INTRODUCTION

An arrangement of numbers $\{a_1, a_2, a_3, \ldots, a_n, \ldots\}$ according to some well defined rule or a set of rules is called a **sequence**. More precisely, we may define a sequence as a function whose domain is some subset of set of natural numbers N of the type $\{1, 2, 3, \ldots, n\} = X$ (say) to some other set of numbers Y. i.e. $f : X \to Y$.

The ordered set of images in Y given by $\{f(1), f(2), f(3)\}, \ldots, f(n)\}$ is the sequence. Sequence containing finite number of terms is called a **finite sequence** and **infinite sequence** if it contains infinite number of terms.

In case $Y = R$, the sequence is a real sequence and if $Y = C$, the sequence is a complex sequence.

If $[a_1, a_2, a_3, \ldots, a_n, \ldots\}$ is a sequence, then the expression $a_1 + a_2 + a_3 + \cdots + a_n + \cdots$ is called the series associated with the sequence.

PROGRESSION

A sequence is said to be a progression if its terms numerically increase (or numerically decrease) continuously.

ARITHMETIC PROGRESSION

A progression $\{a_1, a_2, a_3, .., a_n, \ldots\}$ is called an arithmetic progression (A. P.) if $a_2 - a_1 = a_3 - a_2 = \cdots = a_n - a_{n-1} = \ldots$.

In general $a_{n+1} - a_n$ = constant (say, d) $n \in N$.

The constant difference d is called common difference of A.P. If the first term a_1 of the A.P. be denoted by a then the A.P. is $\{a, a + d, a + 2d, \ldots\}$. Clearly, the general term of A.P. is given by $a_n = a + (n - 1) d, n \in N$.

General characteristics of A.P.

1. If n^{th} term of any sequence in a linear expression in n, then the sequence is an A.P. If a_n is of the form $An + B$, then the common difference is A.
2. For an A.P. $\{a_1, a_2, a_3, \ldots, a_n, \ldots\}$
   (a) $\{a_1 \pm k, a_2 \pm k, a_3 \pm k, \ldots, a_n \pm k, \ldots\}$ is an A.P., where k is a constant.
   (b) $\{ka_1, ka_2, ka_3, \ldots, ka_n, \ldots\}$ is an A.P., where k is a constant.
   (c) $\left\{\frac{a_1}{k}, \frac{a_2}{k}, \frac{a_3}{k}, \ldots, \frac{a_n}{k}, \cdots\right\}$ is an A.P., where $k \neq 0$, a constant.
   (d) $\{a_p, a_{p+q}, a_{p+2q}, \ldots\}$ is an A.P. for any p and q.
3. If $\{a_1, a_2, a_3, \ldots, a_n, \ldots\}$ and $\{b_1, b_2, b_3, \ldots, b_n, \ldots\}$ be two different A.P.'s then $\langle a_1 + b_1, a_2 + b_2, a_n + b_n, \ldots\rangle$ and $\langle a_1 - b_1, a_2 - b_2, a_n - b_n, \ldots\rangle$ are A.P.
4. If three terms to be selected in A.P., choose $a - d, a, a + d$.
5. If four terms to be selected in A.P., choose $a - 3d, a - d, a + d, a + 3d$.
6. The k^{th} term from end of an A.P. = $(n + 1 - k)^{th}$ term from beginning = $a + (n - k) d$.

Alternatively

k^{th} tern from end = $l + (k - 1)(-d)$, where l is the last term.

7. The sum of terms equidistant from beginning and end is constant.

k^{th} term from beginning $\quad t_k = a + (k - 1) d$

k^{th} term from end $\quad T_k = a + (n - k) d$

So, $t_k + T_k = 2a + (n - 1) d \; (= a + l)$ = constant.

Thus, for an A.P. $\{a_1, a_2, a_3, \ldots, a_{n-1}, a_n\}$

$$a_1 + a_n = a_b + a_{n-1} = a_3 + a_{n-2} = \cdots = 2a + (n - 1) d$$

Sum of n terms of an A.P.

Let $\quad S_n = \{a_1 + a_2 + a_3 + \cdots + a_n\}$

Also, $\quad S_n = a_n + a_{n-1} + a_{n-2} + \cdots + a_1$

Adding, we get

$$\begin{aligned} 2S_n &= (a_1 + a_n) + (a_2 + a_{n-1}) + (a_3 + a_{n-2}) + \cdots + (a_n + a_1) \\ &= \{2a + (n - 1)d\} + \{2a + (n - 1) d\} + \{2a + (n - 1) d\} + \cdots + \{2a + (n - 1) d\} \\ &= n\{2a + (n - 1) d\} \end{aligned}$$

$\therefore S_n = \frac{n}{2}\{2a + (n-1)d\}$

Notes:

1. The sum of n terms of an A.P. is a quadratic expression of the form $An^2 + Bn$.
2. If S_n be the expression for sum of n terms, then the n^{th} term is $a_n = S_n - S_{n-1}$, $n > 1$ and $a_1 = S_1$.

ARITHMETIC MEAN

A is said to be arithmetic mean of two numbers a and b, if a, A, b are in A.P. Thus,

$$A - a = b - A \quad \Rightarrow \quad A = \frac{a+b}{2}.$$

Note: If $a_1, a_2, a_3, \ldots, a_n$ be n terms, then their statistical arithmetic mean is defined by

$$A = \frac{a_1 + a_2 + a_3 + \cdots + a_n}{n}$$

Inserting *n* arithmetic means between two terms *a* and *b*

Let $A_1, A_2, A_3, \ldots, A_n$ be inserted between a and b in that order such that $a, A_1, A_2, A_3 \ldots, A_n, b$ is A.P.

Then $b = (n + 2)^{th}$ term $= a + (n + 1)\, d \quad \Rightarrow \quad d = \frac{b-a}{n+1}$.

Thus, n arithmetic means between a and b are as follows :

$$A_1 = a + d = a + \frac{b-a}{n+1} = \frac{an+b}{n+1}$$

$$A_2 = a + 2d = a + \frac{2(b-a)}{n+1} = \frac{a(n-1)+2b}{n+1}$$

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$$A_r = a + rd = a + \frac{r(b-a)}{n+1} = \frac{a(n+1-r)+rb}{n+1}$$

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$$A_n = a + nd = \frac{a+nb}{n+1}$$

We note that

$$A_1 + A_2 + A_3 + \cdots + A_n = n\left(\frac{a+b}{2}\right)$$

That is, sum of n A.M. terms between a and b = n × A.M. of a and b.

Example 1 : Prove that in any arithmetic progression , whose common difference is not equal to zero, the product of two terms equidistant from the extreme terms is the greater the closer these terms are to the middle term.

Solution: Let $\{ a_n \}$ be the A.P., a_k' be the k^{th} term from the end

$$a_k \cdot a_k' = \left[a_1 + (k-1)d\right]\left[a_n - (k-1)d\right]$$

$$= a_1a_n - (k-1)^2 d^2 + (k-1)d(a_n - a_1) = a_1a_n - (k-1)^2 d^2 + (k-1)(n-1)d^2$$

$$\Rightarrow a_k \cdot a_k' = a_1a_n + d^2\left[(k-1)(n-1) - (k-1)^2\right]$$

$$= a_1a_n + d^2(k-1)(n-k)$$

It is enough, if we prove

$P_k = (k-1)(n-k)$ increasing with an increase in k from 1to $\frac{n}{2}$ are $\frac{n+1}{2}$

$P_k = (k-1)(n-k)$, $P_{k+1} = k(n-k-1)$

$\Rightarrow P_{k+1} - P_k = n - 2k$

$\therefore P_{m+1} > P_k$ if $n - 2k > 0$ i.e. if $k < \frac{n}{2}$.

GEOMETRIC PROGRESSION

A progression $\{a_1, a_2, a_3, \ldots, a_n, \ldots\}$ is called a geometric progression (G.P.) if

$$\frac{a_2}{a_1} = \frac{a_3}{a_2} = \cdots = \frac{a_n}{a_{n-1}} = \cdots \ (a_i \neq 0 \ \ \forall \ i = 1, 2, 3, \cdots, n)$$

In general $\frac{a_{n+1}}{a_n}$ = constant (say, r), $n \in N$.

The constant ratio r is known as the common ratio of G.P. If the first term a of the G.P. be denoted by a, then the G.P. is $\{a, ar, ar^2, \ldots\}$. Clearly the general term of A.P. is given by $a_n = ar^{n-1}$, $n \in N$.

General characteristics of G.P.

1. If $\{a_1, a_2, a_3, \ldots, a_n, \ldots\}$ is in G.P., then
   (a) $\{a_1k, a_2k, a_3k, \ldots, a_nk, \ldots\}$, $k \neq 0$, is a G.P.
   (b) $\left\{\frac{a_1}{k}, \frac{a_2}{k}, \frac{a_3}{k}, \cdots, \frac{a_n}{k}, \cdots\right\} k \neq 0$, is a G.P.
   (c) $\left\{\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \cdots, \frac{1}{a_n}, \cdots\right\}$ is a G.P.
   (d) $\{a_p, a_{p+q}, a_{p+2q}, \ldots\}$ is a G.P.
2. If $\{a_1, a_2, a_3, \ldots, a_n, \ldots\}$ and $\{b_1, b_2, b_3, \ldots, b_n, \ldots\}$ be two different geometric progression then $\{a_1b_1, a_2b_2, a_3b_3, \ldots, a_nb_n, \ldots\}$ and $\left\{\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \cdots, \frac{a_n}{b_n}, \cdots\right\}$ are in G.P.
3. If $\{a_1, a_2, a_3, \ldots, a_n, \ldots\}$ is a G.P. of positive terms then $\{\log a_1, \log a_2, \log a_3, \ldots, \log a_n, \ldots\}$ is an A.P. and viceversa.
4. If $\{a_1, a_2, a_3, \ldots, a_n, \ldots\}$ is an A.P. then for any $x > 0$, $\lambda \neq 1$, $\{\lambda^{a_1}, \lambda^{a_2}, \lambda^{a_3}, \cdots, \lambda^{an}, \cdots\}$ is a G.P.
5. If three terms to be selected in G.P., choose them as $\frac{a}{r}, a, ar$.
6. If four terms to be selected in G.P., choose them as $\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$.
7. The k^{th} term from the end in a G.P. = $(n + 1 - k)^{th}$ term from beginning = ar^{n-k}.

Alternatively

k^{th} term from the end $= l\left(\frac{1}{r}\right)^{k-1}$, where l is the last term of the G.P.

8. The product of terms equidistant from beginning and end is constant.
   k^{th} term from beginning $t_k = ar^{k-1}$
   k^{th} term from end $T_k = ar^{n-k}$.
   So, $t_k.T_k = a^2r^{n-1}$ $(= al)$ = constant.

Thus, for a G.P. $\{a_1, a_2, a_3, \ldots, a_{n-1}, a_n, \ldots\}$

$a_1.a_n = a_2.a_{n-1} = a_3.a_{n-2} \ldots = a^2r^{n-1}$

Sum of *n* terms of a G.P.

Let $S_n = a + ar + ar^2 + \cdots + ar^{n-1}$

$r.S_n = ar + ar^2 + \cdots + ar^{n-1} + ar^n$

On subtracting we get

$$(1 - r)S_n = a - ar^n \qquad \Rightarrow \qquad S_n = \frac{a(1-r^n)}{1-r} \quad r \neq 1.$$

In fact it is advisable to use above formula in the following form

$$S_n = \frac{a(1-r^n)}{1-r} \qquad \text{if} \qquad r < 1$$

$$S_n = \frac{a(r^n-1)}{r-1} \qquad \text{if} \qquad r > 1$$

Notes: If $r = 1$, then the G.P. becomes

$S_n = a + a + a + \cdots$ to n terms $= na$

Sum of infinite terms of a G.P.

The sum of n terms has been obtained $S_n = \dfrac{a(1-r^n)}{1-r}$.

If $|r| < 1$, then $\lim\limits_{n\to\infty} r^n = 0$ (if t_n being the nth term of the progression) (i.e., $\lim\limits_{n\to\infty} t_n \to 0$) and then $\lim\limits_{n\to\infty} S_n = \dfrac{a}{1-r}$.

Thus sum of infinite terms of a G.P.

$S = a + ar, ar^2 + \cdots \infty$, $|r| < 1$ is $S = \dfrac{a}{1-r}$.

Notes: (i) If $|r| > 1$, then the sum of an infinite G.P. is **not defined**.

(ii) If S_n be the expression for sum of n terms, then the n^{th} term $t_n = S_n - S_{n-1}$, $n > 1$ and $t_1 = S_1$.

GEOMETRIC MEAN

A positive number G is said to be geometric mean of two positive numbers a and b, if a, G, b are in G.P.

Thus, $\dfrac{G}{a} = \dfrac{b}{G} \Rightarrow G = \sqrt{ab}$.

Note:

If $a_1, a_2, a_3, \ldots, a_n$ be n terms then their statistical geometric mean is defined as

$G = (a_1 a_2 a_3 \cdots a_n)^{1/n}$.

Inserting *n* geometric means between two terms *a* and *b*

Let $G_1, G_2, G_3, \ldots, G_n$ be inserted between a and b in order such that $a, G_1, G_2, G_3, \ldots, G_n, b$ is G.P.

Then $b = (n+2)^{th}$ term $\Rightarrow ar^{n+1} \Rightarrow r = \left(\dfrac{b}{a}\right)^{\frac{1}{n+1}}$.

Thus, n geometric means a and b are as follows:

$$G_1 = ar = a\left(\frac{b}{a}\right)^{\frac{1}{n+1}} = a^{\frac{n}{n+1}} b^{\frac{1}{n+1}}$$

$$G_2 = ar^2 = a\left(\frac{b}{a}\right)^{\frac{2}{n+1}} = a^{\frac{n-1}{n+1}} b^{\frac{2}{n+1}}$$

$$G_3 = ar^3 = a\left(\frac{b}{a}\right)^{\frac{3}{n+1}} = a^{\frac{n-2}{n+1}} b^{\frac{3}{n+1}}$$

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$$G_k = ar^k = a\left(\frac{b}{a}\right)^{\frac{k}{n+1}} = a^{\frac{n+1-k}{n+1}} b^{\frac{k}{n+1}}$$

$$G_n = ar^n = a\left(\frac{b}{a}\right)^{\frac{n}{n+1}} = a^{\frac{1}{n+1}} b^{\frac{n}{n+1}}$$

We wrote that

$$G_1 G_2 G_3 \cdots G_n = (ab)^{n/2} = (\sqrt{ab})^n = G^n$$

That is, product of n G.M. between a and $b = n^{th}$ power of G.M. between a and b.

Example 2 : If three successive terms of a G.P form the sides of a triangle then show that common ratio 'r' satisfies the inequality $\frac{1}{2}\left(\sqrt{5}-1\right) < r < \frac{1}{2}\left(\sqrt{5}+1\right)$.

Solution: Let a, ar, ar^2 be the terms. For triangle formation the necessary and sufficient condition is the sum of any two sides be larger than the third side.

Hence $ar + ar^2 > a$ (assuming $0 < r \le 1$)

$\Rightarrow r^2 + r - 1 > 0$ (since $a > 0$)

$$\Rightarrow \left(r - \frac{-1-\sqrt{5}}{2}\right)\left(r - \frac{-1+\sqrt{5}}{2}\right) > 0$$

$$\Rightarrow \frac{\sqrt{5}-1}{2} < r \le 1 \qquad \text{.. . . . (1)}$$

Consider $r \ge 1$ then $a + ar > ar^2 \Rightarrow r^2 - r - 1 < 0$

$$\Rightarrow \left(r - \frac{1+\sqrt{5}}{2}\right)\left(r - \frac{1-\sqrt{5}}{2}\right) < 0$$

$$\Rightarrow 1 \le r < \frac{1+\sqrt{5}}{2} \qquad \text{. . . . . (ii)}$$

Hence the result.

Alternatively: From (i) if r is replaced by $\frac{1}{r}$ then

we will have $\frac{\sqrt{5}-1}{2} < \frac{1}{r} \le 1$

$\Rightarrow 1 \le r < \frac{\sqrt{5}-1}{2}$ which is same as (ii) .

ARITHMETICO-GEOMETRIC SEQUENCE

Consider an A.P. $\{a, a + d, a + 2d, \ldots\}$ and a G.P. $\{b, br, br^2, \ldots\}$. If a sequence is formed by multiplying the corresponding terms of above two sequences we get

$\{ab, (a + d)\, br, (a + 2d)\, br^2, \ldots.\}$

This sequence is called an arithmetico-geometric sequence (A.G.S).

The general term of this sequence is given by

$t_n = [a + (n - 1)\, d]br^{n-1}$

Summation of *n* terms of an A.G.S.

Let $S_n = ab + (a + d)\, br + (a + 2d)\, br^2 + \cdots + [a + (n - 1)d]\, br^{n-1}$

$rS_n = abr + (a + d)\, br^2 + \cdots + [a + (n - 2)d\,]\, br^{n-1} + [a + (n - 1)d\,]\, br^n$

Subtracting we get

$(1 - r)\, S_n = ab + [bdr + dbr^2 + \cdots + dbr^{n-1}] - [a + (n - 1)d]br^n$

$$= ab + \frac{dbr(1-r^{n-1})}{1-r} - [a+(n-1)d]br^n$$

$$\therefore \quad S_n = \frac{ab}{1-r} + \frac{dbr(1-r^{n-1})}{(1-r)^2} - \frac{[a+(n-1)d]br^n}{1-r}$$

Summation of infinite terms of an A.G.S.

If $M < 1$, then the sum S of infinite terms is

$$S = \lim_{n\to\infty} S_n = \frac{ab}{1-r} + \frac{dbr}{(1-r)^2} \qquad \left[\because \lim_{n\to\infty} r^n = 0 \text{ if } |r| < 1\right]$$

Example 3 : Find the sum of series

$4 - 9x + 16x^2 - 25x^3 + 36x^4 - 49x^5 + \ . \ . \ . + $ to infinite

Solution: Let $S = 4 - 9x + 16x^2 - 25x^3 + 36x^4 - 49x^5 + \ldots..\infty$

$-Sx = -4x + 9x^2 - 16x^3 + 25x^4 - 36x^5 + \ \ldots..\infty$

On subtraction, we get

$S(1+x) = 4 - 5x + 7x^2 - 9x^3 + 11x^4 - 13x^5 + \ldots\infty$

$- S(1+x)x = -4x + 5x^2 - 7x^3 + 9x^4 - 11x^5 + \ldots\infty$

On subtraction, we get

$S(1+x)^2 = 4 - x + 2x^2 - 2x^3 + 2x^4 - 2x^5 + \ldots\infty$

$$= 4 - x + 2x^2(1 - x + x^2 - \ldots\ldots\infty) = 4 - x + \frac{2x^2}{1+x} = \frac{4+3x+x^2}{1+x}$$

$$S = \frac{4+3x+x^2}{(1+x)^3}.$$

HARMONIC PROGRESSION

The sequence $\{a_1, a_2, a_3, \ldots, a_n, \ldots\}$ of non-zero terms is said to be a harmonic progression (H.P.), if the sequence formed by the reciprocals of its terms is an A.P. That is, the sequence $\left\{\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \cdots, \frac{1}{a_n}, \cdots\right\}$ is an A.P. clearly, the standard form of H.P. is $\left\{\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \cdots\right\}$.

Notes:

1. The general term of the harmonic progression $\{a_1, a_2, a_3, \ldots\}$ is given by $a_n = \frac{1}{a+(n-1)d}$, where $a = \frac{1}{a_1}$ and $d = \frac{1}{a_2} - \frac{1}{a_1}$.
2. Corresponding to every H.P. there is an A.P. and vice versa. Therefore problems in H.P. can generally be solved with reference to the corresponding formulas of A.P.
3. There is no formula for finding the sum of n terms of a H.P.

HARMONIC MEAN

A number H is said to be harmonic mean of two non-zero numbers a and b, if a, H, b are in H.P.

Thus $\frac{1}{H} - \frac{1}{a} = \frac{1}{b} - \frac{1}{H} \Rightarrow H = \frac{2ab}{a+b}$

Notes: If $a_1, a_2, a_3, \ldots, a_n$ be n non-zero terms then their statistical harmonic mean is defined as

$$H = \frac{1}{\frac{1}{n}\left[\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right]}.$$

Inserting *n* harmonic means between two terms *a* and *b*

Let $H_1, H_2, H_2, \ldots, H_n$ be inserted between a and b in that order such that $a, H_1, H_2, \ldots, H_n$ b are in H.P. Then

$\frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2}, \cdots, \frac{1}{H_n}, \frac{1}{b}$ are in A.P.

$\therefore \quad \frac{1}{b} = \frac{1}{a} + (n+1)d \quad \Rightarrow \quad d = \frac{a-b}{ab(n+1)}$

Thus, n harmonic means between a and b are as follows :

$$\frac{1}{H_1} = \frac{1}{a} + d = \frac{1}{a} + \frac{a-b}{ab(n+1)} \quad \Rightarrow \quad H_1 = \frac{ab(n+1)}{a+nb}$$

$$\frac{1}{H_2} = \frac{1}{a} + d = \frac{1}{a} + \frac{2(a-b)}{ab(n+1)} \quad \Rightarrow \quad H_2 = \frac{ab(n+1)}{2a+(n-1)b}$$

$$\frac{1}{H_3} = \frac{1}{a} + 3d = \frac{1}{a} + \frac{3(a-b)}{ab(n+1)} \quad \Rightarrow \quad H_3 = \frac{ab(n+1)}{3a+(n-2)b}$$

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$$\frac{1}{H_k} = \frac{1}{a} + kd = \frac{1}{a} + \frac{k(a-b)}{ab(n+1)} \quad \Rightarrow \quad H_k = \frac{ab(n+1)}{ka+(n+1-k)b}$$

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$$\frac{1}{H_n} = \frac{1}{a} + nd = \frac{1}{a} + \frac{n(a-b)}{ab(n+1)} \quad \Rightarrow \quad H_n = \frac{ab(n+1)}{na+b}$$

Example 4 : If a, b, c be in H.P. prove that $\left(\frac{1}{a}+\frac{1}{b}-\frac{1}{c}\right)\left(\frac{1}{b}+\frac{1}{c}-\frac{1}{a}\right)=\frac{4}{ac}-\frac{3}{b^2}$.

Solution: a, b, c are in H.P. $\Rightarrow \frac{2}{b}=\frac{1}{a}+\frac{1}{c}$

Now $\left(\frac{1}{a}+\frac{1}{b}-\frac{1}{c}\right)\left(\frac{1}{b}+\frac{1}{c}-\frac{1}{a}\right)=\left(\frac{1}{b}+\frac{1}{a}-\frac{1}{c}\right)\left(\frac{1}{b}-\left(\frac{1}{a}-\frac{1}{c}\right)\right)$

$$=\left(\frac{1}{b}\right)^2-\left(\frac{1}{a}-\frac{1}{c}\right)^2 = \frac{1}{b^2}-\left[\left(\frac{1}{a}+\frac{1}{c}\right)^2-\frac{4}{ac}\right]$$

$$=\frac{1}{b^2}-\left[\left(\frac{2}{b}\right)^2-\frac{4}{ac}\right] = \frac{4}{ac}-\frac{3}{b^2}.$$

Relation between A.M., G.M., H.M.

If a and b are two positive numbers, then

(i) A, G, and H are in G.P., i.e., $G^2 = AH$

(ii) $A \geq G \geq H$. Equality holds if and only if $a = b$.

(iii) If $a_1, a_2, a_3, \ldots a_n$ are n positive numbers, then for their statistical means $A \geq G \geq H$. Equality holds if and only if $a_1 = a_2 = a_3 = \cdots = a_n$.

SPECIAL SEQUENCE

1. Sum of first n natural numbers

$$\sum_{r=1}^{n} r = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

2. Sum of squares of first n natural numbers

$$\sum_{r=1}^{n} r^2 = 1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

3. Sum of cubes of first n natural numbers

$$\sum_{r=1}^{n} r^3 = 1^3+2^3+3^3+\cdots+n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

4. Sum of sequences using sigma notation

If a sequence is characterized by $\{x_n\}$. Then we write $S_n = x_1 + x_2 + \cdots + x_n = \sum_{r=1}^{n} x_r$ or Σx_n.

Under this notation the above three summations can be denoted by Σn, Σn^2 and Σn^3 respectively.

Suppose the general term of a particular sequence $\{x_n\}$ is given by

$x_n = an^3 + bn^2 + cn + d + kp^n$,

where a, b, c, d, k, p are constants. Then

$S_n = \Sigma x_n = \Sigma(an^3 + bn^2 + cn + d + kp^n)$

$= a\Sigma n^3 + b\Sigma n^2 + c\Sigma n + \Sigma d + k\Sigma p^n$

$$= a\left\{\frac{n(n+1)}{2}\right\}^2 + b\left\{\frac{n(n+1)(2n+1)}{6}\right\} + c\left\{\frac{n(n+1)}{2}\right\} + dn + k\left\{\frac{p(p^n-1)}{p-1}\right\}$$

[$\because \Sigma p^n = p + p^2 + p^3 + \cdots + p^n$, which is a G.P.]

Summation of series using method of difference

1. Consider a series $S_n = a_1 + a_2 + a_3 + \cdots + a_n$.

   Let $a_n - a_{n-1} = t_{n-1}$

   If $\{t_{n-1}\}$, i.e., $t_1, t_2, \ldots, t_{n-1}$ form an A.P. or a G.P. then we find S_n as following:

   $S_n = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n$

   $S_n = a_1 + a_2 + \cdots + a_{n-2} + a_{n-1} + a_n$

   Subtracting, we get

   $0 = a_1 + \{(a_2 - a_1) + (a_3 - a_2) + \cdots + (a_n - a_{n-1})\} - a_n$

   $\Rightarrow \quad a_n = a_1 + \{t_1 + t_2 + \cdots + t_{n-1}\} \qquad \Rightarrow \quad a_n - a_1 + \Sigma t_{n-1}$

   where Σt_{n-1} is can be easily found as it is either an A.P. or a G.P. of $n - 1$ terms. Now, the desired sum S_n can be calculated by $S_n = \Sigma a_n$.

2. Consider a series $S_n = a_1 + a_2 + a_3 + \cdots + a_n$.

   We try to express the general term as the difference of two terms of some other series, i.e.,

   $a_n = b_{n-1} - b_n$ for some series $\{b_n\}$

   Hence, $a_1 + a_2 + a_3 + \cdots + a_n = (b_2 - b_1) + (b_3 - b_2) + (b_4 - b_3) + \cdots + (b_{n-1} - b_n) = b_{n+1} - b_1$

Example 5 : If a, b, c are positive real numbers, then prove that $[(1 + a)(1 + b)(1 + c)]^7 > 7^7 a^4 b^4 c^4$.

Solution: $(1 + a)(1 + b)(1 + c) = 1 + ab + a + b + c + abc + ac + bc$

$\Rightarrow \dfrac{(1+a)(1+b)(1+c)-1}{7} \geq (ab.\ a.\ b.\ c.\ abc.\ ac.\ bc)^{1/7}$ (using AM $\geq$ GM)

$\Rightarrow (1+a)(1+b)(1+c) - 1 > 7(a^4.\ b^4.\ c^4)^{1/7} \Rightarrow (1 + a)(1 + b)(1+c) > 7(a^4.\ b^4.\ c^4)^{1/7}$

$\Rightarrow (1 + a)^7 (1 + b)^7 (1 + c)^7 > 7^7 (a^4.\ b^4.\ c^4)$.