MATHEMATICS

PROGRESSIONS_SYNOPSIS

•	If $t_n = an + b$, then the series so formed is an A.P.
•	If $S_n = an^2 + bn$ then series so formed is an A.P.
•	If every term of an A.P. is increased or decreased by some quantity, the resulting terms will also be in A.P.
•	If every term of an A.P. is multiplied or divided by some non-zero quantity, the resulting terms will also be in A.P.
•	In an A.P. the sum of terms equidistant from the beginning and end is constant and equal to sum of first and last terms.
•	Sum and difference of corresponding terms of two A.P.'s will form an A.P.
•	If terms $a_1, a_2, ..., a_n, a_{n+1}, ..., a_{2n+1}$ are in A.P., then sum of these terms will be equal to $(2n + 1)a_{n+1}$.
•	If terms $a_1, a_2, ..., a_{2n-1}, a_{2n}$ are in A.P. The sum of these terms will be equal to $(2n) . \left(\frac{a_n + a_{n+1}}{2}\right)$
•	The product of the terms equidistant from the beginning and end is constant, and it is equal to the product of the first and the last term.
•	If every term of a G.P. is multiplied or divided by the some non-zero quantity, the resulting progression is a G.P.
•	If a_1, a_2, a_3 ... and b_1, b_2, b_3, ... be two G.P.'s of common ratio r_1 and r_2 respectively, then a_1b_1 , a_2b_2 ... and $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}$ ... will also form a G.P. Common ratio will be r_1r_2 and $\frac{r_1}{r_2}$ respectively.
•	If a_1, a_2, a_3, ... be a G.P. of positive terms, then $\log a_1, \log a_2, \log a_3$, ... will be an A.P. and conversely.
•	If every term of a H.P. is multiplied or divided by some non zero fixed quantity, the resulting progression is a H.P.
•	**INSERTION OF MEANS BETWEEN TWO NUMBERS** Let a and b be two given numbers. (i) Arithmetic Means If three terms are in A.P. then the middle term is called the arithmetic mean

(A.M.) between the other two i.e. if a, b, c are in A.P. then $b=\frac{a+c}{2}$ is the A.M. of a and b.

If a, A_1, A_2 , ... A_n, b are in A.P., then A_1, A_2, ... A_n are called n A.M.'s between a and b.

If d is the common difference, then $b = a + (n + 2 – 1)\, d \Rightarrow d = \frac{b-a}{n+1}$

$A_i = a + id = a + i\,\frac{b-a}{n+1}=\frac{a(n+1-i)+ib}{n+1}$, i = 1, 2, 3, ..., n

Note: The sum of n-A.s, i.e., $A_1 + A_2 + ... + A_n = \frac{n}{2}(a+b)$

(ii) Geometric means

If three terms are in G.P. then the middle term is called the geometric mean (G.M.) between the two. So if a, b, c are in G.P. then $b=\sqrt{ac}$ or $b=-\sqrt{ac}$ corresponding to a & c both are positive or negative respectively.

If a, G_1, G_2 ... G_n, b are in G.P., then G_1, G_2 ... G_n are called n G.M.s between a and b. If r is the common ratio, then $b = a.r^{n+1} \Rightarrow r = \left(\frac{b}{a}\right)^{\frac{1}{(n+1)}}$

$G_i = ar^i = a\left(\frac{b}{a}\right)^{\frac{i}{n+1}} = a^{\frac{n+1-i}{n+1}}.b^{\frac{i}{n+1}}$, i = 1, 2, ..., n

Note: The product of n-G. s i.e., $G_1\, G_2\, ...\, G_n = \left(\sqrt{ab}\right)^n$

(iii) Harmonic mean:

If a & b are two non-zero numbers, then the harmonic mean of a & b is a number H such that a, H, b are in H.P. & $\frac{1}{H}=\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right)$ or $H=\frac{2ab}{a+b}$

If a, H_1, H_2... H_n, b are in H.P., then H_1, H_2 ... H_n are called n between a and b.

If d is the common difference of the corresponding A.P., then

$$\frac{1}{b}=\frac{1}{a}+(n+2-1)d \Rightarrow d=\frac{a-b}{ab(n+1)}$$

$$\frac{1}{H_i}=\frac{1}{a}+id=\frac{1}{a}+i\frac{a-b}{ab(n+1)}, H_i=\frac{ab(n+1)}{b(n-i+1)+ia},\ i=1,2,3,\ ...,n$$

(iv) Term t_{n+1} is the arithmetic, geometric or harmonic mean of t_1 & t_{2n+1} according as the terms t_1, t_{n+1} t_{2n+1} are in A.P., G.P. or H.P. respectively.

- **ARITHMETIC-GEOMETRIC SERIES**

A series whose each term is formed, by multiplying corresponding terms of an A.P. and a G.P., is called an Arithmetic-geometric series.

e.g. $1 + 2x + 3x^2 + 4x^3 + ...$; $a + (a + d)\, r + (a + 2d)r^2 + ...$

(i) Summation of n terms of an Arithmetic-Geometric Series

Let $S_n = a + (a + d)r + (a + 2d)r^2 + \dots + [a + (n - 1)d]\, r^{n-1}$, ,

Multiply by 'r' and rewrite the series in the following way

$rS_n = ar + (a + d)r^2 + (a + 2d)r^3 + \dots + [a + (n - 2)d]r^{n-1} + [a + (n - 1)d\,]r^n$

on subtraction,

$S_n (1 - r) = a + d(r + r^2 + \dots + r^{n-1}) - [a + (n- 1)d]r^n$

or, $S_n(1-r) = a + \dfrac{dr(1-r^{n-1})}{1-r} - [a+(n-1)d].r^n$

or, $S_n = \dfrac{a}{1-r} + \dfrac{dr(1-r^{n-1})}{(1-r)^2} - \dfrac{[a+(n-1)d]}{1-r}.r^n$

(ii) Summation of Infinite Series

If $|r| < 1$, then $(n-1)r^n$, $r^{n-1} \to 0$, as $n \to \infty$. Thus $S_\infty = S = \dfrac{a}{1-r} + \dfrac{dr}{(1-r)^2}$

- **SUM OF MISCELLANEOUS SERIES**

(i) Difference Method

Suppose $a_1, a_2, a_3, \dots\dots\dots\dots$ is a sequence such that the sequence $a_2 - a_1$, $a_3 - a_2$, ... is either an A.P. or G.P. The nth term 'a_n' of this sequence is obtained as follows.

$S = a_1 + a_2 + a_3 + \dots\dots\dots + a_{n-1} + a_n$

$S = \quad a_1 + a_2 + \dots\dots\dots + a_{n-2} + a_{n-1} + a_n$

$\Rightarrow a_n = a_1 + \left[(a_2 - a_1) + (a_3 - a_2) + \dots\dots\dots + (a_n - a_{n-1})\right]$

Since the terms within the brackets are either in an A.P. or in a G.P. we can find the value of a_n, the nth term. We can now find the sum of the n terms of the sequence as $S = \sum_{k=1}^{n} a_k$

(ii) $V_n - V_{n-1}$ Method

Let T_1, T_2, T_3 , ... be the terms of a sequence. If there exists a sequence V_1, V_2, V_3 ...satisfying $T_k = V_k - V_{k-1}$, $k \geq 1$,

then . $S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} (V_k - V_{k-1}) = V_n - V_0$

- **INEQUALITIES**

(i)

Let $a_1, a_2, \dots\dots\dots, a_n$ be n positive real numbers, then we define their arithmetic mean

(A), geometric mean (G) and harmonic mean (H) as $A=\frac{a_1+a_2+.........+a_n}{n}$,

$G=(a_1a_2.........a_n)^{1/n}$ and $H=\frac{n}{\left(\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+.......\frac{1}{a_n}\right)}$

It can be shown that Moreover equality holds at either place if and only if $a_1 = a_2 = ... = a_n$

(ii) Weighted Means

Let $a_1, a_2, ..., a_n$ be n positive real numbers and $w_1, w_2, ..., w_n$ be n positive rational numbers. Then we define weighted Arithmetic mean (A*), weighted Geometric mean (G*) and weighted harmonic mean (H*) as

$$A^* = \frac{a_1w_1+a_2w_2+...+a_nw_n}{w_1+w_2+...+w_n}, \quad G^*=(a_1^{w_1}.a_2^{w_2}...a_n^{w_n})^{\frac{1}{w_1+w_2+...w_n}}$$

and $H^*=\frac{w_1+w_2+...+w_n}{\frac{w_1}{a_1}+\frac{w_2}{a_2}+...+\frac{w_n}{a_n}}$.

A* ³ G* ³ H* More over equality holds at either place if & only if $a_1 = a_2$i ....=.a_n

(iii) Cauchy's Schwartz Inequality:

If $a_1, a_2, ...a_n$ and $b_1, b_2, ..., b_n$ are 2n real numbers, then

$(a_1b_1 + a_2b_2 +.............+ a_n b_n)^2 \; (a_1^2 + a_2^2 +............+ a_n^2)(b_1^2 + b_2^2 +..........+b^2_n)$

with the equality holding if and only if $\frac{a_1}{b_1}=\frac{a_2}{b_2}=..........=\frac{a_n}{b_n}$.

- **ARITHMETIC MEAN OF mth POWER**

Let a_1, a_2 ... , a_n be n positive real numbers and let m be a real number, then

$$\frac{a_1^m+a_2^m+...+a_n^m}{n}\geq\left(\frac{a_1+a_2+...+a_n}{n}\right)^m, \text{ if } m\in R-[0,1].$$

However if (0, 1), then $\frac{a_1^m+a_2^m+...+a_n^m}{n}\leq\left(\frac{a_1+a_2+...+a_n}{n}\right)^m$

Obviously if $m\in\{0,1\}$, then $\frac{a_1^m+a_2^m+...+a_n^m}{n}=\left(\frac{a_1+a_2+...+a_n}{n}\right)^m$

- **KEY POINTS**

A.P.

If a = first term, d = common difference and n is the number of terms, then nth term is denoted by t_n and is given by

	$t_n = a + (n - 1) d$. Sum of first n terms is denoted by S_n and is given by $S_n = \frac{n}{2}[2a + (n-1)d]$ or $S_n = \frac{n}{2}(a + l)$, where l = last term in the series i.e., $l = t_n = a + (n - 1) d$. Arithmetic mean A of any two numbers a and b $A = \frac{a+b}{2}$. Sum of first n natural numbers ($\sum n$) $\sum n = \frac{n(n+1)}{2}$, where $n \in N$. Sum of squares of first n natural numbers ($\sum n^2$) $\sum n^2 = \frac{n(n+1)(2n+1)}{6}$ Sum of cubes of first n natural numbers (Σn^3) $\Sigma n^3 = \left[\frac{n(n+1)}{2}\right]^2$ **G.P.** If a = first term, r = common ratio and n is the number of terms, then n^{th} term, denoted by t_n, is given by $t_n = ar^{n-1}$ Sum of first n terms denoted by S_n is given by $S_n = \frac{a(1-r^n)}{1-r}$ or $\frac{a(r^n-1)}{r-1}$ In case r = 1, $S_n = na$.
•	Sum of infinite terms (S_∞) $S_\infty = \frac{a}{1-r}$ (for $\|r\| < 1 \& r \neq 0$)
•	**H.P.** If a, b are first two terms of an H.P. then $t_n = \frac{1}{\frac{1}{a} + (n-1)\left(\frac{1}{b} - \frac{1}{a}\right)}$ There is no formula for sum of n terms of an H.P.
•	**MEANS** If three terms are in A.P. then the middle term is called the arithmetic mean (A.M. between the other two i.e. if a, b, c are in A.P. then $b = \frac{a+c}{2}$ is the A.M. of a and b.

If a, A_1, A_2 , ... A_n, b are in A.P., then A_1, A_2, ... A_n are called n A.M.'s between a and b.

If d is the common difference, then b = a + (n + 2 – 1) d $\Rightarrow$ d = $\frac{b-a}{n+1}$

A_i **= a + id = a + i** $\frac{b-a}{n+1}=\frac{a(n+1-i)+ib}{n+1}$, **i = 1, 2, 3, ..., n**

If three terms are in G.P. then the middle term is called the geometric mean (G.M.) between the two. So if a, b, c are in G.P. then $b=\sqrt{ac}$ **or** $b=-\sqrt{ac}$ **corresponding to a & c both are positive or negative respectively.**

If a, G_1, G_2 ... G_n, b are in G.P., then G_1, G_2 ... G_n are called n G.M.s between a and b. If r is the common ratio, then b = a.r^{n+1} $\Rightarrow$ r = $\left(\frac{b}{a}\right)^{\frac{1}{(n+1)}}$

G_i **= ar^i =** $a\left(\frac{b}{a}\right)^{\frac{i}{n+1}}=a^{\frac{n+1-i}{n+1}}.b^{\frac{i}{n+1}}$, **i = 1, 2, ..., n**

If a & b are two non-zero numbers, then the harmonic mean of a & b is a number H such that a, H, b are in H.P. & $\frac{1}{H}=\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right)$ or $H=\frac{2ab}{a+b}$

If a, H_1, H_2... H_n, b are in H.P., then H_1, H_2 ... H_n are called n between a and b.

If d is the common difference of the corresponding A.P., then

$$\frac{1}{b}=\frac{1}{a}+(n+2-1)d\Rightarrow d=\frac{a-b}{ab(n+1)}$$

$$\frac{1}{H_i}=\frac{1}{a}+id=\frac{1}{a}+i\frac{a-b}{ab(n+1)};\ \mathbf{H_i}=\frac{ab(n+1)}{b(n-i+1)+ia},\ i=1,2,3,\ldots,n$$

PROGRESSIONS_ASSIGNMENT

1. The third term of a G.P. is 4. The product of the first five terms is

   A) 4^3 B) 4^5 C) 4^4 D) none of these

2. If a, b, c, d are in H.P,, then ab + bc + cd is equal to

   A) 3 ad B) (a + b) (c + d) C)e ac D) none of these

3. If A and G be the A.M. and G.M. respectively between two numbers, then the numbers are

   A) $A \pm \sqrt{G^2 - A^2}$ B) $A \pm \sqrt{A^2 - G^2}$ C) $A \pm \sqrt{A^2 + G^2}$ D) $G \pm \sqrt{A^2 - G^2}$

4. If S be the sum, P the product and R the sum of the reciprocals of n terms of a G.P., then

   $\left(\frac{S}{R}\right)^n =$

   A) P B) P^2 C) P^3 D) $\sqrt{P}$

5. If x, y, z be respectively the pth, qth and rth terms of G.P., then

   log x + (r – p) log y + (p – q) log z =

   A) 0 B) 1 C) –1 D) none

6. The sum of integers from 1 to 100 which are divisible by 2 or 5 is

   A) 300 B) 3050 C) 3200 D) 3250

7. The interior angles of a polygon are in arithmetic progression. The smallest angle is 120^0 and the common difference is 5. The number of sides of the polygon is

   A) 7 B) 9 C) 11 D) 16

8. If $S_n = nP + \frac{1}{2} n(n-1)Q$ where S_n denotes the sum of the first n terms of an A.P., then the common difference is

   A) P + Q B) 2P + 3Q C) 2Q D) Q

9. If $a_1, a_2, a_3 \ldots.. a_n$ are in H.P., then $\frac{a_1}{a_2+a_3+\ldots a_n}, \frac{a_2}{a_1+a_3+\ldots+a_n}, \ldots \frac{a_n}{a_1+a_2+\ldots+a_{n-1}}$ are in

   A) A.P. B) G.P. C) H.P. D) none of these

10. If one geometric mean G and two arithemtic means p and q be inserted between two numbers, then G^2 is equal to:

A) (3p – q) (3q – p) B) (2p – q) (2q – p)

C) (4p – q) (4q – p) D) none of these

11. If the arthimetic mean between a and b is $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$, then n =

A) 0 B) 1 C) –1 D) 1/2

12. The sum upto (2n + 1) terms of the series $a^2 - (a + d)^2 + (a + 2d)^2 - (a + 3d)^2 + ...$ is

A) $a^2 + 3nd^2$ B) $a^2 + 2nad + n(n - 1)d^2$

C) $a^2 + 3nad + n(n - 1)d^2$ D) none of these

13. The positive integer n for which $2 \times 2^2 + 3 \times 2^3 + 4 \times 2^4 + ... + n \times 2^n = 2^{n+10}$ is

A) 510 B) 511 C) 512 D) 513

14. If x > 0 and $\log_2 x + \log_2\left(\sqrt{x}\right) + \log_2\left(\sqrt[4]{x}\right) + \log_2\left(\sqrt[8]{x}\right) + \log_2\left(\sqrt[16]{x}\right) + ... = 4$ then x equals

A) 2 B) 3 C) 4 D) 5

15. If a, b and c are positive real numbers then $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}$ is greater than or equal to

A) 3 B) 6 C) 27 D) none of these

16. In a G.P., $T_2 + T_5 = 216$ and $T_4 : T_6 = 1:4$ and all terms are integers, then its first term is

A) 16 B) 14 C) 12 D) none

17. If $a_1, a_2, a_3, .. a_n$ be an A.P. of non-zero terms then $\frac{1}{a_1a_2}+\frac{1}{a_2a_3}+...+\frac{1}{a_{n-1}a_n}=$

A) $\frac{a}{a_1a_n}$ B) $\frac{n}{a_1a_n}$ C) $\frac{n-1}{a_1a_n}$ D) none

18. If the ratio of sum of m terms and n terms of an A.P. be $m^2 : n^2$, then the ratio of its m^{th} and n^{th} terms will be

A) 2m – : 2n – 1   B) M : n   C) 2m + 1 : 2n + 1   D) none

19. The ratio between the sum of n terms of two A.P.'s is 3n + 8 : 7 n + 15. Then the ratio between their 12^{th} terms is

A) 5 : 7   B) 7 : 16   C) 12 : 11   D) none

20. If A.M. between two numbers is 5 and their G.M. is 4, then the H.M. will be:

A) $\frac{16}{5}$ B) $\frac{14}{5}$ C) $\frac{11}{5}$ D) none of these

21. The A.M. between two numbers b and c is a and the two G.M.s between them are g_1 and g_2. If $g_1^3 + g_2^3 = k\,abc$, then k is equal to

A) 1   B) 2   C) 3   D) 4

22. Between 1 and 31 are inserted m arithmetic means, so that the ratio of the 7th and (m – 1)th means is 5 : 9. Then the value of m is

A) 12   B) 13   C) 14   D)15

23. Sum to n terms of the series $\frac{1}{(1+x)(1+2x)}+\frac{1}{(1+2x)(1+3x)}+\frac{1}{(1+3x)(1+4x)}+..........$ is

A) $\frac{nx}{(1+x)(1+nx)}$ B) $\frac{n}{(1+x)\left[1+(n+1)x\right]}$

C) $\frac{x}{(1+x)(1+(n-1)x)}$ D) none of these

24. $\sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)(n+3)....(n+k)}$ is equal to

A) $\frac{1}{(k-1)\underline{|k-1}}$ B) $\frac{1}{k\underline{|k}}$ C) $\frac{1}{(k-1)\underline{|k}}$ D) $\frac{1}{\underline{|k}}$

25. Sum of the series $S = 1 + \frac{1}{2}(1+2) + \frac{1}{3}(1+2+3) + \frac{1}{4}(1+2+3+4) + ...$ upto 20 terms is

A) 110 B) 111 C) 115 D) 116

26. The sum of first n terms of the series $1^2 + 2 \times 2^2 + 3^2 + 2 \times 4^2 + 5^2 + 2 \times 6^2 + ...$ is $n(n + 1)^2/2$ when n is even. When n is odd the sum of the series is

A) $n^2 (3n + 1)/4$ B) $n^2 \frac{(n+1)}{2}$ C) $n^3 (n - 1)/2$ D) none of these

27. Sum of the series $\sum_{r=1}^{n} \frac{r}{(n+1)!}$ is

A) $1 - \frac{1}{n!}$ B) $1 - \frac{1}{(n+1)!}$ C) $2 - \frac{1}{(n+1)!}$ D) none of these

28. If the A.M. and G.M. of two numbers are 13 and 12 respectively then the two numbers are

A) 8, 12 B) 8, 18 C) 10, 18 D) 12, 18

29. If 2p + 3q + 4r = 15, the maximum value of $p^3 q^5 r^7$ will be

A) 2180 B) $\frac{5^4.3^5}{2^{15}}$ C) $\frac{5^5.7^7}{2^{17}.9}$ D) 2285

30. If $a_1, a_2, ... a_n$ are positive real numbers whose product is a fixed number c, then the minimum value of $a_1 + a_2 + ... a_{n-1} + 2a_n$ is

A) $n(2c)^{1/n}$ B) $(n + 1)c^{1/n}$ C) $2nc^{1/n}$ D) $(n + 1) (2c)^{1/n}$

KEY SHEET

1)	B	2)	A	3)	B	4)	B	5)	A	6)	B	7)	B
8)	D	9)	C	10)	B	11)	A	12)	D	13)	D	14)	C
15)	A	16)	C	17)	C	18)	A	19)	B	20)	A	21)	B
22)	C	23)	B	24)	C	25)	C	26)	B	27)	B	28)	B
29)	C	30)	A										