Q.1 Sol.

To be a multiple of 3, the sum of the digits of the number should be divisible by 3.

Sum of the digits: 1, 2, 3 & 4 $= 1+2+3+4$
$= 10$

Clearly 10 isn't divisible by 3.

$\therefore$ No number can be formed with digits 1, 2, 3 & 4 taking each digit once, (which is multiple of 3)

But all numbers between 2000 & 5000 are of 4 digits.

$\therefore$ We must select 4 digits out of 0, 1, 2, 3, & 4.

The only way to select 4 digits, without taking the combination $\rightarrow$ (1, 2, 3, 4), is to replace ~~of~~ one of the digits of the combination (1, 2, 3, 4) with zero 0.

Therefore,

we conclude that,

0 has to included in all the numbers.

If 0 is one of the digits, we have to select other 3 digits such that the sum of digits is multiple of 3.

Now, taking 3 digits out of 1, 2, 3, 4 and finding their sum,

$1+2+3 = 6$ ✓

$2+3+4 = 9$ ✓

$1+3+4 = 8$ ✗

$1+2+4 = 7$ ✗ → Not multiples of 3

The only combinations are: $(0, 1, 2, 3)$ & $(0, 2, 3, 4)$

Case I $(0, 1, 2, 3)$

The first digit cannot be 0.

The first digit is 1.

If the first digit

1			
	3 ways	2 way	1 way

2nd digit can be taken in 3 ways → 0, 2 or 3

3rd digit can be selected in 2 ways

4^{th} digit can be selected in 1 way

$\therefore$ The total number of ways $= 3 \times 2 \times 1$
$= 6$

Similarly for 2 as 1^{st} digit,

2			
	3 ways	2 ways	1 way

No. of ways $= 3 \times 2 \times 1 = 6$

Similarly for 3 as 1^{st} digit,

3			
	3 ways	2 ways	1 way

No. of ways $= 3 \times 2 \times 1 = 6$

<u>Case II</u> (0, 2, 3, 4)

2			
	3 way	2 ways	1 way

No. of ways when 2 is 1st digit $= 3 \times 2 \times 1$
$= 6$

for 3 as 1st digit,

3			

3 ways 2 ways 1 way

No. of ways = $3 \times 2 \times 1 = 6$

for 4 as 1st digit,

4			

3 ways 2 ways 1 way

No. of ways = $3 \times 2 \times 1 = 6$

Therefore, we see

total no. of ways of forming combinations

$= (6+6+6) + (6+6+6)$

$= 36$

Now, the numbers should be between 2000 and 5000.

So, the 1st digit cannot be 1.

In all other cases, the numbers lie between 2000 & 4000.

$\therefore$ We should not take the case with 1 as first digit.

Number of ways, taking 1 as first digit

$= 6$

$\therefore$ Total number of ways $= 36 - 6$

$= 30$

$\therefore$ Correct option is (B).

Q.2 Sol.

Given function is

$$f(x) = \sqrt{\cos^{-1}\left(\sqrt{\log_4 x}\right) - \frac{\pi}{2}} + \sin^{-1}\left(\frac{1+x^2}{4x}\right)$$

Taking the first term of the function

$$\sqrt{\cos^{-1}\left(\sqrt{\log_4 x}\right) - \frac{\pi}{2}}$$

The above term is defined, only when the quantity whose root is taken is positive.

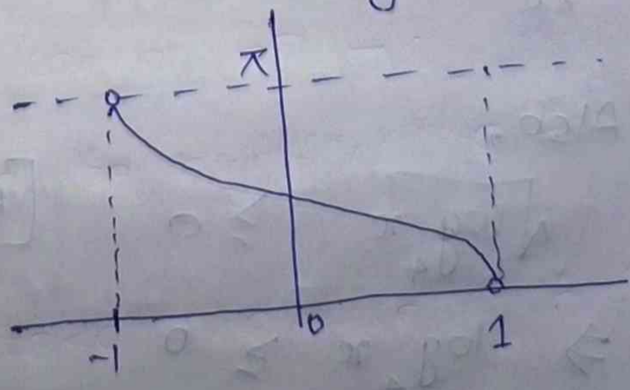
(As the root of -ve number is not defined)

$$\therefore \cos^{-1}\left(\sqrt{\log_4 x}\right) - \frac{\pi}{2} \geq 0$$

$$\Rightarrow \cos^{-1}\left(\sqrt{\log_4 x}\right) \geq \frac{\pi}{2}$$

Above inequality is valid as range of $\cos^{-1} = [0, \pi]$

But we know that the graph of cos inverse function is decreasing,



Q.3 Sol.

Let us consider an arithmeetic progression,

X=(1, 3, 5, 7)

For X+X, we add each element of X to all elements of X.

Let us add first element → 1 to all elements X

1 + 1 = 2

1 + 3 = 4

1 + 5 = 6

1 + 7 = 8

Add 2nd element → 3 to all ements of X

3 + 1 = 4

3 + 3 = 6

3 + 5 = 8

3 + 7 = 10

Add 3rd element → 5 to all elements of X,

5 + 1 = 6

5 + 3 = 8

5 + 5 = 10

5 + 7 = 12

Add 4^{th} element → 7 to all elements of X

$7 + 1 = 8$

$7 + 3 = 10$

$7 + 5 = 12$

$7 + 7 = 14$

Taking the common outcomes once, we get $(X + X)$ as

2, 4, 6, 8, 10, 12, 14

Number of elements = 7

Let us take another A.P

Y = 2, 4, 6, 8, 10

Then, for Y+Y, we add each element to all elements of Y

Adding 2 to all elements, we get,

4, 6, 8, 10, 12

Adding 4 to all elements,

6, 8, 10, 12, 14

Adding 6 to all elements of Y,

8, 10, 12, 14, 16

Adding 8 to all elements of Y,

10, 12, 14, 16, 18

Taking common outcomes once, we have

$(Y+Y) \rightarrow (2, 4, 6, 8, 10, 12, 14, 16, 18)$

Now,

number of elements in X = 4

& number of elements in X + X = 7

Again,

Number of elements in Y = 5

Number of elements in Y + Y = 9

So for number of elements in set A = n

number of elements in set $(A+A) = 2n-1$

Put $n = 4$, $2n-1 = 7$ (for set X)

Put $n = 5$, $2n-1 = 9$ (for set Y)

Thus, if A has n elements &

$A+A$ has $(2n-1)$ elements, A

must be arithmetic progression.

Number of elements in $A = 20$

$39 = 2 \times 20 - 1$

Hence, A is arithmetic progression.

Let the common difference be d.

So, $a_1 = 1 + d$

We know that, n^{th} element in an A.P. is

given by,

$$A_n = a + (n-1)d \qquad \text{——— (1)}$$

where, A_n is n^{th} element

a = first element

n is total number of elements

d is common difference.

Now, 20^{th} element of A is 77.

first element, $a = 1$

$n = 20$

& let the common difference be d.

Putting these values in eq. (1),

$$77 = 1 + (20-1)d$$

$$\Rightarrow d = 4$$

Now, to find $a_1 + a_2 + \dots + a_{18}$

$a_1 = 1 + d = 1 + 4 = 5$

$n = 18$

We know that,
the sum of n elements in AP is given by

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$$

where, S_n = sum of n elements.
n = Number of elements.
a = first element
d = common difference.

$n = 18$, $d = 4$, $a = 5$

$$\therefore (a_1 + a_2 + \dots + a_{18}) = \frac{18}{2}\left[2 \times 5 + (18-1) \times 4\right]$$

$$= 702$$

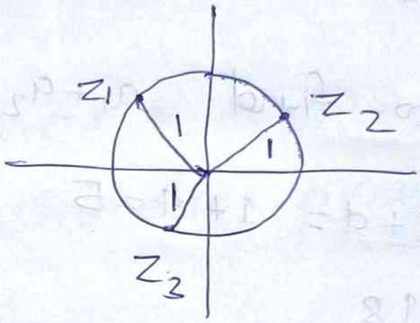
Hence, the correct option is (B).

Q No. 4

Since z_1, z_2 & z_3 have unit modulus.

Their distance from origin = 1

$\Rightarrow$ z_1, z_2 & z_3 lie on a circle of radius 1 unit, centred at origin.

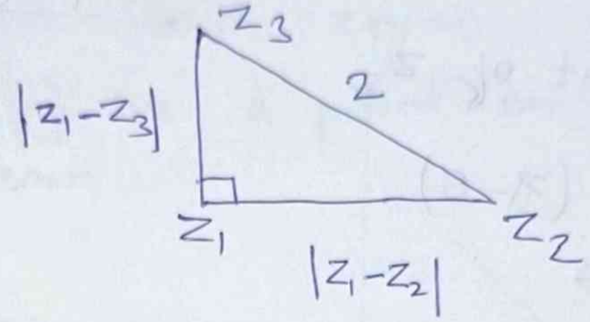


Now,
We know that,
$|z_1 - z_2|$ = distance between points z_1 & z_2
& $|z_1 - z_3|$ = distance between z_1 & z_3.

But given,
$|z_1 - z_2|^2 + |z_1 - z_3|^2 = 4 = 2^2$

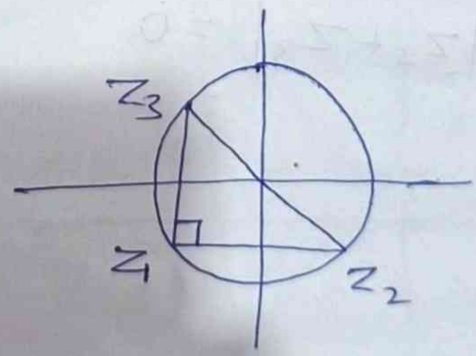
This equation follows Pythagoras theorem where $|z_1 - z_2|$ & $|z_1 - z_3|$ are base & height of right triangle, & where Right angle is formed at point

line joining z_2 to z_1, & line joining z_3 to z_1 meet.



From the above figure,
distance between z_2 & $z_3 = 2$
But, the diameter of given unit circle
$= 2 \times 1 = 2$

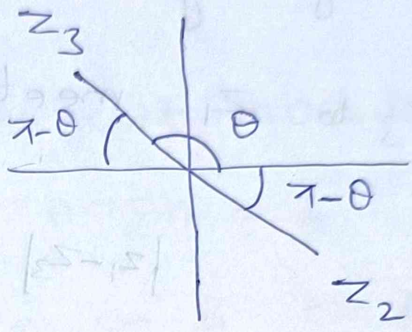
$\therefore$ Points z_2 & z_3 lie on the diameter of the circle.



$\therefore$ z_2 & z_3 are diametrically opposite.

Now, let the argument of z_3 be θ.

Then,



argument of z_2

$= 2\pi - (\pi - \theta)$

$= \pi + \theta$

Representing z_2 & z_3 in polar-coordinates,

$z_3 = |z_3|(\cos\theta + i\sin\theta) = \cos\theta + i\sin\theta$

$z_2 = |z_2|[\cos(\pi+\theta) + i\sin(\pi+\theta)]$

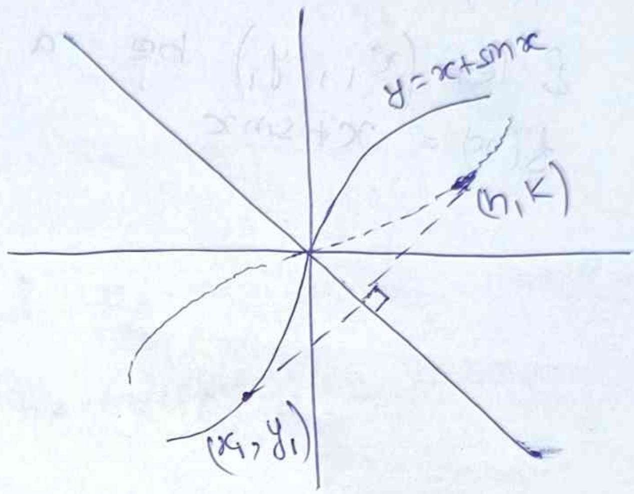
$= -(\cos\theta + i\sin\theta)$

$\Rightarrow z_2 + z_3 = 0$

$\therefore |z_2 + z_3| = 0$

Q No 5

Let (h,k) be the image of the point (x_1, y_1) on $f(x) = x + \sin x$.



$\because$ The image of a point is at the same distance from the line, as that of the point from the line.

$\therefore$ Mid-point of (h,k) & (x_1, y_1) lies on the line $x+y=0$

Mid-point of (h,k) & (x_1, y_1) : $\left(\frac{h+x_1}{2}, \frac{k+y_1}{2}\right)$

It lies on $x+y=0$.

$\therefore \quad \frac{h+x_1}{2} + \frac{k+y_1}{2} = 0$

$\Rightarrow \quad h+k = -(x_1+y_1) \quad \text{—— (1)}$

Also, the line joining the point & its image will be perpendicular to the line.

$\therefore$ Product of their slopes $= -1$

Slope of line joining (h, k) & $(x_1, y_1) = \dfrac{y_1 - k}{x_1 - h}$

& slope of line $y + x = 0$ is -1.

$$\therefore \quad \frac{y_1 - k}{x_1 - h} \times (-1) = -1$$

$$\Rightarrow \quad y_1 - k = x_1 - h$$

$$\Rightarrow \quad h - k = x_1 - y_1 \quad \text{———} \quad (2)$$

From eq① + eq(2),

$$2h = -2y_1$$

$$\therefore \quad h = -y_1$$

$$\text{or} \quad y_1 = -h$$

from eq ① − eq ②,

$$2k = -2x_1$$

$$\Rightarrow \quad x_1 = -k$$

But (x_1, y_1) lies on curve $f(x) = x + \sin x$.

$$\therefore \quad y_1 = x_1 + \sin x_1 \quad \text{———} \quad (3)$$

Putting $y_1 = -h$ & $x_1 = -k$ in eq (3),

$-h = -k + \sin(-k)$

$\Rightarrow -h = -k - \sin k$

$\Rightarrow h = k + \sin k$

This gives us the relation between h & k,

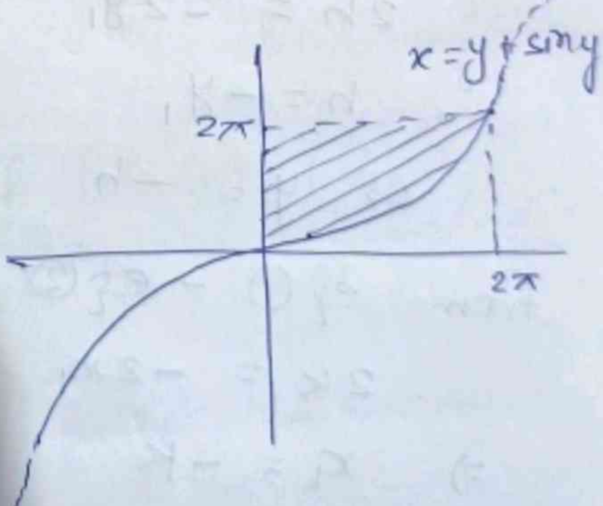
As h is the x-coordinate, &

k is the y-coordinate.

So, replacing h with x, & k with y,

$x = y + \sin y$

This is the image of $f(x) = \sin x + x$ about the line $y + x = 0$



At $x = 2\pi$,

$2\pi = y + \sin y$

taking $y = 2\pi$, $2\pi = 2\pi + \sin 2\pi = 2\pi$

$\therefore$ At $x = 2\pi$, $y = 2\pi$.

Area under curve $x = y + \sin y$ projected on y-axis, from $y = 0$, to $y = 2\pi$, is,

$$= \int_0^{2\pi} (y + \sin y)\, dy$$

$$= \left[\frac{y^2}{2} - \cos y\right]_0^{2\pi} = \left[\frac{(2\pi)^2}{2} - \cos 2\pi\right] - \left[\frac{0^2}{2} - \cos 0\right]$$

$$= 2\pi^2 - 1 + 1 = 2\pi^2$$

Now, we have to find the area formed by

$x = y + \sin y$ on x-axis,

Area of rectangle formed by $x = 0, y = 0$, $x = 2\pi$ & $y = 2\pi$ $= 2\pi \times 2\pi = 4\pi^2$

$\therefore$ Required area = Area of rectange − Area projected on y-axis

$$= 4\pi^2 - 2\pi^2$$
$$= 2\pi^2$$

Acc. to question,

$2\pi^2 = A$

$\therefore$ $\boxed{\frac{A}{\pi^2} = 2}$

Q. No. 6

We know that,

$$f\left[f^{-1}(x)\right] = x$$

Differentiating both sides,

$$f'\left[f^{-1}(x)\right] \cdot \frac{d}{dx}\left[f^{-1}(x)\right] = 1$$

$$\Rightarrow \frac{d\left[f^{-1}(x)\right]}{dx} = \frac{1}{f'\left[f^{-1}(x)\right]}$$

Given,

$$f^{-1}(x) = g(x)$$

$$\therefore \frac{d}{dx}\left[g(x)\right] = \frac{1}{f'\left[f^{-1}(x)\right]} \quad \text{——— (1)}$$

Now,

inverse of a function gives the value of x for a value of y.

So, $f^{-1}(2\pi)$ ~~means~~ finds the value of x for which $y = 2\pi$

$$y = f(x) = 2\pi = (2x - 3\pi)^{25} + \frac{4}{3}x + \cos x$$

At $x = \frac{3\pi}{2}$,

$$f\left(\frac{3\pi}{2}\right) = \left(2 \times \frac{3\pi}{2} - 3\pi\right)^{25} + \frac{4}{3} \times \frac{3\pi}{2} + \cos\frac{3\pi}{2}$$

$$= 2\pi$$

$\therefore \; f\left(\frac{3\pi}{2}\right) = 2\pi$

$\Rightarrow \; f^{-1}(2\pi) = \frac{3\pi}{2} = g(2\pi)$

Putting $x = 2\pi$ in eq(1),

$$\frac{d}{dx}\left[g(2\pi)\right] = \frac{1}{f'\left[f^{-1}(2\pi)\right]}$$

$$\Rightarrow \frac{d}{dx}\left[g(2\pi)\right] = \frac{1}{f'\left(3\frac{\pi}{2}\right)} \quad \text{———} \quad (2)$$

Now,

$$f'(x) = 25(2x - 3\pi)^{24} \times (2) + \frac{4}{3} - \sin x$$

[Apply chain rule for $(2x-3\pi)^{25}$]

$$\Rightarrow f'\left(\frac{3\pi}{2}\right) = 25\left(2 \times \frac{3\pi}{2} - 3\pi\right)^{24} \times (2) + \frac{4}{3} - \sin\left(\frac{3\pi}{2}\right)$$

$$= \frac{4}{3} + 1 = \frac{7}{3}$$

Putting $f'\left(\frac{3\pi}{2}\right)$ in eq(2),

$$\frac{d}{dx}[g(2\pi)] = \frac{1}{7/3} = \frac{3}{7}.$$

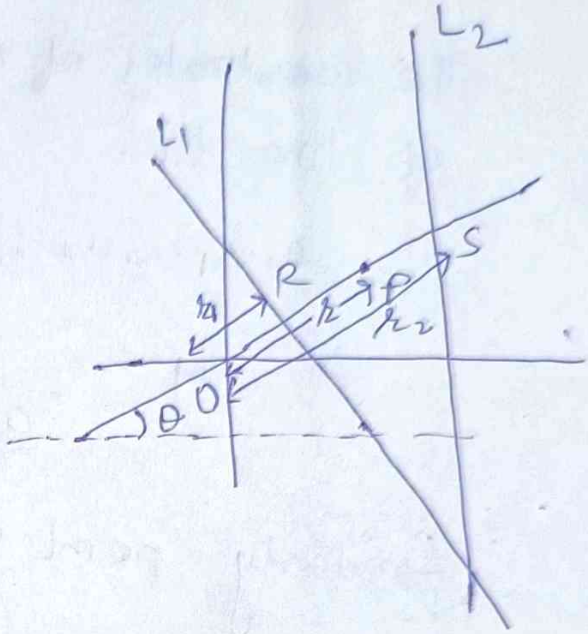
Hence, the correct answer is (A).

Q.No.7

Let the two lines be

$L_1 : a_1x + b_1y + c_1 = 0$

$L_2 : a_2x + b_2y + c_2 = 0$



Let the distance $OR = r_1$

& $OS = r_2$

$OP = r$

and θ be the angle made by variable line with x-axis.

By the parametric form of equation of line which passes through (x_1, y_1) and another point which is at a r distance is :

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r$$

This line passes through origin, R, P & S.

$$\therefore \frac{x-0}{\cos\theta} = \frac{y-0}{\sin\theta} = r_1 \quad (\text{for point O \& R})$$

Coordinates of R : $(r_1 \cos\theta_1 , r_1 \sin\theta_1)$

Coordinates of S : $(r_2 \cos\theta_2 , r_2 \sin\theta_2)$

Coordinates of P : $(r\cos\theta , r\sin\theta)$

The coordinates of R should satisfy the equation of line L_1.

$\therefore \quad a_1 r_1 \cos\theta + b_1 r_1 \sin\theta + c_1 = 0$

$$\Rightarrow r_1 = \frac{-c_1}{a_1\cos\theta + b_1\sin\theta} \quad \text{——— (1)}$$

Similarly point S satisfies L_2.

$$\therefore \quad r_2 = \frac{-c_2}{a_2\cos\theta + b_2\sin\theta} \quad \text{——— (2)}$$

Given,

$$\frac{m+n}{OP} = \frac{m}{OR} + \frac{n}{OS}$$

$$\Rightarrow \frac{m+n}{r} = \frac{m}{r_1} + \frac{n}{r_2} \quad \text{——— (3)}$$

From eq(1), eq(2) & eq(3),

$$\frac{m+n}{r} = \frac{m}{\frac{-c_1}{a_1\cos\theta + b_1\sin\theta}} + \frac{n}{\frac{-c_2}{a_2\cos\theta + b_2\sin\theta}}$$

$$\Rightarrow \frac{m+n}{r} = \frac{m a_1\cos\theta + m b_1\sin\theta}{-c_1} + \frac{a_2 n\cos\theta + n b_2\sin\theta}{-c_2}$$

$$m+n = \frac{m a_1 r\cos\theta + m b_1 r\sin\theta}{-c_1} + \frac{n a_2 r\cos\theta + n b_2 r\sin\theta}{-c_2}$$

But $\left.\begin{aligned} r\cos\theta &= x \ \& \\ r\sin\theta &= y \end{aligned}\right\}$ Parametric form

$$\therefore m+n = \frac{m a_1 x + m b_1 y}{-c_1} + \frac{n a_2 x + n b_2 y}{-c_2}$$

$$\Rightarrow c_1 c_2 (m+n) = -c_2 m a_1 x - c_2 m b_1 y - c_1 n a_2 x - c_1 n b_2 y$$

$$\Rightarrow c_2 m a_1 x + c_2 m b_1 y + c_1 n a_2 x + c_1 n b_2 y + c_1 c_2 (m+n) = 0$$

taking $c_2 m$ & $c_1 n$ as common,

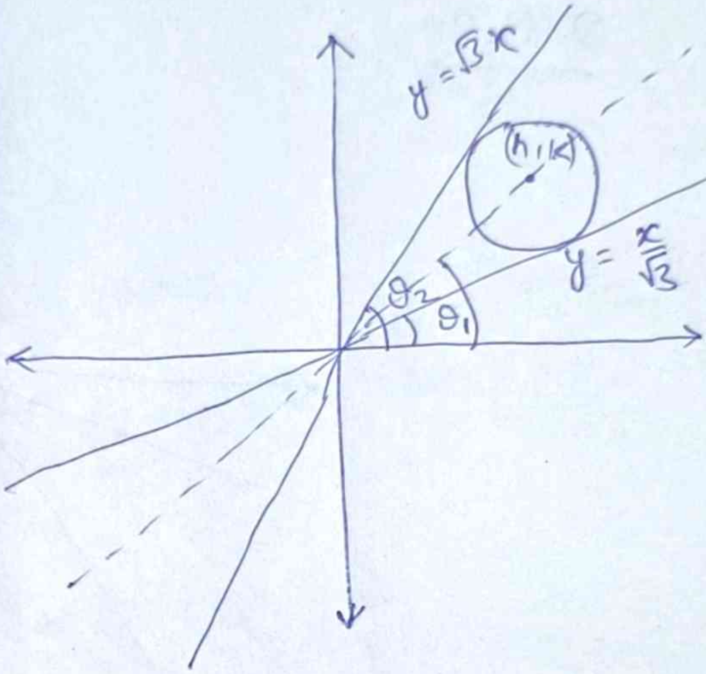
$$\Rightarrow c_2 m (a_1 x + b_1 y + c_1) + c_1 n (a_2 x + b_2 y + c_2) = 0$$

$$\Rightarrow \frac{c_2}{c_1} m (a_1 x + b_1 y + c_1) + n (a_2 x + b_2 y + c_2) = 0$$

Let $\frac{c_1}{c_2} = c$

$$\therefore m (a_1 x + b_1 y + c_1) + cn (a_2 x + b_2 y + c_2) = 0$$

Only option (A) is of this form.



Let (h, k) be the centre of the circle

Slope of line $\left(y = \frac{x}{\sqrt{3}}\right) = \frac{1}{\sqrt{3}}$

$\therefore$ Angle by the line on x-axis $= 30°$

Slope of line, $y = \sqrt{3}x = \sqrt{3}$

$\therefore$ Angle made by the line with x-axis $= 60°$

$\therefore$ Angle between the lines $= 60° - 30° = 30°$

And, angle made the bisector with both lines $= \frac{30°}{2} = 15°$

$\therefore$ Angle made by bisector with x-axis $=$
$30° + 15° = 45°$

$\therefore$ Slope of the bisector $= \tan 45° = 1$

The bisector passes through origin.

$\therefore$ Equation of the bisector: $y = x$

Now,

Centre of a circle lies on the bisector of tangents drawn from a point.

$\therefore$ (h, k) lies on $y = x$

$\therefore$ $h = k$

$\therefore$ The coordinates of centre becomes (h, h)

Now,

we know that,

distance of a point from a line $= \dfrac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$

where, $Ax + Bx + C = 0$ is the equation of the line, and (x_0, y_0) is the point.

$\therefore$ Distance of (h, h) from $y = \sqrt{3}x$

$$= \frac{|h - \sqrt{3}h|}{\sqrt{(\sqrt{3})^2 + 1^2}} = \frac{\sqrt{3}h - h}{2}$$

(the value of modulus is positive, & for positive h, $\sqrt{3}h - h \geq 0$)

h is positive because the circle is in 1^{st} quadrant.

But the radius of circle $= 1$

$\therefore$ (h, h) is at 1 unit distance from the tangents.

$$\therefore \quad \frac{\sqrt{3}h - h}{2} = 1$$

$$\Rightarrow \quad h = \frac{2}{\sqrt{3}-1} = \sqrt{3}+1$$

$\therefore$ The centre of the circle : $(\sqrt{3}+1, \sqrt{3}+1)$

We know that,

if (x_0, y_0) is the centre of a circle and r is the radius, equation of the circle can be written as,

$(x-x_0)^2 + (y-y_0)^2 = r^2$

$\therefore$ Equation of the circle is :

$[x - (\sqrt{3}+1)]^2 + [y - (\sqrt{3}+1)]^2 = 1^2$

$\Rightarrow x^2 + (\sqrt{3}+1)^2 - 2(\sqrt{3}+1)x + y^2 + (\sqrt{3}+1)^2 - 2(\sqrt{3}+1)y = 0$

$\Rightarrow x^2 + y^2 - 2x(\sqrt{3}+1) - 2y(\sqrt{3}+1) + 2(3+1+2\sqrt{3}) = 0$

$\Rightarrow x^2 + y^2 - 2x(\sqrt{3}+1) - 2y(\sqrt{3}+1) + 8 + 4\sqrt{3} = 0$

This matches with option (A).

Q. No. 9

The RHS of the equation is

$12e^{x} - 4e^{2x} - 10$.

Let $t = e^{x}$

The term becomes,

$12t - 4t^2 - 10$

Let $y = 12t - 4t^2 - 10$

$$\frac{dy}{dt} = 12 - 8t$$

$$\frac{dy}{dt} = 0 \quad \text{at} \quad t = \frac{3}{2}$$

$\Rightarrow$ the slope is 0 at $t = 3/2$

Now,

$$\frac{d^2y}{dt^2} = 12 > 0$$

As $\frac{d^2y}{dt^2} > 0$, $t = 3/2$ is local maxima.

$\therefore$ Maximum value of $y = 12t - 4t^2 - 10$ is at $t = 3/2$

At $t = 3/2$,

$$y = 12 \times \frac{3}{2} - 4\left(\frac{3}{2}\right)^2 - 10 = -1$$

This means $(12t - 4t^2 - 10)$ is always less than -1.

$\Rightarrow 12e^{x} - 4e^{2x} - 10 \leq -1$

As modulus is positive,

the LHS is positive.

Now,

$$LHS \geq 0$$

$$RHS \leq -1$$

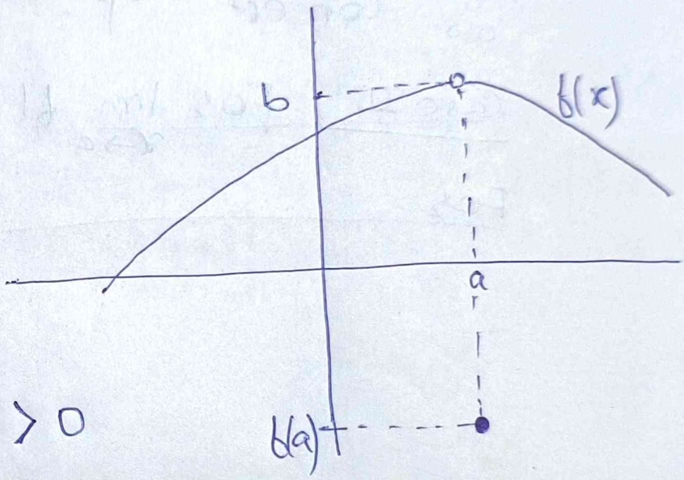
This means $LHS \neq RHS$

This means, the given equation is satisfied for no x.

$\therefore$ Number of roots $= 0$.

$\therefore$ Correct option a)

Q. No. 10



Case I For $\lim_{x \to a} f(x) > 0$

Let us represent an isolated point of discontinuity by the graph above.

In the graph,

$\lim_{x \to a} f(x) = b$, $f(a) \neq b$

So, $x = a$ is an isolated point of discontinuity.

For $|f(x)|$, the values $|f(x)|$ remain the same as that $f(x)$ in vicinity of $x = a$.

$\therefore \lim_{x \to a} |f(x)| = b$

For $|f(x)|$ to be continues, at a,

$\lim_{x \to a} |f(x)| = |f(a)|$

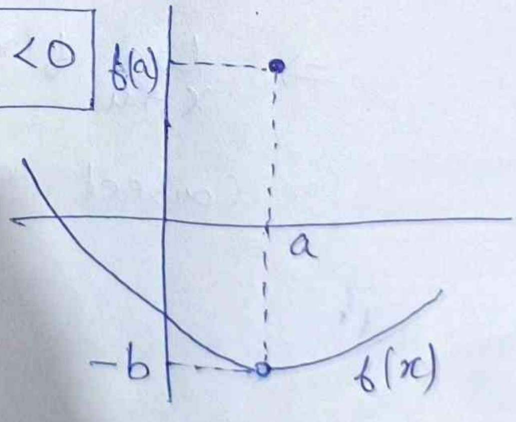
$\Rightarrow |f(a)| = b$

But $f(a) \neq b$

$\therefore f(a) = -b$

$\therefore \lim_{x \to a} f(x) + f(a) = b + (-b) = 0$

Case II For $\boxed{\lim_{x \to a} f(x) < 0}$

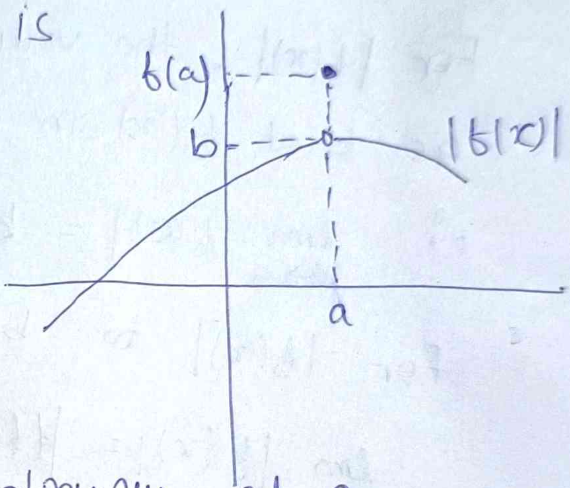


Here,

$\lim_{x \to a} f(x) = -b$, $f(a) \neq -b$

So, $x = a$ is an isolated point of discontinuity.

The graph of $|f(x)|$ is



For $|f(x)|$ to be continuous at a,

$f(a) = b$ (from the graph)

But $b = -\lim\limits_{x \to a} f(x)$

$\therefore f(a) = -\lim\limits_{x \to a} f(x)$

$\Rightarrow \lim\limits_{x \to a} f(x) + f(a) = 0$

$\therefore$ Correct option (B).