BAYES' THEOREM

Introduction:

Revising probability when new information is obtained is an important phase of probability analysis.

- This can be done by using this approach

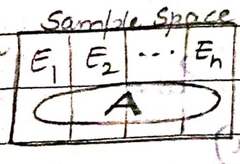
Prior probabilities $P(E_1), P(E_2) \ldots$ → New Info $P(A/E_i)$ → Application of Bayes theorem → posterior probability $P(E_i/A)$

Statement:-

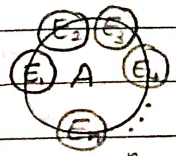
If $E_1, E_2, \ldots E_n$ are mutually disjoint events with $P(E_i) \neq 0$ ($\forall\, i = 1(1)n$) then for any arbitrary event A which is a subset of $\bigcup_{i=1}^{n} E_i$ such that $P(A) > 0$ we have,

$$P(E_i/A) = \frac{P(A/E_i)\,P(E_i)}{\sum_{i=1}^{n} P(A/E_i)\,P(E_i)}$$

(or) $$P(E_i/A) = \frac{P(A/E_i)\,P(E_i)}{P(A)}$$; where $P(A) = \sum_{i=1}^{n} P(A/E_i)\,P(E_i)$



(OR)



Clearly $A \subset \left(\bigcup_{i=1}^{n} E_i\right)$ and E_i's are mutually disjoint.

Examples

1. Suppose that a product is produced in 3 factories X, Y and Z. It is known

$X = 3n$

that factory X produces thrice as many items as factory Y, and that

$Y = n = Z$ Where 'n' is no. of items produced.

factories Y and Z produce the same no. of items. Assume that it

is $\boxed{\text{Known}}$ that 3 percent of the items produced by each of the

$P(\text{defective}|X) = 0.03 = P(\text{defective}|Z)$

factories X and Z are defective while 5 percent of those manufactured

$P(\text{defective}|Y) = 0.05$

by factory Y are defective. All the items produced in the 3 factories are stocked and an item of product is selected at random?

$P(\text{defective})$

Sol (i) What is the probability that this item is defective?

$P(i \mid \text{defective}),\ i = X, Y, Z$

(ii) If an item selected at random is found to be defective, what is the probability that it was produced by factory X, Y and Z respectively?

Z respectively?

Sol^n X : items produced by factory X ; $P(X) = \frac{3n}{3n+n+n} = \frac{3n}{5n} = 0.6.$

Y : — " — " — Y ; $P(Y) = n/5n = 0.2$

Z : — " — " — Z ; $P(Z) = n/5n = 0.2.$

A : defective item.

Given

$P(A|X) = 0.03 = P(A|Z)$; $P(A|Y) = 0.05$

(i) $P(A) = ?$

$$P(A) = P(X)P(A|X) + P(Y)P(A|Y) + P(Z)P(A|Z)$$

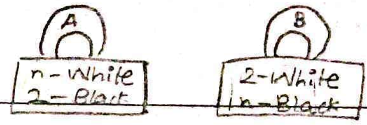
$$= \frac{\cancel{6}^3}{10} \times \frac{3}{\cancel{100}_{50}} + \frac{\cancel{2}^1}{10} \times \frac{5}{\cancel{100}_{50}} + \frac{\cancel{2}^1}{10} \times \frac{3}{\cancel{100}_{50}}$$

$$P(A) = \frac{9}{500} + \frac{5}{500} + \frac{3}{500} = \frac{17}{500} = 0.034$$

(ii) $P(X|A) = \frac{P(A|X)P(X)}{P(A)} = \frac{\frac{3}{\cancel{100}} \times \frac{\cancel{6}^3}{\cancel{10}}}{\frac{17}{\cancel{500}}} = \frac{9}{17}.$

$P(Y|A) = \frac{P(A|Y)P(Y)}{P(A)} = \frac{\frac{\cancel{2}}{\cancel{10}} \times \frac{5}{\cancel{100}}}{\frac{17}{\cancel{500}}} = \frac{5}{17}$

$P(Z|A) = \frac{P(A|Z)P(Z)}{P(A)} = \frac{\frac{3}{\cancel{100}} \times \frac{\cancel{2}^1}{\cancel{10}}}{\frac{17}{\cancel{500}}} = \frac{3}{17}.$ (or) $P(Z|A) = 1 - P(X|A) - P(Y|A)$



2. There are two bags A and B. A contain n white and 2 black balls and B contain 2 white and n black balls. One of the two bags is selected at random and two balls are drawn from it without replacement. If both the balls drawn are white and the probability that the bag A was used to draw the balls is 6/7, find the value of n.

$P(E_1|A) = 6/7$

Soln E_1 : Bag A is selected ; E_2 : Bag B is selected.

A : two balls drawn are white.

$P(E_1) = \frac{1}{2} = P(E_2)$

$P(A|E_1) = \frac{{}^nC_2}{{}^{n+2}C_2} = \frac{n(n-1)}{(n+2)(n+1)}$; $P(A|E_2) = \frac{{}^2C_2}{{}^{n+2}C_2} = \frac{1}{\cancel{(n+2)(n+1)/2!}}$

Given

$P(E_1|A) = 6/7 \rightarrow (1)$

W.k.t

$$P(E_1|A) = \frac{P(A|E_1)P(E_1)}{\sum_{i=1}^{2} P(A|E_i)P(E_i)} = \frac{\frac{n(n-1)}{(n+2)(n+1)}\cdot\frac{1}{2}}{\frac{1}{2}\frac{n(n-1)}{(n+2)(n+1)} + \frac{1}{2}\frac{2}{(n+2)(n+1)}} = \frac{\frac{n(n-1)}{\cancel{(n+2)(n+1)}}\cdot\frac{1}{\cancel{2}}}{\frac{n(n-1)+2}{\cancel{2(n+2)(n+1)}}} = \frac{n(n-1)}{n(n-1)+2}$$

Equating (1) and (2)

$$\frac{n(n-1)}{n(n-1)+2} = \frac{6}{7}$$

$$7n(n-1) = 6n(n-1) + 12$$

$$7n(n-1) - 6n(n-1) = 12$$

$$n(n-1) = 12 \qquad \text{w.k.} \qquad 4 \times 3 = 12$$

$$\therefore \boxed{n = 4}$$

3. A letter is known to have come either from TATANAGAR or from CALCUTTA. On the envelope just two consecutive letters TA are visible. What is the probability that the letter came from CALCUTTA?

Sol^n E_1 : letter came from TATANAGAR ; E_2 : letter came from CALCUTTA

$P(E_1) = \frac{1}{2} = P(E_2)$; A : Event that two consecutive visible letters on the envelope are TA.

$$P(A|E_1) = \frac{2}{8} \quad [\text{a pair of consecutive letters } n-1 = 9-1 = 8]$$

$$P(A|E_2) = \frac{1}{7}. \quad [\text{a pair of consecutive letters } n-1 = 8-1 = 7]$$

$$\therefore P(E_2|A) = \frac{P(A|E_2)P(E_2)}{P(A|E_1)P(E_1) + P(A|E_2)P(E_2)} = \frac{\frac{1}{7} \times \frac{1}{2}}{\frac{2}{8} \times \frac{1}{2} + \frac{1}{7} \times \frac{1}{2}} = \frac{\frac{1}{14}}{\frac{1}{2}\left[\frac{2}{8} + \frac{1}{7}\right]} = \frac{\frac{1}{\cancel{14}}}{\frac{1}{\cancel{2}}\left[\frac{22}{\cancel{56}}\right]} = \frac{\frac{4}{8}}{\frac{\cancel{22}}{11}}$$

$$\therefore P(E_2|A) = \frac{4}{11}$$

4. The chances that doctor A will diagnose a disease X correctly is 60%. The chances that a patient will die by his treatment after correct diagnosis is 40% and the chance of death by wrong diagnosis is 70%. A patient of doctor A, who had disease X, died. What is the chance that his disease was diagnosed correctly

Sol^n. E_1 : disease is correctly diagonised –

$\bar{E_1} = E_2$: – " – not correctly diagonised

A : Patient with disease X will die

$P(E_1) = 0.60$; $P(E_2) = 0.40$

$P(A|E_1) = 0.40$; $P(A|E_2) = 0.70$

To find

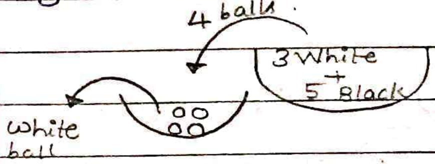
$$P(E_1|A) = \frac{P(A|E_1)\,P(E_1)}{P(E_1)P(A|E_1) + P(E_2)P(A|E_2)} = \frac{\frac{\cancel{40}^{2}}{{}_{5}\cancel{100}} \times \frac{6\cancel{0}}{10\cancel{0}}}{\frac{{}^{2}\cancel{40}}{\cancel{100}_{5}} \times \frac{6\cancel{0}}{10\cancel{0}} + \frac{{}^{2}\cancel{40}}{\cancel{100}_{5}} \times \frac{7\cancel{0}}{10\cancel{0}}} = \frac{\frac{12}{50}}{\frac{12}{50} + \frac{14}{50}}$$

$$= \frac{\frac{12}{\cancel{50}}}{\frac{26}{\cancel{50}}} = \frac{\cancel{12}^{6}}{\cancel{26}_{13}}$$

$$\therefore P(E_1|A) = \frac{6}{13}$$

* New variety of problem

5. From a vessel containing 3 White and 5 Black balls, 4 balls are transferred into an empty vessel. From this vessel a ball is drawn and is found to be white. What is the probability that out of four balls transferred 3 are white and one is black?



Importance of defining events

Sol^n. A : A white ball is obtained in second drawn.

E_1 : 0 white balls and 4 black balls are drawn in 1st draw

E_2 : 1 white ball and 3 black balls are drawn in 1st draw

E_3 : 2 white ball and 2 black balls are drawn in 1st draw

E_4 : 3 white balls & 1 black ball are drawn in 1st draw

To find $P(E_4 | A) - ?$

$$P(E_1) = \frac{{}^3C_0 \times {}^5C_4}{{}^8C_4} = \frac{1 \times \cancel{5}^1}{\cancel{70}_{14}} = \frac{1}{14}$$

$$^8C_4 = \frac{\cancel{8}^{\cancel{2}\,1} \times 7 \times \cancel{6}^{2} \times 5}{\cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}}$$

$$P(E_2) = \frac{{}^3C_1 \times {}^5C_3}{{}^8C_4} = \frac{3 \times (5 \times \cancel{4}^2)/\cancel{2}_1}{70} = \frac{3\cancel{0}}{7\cancel{0}} = \frac{3}{7}$$

$$^8C_4 = 7 \times 10 = 70$$

$$P(E_3) = \frac{{}^3C_2 \times {}^5C_2}{{}^8C_4} = \frac{3 \times \cancel{10}^1}{\cancel{70}\,7} = \frac{3}{7}$$

$$P(E_4) = \frac{{}^3C_3 \times {}^5C_1}{{}^8C_4} = \frac{1 \times \cancel{5}^1}{\cancel{70}_{14}} = \frac{1}{14}$$

$P(A|E_1) = 0$; $P(A|E_2) = \frac{{}^1C_1}{{}^4C_1} = \frac{1}{4}$; $P(A|E_3) = \frac{2}{4} = \frac{1}{2}$; $P(A|E_4) = \frac{3}{4}$.

$$\therefore P(E_4|A) = \frac{P(A|E_4)\,P(E_4)}{\sum_{i=1}^{4} P(A|E_i)P(E_i)} = \frac{\frac{3}{4}\times\frac{1}{14}}{0\times\frac{1}{14}\times\frac{1}{4}\times\frac{3}{7}+\frac{1}{2}\times\frac{3}{7}+\frac{3}{4}\times\frac{1}{14}}$$

$$= \frac{3/4\times14}{3/28 + 3/14 + 3/4\times14} = \frac{\frac{3}{\cancel{4\times14}}}{\frac{6+12+3}{\cancel{4\times14}}} = \frac{\cancel{3}^{1}}{\cancel{21}_{7}} = \frac{1}{7}.$$

$$\therefore P(E_4|A) = \frac{1}{7}$$

\# New problem

6. A and B are two weak students of statistics and their chances of solving a problem in statistics correctly are $\frac{1}{6}$ and $\frac{1}{8}$ respectively. If the probability of their making a common error is $\frac{1}{525}$ and they obtain the same answer (given event), find the probability that their answer is correct.

Sol^n. E_1 = Both A and B solve correctly

E_2 = Exactly one of them solve correctley

E_3 = Neither of them solves the problem correctly

E = They get the same answer

A & B are independently answering.

$P(E_1)$ = A answers correctly & B also answers correctly

$= P(A) \times P(B)$

$= \frac{1}{6} \times \frac{1}{8}$

$\therefore P(E_1) = \frac{1}{48}$

$P(E_2) = P(A)P(\bar{B}) + P(B)P(\bar{A})$

$= \frac{1}{6} \times \frac{7}{8} + \frac{1}{8} \times \frac{5}{6} = \frac{7+5}{48} = \frac{12}{48} = \frac{1}{4}$

$P(E_3) = P(\bar{A})P(\bar{B}) = \frac{5}{6} \times \frac{7}{8} = \frac{35}{48}$

$$\begin{array}{r} 525 \\ +\ 35 \\ \hline 560 \end{array}$$

$P(E|E_1) = 1$; $P(E|E_2) = 0$; $P(E|E_3) = \frac{1}{525}$ (given)

To compute:

→ $P(E_1|E) = \dfrac{P(E|E_1)P(E_1)}{\sum_{i=1}^{3} P(E|E_i)P(E_i)} = \dfrac{1 \times \frac{1}{48}}{1 \times \frac{1}{48} + \frac{35}{48} \times \frac{1}{525}} = \dfrac{\frac{1}{\cancel{48}}}{\frac{525+35}{\cancel{48} \times 525}} = \dfrac{\overset{15}{\cancel{\overset{105}{\cancel{525}}}}}{\underset{16}{\cancel{\underset{112}{\cancel{560}}}}}$

$\therefore P(E_1|E) = \frac{15}{16}$

4. In answering a question on a multiple choice test a student either knows the answer or he guesses. Let p be the probability that he knows the answer and $(1-p)$ the probability that he guesses. Assume that a student who guesses at answer will be correct with probability $1/5$, where 5 is the no. of mcq. What is the conditional probability that a student knew the answer to the question given that he answered it correctly.

Sol^n E_1 : Student knows the answer ; $P(E_1) = p$

E_2 : Student guesses the answer ; $P(E_2) = 1-p$

A : Student gets the correct answer

$P(A|E_1) = 1$; $P(A|E_2) = \frac{1}{5}$

To compute :

$$P(E_1|A) = \frac{P(A|E_1)P(E_1)}{\sum_{i=1}^{2} P(A|E_i)P(E_i)} = \frac{1 \times p}{1 \times p + \frac{1}{5}(1-p)} = \frac{p}{p + \frac{1-p}{5}} = \frac{p}{\frac{5p+1-p}{5}} = \frac{5p}{4p+1}$$

$$\therefore P(E_1/A) = \frac{5p}{4p+1}$$