DEFINITIONS

A circle is the locus of a point which moves in such a way that its distance from a fixed point, called the centre, is always a constant. The distance r from the centre is called the radius of the circle. Twice the radius is known as the diameter d = 2r. The perimeter C of a circle is called the circumference, and is given by $C = \pi d = 2\pi r$.

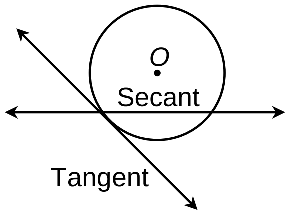
The angle a circle subtends from its centre is a full angle equal to 360^0 or 2π radians.

Secant: A line, which intersects a circle in two distinct points, is called a secant.

Tangent: A line meeting a circle only in one point is called a tangent to the circle at that point.

The point at which the tangent line meets the circle is called the point of contact.

Length of tangent: The length of the line segment of the tangent between a given point and the given point of contact with the circle is called the length of the tangent from the point to the circle.



Equation of a circle

The equation of a circle with (h, k) as its centre and a with radius is $(x - h)^2 + (y - k)^2 = a^2$

when the centre is at the origin, the equation of the circle become $x^2 + y^2 = a^2$.

General equation of a circle is $x^2 + y^2 + 2gx + 2fy + c = 0$ or $(x + g)^2 + (y + f)^2 = g^2 + f^2 - c$. So that its centre is at (– g, – f) and its radius is $\sqrt{g^2 + f^2 - c}$.

The equation of the circle with (x_1, y_1) and (x_2, y_2) as the extremities of arc of its diameter is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2)$.

Example 1 : If the equations of the two diameters of a circle are x + y = 6 and x + 2y = 4 and the radius of the circle is 10, find the equation of the circle.

Solution : Here radius of the circle = 10.

Equations of two diameters say AB and ML of the circle are respectively

$x + y = 6$ ...(i)

and $x + 2y = 4$ ...(ii)

solving (i) and (ii), we get x = 8 and y = –2.

Hence centre of the circle is (8, –2).

Now the equation of the required circle is

$(x - 8)^2 + (y + 2)^2 = 10^2$ or $x^2 + y^2 - 16x + 4y - 32 = 0$.

Position of a Point

The point $P(x_1, y_1)$ lies inside the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ if $S_{(x_1, y_1)} \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c < 0$.

The point $P(x_1, y_1)$ lies on the circle S = 0 if $S_{(x_1, y_1)} \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$.

The point $P(x_1, y_1)$ lies outside the circle S = 0 if $S_{(x_1, y_1)} \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c > 0$.

Example 2 : How are the points (0, 1), (3, 1) and (1, 3) situated with respect to the circle $x^2 + y^2 - 2x - 4y + 3 = 0$?

Solution: Given equation is $x^2 + y^2 - 2x - 4y + 3 = 0$

Let A ≡ (0, 1), B ≡ (3, 1) and C ≡ (1, 3)

For point A(0, 1), $x^2 + y^2 - 2x - 4y + 3 = 0^2 + 1^2 - 2.0 - 4.1 + 3 = 0$

Hence point A lies on the circle

For point B(3, 1), $x^2 + y^2 - 2x - 4y + 3 = 3 > 0$

Hence, point B lies outside the circle.

For point C(1, 3), $x^2 + y^2 - 2x - 4y + 3 = -1 < 0$

Hence point C lies inside the circle.

Two circles

$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and

$S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ be two circles with centre at $A(-g_1, -f_1)$ and $B(-g_2, -f_2)$ and radii r_1 and r_2.

Circle $S_1 = 0$ is outside the circle $S_2 = 0$ (and the circles are non-intersecting), then $AB > r_1 + r_2$.

If the two circles are touching externally, then $AB = r_1 + r_2$.

If the two circles are touching internally, then $AB = |r_1 - r_2|$.

If the circle $S_1 = 0$ is completely inside the circle $S_2 = 0$, then $AB < |r_1 - r_2|$.

Example 3 : Prove that the circle $x^2 + y^2 + 2ax + c^2 = 0$ and $x^2 + y^2 + 2by + c^2 = 0$ touch each other if $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}$.

Solution: Given circles are

$x^2 + y^2 + 2ax + c^2 = 0$ ...(i)

and $x^2 + y^2 + 2by + c^2 = 0$ ...(ii)

Let A and B be the centres of circles (i) and (ii) respectively and r_1 and r_2 be their radii, then

$A \equiv (-a, 0), B \equiv (0, -b), r_1 = \sqrt{a^2 - c^2}, r_2 = \sqrt{b^2 - c^2}$

The two circles (i) and (ii) will touch each other externally or internally according as

$AB = r_1 + r_2$ or $AB = |r_1 - r_2|$

i.e., $AB^2 = (r_1 + r_2)^2$ or $AB^2 = (r_1 - r_2)^2$

Thus the two circles will touch each other if

$AB^2 = (r_1 \pm r_2)^2$ or $a^2 + b^2 = r_1^2 + r_2^2 \pm 2r_1r_2$

or $a^2 + b^2 = a^2 - c^2 + b^2 - c^2 \pm 2\sqrt{a^2 - c^2}\sqrt{b^2 - c^2}$

or $2c^2 = \pm 2\sqrt{a^2 - c^2}\sqrt{b^2 - c^2}$ or $c^2 = \pm\sqrt{a^2 - c^2}\sqrt{b^2 - c^2}$

or $c^4 = (a^2 - c^2)(b^2 - c^2)$ or $c^4 = a^2b^2 - c^2b^2 - a^2c^2 + c^4$ or $c^2b^2 + a^2c^2 = a^2b^2$

or $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}$. [dividing by $a^2b^2c^2$]

A Line and a Circle

The line $ax + by + c = 0$ and the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ intersect in two points whose coordinates are obtained by solving the equations L = 0, S = 0 simultaneously.

The resulting equation, after eliminating x (or y) is a quadratic equation which may have real (distinct or equal) or imaginary roots. If the roots are distinct, the points of intersection are real, if the roots are equal , then line touches the circle. In case of imaginary roots, the line does not cut or touch the circle.

Example 4 : Show that the line $3x - 4y - c = 0$ will meet the circle having centre at (2, 4) and the radius 5 in real and distinct points if $-35 < c < 15$.

Solution: Given line is $3x - 4y - c = 0$ ...(i)

Centre of given circle is (2, 4) and its radius is 5, therefore its equation will be

$(x - 2)^2 + (y - 4)^2 = 5^2$

or $x^2 + y^2 - 4x - 8y - 5 = 0$ ...(ii)

From (i), $y = \frac{1}{4}(3x - c)$. Putting the value of y in (ii), we get

$$x^2 + \frac{1}{16}(3x - c)^2 - 4x - 8.\frac{1}{4}(3x - c) - 5 = 0$$

or $16x^2 + 9x^2 - 6cx + c^2 - 64x - 96x + 32c - 80 = 0$

or $25x^2 - 2(80 + 3c)x + c^2 + 32c - 80 = 0$ ...(iii)

Line (i) will meet the circle (ii) in real and distinct points if discriminant of equation (iii) > 0

i.e., if $4(80 + 3c)^2 - 100(c^2 + 32c - 80) > 0$ or $(80 + 3c)^2 - 25(c^2 + 32c - 80) > 0$

or $6400 + 9c^2 + 480c - 25c^2 - 800c + 2000 > 0$

or $-16c^2 - 320c + 8400 > 0$ or $16c^2 + 320c - 8400 < 0$

or $c^2 + 20c - 525 < 0$

sign scheme for $c^2 + 20c - 525$:

When $c^2 + 20c - 525 = 0$,

$$c = \frac{-20 \pm \sqrt{400 + 2100}}{2} = \frac{-20 \pm 50}{2} = -35, 15$$

Therefore, sign scheme for $c^2 + 20c - 525$ is as follows

$\therefore \quad c^2 + 20c - 525 < 0 \Leftrightarrow -35 < c < 15.$

Tangent and Normal to a Circle

The equation of the tangent to the circle S = 0 at (x_1, y_1) is
$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0.$
The equation of the tangent to the circle $x^2 + y^2 = a^2$ at (x_1, y_1) is
$xx_1 + yy_1 = a^2.$

The line(s) $y = mx \pm a\sqrt{1+m^2}$ is a tangent to the circle (or touches the circle) $x^2 + y^2 = a^2$ for all real values of m.

Example 5 : Find the equation of the normal to the circle $x^2 + y^2 - 2x - 4y + 3 = 0$ at the point (2, 3).

Solution: Given circle is $x^2 + y^2 - 2x - 4y + 3 = 0$ ...(i)

Let $P \equiv (2, 3)$

Let A be the centre of circle (i), then $A \equiv (1, 2)$

Normal at point P(2, 3) of the circle will be line AP.

$$\therefore \quad y - 3 = \frac{2-3}{1-2}(x-2)$$

or $y - 3 = x - 2$ or $x - y + 1 = 0.$

The lines

$y + f = m(x + g) \pm r\sqrt{1+m^2}$, where $r = \sqrt{f^2 + g^2 - c}$, is tangent to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ for all m.

The line $y - b = m(x - a) \pm r\sqrt{1+m^2}$ touches the circle $(x - a)^2 + (y - b)^2 = r^2$ for all m.

The line L = 0 touches the circle S = 0 if the perpendicular from the centre of the circle to the line L = 0 is equal to the radius of the circle.

The length of the tangent to the circle S = 0 from a point $P(x_1, y_1)$ is $\sqrt{S_{(x_1, y_1)}} = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$.

From a point $P(x_1, y_1)$ outside the circle S = 0, two tangents can be drawn to the circle. The equation of the pair of tangents from $P(x_1, y_1)$ to S = 0 is
$SS_1 = T^2$
which $S_1 = S(x_1, y_1)$ and $T = xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$. Any line passing through the centre of the circle is normal to the circle at the point where it meets the circle. The line $y - y_1 = \frac{y_1 + f}{x_1 + g}(x - x_1)$ is normal to the circle S = 0 at the point (x_1, y_1).

Parametric Equation of a Circle

Any point on the circle $x^2 + y^2 = a^2$ may be written as $(a \cos \theta, a \sin \theta)$. $x = a \cos \theta$, $y = a \sin \theta$ are the parametric equations of a circle. The equation of the tangent to the circle at any point is $x \cos \theta + y \sin \theta = a$.
Any point in the circle $(x - a)^2 + (y - b)^2 = r^2$ is $(a + r \cos \theta, B + r \sin \theta)$.

Chords of a Circle

From a point $P(x_1, y_1)$ let us draw two tangents to the circle S = 0 so as to touch it at Q and R. The line QR is called the chord of contact of the two tangents to the circle S = 0. The equation of the chord of contact is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$. If the circle is $x^2 + y^2 = a^2$, then the corresponding equation of the chord of contact is $xx_1 + yy_1 = a^2$. The chord of the circle S = 0 whose midpoint is (x_1, y_1) is $S_1 = T$ i.e. $S(x_1, y_1) = T$
or, $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$.

The equation of the chord of the $x^2 + y^2 = a^2$ with mid point (x_1, y_1) is $x_1^2 + y_1^2 = xx_1 + yy_1$.

Radical Axis of Two Circles

Let $S_1 = 0$ and $S_2 = 0$ be two circles with the coefficients of x^2 and y^2 equal to unity. Let the two circles be non-intersecting the locus of the point P, such that the length of the tangents to the two circle from it are equal, is called the radical axis of the two circles. Equation of the radical axis of $S_1 = 0$, $S_2 = 0$ is $S_1 - S_2 = 0$, which is linear in λ and y and hence represents a straight line.

When the circles are intersecting each other, the radical axis becomes the common chord of the two circles with its equation $S_1 - S_2 = 0$.

When the circles are touching each other (internally or externally) then the common chord becomes the common tangent to the two circles, with its equation $S_1 - S_2 = 0$.

Orthogonal Circles

Let $S_1 = 0$ and $S_2 = 0$ be two circles with radii r_1 and r_2 respectively. The two circles are said to cut orthogonally if the tangents to the two circles at their point of intersection are perpendicular to each other. It means that $AB^2 = r_1^2 + r_2^2$ (where A, B are the centres of the circles)

$\Rightarrow 2f_1f_2 + 2g_1g_2 = c_1 + c_2$.

Example 6 : Obtain the equation of the circle orthogonal to both the circles $x^2 + y^2 + 3x - 5y + 6 = 0$ and $4x^2 + 4y^2 - 28x + 29 = 0$ and whose centre lies on the line $3x + 4y + 1 = 0$.

Solution: Given circles are $x^2 + y^2 + 3x - 5y + 6 = 0$,,,(i)

and $4x^2 + 4y^2 - 28x + 29 = 0$

or $x^2 + y^2 - 7x + \frac{29}{4} = 0$. ...(ii)

Let the required circle be $x^2 + y^2 + 2gx + 2fy + c = 0$ ...(iii)

Since circle (iii) cuts circles (i) and (ii) orthogonally

$\therefore\ 2g\left(\frac{3}{2}\right) + 2f\left(-\frac{5}{2}\right) = c + 6$ or $3g - 5f = c + 6$ ...(iv)

and $2g\left(-\frac{7}{2}\right) + 2f.0 = c + \frac{29}{4}$ or $-7g \equiv c + \frac{29}{4}$ ...(v)

(iv) – (v), we get $10g - 5f = -\frac{5}{4}$ or $40g - 20f = -5$. ...(vi)

Given line is $3x + 4y + 1 = 0$ ...(vii)

Since centre $(-g, -f)$ of circle (iii) lies on line (vii)

$\therefore\ -3g - 4f = -1$ ...(viii)

Solving (vi) and (viii), we get $g = 0$ and $f = \frac{1}{4}$

$\therefore$ from (v), $c = -\frac{29}{4}$

$\therefore$ From (iii), required circle is

$x^2 + y^2 + \frac{1}{2}y - \frac{29}{4} = 0$ or $4(x^2 + y^2) + 2y - 29 = 0$.

Family of Circles

From two points, an infinity of circles can be made to pass. All these circles belong a family. The equation of the family of circles passing through the points of intersection of the line L = 0 and the circle S = 0 is $S + \lambda L = 0$ where λ is an arbitrary constant whose value can be obtained from given geometrical conditions to fix the circle.

The equation of the family of circles passing through the points of intersection of the circles $S_1 = 0$ and $S_2 = 0$ is $S_1 + \lambda S_2 = 0$.

The equation of the family of circles touching a given line $L = ax + by + c = 0$ at the given point (x_1, y_1) is $(x - x_1)^2 + (y - y_1)^2 + \lambda(ax + by + c) = 0$.

The equation of the family of line passing through two given points (x_1, y_1) and (x_2, y_2) is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + \lambda \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0.$$

Example 7 : Find the equation of the circle described on the chord $x \cos \alpha + y \sin \alpha - p = 0$ of $x^2 + y^2 - a^2 = 0$ as diameter.

Solution:

Any such circle is a member of $x^2 + y^2 - a^2 + k (x \cos \alpha + y \sin \alpha - p) = 0$.

If this is to be the circle on the chord as diameter, the centre of the circle

$$\left(-\frac{k}{2}\cos\alpha, -\frac{k}{2}\sin\alpha\right)$$

should lie on $x \cos \alpha + y \sin \alpha - p = 0$

$$\therefore -\frac{k}{2}\cos^2\alpha + \left(-\frac{k}{2}\right)\sin^2\alpha = p \Rightarrow k = -2p.$$

The equation to the required circle is $x^2 + y^2 - a^2 - 2p (x \cos \alpha + y \sin \alpha - p) = 0$.