— LIMITS —

Syllabus

1. *Indeterminate form*
2. *Limits of a Function*
3. *Left and right limits*
4. *To find left/right limit*
5. *Existence of limit*
6. *Theorems of limits*
7. *Methods of evaluation of limits*
8. *Standard limits*
9. *Limits which do not exist*

Total No. of questions in **Limits** are:

Solved examples...27

Level # 1 103

Level # 2 36

Level # 3 31

Level # 4 30

Total No. of questions…...…………………….**227**

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1. Students are advised to solve the questions of exercises (Levels # 1, 2, 3, 4) in the same sequence or as directed by the faculty members.
2. Level #3 is not for foundation course students, it will be discussed in fresher and target courses.

Index : Preparing your own list of Important/Difficult Questions

Instruction to fill

(A) Write down the Question Number you are unable to solve in **column A** below, by Pen.

(B) After discussing the Questions written in **column A** with faculties, strike off them in the manner so that you can see at the time of Revision also, to solve these questions again.

(C) Write down the Question Number you feel are important or good in the **column B**.

EXERCISE NO.	COLUMN :A	COLUMN :B
	Questions I am unable to solve in first attempt	Good/Important questions
Level # 1		
Level # 2		
Level # 3		
Level # 4		

Advantages

1. It is advised to the students that they should prepare a question bank for the revision as it is very difficult to solve all the questions at the time of revision.

2. Using above index you can prepare and maintain the questions for your revision.

KEY CONCEPTS

1. Indeterminate Form

Some times we come across with some functions which do not have definite value corresponding to some particular value of the variable.

For example for the function

$f(x) = \frac{x^2 - 4}{x - 2}$, $f(2) = \frac{4-4}{2-2} = \frac{0}{0}$

which cannot be determined. Such a form is called an Indeterminate form. Some other indeterminate forms are

$0 \times \infty$, 0^o, 1^∞ , $\infty - \infty$, ∞/∞,∞^o, 0/0.

2. Limits of a Function

Let y = f (x) be a function of x and for some particular value of x say x = a, the value of y is indeterminate, then we consider the values of the function at the points which are very near to 'a'. If these values tend to a definite unique number ℓ as x tends to 'a' (either from left or from right) then this unique number ℓ is called the limits of f(x) at x = a and we write it as

$\lim\limits_{x \to a} f(x) = \ell$

Meaning of 'x → a' : Let x be a variable and a be a constant. If x assumes values nearer and nearer to 'a' then we can say ' x tends to a' and we write 'x → a'.

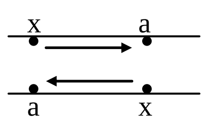
It should be noted that as x → a. we have $x \neq a$.

By ' x tends to a' we mean that

(i) $x \neq a$

(ii) x assumes values nearer and nearer to 'a' and

(iii) we are not spacifying any manner in which x should approach to a.x may approach to a from left or right as shown in figure.



3. Left and Right Limits

If value of a function f(x) tend to a definite unique number when x tends to 'a' from left, then this unique number is called left hand limit (LHL) of f(x) at x = a and we can write it as

$f(a-0)$ or $\lim\limits_{x \to a^-} f(x)$ or $\lim\limits_{x \to a-0} f(x)$

For evaluation

$f(a-0) = \lim\limits_{h \to 0} f(a-h)$

Similarly, we can define right hand limit (RHL) of f(x) at x = a. In this case x tends to 'a' from right. We can write it as

$f(a+0)$ or $\lim\limits_{h \to a^+} f(x)$ or $\lim\limits_{h \to a+0} f(x)$

For evaluation

$f(a+0) = \lim\limits_{h \to 0} f(a+h)$

4. To Find Left/Right Limit

(i) For finding right hand limit of the function we write (x+h) in place of x while for left hand limit we write (x–h) in place of x.

(ii) We replace then x by a in the function so obtained.

(iii) Conclusively we find limit $h \to 0$

5. Existence of Limit

The limit of a function at some point exists only when its left- hand limit and right hand limit at that point exist and are equal. Thus

$\lim\limits_{x \to a} f(x)$ exists $\lim\limits_{x \to a^-} f(x) = \lim\limits_{x \to a^+} f(x) = \ell$

where ℓ is called the limit of the function.

6. Theorems on Limits

The following theorems are very helpful for evaluation of limits–

(i) $\lim\limits_{x \to a} [k\ f(x)] = k \lim\limits_{x \to a} f(x)$, where k is a constant

(ii) $\lim\limits_{x \to a} [f(x) + g(x)] = \lim\limits_{x \to a} f(x) + \lim\limits_{x \to a} g(x)$

(iii) $\lim\limits_{x \to a} [f(x) - g(x)] = \lim\limits_{x \to a} f(x) - \lim\limits_{x \to a} g(x)$

(iv) $\lim\limits_{x \to a} [f(x).g(x)] = \lim\limits_{x \to a} f(x). \lim\limits_{x \to a} g(x)$

(v) $\lim_{x\to a} [f(x)/g(x)] = [\lim_{x\to a} f(x)]/[\lim_{x\to a} g(x)]$ provided $g(x) \neq 0$

(vi) $\lim_{x\to a} f[g(x)] = f[\lim_{x\to a} g(x)]$

(vii) $\lim_{x\to a} [f(x)+k] = \lim_{x\to a} f(x) + k$ where k is a constant

(viii) $\lim_{x\to a} \log\{f(x)\} = \log\{\lim_{x\to a} f(x)\}$

(ix) If $f(x) \le g(x)$ for all x,

then $\lim_{x\to a} f(x) \le \lim_{x\to a} g(x)$

(x) $\lim_{x\to a} [f(x)]^{g(x)} = \{\lim_{x\to a} f(x)\}^{\lim_{x\to a} g(x)}$

(xi) $\lim_{x\to \pm\infty} f(x) = \lim_{x\to 0} f(1/x)$

(xii) $\lim_{x\to 0^+} f(-x) = \lim_{x\to 0^-} f(x)$

7. Methods of Evaluation of Limits

7.1 : When $x \to \infty$

In this case expression should be expressed as a function 1/x and then after removing indeterminant form, (If it is there) replace 1/x by 0.

7. 2 : When $x \in a, a \in R$.

7.2.1 Factorisation method :

If f(x) is of the form $\frac{f(x)}{g(x)}$ and of indeterminate form then this form is removed by factorising g(x) and h(x) and cancel the common factors, then put the value of x.

7.2.2 Rationalisation Method:

In this method we rationalise the factor containing the square root and simplify and we put the value of x.

7.2.3 Expansion method

If $x \to 0$ and there is atleast one function in the given expression which can be expanded then we express numerator and Denominator in the ascending powers of x and remove the common factor there.

The following expansions of some standard functions are given-

(i) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots$

(ii) $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots\ldots$

(iii) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\ldots$

(iv) $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots\ldots$

(v) $a^x = 1 + (x \log a) + \frac{(x \log a)^2}{2!} + \frac{(x \log a)^3}{3!} \ldots$

(vi) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\ldots$

(vii) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\ldots$

(viii) $\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots\ldots$

(ix) $\sin^{-1}x = x + \frac{x^3}{3!} + \frac{9x^5}{5!} + \ldots\ldots$

(x) $\cos^{-1}x = \frac{\pi}{2} - \left(x + \frac{x^3}{3!} + \frac{9x^5}{5!} + \ldots\ldots\right)$

(xi) $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\ldots$

(xii) $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots\ldots$

7.2.4 'L' Hospital rule

If $\lim_{x\to a} \frac{f(x)}{g(x)}$ is of the form , $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Note :

(1) This rule is applicable only when there is $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form.

(2) This rule is applicable only when the limits of the function exists.

8. Some Standard Limits

(i) $\lim\limits_{x\to 0}\dfrac{\sin x}{x} = \lim\limits_{x\to 0}\dfrac{x}{\sin x} = 1;\ \lim\limits_{x\to 0}\sin x = 0$

(ii) $\lim\limits_{x\to 0}\cos x = \lim\limits_{x\to 0}\left(\dfrac{1}{\cos x}\right) = 1$

(iii) $\lim\limits_{x\to 0}\dfrac{\tan x}{x} = \lim\limits_{x\to 0}\dfrac{x}{\tan x} = 1;\ \lim\limits_{x\to 0}\tan x = 0$

(iv) $\lim\limits_{x\to 0}\dfrac{\sin^{-1} x}{x} = \lim\limits_{x\to 0}\dfrac{x}{\sin^{-1} x} = 1$

(v) $\lim\limits_{x\to 0}\dfrac{\tan^{-1} x}{x} = \lim\limits_{x\to 0}\dfrac{x}{\tan^{-1} x} = 1$

(vi) $\lim\limits_{x\to \infty}\left(1+\dfrac{a}{x}\right)^{x} = \lim\limits_{x\to 0}(1+ax)^{1/x} = e^{a}$

(vii) $\lim\limits_{x\to 0}\dfrac{a^{x}-1}{x} = \log_e a\ \ (a>0)$

(viii) $\lim\limits_{x\to 0}\dfrac{e^{x}-1}{x} = 1$

(ix) $\lim\limits_{x\to a}\dfrac{x^{n}-a^{n}}{x-a} = n\,a^{n-1}$

(x) $\lim\limits_{x\to 0}\dfrac{\log(1+x)}{x} = 1$

(xi) $\lim\limits_{x\to 0}\dfrac{(1+x)^{n}-1}{x} = n$

(xii) $\lim\limits_{x\to \infty}\dfrac{\sin x}{x} = \lim\limits_{x\to \infty}\dfrac{\cos x}{x} = 0$

(xiii) $\lim\limits_{x\to \infty}\dfrac{\sin 1/x}{1/x} = 1$

(xiv) $\lim\limits_{x\to \infty} 1/x = 0$

(xv) $\lim\limits_{x\to 0}\dfrac{1}{|x|} = \infty$

(xvi) $\lim\limits_{x\to \infty} a^{x} = \begin{cases} 0, \text{ if } |a| < 1 \\ 1, \text{ if } a = 1 \\ \infty, \text{ if } a > 1 \\ \text{does not exit}, \text{ if } a \le -1 \end{cases}$

(xvii) $\lim\limits_{x\to a}[f(x)]^{g(x)} = e^{\lim\limits_{x\to a} g(x)\{f(x)-1\}}$

9. Some Limits Which do not Exist

(i) $\lim\limits_{x\to 0}\left(\dfrac{1}{x}\right)$

(ii) $\lim\limits_{x\to 0} x^{1/x}$

(iii) $\lim\limits_{x\to 0}\dfrac{|x|}{x}$

(iv) $\lim\limits_{x\to a}\dfrac{|x-a|}{x-a}$

(v) $\lim\limits_{x\to 0}\sin 1/x$

(vi) $\lim\limits_{x\to 0}\cos 1/x$

(vii) $\lim\limits_{x\to 0} e^{1/x}$

(viii) $\lim\limits_{x\to \infty}\sin x$

(ix) $\lim\limits_{x\to \infty}\cos x$