Chinmay Damle's

MATHEMATICS & GUIDANCE ACADEMY

Understanding Calculus (Limits, Derivatives and Integration)

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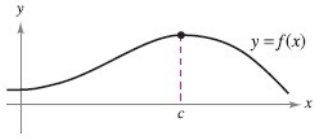
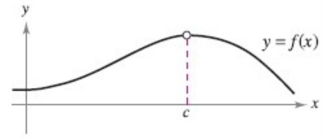
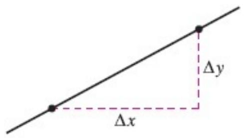
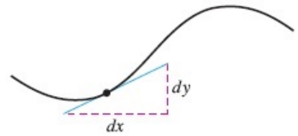
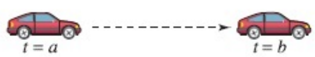
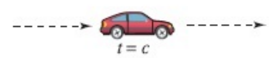
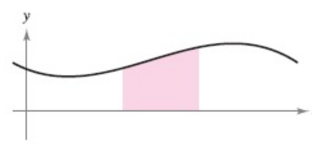
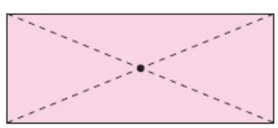
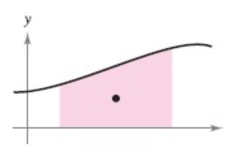
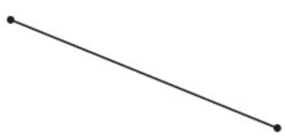
1 Calculus

Calculus is the mathematics of change. For instance, calculus is the mathematics of velocities, acceleration, tangent lines, slopes, areas, volumes, arc lengths, centroids, curvatures and a variety of concepts that have enabled scientists and engineers, and economists to model real life situation. Calculus is introduced to everyone in higher school mathematics.

Although Pre-calculus mathematics which is introduced in schools also deals with velocities, accleration, slopes, etc. Per-calculus mathematics is more static, whereas calculus is more dynamic. One way to answer " What is Calculus?" is to say that calculus is a "limit machine" that involves three stages. The first stage is pre-calculus mathematics, such as the slope of a line or area of rectangle. The second stage is the limiting process and the third stage is a new calculus formulation such as derivative or integral.

The concepts we take for granted today were not accepted by the historical mathematicians. The key ingredient missing in mathematical antiquity was the hairy notion of infinity. Mathematicians and philosophers of the time had an extremely hard time conceptualizing infinitely small or large quantities. Sir Issac Newton and Gottfried Wilhelm Leibniz were the first to individually introduce and develop the concept of Calculus. The development of calculus was built on earlier concepts of instantaneous motion and area under the curve. Although the concept similar to calculus has a long history dating back to 2^{nd} century BC. In modern world calculus is divided as Differential calculus (derivatives) and Integral calculus (integration).

Without Calculus		With Differential Calculus	
Value of $f(x)$ when $x = c$		Limit of $f(x)$ as x approaches c	
Slope of a line		Slope of a curve	
Secant line to a curve		Tangent line to a curve	
Average rate of change between $t = a$ and $t = b$		Instantaneous rate of change at $t = c$	

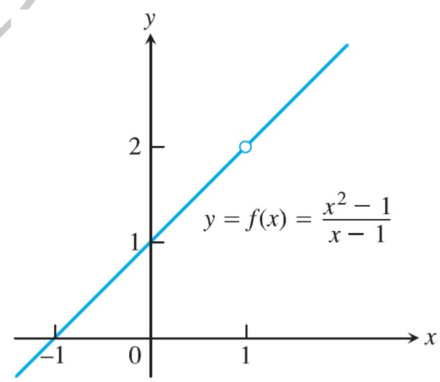
Without Calculus		With Integral Calculus	
Area of a rectangle		Area under a curve	
Work done by a constant force		Work done by a variable force	
Center of a rectangle		Centroid of a region	
Length of a line segment		Length of an arc	

2 Limits

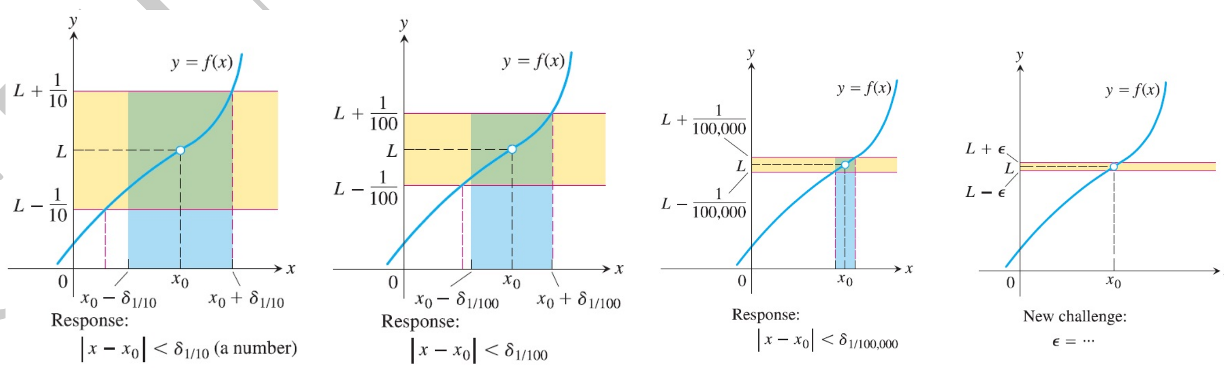
Limit in literal meaning is defined as 'The greatest possible degree of something'.

Let's investigate the behaviour of the function f defined by $f(x) = \frac{x^2-1}{x-1}$ for value of x near 1. The following table gives values of $f(x)$ for values of x close to 1 but not equal to 1.

x, $(x < 1)$	$f(x)$	x, $(x > 1)$	$f(x)$
-3	-2	1.00001	2.00001
-2	-1	1.0001	2.0001
-1	0	1.001	2.001
0	1	1.01	2.01
0.5	1.5	1	2
0.9	1.9	1.5	2.5
0.99	1.99	2	3
0.999	1.999	2.5	3.5
0.9999	1.9999	3	4



Limit of a function is denoted as $\lim_{x \to a} f(x) = L$ and said as the limit of $f(x)$, as x approaches a , equals L.



Limit is mathematically defined as, for any chosen positive integer ϵ, however small, there exists a positive number δ such that, whenever $0 < |x-a| < \delta$, then $|f(x) - L| < \epsilon$.

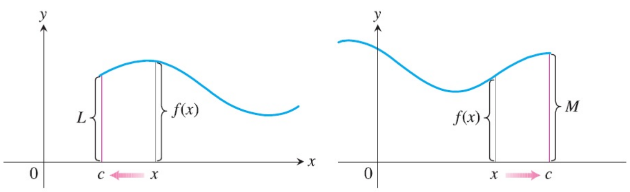
The laws of Limit: If $L, M, k \ and \ c$ are real numbers and $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$ then

1. $\lim_{x\to c}[f(x) \pm g(x)] = L \pm M$
2. $\lim_{x\to c}[k \cdot f(x)] = k \cdot L$
3. $\lim_{x\to c}[f(x) \cdot g(x)] = L \cdot M$
4. $\lim_{x\to c} \frac{f(x)}{g(x)} = L/M, \quad M \neq 0$
5. $\lim_{x\to c}[f(x)]^n = L^n, \quad n$ a positive integer.
6. $\lim_{x\to c}[f(x)]^{(1/n)} = L^{(1/n),} \quad n$ a positive integer.
7. $\lim_{x\to c} P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = P(c) = a_n c^n + a_{n-1}c^{n-1} + \cdots + a_1 c + a_0$
8. If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then $\lim_{x\to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}$
   [JAHAAN JAHAAN x WAHAAN WAHAAN c]

Suppose $Q(c) = 0$, then $(x - c)$ is a factor of $Q(x)$. Thus, if the numerator and denominator of a rational function of x are both zero at $x = c$,they have $(x - c)$ as a common factor common factor.

[While solving Limits first always look at the denominator and make sure it is not getting zero by cancelling out the common factor.]

Two sided Limits: To have a limit L as x approaches c, a function f must be defined on both sides of c and its values $f(x)$ must approach L as x approaches c from either side. Because of this, ordinary limits are called two-sided.



(a) $\lim_{x\to c^+} f(x) = L$ (b) $\lim_{x\to c^-} f(x) = M$

If f fails to have a two-sided limit at c, it may still have a one-sided limit, that is, a limit if the approach is only from one side. If the approach is from the right, the limit is a right-hand limit. From the left, it is a left-hand limit.

A function f(x) has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal.

$\lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x) = \lim_{x\to c} f(x) = L$

$\lim_{x\to a} K = K$	$\lim_{x\to a} x^n = a^n, n \in \mathbb{R}$
$\lim_{x\to a} \frac{x^n - a^n}{x-a} = na^{n-1}, n \in \mathbb{Q}, a > 0$	$\lim_{x\to 0} \frac{(1+x)^n - 1}{x} = n$
$\lim_{x\to 0} \frac{\sin x}{x} = 1 = \lim_{x\to 0} \frac{x}{\sin x} = \lim_{x\to 0} \frac{\tan x}{x}$	$\lim_{x\to 0} \frac{\sin^{-1} x}{x} = 1 = \lim_{x\to 0} \frac{\tan^{-1} x}{x}$
$\lim_{x\to 0} \frac{a^x - 1}{x} = \log_e a, a > 0$	$\lim_{x\to 0} \frac{e^x - 1}{x} = \log_e e = 1$
$\lim_{x\to 0} (1+x)^{1/x} = e$	$\lim_{x\to 0} \frac{\log(1+x)}{x} = 1$
$\lim_{x\to 0} \frac{e^x - 1}{x} = 1$	$\lim_{x\to\infty} (1 + 1/x)^x = e$
$\lim_{x\to\infty} (1 + a/x)^x = e^a$	$\lim_{x\to\infty} e^{f(x)} = e^{\lim_{x\to a} f(x)}$

Examples:

1. Evaluate $\lim_{x\to 1} x^2 + 6x - 1$

   Solution: Let $L = \lim_{x\to 1} x^2 + 6x - 1 = 1^2 + 6(1) - 1 = 6$

2. Evaluate : $\lim_{x\to 1} \frac{x^2 + x - 2}{x^2 - x}$

Solution: (Whenever we are going to solve the problem of Limits always first look at the denominator, assure that it is not getting zero. If it is getting zero modify it. That is cancel the cancellation factor which is present in both numerator and denominator, here $(x-1)$ is the factor)

Let $L = \lim_{x\to 1} \frac{x^2+x-2}{x^2-x} = \lim_{x\to 1} \frac{(x+2)(x-1)}{x(x-1)} = \lim_{x\to 1} \frac{x+2}{x} = \frac{1+2}{1} = 3$

3. $\lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2}$

   Solution: (Tip: Whenever we see square root in the problem first thought that should come to your mind is to take the conjugate.)

   Let $L = \lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2}$

   $L = \lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2} \cdot \frac{\sqrt{t^2+9}+3}{\sqrt{t^2+9}+3} = \lim_{t\to 0} \frac{(t^2+9)-9}{t^2\sqrt{t^2+9}+3} = \lim_{t\to 0} \frac{t^2}{t^2\sqrt{t^2+9}+3} = \lim_{t\to 0} \frac{1}{\sqrt{t^2+9}+3}$

   $. \quad = \frac{1}{\lim_{t\to 0} \sqrt{t^2+9}+3} = \frac{1}{\sqrt{0+9}+3} = \frac{1}{6}$

4. Prove that $\lim_{x\to 0} \frac{|x|}{x}$ does not exist.

   Solution: Right hand Limit (RHL): $\lim_{x\to 0^+} \frac{|x|}{x} = \lim_{x\to 0} \frac{x}{x} = 1$

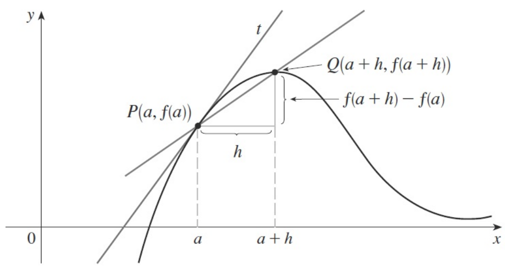
   Left hand Limit (LHL): $\lim_{x\to 0^-} \frac{|x|}{x} = \lim_{x\to 0} \frac{-x}{x} = -1$

   Here, $LHL \neq RHL$ so the limit does not exist.

3 Differentiation(derivatives or rate)

A derivative describes change that is dependent on a variable, such as the change of temperature in a month or the change in price of an item based on the quantity sold. The concept of rate is studied throughout mathematics in different forms. Rates can also be used to describe changes in an environment or physical setting. For example, two hundred additional employees are needed for every 8 per- cent increase in demand for the companys products. When driving along a mountain terrain, a road sign that mentions a 5 percent grade means that there is a change in elevation of five vertical feet for every one hundred horizontal feet. Many scientific, engineering, and human measures are rates. Density is a weight-per-volume measure such as pounds per cubic foot or grams per cubic centimeter. Sound frequencies, such as those associated with musical notes, are expressed as rates in cycles per second. Air pressure, such as tire pressure, is expressed as pounds per square inch. Rate, in mathematics courses through algebra, is often presented as having a constant value. When you read about the speed of an object or a persons work wages, it is assumed that there will not be any change in these values. In such cases, the rate can be represented as the slope of a linear function that describes a total amount.

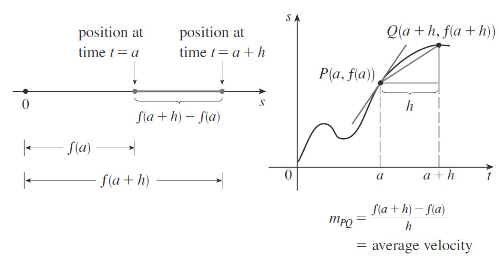
Realistically, rates are often variable, meaning that they change. A car on the highway will not always travel 55 miles per hour because of varying road conditions. If traffic is heavy due to rush hour or an accident, the car will likely slow down at times. Therefore the average rate is sometimes stated in reports. The average (mean) rate can be calculated by finding the slope between beginning and ending points on the graph that represents a total amount. For example, if a car is traveling at a constant speed of 55 miles per hour, then the total distance travelled as a function of time would be a linear function with a slope of 55. However, if the car varies its speed, the total distance function will now look like a curve that does not have a constant slope. If a car travels for three hours on the highway, the average speed can be determined by finding the slope of the line that time equals 0 and 3 hours. According to the slope between the endpoints in the graph in the figure below, the average speed during the three hours is 49 miles per hour, since the change in distance was 147 miles over three hours.



The Tangent Line to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with the slope $m = \lim\limits_{x \to a} \frac{f(x)-f(a)}{x-a}$ provided the limit exists.

In general, suppose an object moves along a straight line according to an equation of motion $s = f(t)$, where s is the displacement (directed distance) of the object from the origin at time t . The function f that describes the motion is called the position function of the object. In the time interval from $t = a$ to $t = a + h$ the change in position is $f(a + h) - f(a)$ The average velocity over this time interval is $average\ velocity = \frac{displacement}{time} = \frac{f(a+h)-f(a)}{h}$.

Now suppose we compute the average velocities over shorter and shorter time intervals [a,a+h]. In other words, we let h approach 0. As in the example of the falling ball, we define the velocity (or instantaneous velocity) $v(a)$ at time $t = a$ to be the limit of these average velocities: $v(a) = \lim\limits_{h \to 0} \frac{f(a+h)=f(a)}{h}$. This means that the velocity at time $t = a$ is equal to the slope of the tangent line at P.



Calculating the derivatives:

The derivative of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

.
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided the limit exists.

A function $y = f(x)$ is differentiable on an open interval (finite or infinite) if it has a derivative at each point of the interval. It is differentiable on a closed interval $[a, b]$ if it is differentiable on the interior (a, b) and if the limits

Right hand Derivative at a: $\lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$

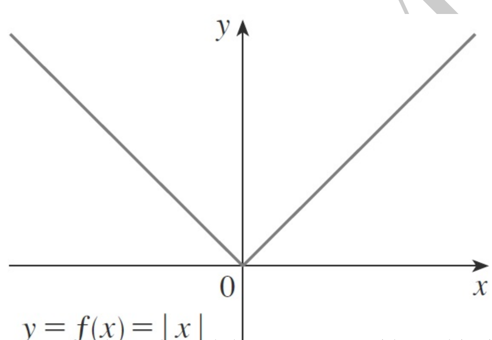
Left hand Derivative at a: $\lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h}$

exists at the end points.

i.e The derivative exist if and only if

$f'(x) =$ Left hand derivative $=$ Right hand derivative

Example: The function $y = |x|$ is differentiable on $(-\infty, 0)$ and $(0, \infty)$ but has no derivative at $x = 0$.



Solution: If $x > 0$ then $|x| = x$ and we can choose h small enough that $x + h > 0$ and hence $|x+h| = x+h$. Similarly for $x < 0$ then $|x| = -x$ and we can choose h small enough that $x + h < 0$ and hence $|x + h| = -(x + h)$.

Right hand derivative :

$$\lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0^+} \frac{|(x+h)| - |(x)|}{h} = \lim_{h \to 0} \frac{(x+h) - (x)}{h} = \lim_{h \to 0} \frac{h}{h} = 1$$

Left hand derivative :

$$\lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0^-} \frac{|(x+h)| - |(x)|}{h} = \lim_{h \to 0} \frac{-(x+h) - [-(x)]}{h} = \lim_{h \to 0} \frac{-h}{h} = -1$$

If a graph contains a sharp point (also known as a cusp), then the function has no derivative at that point.

Rules of Differentiation

1. If $y = f(x) \pm g(x)$ then $dy/dx = f'(x) \pm g'(x)$

   If $y = cf(x)$ then $dy/dx = cf'(x)$

2. If $y = f(x)g(x)$ then $dy/dx = f(x)g'(x) + f'(x)g(x)$

3. If $y = f(x)/g(x)$ then $dy/dx = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$

4. Composite Functions / Chain Rule.

   If $y = f(u)$ and $u = g(x)$ then $dy/dx = dy/du \times du/dx$

5. Parametric form.

   If $y = f(\theta)$ and $x = g(\theta)$ then $dy/dx = \frac{dy/d\theta}{dx/d\theta}$

6. Chain Rule:

   Let f be a real valued function which is a composite functions u and v; i.e., $f = v\ o\ u$. Suppose $t = u(x)$ and if both $\frac{dt}{dx}$ and $\frac{dv}{dt}$ exist, we have $\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx}$.

$y = f(x)$	$dy/dx = f'(x)$	$y = f(x)$	$dy/dx = f'(x)$
x^n	nx^{n-1}	e^x	e^x
a^x	$a^x \log a$	$c(constant)$	0
$\log_e x$	$1/x$	$\sqrt{x}$	$1/2\sqrt{x}$
$\sin x$	$\cos x$	$\cos x$	$-\sin x$
$\csc x$	$-\csc x \cot x$	$\sec x$	$\sec x \tan x$
$\tan x$	$\sec^2 x$	$\cot x$	$-\csc^2 x$
$\sin^{-1} x/a$	$1/\sqrt{a^2 - x^2}$	$\cos^{-1} x/a$	$-1/\sqrt{a^2 - x^2}$
$\tan^{-1} x/a$	$a/a^2 + x^2$	$\cot^{-a} x/a$	$-1/a^2 + x^2$
$\csc^{-1} x/a$	$-a/x\sqrt{x^2 - a^2}$	$\sec^{-1} x/a$	$a/x\sqrt{x^2 - a^2}$

For solving the derivatives problem one must learn by heart all the formulas of Trigonometry and derivatives. Every example of derivative is always based on some formulas. Here are few sample examples for understanding the technique of problem solving

Example: Find the derivative of the function $f(x) = e^{4x} \cdot \sin(x^2)$

Solution: Here we have $f_1(x) = u = e^{4x}$ and $f_2(x) = v = \sin(x^2)$

By seeing the functions we must recall the formulas related to

$\frac{d}{dx} \sin x = \cos x$ and $\frac{d}{dx} e^x = e^x$.

Considering, $f_1(x),\ f_2(x)$ we have $4x$ and x^2 instead of x in both the functions so we should calculate the derivatives of these and multiply to the derivatives of e^x and $\sin x$. i.e We are going to apply the chain rule.

$f_1'(x) = \frac{du}{dx} = \frac{d}{dx}[\sin(x^2)] = \frac{d}{dx}[\sin(x^2)] \cdot \frac{d}{dx}[(x^2)] = \cos(x^2) \cdot 2x$

$f_2'(x) = \frac{dv}{dx} = \frac{d}{dx}[e^{4x}] = \frac{d}{dx}[e^{4x}] \cdot \frac{d}{dx}[(4x)] = e^{4x} \cdot 4$

$$\frac{d}{dx} f(x) = \frac{d}{dx}[f_1(x) \cdot f_2(x)] = f_1(x) f_2'(x) + f_1'(x) f_2(x)$$

$$. \qquad = \frac{d}{dx}(u \cdot v) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$. \qquad = e^{4x} \cdot 2x \cos(x^2) + 4e^{4x} \cdot \sin(x^2) = 2xe^{4x} \cos(x^2) + 4e^{4x} \sin(x^2)$$

$$. \qquad = 2e^{4x}[x \cos(x^2) + 2 \sin(x^2)]$$

Example: Find the derivative of $\frac{\sin(e^{2x})}{x^2}$.

Solution: Consider $f_1(x) = u = \sin(e^{2x}),\ f_2(x) = v = x^2$.

$f_1'(x) = \frac{du}{dx} = \cos(e^{2x}) \cdot e^{2x} 2 = 2e^{2x} \cos e^{2x},\ f_2'(x) = \frac{dv}{dx} = 2x.$

$$\frac{d}{dx}\left[\frac{f_1(x)}{f_2(x)}\right] = \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{f_2(x) f_1'(x) - f_1(x) f_2'(x)}{[f_2(x)]^2} = \frac{v \cdot \frac{du}{dx} - u \cdot \frac{dv}{dx}}{v^2}$$

$$\frac{d}{dx}\left[\frac{\sin(e^{2x})}{x^2}\right] = \frac{x^2 \cdot 2e^{2x} \cos(e^{2x}) - \sin(e^{2x}) \cdot 2x}{(x^2)^2} = \frac{2x^2 e^{2x} \cos(e^{2x}) - 2x \sin(e^{2x})}{x^4} = \frac{2x[xe^{2x} \cos(e^{2x}) - \sin(e^{2x})]}{x^4}$$

$$. \qquad = \frac{2[xe^{2x} \cos(e^{2x}) - \sin(e^{2x})]}{x^3}$$

Example: (Implicit function)

When a relationship between x and y is expressed in a way that it is easy to solve for y and write $y = f(x)$, we say that y is given as an explicit function of x.

In the latter case it is implicit that y is a function of x and we say that the relationship of the second type, above, gives function implicitly. i.e y cannot be written in terms of x.

Find the derivative of $x + \sin x = \cos y$.

Solution: $\frac{d}{dx}[x + \sin x = \cos y] \implies \frac{d}{dx}x + \frac{d}{dx}\sin x = \frac{d}{dx}\cos y$

$\implies 1 + \cos x = (-\sin y)\frac{dy}{dx} \implies \frac{dy}{dx} = \frac{1+\cos x}{(-\sin y)}$

Example: (Parametric form)

Sometimes the relation between two variables is neither explicit nor implicit, but some link of a third variable with each of the two variables, separately, establishes a relation between the first two variables. In such a situation, we say that the relation between them is expressed via a third variable. The third variable is called the parameter. More precisely, a relation expressed between two variables x and y in the form $x = f(t), y = g(t)$ is said to be parametric form with t as a parameter. In order to find derivative of function in such form, we have by chain rule $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$ or $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$.

Find dy/dx, if $x = a\cos\theta,\ y = a\sin\theta$.

$\frac{dx}{d\theta} = -a\sin\theta, \frac{dy}{d\theta} = a\cos\theta$

$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-a\cos\theta}{a\sin\theta} = -\cot\theta$.

Here dy/dx is expressed in terms of the parameter only without directly involving the main variables x and y.

Second Order Derivative:

Let $y = f(x)$. Then $\frac{dy}{dx} = f'(x)$. If $f'(x)$ is differentiable further, we may differentiate again w.r.t. x. Then, the left hand side becomes $\frac{d}{dx}\frac{dy}{dx}$ which is called the second order derivative of y w.r.t. x and is denoted by $\frac{d^2y}{dx^2}$. The second order derivative of $f(x)$ is denoted by $f''(x)$. It is also denoted by D^2y or y'' or y_2 if $y = f(x)$.

Example: Find $\frac{d^2y}{dx^2}$, if $y = x^3 + \tan x$.

Solution: $\frac{dy}{dx} = 3x^2 + \sec^2 x$.

$\frac{d}{dx}(\frac{dy}{dx}) = \frac{d^2y}{dx^2} = \frac{d}{dx}(3x^2 + \sec^2 x) = 6x + 2\sec x \cdot \sec x \tan x$

. $= 6x + 2\sec^2 x \tan x$

3.1 Application of Derivatives

Velocity, density, current, power, and temperature gradient in physics, rate of reaction and compressibility in chemistry, rate of growth and blood velocity gradient in biology, marginal cost and marginal profit in economics, rate of heat flow in geology, rate of improvement of performance in psychology, rate of spread of a rumor in sociologythese are all special cases of a single mathematical concept, the derivative. This is an illustration of the fact that part of the power of mathematics lies in its abstractness. A single abstract mathematical concept (such as the derivative) can have different interpretations in each of the sciences. When we develop the properties of the mathematical concept once and for all, we can then turn around and apply these results to all of the sciences. This is much more efficient than developing properties of special concepts in each separate science. The French mathematician Joseph Fourier put it succinctly: Mathematics compares the most diverse phenomena and discovers the secret analogies that unite them.

Increasing and Decreasing functions (First Derivative test) :

Let f be continuous on $[a, b]$ and differentiable on the open interval (a, b). Then

(a) f is strictly increasing in $[a, b]$, if $f'(x) > 0$ for each $x \in (a, b)$.

(b) f is strictly decreasing in $[a, b]$, if $f'(x) < 0$ for each $x \in (a, b)$.

(a) f is strictly constant in $[a, b]$, if $f'(x) = 0$ for each $x \in (a, b)$.

Example: Find the intervals in which the function f given by

$f(x) = 4x^3 - 6x^2 - 72x + 30$ is strictly increasing and strictly decreasing.

Solution: We have $f(x) = 4x^3 - 6x^2 - 72x + 30$.

$f'(x) = 12x^2 - 12x - 72 = 12(x^2 - x - 6) = 12(x-3)(x+2)$.

Therefore, $f'(x) = 0 \implies (x-3) = 0$ or $(x+2) = 0 \implies x = 3$ or $x = -2$..

The points -2 and 3 divides the real line in the intervals

$(-\infty, -2),\ (-2, 3),\ (3, \infty)$.

$f'(-5) = 12(-5-3)(-5+2) = 288 > 0;$

$f'(0) = 12(0-3)(0+2) - 72 < 0;$

$f'(5) = 12(5-3)(5+2) = 168 > 0$

f is increasing for $x > 3$ upto $infinity$ and $x < -2$ upto $minus\ infinity$.

f is decreasing for $x < 3$ and $x > -2$.

Second Derivative Test:

Let f be a function defined on the interval I and $c \in I$. Let f be twice differentiable at c. Then

1. $x = c$ is a point of local maxima if $f'(c) = 0$ and $f''(c) < 0$
   The value $f(c)$ is local maximum value of f.

2. $x = c$ is a point of local minima if $f'(c) = 0$ and $f''(c) > 0$
   In this case, $f(c)$ is local minimum value of f.

3. The test fails if $f'(c) = 0$ and $f''(c) = 0$.

Example: You have been asked to design a one-liter can shaped like a right circular cylinder . What dimensions will use the least material?

Solution: $r = radius$ and $h = height$ is measured in centimeters.

Volume of can = Volume of cylinder = $\pi r^2 h = 1000,\ 1\ liter = 1000cm^3$

Surface Area of can: $A = 2\pi r^2 h + 2\pi rh$ is the material required.

Here we will ignore the thickness of the material .

We need to optimise the function A for that we express it in one variable say r by $h = 1000/\pi r^2$

$A = 2\pi r^2 + 2\pi rh = 2\pi r^2 + 2\pi r(1000/r^2 h) = 2\pi r^2 + 2000/r$

Our goal is to find thevalue of $r > 0$ that minimizes the value of A.

Since A is differentiable on $r > 0$, an interval with no endpoints, it can have a minimum value only where its first derivative is zero.

$\frac{dA}{dr} = 4\pi r - 2000/r^2 = 0 \implies 4\pi r^3 = 2000 \implies r = (\frac{500}{\pi})^{1/3}$

$\implies r \approx 5.42$

The second derivative : $\frac{d^2A}{dr^2} = 4\pi + 4000/r^3$

For the value of r the second derivative is greater than zero. So according to the second derivative test the value of r is $(\frac{500}{\pi})^{1/3}$.

The value of $h = 1000/\pi r^2 = 1000/[\pi\{(\frac{500}{\pi})^{1/3})\}^2] = 2(\frac{500}{\pi})^{1/3} = 2r$

The one-liter can that uses the least material has height equal to twice the radius, here with $r \approx 5.42cm$ and $h \approx 10.84cm$.

Example: The measured radius of the ball bearing is $0.7in$. If the measurement is correct to within $0.01in$, estimate the propagated error in the Volume V of the ball bearing.

Solution: The formula for the volume of the sphere is $V = \frac{4}{3}\pi r^3$, where r is the radius of the sphere.

$r = 0.7in$; possible error (Δr), $-.001 \leq \Delta r \leq 0.01$

To approximate the propagated error in the volume, differentiate V $w.r.tr$ to obtain $dV/dr = 4\pi r^2$

$\Delta V \approx dV = 4\pi r^2\ dr = 4\pi(0.7)^2(\pm 0.01) \approx \pm 0.06158$ *cubic inch*

So the volume has a propagated error of about 0.06 *cubic inch*

The relative error $= \frac{dV}{V} = \frac{4\pi r^2\ dr}{4\pi r^3} = \frac{3\ dr}{r} \approx \frac{3}{0.7}(\pm 0.01) \approx \pm 0.0429$

The corresponding percent error is approximately 4.29%.

Newton-Raphson method is a technique to approximate the solution to an

equation $f(x) = 0$. A value of x where f is zero is a root of the function f and a solution of the equation $f(x) = 0$.
The formula for calculation of the root is $x_{n+1} = x_n - \frac{f(x_n)}{f'(x)}, f'(x) \neq 0$
(It's a iterative process i.e we start with some initial root and then go on to get the approx root of the function)

Example: Find the the approximate root of the equation
$f(x) = x^2 - 2 = 0$. (Solution of the equation: $x = \sqrt{2} = 1.41421356237309$)
Solution: $f(x) = x^2 - 2$; $f'(x) = 2x$. Using the above formula
$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 2}{2x_n} = \frac{x_n}{2} + \frac{1}{x_n}$
We will start here with $x = 1$ for this corresponding function i.e $x_0 = 1$.
$x_1 = \frac{x_0}{2} + \frac{1}{x_0} = 1/2 + 1/1 = 1.5$
$x_2 = \frac{x_1}{2} + \frac{1}{x_1} = 1.5/2 + 1/1.5 = 1.416667$
$x_3 = \frac{x_2}{2} + \frac{1}{x_2} = 1.416667/2 + 1/1.416667 = 1.414216$

	Error	Number of correct digits
$x_0 = 1$	-0.414213	1
$x_1 = 1.5$	0.085786	1
$x_2 = 1.416667$	0.002454	3
$x_3 = 1.414216$	0.000003	5

4 Integration (Anti-derivative) (Summation)

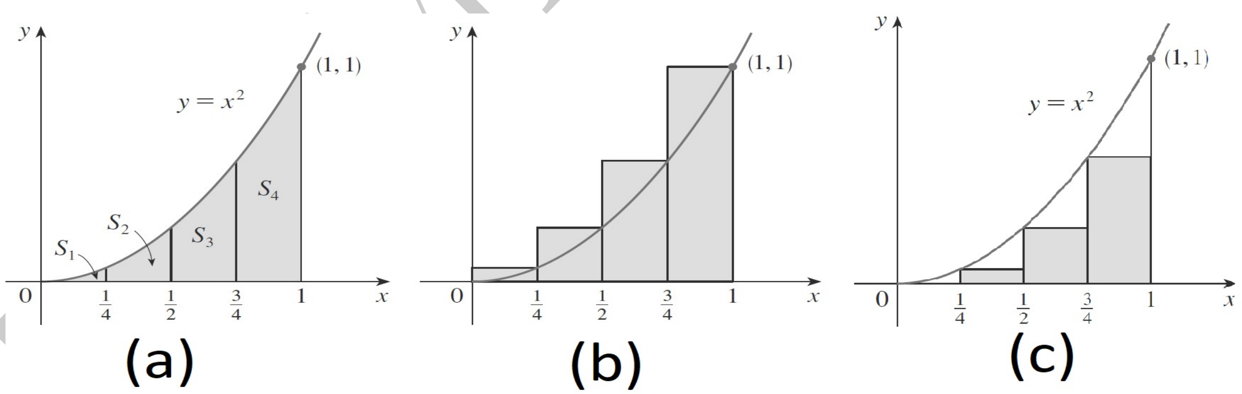
Integration is used to determine a total amount based on a predictable rate pattern, such as a population based on its growth rate, or to represent an accumulation of something such as volume in a tank. It is usually introduced in calculus, but its use and computation can be performed by many calculators or computer programs without taking calculus. Understanding the utility of an integral does not require a background in calculus, but instead a conceptual understanding of rates and area. Many realistic applications of integration that occur in science, engineering, business, and industry cannot be expressed with simple linear functions or geometric formulas. Integration is powerful in such circumstances, because there is not a reliance on constant rates or simple functions to find answers. For example, in many algebra courses, students learn that $distance = rate \times time$. This is true only if the rate of an object always remains the same. In many real-world instances, the rate of an object changes, such as the velocity of an automobile on the road. Cars speed up and slow down according to traffic signals, incidents on the road, and attention to driving. If the velocity of the car can be modelled with a non-linear function, then an integral could help you represent the distance as a function of time, or tell you how far the car has moved from its original position, even if the rate has changed.

A definite integral of a function $f(t)$ is an integral that finds a value based on a set of boundaries. A definite integral can help you determine the total production of textiles based on a specific period of time during the day. For example, suppose a clothes manufacturer recognized that its employees were gradually slowing down as they were sewing clothes, perhaps due to fatigue or boredom. After collecting data on a group of workers, the manufacturer determined that the rate of production of blue jeans, f, can be modelled by the function $f(t) = 6.37e^{-0.04t}$, where t is the number of consecutive hours worked. For the first two hours of work, an expected production amount can be determined by the definite integral, written as $\int_0^2 6.37e^{-0.04t}\ dt$ This information can help managers determine when employees should take breaks so that they can optimize their performance, because they would likely feel more productive when they returned to work.

The definite integral is the key tool in calculus for defining and calculating quantities important to mathematics and science, such as areas, volumes, lengths of curved paths, probabilities, and the weights of various objects, just to mention a few. The idea behind the integral is that we can effectively compute such quantities by breaking them into small pieces and then summing the contributions from each piece. We then consider what happens when more and more, smaller and smaller pieces are taken in the summation process.

Example: Use rectangles to estimate the area under the parabola $y = x^2$ from 0 to 1.

Solution: Let the area under the parabola be A



We first notice that the area of S must be somewhere between 0 and 1 because S is contained in a square with side length 1, but we can certainly do better than that. Suppose we divide S into four strips S_1, S_2, S_3 and S_4

by drawing the vertical lines $x = 1/4, x = 1/2, x = 3/4$ respectively as shown in (a).

We can approximate each strip by a rectangle whose base is the same as the strip and whose height is the same as the right edge of the strip figure (b). In other words, the heights of these rectangles are the values of the function $f(x) = x^2$ at the right end points of the sub-intervals $[0, \frac{1}{4}], [\frac{1}{4}, \frac{1}{2}], [\frac{1}{2}, \frac{3}{4}], [\frac{3}{4}, 1]$. Each rectangle has the width of $\frac{1}{4}$ and height are $(\frac{1}{4})^2, (\frac{1}{2})^2, (\frac{3}{4})^2$ and 1^2.

Summation of the areas of the rectangle be

$A_1 = \frac{1}{4} \cdot (\frac{1}{4})^2 + \frac{1}{4} \cdot (\frac{1}{2})^2 + \frac{1}{4} \cdot (\frac{3}{4})^2 + \frac{1}{4} \cdot (1)^2 = \frac{15}{32} = 0.46875$

Actually the area of the region under the curve $A < \frac{15}{32}$.

Instead of using the rectangles in the figure (b) if we use some smaller rectangles as per figure (c).

$B_1 = \frac{1}{4} \cdot 0^2 + \frac{1}{4} \cdot (\frac{1}{4})^2 + \frac{1}{4} \cdot (\frac{1}{2})^2 + \frac{1}{4} \cdot (\frac{3}{4})^2 = \frac{7}{32} = 0.21875.$

Here the area of the region under the curve $A > \frac{7}{32}$.

$\therefore\ 0.21875 < A < 0.46875$

n	A_n	B_n
10	0.2850000	0.3850000
50	0.3234000	0.3434000
100	0.3283500	0.3383500
1000	0.3328335	0.3338335

Now, if we make n partitions i.e rectangle of the area under the curve and as n approaches infinite we get the exact area under the curve. Let area of the region with n partitions be A_n. here the width of the rectangles is $\frac{1}{n}$ and height is $(\frac{1}{n})^2$.

Now, $A_n = \frac{1}{n} \cdot (\frac{1}{n})^2 + \frac{2}{n} \cdot (\frac{2}{n})^2 + \frac{3}{n} \cdot (\frac{3}{n})^2 + \cdots + \frac{1}{n} \cdot (\frac{n}{n})^2$

. $= \frac{1}{n} \cdot \frac{1}{n^2}[1^2 + 2^2 + 3^2 + \cdots + n^2] = \frac{1}{n^3}[\frac{n(n+1)(2n+1)}{6}] = \frac{(n+1)(2n+1)}{6n^2}$

Area of the region $R = \lim_{n\to\infty} A_n = \lim_{n\to\infty} \frac{(n+1)(2n+1)}{6n^2} = \lim_{n\to\infty} \frac{1}{6}(\frac{n+1}{n})(\frac{2n+1}{n})$

. $= \lim_{n\to\infty} \frac{1}{6}(1 + \frac{1}{n})(2 + \frac{1}{n}) = \frac{1}{6} \cdot 1 \cdot 2 = \frac{1}{3}$

Similarly it can be proved for $B_n = \frac{1}{3}$

Now, if we are going to use the formula $\int x^n \; dx = x^{n+1}/(n+1) + c$ to calculate the integral of the above function $y = f(x) = x^2$. The limits for x are from 0 to 1.

$\int_0^1 x^2 \; dx = [\frac{x^{2+1}}{(2+1)}]_0^1 = [\frac{x^3}{3}]_0^1 = [\frac{1}{3} - \frac{0}{3}] = \frac{1}{3}$.

Considering, the above calculations and from the table we see that the integral is reaching 1/3.

Properties of Integrals.

1. $\int [f(x) \pm g(x)] \; dx = \int f(x) \; dx \pm \int g(x) \; dx$.
2. $\int k \; f(x) \; dx = k \int f(x) \; dx$
3. Fundamental theorem Calculus:

   Let f be a continuous real-valued function defined on a closed interval $[a, b]$. Let F be the function defined, for all x in $[a, b]$, by

   $F(x) = \int f(t) \; dt$.

   $F(x)$ is continuous on $[a, b]$, differentiable on the open interval (a, b), and $F'(x) = f(x)$ for all x in (a, b)
4. Second Fundamental theorem of Calculus:

   Let f be a real-valued function defined on a closed interval $[a, b]$ that admits an anti derivative F on $[a, b]$. That is, f and F are functions such that for all x in $[a, b]$, $f(x) = F'(x)$.

   If f is integrable on $[a, b]$ then

   $\int_a^b f(x) \; dx = F(b) - F(a)$

$\int e^x dx = e^x + c$ $\int 1/x \; dx = \log x + c$ $\int C_1(constant)dx = C_1x + c$	$\int a^x dx = a^x/\log a + c$ $\int x^n dx = \frac{x^{n+1}}{n+1} + c$ $\int (ax + b^n) = \frac{(ax+b)^{n+1}}{(n+1)a} + c$
$\int \sin x \; dx = -\cos x + c$ $\int \tan x \; dx = \ln \lvert \sec x \rvert + c$ $= -\ln \lvert \cos x \rvert + c$ $\int \csc x \; dx = \ln \lvert \csc x - \cot x \rvert + c$ $= \ln \lvert \tan \frac{x}{2} \rvert + c$	$\int \cos x \; dx = \sin x + c$ $\int \cot x \; dx = -\ln \lvert \csc x \rvert$ $= \ln \lvert \sin x \rvert + c$ $\int \sec x \; dx = \ln \lvert \sec x + \tan x \rvert + c$ $= \ln \lvert \tan(\pi/4 + x/2) \rvert + c$
$\int \sec^2 x \; dx = \tan x + c$ $\int \tan x \sec x \; dx = \sec x + c$	$\int \csc^2 x \; dx = -\cot x + c$ $\int \csc x \cot x \; dx = -\csc x + c$
$\int \frac{1}{\sqrt{1-x^2}} \; dx = \sin^{-1} x + c$ $\int \frac{1}{x\sqrt{x^2-1}} \; dx = -\csc^{-1} x + c$ $\int \frac{1}{1+x^2} = \tan^{-1} x + c$	$\int \frac{1}{\sqrt{1-x^2}} \; dx = -\cos^{-1} x + c$ $\int \frac{1}{x\sqrt{x^2-1}} \; dx = \sec^{-1} x + c$ $\int \frac{1}{1+x^2} = -\cot^{-1} x + c$
$\int \frac{1}{\sqrt{a^2-x^2}} \; dx = \sin^{-1} x + c$ $\int \frac{1}{\sqrt{x^2+a^2}} \; dx = \ln \lvert x + \sqrt{x^2 + a^2} \rvert + c$ $\int \frac{1}{x^2-a^2} \; dx = \frac{1}{2a} \ln \lvert \frac{x-a}{x+a} \rvert + c$	$\int \frac{1}{\sqrt{x^2-a^2}} \; dx = \ln \lvert x + \sqrt{x^2 - a^2} \rvert + c$ $\int \frac{1}{x^2+a^2} \; dx = \frac{1}{a} \tan^{-1}(\frac{x}{a}) + c$ $\int \frac{1}{a^2-x^2} \; dx = \frac{1}{2a} \ln \lvert \frac{a+x}{a-x} \rvert + c$
$\int \sqrt{a^2 + x^2} \; dx = \frac{x}{2}\sqrt{a^2 + x^2} +$ $\frac{a^2}{2} \ln \lvert x + \sqrt{a^2 + x^2} \rvert + c$ $\int \sqrt{a^2 - x^2} \; dx = \frac{x}{2}\sqrt{a^2 - x^2} +$ $\frac{a^2}{2} \sin^{-1}(\frac{x}{a}) + c$	$\int \sqrt{x^2 - a^2} \; dx = \frac{x}{2}\sqrt{x^2 - a^2} +$ $\frac{a^2}{2} \ln \lvert x + \sqrt{x^2 - a^2} \rvert + c$ $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a} \sec^{-1}(\frac{x}{a}) + c$
$\int \frac{f'(x)}{f(x)} \; dx = \ln \lvert f(x) \rvert + c$ $\int f'(x)[f(x)]^n \; dx = \frac{[f(x)]^{n+1}}{n+1} + c$	$\int \frac{f'(x)}{\sqrt{f(x)}} \; dx = 2\sqrt{f(x)} + c$ $\int e^x[f(x) + f'(x)] \; dx = e^x f(x) + c$

No.	Type	Method
1.	$\int \frac{ax+b}{cx+d}\ dx$ where $a, b, c, d \in \mathbb{R}$	By adjusting the Numerator try to get $cx + d$ in place of $ax + b$ and then separate and integrate.
2.	$\int (ax+b)(cx+d)^n\ dx$ $n \in \mathbb{Q}$	By adjusting the coefficient try to get $cx + d$ in place of $ax + b$ then separate and integrate
3.	$\int \frac{P(x)}{ax+b}\ dx$, where $P(x)$ is a polynomial of degree ≥ 2	Divide $P(x)$ by $ax + b$ and separate and integrate.
4.	$\int \frac{ax+b}{(cx+d)^n}\ dx$ $n \in \mathbb{Q}$	BY adjusting the coeff. of x try to get $cx + d$ in place of $ax + b$, then separate and integrate
5.	The integrals which contain a single trigonometric term like $\sin^2 x, \cos^3 x$... etc or after simplification single trignometric term is obtained	Using suitable Trignometric formula reduce the degree of trignometric terms to degree 1 and then integrate
6.	Substitution: In this type locate a f^n whose derivative is present or hidden in the problem itself.	Substitute the function whose derivative is present as t make required changes and then integrate
7.	$\int p(x)(ax+b)^n\ dx$	Put $ax + b = t$ make required changes and integrate
8.	$\int \frac{\sin(x\pm a)}{\cos(x\pm a)}\ dx$	By adjusting the constant try to get the denominator angle in numerator then separate and integrate.

9.	$\int \frac{1}{\sin(x\pm a)\cos(x\pm b)}\ dx$ $\int \frac{1}{\cos(x\pm a)\cos(x\pm b)}\ dx$ $\int \frac{1}{\sin(x\pm a)\sin(x\pm b)}\ dx$	Multiply and divide by $\cos(a \pm b) = \cos(x \pm a - (x \pm b))$ Multiply and divide by $\sin(a \pm b) = \sin(x \pm a - (x \pm b))$ Multiply and divide by $\sin(a \pm b) = \sin(x \pm a - (x \pm b))$
10.	$\int \frac{1}{a\sin x + b\cos x}\ dx$	Put $a = r\cos\alpha, b = r\sin\alpha$
11.	$\int \frac{1}{ax^2+bx+c}\ dx$ or $\int \frac{1}{\sqrt{ax^2+bx+c}}\ dx$	By adjusting the constant express $ax^2 + bx + c$ as sum or difference of two terms, then using suitable formulas integrate. $\int \frac{dx}{a^2+x^2}, \int \frac{dx}{a^2-x^2}, \int \frac{dx}{x^2-a^2},$ $\int \frac{dx}{\sqrt{a^2+x^2}}, \int \frac{dx}{\sqrt{a^2-x^2}}, \int \frac{dx}{\sqrt{x^2-a^2}}$
12.	$\int \frac{ax+b}{px^2+qx+r}\ dx$ or $\int \frac{ax+b}{\sqrt{px^2+qx+r}}\ dx$	Let $ax + b = A + B\frac{d}{dx}(px^2 + qx + r)$— (i) Find A and B then divide eq.(i) by denominator then separate and integrate.
13.	$\int \frac{P(x)}{ax^2+bx+c}\ dx$, degree of $P(x) \geq 2$	Divide P(x) by denominator, separate and integrate.
14.	$\int \frac{dx}{c+a\sin^2 x+b\cos^2 x}$ or $\int \frac{dx}{a\sin^2 x+b\cos^2 x}$	Divide Numerator and Denominator by either $\cos^2 x$ or $\sin^2 x$ and put $\tan x = t$ or $\cot x = t$ make the required changes and integrate.
15.	$\int \frac{dx}{a+b\sin x}, \int \frac{dx}{a+b\cos x}$ or $\int \frac{dx}{a+b\sin x+c\cos x}$	Put $\tan\frac{x}{2} = t$
16.	$\int \sqrt{\frac{x}{a\pm x}}\ dx$ or $\int \sqrt{\frac{a\pm x}{x}}\ dx$ or $\int \sqrt{\frac{a\pm x}{a\pm x}}\ dx$	Multiply Nr and Dn by Nr then it reduces to $\int \frac{ax+b}{\sqrt{px^2+qx+r}}$ or $\int \frac{ax+b}{px^2+qx+r}$ and hence can be solved by known method (12)
17.	$\int \frac{a\sin x+b\cos x}{c\sin x+d\cos x}\ dx$ or $\int \frac{ae^x+b}{ce^x+d}$	Let Numerator(Nr), Denominator (Dn) $Nr = A(Dn) + B(\frac{d}{dx}Dn)$. By comparing find A and B then separate and integrate.

18.	Integration by parts [Product Rule] If u and v are integrable functions of x	$\int uv\ dx = u\int v\ dx - \int[\frac{d}{dx}u(\int v\ dx)]\ dx$ L-Logarithmic, I-Inverse, A-Algebraic, T-Trigonometric, E-Exponential. Choose u and v according to hierarchy $LIATE$
19.	$\int \sqrt{px^2+qx+r}\ dx$	By adjusting the coefficient express px^2+qx+r as $\int \sqrt{x^2-a^2}\ dx, \int \sqrt{x^2+a^2}\ dx, \int \sqrt{a^2-x^2}\ dx$
20.	$\int (px+q)\sqrt{ax^2+bx+c}\ dx$	Let $px+q = A + B\frac{d}{dx}(ax^2+bx+c)$. Find A B multiply both sides by $\sqrt{ax^2+bx+c}$ then separate and integrate
21.	Integrals involving odd powers of $\cos x, \sin x, \tan x,$ $\cot x, \csc x, \sec$	Express it as product of even and odd power Ex. $\cos^3 x = \cos^2 x\cos x, \sin^5 x = (\sin^2 x)^2 \sin x$ or directly use the formula for $\sin^n x, \cos^n x$
22.	$\int \frac{x^{1/a}}{x^{1/b}+x^{1/a}}\ dx,$ $\int \frac{(px+q)^{1/a}}{(px+q)^{1/b}+(px+q)^{1/a}}\ dx$	Put $x = t^n$ where n is LCM of a and b
23.	$\int \frac{dx}{(px+q)(ax+b)^{1/n}}$	Put $(ax+b) = t^n$

Partial Fractions.

Provided that the numerator $f(x)$ is of less degree than the relevant denominator, the following identities are typical examples of the form of partial fractions used:

$$\frac{f(x)}{(x+a)(x+b)(x+c)} = \frac{A}{x+a} + \frac{B}{x+b} + \frac{C}{x+c}$$
$$\frac{f(x)}{(x+a)^3(x+b)(x+c)} = \frac{A}{x+a} + \frac{B}{(x+a)^2} + \frac{C}{(x+a)^3} + \frac{E}{x+b} + \frac{F}{x+c}$$
$$\frac{f(x)}{(ax^2+bx+c)(x+d)} = \frac{Ax+B}{(ax^2+bx+c)} + \frac{C}{(x+d)}$$

Example: $\int x^2 \sin x\ dx$

Solution: We are going to solve this integral by using the integration by parts method [type 18].

x^2 is algebraic and $\sin x$ is trigonometric. So according to the method mention in [type 18] first term should be algebraic i.e $u = x^2$ and second trigonometric $v = \sin x$

$I = \int uv\ dx = u \int v\ dx - \int [\frac{d}{dx} u (\int v\ dx)]\ dx$

$[\int \sin x\ dx = -\cos x + c,\ \frac{d}{dx} x^2 = 2x]$

$I = x^2 \int \sin x\ dx - \int [\frac{d}{dx} x^2 (\int \sin x\ dx)]\ dx$

$I = x^2[-\cos x] - \int [2x\{-\cos x\}] dx$

$I = x^2[-\cos x] + 2 \int [x \cos x] dx$ ... [1]

Let $I_1 = x \int \cos x\ dx - \int [\frac{d}{dx} x (\int \cos x\ dx)]\ dx$

$I_1 = x \sin x - \int 1 \cdot \sin x\ dx = x \sin x - [-\cos x] = x \sin x + \cos x$

$I = x^2[-\cos x] + 2[x \sin x + \cos x] + c$... from 1

$\therefore I = \int x^2 \sin x\ dx = -x^2 \cos x + 2x \sin x + 2 \cos x + c.$

Example: $\int \frac{3x-2}{(x+1)^2(x+3)}\ dx$ [Partial Fractions]

Solution: Let $I = \int \frac{3x-2}{(x+1)^2(x+3)}\ dx.$

Here, we seperate the functions into two functions and then integrate i.e

Let, $\frac{3x-2}{(x+1)^2(x+3)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x+3}$

We take LCM on both sides and solve it further by equating the corresponding coefficients and calculate the values of A, B, C.

$\frac{3x-2}{(x+1)^2(x+3)} = \frac{A(x+1)(x+3)}{(x+1)^2(x+3)} + \frac{B(x+3)}{(x+1)^2(x+3)} + \frac{C(x+1)^2}{(x+3)(x+1)^2}$

$3x - 2 = A(x^2 + 4x + 3) + B(x + 3) + C(x^2 + 2x + 1)$

$3x - 2 = (A + C)x^2 + (4A + B + 2C)x + (3A + 3B + C)$

$\implies A + C = 0;\ 4A + B + 2C = 3;\ 3A + 3B + C = -2,$

$\implies A = -C;\ 4(-C) + B + 2C = 3;\ 3(-C) + 3B + C = -2$

$\implies B - 2C = 3; 3B - 2C = -2 \implies B = -5/2;\ C = -11/4;\ A = 11/4$

$\therefore \frac{3x-2}{(x+1)^2(x+3)} = \frac{(11/4)}{x+1} + \frac{(-5/2)}{(x+1)^2} + \frac{-11/4}{x+3}$

$I = \int [\frac{(11/4)}{x+1} + \frac{(-5/2)}{(x+1)^2} + \frac{-11/4}{x+3}]\ dx = \int \frac{(11/4)}{x+1}\ dx + \int \frac{(-5/2)}{(x+1)^2}\ dx + \int \frac{-11/4}{x+3}\ dx$

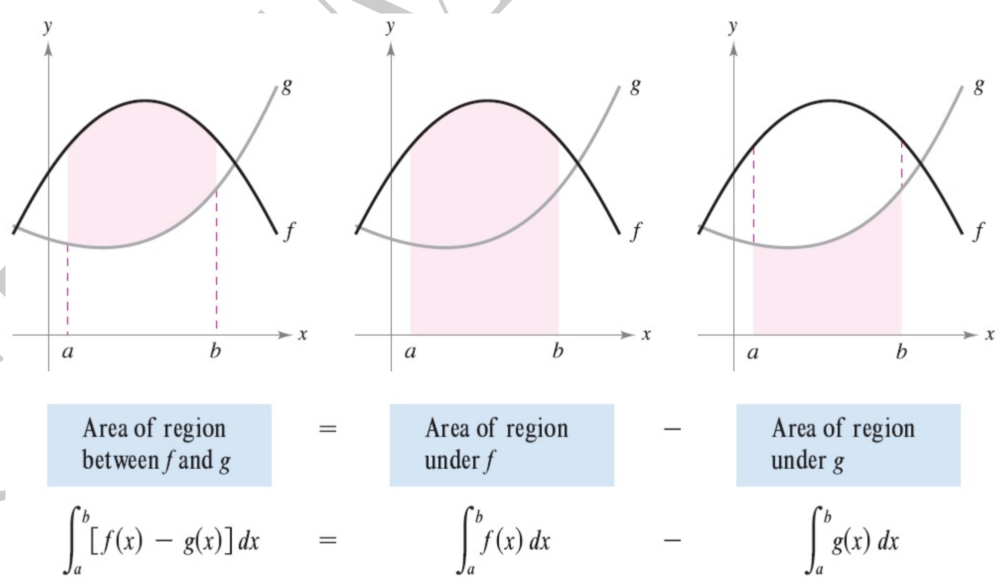
$= (11/4)\int \frac{1}{x+1}\ dx + (-5/2)\int \frac{1}{(x+1)^2}\ dx + (-11/4)\int \frac{1}{x+3}\ dx$
$= (11/4)\log|x+1| + (-5/2)\frac{(x+1)^{-2+1}}{-2+1} + (-11/4)\log|x+3| + c$
$= (11/4)\log|x+1| - (11/4)log|x+3| + \frac{4/2}{x+1} + c$
$= (11/4)\log|\frac{x+1}{x+3}| + \frac{5/2}{x+1} + c.$

Definite Integrals

1. $\int_a^b f(x)\ dx = -\int_b^a f(x)\ dx$
2. $\int_a^b f(x)\ dx = \int_a^b f(t)\ dt$
3. $\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx,\ a < c < b$
4. $\int_a^b f(x)\ dx = \int_a^b f(a+b-x)\ dx$
5. $\int_{-a}^a f(x)dx = 0$ if $f(x)$ is odd.
   . $= \int_{-a}^a f(x)dx$ if $f(x)$ is even.
6. $\int_0^{2a} f(x)dx = \int_0^a f(x)\ dx + \int_0^a f(2a-x)\ dx$

4.1 Application of Integration:

Area of region between two curves:



If f and g are two continuous functions on [a,b] and $g(x) \leq f(x)$ for all $x \in [a, b]$, then the area of the region bounded by the graphs of f and g and the vertical lines $x = a$ and $x = b$ is $\int_a^b [f(x) - g(x)\ dx]$.

Length of the curve:
If f' is continuous on $[a, b]$, then the length (arc length) of the curve $y = f(x)$ from the point $A = (a, f(a))$ to the point $B = (b, f(b))$ is the value of the integral $L = \int_a^b \sqrt{1 + [f'(x)]^2} = \int_a^b \sqrt{1 + [dy/dx]^2}$

Area of the surface: If the function $f(x) \geq 0$ is continuously differentiable on $[a, b]$, the area of the surface generated by revolving the graph of $y = f(x)$ about the $x - axis$ is $S = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2} = \int_a^b 2\pi y\sqrt{1 + [dy/dx]^2}$

Work done: The work done by a variable force $F(x)$ in the direction of motion along the $xaxis$ from $x = a$ to $x = b$ is $W = \int_a^b F(x)\ dx$

Newton's Equation of Motion:
$[v = final\ velocity,\ u = initial\ velocity,\ a = acceleration,\ s = displacement,$ $t = time]$

1. $v = u + at$
2. $s = ut + \frac{1}{2}at^2$
3. $v^2 = u^2 + 2as$

$instantaneous\ velocity\ = v = \frac{ds}{dt}\ \cdots (I)$
$acceleration\ = a = \frac{dv}{dt} = \frac{d}{ds}(\frac{ds}{dt}) = \frac{d^2s}{dt^2}\ \cdots (II)$
from I and II: $\frac{a}{v} = \frac{dv/dt}{ds/dt} = \frac{dv}{ds}$
By cross multiplying we get $v\ dv = a\ ds\ \cdots (III)$

Now, $a = \frac{dv}{dt} = \frac{Final\ velocity - initial\ velocity}{time} = \frac{v-u}{t}$
Cross multiplying we get $at = v - u \implies v = u + at$

Now, if we consider the 2 i.e $s = ut + \frac{1}{2}at^2$ and differentiate with respect to t then we get $\frac{ds}{dt} = u + \frac{1}{2}a\ (2t)$ from I; $v = u + at$

Now, the last equation can be obatined by using III as $v\ dv = a\ ds$
Integrating on both sides with respect to v and s and taking limits as u and v for velocity and 0 and s for displacement
$\int_u^v v\ dv = \int_0^s a\ ds \implies [v^2/2]_u^v = a[s]_0^s \implies \frac{v^2}{2} - \frac{u^2}{2} = a[s-0]$
$\implies \frac{v^2-u^2}{2} = as \implies v^2 - u^2 = 2as \implies v^2 = u^2 + 2as$

Isaac Newton (1643-1727), staunch English Puritan and the Englands champion of math and physics, developed the fundamental concepts of calculus in 1665 and 1666. He organized his ideas into a manuscript in late 1666 and showed it to a few other English mathematicians, but did not publish it. In 1672 to 1676, a German mathematician named Gottfried Leibniz (1646-1716), who started college at 15 and graduated at 17, worked privately on the same problems and came up with similar answers. Leibniz had not heard of Newtons work, and he developed notation and methods that were different from Newtons, but his ideas were essentially the same. Leibniz first published his results in 1684 and 1686; Newton, in 1687. The math debate arose in the late 1690s, when followers of Newton began to accuse Leibniz of having stolen his calculus ideas from Newton. The fact that Leibniz had published first and Newton second might have made this impossible, but Newton and Leibniz had exchanged letters in 1676 and Leibniz had visited London in both 1673 and 1676, so it was not impossible that Leibniz had stolen Newtons ideas merely untrue. Newton and Leibniz actually invented calculus independently, not an uncommon event in science and mathematics. But sharing the accomplishment was not on anyones agenda, especially in a question of national pride. Newton became so angry that he deleted all references to Leibnizs work from his scientific books (except insults). Newton and his followers publicly accused Leibniz of stealing. Leibniz asked the Royal Society of London, the major English scientific club or society of its day, to investigate this damning charge. Newton secretly stage-managed the societys investigation and Leibniz was found guilty. Newton was buried in a cathedral with royal honors and thousands of mourners; Leibnizs funeral was attended only by his secretary. Leibnizs ultimate revenge, however, is that his calculus notation, not Newtons, is used today.

5 Trigonometry Formula

1. $\sin^2\theta + \cos^2\theta = 1$ $\quad 1 + \tan^2\theta = \sec^2\theta$ $\quad 1 + \cot^2\theta = \csc^2\theta$

2. $\sin(A+B) = \sin A\cos B + \cos A\sin B$

   $\sin(A-B) = \sin A\cos B - \cos A\sin B$

   $\cos(A+B) = \cos A\cos B - \sin A\sin B$

   $\cos(A-B) = \cos A\cos B + \sin A\sin B$

3. $\tan(A+B) = \frac{\tan A+\tan B}{1-\tan A\tan B}$ $\quad \tan(A-B) = \frac{\tan A-\tan B}{1+\tan A\tan B}$

4. $\sin(-\theta) = -\sin\theta$ $\quad \cos(-\theta) = \cos\theta$

5. $\sin 2\theta = 2\sin\theta\cos\theta \;=\; \frac{2\tan\theta}{1+\tan^2\theta}$ $\quad \tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$

   $\cos 2\theta = cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}$

6. $\cos^2\theta = \frac{1+\cos 2\theta}{2}$ $\quad \sin^2\theta = \frac{1-\cos 2\theta}{2}$ $\quad \tan^2\theta = \frac{1-\cos 2\theta}{1+\cos 2\theta}$

7. $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$ $\quad \cos 3\theta = 4\cos^3\theta - 3\cos\theta$

   $\tan 3\theta = \frac{3\tan\theta - tan^3\theta}{1-3\tan^2\theta}$

8. $2\sin A\cos B = \sin(A+B) + \sin(A-B)$

   $2\sin B\cos A = \sin(A+B) - \sin(A-B)$

   $2\cos A\cos B = \cos(A+B) + \cos(A-B)$

   $2\sin A\sin B = \cos(A-B) - \cos(A+B)$

9. $\sin C + \sin D = 2\sin(\frac{C+D}{2})\cos(\frac{C-D}{2})$

   $\sin C - \sin D = 2\sin(\frac{C-D}{2})\cos(\frac{C+D}{2})$

   $\cos C + \cos D = 2\cos(\frac{C+D}{2})\cos(\frac{C-D}{2})$

   $\cos C - \cos D = -2\sin(\frac{C+D}{2})\sin(\frac{C-D}{2})$

10. $\sin(n\pi) = 0$ $\quad \cos(n\pi) = (-1)^n$ $\quad \sin(2n\pi) = 0,$ $\quad \cos(2n\pi) = 1$

11. $\pi^c = 180^o$ Radian to degrees conversion

12.

θ	0	30^o	45^o	60^o	90^o	120^o	135^o	150^o	180^o
θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	π
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0
cot	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	$-\frac{1}{\sqrt{3}}$	-1	$-\sqrt{3}$	∞
csc	∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞
sec	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞	-2	$-\sqrt{2}$	$-\frac{2}{\sqrt{3}}$	0

13. Domain and Range of Trignometric functions.

T-Ratio	Domain	Range
$\sin\theta$	$\mathbb{R}$	$[-1,1]$
$\cos\theta$	$\mathbb{R}$	$[-1,1]$
$\tan\theta$	$\mathbb{R}-[(2n+1)\frac{\pi}{2}, n\in\mathbb{Z}]$	$\mathbb{R}$
$\cot\theta$	$\mathbb{R}-[(n)\pi, n\in\mathbb{Z}]$	$\mathbb{R}$
$\csc\theta$	$\mathbb{R}-[(n)\pi, n\in\mathbb{Z}]$	$[-\infty,-1]\cup[1,\infty]$
$\sec\theta$	$\mathbb{R}-[(2n+1)\frac{\pi}{2}, n\in\mathbb{Z}]$	$[-\infty,-1]\cup[1,\infty]$

14. Trignometric Transformations

T-Ratio	$\frac{\pi}{2}-\theta$	$\frac{\pi}{2}+\theta$	$\pi-\theta$	$\pi+\theta$	$\frac{3\pi}{2}-\theta$	$\frac{3\pi}{2}+\theta$	$2\pi-\theta$	$2\pi+\theta$
	$90-\theta$	$90+\theta$	$180-\theta$	$180+\theta$	$270-\theta$	$270+\theta$	$360-\theta$	$360+\theta$
sin	$\cos\theta$	$\cos\theta$	$\sin\theta$	$-\sin\theta$	$-\cos\theta$	$-\cos\theta$	$-\sin\theta$	$\sin\theta$
cos	$\sin\theta$	$-\sin\theta$	$-\cos\theta$	$-\cos\theta$	$-\sin\theta$	$\sin\theta$	$\cos\theta$	$\cos\theta$
tan	$\cot\theta$	$-\cot\theta$	$-\tan\theta$	$\tan\theta$	$\cot\theta$	$-\cot\theta$	$-\tan\theta$	$\tan\theta$
cot	$\tan\theta$	$-\tan\theta$	$-\cot\theta$	$\cot\theta$	$\tan\theta$	$-\tan\theta$	$-\cot\theta$	$\cot\theta$
csc	$\sec\theta$	$\sec\theta$	$\csc\theta$	$-\csc\theta$	$-\sec\theta$	$-\sec\theta$	$-\csc\theta$	$\csc\theta$
sec	$\csc\theta$	$-\csc\theta$	$-\sec\theta$	$-\sec\theta$	$-\csc\theta$	$\csc\theta$	$\sec\theta$	$\sec\theta$