COORDINATE GEOMETRY

The system of Geometry in which a point is specified by means of an ordered number-pair is known as Coordinate Geometry. It enables us to solve geometrical problems by algebraic methods.

Representation of a point

The point P(x, y) lies in:

- the first quadrant iff $x > 0, y > 0$
- the second quadrant iff $x < 0, y > 0$
- the third quadrant iff $x < 0, y < 0$
- the fourth quadrant iff $x > 0, y < 0$.

If the point P(x, y) lies on the x-axis iff $y = 0$ and lies on the y-axis iff $x = 0$.

Distance between two points

The distance between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is

$AB = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

Example # 1

Find the value of x, if the distance between the points (x, –1) and (3, 2) is 5

Solution.

Let P(x ,–1) and Q(3, 2) be the given points. Then PQ = 5 (given)

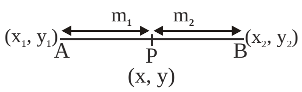
$\sqrt{(x-3)^2 + (-1-2)^2} = 5$

$\Rightarrow$ $(x - 3)2 + 9 = 25$

$\Rightarrow$ **$x = 7$ or $x = -1$.**

Section formula

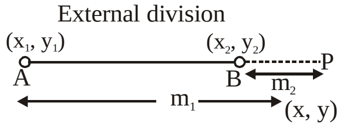
Internal division



$$\frac{AP}{PB} = \frac{m_1}{m_2}$$

$$x = \frac{m_1x_2 + m_2x_1}{m_1 + m_2} \quad y = \frac{m_1y_2 + m_2y_1}{m_1 + m_2}$$

External division



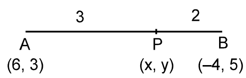
$$x = \frac{m_1x_2 - m_2x_1}{m_1 - m_2} \quad y = \frac{m_1y_2 - m_2y_1}{m_1 - m_2}$$

Example# 2

Find the coordinates of the point which divides the line segment joining the points (6, 3) and (– 4, 5) in the ratio 3 : 2 (i) internally and (ii) externally.

Solution.

Let P (x, y) be the required point.

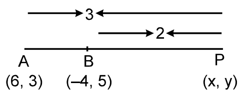


(i) For internal division :

$x = \frac{3 \times -4 + 2 \times 6}{3+2}$ and $y = \frac{3 \times 5 + 2 \times 3}{3+2}$ or x = 0 and $y = \frac{21}{5}$

So the coordinates of P are $\left(0, \frac{21}{5}\right)$

(ii) For external division



$x = \frac{3 \times -4 - 2 \times 6}{3-2}$ and $y = \frac{3 \times 5 - 2 \times 3}{3-2}$

or x = –24 and y = 9

So the coordinates of P are **(–24, 9)**

CENTROID AND AREA OF A TRIANGLE

If $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are the vertices of a triangle ABC, then the centroid G(x, y) is given by

$$x = \frac{x_1 + x_2 + x_3}{3},\ y = \frac{y_1 + y_2 + y_3}{3}.$$

The area of the triangle ABC, i.e.

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

$$= \frac{1}{2}\left[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right].$$

The three points A, B and C are collinear iff $\Delta = 0$

$$\Rightarrow \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_3 - y_1}{x_3 - x_1} \quad \text{or,} \quad \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$$

SLOPE OF A LINE

Slope 'm' of the line joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \tan\theta,$$

where θ is the angle which the line makes the positive x-axis.

LOCUS

A point P(x, y) changes its position on the xy-plane as x or y or both are given different values. x and y may change independently or otherwise under a constraint. When P(x, y) moves under a geometrical constraint or rule, y becomes a function of x and the point P(x, y) is said to trace a locus. The functional relation y = f(x) is called the equation of the path traced by P(x, y), when f(x) is a linear polynomial in x then this locus is a straight line i.e. the equation ax + by + c = 0 represents a straight line. Here $y = -\frac{a}{b}x - \frac{c}{b}$.

Example # 3

Find the locus of the middle points of the segment of a line passing through the point of intersection of the lines ax + by + c = 0 and lx + my + n = 0 and intercepted between the axes.

Solution: Any line (say L = 0) passing through the point of intersection of ax + by + c = 0 and lx + my + n = 0 is (ax+ by + c) + λ(lx + my + n)=0, where λ is any real number.

Point of intersection of L = 0 with axes are $\left(-\frac{c + \lambda n}{a + \lambda l}, 0\right)$ and $\left(0, -\frac{c + \lambda n}{b + \lambda m}\right)$.

Let the mid point be (h, k).

Then $h = -\frac{1}{2}\left(\frac{c+\lambda n}{a+\lambda l}\right)$

and $k = -\frac{1}{2}\left(\frac{c+\lambda n}{b+\lambda m}\right)$.

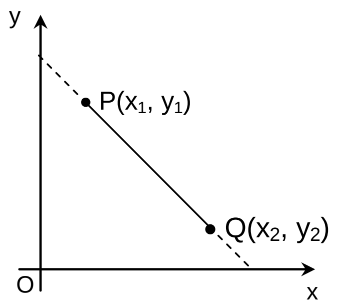
Eliminating λ, we get $\frac{2ah+c}{2hl+n} = \frac{2kb+c}{2km+n}$.

The required locus is: 2(am – lb)xy = (lc - an)x + (nb – mc)y.

FORMS OF THE EQUATION OF LINE

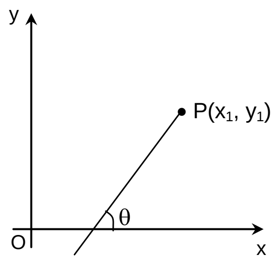
(i) The equation of the straight line passing through the point $P(x_1, y_1)$ and $Q(x_2, y_2)$ is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \quad \text{or,} \quad \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$



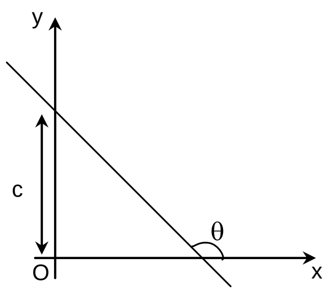
Two-point form

(ii) The equation of the straight line passing through the point $P(x_1, y_1)$ and having slope m (inclined at am angle θ, with m = tanθ, to the positive direction of the x-axis) is $y = y_1 = m(x - x_1)$.



Point-slope form

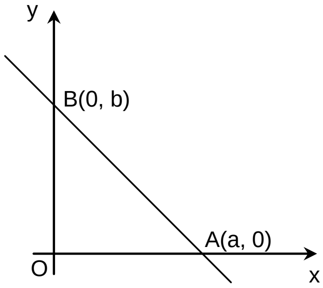
(iii) The equation of the straight line with slope m and y-intercept c on the y-axis is y = mx + c



Slope-intercept form

The equation of the x-axis is y = 0 (m = 0, c = 0) and that of the y-axis is x = 0 ($m \to \infty$).

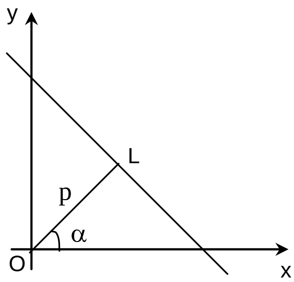
(iv) The equation of the straight line making intercepts a and b on the x-axis and the y-axis respectively is $\frac{x}{a} + \frac{y}{b} = 1$



Intercept form

(v) The equation of the straight line, for which the length of the perpendicular from the origin is p and the perpendicular makes an angle α with the positive x-axis, is

$x \cos\alpha + y \sin\alpha = p$



Normal form

(vi) The equation of the line passing through the point $P(x_1, y_1)$ and making an angle θ with the positive x-axis is

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r \qquad \text{(parametric form)}$$

where r is the distance of any point $(x_1 \pm r\cos\theta, y_1 \pm r\sin\theta)$ on the line from $P(x_1, y_1)$.

Example # 4

Reduce the line 2x - 3y + 5 = 0, in slope-intercept, intercept and normal forms.

Solution: **Slope-Intercept Form:** $y = \frac{2x}{3} + \frac{5}{3}$, $\tan\theta = m = 2/3$, $c = \frac{5}{3}$

Intercept Form: $\frac{x}{\left(-\frac{5}{2}\right)} + \frac{y}{\left(\frac{5}{3}\right)} = 1, a = -\frac{5}{2}, b = \frac{5}{3}$

Normal Form: $-\frac{2x}{\sqrt{13}} + \frac{3y}{\sqrt{13}} = \frac{5}{\sqrt{13}}$

$\sin\alpha = \frac{3}{\sqrt{13}}$, $\cos\alpha = \frac{-2}{\sqrt{13}}$, $p = \frac{5}{\sqrt{13}}$

Position of a point w.r.t. a line

A point $A(x_1, y_1)$ lies on the line $ax + by + c = 0$ if $ax_1 + by_1 + c = 0$.

The points $A(x_1, y_1)$ and $B(x_2, y_2)$ lie on the same side of the line $ax + by + c = 0$ if $ax_1 + by_1 + c = 0$ and $ax_2 + by_2 + c = 0$ have the same sign.

The points $A(x_1, y_1)$ and $B(x_2, y_2)$ lies on the opposite sides of the line $ax + by + c = 0$ if $ax_1 + by_1 + c = 0$ and $ax_2 + by_2 + c = 0$ are of opposite signs.

Example # 5

Find the range of θ in the interval $(0, \pi)$ such that the points (3, 5) and $(\sin\theta, \cos\theta)$ lie on the same side of the line x + y - 1 = 0.

Solution: Here 3 + 5 - 1 = 7 > 0

Hence $\sin\theta + \cos\theta - 1 > 0$

$\Rightarrow \sin(\pi/4 + \theta) > 1/\sqrt{2} \Rightarrow \pi/4 < \pi/4 + \theta < 3\pi/4 \Rightarrow 0 < \theta < \pi/2$.

Angle between two lines

(i) Let θ be the acute angle, between two straight lines $y = m_1x + c_1$ and $y = m_2x + c_2$. Then

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right|$.

The lines are parallel if $\tan\theta = 0 \Rightarrow m_1 = m_2$.

(ii) The lines are perpendicular if $\theta = 90° \Rightarrow m_1m_2 = -1$.

Any line parallel to the line ax + by + c = 0 has the equation ax + by = k, where k is an arbitrary constant to be obtained from the given geometrical constraints.

(iii) Any line perpendicular to the line ax + by + c = 0 as the equation $bx - ay = \lambda$, where λ is an arbitrary constant to be obtained from the given geometrical constraints.

Example # 6

Find the equation to the straight line which is perpendicular bisector of the line segment AB, where A, B are (a, b) and (a′, b′) respectively.

Solution: Equation of AB is $y - b = \frac{b' - b}{a' - a}(x - a)$

i.e. $y(a' - a) - x(b' - b) = a'b - ab'$.

Equation to the line perpendicular to AB is of the form

$(b' - b)y + (a' - a)x + k = 0$ ...(1)

Since the midpoint of AB lies on (1)

$(b' - b)\left(\frac{b + b'}{2}\right) + (a' - a)\left(\frac{a + a'}{2}\right) + k = 0$.

Hence the required equation of the straight line is

$2(b' - b)y + 2(a' - a)x = (b'^2 - b^2 + a'^2 - a^2)$.

Intersection and family of lines

(i) The point of intersection of two lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ is obtained by solving these equations for x and y.

(ii) The equation of any line passing through the intersection of these lines is $a_1x + b_1y + c_1 + \lambda(a_2x + b_2y + c_2) = 0$ where λ is an arbitrary constant to be obtained by using additional geometrical constraints. This equation also represents for any value of λ, a family of straight lines passing through the intersection of the fixed lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$.

(iii) The three lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$ are concurrent if

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

or, the point of intersection of any two of these lines lies on the third line.

Example # 7

Show that all the chords of the curve $3x^2 - y^2 - 2x + 4y = 0$, which subtend a right angle at the origin pass through a fixed point. Find that point.

Solution: Let the equation of chord be lx + my = 1.

So equation of pair of straight line joining origin to the points of intersection of chord and curve.

$3x^2 - y^2 - 2x(lx + my) + 4y(lx + my) = 0$, which subtends right angle at origin.

$\Rightarrow (3 - 2l + 4m - 1) = 0 \Rightarrow l = 2m + 1$

Hence chord becomes $(2m + 1)x + my = 1$

$\underbrace{x - 1}_{L_1} + m\underbrace{(2x + y)}_{L_2} = 0$

Which will pass through point of intersection of $L_1 = 0$ and $L_2 = 0$

$\Rightarrow x = 1, y = -2$. Hence fixed point is $(1, -2)$.

Distance of a point from a line

The perpendicular distance of the point $P(x_1, y_1)$ from the line $ax + by + c = 0$ is $\left|\dfrac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$.

Bisector of the angle between two lines

The equations of the bisectors of the angle between the lines $ax_1 + by_1 + c_1 = 0$, $ax_2 + by_2 + c_2 = 0$ are $\dfrac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm\dfrac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$.

These are two perpendicular lines; one represents the bisector of the acute angle and the other the bisector of the obtuse angle. If for a point (α, β) the expression $a_1\alpha + b_1\beta + c_1$ and $a_2\alpha + b_2\beta + c_2$ are of the same sign, then the equation of the bisector of the angle containing the point (α, β) is

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}.$$

The bisector of the angle containing the origin is also the bisector of the acute angle between the lines if $c_1c_2 > 0$ and $a_1a_2 + b_1b_2 < 0$.

Example # 8

For the straight lines $4x + 3y - 6 = 0$ and $5x + 12y + 9 = 0$, find the equation of the

(i) bisector of the obtuse angle between them.

(ii) bisector of the acute angle between them.

(iii) bisector of the angle which contains (1, 2).

Solution: Equations of bisectors of the angles between the given lines are

$$\frac{4x + 3y - 6}{\sqrt{4^2 + 3^2}} = \pm\frac{5x + 12y + 9}{\sqrt{5^2 + 12^2}} \Rightarrow 9x - 7y - 41 = 0 \text{ and } 7x + 9y - 3 = 0.$$

If θ is the acute angle between the line $4x + 3y - 6 = 0$ and the bisector $9x - 7y - 41 = 0$, then $\tan\theta = \left|\dfrac{-\frac{4}{3} - \frac{9}{7}}{1 + \left(\frac{-4}{3}\right)\frac{9}{7}}\right| = \dfrac{11}{3} > 1.$

Hence

(i) The bisector of the obtuse angle is $9x - 7y - 41 = 0$

(ii) The bisector of the acute angle is $7x + 9y - 3 = 0$

(iii) The bisector of the angle containing the origin

$$\frac{-4x - 3y + 6}{\sqrt{(-4)^2 + (-3)^2}} = \frac{5x + 12y + 9}{\sqrt{5^2 + 12^2}} \Rightarrow 7x + 9y - 3 = 0$$

(i) For the point (1, 2), $4x + 3y - 6 = 4 \times 1 + 3\times2 - 6 > 0$

$5x + 12y + 9 = 5 \times 1 + 12 \times 2 + 9 > 0$

Hence equation of the bisector of the angle containing the point (1, 2) is $\dfrac{4x + 3y - 6}{5} = \dfrac{5x + 12y + 9}{13}$

$\Rightarrow 9x - 7y - 41 = 0.$

Alternative:

Making C_1 and C_2 positive in the given equations, we get

$-4x - 3y + 6 = 0$ and $5x + 12y + 9 = 0$

Since $a_1a_2 + b_1b_2 = -20 - 36 = -56 < 0$, so the origin will lie in the acute angle. Hence bisector of the acute angle is given by

$$\frac{-4x - 3y + 6}{\sqrt{4^2 + 3^2}} = \frac{5x + 12y + 9}{\sqrt{5^2 + 12^2}}$$ i.e. $7x + 9y - 3 = 0$

Similarly bisector of obtuse angle is $9x - 7y - 41 = 0$.

SOME DEFINITIONS:

Incentre

The incentre of a triangle ABC is the point of intersection of the internal bisectors of the angles of the triangle.

Circumcentre

The circumcentre of a triangle ABC is the point of intersection of the right bisectors of the sides of the triangle. Circumcentre of a right angled triangle is the mid-point of the hypotenuse.

Orthocentre

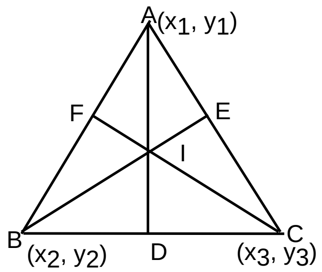
The orthocenter of a triangle ABC is the point of intersection of the perpendicular lines (altitudes) from the vertices to the opposite sides of the triangle. Orthocentre of a triangle ABC, right angled at A, is A.

Note: (i) In an equilateral triangle, the centroid, orthocentre, incentre and circumcentre coincide.

(ii) In a triangle ABC, the orthocentre H, the circumcentre O and the centroid. G are collinear where G divides OH in the ratio 1 : 2.

Example # 9

Prove that the incentre of the triangle whose vertices are given by $A(x_1,y_1)$,$B(x_2,y_2)$, $C(x_3, y_3)$ is $\left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c}\right)$ where a, b, and c are the sides opposite to the angles A, B and C respectively.



Solution: By geometry, we know that $\frac{BD}{DC} = \frac{AB}{AC}$ (since AD bisects $\angle A$).

If the lengths of the sides AB, BC and AC are c, a and b respectively, then $\frac{BD}{DC} = \frac{AB}{AC} = \frac{c}{b}$

$\Rightarrow$ Coordinates of D are $\left(\frac{bx_2 + cx_3}{b+c}, \frac{by_2 + cy_3}{b+c}\right)$.

Since $\frac{BD}{DC} = \frac{c}{b}$, $BD = \frac{ac}{b+c}$

B bisects $\angle B$. Hence $\frac{ID}{IA} = \frac{BD}{BA} = \frac{\left(\frac{ac}{b+c}\right)}{c} = \frac{a}{c+b}$

Let the coordinates of I be $(\bar{x}, \bar{y})$.

Then $\bar{x} = \frac{ax_1 + bx_2 + cx_3}{a+b+c}$, $\bar{y} = \frac{ay_1 + by_2 + cy_3}{a+b+c}$ (using section formula).

PAIR OF LINES

(i) The second degree equation $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$ represents a pair of lines if $h^2 \geq ab$ and $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ or, $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$.

Their point of intersection is $\left(\frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2}\right)$.

The angle θ between these lines is given by $\tan\theta = \pm\frac{\sqrt{h^2 - ab}}{a + b}$ so that the lines are **parallel** or **coincident** if $h^2 = ab$ and **perpendicular** to each other if $a + b = 0$.

(ii) The homogeneous equation $ax^2 + 2hxy + by^2 = 0$ represents a pair of lines passing through the origin. If $y = m_1x$ and $y = m_2x$ are two straight lines represented by $ax^2 + 2hxy + by^2 = 0$, then $m_1 + m_2 = -\frac{2h}{b}$ and $m_1m_2 = \frac{a}{b}$.

(iii) The equation of the pair of bisectors of the angles between the above pair of lines is $\frac{x^2 - y^2}{a - b} = \frac{xy}{h}$.

The equation of pair of lines through the origin and perpendicular to pair of lines given by $ax^2 + 2hxy + by^2 = 0$ is $bx^2 - 2hxy + ay^2 = 0$.

(iv) The equation of the pair of lines joining the origin and the points of intersection of a curve and a line is obtained by making the equation of the curve homogeneous with the help of the equation of the line.

If $y = mx + c$ be a straight line and a curve be $ax^2 + 2hxy + by^2 + 2gx + 2fy + k = 0$ and the line cuts the curve at points A and B, then the joint equation of OA and OB is

$$ax^2 + 2hxy + by^2 + (2gx + 2fy)\left(\frac{y - mx}{c}\right) + k\left(\frac{y - mx}{c}\right)^2 = 0.$$

Example # 10

The chord $\sqrt{6}\,y = \sqrt{8}\,px + \sqrt{2}$ of the curve $py^2 + 1 = 4x$ subtends a right angle at origin then find the value of p.

Solution: $\sqrt{3}\,y - 2px = 1$ is the given chord. Homogenizing the equation of the curve, we get,

$py^2 - 4x(\sqrt{3}\,y - 2px) + (\sqrt{3}\,y - 2px)^2 = 0$

$\Rightarrow (4p^2 + 8p)x^2 + (p + 3)y^2 - 4\sqrt{3}\,xy - 4\sqrt{3}\,pxy = 0$

Now, angle at origin is 90^0

$\therefore$ coefficient of x^2 + coefficient of $y^2 = 0$

$\therefore 4p^2 + 8p + p + 3 = 0 \Rightarrow 4p^2 + 9p + 3 = 0$

$\therefore p = \frac{-9 \pm \sqrt{81 - 48}}{8} = \frac{-9 \pm \sqrt{33}}{8}$.