Energy Quantization from Phase Space Area

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May 2025

1. Introduction

We derive the semiclassical energy levels of a particle in the one-dimensional potential

$$V(x) = a|x|^n, \quad a > 0,\ n > 0$$

using the WKB approximation. This method involves computing the integral of classical momentum over the classically allowed region between the turning points of the potential.

2. WKB Quantization Condition

The WKB quantization rule for bound states is:

$$\int_{x_1}^{x_2} p(x)\,dx = \left(n + \frac{1}{2}\right)\pi\hbar$$

$$\int_{x_1}^{x_2} \sqrt{2m(E - V(x))}\,dx = \left(n + \frac{1}{2}\right)\pi\hbar,$$

where x_1 and x_2 are the classical turning points satisfying $E = V(x)$.

Remark : The factor $\frac{1}{2}$ is more in appearance than in substance, since WKB approximation holds in semi-classical approximation, whereby mostly n needs to be large.

3. Classical Turning Points

For the potential $V(x) = a|x|^n$, the turning points are:

$$E = a|x|^n \quad \Rightarrow \quad x = \pm x_0, \quad x_0 = \left(\frac{E}{a}\right)^{1/n}.$$

4. WKB Integral

Due to the even nature of the potential, the integral simplifies to:

$$\int_{-x_0}^{x_0} \sqrt{2m(E - a|x|^n)}\,dx = 2\int_0^{x_0} \sqrt{2m(E - ax^n)}\,dx.$$

We change variables:

$$x = x_0 u, \quad dx = x_0 du, \quad x_0 = \left(\frac{E}{a}\right)^{1/n}.$$

Then:

$$E - ax^n = E(1 - u^n),$$

and the integral becomes:

$$2\sqrt{2mE} \cdot x_0 \int_0^1 \sqrt{1 - u^n}\, du.$$

Substituting for x_0, we have:

$$\int_{-x_0}^{x_0} \sqrt{2m(E - V(x))}\, dx = 2\sqrt{2mE} \left(\frac{E}{a}\right)^{1/n} \int_0^1 \sqrt{1 - u^n}\, du.$$

5. Energy Quantization

The WKB condition becomes:

$$2\sqrt{2mE} \left(\frac{E}{a}\right)^{1/n} \int_0^1 \sqrt{1 - u^n}\, du = \left(n + \frac{1}{2}\right) \pi\hbar.$$

Let $I_n = \int_0^1 \sqrt{1 - u^n}\, du$, then:

$$\sqrt{2m} E^{\frac{1}{2} + \frac{1}{n}} a^{-1/n} \cdot 2I_n = \left(n + \frac{1}{2}\right) \pi\hbar.$$

Solving for E, we find:

$$E^{\frac{1}{2} + \frac{1}{n}} = \frac{(n + \frac{1}{2})\pi\hbar \cdot a^{1/n}}{2\sqrt{2m} \cdot I_n},$$

which implies:

$$\boxed{E_n \propto \left(n + \tfrac{1}{2}\right)^{\frac{2n}{n+2}}.}$$

6. An Intelligent observation

The WKB quantization rule for bound states is:

$$\int_{x_1}^{x_2} p(x)\, dx = \left(n + \frac{1}{2}\right) \pi\hbar$$

where x_1 and x_2 are the classical turning points satisfying $E = V(x)$.

Now, the integral is only over the phase space of the system.

For a potential $V(x) = a|x|^n$, the energy(E) conservation equation says,

$$\frac{p^2}{2m} + a|x|^n = E$$

We obtain the limiting value of the area under the phase space.

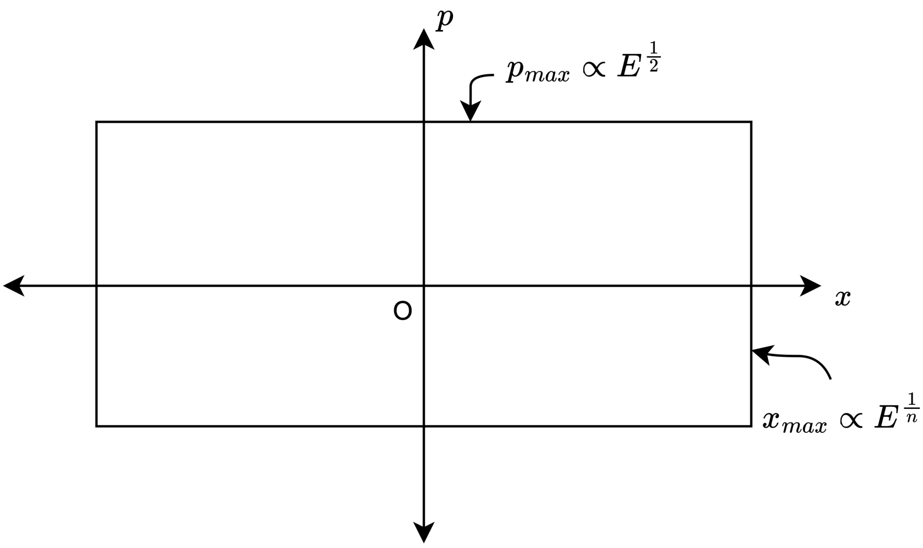

Figure 1: Phase space diagram

- The maximum value of p is
$$\frac{p_{max}^2}{2m} = E \implies p_{max} \propto E^{\frac{1}{2}}$$
This corresponds to the width of the box within which the phase space lies.

- The maximum value of x is
$$ax_{max}^n = E \implies x_{max} \propto E^{\frac{1}{n}}$$
This corresponds to the height of the box within which the phase space lies.

So, the extremum area under the curve (only the relevant energy scaling is)

$$Area_{max} = \int_{E\ scale} p(x)dx \propto P_{max} * x_{max} \propto E^{\frac{1}{2}} * E^{\frac{1}{n}} \propto E^{\frac{1}{2}+\frac{1}{n}}$$

This implies

$$E^{\frac{1}{2}+\frac{1}{n}} \propto \left(n + \frac{1}{2}\right) \quad \text{using WKB Energy Quantization}$$

Therefore,

$$\boxed{E_n \propto \left(n + \tfrac{1}{2}\right)^{\frac{2n}{n+2}}}$$

Same as we got before. Just our laziness, and intelligence might save efforts to get the answer, without solving any integral.

7. Special Cases

- $n = 2$: Harmonic oscillator $\Rightarrow E_n \propto (n + \frac{1}{2})$
- $n = 1$: Linear potential $\Rightarrow E_n \propto (n + \frac{1}{2})^{2/3}$
- $n \to \infty$: Infinite square well $\Rightarrow E_n \propto n^2$

8. Conclusion

Using the WKB approximation, we derived the scaling behavior of energy levels for a wide class of symmetric power-law potentials. The quantized energy levels exhibit the universal form:

$$E_n \propto \left(n + \tfrac{1}{2}\right)^{\frac{2n}{n+2}},$$

which interpolates smoothly between the harmonic oscillator and infinite square well limits.