Matrices

Prerequisites: Adding, subtracting, multiplying and dividing numbers; elementary row operations.

Maths Applications: Solving systems of equations; describing geometric transformations; deriving addition formulae.

Real-World Applications: Balancing chemical equations; flight stopover information; currents in electrical circuits; formulation of fundamental physical laws.

Basic Definitions

Definition:

A **matrix** is a rectangular array of numbers (aka **entries** or **elements**) in parentheses, each entry being in a particular **row** and **column**.

Definition:

The **order** of a matrix is given as $m \times n$ (read **m by n**), where m is the number of rows and n the number of columns and is written as,

$$A \equiv (a_{ij})_{m \times n} \stackrel{def}{=} \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \\ a_{m1} & a_{m2} & a_{m3} & & a_{mn} \end{pmatrix}$$

The element in row i and column j of a matrix is written as a_{ij} and called the **(i, j) th entry of A**.

In this course, we will deal almost exclusively with matrices that have orders 2×2 and 3×3.

Definition:

The **main diagonal** (aka **leading diagonal**) of any matrix is the set of entries a_{ij} where $i = j$.

Special cases arise when either $m = 1$ or $n = 1$.

Definition:

A **row matrix** is a $1 \times n$ matrix and is written as,

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1(n-1)} & a_{1n} \end{pmatrix}$$

A **column matrix** is a $m \times 1$ matrix and is written as,

$$\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{(m-1)1} \\ a_{m1} \end{pmatrix}$$

The case when $m = n$ is a very important one.

Definition:

A **square matrix (of order m × m)** is a matrix with the same number of rows as columns (equal to m) and is written as,

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \\ a_{m1} & a_{m2} & & a_{mm} \end{pmatrix}$$

Definition:

The **identity matrix (of order m)** is the $m \times m$ matrix all of whose entries are 0 apart from those on the main diagonal, where they all equal 1,

$$I_m \overset{def}{=} \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & & 1 \end{pmatrix}$$

Definition:

The **zero matrix (of order m × n)** is the $m \times n$ matrix all of whose entries are 0,

$$O_{m \times n} \overset{def}{=} \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & & 0 \end{pmatrix}$$

For a square zero matrix, sometimes the notation O_m is used.

Matrix Algebra

Addition, Subtraction and Scalar Multiplication

Definition:

The **matrix sum of *A* and *B*** is obtained by adding corresponding entries of A and B,

$$(a + b)_{ij} \overset{def}{=} a_{ij} + b_{ij}$$

The **matrix difference of *A* and *B*** is obtained by subtracting the entries of B from the corresponding ones in A,

$$(a - b)_{ij} \overset{def}{=} a_{ij} - b_{ij}$$

Note that matrix addition and subtraction only makes sense if A and B both have the same order.

Example 1

Add the matrices $A = \begin{pmatrix} 0 & -3 \\ 5 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 13 & 1 \\ -4 & 5 \end{pmatrix}$.

As the matrices have the same order, they can be added.

$$A + B = \begin{pmatrix} 0 & -3 \\ 5 & 2 \end{pmatrix} + \begin{pmatrix} 13 & 1 \\ -4 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 0 + 13 & -3 + 1 \\ 5 + (-4) & 2 + 5 \end{pmatrix}$$

$$= \begin{pmatrix} 13 & -2 \\ 1 & 7 \end{pmatrix}$$

Example 2

Find the difference $P - Q$ where $P = \begin{pmatrix} 4 & -2 & 0 \\ 1 & 0 & 37 \end{pmatrix}$ and $Q = \begin{pmatrix} -4 & -2 & 2 \\ -8 & 3 & 6 \end{pmatrix}$.

$$P - Q = \begin{pmatrix} 4 & -2 & 0 \\ 1 & 0 & 37 \end{pmatrix} - \begin{pmatrix} -4 & -2 & 2 \\ -8 & 3 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 0 & -2 \\ 9 & -3 & 31 \end{pmatrix}$$

Definition:

The **scalar multiplication of *A* by *k*** ($k \in \mathbb{R}$) is obtained by multiplying each entry of A by k,

$$(ka)_{ij} \overset{def}{=} ka_{ij}$$

Example 3

If $A = \begin{pmatrix} 4 & -4 & 0 \\ 5 & 6 & 1 \\ 8 & 9 & -1 \end{pmatrix}$, calculate $\frac{1}{2}A$.

$$\frac{1}{2}A = \frac{1}{2}\begin{pmatrix} 4 & -4 & 0 \\ 5 & 6 & 1 \\ 8 & 9 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 4/2 & -4/2 & 0/2 \\ 5/2 & 6/2 & 1/2 \\ 8/2 & 9/2 & -1/2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -2 & 0 \\ \frac{5}{2} & 3 & \frac{1}{2} \\ 4 & \frac{9}{2} & -\frac{1}{2} \end{pmatrix}$$

Matrix Multiplication

Definition:

The **matrix product of A and B**, where A is of order $m \times n$ and B is of order $n \times p$ is obtained by the following prescription,

$$(ab)_{ij} \stackrel{def}{=} \sum_{k=1}^{n} a_{ik} b_{kj}$$

$$(1 \le i \le m \text{ and } 1 \le j \le p)$$

Note that the number of columns of A must equal the number of rows of B. To see how to use the horrible prescription, split up A into rows and B into columns; then taking the ' scalar product ' of the i^{th} row of A with the j^{th} column of B gives the $(i, j)^{th}$ entry of AB.

Example 4

Calculate AB when $A = \begin{pmatrix} 1 & -2 & 0 \\ 8 & 0 & 1 \\ -2 & 4 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 0 \\ 1 & 2 \\ -3 & 4 \end{pmatrix}$.

As the number of columns of A equals the number of rows of B, the product AB makes sense.

$$AB = \begin{pmatrix} 1 & -2 & 0 \\ 8 & 0 & 1 \\ -2 & 4 & 6 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 1 & 2 \\ -3 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 1\times3+(-2)\times1+0\times(-3) & 1\times0+(-2)\times2+0\times4 \\ 8\times3+0\times1+1\times(-3) & 8\times0+0\times2+1\times4 \\ (-2)\times3+4\times1+6\times(-3) & (-2)\times0+4\times2+6\times4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -4 \\ 21 & 4 \\ -20 & 32 \end{pmatrix}$$

In general, AB is not the same matrix as BA.

Example 5

Show that $AB \neq BA$ for the matrices $A = \begin{pmatrix} 0 & -3 \\ 5 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$.

First AB,

$$AB = \begin{pmatrix} 0 & -3 \\ 5 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

$$AB = \begin{pmatrix} 0 & -6 \\ 5 & 4 \end{pmatrix}$$

Next BA,

$$BA = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & -3 \\ 5 & 2 \end{pmatrix}$$

$$BA = \begin{pmatrix} 0 & -3 \\ 10 & 4 \end{pmatrix}$$

<u>Definition:</u>

When forming AB, **B is pre-multiplied by A** and **A is post-multiplied by B**.

<u>Definition:</u>

The **transpose of A** (with the order of A being $m \times n$), denoted by A^{T} (sometimes A'), is the $n \times m$ matrix obtained by interchanging the rows and columns of A,

$$(a^{\mathrm{T}})_{ij} \overset{def}{=} a_{ji}$$

<u>Example 6</u>

Find the transpose of $P = \begin{pmatrix} 3 & -7 & 0 \\ 5 & 9 & 46 \end{pmatrix}$.

$$P^{\mathrm{T}} = \begin{pmatrix} 3 & 5 \\ -7 & 9 \\ 0 & 46 \end{pmatrix}$$

<u>Definition:</u>

A (square) matrix A can be multiplied by itself any number of times, giving the **n^{th} power of A**,

$$A^n \overset{def}{=} \underbrace{A \times A \times A \times \ldots \times A}_{n \text{ times}}$$

Basic Properties of Matrices

- $A + B = B + A$
- $(A + B) + C = A + (B + C)$
- $k(A + B) = kA + kB$
- $(A + B)^{T} = A^{T} + B^{T}$
- $(A^{T})^{T} = A$
- $(kA)^{T} = kA^{T}$
- $A(BC) = (AB)C$
- $A(B + C) = AB + AC$
- $(AB)^{T} = B^{T}A^{T}$
- $A^{m}A^{n} = A^{m+n} = A^{n}A^{m}$

There are 3 important properties that are worth singling out separately.

- $A + O = O + A = A$
- $AI = AI = A$
- $AO = AO = O$

Thus, the identity and zero matrices behave like the numbers 1 and 0 respectively in ordinary arithmetic and algebra.

<u>Example 7</u>

For the matrix $A = \begin{pmatrix} 2 & 5 \\ 1 & 16 \end{pmatrix}$, show that $A^{2} = pA + qI_2$, stating the values of the integers p and q. Hence write A^{3} in the form $gA + hI_2$, stating the values of g and h.

$$A^2 = \begin{pmatrix} 2 & 5 \\ 1 & 16 \end{pmatrix} \begin{pmatrix} 2 & 5 \\ 1 & 16 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 9 & 90 \\ 18 & 261 \end{pmatrix}$$

Next,

$$pA + qI_2 = p\begin{pmatrix} 2 & 5 \\ 1 & 16 \end{pmatrix} + q\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$pA + qI_2 = \begin{pmatrix} 2p + q & 5p \\ p & 16p + q \end{pmatrix}$$

Equating the 2 matrices gives,

$$\begin{pmatrix} 2p + q & 5p \\ p & 16p + q \end{pmatrix} = \begin{pmatrix} 9 & 90 \\ 18 & 261 \end{pmatrix}$$

This immediately gives $p = 18$; substituting this into one of the other equations involving q gives $q = -27$. Thus,

$$A^2 = 18A - 27I_2$$

For the second part,

$$\begin{aligned} A^3 &= A^2 A \\ &= (18A - 27I_2)\,A \\ &= 18A^2 - 27A \\ &= 18\,(18A - 27I_2) - 27A \\ &= 324A - 486I_2 - 27A \\ &= 297A - 486I_2 \end{aligned}$$

Hence, $g = 297$ and $h = -486$.

Example 8

Show that $(kABC)^T = k\,C^T B^T A^T$.

$$\begin{aligned}(kABC)^T &= k\,(ABC)^T \\ &= k\,((AB)C)^T \\ &= k\,C^T (AB)^T \\ &= k\,C^T (B^T A^T) \\ &= k\,C^T B^T A^T\end{aligned}$$

Make sure you can justify each equality in Example 8.

One matrix property that has no counterpart in ordinary arithmetic and algebra is the fact that the product of 2 matrices can be zero without either of the matrices being the zero matrix.

Example 9

Calculate AB for the matrices $A = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 3 & 4 \end{pmatrix}$.

$$\begin{aligned}AB &= \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 3 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 0+0 & 0+0 \\ 0+0 & 0+0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\end{aligned}$$

Special Types of Matrices

Symmetric and Skew-Symmetric Matrices

Definition:

A matrix A is **symmetric** if,

$$A^{\mathrm{T}} = A$$

Note that a symmetric matrix must be square.

Example 10

Show that the matrix $A = \begin{pmatrix} 1 & -2 & 4 \\ -2 & 0 & 17 \\ 4 & 17 & 6 \end{pmatrix}$ is symmetric.

$$A^{\mathrm{T}} = \begin{pmatrix} 1 & -2 & 4 \\ -2 & 0 & 17 \\ 4 & 17 & 6 \end{pmatrix}^{\mathrm{T}}$$

Interchanging the rows and columns of A gives,

$$A^{\mathrm{T}} = \begin{pmatrix} 1 & -2 & 4 \\ -2 & 0 & 17 \\ 4 & 17 & 6 \end{pmatrix}$$

$$A^{\mathrm{T}} = A$$

Hence, as $A^{\mathrm{T}} = A$, A is symmetric.

Definition:

A matrix A is **skew-symmetric** (aka **anti-symmetric**) if,

$$A^{\mathrm{T}} = -A$$

Note that a skew-symmetric matrix must be square and have all diagonal entries equal to 0.

Example 11

Show that the sum of 2 skew-symmetric matrices is skew-symmetric.

Let the 2 skew-symmetric matrices be A and B. Then,

$$(A + B)^{\mathrm{T}} = A^{\mathrm{T}} + B^{\mathrm{T}}$$

$$(A + B)^{\mathrm{T}} = (-A) + (-B)$$

$$(A + B)^{\mathrm{T}} = -(A + B)$$

Hence, as the sum $A + B$ satisfies the skew-symmetric condition, the sum of 2 skew-symmetric matrices is skew-symmetric.

Orthogonal Matrices

Definition:

A square matrix A (of order $n \times n$) is **orthogonal** if,

$$A^{\mathrm{T}} A = I_n$$

Example 12

Show that $R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ is an orthogonal matrix.

$$R^{\mathrm{T}} R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \sin\theta\cos\theta \\ \sin\theta\cos\theta - \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= I_2$$

Hence, as $R^T R$ equals the identity matrix, R is orthogonal.

Determinants

The solution to the 1 × 1 system,

$$ax = b$$

is,

$$x = \frac{b}{a}$$

assuming $a \neq 0$.

The solution to the 2 × 2 system of equations,

$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned}$$

is,

$$x = \frac{de - bf}{ad - bc}$$

$$y = \frac{af - ce}{ad - bc}$$

assuming $ad - bc \neq 0$.

The solution to the 3 × 3 system of equations,

$$\begin{aligned} ax + by + cz &= j \\ dx + ey + fz &= k \\ gx + hy + iz &= l \end{aligned}$$

is (this takes a lot more effort),

$$x = \frac{j(ei - fh) - k(bi - ch) + l(bf - ce)}{a(ei - fh) - b(di - fg) + c(dh - eg)}$$

$$y = \frac{j(fg - di) - k(cg - ai) + l(cd - af)}{a(ei - fh) - b(di - fg) + c(dh - eg)}$$

$$z = \frac{j(dh - ge) - k(ah - bg) + l(ae - bd)}{a(ei - fh) - b(di - fg) + c(dh - eg)}$$

assuming $a(ei - fh) - b(di - fg) + c(dh - eg) \neq 0$.

In each of these solutions, we require the denominators to be non-zero. The denominators that arise in the solutions have a pattern (not necessarily that obvious !) and a special name. We first need some definitions.

<u>Definition:</u>

A **permutation**, denoted by σ , of an ordered set of numbers (1, 2, 3, … , n) is a rearrangement of those numbers.

An **even permutation** is one where the rearrangement involves an even number of consecutive switches starting from the original numbers.

An **odd permutation** is one where the rearrangement involves an odd number of consecutive switches starting from the original numbers.

<u>Definition:</u>

The **sign of a permutation** σ , denoted by sign σ , is defined to be $+1$ for an even permutation and -1 for an odd permutation.

<u>Example 13</u>

For the case $n = 3$, the permutation (1, 2, 3) $\xrightarrow{\sigma}$ (2, 1, 3) is odd, as the result of σ involves only 1 switch ($1 \leftrightarrow 2$) between consecutive numbers of (1, 2, 3). The sign of this permutation is -1.

Example 14

Again, for the case $n = 3$, the permutation $(1, 2, 3) \xrightarrow{\sigma} (3, 1, 2)$ is even, as the result of σ involves 4 switches between consecutive numbers of (1, 2, 3). The switches are $(1, 2, 3) \rightarrow (2, 1, 3) \rightarrow (2, 3, 1) \rightarrow (3, 2, 1) \rightarrow (3, 1, 2)$. For this permutation, sign $\sigma = +1$.

A permutation can be thought of as a function. For example, in Example , 1 gets sent to 3 would be written as $\sigma(1) = 3$.

Definition:

The **determinant of an $n \times n$ matrix** is given by the **Leibniz formula**,

$$\det(A) \equiv |A| \overset{def}{=} \left(\sum_{\sigma}(\text{sign } \sigma)\right)\left(\prod_{i=1}^{n} a_{i\,\sigma(i)}\right)$$

Fortunately, this cryptic formula does not need to be remembered; an alternative formula that is more practical will be given below.

Definition:

For an $n \times n$ matrix A, the **minor** of entry a_{ij} is the determinant (denoted M_{ij}) of the $(n - 1) \times (n - 1)$ matrix formed from A by deleting the i^{th} row and j^{th} column of A.

Definition:

The **cofactor** of entry a_{ij} is the quantity

$$C_{ij} \overset{def}{=} (-1)^{i+j} M_{ij}$$

It can be shown that the Leibniz formula for determinants can be written using cofactors in the form given in the following theorem.

<u>Theorem:</u>

The determinant of an $n \times n$ matrix is given by the **Laplace expansion formula**,

$$\det(A) = \sum_{j=1}^{n} a_{ij} C_{ij} \qquad (i = 1, 2, 3, \ldots, n)$$

The Laplace expansion formula expresses the determinant of a matrix in terms of smaller determinants. For satisfaction and reassurance, the following theorems should be proven using the Laplace expansion formula.

<u>Theorem:</u>

The determinant of a 1×1 matrix is,

$$|A| \equiv |(a)| = a$$

<u>Theorem:</u>

The determinant of a 2×2 matrix is,

$$|A| \equiv \left|\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right| = ad - bc$$

<u>Theorem:</u>

The determinant of a 3×3 matrix is,

$$|A| \equiv \left|\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\right| = a\left|\begin{pmatrix} e & f \\ h & i \end{pmatrix}\right| - b\left|\begin{pmatrix} d & f \\ g & i \end{pmatrix}\right| + c\left|\begin{pmatrix} d & e \\ g & h \end{pmatrix}\right|$$

Notice that these are precisely the expressions for the denominators for the systems at the start of this section.

<u>Example 15</u>

Calculate the determinant of the matrix $B = \begin{pmatrix} 1 & -4 \\ 2 & 6 \end{pmatrix}$.

$$\det(B) = \left|\begin{pmatrix} 1 & -4 \\ 2 & 6 \end{pmatrix}\right|$$

$$= 1.6 - (-4).2$$

$$= 14$$

<u>Example 16</u>

Calculate the determinant of the matrix $F = \begin{pmatrix} 1 & -2 & 7 \\ 6 & 0 & -1 \\ -3 & -10 & 4 \end{pmatrix}$.

$$\det(F) = \left|\begin{pmatrix} 1 & -2 & 7 \\ 6 & 0 & -1 \\ -3 & -10 & 4 \end{pmatrix}\right|$$

$$= 1\left|\begin{pmatrix} 0 & -1 \\ -10 & 4 \end{pmatrix}\right| - (-2)\left|\begin{pmatrix} 6 & -1 \\ -3 & 4 \end{pmatrix}\right| + 7\left|\begin{pmatrix} 6 & 0 \\ -3 & -10 \end{pmatrix}\right|$$

$$= 1(0 - 10) + 2(24 - 3) + 7(-60 - 0)$$

$$= -388$$

<u>Example 17</u>

Solve the equation $\left|\begin{pmatrix} 2 & -4 \\ 3x & 5 \end{pmatrix}\right| = 21$ for x.

$$\left|\begin{pmatrix} 2 & -4 \\ 3x & 5 \end{pmatrix}\right| = 21$$

$$10 + 12x = 21$$

$$12x = 11$$

$$x = \frac{11}{12}$$

Properties of Determinants

The following properties hold for $n \times n$ matrices and $k \in \mathbb{R}$.

- $\det(AB) = \det(A)\det(B)$
- $\det(kA) = k^n \det(A)$
- $\det(A^T) = \det(A)$

Inverse of a Matrix

Inverse Matrices

Of all the operations that have been described for matrices, that of division has not been mentioned. The closest concept of ' division of matrices ' involves the following.

Definition:

An $n \times n$ matrix A has an **inverse** if there is a matrix (denoted A^{-1}) such that,

$$AA^{-1} = A^{-1}A = I_n$$

If a matrix A has an inverse, then A is said to be **invertible** (aka **non-singular**).

If a matrix does not have an inverse, then it is **non-invertible** (aka **singular**).

Theorem:

A matrix has only 1 inverse.

Definition:

The **cofactor matrix** of a square matrix A is the matrix C whose $(i, j)^{\text{th}}$ entry is C_{ij}.

Definition:

The **adjugate** (aka **classical adjoint**) of a square matrix A is the transpose of the cofactor matrix C,

$$\text{adj}(A) \overset{def}{=} C^{\text{T}}$$

Theorem:

The inverse of a matrix A is given by,

$$A^{-1} = \frac{\text{adj}(A)}{\det(A)}$$

Theorem:

A matrix is invertible iff $\det(A) \neq 0$.

This theorem implies that a matrix is non-invertible iff $\det(A) = 0$.

Example 18

Show that $A = \begin{pmatrix} 1 & 0 & 4 \\ -2 & 0 & 17 \\ 4 & 17 & 6 \end{pmatrix}$ is invertible.

$$\det(A) = \left| \begin{pmatrix} 1 & 0 & 4 \\ -2 & 0 & 17 \\ 4 & 17 & 6 \end{pmatrix} \right|$$

$$= 1 \left| \begin{pmatrix} 0 & 17 \\ 17 & 6 \end{pmatrix} \right| - 0 \left| \begin{pmatrix} -2 & 17 \\ 4 & 6 \end{pmatrix} \right| + 4 \left| \begin{pmatrix} -2 & 0 \\ 4 & 17 \end{pmatrix} \right|$$

$$= 1(0 - 289) + 4(-34 - 0)$$

$$= -425$$

Hence, as det $(A) \neq 0$, A is invertible.

Example 19

Determine the values of k for which the matrix $\begin{pmatrix} k & -2 \\ -2 & k+5 \end{pmatrix}$ is singular.

For singularity, we require the determinant of the given matrix to be 0.

$$\left| \begin{pmatrix} k & -2 \\ -2 & k+5 \end{pmatrix} \right| = 0$$

$$k(k+5) - (-2).(-2) = 0$$

$$k^2 + 5k + 4 = 0$$

$$(k+1)(k+4) = 0$$

Hence, the given matrix is singular when $k = -1$ and $k = -4$.

Example 20

If the matrix A satisfies the equation $A^2 = 18A - 27I_2$, show (without explicitly calculating A^{-1}) that $A^{-1} = DA + EI_2$, stating the values of D and E.

$$A^2 = 18A - 27I_2$$

Multiplying (doesn't matter whether post or pre, as the only matrices involved are A and I_2) this equation throughout by A^{-1} gives,

$$A^{-1}A^2 = A^{-1}(18A - 27I_2)$$

Performing the multiplications and simplifying gives,

$$A = 18I_2 - 27A^{-1}$$

Rearranging and solving for A^{-1} gives,

$$A^{-1} = -\frac{1}{27}A + \frac{2}{3}I_2$$

Hence, $D = -\frac{1}{27}$ and $E = \frac{2}{3}$.

Inverse and Systems of Equations

An $n \times n$ system of equations with coefficient matrix A and solution represented by the column matrix x can be written as,

$$A\,x = b$$

where b is the RHS of the system of equations. Assuming the inverse A^{-1} exists, multiplying the above equation by A^{-1} gives,

$$A^{-1}A\,x = A^{-1}b$$

$$x = A^{-1}b$$

In other words, the following theorem holds.

Theorem:

The $n \times n$ system of equations,

$$A\,x = b$$

has a solution if A^{-1} exists; the solution is then obtained by calculating,

$$x = A^{-1}b$$

This result is stated differently in terms of determinants in the following theorem.

Corollary:

A system of n equations in n unknowns has a solution if the determinant of the coefficient matrix is non-zero.

Example 21

Determine the value of k for which the following system of equations has a solution.

$$\begin{aligned} x &+ y + z = 1 \\ 2x &- ky + 3z = 0 \\ x &+ 4y - z = 3 \end{aligned}$$

The coefficient matrix is,

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & k & 3 \\ 1 & 4 & -1 \end{pmatrix}$$

The determinant of the coefficient matrix is,

$$\left|\begin{pmatrix} 1 & 1 & 1 \\ 2 & k & 3 \\ 1 & 4 & -1 \end{pmatrix}\right| = 1\left|\begin{pmatrix} k & 3 \\ 4 & -1 \end{pmatrix}\right| - 1\left|\begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}\right| + 1\left|\begin{pmatrix} 2 & k \\ 1 & 4 \end{pmatrix}\right|$$

$$= (-k - 12) - (-2 - 3) + (8 - k)$$

$$= -2k - 15$$

According to the previous theorem, the system of equations has a solution when $-2k - 15 \neq 0$, i.e. when $k \neq -\frac{15}{2}$.

In practice, the formula for the inverse involving the adjugate is difficult to use from scratch. Fortunately, for the cases we are interested in, there are other approaches.

Inverse of a 2 x 2 Matrix

Theorem:

The inverse of the 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is,

$$A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

This is just the adjugate formula for a 2×2 matrix, but it is easier to remember in this form.

Example 22

Find the inverse of the matrix $\begin{pmatrix} 7 & -3 \\ 4 & 2 \end{pmatrix}$.

Using the formula in the above theorem gives,

$$\begin{pmatrix} 7 & -3 \\ 4 & 2 \end{pmatrix}^{-1} = \frac{1}{7.2 - (-3).4}\begin{pmatrix} 2 & 3 \\ -4 & 7 \end{pmatrix}$$

$$\begin{pmatrix} 7 & -3 \\ 4 & 2 \end{pmatrix}^{-1} = \frac{1}{26}\begin{pmatrix} 2 & 3 \\ -4 & 7 \end{pmatrix}$$

It is ok to leave the answer in this form, instead of actually performing the scalar multiplication; there is no virtue in complicating an already ugly answer (technically, the answer in the form above is fully simplified, as the $\frac{1}{26}$ has been factorised out).

Example 23

Solve the following equation for x and y.

$$\begin{pmatrix} 2 & 1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \end{pmatrix}$$

Multiplying this equation by $\begin{pmatrix} 2 & 1 \\ -2 & 3 \end{pmatrix}^{-1}$, which equals $\frac{1}{8}\begin{pmatrix} 3 & -1 \\ 2 & 2 \end{pmatrix}$ (check !), gives,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{8}\begin{pmatrix} 3 & -1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 8 \\ 16 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Performing the matrix multiplication gives $x = 1$ and $y = 6$. Check that these values work in the original equation.

Inverse of a 3 x 3 Matrix

Recall that the $n \times n$ system $A\,x = b$ can be solved by calculating $x = A^{-1}b$. Picking b to be $b_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $b_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}$, …, $b_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ n \end{pmatrix}$ shows that the solution vector x picks out the columns of A^{-1} (b_1 picks out the first column of A^{-1} etc.). Alternatively, the solution vectors x can be obtained by row-reducing the augmented matrices,

$$\left(A \mid b_1\right)$$

$$\left(A \mid b_2\right)$$

$$\vdots$$

$$\left(A \mid b_n\right)$$

so that the LHS of each gives the identity matrix. According to the first way of obtaining the RHS, what is left on each RHS will be the columns of A^{-1}.

These n calculations can be performed in one go by forming a giant Augmented matrix consisting of A on the LHS, but now b_1, b_2, …, b_n written beside each other so that the RHS is effectively I_n.

Theorem:

The inverse of A can be found by row-reducing the extended Augmented matrix,

$$\left(A \mid I_n\right)$$

into,

$$\left(I_n \mid B\right)$$

Then $B = A^{-1}$.

This theorem is mainly used for a 3 $\times$ 3 matrix A.

Example 24

Find the inverse of $N = \begin{pmatrix} 1 & 0 & 4 \\ -2 & 0 & 7 \\ 4 & 1 & 6 \end{pmatrix}$.

First set up the big-daddy Augmented matrix,

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ -2 & 0 & 7 & 0 & 1 & 0 \\ 4 & 1 & 6 & 0 & 0 & 1 \end{array}\right)$$

Then use EROs to reduce this matrix so that the LHS becomes I_3; the RHS will then be N^{-1}.

$R_2 \to R_2 + 2R_1$
$R_3 \to R_3 - 4R_1$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 11 & 2 & 1 & 0 \\ 0 & 1 & -2 & -4 & 0 & 1 \end{array}\right)$$

$R_2 \leftrightarrow \frac{1}{11}R_2$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{2}{11} & \frac{1}{11} & 0 \\ 0 & 1 & -2 & -4 & 0 & 1 \end{array}\right)$$

$R_2 \leftrightarrow R_3$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -2 & -4 & 0 & 1 \\ 0 & 0 & 1 & \frac{2}{11} & \frac{1}{11} & 0 \end{array}\right)$$

$R_2 \to R_2 + 2R_3$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -\frac{40}{11} & \frac{2}{11} & 1 \\ 0 & 0 & 1 & \frac{2}{11} & \frac{1}{11} & 0 \end{array}\right)$$

$R_1 \to R_1 - 2R_3$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{7}{11} & -\frac{2}{11} & 0 \\ 0 & 1 & 0 & -\frac{40}{11} & \frac{2}{11} & 1 \\ 0 & 0 & 1 & \frac{2}{11} & \frac{1}{11} & 0 \end{array}\right)$$

To get the inverse into a slightly slicker form, factorise out the fraction,

$$N^{-1} = \frac{1}{11}\begin{pmatrix} 7 & -2 & 0 \\ -40 & 2 & 11 \\ 2 & 1 & 0 \end{pmatrix}$$

Properties of Inverse Matrices

- $(AB)^{-1} = B^{-1}A^{-1}$
- $(A^{-1})^{T} = (A^{T})^{-1}$
- $\det(A^{-1}) = \dfrac{1}{\det(A)}$
- $(kA)^{-1} = \dfrac{1}{k}A^{-1}$

Transformation Matrices

Definition:

A **(geometrical) transformation** is a way of changing points in space.

In this course, we will focus on transformations in the xy - plane. A transformation is described by a function.

Definition:

A **linear transformation in the plane** is a function that sends a point P (x, y) to a point Q $(ax + by, cx + dy)$ $(a, b, c, d \in \mathbb{R})$.

Geometrical transformations can be described using matrices.

Theorem:

A linear transformation in the plane can be described as a matrix equation,

$$\begin{pmatrix} x' \\ y' \end{pmatrix} \stackrel{def}{=} \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

The matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is called the **transformation matrix**.

A transformation matrix can be determined by looking at where the geometrical transformation sends the points (0, 1) and (1, 0).

Example 25

Find the matrix associated with the transformation $(x, y) \rightarrow (x', y')$ where $x' = 2x + 5y$ and $y' = x - 3y$.

Thinking in terms of matrices and vectors,

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

shows that $x' = ax + by$ and $y' = cx + dy$. Comparing this with $x' = 2x + 5y$ and $y' = x - 3y$ gives $a = 2$, $b = 5$, $c = 1$, and $d = -3$. So, the required transformation matrix is $\begin{pmatrix} 2 & 5 \\ 1 & -3 \end{pmatrix}$.

Example 26

Find the image of the point P $(3, -1)$ under the transformation associated with the matrix $\begin{pmatrix} 1 & -2 \\ 3 & 0 \end{pmatrix}$.

The image of P is given by P ' (x', y') where,

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

This gives $x' = 5$ and $y' = 9$, Thus, the image of P is P ' (5, 9).

Invariant Points

Some points are left unchanged under a transformation. If x is a coordinate vector of a point and T the transformation matrix, then the following definition can be made.

Definition:

An **invariant point** is a point whose coordinates stay the same after a transformation,

$$Tx = x$$

Example 27

Find the invariant points under the transformation described by the matrix $\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$.

The matrix equation,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

yields the equations $x = -x + 2y$ and $y = 2x + 3y$, which simplify to $x = y$ and $x = -y$. Solving these gives $x = 0$, and $y = 0$. Hence, (0, 0) is the only invariant point under this transformation.

There are some standard transformations that must be known.

Reflection in the x - axis

Theorem:

A reflection in the x-axis is described by the transformation matrix,

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Reflection in the y – axis

Theorem:

A reflection in the y-axis is described by the transformation matrix,

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Reflection in the Line y = x

Theorem:

A reflection in the line $y = x$ is described by the transformation matrix,

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Anticlockwise rotation about the origin

Theorem:

An anticlockwise rotation of angle θ about the origin is described by the transformation matrix,

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Dilatation (Scaling)

Definition:

A **dilatation** (aka **scaling** or **homothety**) is a transformation that scales each coordinate of a point by the same amount.

Theorem:

A dilatation is described by the transformation matrix,

$$\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \qquad (k \in \mathbb{R})$$

If $k > 1$, the dilatation is an enlargement, whereas if $k < 1$, the dilatation is a reduction. If $k < 0$, then the dilatation inverts

Transformation matrices are usually combined to yield a resultant transformation.

Example 28

Find the transformation matrix associated with a reflection in the x-axis followed by anti-clockwise rotation of 90° about the origin.

The first transformation is the reflection, which we will label as T_R, and the second is the rotation, labelled as T_θ. Taking a generic point P, the first transformation will be T_R(P) and the second one will be $T_\theta(T_R(\text{P}))$. In other words, we have to perform matrix multiplication (and remember that, in general, the order of multiplication matters). So, the required transformation is,

$$T_\theta T_R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Notice that this is the same as reflecting in the line $y = x$.

Example 29

Find the image of the point P (1, 2) after a reflection in the line $y = x$ followed by the dilatation $\begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix}$.

The image of P will be given by,

$$\begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Performing this multiplication gives the vector $\begin{pmatrix} 14 \\ 7 \end{pmatrix}$. So, the image of P is P ' (14, 7).

Transformation of Loci

It is important to know where sets of points, for example, those on a curve, get mapped to under a transformation.

Example 30

Find the equation of the image of the curve with equation $y = x^2$ under the transformation with associated matrix $\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$.

Let the image of a point P (x, y) be P ' (x', y'). Then,

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}^{-1} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3x' + 2y' \\ 2x' - y' \end{pmatrix}$$

Substituting the above expressions for x and y into $y = 3x^2$ gives the image curve equation in implicit form as $2x' - y' = 3(-3x' + 2y')^2$.