If G is an open connected set in $\mathbb{C}$ and $f : G \to \mathbb{C}$ is continuous function such that

$$z = \exp f(z)$$

for all $z \in G$, then f is a branch of logarithm.

Cauchy–Riemann equation—

If $\mu = \mu(x, y)$

and $v = v(x, y)$; then

$$\frac{\delta\mu}{\delta x} = \frac{\delta v}{\delta y}$$

and $$\frac{\delta\mu}{\delta y} = -\frac{\delta v}{\delta x}$$

Harmonic function—If $\mu(x, y)$ and $\dfrac{\delta^2\mu}{\delta x^2} + \dfrac{\delta^2\mu}{\delta y^2} = 0$.

Harmonic Conjugate—If $\mu : G \to \mathbb{R}$ is harmonic and $r : G \to \mathbb{R}$ such that $f = \mu + iv$ is analytic in G, then v is the harmoinc conjugate of μ.

Some Important Theorems

1. If $f : G \to \mathbb{C}$ is differentiable at a point $a \in G$. Then f is continuous at a.
2. If f and g are analytic on G and Ω respectively and suppose that f (G) C Ω, then gof is analytic on G and.
   $(gof)'(z) = g'(f(z)) f'(z)$ for all $z \in G$.
3. If f and g analytic in G. Then fg and $f + g$ are also analytic.
4. If f and g are analytic in G and g does not vanish in G, then $f/_g$ is analytic.
5. If $f(z) = \sum_{n=0}^{\infty} a_n (z - a)^n$ have radius of convergence $R > 0$, then
   (a) For each $K \geq 1$ the series $\sum_{n=0}^{\infty} n(n-1)\ldots(n-K+1)a_n (z-a)^{n-k}$ has radius of convergence R.
   (b) For $n \geq 0$, $a_n = \frac{1}{n!} f(n)(a)$.
6. If the series $\sum_{n=0}^{\infty} a_n (z - a)^n$ has radius of convergence $R = 0$. Then $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$ is analytic in open boll $\beta(a, R)$
7. If G is open and connected and $f : G \to \mathbb{C}$ is differentiable with $f'(z) = 0$ for all $z \in G$, then f is constant.
8. If $G \subset \mathbb{C}$ is open and connected and f is a branch of log z on G, then the totality of branches of log z are the functions $f(z) + 2\pi ki$, $K \in \mathbb{Z}$.
9. If G and Ω are open subsets of $\mathbb{C}$. Suppose f : (F) $\to \mathbb{C}$ and $g : \Omega \to \mathbb{C}$ are continuous functions such that $f(G) \subset \Omega$ an $g(f(z)) = z$ for all $z \in G$. If g is differentiable and $g'(z) \neq 0$, f is differentiable and $f'(z) = \frac{1}{g(f(z))}$, then if g is analytic f is analytic too.
10. A branch of the logarithm function is analytic and its derivative is z^{-1}.
11. If μ and v are real valued function defined on a region G and suppose that μ and v have continuous partial derivatives. Then $f : G \to \mathbb{C}$ defined by $f(z) = \mu(z) + iv(z)$ is analytic iff μ and v satisfies Cauchy-Riemann equations.
12. Suppose G is either the whole plane $\mathbb{C}$ or some open disk. If $\mu : G \to \mathbb{R}$ is harmonic function. Then μ has a harmonic conjugate.

The Conformal Mappings Theorem

1. Let G be a region and f be an analytic function on G with zeros, $a_1, a_2, \ldots\ldots a_n$. If y is a closed rectifiable curve in G which does not pass through any point a_k and If $y = 0$, then
   $$\frac{1}{2\pi i}\int \frac{f'(z)}{f(z)} dz = \sum_{k=1}^{m} n(y, a_k)$$
2. If G is a region, f is analytic function on G with $a_1, \ldots, a_k \in G$ and y is a closed rectifiable curve in G which does not pass through any point a_k and if $y = 0$ an $f(z) = \alpha$, then
   $$\frac{1}{2\pi i}\int \frac{f'(z)}{f(z) - \alpha} dz = \sum_{k=1}^{m} n(y, a_k)$$
3. If f is analytic in $B(a, R)$ and let $\alpha = f(a)$. If $f(z) - \alpha$ has a zero of order m at $z = a$, then there is an $\in > 0$ and $\delta > 0$ such that $|\delta - \alpha| < \delta$, then equation $f(z) = y$ has exactly m simple roots in $B(a, \in)$.
4. Conformal mapping theorem—Let G be a region and suppose f is a non constant analytic function on G. Then for any conformal set U in G, f(U) is conformal.
5. If $f : G \to \mathbb{C}$ is one analytic and $f(G) = \Omega$, then $f^{-1} : \Omega \to \mathbb{C}$ is analytic and $f^{-1}(w) = [f(z)]^{-1}$ where $w = f(z)$
6. Goursat is theorem—Let G be an conformal set and $f : G \to \mathbb{C}$ be differentiable function; then f is analytic on G.

The Cauchy's Integral Theorem and Formula

1. If y is a rectifiable curve and suppose ϕ is a function defined and continuous on $\{y\}$. For each $m \geq 1$, Let $F_m(z) = \int_y \phi(w)(w - z)^{-m} dw$ for $z \notin y$. Then F_m is analytic on $\mathbb{C} - \{y\}$ and $F_m(z) = mF_{m+1}(z)$ for each m.
2. Cauchy's integral formula (first version)—
   Let G be an open subset of the plane and $f : G \to \mathbb{C}$ an analytic function. If y is a closed rectifiable curve in G. Such that $n(y, w) = 0$ for all $w \in \mathbb{C} \ldots G$, then for $a \in G - \{y\}$.
   $$n(y : a) = \frac{1}{2\pi i}\int_y \frac{f(g)}{z - a} dz.$$

3. Cauchy's integral formula (second version)— Let G be an open subset of the plane and f : G $\to \mathbb{C}$ an analytic function. If $y_1, \ldots\ldots, y_m$ are closed rectifiable curves in G such that $n(y_i, w) + \ldots + n(y_m, w) = 0$ for all $w \in \mathbb{C} \to G$ then for $a \in G - \{y\}$

$$f(a) = \sum_{k=1}^{m} n(y_k;a)$$
$$= \sum_{k=1}^{m} \frac{1}{2\pi i} \int_{y_k} \frac{f(z)}{z-a}\, dz$$

4. Cauchy's theorem (First version)—

If G is an open subset of the plane and f : G $\to \mathbb{C}$ an analytic function. If $y_1, \ldots, y_m$ are closed rectifiable curves in G such that $n(y_1; w) + n(y_2; w) + \ldots\ldots + n(y_m; w) = 0$ for all $w \in \mathbb{C} \to G$, then

$$\sum_{k=1}^{m} \int_{y_k} f = 0$$

5. Let G be an open subset of the plane and f : G $\to \mathbb{C}$ an analytic function. If $y_1 \ldots\ldots y_n$ are closed rectifiable curves in G such that $n(y_1; w) + n(y_2; w) + \ldots\ldots + n(y_m; w) = 0$ for all $w \in \mathbb{C} - G$, then $a \in G - \{y\}$ and $K \geq 1$.

6. Let G be an open set and $f : G \to \mathbb{C}$ an analytic function. If y is closed rectifiable curve in G such that $n(y; w) = 0$ for all $w \in \mathbb{C}$ — G, then for $a \in G - \{y\}$

$$f^{(K)}(a)\, n(y; a) = \frac{\lfloor K}{2\pi i} \int_{y} \frac{f(z)}{(z-a)^{k+1}}\, dz$$

7. Morera's Theorem—

Let G be an region and $f : G \to \mathbb{C}$ be an analytic function such that $\int_T f = 0$ for every T, a Triangular path in G; then f is analytic in G.

Some Important Theorems

1. If G is an open set which is a star shaped, if y_0 is the curve which is constantly equal to a then every closed rectifiable curve in G is homotopic to y_0

2. Cauchy's theorem (Second version)—If f : G $\to \mathbb{C}$ is and analytic function and y is a closed rectifiable curve in G such that y — 0,

then $\int_y f = 0$

3. Cauchy's theorem (Third version)—If y_0 and y_1 are two closed rectifiable curves in G and $y_0 \sim y_1$, then $\int_{y_0} f = \int_{y_1} f$ for every function f analytic on G.

4. If y is a closed rectifiable curve in G such that $y \sim 0$ then $n(y : w) = 0$ for all $w \in \mathbb{C} \to G$

5. Cauchy's theorem (Fourth version)—If G is simply connected then $\int_y f = 0$ for every closed rectifiable curve and every analytic function f.

6. If G is simply connected and $f : G \to \mathbb{C}$ is anlytic in G then f has a primitive in G.

7. If G is simply connected and $f : G \to \mathbb{C}$ an analytic function in G such that $f(z) \neq 0$ for any $z \in G$, then there is an analytic function $g : G \to \mathbb{C}$ such that $f(z) = \exp g(z)$. If $z_0 \in G$ and $e^{w_0} = f(z_0)$, we have $g(z_0) = w_0$.

The Liouville's theorem—If f is a bounded entire function then f is constant.

Given f a bounded function

$\therefore \quad |f(z)| \leq M$ for all $z \in \mathbb{C}$.

By Cauchy's estimate theorem since f is bounded and analytic we have,

$$|f^{(n)}(z)| \leq \frac{LnM}{R}$$

$\Rightarrow \quad |f'(z)| \leq \frac{M}{R}$, since R is arbitrary,

we have

$$|f'(z)| = 0$$

$\therefore f(z)$ is constant.

The Maximum Modulus Principle

1. Maximum modulus principle (First version)—If f is analytic in a region G and $a \in G$ with $|f(a)| \geq |f(z)|$ for all $z \in G$, then f must be a constant function.

2. Maximum modulus principle (Second version)—If G is a bounded open set in $\mathbb{C}$ and f is continuous function on $\bar{G}$ which is analytic in G then $\max\{|f(z)| : z \in \bar{G}\} = \max\{|f(z)| : z \in \delta G\}$

3. Maximum modulus principle (Third version)—If G is a region in $\mathbb{C}$ and f an analytic function on G suppose M is a constant such that $\lim_{z \to a} |f(z)| \leq M$ for all a

$\in \delta_\infty$ G (δ_∞ G is the boundary of G in $\mathbb{C}_\infty$) then $|f(z)| \leq$ M for all $z \in$ G.

4. Schwarz's lemma—If D = $\{z : |z| < 1\}$ and suppose f is analytic on D with
   (a) $|f(z)| \leq i$ for $z \in$ D
   (b) $f(0) = 0$
   Then $|f'(0)| \leq 1$ and $|f(z)| \leq |z|$ for all z in the disk D. Moreover if : $f'(0) = 1$ or $|f(z)| = |z|$ for some $z \neq 0$, then there is a constant C, $|c| = 1$, such that
   $$f(w) = cw \text{ for } w \in \text{D}.$$
5. If $|a| < 1$ then ϕ_a is a one-one map of D = $\{z : |z| < 1\}$ onto itself, the inverse of ϕ_a is ϕ_{-ar}. Further more, ϕ_a maps δD onto δD, $\phi_a(a) = 0$, $\phi'_a(0) = 1 - |a|^2$ and $\phi'_a(a) = (1 - |a|^2)^{-1}$
6. If f : D $\to$ D is one one analytic map of D onto itself and suppose $f(a) = 0$, then there is a complex number C with $|c| = 1$ such that $f = c\,\phi_a$.
7. A function $f : [a, b] \to$ R is convex iff the set A = $\{(x, y) : a \leq x \leq b \text{ and } f(x) \leq y\}$ is convex.
8. A function $f : [a, b] \to$ R is convex iff for any points $x_1 \ldots\ldots x_n$ in $[a, b]$ and real numbers t_1, ... $t_n \geq 0$ with $\sum_{k=1}^{n} t_k = 1$
   $$f\left(\sum_{k=1}^{n} t_k x_k\right) \leq \sum_{k=1}^{n} t_k f(x_k)$$
9. A set A $\subset \mathbb{C}$ is convex iff for any points $z_1, \ldots\ldots z_n \in$ A and real numbers $t_1 \ldots\ldots t_n \geq 0$ with
   $$\sum_{k=1}^{n} t_k = 1 \text{ and } \sum_{k=1}^{n} t_k z_k \in \text{A}$$
10. A differentiable function f on $[a, b]$ is convex iff f is increasing.
11. If $a < b$ and G is the vertical slrip $\{x + iy : a < x < b\}$. Suppose $f : \overline{\text{G}} \to \mathbb{C}$ is continuous and f is analytic in G. If M : $[a, b] \to \mathbb{R}$ then M(x) = sup $\{|f(x + iy)| : -\infty < y < \infty\}$ and $|f(z)| <$ B for all $z \in$ G, then log M(x) is a convex function.
12. **Hadamard's three circle theorem**—
   If $0 < \text{R}_1 < \text{R}_2 < \infty$ and suppose f is analytic on ann (0; R$_1$, R$_2$). If R$_1 < r <$ R$_2$, define M(r) = max $\{|f(re^{i\theta}| : 0 \leq 0 \leq 2\pi\}$. Then for $\text{R}_1 < r_1 \leq r \leq r_2 < \text{R}_2$,
   $$\log \text{M}(r) \leq \frac{\log r_2 - \log r}{\log r_2 - \log r_1} \log \text{M}(r_1) + \frac{\log r - \log r_1}{\log r_2 - \log r_1} \log \text{M}(r_2)$$
13. **Phragmen–Lindel of theorem**—
   If G is simply connected region and f is an analytic function on G. Suppose there is an analytic function ϕ : G $\to \mathbb{C}$ which never vanishes and is bounded on G. If M is a constant and δ_∞G = A $\cup$ B such that
   (a) for every $a \in$ A, $\lim_{z \to a}$ sup $|f(z)| \leq$ M.
   (b) for every $b \in$ B and $n > 0$; $\lim_{z \to a}$ sup $|f(z)|$ $|\phi(z)|^n \leq$ M, then $|f(z)| \leq$ M for all $z \in$ G.
14. If $a \geq \frac{1}{2}$ and G = $\{z : |arg\, z| < \frac{\pi}{2a}\}$. Suppose f is analytic on G and there is a constant M such that $\lim_{z \to w}$ sup $|f(z)| \leq$ M for all $w \in \delta$G.
   If there are positive constant P and $b < a$ such that $|f(z)| \leq$ P exp ($|z|$ b) for all z with $|z|$ sufficiently large then $|f(z)| \leq$ M for all $z \in$ G.

Taylor and Laurent Series—

Taylor Series—A taylor series of a function $f(z)$ is

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

where $$a_n = \frac{1}{\underline{|n}} f(n)(z_0)$$

or $$a_n = \frac{1}{2\pi i} \phi_c \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi$$

with positive orientation around a simple closed contour c that contains z_0 in its interior and $f(z)$ is analytic on and everywhere inside c.

Taylor's series with reminder R$_n$—

$$f(z) = \sum_{k=0}^{n} \frac{f(k)(z_0)}{\underline{|K}} (z - z_0)^K + \text{R}_n(z)$$

where $$\text{R}_n(z) = \frac{(z - z_0)^{n+1}}{2\pi i} \oint_c \frac{f(\xi)}{(\xi - z_0)^{n+1}} (\xi - z)\, d\xi$$

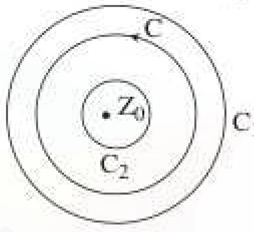
Maclaurin series—A Taylor series with centre $z_0 = 0$.

Laurent Series—If $f(z)$ is analytical on two co-centric circles c_1 and c_2 with centre z_0 and in the annulus between them then $f(z)$ can be represented by the laurent series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=0}^{\infty} \frac{b_n}{(z - z_0)^n}$$
$$= \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$$

Consisting of non-negative powers and negative power (principle part). The coefficients of this series are the integrals.

$$c_n = a_n = \frac{1}{2\pi i} \oint_c \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi$$



and $\quad c_{-n} = b_n = \frac{1}{2\pi i} \oint_c (\xi - z_0)^{n-1} f(\xi) d\xi$

with positive orientation around any simple closed contoure, that lies in the annulus and encircles the inner circle.

Th.—The Laurent series of a given analytic function in its annulus of convergence is unique. However $f(z)$ may have different laurent series in two annuli with the same centre.

Residue theorem and applications for evaluating real integrals—

Residue—If $f(z)$ is analytic and have non removable singularity at $z = z_0$

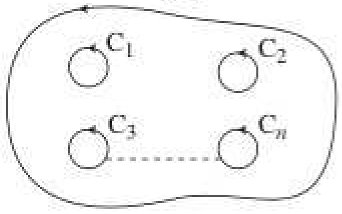
Then $f(z)$ has the laurent series representation

$$f(z) = \sum_{-\infty}^{\infty} c_n (z - z_0)^n$$

The coefficient c_{-1} of $\frac{1}{(z - z_0)}$ is called the residue of f at z_0, *i.e.* $\underset{z=z_0}{\text{Res}} f(z) = c_{-1}$

Cauchy's Residue theorem—Let D be a simple connected domain, and let c be a simple closed positively oriented contour that lies in D. If f is analytic inside c and on c, except at the points $z_1, z_2, \ldots z_n$ that lies inside c, then

$$\oint_c f(z) dz = 2\pi i \sum_{k=1}^{n} \text{Res} f(z)\, z = z_0$$



Residues at Singularities

1. If $f(z)$ has a removable singularity at z_0, then $a_n = 0$ for $n = 1, 2, \ldots$ and $\underset{z=z_0}{\text{Res}} f(z) = 0$
2. If $f(z)$ has a simple pole at z_0,
$$\underset{z=z_0}{\text{Res}} f(z) = \lim_{z \to z_0} f(z)(z - z_0)$$
3. If $f(z)$ has a pole of order 2 at z_0,
$$\underset{z=z_0}{\text{Res}} f(z) = \lim_{z \to z_0} \frac{d}{dz} (z - z_0)^2 f(z)$$
4. If $f(z)$ has a pole of order K at z_0,
$$\underset{z=z_0}{\text{Res}} f(z) = \frac{1}{\lfloor K - 1} \lim_{z \to z_0} \frac{d^{K-1}}{dz^{K-1}} (z - z_0)^K f(z)$$
5. If $f(z)$ and $g(z)$ have an isolated singularity at z_0, then
$$\underset{z=z_0}{\text{Res}} [(f + g)(z)] = \underset{z=z_0}{\text{Res}} f(z) + \underset{z=z_0}{\text{Res}} g(z)$$
6. If f and g are analytic at z_0, $f(z_0) \neq 0$, $g(z_0)$ has simple zero, then
$$\underset{z=z_0}{\text{Res}} [(f/g)(z)] = \frac{f(z_0)}{g'(z_0)}.$$

Some Solved Examples

Example 1. Let $\mu - v = (x - y)(x^2 + 4xy + y^2)$ and $f(z) = \mu + iv$ is an analytic function. Find $f(z)$ in terms of z.

Solution : Given

$$f(z) = \mu + iv,$$
then $\quad if(z) = i\mu - v$
$$\Rightarrow \quad (1 + i) f(z) = \mu - v + i(\mu + v)$$
$$= U + iv \text{ (Say)}$$
since $\quad U = \mu - v$
$$= (x - y)(x^2 + 4xy + y^2)$$
$$\Rightarrow \quad \frac{\delta U}{\delta x} = \frac{\delta \mu}{\delta x} - \frac{\delta v}{\delta x}$$
$$= x^2 + 4xy + y^2 + (x - y)(2x + 4y)$$
$$= 3x^2 + 6xy - 3y^2 = \phi_1(x, y)$$
and $\quad \frac{\delta U}{\delta y} = \frac{\delta \mu}{\delta y} - \frac{\delta v}{\delta y}$
$$= -(x^2 + 4xy + y^2) + (x - y)(4x + 2y)$$
$$= 3x^2 - 6xy - 3y^2 = \phi_2(x, y)$$

Since, $(1 + i) f(z)$

$$= \int [\phi_1(z, 0) - i\,\phi_2(z, 0)]dz + c$$
$$= \int (3z^2 - i\,3z^2)dz + c$$
$$= \int 3(1 - i)z^2 dz + c$$
$$= (1 - i)z^3 + c$$
$$\therefore \quad f(z) = \frac{1-i}{1+i}z^3 + \frac{c}{1+i}$$
$$= -i z^3 + d.$$

Example 2. Prove that if $f : G \to \mathbb{C}$ is analytic and $a \in G$, then

$$f(z) = \sum_{n=0}^{\infty} a_n (z - a)^n,$$

for $|z - a| < R$ where $R = d(a, \delta G)$

Solution : Since $R = d(a, \delta G)$ we have an open ball centered at $a \in G$. such that $B(a, R) \subset G$.

If f is analytic in open ball $B(a, R)$ then it can be expressed as

$$f(z) = \sum_{n=0}^{\infty} a_n (z - a)^n,$$

for $|z - a| < R$ where

$$a_n = \frac{1}{\underline{|n}} f(n)(a).$$

Example 3. Evaluate the following integral $\int_c \tan z\, dz$ where c is the circle $|z| = 2$

Solution : The poles of $f(z) = \dfrac{\sin z}{\cos z}$ are $\cos z = 0$, *i.e.* $z = (2n + 1)\frac{\pi}{2}$, $n = 0, \pm 1, \pm 2, \ldots$, $z = \frac{\pi}{2}$ and $-\frac{\pi}{2}$ only poles that are within the given circle.

$$\therefore \text{Res} f(\pi/2) = \lim_{z \to \frac{\pi}{2}} \frac{\sin z}{\frac{d}{dz}(\cos z)}$$
$$= \lim_{z \to \frac{\pi}{2}} \left(\frac{\sin z}{-\sin z}\right) = -1$$

Similarly

$$\text{Res} f\left(-\frac{\pi}{2}\right) = \lim_{z \to -\frac{\pi}{2}} \frac{\sin z}{\frac{d}{dz}(\cos z)} = -1$$

Hence, by residue theorem.

$$\int_c f(z)\, dz = 2\pi i \left\{\text{Res} f\left(\frac{\pi}{2}\right) + \left\{\text{Res.} f\left(-\frac{\pi}{2}\right)\right\}\right.$$
$$= 2\pi i(-1 - 1) = -4\pi i.$$

Example 4. Prove that $\int \frac{ds}{y}$ is an invariant with respect to the transformation $z = \dfrac{az + b}{cz + d}$ where a, b, c, d satisfies $ad - bc = 1$ and $ds = \sqrt{(dx^2 + dy^2)}$

Solution : From

$$z = \frac{az + b}{cz + d} \text{ we have}$$
$$z = \frac{b - dz}{cz - a} \qquad \ldots(1)$$

Differentiating (1), we get

$$dz = \frac{ad - bc}{(cz - a)^2} dz$$
$$\Rightarrow \quad dz = \frac{dz}{(cz - a)^2} \qquad [\because ad - bc = 1]$$

And so,
$$ds = \sqrt{(dx^2 + dy^2)}$$
$$= |dz| = \frac{|dz|}{|cz - a|^2}$$
$$= \frac{d\sigma}{|cz - a|^2} \qquad \ldots(2)$$

Also,
$$2iy = z - \bar{z}$$
$$= \frac{b - dz}{cz - a} - \frac{b - d\bar{z}}{c\bar{z} - a}$$
$$= \frac{(ad - bc)(z - \bar{z})}{|cz - a|^2}$$

or
$$2iy = \frac{2iy}{|cz - a|^2}$$
$$[\because ad - bc = 1, z = x + iy]$$

or
$$\frac{v}{y} = \frac{1}{|cz - a|^2}$$

$\therefore$ (2) given,

$$d\sigma = \frac{v}{y} d\sigma$$

or
$$\int \frac{ds}{y} = \int \frac{d\sigma}{y}$$

Thus $\int \frac{ds}{y}$ is invariant under the given transformation.

Example 5. Evaluate $\int_c \frac{dz}{z-a}$ where c represents the circle $|z-a| = r$.

Solution : Here for the circle $|z - a| = r$

$$z = a + re^{\theta}$$

where $0 \leq \theta \leq 2\pi$

$$\therefore \quad dz = ire^{i\theta}\, d\theta$$

Hence $$\int_y \frac{dz}{z-a} = \int_{\theta}^{2\pi} \frac{ire^{i\theta}}{re^{i\theta}}\, d\theta = 2\pi i$$

Example 6. Evaluate the residue of $\frac{z^2}{z^2 + a^2}$ at $z = ia$.

Solution :

Here $$f(z) = \frac{z^2}{z^2 + a^2} = \frac{z^2}{(z+ia)(z-ia)}$$

$z = ia$ is a simple pole of $f(z)$

Residue at $z = ia$ is $\lim_{z \to ia} (z - ia) f(z)$

$$= \lim_{z \to ia} (z - ia) \frac{z^2}{(z-ia)(z+ia)}$$

$$= \lim_{z \to ia} \frac{z^2}{z + ia} = \frac{-a^2}{2ia} = \frac{1}{2} ia.$$

Example 7. Evaluate the analytic function

$$f(z) = \mu + iv, \text{ if}$$

$$\mu - v = \frac{\cos x + \sin x - e^{-y}}{2(\cos x - \cosh y)}$$

and $$f\left(\frac{\pi}{2}\right) = 0$$

Solution :

Here $$\mu - v = \frac{\cos x + \sin x - e^{-y}}{2(\cos x - \cos hy)}$$

$$\therefore \frac{\delta\mu}{\delta x} - \frac{\delta v}{\delta x}$$

$$= \frac{(\sin x - \cos x)\cosh y + 1 - e^{-y} \sin x}{2(\cos x - \sin hy)^2} \quad \ldots(1)$$

and $$\frac{\delta\mu}{\delta y} - \frac{\delta v}{\delta y} = \frac{(\cos x - \cosh y)e^{-y} + (\cos x + \sin x - e^{-y})\sinh y}{2(\cos x - \cosh y)^2}$$

Since, $f(z) = \mu + iv$ is analytic

$$\therefore \quad \frac{\delta\mu}{\delta x} = +\frac{\delta v}{\delta y}$$

and $$\frac{\delta v}{\delta x} = -\frac{\delta\mu}{\delta y} \text{ gives}$$

$$-\frac{\delta v}{\delta x} - \frac{\delta\mu}{\delta x}$$

$$= +\frac{e^{-y}(\cos x - \cosh y - \sinh y)}{2(\cos x - \cosh y)^2} \quad \ldots(2)$$

Subtructing (2) from (1), we get

$$2\frac{\delta\mu}{\delta x} = \frac{(\sin x - \cos x)(\sin x + \cos x)\sinh y + 1 - e^{-y}(\sin x + \cos x - \cosh y - \sinh y)}{2(\cos x - \cosh y)^2}$$

Adding (1) and (2), we have

$$\frac{(\sin x - \cos x)\cosh y - (\sin x + \cos x)\sinh y + 1 + e^{-y}(-\sin x + \cos x - \cosh y - \sinh y)}{2(\cos x - \cosh y)^2}$$

$$\Rightarrow \quad f'(z) = \frac{\delta\mu}{\delta x} + i\frac{\delta v}{\delta x} = \frac{1 - \cos z}{2(1 - \cos z)^2}$$

[Putting $x = z$ and $y = 0$]

$$= \frac{1}{2(1 - \cos z)} = \frac{1}{4\sin^2 \frac{z}{2}}$$

$$= \frac{1}{4} \operatorname{cosec}^2 \frac{z}{2}$$

Integrating,

$$f(z) = -\frac{1}{2} \cot \frac{z}{2} + c$$

Since $$f\left(\frac{\pi}{2}\right) = 0,$$

$$0 = -\frac{1}{2} \cot \frac{\pi}{4} + c,$$

where $c = \frac{1}{2}$

$$\therefore \quad f(z) = \frac{1}{2}\left(1 - \cot \frac{z}{2}\right)$$

Example 8. Evaluate $\int_c \frac{z^2 - z + 1}{z - 1} dx$ where C is the circle (i) $|z| = 1$, (ii) $|z| = \frac{1}{2}$.

Solution :

(i) $\because \int_C \frac{f(z)dz}{z-a} = \int_C \frac{z^2 - z + 1}{z - 1} dz$ and

$$f(z) = z^2 - z + 1 \text{ and } a = 1$$

Since $f(z)$ is analytic within and on circle C : $|z| = 1$ and $a = 1$ lies on C.

$\therefore$ By Cauchy's integral formula.

$$\frac{1}{2\pi i}\int_C \frac{f(z)}{z-a} = f(a) = 1,$$

i.e. $\int_C \frac{z^2 - z + 1}{z-1} dz = 2\pi i$

(ii) Here $a = 1$ lies out side the circle C : $|z| = y_2$. So $\frac{z^2 - z + 1}{z - 1}$ is analytic everywhere within C.

$\therefore$ By Cauchy's theorem $\int_C \frac{z^2 - z + 1}{z-1} dz = 0$

Example 9. Prove that the transformation $w = \frac{iz+2}{4z+i}$ maps the real axis in the z-plane into a circle in the w-plane.

Solution : The given transformation

$$w = \frac{iz+2}{4z+i}$$

$$\Rightarrow \quad z = \frac{2-iw}{4w-i}$$

The equation of the real axis in the plane is $z - \bar{z} = 0$ substituting for z and $\Sigma$the transformation equation is

$$\frac{2-iw}{4w-i} - \frac{2-i\bar{w}}{4\bar{w}+i} = 0$$

or $8\bar{w} + 2i - 4iw\bar{w} + w - 8w + 2i - 4iw\bar{w} - \bar{w} = 0$

or $8iw\bar{w} + 7(w - \bar{w}) - 4i = 0$

or $8i(\mu^2 + v^2) + 14iv - 4i = 0$

or $\mu^2 + v^2 + \frac{7}{4}v - \frac{1}{2} = 0$

which is the equation of a circle in the w-plane.

Example 10. If $0 < |z-1| < 2$, expand

$$f(z) = \frac{z}{(z-1)(z-3)}$$

Solution :

Here $f(z) = \frac{z}{(z-1)(z-3)}$

$$= -\frac{1}{2(z-1)} + \frac{3}{2}\frac{1}{(z-3)}$$

Putting $z - 1 = \mu$, we have $0 < |\mu| < 2$ and

$$f(z) = -\frac{1}{2\mu} + \frac{3}{2(\mu-2)}$$

$$= -\frac{1}{2\mu} - \frac{3}{4}\left(1 - \frac{\mu}{2}\right)^{1}$$

$$= -\frac{1}{2\mu} - \frac{3}{4}\left(1 + \frac{\mu}{2} + \frac{\mu^2}{2^2} + \frac{\mu^3}{2^3} + \dots\right)$$

$$= -\frac{1}{2\mu} - \frac{3}{4}\sum_{n=0}^{\infty}\left(\frac{\mu}{2}\right)^n$$

$$= -\frac{1}{2(z-1)} \frac{-3}{4}\sum_{n=0}^{\infty}\left(\frac{z-1}{2}\right)^n$$

Example 11. Evaluate $\int_y \frac{z+2}{z} dz$, where y is the semicircle $z = 2e^{it}, 0 \le t \le \pi$

Solution :

Here $z = 2e^{it}, 0 \le t \le \pi$

$\Rightarrow \quad dz = 2ie^{it}dt$

Hence $\int_y \frac{z+2}{z}dz = \int_0^{\pi} \frac{(2e^{it}+2)}{2\, e^{it}} 2ie^{it}\, dt$

$$= 2i\int_0^{\pi}(1 + e^{-it})\, e^{it}\, dt$$

$$= 2i\int_0^{\pi}(e^{it} + 1)dt$$

$$= 2i\left[\frac{e^{it}}{i} + t\right]_0^{\pi}$$

$$= 2i\left(\frac{e^{i\pi}}{i} - \frac{1}{i} + \pi\right)$$

$$= 2\, e^{i\pi} - 2 + 2i\pi$$

$$= -2 + 4i\pi$$

Example 12. Prove that for

$0 < |z| < 4, \frac{1}{4z - z^2} = \sum_{\mu=0}^{\infty} \frac{z^{n-1}}{4^{n+1}}$

Solution : We have $0 < |z| < 4 \Rightarrow \frac{|z|}{4} < 1$

$$\therefore \frac{1}{4z - z^2} = \frac{1}{4z\left(1 - \frac{z}{4}\right)}$$

$$= \frac{1}{4z}\left(1 - \frac{z}{4}\right)^{-1} = \frac{1}{4z}\left[1 + \frac{z}{4} + \left(\frac{z}{4}\right)^2 + \left(\frac{z}{4}\right)^3 + \dots\right]$$

$$= \frac{z^{-1}}{4} + \frac{1}{4^2} + \frac{z}{4^3} + \frac{z^2}{4^4} + \ldots$$

$$= \sum_{n=0}^{\infty} \frac{z^{n-1}}{4^{n+1}}.$$

Example 13. Evaluate the integral

$\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2 \ (z-2)} dz$, where C is the circle $|z| = 3$.

Soltuion :

$$\text{Here} \quad f(z) = \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2 (z-2)}$$

is analytic within the circle $|z| = 3$ except at the poles $z = 1$ and $z = 2$.

Since $z = 1$ is a pole of order 2.

$$\therefore \text{Res.} f(1) = \frac{1}{1!}\left[\frac{d}{dz}(z-1)^2 f(z)\right]_{z=1}$$

$$= \left[\frac{d}{dz}\left(\frac{\sin \pi z^2 + \cos \pi z^2}{z-2}\right)\right]_{z=1}$$

$$= \left[\frac{(z-2)(2\pi z \cos \pi z^2 - 2\pi z \sin \pi z^2) - (\sin \pi z^2 + \cos \pi z^2)}{(z-2)^2}\right]_{z=1}$$

$$= (-1)(-2\pi) - (-1) = 2\pi + 1$$

$$\text{and} \quad \text{Res} f(2) = \lim_{z \to 2} [(z-2) f(z)]$$

$$= \lim_{z \to 2} \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2} = 1$$

By residue theorem

$$\int_C f(z) dz = 2\pi i [\text{Res} f(1) + \text{Res} f(2)]$$

$$= 2\pi i (2\pi + 1 + 1)$$

$$= 4\pi (\pi + 1)^i.$$

Example 14. If $\mu = \frac{\sin 2x}{\cos 2y + \cos 2x}$, find the corresponding analytic function

$$f(z) = \mu + iv$$

Solution :

$$\text{Here} \quad \mu = \frac{\sin 2x}{\cos 2y + \cos 2x}$$

$$\therefore \quad \frac{\delta \mu}{\delta x} = \frac{2 \cos 2x (\cosh 2y + \cos 2x) - \sin 2x (-2 \sin 2x)}{(\cos 2y + \cos 2x)^2}$$

$$= \frac{2 + 2 \cos 2x \cos 2y}{(\cosh 2y + \cos 2x)^2} = g_1(x,y)$$

$$\text{and} \quad \frac{\delta \mu}{\delta y} = \frac{-2 \sin 2x \sinh 2y}{(\cosh 2y + \cos 2x)^2} = g_2(x,y)$$

The function $f(z)$ is given by

$$f(z) = \int [g_1(z, 0) - i g_2(z, 0)] dz + c$$

$$= \int \left[\frac{2 + 2 \cos 2z}{(1 + \cos 2z)^2} - i0\right] dz + c$$

$$= \int \frac{2dz}{1 + \cos 2z} + c$$

$$= \int \sec^2 z \, dz + c$$

$$= \tan z + c$$

Example 15. Find the bilinear transformation which maps the points $z_1 = 2$, $z_2 = i$ and $z_3 = -2$, into the points $w_1 = 1$, $w_2 = i$ and $w_3 = -1$ respectively.

Solution : The bilinear transformation which transforms z_1, z_2, z_3 respectively into w_1, w_2, w_3 is given by

$$\frac{(w - w_1)(w_2 - w_3)}{(w_1 - w_2)(w_3 - w)} = \frac{(z - z_1)(z_2 - z_3)}{(z_1 - z_2)(z_3 - z)}$$

Substituting $z_1 = 2$, $z_2 = i$, $z_3 = -2$, $w_1 = 1$, $w_2 = i$, $w_3 = -1$ we have

$$\frac{(w-1)(i+1)}{(1-i)(-1-w)} = \frac{(z-2)(i+2)}{(2-i)(-2-z)}$$

$$\text{or} - \frac{(w-1)(1+i)^2}{(w+1)(1-i)(1+i)} = - \frac{(z-2)(2+i)^2}{(z+2)(2-i)(2+i)}$$

$$\text{or} \quad \frac{(w-1).2i}{(w+1).2} = \frac{(z-2)(3+4i)}{(z+2).5}$$

$$\text{or} \quad \frac{w-1}{w+1} = \frac{(z-2)(4-3i)}{5(z+2)}$$

$$\text{or} \quad \frac{2w}{2} = \frac{(z-2)(4-3i) + 5(z+2)}{5(z+2) - (z-2)(4-3i)}$$

$$\text{or} \quad w = \frac{3(3-i)z + 2(1+3i)}{(1+3i)z + 6(3-i)}$$

$$= \frac{3z + \left[\frac{2(1+3i)}{(3-i)}\right]}{\left[\frac{(1+3i)}{(3-i)}\right] z + 6}$$

$$\text{or} \quad w = \frac{3z + 2i}{iz + 6}.$$

The required transformation.

Example 16. Let $f(z) = \mu + iv$ is an analytic function of $z = x + iv$ and $\mu - v = \frac{e^y - \cos x + \sin x}{\cosh y - \cos x}$ find $f(z)$, given

$$f\left(\frac{\pi}{2}\right) = \frac{3+i}{2}$$

Solution :

Given $f(z) = \mu + iv$,

then if $(z) = i\mu + v$

Adding $f(z) + if(z)$, we have

$$f(z) + if(z) = \mu - v + i(\mu + v)$$
$$= \mu - iv \text{ (say)}$$

Since $\quad U = \mu - v$

$$= \frac{e^y - \cos x + \sin x}{\cosh y - \cos x}$$
$$= \frac{\cosh y + \sinh y - \cos x + \sin x}{\cosh y - \cos x}$$
$$= 1 + \frac{\sinh y + \sin x}{\cosh y - \cos x}$$
$$\frac{\delta U}{\delta x} = -\frac{(\sinh y + \sin x)\sin x}{(\cosh y - \cos x)^2}$$
$$= g_1(x, y)$$
$$\Rightarrow g_1(z, 0) = \frac{\cos z(1 - \cos z) - \sin^2 z}{(1 - \cos z)^2}$$
$$= \frac{\cos z - 1}{(1 - \cos z)^2} = \frac{-1}{1 - \cos z}$$
$$= -\frac{1}{2}\operatorname{cosec}^2 \frac{1}{2} z.$$

and
$$\frac{\delta U}{\delta y} = \frac{\cosh y(\cosh y - \cos x) - \sinh y(\sinh y + \sin x)}{(\cosh y - \cos x)^2}$$
$$= g_2(x, y)$$
$$\Rightarrow \phi_2(z, 0) = \frac{1 - \cos z}{(1 - \cos z)^2} = \frac{1}{1 - \cos z}$$
$$= -\frac{1}{2}\operatorname{cosec}^2 \frac{1}{2} z.$$

Since $(1 + i) f(z)$

$$= \int [g_1(z, 0) - i g_2(z, 0)]dz + c$$
$$= \int \left(-\frac{1}{2}\operatorname{cosec}^2 \frac{1}{2} z - i\frac{1}{2}\operatorname{cosec}^2 \frac{1}{2} z\right)dz + c$$
$$= -\frac{1}{2}(1 + i)\int \operatorname{cosec}^2 \frac{1}{2} z dz + c$$
$$= (1 + i)\cot \frac{1}{2} z + c.$$
$$\Rightarrow \quad f(z) = \cot \frac{1}{2} z + \frac{c}{1 + i}$$
$$\therefore \quad f\left(\frac{\pi}{2}\right) = \frac{3 - i}{2}$$
$$\Rightarrow \quad \frac{c}{1 + i} = f\left(\frac{\pi}{2}\right) - \cot \frac{\pi}{4}$$
$$= \frac{3 - i}{2} - 1 = \frac{1 - i}{2}$$

which gives $c = \frac{1}{2}(1 - i)$

Hence $f(z) = \cot \frac{1}{2} z + \frac{1}{2}(1 - i)$.

Example 17. Find the bilinear transformation which maps the points $z = 1, i = -1$ onto the points $w = i, 0 = -i$

Hence find the image of $|z| < 1$.

Solution : Let the points $z_1 = 1$, $z_2 = i$, $z_3 = -1$ and $z_4 = z$ map onto the points $w_1 = i$, $w_2 = 0$, $w_3 = -i$ and $w_4 = w$.

Since the cross-ratio remains unchanged under a bilinear transformation

$$\therefore \frac{(1 - i)(-1 - z)}{(1 - z)(-1 - i)} = \frac{(i - 0)(-i - w)}{(i - w)(-i - 0)}$$

or
$$\frac{w + i}{w - i} = \frac{(z + 1)(1 - i)}{(z - 1)(1 + i)}$$

By Componendo, dividendo, we get

$$\frac{2w}{2i} = \frac{(z + 1)(1 - i) + (z - 1)(1 + i)}{(z + 1)(1 - i) - (z - 1)(1 + i)}$$

or
$$w = \frac{1 + iz}{1 - iz}$$

which is the required bilinear transformation

Since
$$z = i\frac{1 - w}{1 + w}$$
$$\therefore \left|\frac{i(1 - w)}{1 + w}\right| = |z| < 1$$

or $\quad |i|\,|1 - w| < |1 + w|$

or $\quad |1 - \mu - iv| < |1 + \mu + iv| \qquad [\because |i| = 1]$

or $\quad (1 - \mu)^2 + v^2 < (1 + \mu)^2 + v^2$

which reduces to $\mu > 0$.

Hence the interior of the circle $x^2 + y^2 = 1$ in the z-plane is maped onto the entire half of the w-plane to the right of the imaginary axis.

Example 18. Prove that if w_1, w_2, w_3, w_4 is the images of the four distinct points z_1, z_2, z_3, z_4 in the plane under a bilinear transformation.

Then $(w_1, w_2, w_3, w_4) = (z_1, z_2, z_3, z_4)$

Solution : Let the bilinear transformation be

$$w = T(z)$$
$$= \frac{az + b}{cz + d}\,(ad - bc \neq 0) \qquad \ldots(1)$$

Since w_1, w_2, w_3, w_4 are the images of z_1, z_2, z_3, z_4 resp. we have

$$w_1 = \frac{az_1 + b}{cz_1 + d},$$

$$w_2 = \frac{az_2 + b}{cz_2 + d}$$

$$\Rightarrow w_1 - w_2 = \frac{(ad - bc)(z_1 - z_2)}{(cz_1 + d)(cz_2 + d)}$$

Similarly, we have

$$w_2 - w_3 = \frac{(ad - bc)(z_2 - z_3)}{(cz_2 + d)(cz_3 + d)}$$

$$w_3 - w_4 = \frac{(ad - bc)(z_3 - z_4)}{(cz_3 + d)(cz_4 + d)}$$

$$w_4 - w_1 = \frac{(ad - bc)(z_4 - z_1)}{(cz_4 + d)(cz_1 + d)}$$

$$\Rightarrow \frac{(w_1 - w_2)(w_3 - w_4)}{(w_2 - w_3)(w_4 - w_1)} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)}$$

$$\Rightarrow (w_1, w_2, w_3, w_4) = (z_1, z_2, z_3, z_4).$$

Example 19. Expand $\frac{(z-2)(z+2)}{(z+1)(z+4)}$ for

(i) $|z| < 1$, (ii) $1 < |z| < 4$.

Solution :

$$\text{Here } f(z) = \frac{(z-2)(z+2)}{(z+1)(z+4)}$$

$$= 1 - \frac{5z + 8}{(z+1)(z+4)}$$

$$= 1 - \frac{1}{z+1} - \frac{4}{z+4}$$

(i) $|z| < 1$, we have

$$f(z) = 1 - (1+z)^{-1} - \left(1 + \frac{z}{4}\right)^{-1}$$

$$= 1 - (1 - z + z^2 - z^3 + \ldots) - \left[1 - \frac{z}{4} + \left(\frac{z}{4}\right)^2 - \left(\frac{z}{4}\right)^3 + \ldots\right]$$

$$= -1 + \left(1 + \frac{1}{4}\right)z + \left(-1 - \frac{1}{4^2}\right)z^2 + \left(1 + \frac{1}{4^3}\right)z^3 + \ldots$$

$$= -1 - \frac{5}{4}z - \frac{17}{16}z^2 - \frac{65}{64}z^3 - \ldots\ldots$$

The required expansion.

(ii) $1 < |z| < 4$. then $\frac{1}{|z|} < 1$ and $\frac{|z|}{4} < 1$

$$\therefore \quad f(z) = 1 - \frac{1}{z}\left(1 + \frac{1}{z}\right)^{-1} - \left(1 + \frac{z}{4}\right)^{-1}$$

$$= 1 - \frac{1}{z}\left[1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \ldots\right] - \left[1 - \frac{z}{4} + \frac{z^2}{4^2} - \frac{z^3}{4^3} + \ldots\right]$$

$$= -\ldots + \frac{1}{z^4} - \frac{1}{z^3} + \frac{1}{z^2} - \frac{1}{z} + \frac{z}{4} - \frac{z^2}{4^2} + \frac{z^3}{4^3} \ldots\ldots$$

The required expansion.

Example 20. What are the residues of the function $\frac{\cot \pi z}{(z-a)^2}$?

Solution :

$$\text{Here } f(z) = \frac{\cot \pi z}{(z-a)^2} = \frac{\cos \pi z}{\sin \pi z\,(z-a)^2}$$

Poles of $f(z)$ are given by $(z-a)^2 \sin \pi z = 0$

so $(z-a)^2 = 0 \Rightarrow z = a$ is a pole of order two and $\sin \pi z = 0 \Rightarrow \pi z = n\pi \Rightarrow z = n$ where $n \in \text{I}$

$z = n$, $(n \in \text{I})$ are simple poles of $f(z)$ if n is finite $z = \infty$ is the limit point of these poles so $z = \infty$ is the non-isolated essential singularity.

Residue at $(z = a)$ is $\frac{1}{1!}f'(a)$

where $f(z) = \cot \pi z = -\pi \operatorname{cosec}^2 \pi a$

Residue at $(z = n)$ is $\left[\frac{f(z)}{g'(z)}\right]_{z=n}$

(where $g(z) = \sin \pi z$)

$$= \left[\frac{\frac{\cos \pi z}{(z-a)^2}}{\pi \cot \pi z}\right]_{z=n}$$

$$= \frac{1}{\pi (n-a)^2}.$$

Example 21. Show that both the transformation $w = \frac{z+i}{z-i}$ and $w = \frac{i+z}{i-z}$ transform $|w| \leq 1$ into the lower half plane $\text{I}(z) \leq 0$.

Solution :

$$\text{Here} \quad w = \frac{z+i}{z-i}$$

$$\Rightarrow \quad \overline{w} = \frac{\overline{z} - i}{\overline{z} + i}$$

$$\therefore \quad w\bar{w} - 1 = \frac{z+i}{z-i}\,\frac{\bar{z}-i}{\bar{z}+i} - 1$$

$$= \frac{(z+i)(\bar{z}-i) - (z-i)(\bar{z}+i)}{(z-i)(\bar{z}+i)}$$

$$= \frac{-2i(z-\bar{z})}{|z-i|^2} = \frac{4\mathrm{I}(z)}{|z-i|^2}$$

$$= |\because z - \bar{z} = 2i\,\mathrm{I}(z)|$$

Taking $w = \frac{i+z}{i-z}$, we have

$$w = \bar{w} - 1$$

Hence for both the transformations $|w^2| - 1 \leq 0$

i.e., $|w| \leq 1$ gives $\mathrm{I}(z) \leq 0$

i.e., The boundary of the circle $|w| = 1$ corresponds to real axis in the z-plane and the interior of the circle transforms into the lower half z-plane.

Example 22. Prove that function $f(z) = z^n$, where n is a positive integer is an analytic function.

Solution : We have

$$f(z) = z^n$$

Now $$f(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(z+\Delta z)^n - z^n}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{z^n + nz^{n-1}\Delta z + \frac{1}{2} n(n-1) z^{n-2} (\Delta z)^2 + \ldots + \Delta^n z - z^n}{\Delta z}$$

(by Binomial theorem)

$$= \lim_{\Delta z \to 0} (nz^{n-1} + \frac{1}{2} n(n-1) z^{n-2} \Delta z + \ldots\ldots + \Delta^{n-1} z)$$

$$= nz^{n-1}$$

which exists for all finite values of z.

Hence $f(z)$ is an analytic function.

Example 23. If λ is real, a, b are complex numbers such that $|a| > |b|$, show that the bilinear transformation $w = e^{i\lambda} \frac{az+b}{\bar{a}+\bar{b}z}$ maps the inside of the circle $|z| = 1$ on the inside of the circle $|w| = 1$

Solution : We have

$$w\bar{w} - 1 = e^{i\lambda} \frac{az+b}{\bar{a}+\bar{b}z}\, e^{-i\lambda} \frac{\bar{a}\bar{z}+\bar{b}}{a+b\bar{z}} - 1$$

$$= \frac{(az+b)(\bar{a}\bar{z}+\bar{b}) - (\bar{a}+\bar{b}z)(a+b\bar{z})}{(\bar{a}+\bar{b}z)(a+b\bar{z})}$$

$$= \frac{(a\bar{a} - b\bar{b}) - (z\bar{z} - 1)}{|a+b\bar{z}|^2}$$

$$= \frac{(|a|^2 - |b|^2)(|z|^2 - 1)}{|a+b\bar{z}|^2}$$

If $|z| < 1$, then $w\bar{w} - 1 < 0$, because $|a| > |b|$.

Hence $|z| < 1$ corresponds to $|w| < 1$. *i.e.*, the interiors of the two circles correspond.

Example 24. Show that the function e^{-1/z^2} has no singularities.

Solution : We have $f(z) = e^{-1/z^2}$

Zeros of $f(z)$ are given by

$$e^{-1/z^2} = 0$$

or $$z^2 = 0$$

$\therefore$ $z = 0$ is a zero of order two

Since zeros have no limit point,

$\therefore$ There is no singularity of $f(z)$.

Here the poles of $f(z)$ are given by

$e^{-1/z^2} = 0$ which is not possible.

$\therefore$ There exist no poles

Hence e^{-1/z^2} has no singularities.

Example 25. If $w = \left(\frac{z-c}{z+c}\right)^2$ where $c > 0$, find the area of the z-plane of which the upper half of the w-plane is the conformal representation.

Solution : Let $w = \mu + iv$ and $z = x + iv$, then

$$\mu = \frac{(z^2 + y^2 - c^2)^2 - 4c^2y^2}{\{(x+c)^2 + y^2\}^2} \quad \ldots(1)$$

and $$v = \frac{4cy(x^2 + y^2 - c^2)}{\{(x+c)^2 + y^2\}^2} \quad \ldots(2)$$

from (2) $v < 0$ if

(i) y and $x^2 + y^2 - c^2$ are both positive, *i.e.*, for the points in the upper half z-plane and exterior of the circle $|z| = c$, or

(ii) y and $x^2 + y^2 - c^2$ are both negative, *i.e.*, for the points in the lower half z-plane and interior of the circle $|z| = c$.

Thus interior of the circle $|z| = c$ in the lower half and its exterior in the upper half both separately correspond to the upper half of the w-plane.

Examples 26. Prove that the function $\mu = x^3 - 3xy^2$ is harmonic and find the corresponding analytic function.

Solution : We have $\mu = x^3 - 3xy^2$

$$\frac{\delta\mu}{\delta x} = 3x^2 - 3y^2,$$

$$\frac{\delta^2\mu}{\delta x^2} = 6x.$$

$$\frac{\delta\mu}{\delta x} = -6xy \text{ and } \frac{\delta^2\mu}{\delta y^2} = -6x.$$

Now $\frac{\delta^2\mu}{\delta x^2} + \frac{\delta^2\mu}{\delta y^2} = 0$, so that μ satisfies Laplace's equation.

Also since first and second order partial derivatives of μ are continuous functions of x and y.

Thus μ is a harmonic function.

Example 27. Prove that $f(z) = \sin x (\cosh y) + i \cos (\sinh y)$ is continuous as well as analytic every where.

Solution : Let $\mu (x, y) = \sin x (\cosh y)$ and $v (x, y) = \cos x (\sinh y)$.

Here μ and v are both rational functions of x and y, μ and v are both continuous every where.

Hence $f(z)$ is continuous every where.

Here $$\frac{\delta\mu}{\delta v} = \cos x (\cosh y)$$

$$\frac{\delta\mu}{\delta y} = \sin x (\sinh y)$$

and $$\frac{\delta v}{\delta x} = -\sin x (\sinh y)$$

$$\frac{\delta v}{\delta y} = \cos x (\cosh y).$$

$$\therefore \quad \frac{\delta\mu}{\delta n} = \frac{\delta v}{\delta y}, \frac{\delta v}{\delta x} = -\frac{\delta\mu}{\delta y}$$

$\therefore$ μ and v satisfy Cauchy-Riemann equations.

Thus $f(z)$ is analytic every where.

Example 28. Show that the function $f(z) = xy + iy$ is everywhere continuous but is not analytic.

Solution : Let $f(z) = \mu(x, y) + iv(x, y)$

Then $\mu(x, y) = xy$

and $v(x, y) = y$

Since μ and v are polynomials in x and y, therefore, they are continuous at each point.

Hence $f(z)$ is every where continuous

Here $$\frac{\delta\mu}{\delta x} = y,$$

$$\frac{\delta\mu}{\delta y} = x,$$

$$\frac{\delta v}{\delta x} = 0,$$

$$\frac{\delta v}{\delta y} = 1$$

and $$\frac{\delta\mu}{\delta x} \neq \frac{\delta v}{\delta y},$$

$$\frac{\delta\mu}{\delta y} \neq -\frac{\delta v}{\delta x},$$

so that Cauchy-Riemann equations are not satisfied.

Hence $f(z)$ is not analytic.

OBJECTIVE TYPE QUESTIONS

1. $(1 + i)^{10} + (1 - i)^{10} =$
   (A) -1 (B) 1
   (C) 0 (D) 2

2. If $z = \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{35} + \left(\frac{-1}{2} - i\frac{\sqrt{3}}{2}\right)^{200}$, then—
   (A) $\text{Re}(z) < 0$ (B) $\text{Re}(z) = 0$
   (C) $\text{Re}(z) > 0$ (D) None of these

3. The product of two complex numbers (a, b) and (c, d) is—
   (A) $(ac + bd, ad + bc)$
   (B) $(ac - bd, ad + bc)$
   (C) $(ac + bd, ad - bc)$
   (D) $(ad + bc, ac - bd)$

4. Suppose a $a'b$ and b' are real numbers, then $\frac{(a + ib)}{(a' + ib')}$ will also be real number, if—
   (A) $ab - a'b' = 0$ (B) $ab' - a'b = 0$
   (C) $aa' - bb' = 0$ (D) $ab' - a'b = 0$

5. The conjugate of $(1 + i)^2$ is given by—
   (A) $(1 - i)^{-2}$ (B) $(1 + i)^{-1}$
   (C) $-2i$ (D) $2i$

6. Which of the following is true for complex number $\mathbb{C}$—
   (A) $\alpha + i\beta = \gamma + i\delta$, if $\alpha = \gamma$ and $\beta = \delta$
   (B) $\alpha + i\beta > 0 + i\beta$, if $\alpha > 0$ and $\beta < 0$
   (C) Transitivity low holds in $\mathbb{C}$
   (D) Trichotomy low holds in $\mathbb{C}$

7. Which of the following is false ?
   (A) $\text{Re}(z_1 + z_2) = \text{Re}(z_1) + \text{Re}(z_2)$
   (B) $|z_1 + z_2| \geq |z_1| + |z_2|$
   (C) $|z_1 z_2| = |z_1|\,|z_2|$
   (D) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$
   (E) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\{|z_1|^2 + |z_2|^2\}$

8. If $x = -2 - \sqrt{3i}$, then the value of $2x^4 + 5x^3 + 7x^2 + 41$ is—
   (A) $4 + \sqrt{3}i$ (B) $4 - \sqrt{3}i$
   (C) $\sqrt{3} + 4i$ (D) $\sqrt{3} - 4i$

9. The real part of exp (exp $i\theta$) is—
   (A) $e^{\cos\theta}$ (B) $e^{\cos\theta}\sin(\sin\theta)$
   (C) $e^{\cos\theta}\cos(\sin\theta)$ (D) $e^{\cos\theta}\cos(\cos\theta)$

10. The correct polar form of the complex number $1 - i$ is—
    (A) $\sqrt{2}\,e^{\pi/4i}$ (B) $e^{-\pi/4i}$
    (C) $\sqrt{2}\,e^{-\pi/4i}$ (D) $e^{\pi/4i}$

11. Let c be any complex numbers. Let for any $z = (x, y)$ in c, $z \cdot \bar{z} = z$, then $\bar{z}$ is equal to—
    (A) (0, 0) (B) (1, 0)
    (C) (0, 1) (D) (1, 1)

12. The value of $\left(\dfrac{\cos\theta + i\sin\theta}{\cos\theta - i\sin\theta}\right)^4$ is—
    (A) 1 (B) 0
    (C) $\cos 4\theta - i\sin 4\theta$ (D) $\cos 8\theta + i\sin 8\theta$

13. $(\sin\theta + i\cos\theta)^6$—
    (A) $\sin 6\theta + i\cos 6\theta$
    (B) $\cos 6\theta - i\sin 6\theta$
    (C) $-\cos 6\theta + i\sin 6\theta$
    (D) $\sin 6\theta - i\cos 6\theta$

14. If 1, w, w^2 are the cube roots of unity then $(x - y)(x - wy)(x - w^2y)$ is equal to—
    (A) $x - y$ (B) $x^2 - y^2$
    (C) $x^3 - y^3$ (D) $x^3 + y^3$

15. The reciprocal $a + ib$ is equal to—
    (A) $\dfrac{1}{a^2 + b^2} - ib$
    (B) $\dfrac{ib}{a + ib} - a$
    (C) $\dfrac{a^2}{a^2 + b^2} - \dfrac{b^2 i}{a^2 + b^2}$
    (D) $\dfrac{a}{a^2 + b^2} - \dfrac{bi}{a^2 + b^2}$

16. arg $(-1 + \sqrt{3i})$ equals—
    (A) $\dfrac{\pi}{3}$ (B) $\dfrac{\pi}{6}$
    (C) $\dfrac{2\pi}{3}$ (D) $\dfrac{5\pi}{6}$

17. The necessary condition that the points A, B, C representing numbers z_1, z_2, z_3 respectively on the Argand plane be the vertices of an equilateral triangle in that—
    (A) $\dfrac{1}{z_1 - z_2} = \dfrac{1}{z_3 - z_1} + \dfrac{1}{z_2 - z_3}$
    (B) $\dfrac{1}{z_1 - z_2} = \dfrac{1}{z_1 - z_3} + \dfrac{1}{z_2 - z_3}$
    (C) $\dfrac{1}{z_1 - z_2} = \dfrac{1}{z_1 - z_3} + \dfrac{1}{z_3 - z_2}$
    (D) None of these

18. If 1, w, w^2 are the cube root of unity, then the roots of $(x - 1)^3 + 8 = 0$ are—
    (A) $-1, -1, -1$
    (B) $-1, 1 + 2w, 1 + 2w^2$
    (C) $1, w, 2w$
    (D) $-1, 1 - 2w, 1 - 2w^2$

19. Complex form of $\sqrt{3 + 4i}$ is—
    (A) $\sqrt{3 + i}$ (B) $2 - i$
    (C) $2 + i$ (D) $\sqrt{3} - i$

20. If cube root of $a + ib$ is $x + iy$, then $\text{H}(x^2 - y^2)$ is—
    (A) $\dfrac{a}{x} + \dfrac{b}{y}$ (B) $\dfrac{x}{a} + \dfrac{y}{b}$
    (C) $ax + by$ (D) $ax - by$

21. If w be an imaginary cube root of unity, then $(1 - w + w^2)^5 + (1 + w - w^2)^5$ is equal to—
(A) 64 (B) 32
(C) 16 (D) 8

22. The complex numbers $z_1 = 1 + 2i$, $z_2 = 4 - 2i$ and $z_3 = 1 - 6i$ form the vertices of *a*—
(A) Right angled triangle
(B) Isosceles triangle
(C) Equilateral triangle
(D) Scalene triangle

23. If m and n are integers, then the value of the complex number $\log_i i$ is given by—
(A) $\frac{4m+1}{4n+1}$ (B) $e^{\frac{4m+1}{4n+1}}$
(C) $\log \frac{4m+1}{4n+1}$ (D) 1

24. The amplitude of the complex number $\left(\frac{1}{1-2i} + \frac{3}{1+i}\right)\left(\frac{3+4i}{2-4i}\right)$ is given by—
(A) $\tan^{-1} 6$ (B) $\tan^{-1} 9$
(C) $\tan^{-1} 3$ (D) $\tan^{-1} \frac{3}{2}$

25. The real part of the complex number $(1 + i)^n$ is—
(A) $2^{n/2} \cos \frac{n\pi}{4}$ (B) $2^n \cos \frac{n\pi}{2}$
(C) $2^{-n/2} \cos n\pi$ (D) $2^{-n} \cos \frac{n\pi}{2}$

26. Let z_1 and z_2 are two complex numbers with α and β as their principal arguments such that $\alpha + \beta > \pi$, then $\text{Arg}(z_1 \cdot z_2)$ is given by—
(A) $\alpha + \beta$ (B) $\alpha + \beta - \pi$
(C) $\alpha + \beta + \pi$ (D) $\alpha + \beta - 2\pi$

27. The locus of the complex number satisfying $\arg \frac{z-1}{z+1} = \frac{\pi}{3}$ is a—
(A) Straight line (B) Circle
(C) Parabola (D) Hyperbola

28. If w is an imaginary cube root of unity, $x = a + b$, $y = aw + bw^2$ and $z = aw^2 + bw$, then xyz equals—
(A) $a + b$ (B) $a^2 + b^2$
(C) $a^4 + b^4$ (D) $a^3 + b^3$

29. Principal value of argument of (cos 1200° + i sin 1200°) is—
(A) 300° (B) 120°
(C) – 150° (D) 180°

30. If z_1 and z_2 are complex numbers, then amp $(z_1 . z_2)$ is equal to—
(A) amp (z_1) + amp (z_2)
(B) amp (z_1) amp (z_2)
(C) amp (z_1) – amp (z_2)
(D) $\frac{\text{amp }(z_1)}{\text{amp }(z_2)}$

31. The value of $\arg(z) + \arg(\bar{z})$, $(z \neq 0)$ is—
(A) 0 (B) π
(C) $\frac{\pi}{2}$ (D) $\frac{\pi}{4}$

32. If $z = x + iy$, then the number of solutions of the equation $z^2 = \bar{z}$ is—
(A) One (B) Two
(C) Four (D) Infinite

33. If $\frac{4+3i}{3-4i} = x + iy$, then $\frac{x}{y}$ is equal to—
(A) 0 (B) 1
(C) $\frac{4}{3}$ (D) $\frac{4}{5}$

34. If $|z - 1| = 2$, then the value of $z\bar{z} - z - \bar{z}$ is—
(A) 4 (B) 2
(C) 1 (D) 3

35. The solution of the equation $|z| - z = 1 + 2i$ is—
(A) $1 - 2i$ (B) $2 - \frac{3}{2}i$
(C) $\frac{3}{2} + 2i$ (D) $\frac{3}{2} - 2i$

36. If $|z| = |z - 1|$, then—
(A) $\text{Re}(z) = 1$ (B) $\text{Re}(z) = \frac{1}{2}$
(C) $\text{Im}(z) = 1$ (D) $\text{Im}(z) = \frac{1}{2}$

37. If $\left|\frac{z-5i}{z+5i}\right| = 1$, then $z = x + iy$ lie on—
(A) The real axis

(B) The straight line $x = 5$
(C) The straight line $y = 5$
(D) A circle passing through origin

38. If $2 \cos \alpha_1 = a + \frac{1}{a}$, $2 \cos \alpha_2 = b + \frac{1}{b}$ etc, then $abc + \frac{1}{abc}$ …… will be given by—
(A) $\cos (2\alpha_1 + 2\alpha_2 + \ldots\ldots)$
(B) $2 \cos (\alpha_1 + \alpha_2 + \ldots\ldots)$
(C) $2 \sin (\alpha_1 + \alpha_2 + \ldots\ldots)$
(D) $\sin (2\alpha_1 + 2\alpha_2 + \ldots\ldots)$

39. If α, β are the roots of the equation $x^2 - 2x + 4 = 0$, $\alpha^n + \beta^n$ is equal to—
(A) $2^{n+1} \cos\left(\frac{n\pi}{3}\right)$
(B) $2^n \cos\left(\frac{n\pi}{3}\right)$
(C) $2^{n+1} \sin\left(\frac{n\pi}{3}\right)$
(D) $2^n \sin\left(\frac{n\pi}{3}\right)$

40. If the imaginary part of $\frac{2z+1}{iz+1}$ is -2, then the locus of a point representing z, is a—
(A) Circle (B) Straight line
(C) Parabola (D) None of these

41. For complex number z,
$|z + 5|^2 + |z - 5|^2 = 75$ represents—
(A) A circle (B) An ellipse
(C) A triangle (D) A straight line

42. If θ is a positive acute angle, then real and imaginary parts of $\cos^{-1} (\cos \theta + i \sin \theta)$ are—
(A) $\sin^{-1} (\sqrt{\sin \theta}) + i \log \{\sqrt{1 + \sin \theta} - \sqrt{\sin \theta}\}$
(B) $\cos^{-1} (\sqrt{\sin \theta}) + i \log \{\sqrt{1 + \cos \theta} - \sqrt{\cos \theta}\}$
(C) $\sin^{-1} (\sqrt{\cos \theta}) + i \log \{\sqrt{1 + \sin \theta} - \sqrt{\cos \theta}\}$
(D) None of these

43. If $(1 + i \tan \alpha)^{1 + i \tan \beta}$ has only real values, one of them is given by—
(A) $(\sec \beta)^{\sec^2 \alpha}$ (B) $(\sec \alpha)^{\sec \beta}$
(C) $(\sec \alpha)^{\sec^2 \beta}$ (D) None of these

44. If y_0 and y_1 are two rectifiable curves in G and y_0 and y_1 are fixed-end-point homotopic, then for any analytic function f in G—
(A) $\int_{y_0} f - \int_{y_1} f = 0$ (B) $\int_{y_0} f + \int_{y_1} f = 0$
(C) $\int_{y_0} f \int_{y_1} f = 0$ (D) None of these

45. If G is simply connected, then for every curve $y \in G$, $\int_y f = 0$—
(A) For every function f
(B) For every non-analytic function f
(C) For every analytic function f
(D) None of these

46. If G is simply connected, then $\int_y f = 0$ for—
(A) Every rectifiable curve
(B) Every closed rectifiable curve y
(C) Every function f
(D) None of these

47. If f and g are continuous functions on $[a, b]$ and y and σ are the function of bounded variation on $[a, b]$, then—
(A) $\int_a^b (f + g)dy = \int_a^b f\, dy$
(B) $\int_a^b (f + g)\, dy = \int_a^b g\, dy$
(C) $\int_a^b (f + g)dy = \int_a^b f\, dy = \int_a^b g\, dy$
(D) None of these

48. If y piecewise smooth and $f : [a, b] \to \mathbb{C}$ is continuous, then—
(A) $\int_a^b f dy = \int_a^b f(t) y'(t) dt$
(B) $\int_a^b f dy = \int_a^b f(t)\, y(t) dt$
(C) $\int_a^b f dy = \int_a^b f'(t)\, y(t) dt$
(D) None of these

49. If y is a rectifiable curve and f is continuous function on $\{y\}$, then—
(A) $\int_y f = -\int_y f$
(B) $\int_y f\, dt = -\int_{-y} f\, dt$
(C) $\int_y f\, dt = -\int_y f\, dt$
(D) $\int_y f\, dt = -\int_{-y} f\, dt$

50. If y is a rectifiable curve and f is continuous function on $\{y\}$, then if $c \in \mathbb{C}$—
   (A) $\int_y f(z)dz = \int_{y+c} f(z-c)dz$
   (B) $\int_y f(z)dz = \int_y f(z-c)dz$
   (C) $\int_y f(z)dz = \int_y f(c)dz$
   (D) None of these

51. If G is simply connected and $f : G \to \mathbb{C}$ is analytic in G, then—
   (A) f has a primitive in G
   (B) f has no primitive in G
   (C) f is constant in G
   (D) None of these

52. If G is an open set then curve y is homologous to zero if for all $w \in \mathbb{C} - G$—
   (A) $n(y ; w) = 0$ (B) $n(y ; w) = 1$
   (C) $n(y ; w) = 2$ (D) $n(y ; w) = 4$

53. If G is a region and f is non-constant analytic function on G. Then open mapping theorem states, for any open set $\cup \subset G$—
   (A) F(U) is closed (B) F(U) is open
   (C) F(U) = U (D) None of these

54. If G is an open set and $f : G \to \mathbb{C}$ is differentiable function, then—
   (A) f is analytic on G
   (B) f is non-analytic on G
   (C) f is constant on G
   (D) None of these

55. If function $f(z)$ has an isolated singularities at $z = a$, then $z = a$ has removable singularity if—
   (A) $\lim_{z \to a} (z - a) = 0$
   (B) $\lim_{z \to a} f(z) = 0$
   (C) $\lim_{z \to a} (z - a) f(z) = 0$
   (D) None of these

56. Series Σa_n converges absolutely if—
   (A) $\Sigma |a_n|$ converges (B) Σa_n converges
   (C) $\Sigma |a_n|$ diverges (D) None of these

57. If f and g are analytic function, then—
   (A) $\frac{f}{g}$ is always analytic
   (B) $\frac{f}{g}$ is analytic when ever $g(x) \neq 0$
   (C) $\frac{f}{g}$ is analytic whenever $f(x) \neq 0$
   (D) None of these

58. A function $f(z + c) = f(z)$, where c is any number, then f is—
   (A) A periodic function
   (B) Periodic function with period C
   (C) Periodic function with period z
   (D) None of these

59. If G is open connected set in $\mathbb{C}$ and $f : G \to \mathbb{C}$ is a continuous function. Then f is a branch of logarithm if $z \in G$—
   (A) $z = \sin f(z)$ (B) $z = \cos f(z)$
   (C) $z = \exp f(z)$ (D) $z = f(z)$

60. If $w' = T_1(z) = \frac{z+2}{z+3}$, then $T_1^{-1}(z)$ is—
   (A) $\frac{2-3w}{w+1}$ (B) $\frac{2-3w}{w-1}$
   (C) $\frac{1}{w+3}$ (D) None of these

61. What is the radius of convergence for power series $f(z) = \Sigma \frac{1}{n\rho} z^n$?
   (A) 1 (B) 2
   (C) 0 (D) ∞

62. $f(z) = \frac{\sin z}{(z-\pi)^2}$ have the pole of order—
   (A) 1 (B) 2
   (C) 3 (D) 0

63. If y is a rectifiable curve and f is a continuous function on $\{y\}$, then—
   (A) $\int_y f(z)dz \leq \int_y |f(z)|\,|dz|$
   (B) $\int_y f(z)dz \geq \int_y |f(z)|\,|dz|$
   (C) $\int_y f(z)dz = \int_y |f(z)|\,|dz|$
   (D) None of these

64. G is open set in $\mathbb{C}$, y is a closed rectifiable path in G and $f : G \to \mathbb{C}$ is a continuous function, then—
   (A) $\int_y f(z)dz = 0$ (B) $\int_y f(z)dz > 0$
   (C) $\int_y f(z)dz < 0$ (D) None of these

65. An analytic function is—
   (A) Infinitely differentiable

(B) Finitely differentiable
(C) Not differentiable
(D) None of these

66. If $\phi : [a, b] \times [c, d] \to \mathbb{C}$ is a continuous function and $g : [c, d] \to \mathbb{C}$ such that $g(t) = \int_a^b \phi(s, t)ds$, then—
(A) g is not a continuous function
(B) g is a continuous function
(C) g is an increasing function
(D) g is a decreasing function

67. If is a analytic and $f'(z) \neq 0$, then—
(A) f is non-conformal mapping
(B) f is a conformal mapping
(C) f is a constant function
(D) None of these

68. A mapping $S(z) = \dfrac{az + b}{cz + d}$ which is a linear functionl transformation is mobius transformation, if—
(A) $ad - bc \neq 0$ (B) $ad - bc = 0$
(C) $ab - cd \neq 0$ (D) None of these

69. For any point z_1, if z_2, z_3, z_4 are distinct points and T is any mobius transformation then the cross ratio (z_1, z_2, z_3, z_4) is equal to—
(A) (Tz_1, Tz_2, z_3, z_4)
(B) (Tz_1, Tz_2, Tz_3, z_4)
(C) (Tz_1, Tz_2, Tz_3, Tz_4)
(D) None of these

70. If $z_1 \neq z_2 \neq z_3 \neq z_4$ in $\mathbb{C}_\infty$. Then cross ratio (z_1, z_2, z_3, z_4) is a real number if z_1, z_2, z_3, z_4 lies on—
(A) Triangle (B) A parabola
(C) A circle (D) A hyperbola

71. The mobius transform takes—
(A) Circles in to line
(B) Circle into circle
(C) Circle into square
(D) None of these

72. If $z = a$ is an isolated singularity of f and $f(z) = \sum_{-\infty}^{\infty} a_n (z - a)^n$ is its Laurent expansion in ann $(a', 0, R)$. Then if $a_n = 0$ for $n \leq -1$, $z = a$ is—
(A) A pole of order m
(B) An essential singularity
(C) A removable singularity
(D) None of these

73. If G is a region and $f : G \to \mathbb{C}$ is continuous function such that $\int_y f = 0$ for every closed path y in G. Then—
(A) f is analytic in G
(B) f is continuous in G
(C) f is non-analytic in G
(D) f is discontinuous in G

74. If the series $\sum_{n=0}^{\infty} a_n (z - a)^n$ has radius of convergence $R > 0$, then $f(z) = \sum_{n=0}^{\infty} a_n (z - a)^n$ is—
(A) Analytic in ball B $(a; R)$
(B) Analytic outside ball B$(a; R)$
(C) Non-analytic in ball B$(a; R)$
(D) None of these

75. The following statement is false for complex number z—
(A) $\text{Re}\, z = \frac{1}{2}(z + \bar{z})$ (B) $\text{Im}\, z = \frac{1}{2i}(z + \bar{z})$
(C) $|\bar{z}| = |z|$ (D) $|z^2| = z$

76. If $z = a$ is an isolated singularity of f, then a is the pole of f. If—
(A) $\lim_{z \to a} |f(z)| = 0$ (B) $\lim_{z \to a} |f(z)| = a$
(C) $\lim_{z \to a} |f(z)| = \infty$ (D) None of these

77. If $z = a$ is an isolated singularity of f and $f(z) = \sum_{-\infty}^{\infty} a_n (z - a)^n$ is its Laurent expansion in *ann* $(a;\ 0;\ R)$. Then $z = a$ is a removable singularity, if—
(A) $a_n = 0, n \leq -1$ (B) $a_n \neq 0, n \leq -1$
(C) $a_n = 0, n \geq -1$ (D) $a_n \neq 0, n \geq -1$

78. If $z = a$ is an isolated singularity of f and $f(z) = \sum_{-\infty}^{\infty} a_n(z - a)^n$ is its Laurent expansion in *ann* $(a; 0, R)$. Then $z = a$ is a pole of order m, then—
(A) $a_{-m} \neq 0$ and $a_n = 0$ for $n \leq -(m + 1)$
(B) $a_{-m} = 0$ and $a_n \neq 0$ for $n \leq -(m + 1)$

(C) $a_{-m} = 0$ and $a_n = 0$ for $m \le -(m+1)$

(D) None of these

79. If $z = a$ is an isolated singularity of f and $f(z) = \sum_{-\infty}^{\infty} a_n (z-a)^n$ is its Laurent expansion in $ann\ (a;\ 0,\ R)$. Then $z = a$ is an essential singularity if—

(A) $a_n \neq 0$ for all integers n

(B) $a_n = 0$ for all integers n

(C) $a_n \neq 0$ for infinitely many negative integers n

(D) $a_n \neq 0$ for infinitely many positive integers n

80. If $z = a$ is an isolated singularity of f and $f(z) = \sum_{-\infty}^{\infty} a_n (z-a)^n$ is its Laurent expansion in $ann\ (a;\ 0,\ R)$. Also if $a_n \neq 0$ for infinitely many negative integers n, then—

(A) $z = a$ is a removable singularity

(B) $z = a$ is a pole of order m

(C) $z = a$ is a an essential singularity

(D) None of these

81. If f have an isolated singularity at $z = a$ and $f(z) = \sum_{-\infty}^{\infty} a_n (z-a)^n$ is its Laurent expansion about $z = a$. Then residue of f at $z = a$ is—

(A) a_{-1} (B) a_0

(C) a_{-2} (D) a_1

82. If f has a pole of order m at $z = a$ and $g(z) = (z-a)^m f(z)$, then—

(A) $\text{Res}\,(f\,;a) = \dfrac{1}{\lfloor m-1} g\,(m-1)\,(a)$

(B) $\text{Res}\,(f\,;a) = g^{(m-1)}\,(a)$

(C) $\text{Res}\,(f\,;a) = \dfrac{1}{\lfloor m-1} g(a)$

(D) None of these

83. If $z = a$ is an isolated singularity of f and $f(z) = \sum_{-\infty}^{\infty} a_n (z-a)^n$ is its Laurent expansion in $ann\ (a',\ 0,\ R)$. Also if $a_{-m} \neq 0$ and $a_n = 0$ for $n \le (m+1)$, then—

(A) $z = a$ is a removable singularity

(B) $z = a$ is a pole of order m

(C) $z = a$ is a an essential singularity

(D) None of these

84. If $T_1(z) = \dfrac{z+2}{z+3}$ and $T_2(z) = \dfrac{z}{z+1}$; then $T_2T_z(z)$ is—

(A) $z+2$ (B) $\dfrac{2}{2z+5}$

(C) $\dfrac{z+2}{2z+5}$ (D) None of these

85. The radius of convergence of the power series $f(z) = \Sigma \dfrac{n+1}{(n+2)\,(n+3)} z^n$ is—

(A) 1 (B) 2

(C) 3 (D) 4

86. If $f(z) = \dfrac{1-e^z}{1+e^z}$, then at $z = \infty$, $f(z)$ have—

(A) Pole

(B) Removable singularity

(C) Isolated singularity

(D) Non-isolated singularity

87. A mapping $S(z)$ is called linear transformation if—

(A) $S(z) = \dfrac{az}{cz}$ (B) $S(z) = az$

(C) $S(z) = \dfrac{az+b}{cz+d}$ (D) None of these

88. T is a circle through points z_2, z_3, z_4. The points z, $z^* \in \mathbb{C}_\infty$ are symmetric with respect to T if—

(A) $(z^*, z_2, z_3, z_4) = (z_1, z_2, z_3, z_4)$

(B) $(z^*, z_2, z_3, z_4) = \overline{(z_1, z_2, z_3, z_4)}$

(C) $z^*, z_2 = z, z_3$

(D) None of these

89. If (z_1, z_2, z_3) is an orientation of T, then right side of Γ with respect to (z_1, z_2, z_3) is—

(A) $\{z : \text{Im}\,(z_1, z_2, z_3) < 0\}$

(B) $\{z : \text{Im}(z, z_1, z_2, z_3) > 0\}$

(C) $\{z : \text{Re}\,(z, z_1, z_2, z_3) < 0\}$

(D) None of these

90. If $(1, 0, \infty)$ is the orientation of $\mathbb{R}$, then the cross ratio—

(A) $(z, 1, 0, \infty) = 1$ (B) $(z, 1, 0, \infty) = 0$

(C) $(z, 1, 0, \infty) = \infty$ (D) $(z, 1, 0, \infty) = z$

91. If f is an entire function, then—
(A) f has a power series expression
(B) f has not a power series expression
(C) f is constant
(D) None of these

92. If f is a bounded entire function, then—
(A) f is constant
(B) f is equal to zero
(C) f is increasing function
(D) f is decreasing function

93. If $y : [0, 1] \to \mathbb{C}$ is a closed rectifiable curve and $a \notin \{y\}$, then $\frac{1}{2\pi i}\int_y \frac{dz}{z-a}$ is—
(A) An integer (B) Rational number
(C) Real number (D) Complex number

94. If y is a closed rectifiable curve in $\mathbb{C}$, then for $a \notin \{y\}$, the index of y with respect to point a is—
(A) $n(y\,;a) = \frac{1}{2\pi i}\int_y z\,dz$
(B) $n(y\,;a) = \frac{1}{2\pi i}\int_y (z-a)^{-1}dz$
(C) $n(y\,;a) = \frac{1}{2\pi i}\int_y a\,dz$
(D) None of these

95. If $G \subset \mathbb{C}$ and G is open and connected. Also if f is a branch of log z on G, then the totality of branches of log z are the functions—
(A) $f(z)$ (B) $f((z) + 2n\pi i$
(C) $f(z)$ + const. (D) None of these

96. A branch of logarithm function is—
(A) Continuous function
(B) Differential function
(C) Analytic function
(D) None of these

97. A derivative of branch of logarithm function is—
(A) z (B) $\frac{1}{z}$
(C) 0 (D) None of these

98. The Cauchy–Riemann equation are if $\mu = \mu(x, y)$ and $v(x, y) = v$
(A) $\frac{\delta\mu}{\delta x} = \frac{\delta v}{\delta y}, \frac{\delta\mu}{\delta y} = -\frac{\delta v}{\delta x}$
(B) $\frac{\delta^2\mu}{\delta x^2} = \frac{\delta^2\mu}{\delta y^2}$
(C) $\frac{\delta^3 v}{\delta x^3} = \frac{\delta^2\mu}{\delta x^2}$
(D) None of these

99. A function $\mu = \mu(x, y)$ is harmonic if—
(A) μ have continuous second derivative
(B) μ have continuous second derivative and $\frac{\delta^2\mu}{\delta x^2} + \frac{\delta^2\mu}{\delta y^2} = 0$
(C) $\frac{\delta^2\mu}{\delta x^2} + \frac{\delta^2\mu}{\delta y^2} = 0$
(D) None of these

100. If y and σ are closed rectifiable curve having same initial points, then for ever $a \notin \{y\}$—
(A) $n(y\,;a) = -n(y\,;a)$
(B) $n(y\,;a) = n(-y\,;a)$
(C) $n(y\,;a) = -n(-y\,;a)$
(D) None of these

101. If y and σ are closed rectifiable curves having same initial points, then for every $a \notin \{y\} \cup \{\sigma\}$—
(A) $m(y + \sigma\,;a) = n(y\,;a)$
(B) $m(y + \sigma; a) = n(\sigma\,;a)$
(C) $m(y + \sigma; a) = n(y\,;a) + n(\sigma\,;a)$
(D) None of these

102. If y_0 and y_1 are two closed rectifiable curves in G and y_0 is homotopic to y_1, then for every analytic function f on G—
(A) $\int_{y_0} f \neq \int_{y_1} f$ (B) $\int_{y_0} f = \int_{y_1} f$
(C) $\int_{y_0} f + \int_{y_1} f = 0$ (D) None of these

103. If y is a closed rectifiable curve in G and y in homotopic to 0, then for all $\omega \in \mathbb{C} - G$—
(A) $n(y\,;\omega) = 1$ (B) $n(y\,;\omega) = 2$
(C) $n(y\,;\omega) = 0$ (D) $n(y\,;\omega) = 3$

104. If $T_1(z) = \frac{z+2}{z+3}$ and $T_2(z) = \frac{z}{z+1}$, then $T_2^{-1}T_1(z)$ is—
(A) $z + 3$ (B) $z + 2$
(C) $z + 6$ (D) $z - 3$

105. If $T_1(z) = \frac{z+2}{z+3}$ and $T_2(z) = \frac{z}{z+1}$, then $T_1T_2(z)$ is equal to—

(A) $\frac{3z+2}{4z+2}$ (B) $\frac{2}{2z+1}$

(C) $\frac{3z}{4z-2}$ (D) None of these

106. If $\Sigma \frac{z^n}{n!}$ is a power series, its radius of convergence is—

(A) 0 (B) ∞

(C) 1 (D) n

107. For the power series $\Sigma \frac{n!}{n^n}, z^n$ the radius of convergence is—

(A) e (B) 1

(C) ∞ (D) Zero

108. If (z_1, z_2, z_3) is an orientation of T, then left side of T with respect to (z_1, z_2, z_3) is—

(A) $\{z : \text{Im}\,(z, z_1, z_2, z_3) < 0\}$

(B) $\{z : \text{Im}\,(z, z_1, z_2, z_3) > 0\}$

(C) $\{z : \text{Im}\,(z, z_1, z_2, z_3) = 0\}$

(D) None of these

109. If $y : [a, b] \to \mathbb{C}$ is of bounded variation, then and P, Q are partition of $[a, b]$. If $P \subset Q$ then—

(A) $v\,(y\,;P) \le v\,(y\,;Q)$

(B) $v\,(y\,;Q) \le v\,(y\,;P)$

(C) $v\,(y\,;P) = v\,(y\,;Q)$

(D) None of these

110. If $y : [a, b] \to \mathbb{C}$ and P is any partition of $[a, b]$. Then variation of y, $v\,(y)$ is equal to—

(A) $\inf\,\{v\,(y\,;P)\}$

(B) $\sup\,\{v\,(y\,;P)\}$

(C) $\max\,\{v\,(y\,;P)\}$

(D) $\min\,\{v\,(y\,;P)\}$

111. If $y : [a, b] \to \mathbb{C}$ and y is of bounded variation then total variation $v\,(y)$ is—

(A) $v\,(y) \le \int_b^i |y'(t)|dt$

(B) $v\,(y) \ge \int_b^i |y'(t)|dt$

(C) $v\,(y) = \int_b^i |y'(t)|dt$

(D) None of these

112. If y is a rectifiable curve in $\mathbb{C}$ and F_n and F are continuous function on $\{y\}$. Also if $F = \mu - \lim F_n$ on $\{y\}$, then—

(A) $\int_y F = \lim \int_y F_n$ (B) $\int_y F = \int_y F_n$

(C) $\int_y F \ne \lim \int_y F_n$ (D) $\int_y F \ne \int_y F_n$

113. If $f : G \to \mathbb{C}$ is analytic and $R = d\,(a, \delta G)$ then for $a \in G$ and $|z - a| < R$—

(A) $f\,(z) = a_n\,(z-a)^n$

(B) $f\,(z) = \sum_{n=0}^{\infty} a_n\,(z-a)^n$

(C) $f\,(z) = \Sigma\,(z-a)^n$

(D) None of these

114. If f is analytic in a ball $B(a\,;R)$ and $|f(z)| \le M$, $\forall\, a \in R\,(a\,;R)$, then—

(A) $|f^{(n)}\,(a)| \le \frac{n!.M}{R^n}$ (B) $|f^{(n)}\,(a)| \ge \frac{n!\,M}{R^n}$

(C) $|f^{(n)}\,(a)| = n!\,M$ (D) None of these

115. If f is analytic in a disk $B(a\,;R)$ and y is closed rectifiable curve in $B(a\,;R)$. Then—

(A) $\int_y F = 0$ (B) $\int_y F \ne 0$

(C) $\int_y F = R$ (D) None of these

116. If series Σa_n converges absolutely, then—

(A) $\Sigma\,a_n$ converges

(B) $\Sigma\,a_n$ does not converges

(C) $\Sigma\,a_n$ diverges

(D) None of these

117. If $f : G \to \mathbb{C}$ is differentiable at a point $a \in G$, then—

(A) f is discontinuous at a

(B) f is continuous at a

(C) f is constant at a

(D) None of these

118. If f is analytic function in some domain, then in that domain—

(A) f is continuous only

(B) f is differentiable only

(C) f is continuous and differentiable both

(D) None of these

119. $f\,(z) = \sum_{n=0}^{\infty} a_n\,(z-a)^n$ have radius of convergence $R > 0$, then—

(A) For $n \le 0$, $a_n = \frac{1}{n!} f^{(n)}\,(a)$

(B) For $n \ge 0$, $a_n = 0$

(C) For $n \geq 0, a_n = \frac{1}{n!} f^{(n)}(a)$

(D) None of these

120. If $f : G \rightarrow \mathbb{C}$ is differentiable with $f'(z) = 0$ for all $z \in G$ and G is open and connected, then—
   (A) f is constant function
   (B) f is increasing function
   (C) f is decreasing function
   (D) None of these

121. A path in same region is—
   (A) Continuous function
   (B) Discontinuous function
   (C) Differentiable function
   (D) None of these

122. A path is said to be smooth path if—
   (A) It is continuous function
   (B) It is a continuous and differentiable function
   (C) Differentiable function
   (D) None of these

123. A path y is piecewise smooth in interval $[a, b]$ if in a partition P of $[a, b]$—
   (A) In each sub-interval y is continuous
   (B) In each sub-interval y is smooth path
   (C) In each sub-interval y is a path
   (D) None of these

Answers with Hints

1. (C) $(1+i)^{10} + (1-i)^{10}$

$$= \left[\sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\right]^{10} + \left[\sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)\right]^{10}$$

$$= 2^5\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^{10} + 2^5\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)^{10}$$

$$= 2^5\left[\left(\cos\frac{10\pi}{4} + i\sin\frac{10\pi}{4}\right) + \left(\cos\frac{10\pi}{4} - i\sin\frac{10\pi}{4}\right)\right]$$

$$= 2^5\left[2\cos\frac{10\pi}{4}\right]$$

$$= 2^6\cos\left(\frac{5\pi}{2}\right) = 2^6 \times 0 = 0$$

2. (A)

$$z = \left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)^{35} + \left(\cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3}\right)^{200}$$

$$= \left(\cos\frac{70\pi}{3} + i\sin\frac{70\pi}{3}\right) + \left(\cos\frac{400\pi}{3} - i\sin\frac{400\pi}{3}\right)$$

$$= \left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right) + \left(\cos\frac{4\pi}{3} - i\sin\frac{4\pi}{3}\right)$$

$$= 2\cos\frac{4\pi}{3} = -2\cos\frac{\pi}{3}$$

$$= -2 \times \frac{1}{2} = -1.$$

$\therefore$ Re$(z) < 0$

3. (B)

4. (D) $$\frac{a+ib}{a'+ib'} = \frac{a+ib}{a'+ib'} \times \frac{a'-ib'}{a'-ib'} = \frac{(aa'+bb') + i(a'b - ab')}{(a')^2 + (b')^2}$$

This number will be real, if its imaginary part is zero

$\therefore a'b - ab' = 0.$

5. (C) $(1+i)^2 = 1 + i^2 + 2i = 1 + (-1) + 2i = 2i$

$\therefore$ Conjugate of $(1+i)^2$ = conjugate of $2i = -2i$

6. (A) Since the set of complex number does not possess order relation. Hence (B), (C) and (D) are wrong.

7. (B) The relation $|z_1 + z_2| \geq |z_1| + |z_2|$ is false. The correct relation is $|z_1 + z_2| \leq |z_1| + |z_2|$.

8. (B) Given that

$$x = -2 - \sqrt{3}i$$

$$\therefore \quad x + 2 = -\sqrt{3}i$$

$$\therefore \quad (x+2)^2 = (-\sqrt{3}i)^2$$

$$\Rightarrow \quad x^2 + 4x + 4 = -3$$

$$\Rightarrow \quad x^2 + 4x + 7 = 0$$

Dividing $2x^4 + 5x^3 + 7x^2 + 41$ by $x^2 + 4x + 7$
we get $2x^4 + 5x^3 + 7x^2 + 41$
$= (2x^2 - 3x + 5)(x^2 + 4x + 7) + x + 6$
$= (2x^2 - 3x + 5)(0) + (-2 - \sqrt{3}i) + 6$
$= 4 - \sqrt{3}i$

9. (C) $\exp(\exp i\theta) = e^{e^{i\theta}} = e^{\cos\theta + i\sin\theta}$
$= e^{\cos\theta}.e^{i\sin\theta}$
$= e^{\cos\theta}\{\cos(\sin\theta) + i\sin(\sin\theta)\}$

$\therefore$ Real (Exp (exp $i\,\theta$)
$= e^{\cos\theta}.\cos(\sin\theta)$

10. (C) Let
$1 - i = r(\cos\theta + i\sin\theta)$
$\therefore\ r\cos\theta = 1,$
$r\sin\theta = -1$
$\therefore\ r^2\cos^2\theta + r^2\sin^2\theta = 1 + 1$
$\therefore\ r^2 = 2$
$\Rightarrow\ r = \sqrt{2}$
and $\tan\theta = \dfrac{r\sin\theta}{r\cos\theta} = -1$
$\Rightarrow\ \theta = -\dfrac{\pi}{4}$
$\therefore\ 1 - i = \sqrt{2}\left(\cos\dfrac{\pi}{4} - i\sin\dfrac{\pi}{4}\right)$
$= \sqrt{2}\,e^{-i\pi/4}$

11. (B) Given that
$z = (x, y)$
i.e., $z = x + iy$
$\therefore\ \bar{z} = x - iy$
and $z\bar{z} = z$
$\Rightarrow\ \bar{z} = 1$

12. (D)
$\left(\dfrac{\cos\theta + i\sin\theta}{\cos\theta - i\sin\theta}\right)^4$
$= (\cos\theta + i\sin\theta)^4.(\cos\theta - i\sin\theta)^{-4}$
$= (\cos\theta + i\sin\theta)^4.(\cos\theta + i\sin\theta)^4$
$= (\cos\theta + i\sin\theta)^8$
$= \cos 8\theta + i\sin 8\theta.$

13. (C) $(\sin\theta + i\sin\theta)^6$
$= \left[\cos\left(\dfrac{\pi}{2} - \theta\right) + i\sin\left(\dfrac{\pi}{2} - \theta\right)\right]^6$
$= \cos 6\left(\dfrac{\pi}{2} - \theta\right) + i\sin 6\left(\dfrac{\pi}{2} - \theta\right)$
$= -\cos 6\theta + i\sin 6\theta$

14. (C) $(x - y)(x - wy)(x - w^2y)$
$= x^3 - (1 + w + w^2)x^2y + (1 + w + w^2)xy^2 - w^3y^3$
$= x^3 - (0)x^2y + (0)xy^2 - (1)y^3$
$= x^3 - y^3$ $\quad [\because 1 + w + w^2 = 0$ and $w^3 = 1]$

15. (D) The reciprocal of $a + ib$ is $\dfrac{1}{a + ib}$
$\therefore\ \dfrac{1}{a + ib} = \dfrac{1}{a + ib} \times \dfrac{a - ib}{a - ib}$
$= \dfrac{a - ib}{a^2 + b^2}$
$= \dfrac{a}{a^2 + b^2} - \dfrac{ib}{a^2 + b^2}$

16. (C)

17. (C) Let $\alpha = z_2 - z_3$, $\beta = z_3 - z_1$, $\gamma = z_1 - z_2$
then, $\alpha + \beta + \gamma = 0$ ...(1)
Since ABC is an equilateral triangle.
$\therefore$ AB = BC = CA
$\Rightarrow\ |z_2 - z_3|^2 = |z_3 - z_1|^2 = |z_1 - z_2|^2$
$\Rightarrow\ \alpha\bar{\alpha} = \beta\bar{\beta} = \gamma\bar{\gamma} = K$ $\quad [\because |z|^2 = z\bar{z}]$
From (1)
$\bar{\alpha} + \bar{\beta} + \bar{\gamma} = 0$
$\dfrac{K}{\alpha} + \dfrac{K}{\beta} + \dfrac{K}{\gamma} = 0$
or, $\dfrac{1}{z_2 - z_3} + \dfrac{1}{z_3 - z_1} + \dfrac{1}{z_1 - z_2} = 0$
or, $\dfrac{1}{z_1 - z_2} = \dfrac{1}{z_1 - z_3} + \dfrac{1}{z_3 - z_2}$

18. (D) $(x - 1)^3 + 8 = 0$
$\Rightarrow\ (x - 1)^3 = -8$
$\therefore\ x - 1 = (-8)^{1/3}$
$= -2, -2w, -2w^2$
Hence $x = -1, 1 - 2w, 1 - 2w^2$

19. (C) Let
$\sqrt{3 + 4i} = x + i$
Then $3 + 4i = x^2 - y^2 + 2ixy$
$\therefore\ x^2 - y^2 = 3$ and $xy = 2$
Solving these, we get
$x = \pm 2, y = \pm 1$

Thus, we have $x = 2, y = 1$ or $x = -2, y = -1$

$\therefore \quad \sqrt{3+4i} = 2 + i$

20. (A) Given that

$$(a + ib)^{1/3} = x + iy$$

$$\Rightarrow \quad (a + ib) = (x + iy)^3$$

$$\therefore \quad a + ib = (x^3 - 3xy^2) + i(3x^2y - y^3)$$

Comparing real and imaginary parts, we get

$$\therefore \quad x^3 - 3xy^2 = a;$$

$$3x^2y - y^3 = b$$

or, $$x(x^2 - 3y^2) = a,$$

$$y(3x^2 - y^2) = b$$

or, $$x^2 - 3y^2 = \frac{a}{x},$$

$$3x^2 - y^2 = \frac{b}{y}$$

$$\therefore \quad x^2 - 3y^2 + 3x^2 - y^2 = \frac{a}{x} + \frac{b}{y}$$

or $$4(x^2 - y^2) = \frac{a}{x} + \frac{b}{y}$$

21. (B) $(1 - w + w^2)^5 + (1 + w - w^2)^5$

$$= (-w - w)^5 + (-w^2 - w^2)^5$$

$$= -32w^5 - 32w^{10} = -32w^2 - 32w$$

$$= -32(w + w^2) = -32.(-1)$$

$$= 32$$

22. (B) $z_1 = (1, 2), z_2 = (4, -2), z_3 = (1, -6)$

Here distance between z_1 and z_2

= distance between z_2 and $z_3 = 5$

But distance between z_1 and $z_3 = 8$

Hence, z_1, z_2 and z_3 forms a isosceles triangle.

23. (D) We know that

$$\log_a a = 1$$

$$\therefore \quad \log_i i = 1$$

24. (B) $$z = \left(\frac{1}{1-2i} + \frac{3}{1+i}\right)\left(\frac{3+4i}{2-4i}\right)$$

$$= \frac{(4-5i)}{(1-2i)(1+i)} \times \left(\frac{3+4i}{2-4i}\right)$$

$$= \frac{32+i}{2(1-7i)}$$

$$= \frac{32+i}{2(1-7i)} \times \left(\frac{1+7i}{1+7i}\right)$$

$$= \frac{25+225i}{100} = \frac{1}{4} + \frac{9}{4}i$$

$$\therefore \quad \text{Amp } z = \tan^{-1}\frac{\frac{9}{4}}{\frac{1}{4}} = \tan^{-1} 9$$

25. (A) $$(1+i)^n = \left[\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^n$$

$$= 2^{n/2}\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right)$$

$\therefore$ Real part of $(1+i)^n = 2^{n/2}\cos\frac{n\pi}{4}$

26. (D)

27. (B) Let $z = x + iy$

$$\therefore \quad \frac{z-1}{z+1} = \frac{(x-1)+iy}{(x+1)+iy}$$

$$\Rightarrow \frac{(x-1)+iy}{(x+1)+iy} \times \frac{(x+1)-iy}{(x+1)-iy}$$

$$= \frac{(x^2+y^2-1)+2iy}{(x+1)^2+y^2}$$

$$\therefore \quad \text{Arg}\left(\frac{z-1}{z+1}\right) = \frac{\pi}{3}$$

$$\Rightarrow \tan^{-1}\frac{2y}{x^2+y^2-1} = \frac{\pi}{3}$$

$$\Rightarrow \frac{2y}{x^2+y^2-1} = \tan\frac{\pi}{3} = \sqrt{3}$$

$$\Rightarrow x^2 + y^2 - \frac{2}{\sqrt{3}}y - 1 = 0$$

which is a circle.

28. (D) $$xyz = (a+b)(aw+bw^2)(aw^2+bw)$$

$$= [a^2w + b^2w^2 + ab(w+w^2)](aw^2+bw)$$

$$= [a^2w + b^2w^2 - ab](aw^2+bw)$$

$$= a^3w^3 + b^3w^3$$

$$= a^3 + b^3 \qquad [\because w^3 = 1]$$

29. (B) 30. (A)

31. (A) Let $z = x + iy$

$$\Rightarrow \quad \bar{z} = x - iy$$

$\therefore \arg(z) + \arg(\bar{z})$

$$= \tan^{-1}\left(\frac{y}{x}\right) + \tan^{-1}\left(\frac{-y}{x}\right)$$

$$= \tan^{-1}\left(\frac{y}{x}\right) - \tan^{-1}\left(\frac{y}{x}\right) = 0$$

32. (C) $z^2 = \bar{z}$

$\Rightarrow (x+iy)^2 = x - iy$

$\Rightarrow x^2 - y^2 + 2ixy = x - iy$

Comparing real and imaginary parts of both side, we get

$x^2 - y^2 = x$ ...(i)

$2xy = -y$ ...(ii)

From (ii) $y = 0$ and $x = -\frac{1}{2}$

$\therefore$ From (i), when $y = 0, x = \pm 1$, when $x = \frac{-1}{2}$, $y = \pm\frac{\sqrt{3}}{2}$

$\therefore$ (0, 1), (0, – 1), $\left(\frac{-1}{2}, \frac{\sqrt{3}}{2}\right)$ and $\left(\frac{-1}{2}, -\frac{\sqrt{3}}{2}\right)$ satisfy the *eg.*

(i) and (ii)

$\therefore$ There are four solutions.

33. (A) $\frac{4+3i}{3-4i} = x + iy$

$\therefore 4 + 3i = 3x + 4y + (3y - 4x)i$

$\therefore 3x + 4y = 4$ ...(i)

and $3y - 4x = 3$ ...(ii)

Solving (i) and (ii), we get

$x = 0, y = 1$

34. (D) Given that, $|z - 1| = 2$

$\therefore |z-1|^2 = 4$

$\therefore (z-1)(\bar{z}-1) = 4$

$\Rightarrow z\bar{z} - z - \bar{z} + 1 = 4$

$\Rightarrow z\bar{z} - \bar{z} - z = 3$

35. (D)

36. (B) Sol. $|z| = |z-1|$

$|x+iy| = |(x-1)+iy|$

$\Rightarrow x^2 + y^2 = (x-1)^2 + y^2$

$\Rightarrow 2x = 1$

$\Rightarrow x = \frac{1}{2}$

$\Rightarrow$ Real of $z = \frac{1}{2}$

37. (D) $\left|\frac{z-5i}{z+5i}\right| = 1$

$\Rightarrow |z-5i| = |z+5i|$

$\Rightarrow |x+(y-5)i| = |x+(y+5)i|$

$\Rightarrow x^2 + (y-5)^2 = x^2 + (y+5)^2$

$\Rightarrow y = 0$

38. (B) $2\cos\alpha_1 = a + \frac{1}{a}$

$\Rightarrow a^2 - 2a\cos\alpha_1 + 1 = 0$

$$\Rightarrow a = \frac{2\cos\alpha_1 \pm \sqrt{4\cos^2\alpha_1 - 4}}{2} = \cos\alpha_1 \pm i\sin\alpha_1$$

Similarly from $2\cos\alpha_2 = b + \frac{1}{b}$

$b = \cos\alpha_2 \pm i\sin\alpha_2$ etc.

$abc\ldots\ldots + \frac{1}{abc\ldots}$

$$= (\cos\alpha_1 + i\sin\alpha_1)(\cos\alpha_2 + i\sin\alpha_2)\ldots\ldots + \frac{1}{(\cos\alpha_1 + i\sin\alpha_1)(\cos\alpha_2 + i\sin\alpha_2)}$$

$$= \{\cos(\alpha_1 + \alpha_2 + \ldots) + i\sin(\alpha_1 + \alpha_2 + \ldots)\} + \frac{1}{\cos(\alpha_1 + \alpha_2 + \ldots) + i\sin(\alpha_1 + \alpha_2 + \ldots)}$$

$$= \{\cos(\alpha_1 + \alpha_2 + \ldots) + i\sin(\alpha_1 + \alpha_2 + \ldots)\} + \{\cos(\alpha_1 + \alpha_2 + \ldots) - i(\sin(\alpha_1 + \alpha_2 + \ldots)$$

$= 2\cos(\alpha_1 + \alpha_2 + \ldots)$

39. (A) $x^2 - 2x + 4 = 0$

$$\Rightarrow x = \frac{2 \pm \sqrt{4-16}}{2} = 2\left(\frac{1}{2} \pm \frac{\sqrt{3}}{2}i\right) = 2\left(\cos\frac{\pi}{3} \pm i\sin\frac{\pi}{3}\right)$$

Let $\alpha = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

and $\beta = 2\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$

$$\therefore \alpha^n + \beta^n = 2^n\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^n + 2^n\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^n$$

$$= 2^n\left[\left(\cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3}\right) + \left(\cos\frac{n\pi}{3} - i\sin\frac{n\pi}{3}\right)\right]$$

40. (B) $z = x + iy$

$$\therefore \frac{2z+1}{iz+1} = \frac{(2x+1)+2iy}{i(x+iy)+1}$$
$$= \frac{(2x+1)+2iy}{(-y+1)+ix}$$
$$= \frac{(2x+1)+2iy}{(-y+1)+ix} \times \frac{(-y+1)-ix}{(-y+1)+ix}$$
$$= \frac{(2x-y+1)-i(2x^2+2y^2+x-2y)}{(-y+1)^2+x^2}$$

Given that $\text{Im}\left(\frac{2z+1}{iz+1}\right) = -2$

$$\Rightarrow \frac{-(2x^2+2y^2+x-2y)}{x^2+y^2-2y+1} = -2$$

$\Rightarrow x + 2y = 2$, which is a straight line.

41. (A) $|z+5|^2 + |z-5|^2 = 75$

$\Rightarrow |(x+5)+iy|^2 + |(x-5)+iy|^2 = 75$

$\Rightarrow \quad 2x^2 + 2y^2 = 25$

which is a circle.

42. (D) Let $\cos^{-1}(\cos\theta + i\sin\theta) = x + iy$

or $\cos\theta + i\sin\theta$
$$= \cos(x+iy)$$
$$= \cos x . \cos iy - \sin x \sin yi$$
$$= \cos x \cosh y - \sin x \sinh y$$

Equating real and imaginary parts on both sides, we get

$$\cos\theta = \cos x \cosh y \qquad \ldots(i)$$
$$\sin\theta = -\sin x \sinh y \qquad \ldots(ii)$$

From (i) and (ii), we get

$$\frac{\cos^2\theta}{\cos^2 x} - \frac{\sin^2\theta}{\sin^2 x} = \cosh^2 y - \sinh^2 y = 1$$

or, $\cos^2\theta \sin^2 x - \sin^2\theta \cos^2 x = \cos^2 x \sin^2 x$

or, $\cos^2\theta \sin^2 x - \sin^2\theta (1 - \sin^2 x)$
$$= (1 - \sin^2 x)\sin^2 x$$

or, $\sin^4 x + \sin^2 x(\cos^2\theta + \sin^2\theta - 1) - \sin^2\theta = 0$

or, $\sin^4 x - \sin^2\theta = 0$

or, $\sin^2 x = \sin\theta$,

or $\sin x = \sqrt{\sin\theta}$

or, $x = \sin^{-1}\sqrt{\sin\theta}$

From (ii) $\sin\theta = -\sin x \sinh y$

or, $\sinh y = -\sqrt{\sin\theta}$

or, $y = \sinh[-\sqrt{\sin\theta}]$
$$= \log_e[-\sqrt{\sin\theta} + \sqrt{(\sin\theta + 1)}]$$

Hence, $\cos^{-1}(\cos\theta + i\sin\theta)$
$$= x + iy$$
$$= \sin^{-1}(\sqrt{\sin\theta}) + i\log_e[-\sqrt{\sin\theta} + \sqrt{\sin\theta + 1}]$$

43. (B)	44. (A)	45. (C)	46. (B)	47. (C)
48. (A)	49. (B)	50. (A)	51. (A)	52. (A)
53. (B)	54. (A)	55. (C)	56. (A)	57. (B)
58. (B)	59. (C)			

60. (B) Here $w = \frac{z+2}{z+3}$

$$\Rightarrow \quad z = \frac{2-3w}{w-1}$$
$$\therefore \quad T_1^{-1}(w) = \frac{2-3w}{w-1}$$

61. (A) Here $a_n = \frac{1}{n^p}$

and $a_{n+1} = \frac{1}{(n+1)^p}$

$\therefore$ Radius of convergence,

$$R = \lim_{n\to\infty} \frac{a_n}{a_{n+1}}$$
$$= \lim_{n\to\infty} \frac{(n+1)^p}{n^p} = 1$$

62. (C) Let $f(z) = \frac{\sin z}{(z-\pi)^2}$

Then singularities of $f(z)$ are given by

$$(z-\pi)^2 = 0$$

$\Rightarrow z = \pi$ is a pole of order two of $f(z)$.

63. (A)	64. (A)	65. (A)	66. (B)	67. (B)
68. (A)	69. (C)	70. (C)	71. (B)	72. (C)
73. (A)	74. (A)	75. (D)	76. (C)	77. (A)
78. (A)	79. (C)	80. (C)	81. (A)	82. (A)
83. (B)				

84. (C) $T_2T_1(z) = T_2\left(\frac{z+2}{z+3}\right)$

$$= \frac{\frac{z+2}{z+3}}{\frac{z+2}{z+3}+1} = \frac{z+2}{2z+5}$$

85. (A) Here $a_n = \frac{n+1}{(n+2)(n+3)}$

and $a_{n+1} = \frac{n+2}{(n+3)(n+4)}$

∴ The radius of convergence,

$$R = \lim_{n\to\infty} \frac{a_n}{a_{n+1}}$$

$$= \lim_{n\to\infty} \frac{n+1}{(n+2)(n+3)} \cdot \frac{(n+3)(n+4)}{(n+2)}$$

$$= \lim_{n\to\infty} \frac{(n+1)(n+4)}{(n+2)^2} = 1$$

86. (D) Here $f(z) = \dfrac{1-e^z}{1+e^z}$

Poles of $f(z)$ are obtained by equating to zero the denominator of $f(z)$. *i.e.*

$$1 + e^z = 0$$

$$\Rightarrow \quad e^z = -1 = e^{2\pi n i + \pi i}$$

$$\Sigma = (2n+1)\pi i,$$

where n is any integer

Hence $z = (2n + 1)\pi i$ ($n \in$ I) are the simple poles of $f(z)$.

Obviously $z = \infty$ is a limit point of these poles

∴ $z = \infty$ is a non-isolated essential singularity.

87. (C) 88. (B) 89. (B) 90. (D) 91. (A)
92. (A) 93. (A) 94. (B) 95. (B) 96. (C)
97. (B) 98. (A) 99. (B) 100. (C) 101. (C)
102. (B) 103. (C)

104. (B) $T_2T_1(z) = T_2^{-1}\left(\dfrac{z+2}{z+3}\right)$

$$= \frac{\dfrac{z+2}{z+3}}{\dfrac{z+2}{z+3} - 1} = z + 2$$

105. (A) $T_2T_1(z) = T_1\left(\dfrac{z}{z+1}\right)$

$$= \frac{\dfrac{z}{z+1} + 2}{\dfrac{z}{z+1} + 3} = \frac{3z+2}{4z+3}$$

106. (B) We have

$$a_n = \frac{1}{n!},\ a_{n+1} = \frac{1}{(n+1)!}$$

$$R = \lim_{n\to\infty} \frac{a_n}{a_{n+1}}$$

$$= \lim_{n\to\infty} \frac{(n+1)!}{n!}$$

$$= \lim_{n\to\infty} (n+1) = \infty$$

107. (A) We have

$$a_n = \frac{n!}{n^n}\ a_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$$

$$R = \lim_{n\to\infty} \frac{a_n}{a_{n+1}}$$

$$= \lim_{n\to\infty} \frac{n!}{n^n} \cdot \frac{(n+1)^{n+1}}{(n+1)!}$$

$$= \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$$

108. (A) 109. (A) 110. (B) 111. (C) 112. (A)
113. (B) 114. (A) 115. (A) 116. (A) 117. (B)
118. (C) 119. (C) 120. (A) 121. (A) 122. (B)
123. (B)

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