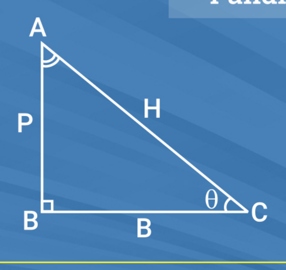
TRIGONOMETRY RATIO

"Pandit Badri Prasad Bole Hari Hari"

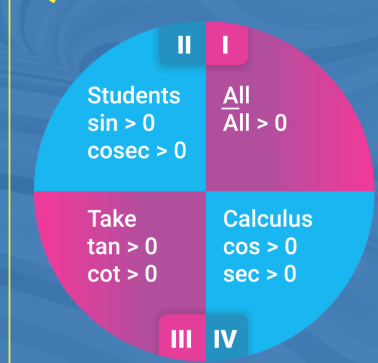


sin	cos	tan	cot	sec	cosec
P	B	P	B	H	H
H	H	B	P	B	P

Value

θ	0	π/6	π/4	π/3	π/2
sinθ	0	1/2	1/√2	√3/2	1
cosθ	1	√3/2	1/√2	1/2	0
tanθ	0	1/√3	1	√3	N.D.
cotθ	N.D.	√3	1	1/√3	0
secθ	1	2/√3	√2	2	N.D.
cosecθ	N.D.	2	√2	2/√3	1

Quadrant



(90° + θ) Reduction

$\sin(90°+\theta)=\cos\theta$ | $\cot(90°+\theta)=-\tan\theta$
$\cos(90°+\theta)=-\sin\theta$ | $\sec(90°+\theta)=-\csc\theta$
$\tan(90°+\theta)=-\cot\theta$ | $\csc(90°+\theta)=\sec\theta$

"Complementary angles are those whose sum is 90°"

(360° − θ) or (2π − θ) Reduction

$\sin(2\pi-\theta)=\sin(-\theta)=-\sin\theta$
$\cos(2\pi-\theta)=\cos(-\theta)=\cos\theta$
$\tan(2\pi-\theta)=\tan(-\theta)=-\tan\theta$

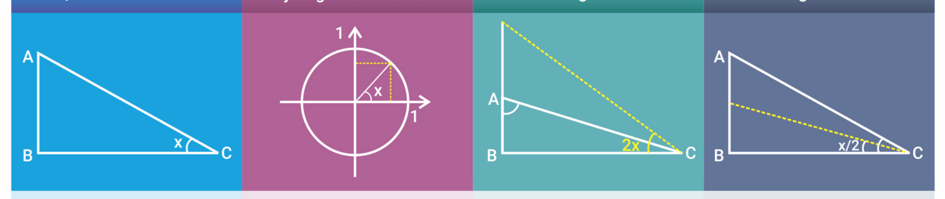
(180° + θ) Reduction

$\sin(180°+\theta)=-\sin\theta$ | $\cot(180°+\theta)=\cot\theta$
$\cos(180°+\theta)=-\cos\theta$ | $\csc(180°+\theta)=-\csc\theta$
$\tan(180°+\theta)=\tan\theta$ | $\sec(180°+\theta)=-\sec\theta$

$\cot(-\theta)=-\cot\theta$
$\csc(-\theta)=-\csc\theta$
$\sec(-\theta)=\sec\theta$

TRIGONOMETRIC IDENTITIES

1 Quotient Identities | 2 Pythagorean Identities | 3 Double Angle Identities | 4 Half Angle Identities



1. Quotient Identities

$\tan x=\frac{\sin x}{\cos x}$
$\cot x=\frac{\cos x}{\sin x}$
$\sec x=\frac{1}{\cos x}$
$\operatorname{cosec} x=\frac{1}{\sin x}$

2. Pythagorean Identities

$\sin^2x+\cos^2x=1$
$\sec^2x-\tan^2x=1$
$\operatorname{cosec}^2x-\cot^2x=1$

3. Double Angle Identities

$\sin 2x=2\sin x\cos x$
$\cos 2x=\cos^2x-\sin^2x$
$\cos 2x=2\cos^2x-1$
$\cos 2x=1-2\sin^2x$
$\cos 2x=\frac{1-\tan^2x}{1+\tan^2x}$
$\tan 2x=\frac{2\tan x}{1-\tan^2x}$

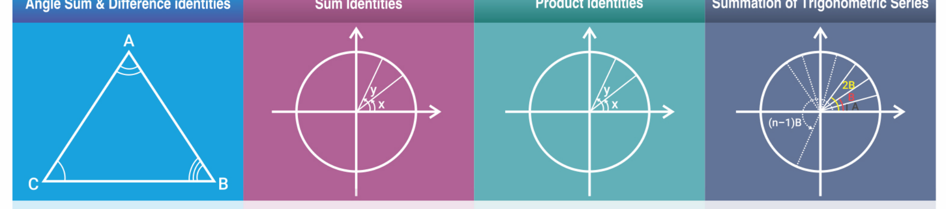
4. Half Angle Identities

$\sin\frac{x}{2}=\pm\sqrt{\frac{1-\cos x}{2}}$
$\cos\frac{x}{2}=\pm\sqrt{\frac{1+\cos x}{2}}$
$\tan\frac{x}{2}=\pm\sqrt{\frac{1-\cos x}{1+\cos x}}$
$\tan\frac{x}{2}=\frac{1-\cos x}{\sin x}$
$\tan\frac{x}{2}=\frac{\sin x}{1+\cos x}$

TRIGONOMETRIC IDENTITIES

Part-II

5 Angle Sum & Difference Identities | 6 Sum Identities | 7 Product Identities | 8 Summation of Trigonometric Series



5. Angle Sum & Difference Identities

$\sin(A\pm B)=\sin A\cos B\pm\cos A\sin B$
$\cos(A\pm B)=\cos A\cos B\mp\sin A\sin B$
$\tan(A\pm B)=\frac{\tan A\pm\tan B}{1\mp\tan A\tan B}$
$\cot(A\pm B)=\frac{\cot A\cot B\mp1}{\cot B\pm\cot A}$

6. Sum Identities

$\sin C+\sin D=2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$
$\sin C-\sin D=2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$
$\cos C+\cos D=2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$
$\cos C-\cos D=-2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

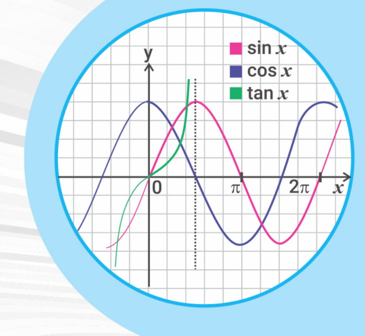
7. Product Identities

$2\sin A\cos B=[\sin(A+B)+\sin(A-B)]$
$2\cos A\sin B=[\sin(A+B)-\sin(A-B)]$
$2\cos A\cos B=[\cos(A+B)+\cos(A-B)]$
$2\sin A\sin B=[\cos(A-B)-\cos(A+B)]$

8. Summation of Trigonometric Series

$\sin A+\sin(A+B)+\sin(A+2B)+\ldots\ldots+\sin(A+(n-1)B)=\frac{\sin nB/2}{\sin B/2}\sin\left(A+\frac{n-1}{2}B\right)$
$\cos A+\cos(A+B)+\cos(A+2B)+\ldots\ldots+\cos(A+(n-1)B)=\frac{\sin nB/2}{\sin B/2}\cos\left(A+\frac{n-1}{2}B\right)$

TRIGONOMETRIC EQUATIONS



Principal Solution

The solutions of a trigonometric equation which lie in the interval [0, 2π] are called principal solutions.

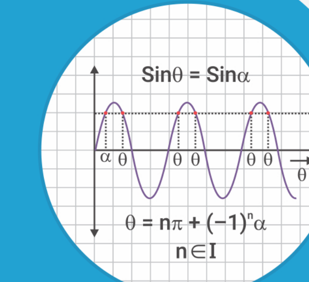
Eg: $\sin x=\frac{1}{2}\Rightarrow x=\frac{\pi}{6},\frac{5\pi}{6},\frac{9\pi}{6}$......

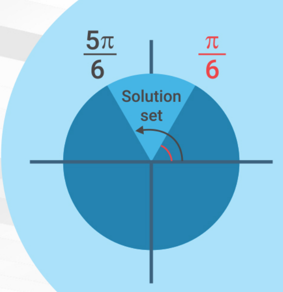
But, principal solution of

$\sin x=\frac{1}{2}$ are $\frac{\pi}{6},\frac{5\pi}{6}\in[0,2\pi]$

General Solution

$\sin\theta=\sin\alpha\Rightarrow\theta=n\pi+(-1)^n\alpha,\ \alpha\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]n\in I$
$\cos\theta=\cos\alpha\Rightarrow\theta=2n\pi\pm\alpha,\ \alpha\in[0,\pi],\ n\in I$
$\tan\theta=\tan\alpha\Rightarrow\theta=n\pi+\alpha,\ \alpha\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)n\in I$
$\sin^2\theta=\sin^2\alpha\Rightarrow\theta=n\pi\pm\alpha,\ n\in I$
$\cos^2\theta=\cos^2\alpha\Rightarrow\theta=n\pi\pm\alpha,\ n\in I$
$\tan^2\theta=\tan^2\alpha\Rightarrow\theta=n\pi\pm\alpha,\ n\in I$; α is called one principal angle.



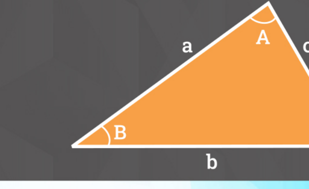


Trigonometric Inequalities

Eg: $\sin x>\frac{1}{2}\Rightarrow\frac{\pi}{6}<x<\frac{5\pi}{6}$

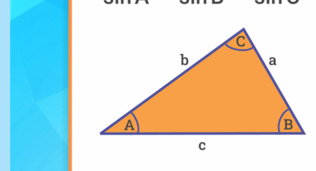
SOLUTION OF TRIANGLE

Part-II



LAW OF SINES

$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$



LAW OF COSINES

$a^2=b^2+c^2-2bc\cos A$
$b^2=a^2+c^2-2ac\cos B$
$c^2=a^2+b^2-2ab\cos C$

PROJECTION FORMULA

$a=b\cos C+c\cos B$
$b=c\cos A+a\cos C$
$c=a\cos B+b\cos A$

AREA OF TRIANGLE

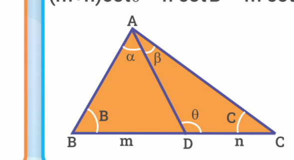
Area $=\frac{1}{2}ab\sin C$
Area $=\frac{1}{2}bc\sin A$
Area $=\frac{1}{2}ca\sin B$

NAIPER'S ANALOGY

$\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}$
$\tan\frac{C-A}{2}=\frac{c-a}{c+a}\cot\frac{B}{2}$
$\tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2}$

M-N THEOREM

$(m+n)\cot\theta=m\cot\alpha-n\cot\beta$
$(m+n)\cot\theta=n\cot B-m\cot C$



TRIGONOMETRIC HALF ANGLES

$\sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}$ | $\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}$ | $\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$

$\sin\frac{B}{2}=\sqrt{\frac{(s-c)(s-a)}{ca}}$ | $\cos\frac{B}{2}=\sqrt{\frac{s(s-b)}{ca}}$ | $\tan\frac{B}{2}=\sqrt{\frac{(s-c)(s-a)}{s(s-b)}}$

$\sin\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{ab}}$ | $\cos\frac{C}{2}=\sqrt{\frac{s(s-c)}{ab}}$ | $\tan\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$

where, $s=\frac{a+b+c}{2}$

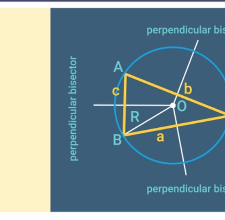
PROPERTIES OF TRIANGLES AND CIRCLES CONNECTED WITH THEM

Part-II

R→Circumradius of ΔABC.

$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=2R;\ \Delta=\frac{abc}{4R}$

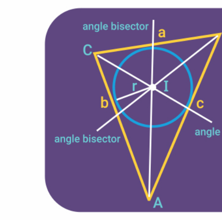
$a\cos B\cos C+b\cos C\cos A+c\cos A\cos B=\frac{\Delta}{R}$



r→Inradius of ΔABC.

$r=\frac{a\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}}=\frac{b\sin\frac{C}{2}\sin\frac{A}{2}}{\cos\frac{B}{2}}=\frac{c\sin\frac{A}{2}\sin\frac{B}{2}}{\cos\frac{C}{2}}$

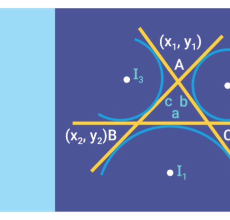
$r=(s-a)\tan\frac{A}{2}=(s-b)\tan\frac{B}{2}=(s-c)\tan\frac{C}{2};\ r=\frac{\Delta}{s};\ s=\frac{a+b+c}{2}$

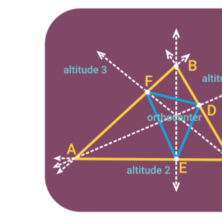


r_1, r_2, r_3 → are radii of excircles of ΔABC.

$r_1=\frac{\Delta}{s-a};\ r_2=\frac{\Delta}{s-b};\ r_3=\frac{\Delta}{s-c};\ s=\frac{a+b+c}{2}$

$r_1=s\tan\frac{A}{2};\ r_2=s\tan\frac{B}{2};\ r_3=s\tan\frac{C}{2}$





ORTHOCENTRE AND PEDAL TRIANGLE

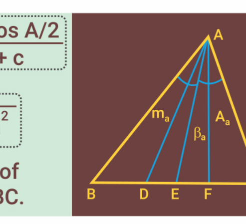
ORTHOCENTRE - Point of intersection of 3 altitudes.
PEDAL - Pedal triangle is formed by joining the feet of altitudes.
Area of ΔDEF = 4Δ cosA cosB cosC
where Δ is the area of triangle ABC.

LENGTH OF ANGLE BISECTOR MEDIAN & ALTITUDE

Length of an angle bisector from the angle A = $\beta_a=\frac{2bc\cos A/2}{b+c}$

Length of median from the angle A = $m_a=\frac{1}{2}\sqrt{2b^2+2c^2-a^2}$

Length of altitude from the angle A = $A_a=\frac{2\Delta}{a}$ Δ → Area of triangle ABC.



COMPOUND ANGLES

BASIC TRIGONOMETRIC IDENTITIES

- $\sin^2\theta+\cos^2\theta=1;\ -1\le\sin\theta\le1;\ -1\le\cos\theta\le1\ \forall\ \theta\in R$
- $\sec^2\theta-\tan^2\theta=1;\ |\sec\theta|\ge1\ \forall\ \theta\in R-\{(2n-1)\pi/2, n\in I\}$
- $\operatorname{cosec}^2\theta-\cot^2\theta=1;\ |\operatorname{cosec}\theta|\ge1\ \forall\ \theta\in R-\{n\pi, n\in I\}$

IMPORTANT RATIOS

- $\sin n\pi=0;\ \cos n\pi=(-1)^n;\ \tan n\pi=0$ where $n\in I$
- $\sin\frac{(2n+1)\pi}{2}=(-1)^n$ & $\cos\frac{(2n+1)\pi}{2}=0$ where $n\in I$
- $\sin\frac{\pi}{12}=\frac{\sqrt3-1}{2\sqrt2}=\cos\frac{5\pi}{12};\ \cos\frac{\pi}{12}=\frac{\sqrt3+1}{2\sqrt2}=\sin\frac{5\pi}{12}$
- $\tan\frac{\pi}{12}=\frac{\sqrt3-1}{\sqrt3+1}=2-\sqrt3=\cot\frac{5\pi}{12}$
- $\tan\frac{5\pi}{12}=\frac{\sqrt3+1}{\sqrt3-1}=2+\sqrt3=\cot\frac{\pi}{12}$
- $\sin\frac{\pi}{10}=\frac{\sqrt5-1}{4}$ & $\cos\frac{\pi}{5}=\frac{\sqrt5+1}{4}$

TRIGONOMETRIC FUNCTIONS OF SUM OR DIFFERENCE OF TWO ANGLES

- $\sin(A\pm B)=\sin A\cos B\pm\cos A\sin B$
- $\cos(A\pm B)=\cos A\cos B\mp\sin A\sin B$
- $\sin^2A-\sin^2B=\cos^2B-\cos^2A=\sin(A+B)\cdot\sin(A-B)$
- $\cos^2A-\sin^2B=\cos^2B-\sin^2A=\cos(A+B)\cdot\cos(A-B)$
- $\tan(A\pm B)=\frac{\tan A\pm\tan B}{1\mp\tan A\tan B}$
- $\cot(A\pm B)=\frac{\cot A\cot B\mp1}{\cot B\pm\cot A}$

FACTORISATION OF THE SUM OR DIFFERENCE OF TWO SINES OR COSINE

- $\sin C+\sin D=2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$
- $\sin C-\sin D=2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$
- $\cos C+\cos D=2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$
- $\cos C-\cos D=-2\sin\frac{C+D}{2}\sin\frac{C-D}{2}$

TRANSFORMATION OF PRODUCT INTO SUM OR DIFFERENCE OF SINES & COSINES

- $2\sin A\cos B=\sin(A+B)+\sin(A-B)$
- $2\cos A\sin B=\sin(A+B)-\sin(A-B)$
- $2\cos A\cos B=\cos(A+B)+\cos(A-B)$
- $2\sin A\sin B=\cos(A-B)-\cos(A+B)$

MULTIPLE ANGLES AND HALF ANGLES

- $\sin 2A=2\sin A\cos A$
- $\cos 2A=\cos^2A-\sin^2A=2\cos^2A-1=1-2\sin^2A$;
  $2\cos^2A=1+\cos 2A,\ 2\sin^2A=1-\cos 2A$
- $\tan^2A=\frac{1-\cos 2A}{1+\cos 2A}$ • $\tan 2A=\frac{2\tan A}{1-\tan^2A}$
- $\sin 2A=\frac{2\tan A}{1+\tan^2A};\ \cos 2A=\frac{1-\tan^2A}{1+\tan^2A}$
- $\sin 3A=3\sin A-4\sin^3A$
- $\cos 3A=4\cos^3A-3\cos A$
- $\tan 3A=\frac{3\tan A-\tan^3A}{1-3\tan^2A}$

THREE ANGLES

(a) $\tan(A+B+C)=\frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan B\tan C-\tan C\tan A}$

NOTE IF

- $A+B+C=\pi$ then $\tan A+\tan B+\tan C=\tan A\tan B\tan C$
- $A+B+C=\frac{\pi}{2}$ then $\tan A\tan B+\tan B\tan C+\tan C\tan A=1$

(b) If $A+B+C=\pi$ then;

- $\sin 2A+\sin 2B+\sin 2C=4\sin A\sin B\sin C$
- $\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

MAXIMUM & MINIMUM VALUES OF TRIGONOMETRIC FUNCTIONS

- Min. value of $a^2\tan^2\theta+b^2\cot^2\theta=2ab$
- Max. and Min. value of $a\cos\theta+b\sin\theta$ are $\sqrt{a^2+b^2}$ and $-\sqrt{a^2+b^2}$

SUM OF SINES OR COSINES OF N ANGLES

$\sin\alpha+\sin(\alpha+\beta)+\sin(\alpha+2\beta)+\ldots+\sin(\alpha+\overline{n-1}\,\beta)$

$=\frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\sin\left(\alpha+\frac{n-1}{2}\beta\right)$

$\cos\alpha+\cos(\alpha+\beta)+\cos(\alpha+2\beta)+\ldots+\cos(\alpha+\overline{n-1}\,\beta)$

$=\frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\cos\left(\alpha+\frac{n-1}{2}\beta\right)$

SOME TIPS TO BE LEARN

(A) Continues product Series

- $\sin\left(\frac{\pi}{3}-A\right)\sin A\sin\left(\frac{\pi}{3}+A\right)=\frac{1}{4}\sin 3A$
- $\cos\left(\frac{\pi}{3}-A\right)\cos A\cos\left(\frac{\pi}{3}+A\right)=\frac{1}{4}\cos 3A$
- $\tan\left(\frac{\pi}{3}-A\right)\tan A\tan\left(\frac{\pi}{3}+A\right)=\tan 3A$
- $\cos\theta\cos2\theta\cos2^2\theta\ldots\ldots\cos2^{n-1}\theta=\frac{\sin2^n\theta}{2^n\sin\theta}$

(B) Extra formulas useful while solving problems

- $\cot A-\tan A=2\cot 2A$
- $\tan 3A\tan 2A\tan A=\tan 3A-\tan 2A-\tan A$
- $(1+\tan A)(1+\tan(45°-A))=2$
- $(1-\cot A)(1-\cot(45°-A))=2$

TRIGONOMETRIC EQUATION

- If $\sin\theta=\sin\alpha\Rightarrow\theta=n\pi+(-1)^n\alpha$
  where $\alpha\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right], n\in I$
- If $\cos\theta=\cos\alpha\Rightarrow\theta=2n\pi\pm\alpha$
  where $\alpha\in[0,\pi], n\in I$
- If $\tan\theta=\tan\alpha\Rightarrow\theta=n\pi+\alpha$
  where $\alpha\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right), n\in I$
- If $\sin^2\theta=\sin^2\alpha\Rightarrow\theta=n\pi\pm\alpha$
- If $\cos^2\theta=\cos^2\alpha\Rightarrow\theta=n\pi\pm\alpha$
- If $\tan^2\theta=\tan^2\alpha\Rightarrow\theta=n\pi\pm\alpha$

SOLUTION OF TRIANGLE

SINE FORMULA

In any triangle ABC, $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=2R$,
(Radius of Circle which circumscribe the triangle)

COSINE FORMULA

- $\cos A=\frac{b^2+c^2-a^2}{2bc}$

PROJECTION FORMULA

- $a=b\cos C+c\cos B$ • $b=c\cos A+a\cos C$
- $c=a\cos B+b\cos A$

NAPIER'S ANALOGY = TANGENT RULE

$\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}$

TRIGONOMETRIC FUNCTIONS OF HALF ANGLES

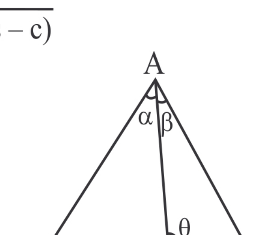
- $\sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}$ • $\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}$
- $\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}=\frac{\Delta}{s(s-a)}$

where $s=\frac{a+b+c}{2}$ & Δ = area of triangle.

Area of triangle $=\sqrt{s(s-a)(s-b)(s-c)}$

m - n RULE

In any triangle,
$(m+n)\cot\theta=m\cot\alpha-n\cot\beta$
$=n\cot B-m\cot C$



$\Delta=\frac{1}{2}ab\sin C=\frac{1}{2}bc\sin A=\frac{1}{2}ca\sin B$

Δ = area of triangle ABC

$R=\frac{abc}{4\Delta}$; Where R is the radius of circumcircle.

RADIUS OF THE INCIRCLE 'r' IS GIVEN BY

- $r=\frac{\Delta}{s}$ where $s=\frac{a+b+c}{2}$
- $r=(s-a)\tan\frac{A}{2}=(s-b)\tan\frac{B}{2}=(s-c)\tan\frac{C}{2}$
- $r=\frac{a\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}}$ & so on
- $r=4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

RADIUS OF THE EX-CIRCLES r_1, r_2 & r_3 ARE GIVEN BY

- $r_1=\frac{\Delta}{s-a};\ r_2=\frac{\Delta}{s-b};\ r_3=\frac{\Delta}{s-c}$
- $r_1=s\tan\frac{A}{2};\ r_2=s\tan\frac{B}{2};\ r_3=s\tan\frac{C}{2}$
- $r_1=\frac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}$ & so on
- $r_1=4R\sin\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\cos\frac{C}{2}$
- $r_2=4R\sin\frac{B}{2}\cdot\cos\frac{A}{2}\cdot\cos\frac{C}{2}$
- $r_3=4R\sin\frac{C}{2}\cdot\cos\frac{A}{2}\cdot\cos\frac{B}{2}$

LENGTH OF ANGLE BISECTOR & MEDIANS

If m_a and β_a are the lengths of a median and angle bisector from the angle A then,

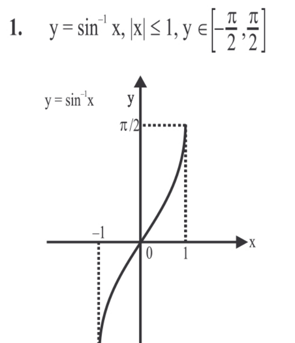
- $m_a=\frac{1}{2}\sqrt{2b^2+2c^2-a^2}$ and $\beta_a=\frac{2bc\cos\frac{A}{2}}{b+c}$

Note that $m_a^2+m_b^2+m_c^2=\frac{3}{4}(a^2+b^2+c^2)$

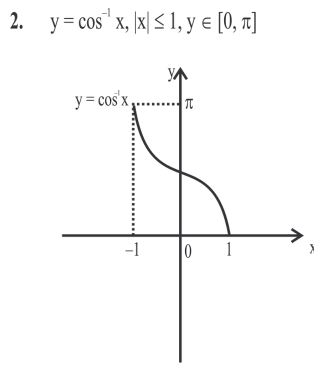
INVERSE TRIGONOMETRIC FUNCTION

DOMAIN, RANGE AND GRAPH OF INVERSE TRIGONOMETRIC FUNCTION

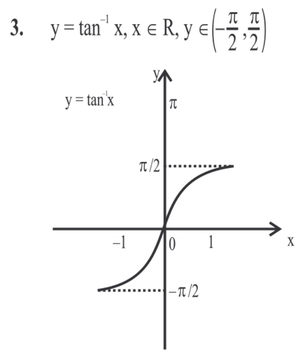
1. $y=\sin^{-1}x,\ |x|\le1,\ y\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$



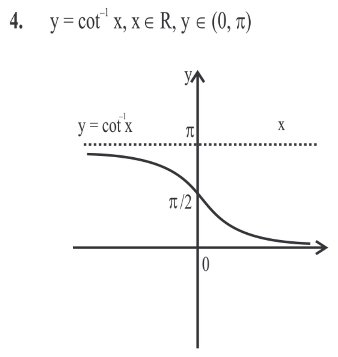
2. $y=\cos^{-1}x,\ |x|\le1,\ y\in[0,\pi]$



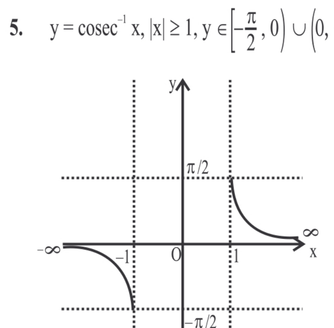
3. $y=\tan^{-1}x,\ x\in R,\ y\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$



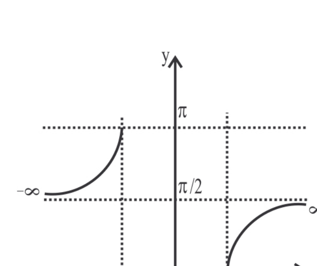
4. $y=\cot^{-1}x,\ x\in R,\ y\in(0,\pi)$



5. $y=\operatorname{cosec}^{-1}x,\ |x|\ge1,\ y\in\left[-\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right]$



6. $y=\sec^{-1}x,\ |x|\ge1,\ y\in\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]$



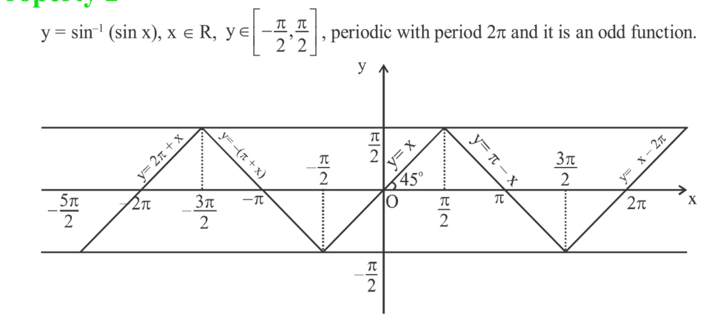
PROPERTIES OF INVERSE TRIGONOMETRIC FUNCTION

Property 1

1. $y=\sin(\sin^{-1}x)=x,\ x\in[-1,1],\ y\in[-1,1]$
2. $y=\cos(\cos^{-1}x)=x,\ x\in[-1,1],\ y\in[-1,1]$
3. $y=\tan(\tan^{-1}x)=x,\ x\in R,\ y\in R$
4. $y=\cot(\cot^{-1}x)=x,\ x\in R,\ y\in R$
5. $y=\operatorname{cosec}(\operatorname{cosec}^{-1}x)=x,\ |x|\ge1,\ |y|\ge1$
6. $y=\sec(\sec^{-1}x)=x,\ |x|\ge1,\ |y|\ge1$

Property 2

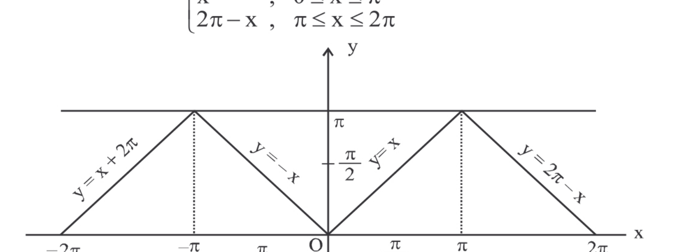
1. $y=\sin^{-1}(\sin x),\ x\in R,\ y\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, periodic with period 2π and it is an odd function.



$\sin^{-1}(\sin x)=\begin{cases}2\pi+x, & -\frac{5\pi}{2}\le x\le-\frac{3\pi}{2}\\ -\pi-x, & -\frac{3\pi}{2}\le x\le-\frac{\pi}{2}\\ x, & -\frac{\pi}{2}\le x\le\frac{\pi}{2}\\ \pi-x, & \frac{\pi}{2}\le x\le\frac{3\pi}{2}\\ x-2\pi, & \frac{3\pi}{2}\le x\le\frac{5\pi}{2}\end{cases}$

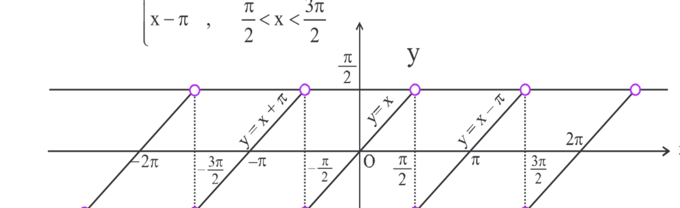
2. $y=\cos^{-1}(\cos x),\ x\in R,\ y\in[0,\pi]$, periodic with period 2π and it is an even function.

$\cos^{-1}(\cos x)=\begin{cases}2\pi+x, & -2\pi\le x\le-\pi\\ -x, & -\pi\le x\le0\\ x, & 0\le x\le\pi\\ 2\pi-x, & \pi\le x\le2\pi\end{cases}$



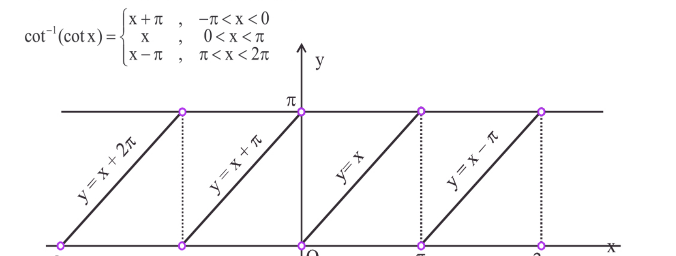
3. $y=\tan^{-1}(\tan x)$
$x\in R-\left\{(2n-1)\frac{\pi}{2}, n\in I\right\};\ y\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, periodic with period π and it is an odd function.

$\tan^{-1}(\tan x)=\begin{cases}x+\pi, & -\frac{3\pi}{2}<x<-\frac{\pi}{2}\\ x, & -\frac{\pi}{2}<x<\frac{\pi}{2}\\ x-\pi, & \frac{\pi}{2}<x<\frac{3\pi}{2}\end{cases}$

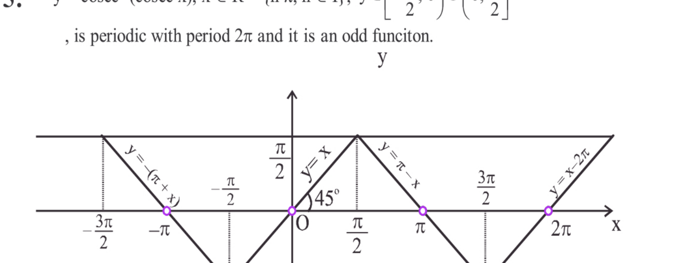


4. $y=\cot^{-1}(\cot x),\ x\in R-\{n\pi, n\in I\},\ y\in(0,\pi)$, periodic with period π and neither even nor odd function.

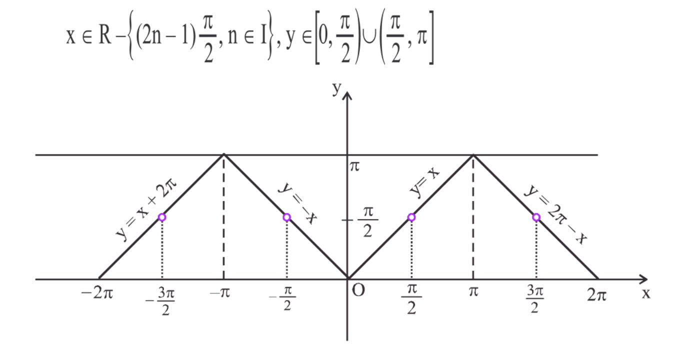
$\cot^{-1}(\cot x)=\begin{cases}x+\pi, & -\pi<x<0\\ x, & 0<x<\pi\\ x-\pi, & \pi<x<2\pi\end{cases}$



5. $y=\operatorname{cosec}^{-1}(\operatorname{cosec}x),\ x\in R-\{n\pi, n\in I\},\ y\in\left[-\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right]$, is periodic with period 2π and it is an odd function.



6. $y=\sec^{-1}(\sec x)$, y is periodic with period 2π and it is an even function
$x\in R-\left\{(2n-1)\frac{\pi}{2}, n\in I\right\},\ y\in\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]$



Property 3

1. $\sin^{-1}(-x)=-\sin^{-1}(x)$, for all $x\in[-1,1]$
2. $\cos^{-1}(-x)=\pi-\cos^{-1}x$, for all $x\in[-1,1]$
3. $\tan^{-1}(-x)=-\tan^{-1}x$, for all $x\in R$
4. $\operatorname{cosec}^{-1}(-x)=-\operatorname{cosec}^{-1}x$, for all $x\in(-\infty,-1]\cup[1,\infty)$
5. $\sec^{-1}(-x)=\pi-\sec^{-1}x$, for all $x\in(-\infty,-1]\cup[1,\infty)$
6. $\cot^{-1}(-x)=\pi-\cot^{-1}x$, for all $x\in R$

Property 4

1. $\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$, for all $x\in[-1,1]$
2. $\tan^{-1}x+\cot^{-1}x=\frac{\pi}{2}$, for all $x\in R$
3. $\sec^{-1}x+\operatorname{cosec}^{-1}x=\frac{\pi}{2}$, for all $x\in(-\infty,-1]\cup[1,\infty)$

Property 5

1. $\operatorname{cosec}^{-1}x=\sin^{-1}\frac{1}{x};\ |x|\ge1$
2. $\sin^{-1}x=\operatorname{cosec}^{-1}\frac{1}{x};\ |x|\le1,\ x\ne0$
3. $\sec^{-1}x=\cos^{-1}\frac{1}{x};\ |x|\ge1$
4. $\cos^{-1}x=\sec^{-1}\frac{1}{x};\ |x|\le1,\ x\ne0$
5. $\cot^{-1}x=\tan^{-1}\frac{1}{x};\ x>0\ =\pi+\tan^{-1}\frac{1}{x};\ x<0$

Property 6

$\tan^{-1}x+\tan^{-1}y=\begin{cases}\tan^{-1}\left(\frac{x+y}{1-xy}\right), & \text{if } xy<1\\ \pi+\tan^{-1}\left(\frac{x+y}{1-xy}\right), & \text{if } x>0, y>0 \text{ and } xy>1\\ -\pi+\tan^{-1}\left(\frac{x+y}{1-xy}\right), & \text{if } x<0, y<0 \text{ and } xy>1\end{cases}$

$\tan^{-1}x-\tan^{-1}y=\begin{cases}\tan^{-1}\left(\frac{x-y}{1+xy}\right), & \text{if } xy>-1\\ \pi+\tan^{-1}\left(\frac{x-y}{1+xy}\right), & \text{if } x>0, y<0 \text{ and } xy<-1\\ -\pi+\tan^{-1}\left(\frac{x-y}{1+xy}\right), & \text{if } x<0, y>0 \text{ and } xy<-1\end{cases}$

Property 7

(1) $\sin^{-1}x+\sin^{-1}y$

$=\begin{cases}\sin^{-1}\left[x\sqrt{1-y^2}+y\sqrt{1-x^2}\right], & \text{if } -1\le x, y\le1 \text{ and } x^2+y^2\le1 \text{ or if } xy<0 \text{ and } x^2+y^2>1\\ \pi-\sin^{-1}\left[x\sqrt{1-y^2}+y\sqrt{1-x^2}\right], & \text{if } 0<x, y\le1 \text{ and } x^2+y^2>1\\ -\pi-\sin^{-1}\left[x\sqrt{1-y^2}+y\sqrt{1-x^2}\right], & \text{if } -1\le x, y<0 \text{ and } x^2+y^2>1\end{cases}$

(2) $\sin^{-1}x-\sin^{-1}y$

$=\begin{cases}\sin^{-1}\left[x\sqrt{1-y^2}-y\sqrt{1-x^2}\right], & \text{if } -1\le x, y\le1 \text{ and } x^2+y^2\le1 \text{ or if } xy>0 \text{ and } x^2+y^2>1\\ \pi-\sin^{-1}\left[x\sqrt{1-y^2}-y\sqrt{1-x^2}\right], & \text{if } 0<x\le1, -1\le y\le0 \text{ and } x^2+y^2>1\\ -\pi-\sin^{-1}\left[x\sqrt{1-y^2}-y\sqrt{1-x^2}\right], & \text{if } -1\le x<0, 0<y\le1 \text{ and } x^2+y^2>1\end{cases}$

Property 8

(1) $\cos^{-1}x+\cos^{-1}y=\begin{cases}\cos^{-1}\{xy-\sqrt{1-x^2}\sqrt{1-y^2}\}, & \text{if } -1\le x, y\le1 \text{ and } x+y\ge0\\ 2\pi-\cos^{-1}\{xy-\sqrt{1-x^2}\sqrt{1-y^2}\}, & \text{if } -1\le x, y\le1 \text{ and } x+y\le0\end{cases}$

(2) $\cos^{-1}x-\cos^{-1}y=\begin{cases}\cos^{-1}\{xy+\sqrt{1-x^2}\sqrt{1-y^2}\}, & \text{if } -1\le x, y\le1 \text{ and } x\le y\\ -\cos^{-1}\{xy+\sqrt{1-x^2}\sqrt{1-y^2}\}, & \text{if } -1\le y\le0, 0<x\le1 \text{ and } x\ge y\end{cases}$

Property 9 : If $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\tan^{-1}\left[\frac{x+y+z-xyz}{1-xy-yz-zx}\right]$
if, x > 0, y > 0, z > 0 & xy + yz + zx < 1

Note : (i) If $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\pi$ then $x+y+z=xyz$
(ii) If $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\frac{\pi}{2}$ then $xy+yz+zx=1$

Property 10

1. $2\sin^{-1}x=\begin{cases}\sin^{-1}(2x\sqrt{1-x^2}), & \text{if } -\frac{1}{\sqrt2}\le x\le\frac{1}{\sqrt2}\\ \pi-\sin^{-1}(2x\sqrt{1-x^2}), & \text{if } \frac{1}{\sqrt2}\le x\le1\\ -\pi-\sin(2x\sqrt{1-x^2}), & \text{if } -1\le x\le-\frac{1}{\sqrt2}\end{cases}$

2. $2\cos^{-1}x=\begin{cases}\cos^{-1}(2x^2-1), & \text{if } 0\le x\le1\\ 2\pi-\cos^{-1}(2x^2-1), & \text{if } -1\le x\le0\end{cases}$

3. $2\tan^{-1}x=\begin{cases}\tan^{-1}\left(\frac{2x}{1-x^2}\right), & \text{if } -1<x<1\\ \pi+\tan^{-1}\left(\frac{2x}{1-x^2}\right), & \text{if } x>1\\ -\pi+\tan^{-1}\left(\frac{2x}{1-x^2}\right), & \text{if } x<-1\end{cases}$

4. $2\tan^{-1}x=\begin{cases}\sin^{-1}\left(\frac{2x}{1+x^2}\right), & \text{if } -1\le x\le1\\ \pi-\sin^{-1}\left(\frac{2x}{1+x^2}\right), & \text{if } x>1\\ -\pi-\sin^{-1}\left(\frac{2x}{1+x^2}\right), & \text{if } x<-1\end{cases}$

5. $2\tan^{-1}x=\begin{cases}\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right), & x\ge0\\ -\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right), & x<0\end{cases}$

Property 11

(1) $3\sin^{-1}x=\begin{cases}\sin^{-1}(3x-4x^3), & \text{if } -\frac{1}{2}\le x\le\frac{1}{2}\\ \pi-\sin^{-1}(3x-4x^3), & \text{if } \frac{1}{2}<x\le1\\ -\pi-\sin^{-1}(3x-4x^3), & \text{if } -1\le x<-\frac{1}{2}\end{cases}$

(2) $3\cos^{-1}x=\begin{cases}\cos^{-1}(4x^3-3x), & \text{if } \frac{1}{2}\le x\le1\\ 2\pi-\cos^{-1}(4x^3-3x), & \text{if } -\frac{1}{2}\le x\le\frac{1}{2}\\ 2\pi+\cos^{-1}(4x^3-3x), & \text{if } -1\le x\le-\frac{1}{2}\end{cases}$

(3) $3\tan^{-1}x=\begin{cases}\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right), & \text{if } -\frac{1}{\sqrt3}<x<\frac{1}{\sqrt3}\\ \pi+\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right), & \text{if } x>\frac{1}{\sqrt3}\\ -\pi+\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right), & \text{if } x<-\frac{1}{\sqrt3}\end{cases}$

Property 12

(1) $\sin^{-1}x=\cos^{-1}\sqrt{1-x^2}=\tan^{-1}\frac{x}{\sqrt{1-x^2}}$ $0\le x\le1$
$=\cot^{-1}\frac{\sqrt{1-x^2}}{x}=\sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)=\operatorname{cosec}^{-1}\left(\frac{1}{x}\right)$

(2) $\cos^{-1}x=\sin^{-1}\sqrt{1-x^2}=\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$ $0\le x\le1$
$=\cot^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)=\sec^{-1}\frac{1}{x}=\operatorname{cosec}^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$

(3) $\tan^{-1}x=\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)=\cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$
$=\cot^{-1}\left(\frac{1}{x}\right)=\sec^{-1}\sqrt{1+x^2}=\operatorname{cosec}^{-1}\left(\frac{\sqrt{1+x^2}}{x}\right)$
$x\ge0$

Property 13

If $x_1, x_2, \ldots, x_n\in R$, then
$\tan^{-1}x_1+\tan^{-1}x_2+\ldots+\tan^{-1}x_n=\tan^{-1}\left(\frac{S_1-S_3+S_5-S_7+\ldots}{1-S_2+S_4-S_6+\ldots}\right)$

where S_k = Sum of the products of $x_1, x_2, \ldots, x_n$ taken k at a time.

IMPORTANT RESULTS

(i) $\tan^{-1}1+\tan^{-1}2+\tan^{-1}3=\pi$
(ii) $\tan^{-1}1+\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{3}\right)=\frac{\pi}{2}$
(iii) $\frac{\tan^{-1}1+\tan^{-1}2+\tan^{-1}3}{\cot^{-1}1+\cot^{-1}2+\cot^{-1}3}=2$
(iv) $\sin^{-1}x+\sin^{-1}y+\sin^{-1}Z=\frac{3\pi}{2}$, $x=y=z=1$
(v) $\cos^{-1}x+\cos^{-1}y+\cos^{-1}Z=3\pi$, $x=y=z=-1$

SIMPLIFIED INVERSE TRIGONOMETRIC FUNCTION

1. $\sin^{-1}\frac{2x}{1+x^2}=\begin{cases}2\tan^{-1}x, & \text{if } -1\le x\le1\\ \pi-2\tan^{-1}x, & \text{if } x\ge1\\ -\pi-2\tan^{-1}x, & \text{if } x\le-1\end{cases}$

2. $\cos^{-1}\frac{1-x^2}{1+x^2}=\begin{cases}2\tan^{-1}x, & \text{if } x\ge0\\ -2\tan^{-1}x, & \text{if } x<0\end{cases}$

3. $\tan^{-1}\frac{2x}{1-x^2}=\begin{cases}\pi+2\tan^{-1}x, & \text{if } x<-1\\ 2\tan^{-1}x, & \text{if } -1<x<1\\ 2\tan^{-1}x-\pi, & \text{if } x>1\end{cases}$