Chapter 15

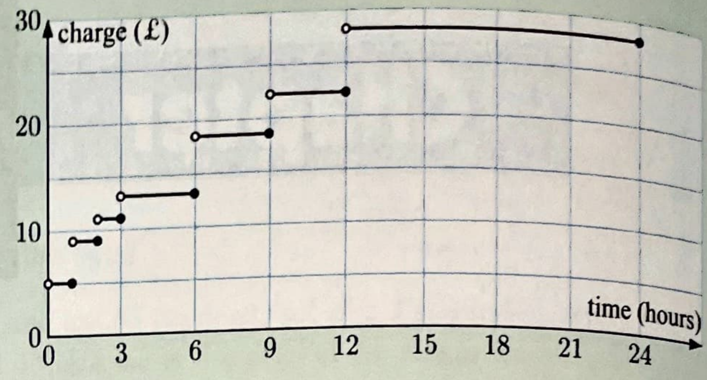
Functions

Contents:

OPENING PROBLEM

The charges for parking a car in a short-term car park at an airport are shown in the table and graph below. The total charge is *dependent* on the length of time t the car is parked.

Car park charges	
Time t (hours)	*Charge*
$0 < t \leqslant 1$	£5.00
$1 < t \leqslant 2$	£9.00
$2 < t \leqslant 3$	£11.00
$3 < t \leqslant 6$	£13.00
$6 < t \leqslant 9$	£18.00
$9 < t \leqslant 12$	£22.00
$12 < t \leqslant 24$	£28.00

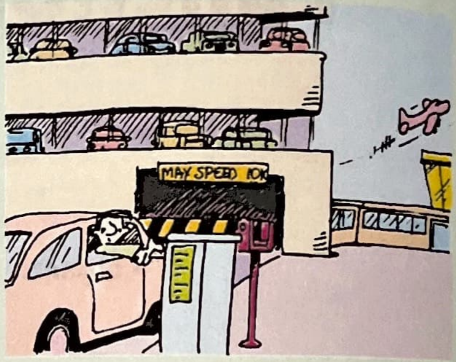


Things to think about:

a What values of *time* are illustrated in the graph?

b What are the possible charges?

c What feature of the graph ensures that there is only one charge for any given time?



In the course so far, we have studied several different relationships between variables. In particular, for two variables x and y:

- A **linear function** is a relationship which can be expressed in the form $y = ax + b$ where a, b are constants, $a \neq 0$.
- A **quadratic function** is a relationship which can be expressed in the form $y = ax^2 + bx + c$ where a, b, c are constants, $a \neq 0$.

In the **Opening Problem** we see another type of relationship, between the two variables *time* and *charge*. We call this a **piecewise function** because its graph has several sections.

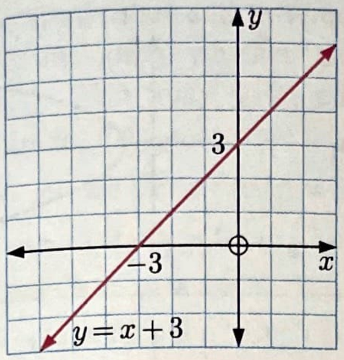
In this Chapter we explore what it really means for the relationship between two variables to be called a **function**. We will then explore properties of functions which will help us work with and understand them.

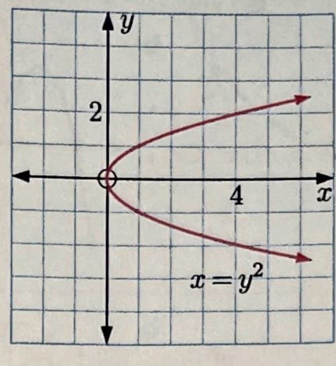
A RELATIONS AND FUNCTIONS

A **relation** between variables x and y is any set of points in the (x, y) plane. We say that the points *connect* the two variables.

A relation is often expressed in the form of an **equation** connecting the **variables** x and y.

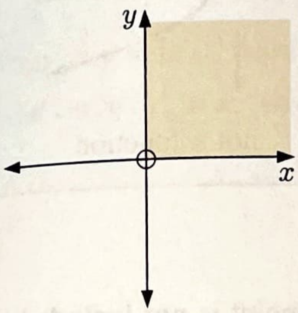
For example, $y = x + 3$ and $x = y^2$ are the equations of two relations. Each equation generates a set of ordered pairs, which we can graph:





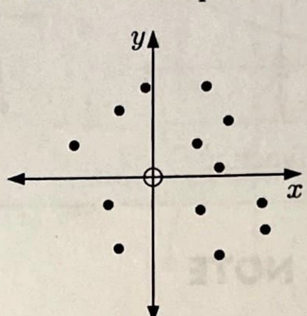
However, not all relations can be defined by an equation. Below are two examples:

(1)



The set of all points in the first quadrant is the relation $x > 0,\ y > 0$.

(2)



These 13 points form a relation. It can be described as a finite set of points, but not by an equation.

FUNCTIONS

A **function** is a relation in which no two different ordered pairs have the same x-coordinate or first component.

We can see from this definition that a function is a special type of relation.

Every function is a relation, but not every relation is a function.

ALGEBRAIC TEST FOR FUNCTIONS

Suppose a relation is given as an equation. If the substitution of any value for x results in at most one value of y, then the relation is a function.

For example:

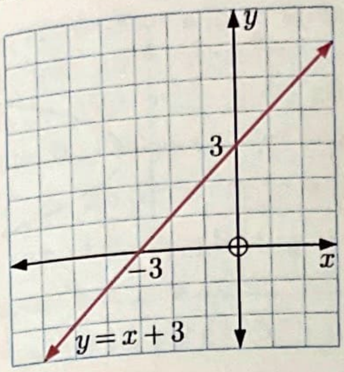
- $y = 3x - 1$ is a function, since for any value of x there is only one corresponding value of y
- $x = y^2$ is not a function, since if $x = 4$ then $y = \pm 2$.

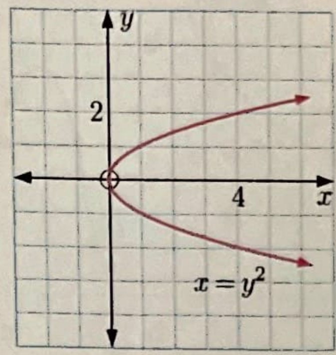
GEOMETRIC TEST OR VERTICAL LINE TEST FOR FUNCTIONS

Suppose we draw all possible vertical lines on the graph of a relation.

- If each line cuts the graph at most once, then the relation is a function.
- If at least one line cuts the graph more than once, then the relation is not a function.

For example, $y = x + 3$ and $x = y^2$ are the equations of two relations. Each equation generates a set of ordered pairs, which we can graph:





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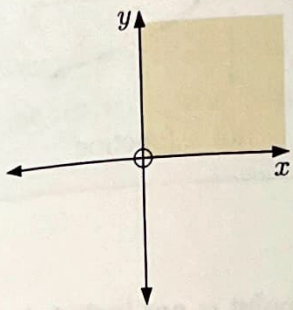
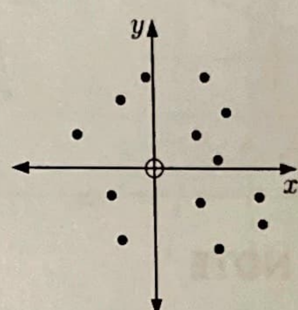


The set of all points in the first quadrant is the relation $x > 0,\ y > 0$.

(2)



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- If each line cuts the graph at most once, then the relation is a function.
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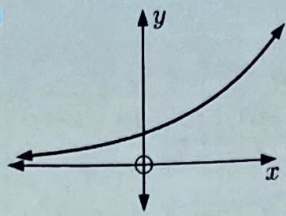
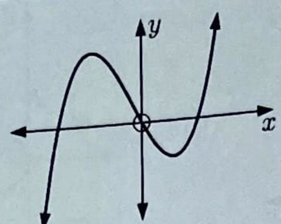
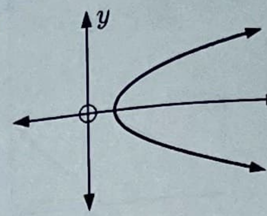
Self Tutor

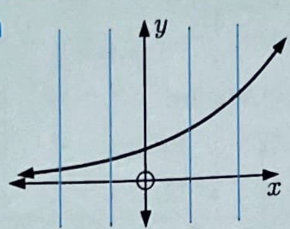
Example 1

Which of the following relations are functions?

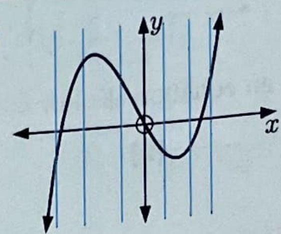
a


b


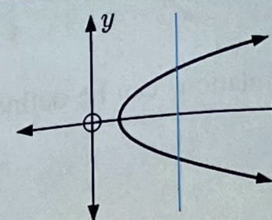
c


a


a function

b


a function

c


not a function

DEMO

GRAPHICAL NOTE

- If a graph contains a small **open circle** such as —o—, this point is **not included.**
- If a graph contains a small **filled-in circle** such as ——•, this point is **included.**
- If a graph contains an **arrowhead** at an end such as ——▸, then the graph continues indefinitely in that general direction, or the shape may repeat as it has done previously.

EXERCISE 15A

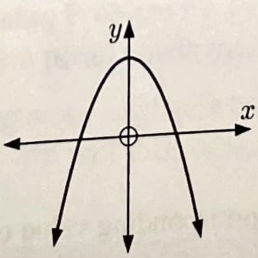
1 Which of the following sets of ordered pairs are functions? Explain your answers.

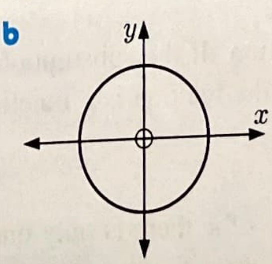
a $\{(1, 3), (2, 3), (3, 1), (4, 2)\}$ **b** $\{(2, 1), (3, 1), (-1, 2), (2, 0)\}$

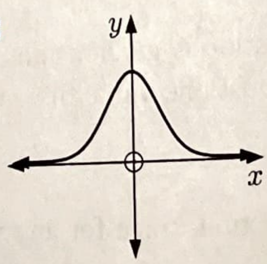
2 Use algebraic methods to decide whether these relations are functions. Explain your answers.

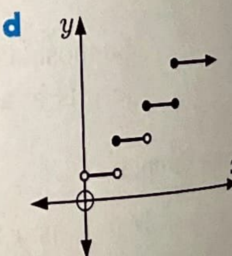
a $y = x^2 - 9$ **b** $x + y = 9$ **c** $x^2 + y^2 = 9$

3 Use the vertical line test to determine which of the following relations are functions:

a


b


c


d


4 The managers of a new amusement park are discussing the schedule of ticket prices. Maurice suggests the table alongside. Explain why this relation between *age* and *cost* is not a function, and discuss the problems that this will cause.

Age	*Cost*
0 - 2 years (infants)	\$0
2 - 16 years (children)	\$20
16+ years (adults)	\$30

5 Is it possible for a function to have more than one y-intercept? Explain your answer.

6 Is the graph of a straight line always a function? Give evidence to support your answer.

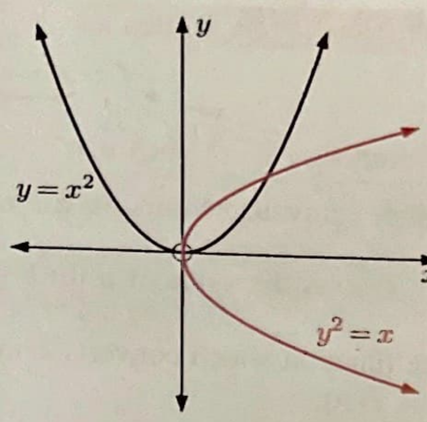
7 The graph alongside shows the curves $y = x^2$ and $y^2 = x$.

a Discuss the similarities and differences between the curves, including whether each curve is a function. You may also consider what transformation(s) map one curve onto the other.

b Using $y^2 = x$, we can write $y = \pm\sqrt{x}$.

i What part of the graph of $y^2 = x$ corresponds to $y = \sqrt{x}$?

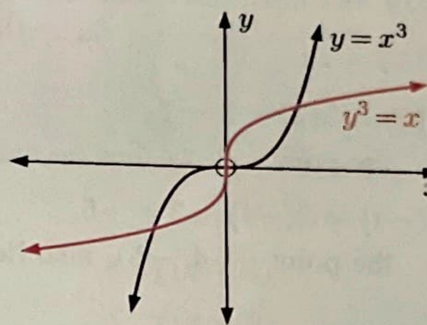
ii Is $y = \sqrt{x}$ a function? Explain your answer.



8 The graph alongside shows the curves $y = x^3$ and $y^3 = x$.

a Explain why both of these curves are functions.

b For the curve $y^3 = x$, write y as a function of x.



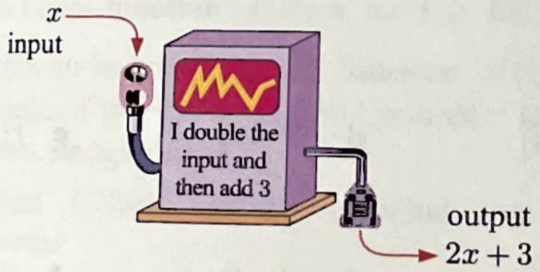
DISCUSSION

In the **Opening Problem**:

- Is the relation describing the car park charges a function?
- If we know the *time* somebody parked for, can we determine the exact *charge* they need to pay?
- If we know the *charge* somebody pays, can we determine the exact *time* they have parked for?

B FUNCTION NOTATION

Function **machines** are sometimes used to illustrate how functions behave.



If 4 is the input fed into the machine, the output is $2(4) + 3 = 11$.

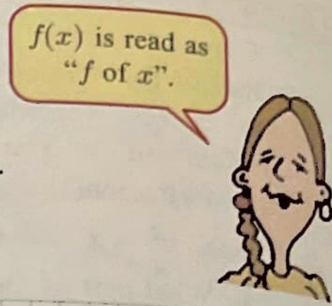
The above "machine" has been programmed to perform a particular function. If we use f to represent that particular function, we can write "f is the function that will convert x into $2x + 3$."

So, f would convert 2 into $2(2) + 3 = 7$ and
-4 into $2(-4) + 3 = -5$.

This function can be written as:

$$f : x \mapsto 2x + 3$$

function f such that x is converted into $2x + 3$



Two other equivalent forms we use are $f(x) = 2x + 3$ and $y = 2x + 3$.

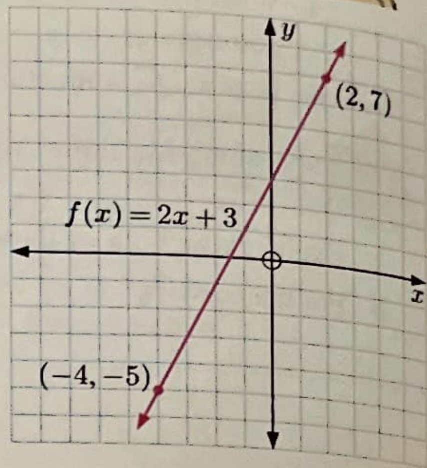
$f(x)$ is the value of y for a given value of x, so $y = f(x)$.

f is the function which converts x into $f(x)$, so we write $f : x \mapsto f(x)$.

$y = f(x)$ is sometimes called the **function value** or **image** of x.

For $f(x) = 2x + 3$:

- $f(2) = 2(2) + 3 = 7$
  $\therefore$ the point $(2, 7)$ lies on the graph of the function.
- $f(-4) = 2(-4) + 3 = -5$
  $\therefore$ the point $(-4, -5)$ also lies on the graph.



Example 2 **Self Tutor**

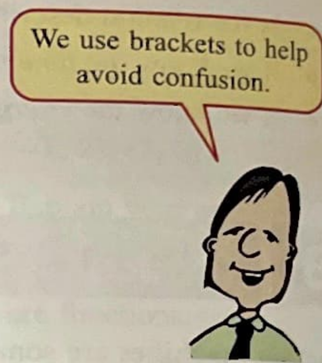
If $f : x \mapsto 2x^2 - 3x$, find the value of:

a $f(5)$ **b** $f(-4)$

$f(x) = 2x^2 - 3x$

a $f(5) = 2(5)^2 - 3(5)$ {replacing x with (5)}
$= 2 \times 25 - 15$
$= 35$

b $f(-4) = 2(-4)^2 - 3(-4)$ {replacing x with (−4)}
$= 2(16) + 12$
$= 44$



EXERCISE 15B

1 If $f(x) = 3x - x^2 + 2$, find the value of:

a $f(0)$ **b** $f(3)$ **c** $f(-3)$ **d** $f(-7)$ **e** $f(\frac{3}{2})$

2 If $g : x \mapsto x - \dfrac{4}{x}$, find the value of:

a $g(1)$ **b** $g(4)$ **c** $g(-1)$ **d** $g(-4)$ **e** $g(-\frac{1}{2})$

3 Suppose $G(x) = \dfrac{2x + 3}{x - 4}$.

a Evaluate: **i** $G(2)$ **ii** $G(0)$ **iii** $G(-\frac{1}{2})$

b Find a value of x such that $G(x)$ does not exist.

c Find x such that $G(x) = -3$.

Example 3 Self Tutor

If $f(x) = 5 - x - x^2$, find in simplest form:

a $f(-x)$ **b** $f(x+2)$ **c** $f(x-1) - 5$

a $f(-x) = 5 - (-x) - (-x)^2$ {replacing x with $(-x)$}
$= 5 + x - x^2$

b $f(x+2) = 5 - (x+2) - (x+2)^2$ {replacing x with $(x+2)$}
$= 5 - x - 2 - [x^2 + 4x + 4]$
$= 3 - x - x^2 - 4x - 4$
$= -x^2 - 5x - 1$

c $f(x-1) - 5 = (5 - (x-1) - (x-1)^2) - 5$ {replacing x with $(x-1)$}
$= 5 - x + 1 - (x^2 - 2x + 1) - 5$
$= -x^2 + x$

4 If $f(x) = 7 - 3x$, find in simplest form:

a $f(a)$ **b** $f(-a)$ **c** $f(a+3)$
d $f(2a)$ **e** $f(x+2)$ **f** $f(x+h)$

5 If $F(x) = 2x^2 + 3x - 1$, find in simplest form:

a $F(x+4)$ **b** $F(2-x)$ **c** $F(-x)$
d $F(x^2)$ **e** $F(3x)$ **f** $F(x+h)$

6 If $f(x) = x^2$, find in simplest form:

a $f(3x)$ **b** $f\left(\frac{x}{2}\right)$ **c** $3\,f(x)$ **d** $2\,f(x-1) + 5$

7 If $f(x) = \frac{1}{x}$, find in simplest form:

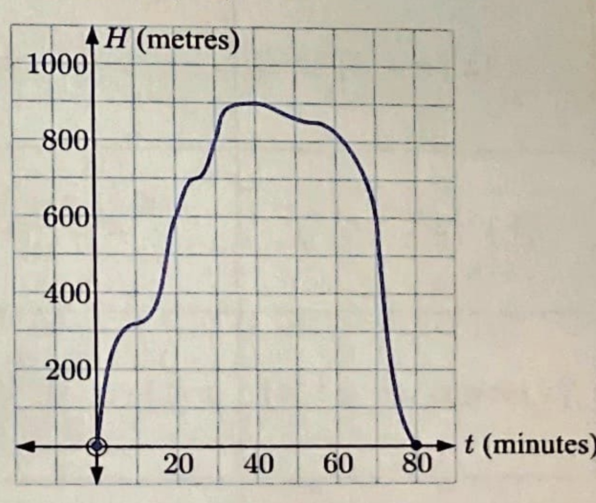
a $f(-x)$ **b** $f(\frac{1}{2}x)$ **c** $2\,f(x) + 3$ **d** $3\,f(x-1) + 2$

8 f represents a function. Explain the difference in meaning between f and $f(x)$.

9 On the same set of axes, draw the graphs of three different functions $f(x)$ such that $f(2) = 1$ and $f(5) = 3$.

10 Find a linear function $f(x) = ax + b$ for which $f(2) = 1$ and $f(-3) = 11$.

11 For a hot air balloon ride, the function $H(t)$ gives the height of the balloon after t minutes. Its graph is shown alongside.



a Find $H(30)$, and explain what your answer means.

b Find the values of t such that $H(t) = 600$. Interpret your answer.

c For what values of t was the height of the balloon recorded?

d What range of heights was recorded for the balloon?

12 Given $f(x) = ax + \frac{b}{x}$, $f(1) = 1$, and $f(2) = 5$, find constants a and b.

13 The quadratic function $T(x) = ax^2 + bx + c$ has the values $T(0) = -4$, $T(1) = -2$, and $T(2) = 6$. Find a, b, and c.

14 The value of a photocopier t years after purchase is given by $V(t) = 9000 - 900t$ pounds.

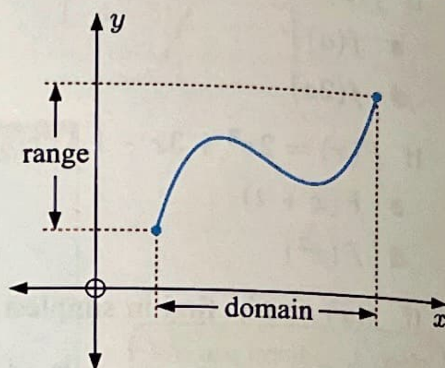
- **a** Find $V(4)$, and state what $V(4)$ means.
- **b** Find t when $V(t) = 3600$, and explain what this means.
- **c** Find the original purchase price of the photocopier.
- **d** For what values of t is it reasonable to use this function?

C DOMAIN AND RANGE

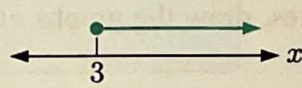
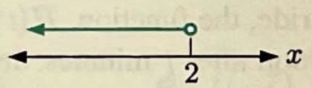
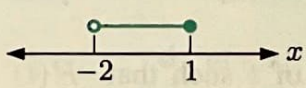
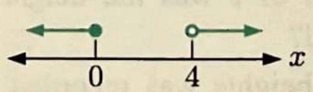
We have seen that a relation is a set of points which connects two variables.

The **domain** of a relation is the set of values which the variable on the horizontal axis can take. This variable is usually x.

The **range** of a relation is the set of values which the variable on the vertical axis can take. This variable is usually y.



The domain and range of a relation can be described in several ways. Examples are given in the table below:

Set notation	*Bracket notation*	*Number line graph*	*Meaning*
$\{x \mid x \geqslant 3\}$	$x \in [3, \infty[$		the set of all x such that x is greater than or equal to 3
$\{x \mid x < 2\}$	$x \in]-\infty, 2[$		the set of all x such that x is less than 2
$\{x \mid -2 < x \leqslant 1\}$	$x \in]-2, 1]$		the set of all x such that x is between -2 and 1, including 1
$\{x \mid x \leqslant 0$ or $x > 4\}$	$x \in]-\infty, 0]$ or $]4, \infty[$		the set of all x such that x is less than or equal to 0, or greater than 4

DOMAIN AND RANGE OF FUNCTIONS

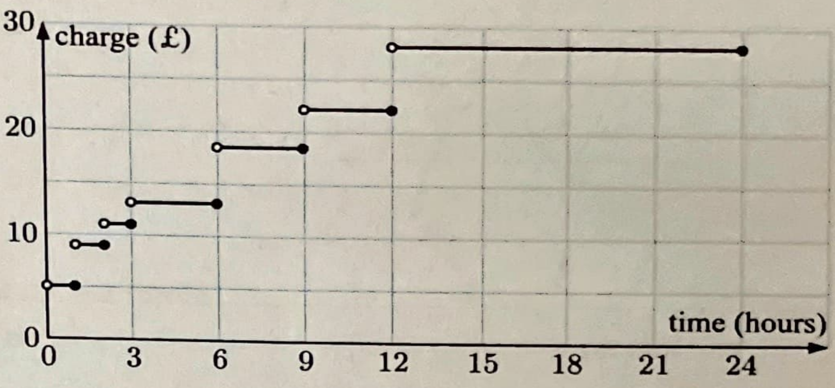
To find the domain and range of a function, we can observe its graph. For example:

(1) In the **Opening Problem**, the car park charges function is defined for all times t such that $0 < t \leqslant 24$.

$\therefore$ the domain is $\{t \mid 0 < t \leqslant 24\}$.

The possible charges are £5, £9, £11, £13, £18, £22, and £28.

$\therefore$ the range is $\{5, 9, 11, 13, 18, 22, 28\}$.



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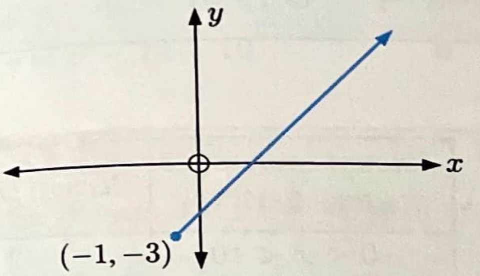
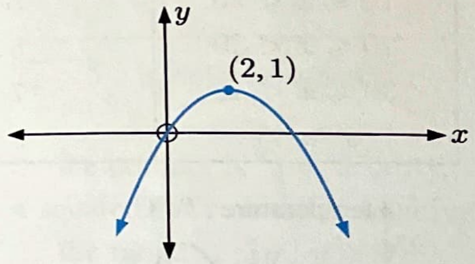


All values of $x \geqslant -1$ are included, so the domain is $\{x \mid x \geqslant -1\}$.

All values of $y \geqslant -3$ are included, so the range is $\{y \mid y \geqslant -3\}$.

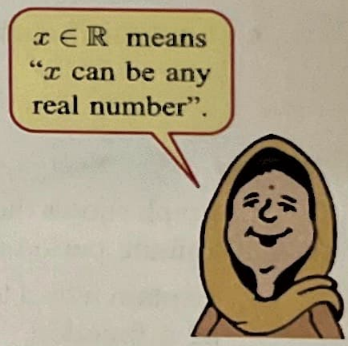
(3)



x can take any value, so the domain is

$\{x \in \mathbb{R}\}$ or $x \in \mathbb{R}$.

y cannot be > 1, so the range is $\{y \mid y \leqslant 1\}$.



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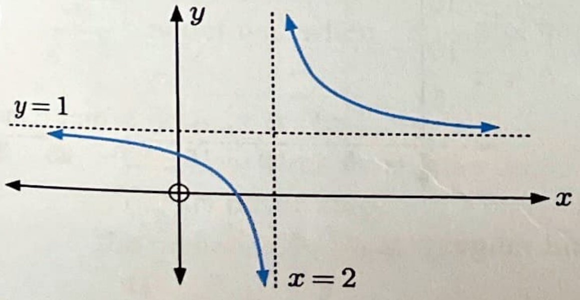


x can take all values except 2, so the domain is $\{x \mid x \neq 2\}$ or $x \neq 2$.

y can take all values except 1, so the range is $\{y \mid y \neq 1\}$ or $y \neq 1$.

To fully describe a function, we need both a rule *and* a domain.

For example, we can specify $f(x) = x^2$ where $x \geqslant 0$.

If a domain is not specified, we use the **natural domain**, which is the largest part of $\mathbb{R}$ for which $f(x)$ is defined.

Some examples of natural domains are shown in the table opposite.

$f(x)$	*Natural domain*
x^2	$x \in \mathbb{R}$
$\sqrt{x}$	$x \geqslant 0$
$\frac{1}{x}$	$x \neq 0$
$\frac{1}{\sqrt{x}}$	$x > 0$

Click on the icon to obtain software for finding the natural domain and range of different functions.

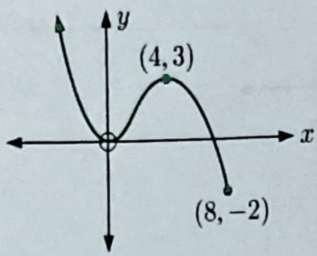
DOMAIN AND RANGE

Example 4

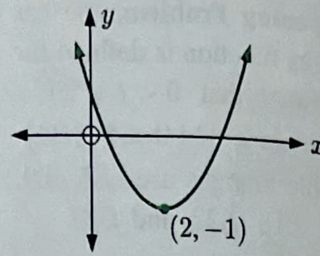
Self Tutor

For each of the following graphs, state the domain and range:

a



b



a Domain is $\{x \mid x \leqslant 8\}$.
Range is $\{y \mid y \geqslant -2\}$.

b Domain is $\{x \in \mathbb{R}\}$.
Range is $\{y \mid y \geqslant -1\}$.

EXERCISE 15C

1 A driver who exceeds the speed limit receives demerit points as shown in the table.

a Draw a graph to display this information.

b Is the relation a function? Explain your answer.

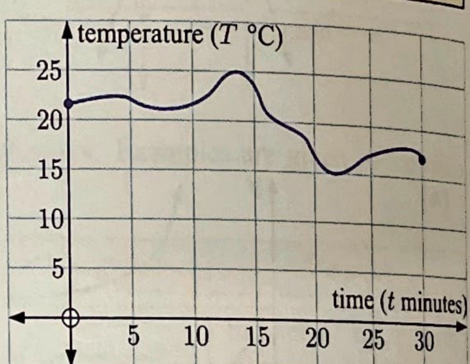
c Find the domain and range of the relation.

Amount over speed limit (x km h^{-1})	*Demerit points* (y)
$0 < x < 10$	2
$10 \leqslant x < 20$	3
$20 \leqslant x < 30$	5
$30 \leqslant x < 45$	7
$x \geqslant 45$	9

2 This graph shows the temperature in Barcelona over a 30 minute period as the wind shifts.

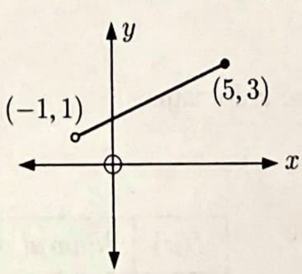
a Explain why a temperature graph like this must be a function.

b Find the domain and range of the function.

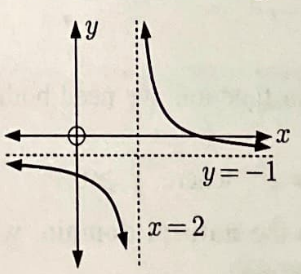


3 For each of the following graphs, find the domain and range:

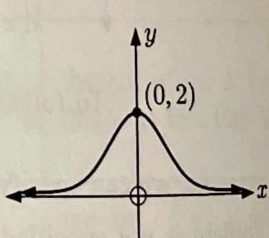
a



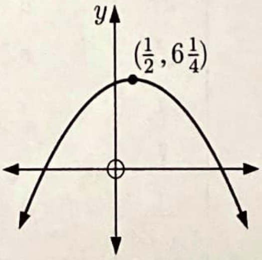
b



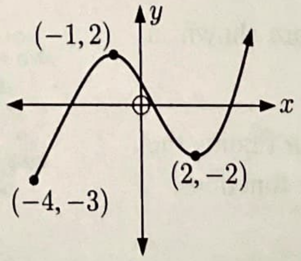
c



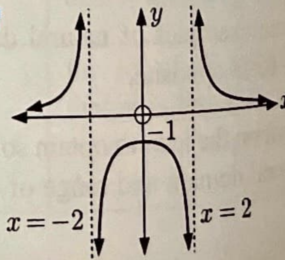
d



e



f



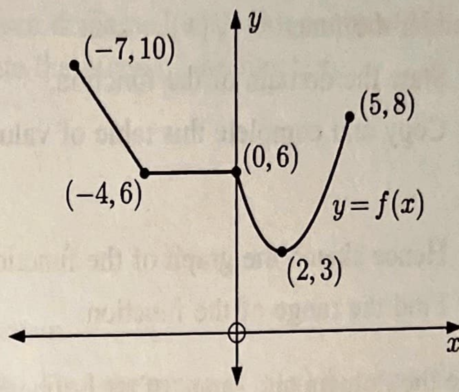
4 Consider the graph of $y = f(x)$ alongside. Decide whether each statement is true or false:

a -5 is in the domain of f.

b 2 is in the range of f.

c 9 is in the range of f.

d $\sqrt{2}$ is in the domain of f.



5 Use quadratic theory to find the range of each function:

a $y = x^2$

b $y = -x^2$

c $y = x^2 + 2$

d $y = -2(x+3)^2$

e $y = 1 - (x-2)^2$

f $y = (2x+1)^2 + 3$

g $y = x^2 - 7x + 10$

h $y = -x^2 + 2x + 8$

i $f : x \mapsto 5x - 3x^2$

Example 5 Self Tutor

State the domain and range of each of the following functions:

a $f(x) = \sqrt{x-5}$

b $f(x) = \dfrac{1}{x-5}$

c $f(x) = \dfrac{1}{\sqrt{x-5}}$

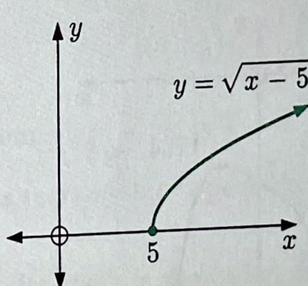
a $\sqrt{x-5}$ is defined when $x - 5 \geqslant 0$

$\therefore\ x \geqslant 5$

$\therefore$ the domain is $\{x \mid x \geqslant 5\}$.

A square root cannot be negative.

$\therefore$ the range is $\{y \mid y \geqslant 0\}$.



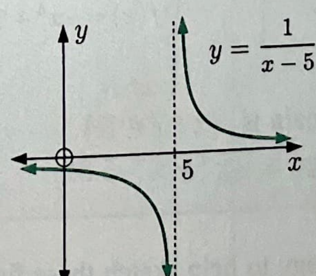
b $\dfrac{1}{x-5}$ is defined when $x - 5 \neq 0$

$\therefore\ x \neq 5$

$\therefore$ the domain is $\{x \mid x \neq 5\}$.

No matter how large or small x is, $y = f(x)$ is never zero.

$\therefore$ the range is $\{y \mid y \neq 0\}$.



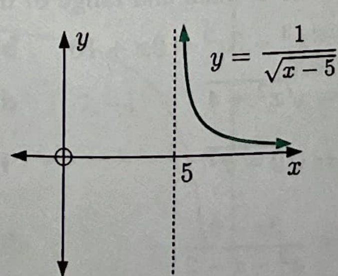
c $\dfrac{1}{\sqrt{x-5}}$ is defined when $x - 5 > 0$

$\therefore\ x > 5$

$\therefore$ the domain is $\{x \mid x > 5\}$.

$y = f(x)$ is always positive and never zero.

$\therefore$ the range is $\{y \mid y > 0\}$.



6 Consider the function $f(x) = \sqrt{x}$.

a State the domain of the function.

b Copy and complete this table of values:

x	0	1	4	9	16
$f(x)$					

DOMAIN AND RANGE

c Hence sketch the graph of the function.

d Find the range of the function.

7 State the domain and range of each function:

a $f(x) = \sqrt{x+6}$ **b** $f : x \mapsto \dfrac{1}{x^2}$ **c** $f(x) = \dfrac{1}{x+1}$

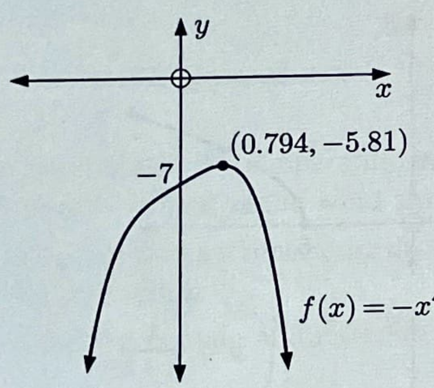
d $y = -\dfrac{1}{\sqrt{x}}$ **e** $f : x \mapsto \dfrac{1}{3-x}$ **f** $f : x \mapsto \sqrt{4-x}$

Example 6 Self Tutor

Use technology to help sketch these functions. Locate any turning points. Hence state the domain and range of the function.

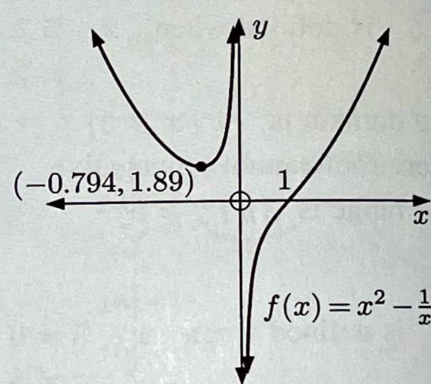
a $f(x) = -x^4 + 2x - 7$ **b** $f(x) = x^2 - \dfrac{1}{x}$

a



The domain is $\{x \mid x \in \mathbb{R}\}$.
The range is $\{y \mid y \leqslant -5.81\}$.

b



The domain is $\{x \mid x \neq 0\}$.
The range is $\{y \mid y \in \mathbb{R}\}$.

8 Use technology to help sketch these functions. Locate any turning points. Hence state the domain and range of the function.

GRAPHING PACKAGE

a $f(x) = x^3 - 3x^2 - 9x + 10$ **b** $f(x) = x^4 + 4x^3 - 16x + 3$

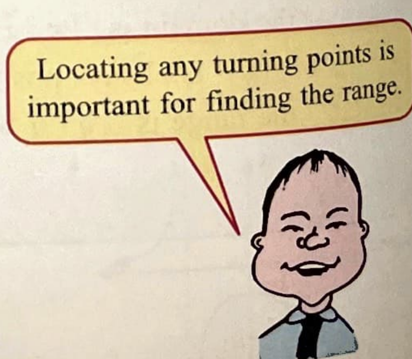
c $f(x) = \sqrt{x^2 + 4}$ **d** $f(x) = \sqrt{x^2 - 4}$

e $f(x) = \sqrt{9 - x^2}$ **f** $f(x) = \dfrac{x+4}{x-2}$

g $f(x) = \dfrac{3x - 9}{x^2 - x - 2}$ **h** $f(x) = x + \dfrac{1}{x}$

i $f(x) = x^2 + \dfrac{1}{x^2}$ **j** $f(x) = x^3 + \dfrac{1}{x^3}$

k $f(x) = 3^x$ **l** $f(x) = x\,2^{-x}$



9 Use technology to sketch these functions on their given domain. Locate the points at the end(s) of the domain, as well as any turning points. Hence state the range of the function.

a $y = -x^4 + 2x^3 + 5x^2 + x + 2, \quad 0 \leqslant x \leqslant 4$

b $y = -2x^4 + 5x^2 + x + 2, \quad -2 \leqslant x \leqslant 2$

c $y = \dfrac{1}{1+2^{-x}}, \quad x > 0$

10 The function $f(x) = \sqrt{x^2 + 5x + k}$ has natural domain $x \in \mathbb{R}$.

a Find the possible values of k.

b Find the range of $f(x)$ in terms of k.

11 State the domain and range of each relation:

a $\{(x, y) \mid x^2 + y^2 = 4\}$

b $\{(x, y) \mid x^2 + y^2 = 4, \; x \in \mathbb{Z}\}$

D RATIONAL FUNCTIONS

Linear and quadratic functions are the first members of a family called the **polynomials.** The polynomials can all be written in the form $y = a_0 + a_1x + a_2x^2 + a_3x^3 + ...$

When a polynomial is divided by another polynomial, we call it a **rational function.**

In this Section we consider only the simplest cases of a linear function divided by another linear function.

RECIPROCAL FUNCTIONS

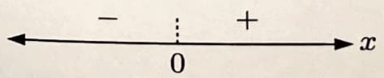
A **reciprocal function** is a function of the form $y = \dfrac{k}{x}, \; k \neq 0.$

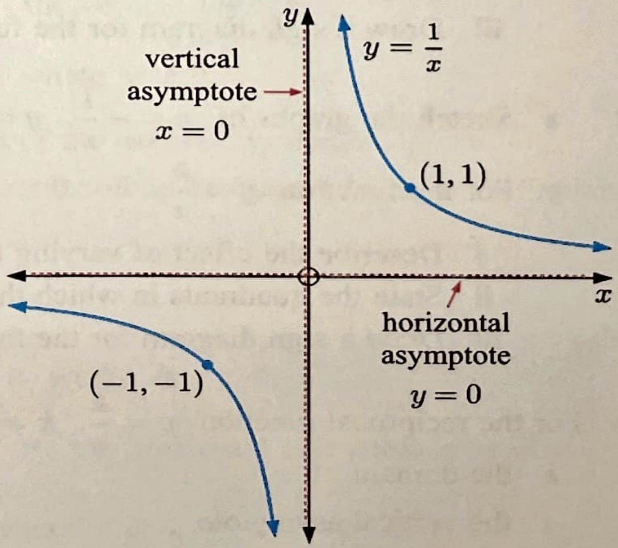
The graph of a reciprocal function is called a **rectangular hyperbola.**

The simplest example of a reciprocal function is $f(x) = \dfrac{1}{x}$. Its graph is shown below.

Notice that:

- The graph has two branches.
- $y = \dfrac{1}{x}$ is undefined when $x = 0$, so the domain is $\{x \mid x \neq 0\}$.
  On a sign diagram, we indicate this value with a dashed line.



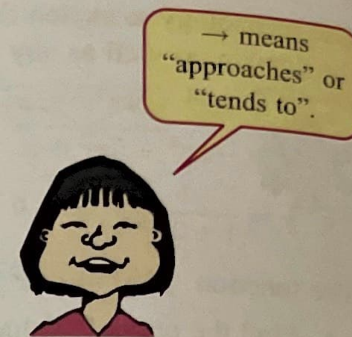


- The graph includes two **asymptotes**, which are lines the graph approaches but never reaches.
  - $x = 0$ is a **vertical asymptote**.

    We write: as $x \to 0^-$, $\dfrac{1}{x} \to -\infty$

    as $x \to 0^+$, $\dfrac{1}{x} \to \infty$

    When "as $x \to 0^+$, $\dfrac{1}{x} \to \infty$" is read out loud, we say "as x tends to zero from the right, $\dfrac{1}{x}$ tends to infinity."



- $y = 0$ is a **horizontal asymptote**.

We write: as $x \to \infty$, $\frac{1}{x} \to 0^+$

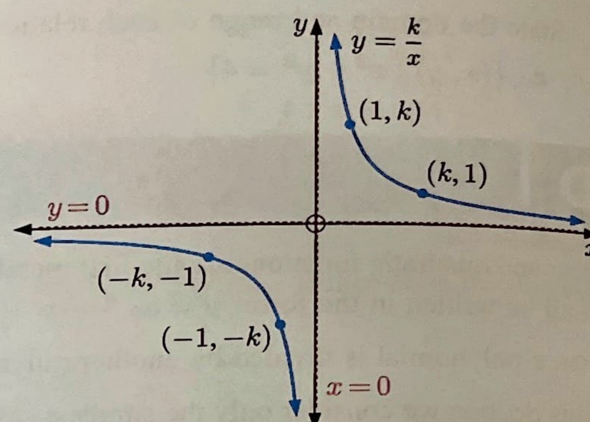
as $x \to -\infty$, $\frac{1}{x} \to 0^-$

When "as $x \to \infty$, $\frac{1}{x} \to 0^+$" is read out loud,

we say "as x tends to infinity, $\frac{1}{x}$ tends to zero from above."

When sketching the graph of a reciprocal function, it is useful to determine some points which lie on the graph.

The reciprocal function $y = \frac{k}{x}$ passes through the points $(1, k)$, $(k, 1)$, $(-1, -k)$, and $(-k, -1)$.



EXERCISE 15D.1

DEMO

1 **a** Sketch the graphs of $y = \frac{1}{x}$, $y = \frac{2}{x}$, and $y = \frac{4}{x}$ on the same set of axes.

b For the function $y = \frac{k}{x}$, $k > 0$:

- **i** Describe the effect of varying k.
- **ii** State the quadrants in which the graph lies.
- **iii** Draw a sign diagram for the function.

2 **a** Sketch the graphs of $y = -\frac{1}{x}$, $y = -\frac{2}{x}$, and $y = -\frac{4}{x}$ on the same set of axes.

b For the function $y = \frac{k}{x}$, $k < 0$:

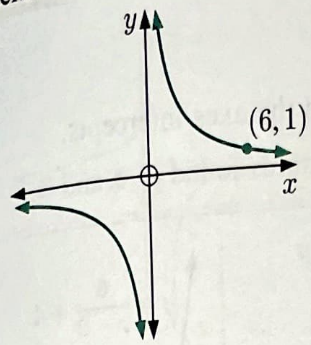
- **i** Describe the effect of varying k.
- **ii** State the quadrants in which the graph lies.
- **iii** Draw a sign diagram for the function.

3 For the reciprocal function $y = \frac{k}{x}$, $k \neq 0$, state:

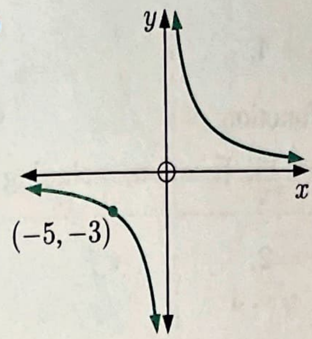
a the domain
b the range
c the vertical asymptote
d the horizontal asymptote.

4 Determine the equation of each reciprocal function:

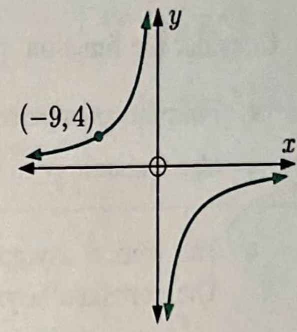
a



b



c



RATIONAL FUNCTIONS OF THE FORM $y = \dfrac{ax+b}{cx+d}$, $c \neq 0$

We now consider the rational functions which result when a linear function is divided by another linear function.

The graphs of these rational functions also have horizontal and vertical asymptotes.

INVESTIGATION — RATIONAL FUNCTIONS

What to do:

GRAPHING PACKAGE

1 Use technology to examine graphs of the following functions. For each graph:

i State the domain. **ii** Write down the equations of the asymptotes.

a $y = -1 + \dfrac{3}{x-2}$ **b** $y = \dfrac{3x+1}{x+2}$ **c** $y = \dfrac{2x-9}{3-x}$

2 Experiment with functions of the form $y = \dfrac{b}{cx+d} + a$ where $b, c \neq 0$.

For an equation of this form, state the equation of:

a the horizontal asymptote **b** the vertical asymptote.

3 Experiment with functions of the form $y = \dfrac{ax+b}{cx+d}$ where $c \neq 0$.

a For an equation of this form, state the equation of the vertical asymptote.

b Can you see how to quickly write down the equation of the horizontal asymptote? Explain your answer.

- For a function written in the form $y = \dfrac{b}{cx+d} + a$ where $b, c \neq 0$:
  - the vertical asymptote is $x = -\dfrac{d}{c}$
  - the horizontal asymptote is $y = a$.
- For a function written in the form $y = \dfrac{ax+b}{cx+d}$ where $c \neq 0$:
  - the vertical asymptote is $x = -\dfrac{d}{c}$
  - the horizontal asymptote is $y = \dfrac{a}{c}$.

Example 7 Self Tutor

Consider the function $y = \dfrac{6}{x-2} + 4$.

a Find the asymptotes of the function.

b Find the axes intercepts.

c Use technology to help sketch the function, including the features found in **a** and **b**.

a The vertical asymptote is $x = 2$.
The horizontal asymptote is $y = 4$.

b When $y = 0$, $\dfrac{6}{x-2} = -4$

$\therefore\ -4(x-2) = 6$

$\therefore\ -4x + 8 = 6$

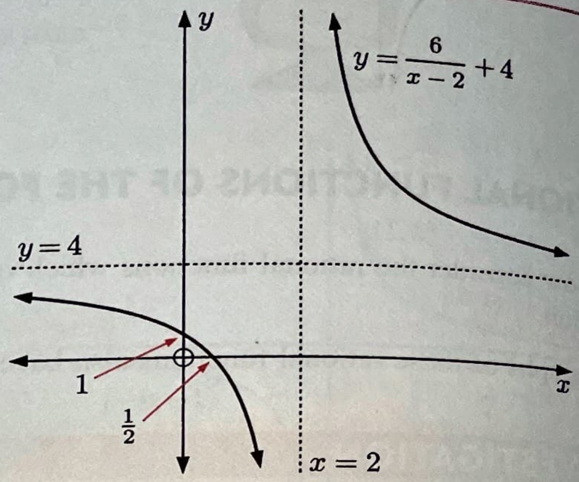
$\therefore\ -4x = -2$

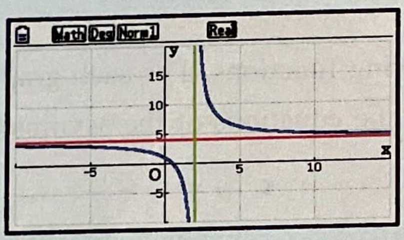
$\therefore\ x = \frac{1}{2}$

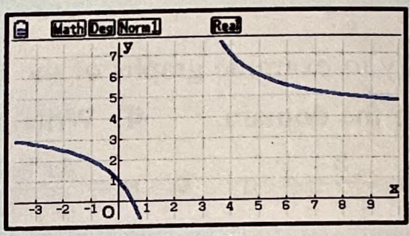
When $x = 0$, $y = \frac{6}{-2} + 4 = 1$

So, the x-intercept is $\frac{1}{2}$ and the y-intercept is 1.

c







EXERCISE 15D.2

1 For each of the following functions:

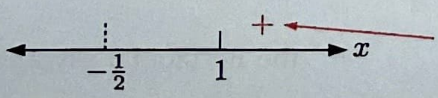
- **i** Find the equations of the asymptotes.
- **ii** State the domain and range.
- **iii** Find the axes intercepts.
- **iv** Discuss the behaviour of the function as it approaches its asymptotes.
- **v** Sketch the graph of the function.

a $f(x) = \dfrac{3}{x-2}$ **b** $f : x \mapsto 2 + \dfrac{1}{x-3}$ **c** $f(x) = 2 - \dfrac{3}{x+1}$

Example 8 Self Tutor

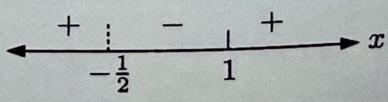
Draw a sign diagram for $\dfrac{x-1}{2x+1}$.

$\dfrac{x-1}{2x+1}$ is zero when $x = 1$ and undefined when $x = -\frac{1}{2}$.



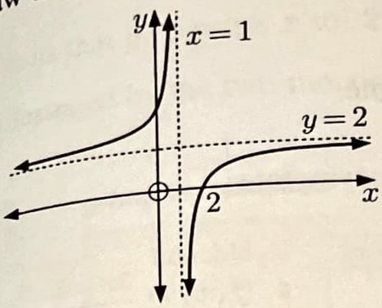
When $x = 10$, $\dfrac{x-1}{2x+1} = \dfrac{9}{21} > 0$

Since $(x-1)$ and $(2x+1)$ are single factors, the signs alternate.

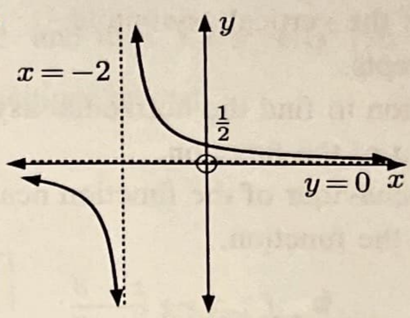


2 Draw the sign diagram for:

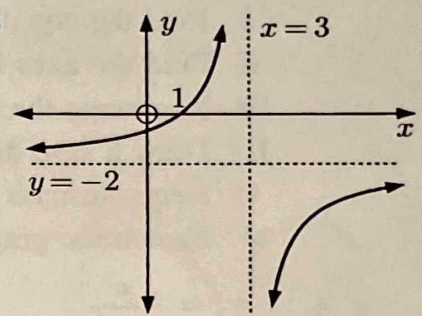
a



b



c



3 Draw a sign diagram for:

a $\dfrac{x+2}{x-1}$ b $\dfrac{x}{x+3}$ c $\dfrac{x+1}{x+5}$ d $\dfrac{x-2}{2x+1}$

e $\dfrac{2x+3}{4-x}$ f $\dfrac{4x-1}{2-x}$ g $\dfrac{3x}{x-2}$ h $\dfrac{-8x}{3-x}$

Example 9 Self Tutor

Consider the function $f(x) = \dfrac{2x+1}{x-1}$.

a Find the vertical asymptote of the function. b Find the axes intercepts.

c Rearrange the function to find the horizontal asymptote.

d Draw a sign diagram of the function.

e Hence discuss the behaviour of the function near the asymptotes.

f Sketch the function, showing the features you have found.

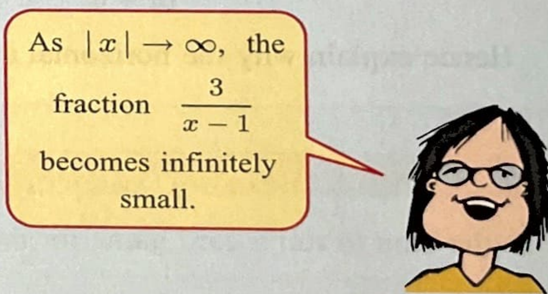
a The vertical asymptote is $x = 1$.

b $f(0) = \dfrac{1}{-1} = -1$, so the y-intercept is -1.

$f(x) = 0$ when $2x + 1 = 0$

$\therefore x = -\frac{1}{2}$

$\therefore$ the x-intercept is $-\frac{1}{2}$.



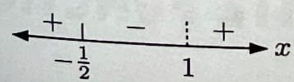
c $f(x) = \dfrac{2x+1}{x-1}$

$= \dfrac{2(x-1)+3}{x-1}$

$= 2 + \dfrac{3}{x-1}$

$\therefore$ the horizontal asymptote is $y = 2$.

d



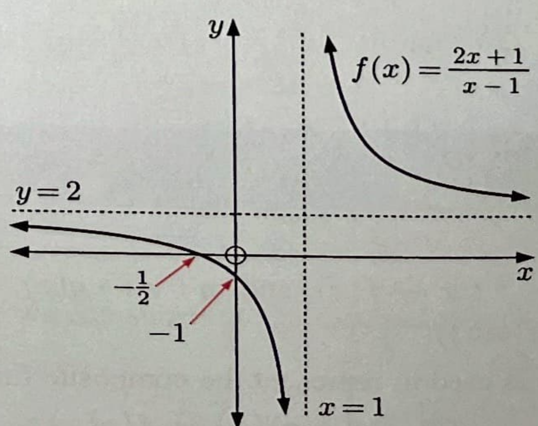
e As $x \to 1^-$, $f(x) \to -\infty$

As $x \to 1^+$, $f(x) \to \infty$

As $x \to -\infty$, $f(x) \to 2^-$

As $x \to \infty$, $f(x) \to 2^+$

f



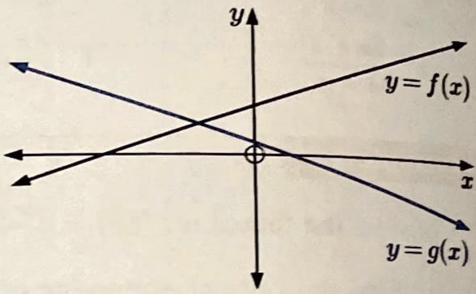
4 For each of the following functions:

i Find the equation of the vertical asymptote.
ii Find the axes intercepts.
iii Rearrange the function to find the horizontal asymptote.
iv Draw a sign diagram of the function.
v Hence discuss the behaviour of the function near its asymptotes.
vi Sketch the graph of the function.

a $f(x)=\frac{x}{x-1}$ **b** $f: x \mapsto \frac{x+3}{x-2}$ **c** $f(x)=\frac{3x-1}{x+2}$

d $f(x)=-\frac{2x+1}{x-3}$ **e** $f: x \mapsto \frac{2x+4}{3-x}$ **f** $f(x)=\frac{x+3}{2x-1}$

5 The graph alongside shows the linear functions $f(x)$ and $g(x)$.
Copy the graph, and on the same set of axes, graph $y=\frac{f(x)}{g(x)}$. Indicate clearly where any x-intercepts and asymptotes occur.



6 Consider the function $y=\frac{ax+b}{cx+d}$, where a, b, c, d are constants and $c \neq 0$.

a State the domain of the function.
b State the equation of the vertical asymptote.
c Find the axes intercepts.
d Show that for $c \neq 0$, $\frac{ax+b}{cx+d}=\frac{a}{c}+\frac{b-\frac{ad}{c}}{cx+d}$.

Hence explain why the horizontal asymptote is $y=\frac{a}{c}$.

ACTIVITY

Click on the icon to run a card game for rational functions.

GAME