Advanced Worksheet: Challenging Locus Problems in the Complex Plane

Understanding complex number loci is essential for solving advanced geometry and algebra problems, such as those found in IIT JEE Main and Advanced exams. The locus of a complex number can represent various geometric shapes, including circles, ellipses, hyperbolas, and lines determined by arguments and moduli. Mastery of these concepts helps solve intricate problems involving distances, angles, and geometric places in the complex plane.

Fill in the Blank: Fill in the blank with the correct words.

1. The locus defined by $|z-a|+|z-b|=k$ (where $k>|a-b|$) represents an ____________ with foci at a and b.
2. The equation $|z-a|-|z-b|=d$ (where $d<|a-b|$) describes a ____________ with foci at a and b.
3. The locus $\arg\left(\frac{z-a}{z-b}\right)=\theta$ is a ____________ passing through a and b, making a constant angle θ at each point.
4. The locus $|z-a|=r|z-b|$ (for r ⊟ 1) is a ____________, except when $r=1$, in which case it is a ____________.
5. The equation $|z-1|+|z+1|=4$ represents an ellipse with major axis length ____________.

Word Bank:
ellipse, hyperbola, circle, straight line, major axis, 4, bisector, locus, 2

Multiple Choice Questions: Choose the correct answer from the choices for each question.

1. The locus $|z-3|+|z+3|=10$ describes:
   - a) A circle
   - b) An ellipse with foci at 3 and -3
   - c) A hyperbola with foci at 3 and -3
   - d) A straight line
2. The locus $|z-2|-|z+2|=1$ is:
   - a) An ellipse
   - b) A straight line
   - c) A hyperbola with foci at 2 and -2
   - d) A circle
3. The locus $\arg\left(\frac{z-i}{z+i}\right)=\frac{\pi}{2}$ is:
   - a) A circle passing through i and $-i$
   - b) The perpendicular bisector of the segment joining i and $-i$
   - c) A straight line passing through i and $-i$ and making an angle of $\frac{\pi}{2}$ at all points
   - d) A pair of lines
4. If $|z-1|=2|z+1|$, the locus is:
   - a) A circle not passing through 1 and -1

- b) A straight line
- c) A circle passing through 1 and −1
- d) A hyperbola

5. The equation $\arg(z - 1) = \frac{\pi}{4}$ represents:
   - a) A ray starting at 1 making 45° with the positive real axis
   - b) A circle centered at 1
   - c) The real axis
   - d) The imaginary axis

Open-Ended Questions: Answer the following questions in complete sentences.

1. Derive the equation of the locus of the point z in the complex plane such that the sum of the distances from z to two fixed points a and b is constant, and interpret its geometric meaning.
2. Explain the geometric locus represented by $\arg\left(\frac{z-a}{z-b}\right) = \theta$ for arbitrary points a and b and real θ.
3. Find and describe the locus of all points z such that $|z - 1| = 2|z + 1|$. What kind of conic section does this represent? Derive its equation in Cartesian form.

ANSWER KEY

Important: For all math-related questions, teachers should always review AI-generated answers for accuracy before distributing to students.

Fill in the Blank

1. ellipse
2. hyperbola
3. circle
4. circle, straight line
5. 4

Multiple Choice

1. b) An ellipse with foci at 3 and −3
2. c) A hyperbola with foci at 2 and −2
3. c) A straight line passing through i and $-i$ and making an angle of $\frac{\pi}{2}$ at all points
4. a) A circle not passing through 1 and −1
5. a) A ray starting at 1 making 45° with the positive real axis

Open-Ended Questions (Sample Answers)

1. The equation $|z - a| + |z - b| = k$ (where $k > |a - b|$) describes the locus of all points z such that the sum of distances from z to the fixed points a and b is constant. Geometrically, this locus is an ellipse with foci at a and b and major axis length k.
2. The equation $\arg\left(\frac{z-a}{z-b}\right) = \theta$ describes the locus of points z such that the angle subtended at z by the segment joining a and b is constant, specifically θ. This locus is a circle (called the circle of Apollonius) passing through a and b, except when $\theta = 0$ or π, where it becomes a straight line.

3. Given $|z - 1| = 2|z + 1|$, let $z = x + iy$. Then $\sqrt{(x - 1)^2 + y^2} = 2\sqrt{(x + 1)^2 + y^2}$. Squaring both sides and simplifying leads to the equation $(x - 1)^2 + y^2 = 4[(x + 1)^2 + y^2]$, which simplifies to $3x^2 + 3y^2 + 10x - 3 = 0$. This is the equation of a circle not passing through 1 and −1.