

PERMUTATION AND COMBINATION STRAIGHT LINES

INTRODUCTION

Study of permutation and combination will introduce to us the methods to count the number of ways of doing a given work without actual counting. For example

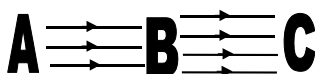
- (i) If the product of the digits of a 3 digit number is 10, then the possibilities of the digits are 1, 2, 5 only. Hence there are 6 possible numbers (125, 152, 215, 251, 512, 521). If the number is made of 7 digits and the product of the digits is 420, then finding the number of all possible numbers is difficult by actual counting. But after studying this topic the task will become simple.
- (ii) If a person has three shirts S_1, S_2, S_3 and two pants P_1, P_2 , then he can choose one shirt and one pant in 6 ways : $S_1P_1, S_1P_2, S_2P_1, S_2P_2, S_3P_1, S_3P_2$. Here we had to count all the cases but after the study of this topic one can find the number of ways easily even if their number is very large, without actual counting.

FUNDAMENTAL PRINCIPLE OF COUNTING

Multiplication rule or Rule of Product : Let a work C can be done in two parts i.e. A and B. A can be done in m ways and B can be done in n ways. Then the work C can be done in $m \times n$ ways. A similar treatment can be done when the work C completes only after completion of three or four or more parts.

Example 1 : A person wants to go from station A to station C via station B. There are three routes from A to B and four routes from B to C. In how many ways can he travel from A to C?

Solution: A \rightarrow B in 3 ways
B \rightarrow C in 4 ways
 \Rightarrow A \rightarrow C in $3 \times 4 = 12$ ways



Remark:

The rule of product is applicable only when the number of ways of doing each part is independent of each other i.e. corresponding to any method of doing the first part, the other part can be done by any method

Addition Rule : Let a work C is completed when either of works A or B (or more) has been done. Let A can be done in m ways and B can be done in n ways, then the work C can be done in $(m + n)$ ways.

- Number of permutations of n different things taken all together = $n!$
- Number of permutations of n different things taken r together

$$= \frac{n!}{(n-r)!} = {}^n P_r.$$

If $r > n$, then ${}^n P_r = 0$.

(Note: $0! = 1$.)

- Number of permutations of n things taken all together when p of them is alike and of one kind, q of another kind and alike, r of third kind and alike and the rest are different

$$= \frac{n!}{p!q!r!}$$

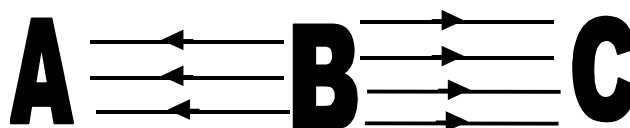
- Number of permutations of n different things taken r at a time when each thing can be repeated r times

$$= n^r$$

$${}^n P_r = r \cdot {}^{n-1} P_{r-1} + {}^{n-1} P_r$$

Example 2 : A person wants to leave station B. There are three routes from station B to A and four routes from B to C. In how many ways can he leave the station B.

Solution: B \rightarrow A in 3 ways
B \rightarrow C in 4 ways
 \Rightarrow He can leave station B in $3 + 4 = 7$ ways.



PERMUTATION

For putting n things at n places, there is simply one way to complete the task. But if the order of things is considered, then there are several ways. In this case, we use the word arrangement and the process is termed as permutation.

Example 3 : How many number can be formed using the digits 7, 8 and 9 without using any mathematical symbol (e.g. 798 , 79^8 , 7^{98} , 7^{89}).

Solution : Number of numbers of each form abc , ab^c , a^{bc} and a^{b^c} are 3P_3 . Therefore total numbers = $4 \times {}^3P_3 = 4 \times 3 \times 2 \times 1 = 24$.

COMBINATION

If in the above case we use the word 'selection' instead of 'arrangement', then the order of things is of no importance and the process is termed as combination.

While dealing with permutation and combination, there is fundamental principal of counting to be acquainted with.

Example 4 : There are two boys B_1 and B_2 . B_1 has n_1 different toys and B_2 has n_2 different toys. Find the number of ways in which B_1 and B_2 can exchange their toys in such a way that after exchanging they still have same number of toys but not the same set.

Solution: Total number of toys = $n_1 + n_2$
 Now let us keep all toys at one place and ask B_1 to pick up any n_1 toys out of these $n_1 + n_2$ toys. He can do it in ${}^{n_1+n_2}C_{n_1}$ ways Out of these ways there is one way when he picks up those n_1 toys which he was initially having.
 Thus required number of ways are ${}^{n_1+n_2}C_{n_1} - 1$.

CIRCULAR PERMUTATIONS

(i). When view is one sided i.e. clockwise and anticlockwise arrangements are different.

- Number of circular permutations of n things taken all together
 $= (n - 1)!$
- Number of circular permutations of n things taken r at a time
 $= \frac{1}{r} {}^n p_r$

(ii). When view is two sided i.e. clockwise and anticlockwise arrangements are same.

- Number of circular permutations of n things taken all together
 $= \frac{1}{2}(n - 1)!$
- Number of circular permutations of n things taken r at a time
 $= \frac{1}{2r} {}^n p_r$.
- Number of combinations of n different things taken all together = 1
- Number of combinations of n different things taken r at a time
 $= {}^n C_r = \frac{n!}{r!(n-r)!} \quad (r \leq n).$

$${}^n C_r = {}^n C_{n-r}$$

If ${}^n C_x = {}^n C_y$, then either $x = y$ or $x + y = n$

$${}^n C_r + {}^n C_{r-1} = {}^{n+1} C_r$$

$$r {}^n C_r = n {}^{n-1} C_{r-1}.$$

- Number of combinations of n different things taken r at a time when p particular things are always included
 $= {}^{n-p} C_{r-p}$
- Number of combinations of n different things taken r at a time when p particular things are always to be excluded
 $= {}^{n-p} C_r$

- Number of combinations of n different things taken one or more at a time
 $= 2^n - 1$.
- Number of selections of one or more things from p identical things of one type, q identical things of another type, r identical things of third type and rest is different things
 $= (p + 1)(q + 1)(r + 1) 2^s - 1$.

Example 5 : Find the number of ways of arranging 4 boys and 4 girls in a circle so that boys and girls are alternate.

Solution : First of all we arrange 4 girls in a circle, whose number of ways is $|3 = 6$. For any of their arrangement, the four boys can be arranged (at alternate positions) in $|4$ ways (as after arranging the girls, the positions in the circle are labeled). Thus totally there are $|3 \times |4 = 144$ ways.

ALL POSSIBLE SELECTIONS

Selection from distinct objects: The number of selections from n different objects, taken at least one
 $= {}^n C_1 + {}^n C_2 + {}^n C_3 + \dots + {}^n C_n = 2^n - 1$.

In other words, for every object, we have two choices i.e. either select or reject in a particular group. Total number of choices (all possible selections) = $2 \cdot 2 \cdot 2 \dots n$ times = 2^n .

But this also includes the case when none of them is selected and the number of such cases = 1.

Hence the number of selections, when at least one is selected = $2^n - 1$.

Selection from identical objects:

- The number of selections of r objects out of n identical objects is 1.
- Total number of selections of zero or more objects from n identical objects is $n + 1$.
- The total number of selections of at least one out of $a_1 + a_2 + a_3 + \dots + a_n$ objects, where a_1 are alike (of one kind), a_2 are alike (of second kind) and so on $\dots a_n$ are alike (of n th kind), is $[(a_1 + 1)(a_2 + 1)(a_3 + 1) \dots (a_n + 1)] - 1$.

Selection when both identical and distinct objects are present:

The number of selections taking at least one out of $a_1 + a_2 + a_3 + \dots + a_n + k$ objects, where a_1 are alike (of one kind), a_2 are alike (of second kind) and so on $\dots a_n$ are alike (of n th kind), and k are distinct = $[(a_1 + 1)(a_2 + 1)(a_3 + 1) \dots (a_n + 1)] 2^k - 1$.

Example 6. Let a person have 3 coins of 25 paise, 4 coins of 50 paise and 2 coins of 1 rupee. Then, in how many ways can he give none or some coins to a beggar? Further find the number of ways so that

- he gives at least one coin of one rupee.
- he gives at least one coin of each kind.

Solution: Total number of ways of giving none or some coins is

$$(3 + 1)(4 + 1)(2 + 1) = 60 \text{ ways}$$

(i) Number of ways of giving at least one coin of one rupee

$$= (3 + 1)(4 + 1) \times 2 = 40$$

(ii) Number of ways of giving at least one coin of each kind

$$= 3 \times 4 \times 2 = 24.$$

Total number of divisors of a given natural number:

To find number of divisors of a given natural number greater than 1 we can write n as

$$n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \dots p_n^{\alpha_n}$$

where p_1, p_2, \dots, p_n are distinct prime numbers and $\alpha_1, \alpha_2, \dots, \alpha_n$ are positive integers. Now any divisor of n will be of the form $d = p_1^{\beta_1} p_2^{\beta_2} \dots p_n^{\beta_n}$ (where $0 \leq \beta_i \leq \alpha_i, \beta_i \in \mathbb{I}, \forall i = 1, 2, 3, \dots, n$)

Here number of divisors will be equal to numbers of ways in which we can choose β_i 's which can be done in $(\alpha_1 + 1)(\alpha_2 + 1) \dots (\alpha_n + 1)$ ways. e.g. Let $n = 360 \Rightarrow n = 2^3 \cdot 3^2 \cdot 5$

$$\Rightarrow \text{No. of divisors of } 360 = (3 + 1)(2 + 1)(1 + 1) = 24$$

Sum of all the divisors of n is given by

$$\left(\frac{p_1^{\alpha_1+1} - 1}{p_1 - 1}\right) \cdot \left(\frac{p_2^{\alpha_2+1} - 1}{p_2 - 1}\right) \cdot \left(\frac{p_3^{\alpha_3+1} - 1}{p_3 - 1}\right) \cdots \left(\frac{p_n^{\alpha_n+1} - 1}{p_n - 1}\right)$$

$$\text{As in above case, sum of all the divisors} = \left(\frac{2^4 - 1}{2 - 1}\right) \left(\frac{3^3 - 1}{3 - 1}\right) \left(\frac{5^2 - 1}{5 - 1}\right) = 1170$$

Remark: The number of factors of a given natural number 'n' will be odd if and only if 'n' is a perfect square.

DIVISION AND DISTRIBUTION OF OBJECTS (with fixed number of objects in each group)

Into groups of unequal size (different number of objects in each group):

(a) Number of ways in which n distinct objects can be divided into r unequal groups containing a_1 objects in the first group, a_2 objects in the second group and so on

$$= {}^n C_{a_1} \cdot {}^{n-a_1} C_{a_2} \cdot {}^{n-a_1-a_2} C_{a_3} \cdots {}^{a_r} C_{a_r} = \frac{n!}{a_1! a_2! a_3! \cdots a_r!}$$

Here $a_1 + a_2 + a_3 + \dots + a_r = n$.

(b) Number of ways in which n distinct objects can be distributed among r persons such that first person gets a_1 objects, 2nd person gets a_2 objects..., rth person gets a_r objects

$$= \frac{n! r!}{a_1! a_2! a_3! \cdots a_r!}$$

Explanation: Let us divide the task into two parts. In the first part, we divide the objects into groups. In the second part, these r groups can be assigned to r persons in $r!$ ways.

Into groups of equal size (each group containing same number of objects):

(a) Number of ways in which $m \times n$ distinct objects can be divided equally into n groups (unmarked) =

$$\frac{(mn)!}{(m!)^n n!}$$

(b) Number of ways in which $m \times n$ different objects can be distributed equally among n persons (or numbered groups) = (number of ways of dividing into groups) \times (number of groups)! =

$$\frac{(mn)! n!}{(m!)^n n!} = \frac{(mn)!}{(m!)^n}$$

Derangements: If n things are arranged in a row, the number of ways in which they can be deranged so that none of them occupies its original place is

$$= n! \sum_{r=0}^n (-1)^r \frac{1}{r!}$$

Example 7: Suppose 4 letters are taken out of 4 different envelopes. In how many ways, can they be reinserted in the envelopes so that no letter goes in to its original envelope?

Solution: Using the formula for the number of derangements that are possible out of 4 letters in 4 envelopes, we get the number of ways as :

$$4! \left(1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right) = 24 \left(1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24}\right) = 9.$$

USE OF MULTINOMIAL THEOREM IN PERMUTATION AND COMBINATION

The general term in the expansion of $(x_1 + x_2 + x_3 + \dots + x_m)^n$... (1) is

$\frac{n!}{\alpha_1! \alpha_2! \dots \alpha_m!} x_1^{\alpha_1} x_2^{\alpha_2} \dots x_m^{\alpha_m}$... (2) where

$$\alpha_1 + \alpha_2 + \dots + \alpha_m = n \quad \dots (3)$$

- Number of terms in the expansion of (1) is number of non-negative integral solutions of (3) i.e. number of ways of distributing n identical things among m persons when each person can get zero or more things = ${}^{m+n-1} C_n$ or ${}^{m+n-1} C_{m-1}$

Now in $(1 + ax)^n$ coeff. of x^r

\Rightarrow number of ways of selecting r a's out of n a's, which is same as coeff. of x^r in $(1 + x)^n$.

We can use this concept while dealing with problems of permutation and combination. Let out of n things p are of one type and rest are different. Now we have to select r things out of these n things. For this, we to select one from p alike things and $(n - 1)$ from rest different things or 2 from p alike and $(n - 2)$ from rest different things and so on i.e.

Coeff. of x^r in $(1 + x + x^2 + \dots + x^p)(1 + x)^{n-p}$

In same way number of solutions (integral solution) of $x_1 + x_2 + x_3 + \dots + x_m = n$, where

$a_1 \leq x_1 \leq b_1; \dots; a_m \leq x_m \leq b_m$ is coefficient of x^n in the expansion of

$$(x^{a_1} + x^{a_1+1} + \dots + x^{b_1})(x^{a_2} + x^{a_2+1} + \dots + x^{b_2}) \dots (x^{a_m} + x^{a_m+1} + \dots + x^{b_m})$$

- Number of combinations of r things out of n things of which p are alike and are of one kind, q are alike and of second kind and rest $(n - p - q)$ are different

$$= \text{coeff. of } x^r \text{ in } (1 + x + x^2 + \dots + x^p)(1 + x + x^2 + \dots + x^q)(1 + x)^{n-p-q}$$

- Number of ways in which r identical things can be distributed among n persons when each person can get zero or more things

$$= \text{coeff. of } x^r \text{ in } (1 + x + x^2 + \dots + x^r)^n$$

$$= {}^{n+r-1}C_r$$

- Number of non-negative integral solutions of the equation $x + 2y + 3z + 4w = n$
= coefficient of permutations of r things out of n things of which p are of one kind, q are of second kind and so on

$$= r! \cdot \text{coeff. of } x^r \text{ in } \left[\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^p}{p!} \right) \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^q}{q!} \right) \dots \right]$$

- Number of rectangles of any size in a square of size $n \times n = \sum_{r=1}^n r^3$ and number of squares = $\sum_{r=1}^n r^2$.

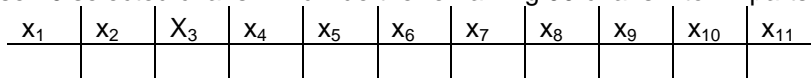
- Number of ways of distribution of n distinct balls in r distinct boxes when order is considered
= $n! {}^{n-1}C_{r-1}$ if blank boxes are not allowed
= $n! {}^{n+r-1}C_{r-1}$ if blank boxes are allowed.

- Number of ways of distribution of n identical balls into r distinct boxes
= ${}^{n-1}C_{r-1}$ if blank boxes are not allowed
= ${}^{n+r-1}C_{r-1}$ if blank boxes are allowed.

- Number of ways of distribution of n distinct balls into r distinct boxes when order is not considered
= r^n if blank boxes are allowed
= $n!$ coefficient x^n in $(e^x - 1)^r$
= $r^n - {}^rC_1 (r-1)^n + {}^rC_2 (r-2)^n - \dots + (-1)^{r-1} {}^rC_{r-1}$.

Example 8. Find the number of ways in which 10 girls and 90 boys can sit in a row having 100 chairs such that no girls sit at the either end of the row and between any two girls, at least five boys sit.

Solution: First we select 10 chairs which will be occupied by 10 girls under the given condition. Now these 10 selected chairs will divide the remaining 90 chairs into 11 parts



Therefore no of ways of selecting 10 chairs

$$\equiv \text{No of solutions of } x_1 + x_2 + x_3 + \dots + x_{11} = 90$$

under the condition $x_1, x_{11} \geq 1, x_2, x_3, x_4, \dots, x_{10} \geq 5$

$$= \text{coeff. of } t^{90} \text{ in } (t + t^2 + t^3 + \dots)^2 (t^5 + t^6 + \dots)^9$$

$$= \text{coeff. of } t^{43} \text{ in } (1 + t + t^2 + \dots)^{11}$$

$$= \text{coeff. of } t^{43} \text{ in } (1 - t)^{-11} = {}^{53}C_{43}$$

Hence the required number of ways is ${}^{53}C_{43} \times 10! \times 90!$.