BINOMIAL EXPRESSION

Any algebraic expression consisting of only two terms is known as a binomial expression.

BINOMIAL THEOREM

Such formula by which any power of a binomial expression can be expanded in the form of a series is known as binomial theorem. For a positive integer n the expansion is given by

$$(a + x)^n = {}^nC_0a^n + {}^nC_1a^{n-1}x + {}^nC_2a^{n-2}x^2 + \cdots + {}^nC_nx^n$$

where ${}^nC_0$, ${}^nC_1$, ${}^nC_2$, ⋯ , ${}^nC_n$ are called the binomial coefficients. The value ${}^nC_r$ is defined as

$${}^nC_r = \frac{n!}{r!(n-r)!} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{1\cdot2\cdot3\cdots r}$$

Similarly $(a - x)^n = {}^nC_0a^n - {}^nC_1a^{n-1}x + {}^nC_2a^{n-2}x^2 + \cdots + (-1)^n\,{}^nC_nx^n$

Example 1 : Expand $\left(x + \frac{1}{x}\right)^7$.

Solution : $\left(x + \frac{1}{x}\right)^7 = {}^7C_0x^7 + {}^7C_1x^6\,\frac{1}{x} + {}^7C_2x^5\,\frac{1}{x^2} + {}^7C_3x^4\,\frac{1}{x^3} + {}^7C_4x^3\,\frac{1}{x^4}$

$+{}^7C_5x^2\,\frac{1}{x^5} + {}^7C_6\,x\,\frac{1}{x^6} + {}^7C_7\,\frac{1}{x^7}$

$= x^7 + 7x^5 + 21x^3 + 35\,x + \frac{35}{x} + \frac{21}{x^3} + \frac{7}{x^5} + \frac{1}{x^7}$.

GENERAL TERM IN THE EXPANSION

The general term in the expansion of $(a + x)^n$ is $(r + 1)^{th}$ term given by $t_{r+1} = nC_ra^{n-r}x^r$. Similarly the general term in the expansion of $(x + a)^n$ is given by $t_{r+1} = {}^nC_rx^{n-r}a^r$. The terms are considered from the beginning.

Note:

(i) The $(r + 1)^{th}$ term from the end = $(n - r + 1)^{th}$ term from the beginning.

(ii) The binomial coefficients in the expansion of $(a + x)^n$ equidistant from the beginning and the end are equal.

(iii) Middle term of $(a + x)^n$:

(a) is $\left(\frac{n}{2}+1\right)^{th}$ term, when n is even

(b) is $\left(\frac{n+1}{2}\right)^{th}$ term and $\left(\frac{n+3}{2}\right)^{th}$ term, when n is odd

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Example 2 : Find the co–efficient of x^{24} in $\left(x^2 + \frac{3a}{x}\right)^{15}$.

Solution : General term ((r+1) th term) in $\left(x^2 + \frac{3a}{x}\right)^{15}$

$= {}^{15}C_r(x^2)^{15-r}\left(\frac{3a}{x}\right)^r = {}^{15}C_r\,x^{30-2r}\,\frac{3^r\,a^r}{x^r} = {}^{15}C_r\,3^ra^r\,x^{30-3r}$

If this term contains x^{24}. Then $30-3r = 24 \Rightarrow 3r = 6 \Rightarrow r = 2$

Therefore, the co–efficient of $x^{24} = {}^{15}C_2 \times 9a^2$.

GREATEST BINOMIAL COEFFICIENT

The greatest binomial coefficient is the binomial coefficient of middle term.

Greatest binomial coefficient in $(1 + x)^n$

(i) is ${}^{n}C_{n/2}$ when n is even

(ii) ${}^{n}C_{\frac{n+1}{2}}$ and ${}^{n}C_{\frac{n-1}{2}}$ when n is odd

GREATEST TERM

To determine the numerically greatest term (absolute term) in the expansion of $(a + x)^n$, where n is a positive integer.

$$\left|\frac{T_{r+1}}{T_r}\right| = \left|\frac{{}^{n}C_r a^{n-r} x^r}{{}^{n}C_{r-1} a^{n-r+1} \cdot x^{r-1}}\right| = \left|\frac{{}^{n}C_r}{{}^{n}C_{r-1}}\right|\left|\frac{x}{a}\right| = \left|\frac{n+1}{r} - 1\right|\left|\frac{x}{a}\right|$$

Thus $|T_{r+1}| > |T_r|$ if $\left(\frac{n+1}{r} - 1\right)\left|\frac{x}{a}\right| > 1$

$\Rightarrow \quad r < \dfrac{n+1}{1 + \left|\frac{a}{x}\right|}$ …(1)

$\left(\frac{n+1}{r} - 1\right)$ must be positive since $n > r$. Thus T_{r+1} will be the greatest term if r has the greatest value consistent with inequality (1).

Example 3 : Find the greatest term in the expansion of $(2 + 3x)^9$ if $x = 3/2$.

Solution: $\dfrac{T_{r+1}}{T_r} = \left(\dfrac{n-r+1}{r}\right)\left(\dfrac{3x}{2}\right)$

$= \left(\dfrac{10\text{-}r}{r}\right)\left(\dfrac{3x}{2}\right), \left(\text{where } x = \dfrac{3}{2}\right)$

$= \left(\dfrac{10\text{-}r}{r}\right)\left(\dfrac{3}{2}\right)\left(\dfrac{3}{2}\right) = \dfrac{10\text{-}r}{r} \cdot \dfrac{9}{4}$

$\dfrac{T_{r+1}}{T_r} = \dfrac{90 - 9r}{4r}$

Therefore $T_{r+1} \geq T_r$ if,

$90 - 9r \geq 4r \Rightarrow 90 \geq 13r$

$r \leq \dfrac{90}{13}$, r being an integer, hence r = 6.

$T_{r+1} = T_7 = T_{6+1} = {}^{9}C_6\, (2)^3\, (3x)^6 = \dfrac{3^{13}.7}{2}$.

PROPERTIES OF BINOMIAL COEFFICIENT

For sake of convenience the coefficients ${}^{n}C_0$, ${}^{n}C_1$, ⋯ , ${}^{n}C_n$ are usually denoted by C_0, C_1, ⋯ , C_n respectively.

$$(1 + x)^n = C_0 + C_1x + C_2x^2 + \cdots + C_nx^n$$

Putting $x = 1$, we get $C_0 + C_1 + C_2 + \cdots + C_n = 2^n$.

Putting $x = -1$, we get $C_0 + C_2 + C_4 + \cdots = C_1 + C_3 + C_5 = 2^{n-1}$.

Putting $x = 1$ and -1 and adding, we get $C_0 + C_2 + C_4 + \cdots = 2^{n-1}$.

Putting $x = 1$ and -1 and subtracting, we get $C_1 + C_3 + C_5 + \cdots = 2^{n-1}$.

Putting $x = i$ and equating real part, we get $C_0 - C_2 + C_4 \cdots = 2^{n/2} \cos \dfrac{n\pi}{4}$.

Putting $x = i$ and equating imaginary part, we get $C_1 - C_3 + C_5 \;\cdots = 2^{n/2} \sin \dfrac{n\pi}{4}$.

Notes:

(i) **Differentiation:** When the terms in an identity are the product of a numerical (natural number) and a binomial coefficient, then differentiation is used.

(ii) **Integration:** When the numerical (natural number) occurs as the denominator of the binomial coefficient, integration is used.

(iii) **Multiplication of binomial expansion:** When each term is summation contains the product of two binomial coefficients or square of binomial coefficient, multiplication of binomial coefficient is used.

Example 4 : If $(1+x)^n = \sum_{r=0}^{n} {}^nC_r x^r$, then prove that $C_0^2 + \frac{C_1^2}{2} + \frac{C_2^2}{3} + \cdots + \frac{C_n^2}{n+1} = \frac{(2n+1)!}{\left[(n+1)!\right]^2}$.

Solution: Given $(1+x)^n = C_0 + C_1x + C_2x^2 + \cdots + C_nx^n$ …. (1)

Integrating w.r.t. x between the limits 0 and x we get

$$\left[\frac{(1+x)^{n+1}}{n+1}\right]_0^x = \left[C_0x + C_1\frac{x^2}{2} + C_2\frac{x^3}{3} + \cdots + C_n\frac{x^{n+1}}{n+1}\right]_0^x$$

$$\frac{(1+x)^{n+1}}{n+1} - \frac{1}{n+1} = C_0x + C_1\frac{x^2}{2} + C_2\frac{x^3}{3} + \cdots + C_n\frac{x^{n+1}}{n+1} \quad \text{…. (2)}$$

Also $(1+x)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \cdots + C_n$ …. (3)

Multiplying (2) and (3) and equating coefficient of x^{n+1} of both sides we get

$$C_0^2 + \frac{C_1^2}{2} + \frac{C_2^2}{3} + \cdots + \frac{C_n^2}{n+1} = \frac{{}^{2n+1}C_{n+1} - 0}{n+1} = \frac{(2n+1)!}{\left[(n+1)!\right]^2}.$$
