Module – 4 Analytic Functions

Definition of Analytic Function - Cauchy Riemann equations - Properties of analytic functions - Determination of analytic function using Milne Thomson's method - Conformal mappings: Magnification, Rotation, Inversion, Reflection - Bilinear Transformation - Cauchy's integral theorem (without proof) - Cauchy's integral theorem applications - Application of Bilinear transformation and Cauchy's Integral in Engineering.

Analytic function (or) Holomorphic function (or) Regular function.

A function is said to be analytic at a point if its derivative exists not only at that point but also in some neighbourhood of that point.

Entire (or) an Integral function.

A function which is analytic everywhere in the finite plane except at z = ∞ is called an entire function.
Example: e^z, sinz, coshz.

Necessary conditions for f(z) to be analytic.

The necessary conditions for a complex function f(z) = u(x,y) + i v(x,y) to be analytic in a region R are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad and \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad \text{(i.e) C – R equations.}$$

Sufficient conditions for f (z) to be analytic.

If the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial u}{\partial y}$ exist and continuous in D and satisfies the conditions

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad and \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$. Then the function f(z) is analytic in a domain D.

Harmonic function.

Any function which possess continuous second order partial derivatives and which satisfies Laplace equation is called a harmonic function. (i.e) If $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$, then f is harmonic then

Show that the function $u = 2x - x^3 + 3xy^2$ is harmonic.

Solution: Given $u = 2x - x^3 + 3xy^2$

$u_x = 2 - 3x^2 + 3y^2 \qquad u_y = 6xy$

$u_{xx} = -6x \qquad u_{yy} = 6x$

$u_{xx} + u_{yy} = -6x + 6x = 0$

Hence u is harmonic

Show that the function $u=\frac{1}{2}\log(x^2+y^2)$ is harmonic and determine its conjugate. Also find f(z).

Given $u=\frac{1}{2}\log\left(x^2+y^2\right)$

$$\frac{\partial u}{\partial x}=\frac{1}{2}\cdot\frac{1}{x^2+y^2}(2x)=\frac{x}{x^2+y^2}$$

$$\frac{\partial u}{\partial y}=\frac{1}{2}\cdot\frac{1}{x^2+y^2}(2y)=\frac{y}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{\left(x^2+y^2\right)-x(2x)}{\left(x^2+y^2\right)^2}=\frac{y^2-x^2}{\left(x^2+y^2\right)^2}$$

$$\frac{\partial^2 u}{\partial y^2}=\frac{\left(x^2+y^2\right)(1)-2y^2}{\left(x^2+y^2\right)^2}=\frac{x^2-y^2}{\left(x^2+y^2\right)^2}$$

$$\therefore \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{y^2-x^2+x^2-y^2}{\left(x^2+y^2\right)^2}=0$$

Hence u is harmonic function

To find conjugate of u

$$\phi_1(x,y)=\frac{\partial u}{\partial x}=\frac{x}{x^2+y^2}$$

$$\phi_1(z,o)=\frac{1}{z}$$

$$\phi_2(x,y)=\frac{\partial u}{\partial y}=\frac{y}{x^2+y^2}$$

$$\phi_2(z,o)=0$$

By Milne Thomson Methods

$$f'(z)=\phi_1(z,o)-i\phi_2(z,o)$$

$$\int f'(z)\,dz=\int\frac{1}{z}dz+0$$

$$=\log z+c$$

$$f(z)=\log re^{i\theta}$$

$$f(z)=u+iv=\log r+i\theta$$

$u=\log r$, $v=\theta$

$u=\log\sqrt{x^2+y^2}$ $\quad\left[\therefore r^2=x^2+y^2,\ \theta=\tan^{-1}\left(\frac{y}{x}\right)\right]$

$v=\tan^{-1}\left(\frac{y}{x}\right)$ $\therefore$ Conjugate of u is $\tan^{-1}\left(\frac{y}{x}\right)$.

Conformal transformation.

A mapping or transformation which preserves angles in magnitude and in direction between every pair of curves through a point is said to be conformal transformation.

Isogonal transformation.

A transformation under which angles between every pair of curves through a point are preserved in magnitude but altered in sense is said to be isogonal at that point.

Bilinear transformation (or) Mobius transformation (or) linear fractional transformation.

The transformation $w=\frac{az+b}{cz+d}$, $ad-bc\neq 0$ where a, b,c,d are complex numbers is called a bilinear transformation. This is also called as Mobius or linear fractional transformation.

Cross Ratio.

The cross ratio of four points z_1,z_2,z_3,z_4 is given by $\frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)}$.

Show that $f(z)=|z|^2$ is differentiable at $z=0$ but not analytic at $z=0$.

Solution: Let $z=x+iy$ and $\bar{z}=x-iy$

$$|z|^2=z\bar{z}=x^2+y^2$$

$$f(z)=|z|^2=(x^2+y^2)+i0$$

$$u=x^2+y^2\ ,\quad v=0$$

$$u_x=2x\ ,\quad v_x=0$$

$$u_y=2y\ ,\quad v_y=0$$

So the C-R equations $u_x=v_y$ and $u_y=-v_x$ are not satisfied everywhere except at $z=0$. So f (z) may be differentiable only at $z=0$. Now $u_x=2x$, $v_y=0$ and $u_y=2y$, $v_x=0$ are continuous everywhere and in particular at (0, 0). So f (z) is differentiable at $z=0$ only and not analytic.

Obtain the invariant points of the transformation $w=\frac{z-1}{z+1}$

Solution: Given: $w=\frac{z-1}{z+1}$

The invariant points are obtained by replacing w by z.

$$\text{i.e, } z=\frac{z-1}{z+1}\Rightarrow z^2+1=0\ \therefore z=\pm i$$

Can $v=\tan^{-1}\left(\frac{y}{x}\right)$ be the imaginary part of an analytic function? If so construct an analytic function f(z) = u + iv, taking v as the imaginary part and hence find u.

Solution:

Let $v=\tan^{-1}\left(\frac{y}{x}\right)$

$$v_x = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{-y}{x^2}\right) = \frac{-y}{x^2+y^2};\ v_{xx} = -\left(\frac{(x^2+y^2).0 - y(2x)}{\left(x^2+y^2\right)^2}\right) = \frac{2xy}{\left(x^2+y^2\right)^2}$$

$$v_y = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2};\quad v_{yy} = -\left(\frac{(x^2+y^2).0 - x(2y)}{\left(x^2+y^2\right)^2}\right) = \frac{-2xy}{\left(x^2+y^2\right)^2}$$

$v_{xx} + v_{yy} = 0 \Rightarrow$ *v is harmonic and hence v can be the imaginary part of an analytic function.*

By Milne's method, $f(z) = \int \{v_y(z,0) + iv_x(z,0)\}dz + c$

$$v_x = \frac{-y}{x^2+y^2};\ v_x(z,0) = 0;$$

$$v_y = \frac{x}{x^2+y^2};\ v_y(z,0) = \frac{1}{z}$$

$$f(z) = \int \frac{dz}{z} + c = \log z + c = \log r + i\theta + c_1 + ic_2 \quad (\because z = re^{i\theta})$$

$$= \underbrace{\left(\frac{1}{2}\log(x^2+y^2) + c_1\right)}_{u} + i\underbrace{\tan^{-1}\left(\frac{y}{x}\right)}_{v} \qquad \left(\because r = \sqrt{x^2+y^2}\ \&\ \theta = \tan^{-1}\left(\frac{y}{x}\right)\right)$$

$(c_2 = 0)$

$$\therefore u = \frac{1}{2}\log(x^2+y^2) + c_1$$

Prove that $u = x^2 - y^2$ & $v = \dfrac{-y}{x^2+y^2}$ are harmonic functions but not harmonic conjugate.

Solution:

$$u = x^2 - y^2 \qquad\qquad v = \frac{-y}{x^2+y^2}$$

$$u_x = 2x \qquad\qquad v_x = \frac{2xy}{\left(x^2+y^2\right)^2}$$

$$u_y = -2y \qquad\qquad v_y = \frac{y^2 - x^2}{\left(x^2+y^2\right)^2}$$

$$u_{xx} = 2 \qquad\qquad v_{xx} = \frac{2y(y^2 - 3x^2)}{\left(x^2+y^2\right)^3}$$

$$u_{yy} = -2 \qquad\qquad v_{yy} = \frac{2y(3x^2 - y^2)}{\left(x^2+y^2\right)^3}$$

$$\therefore u_{xx} + u_{yy} = 0 \qquad\qquad v_{xx} + v_{yy} = 0$$

Hence u and v are harmonic.

But $u_x \neq v_y$ & $v_x \neq -u_y$

C-R equations are not satisfied. Hence u+iv are not an analytic function. So they are not harmonic conjugate.

Prove that $w=\dfrac{z}{z+a}$ where $a \neq 0$ is analytic whereas $w=\dfrac{\bar{z}}{\bar{z}+a}$ is not analytic.

Solution:

$$w=\frac{z}{z+a}=\frac{x+iy}{x+iy+a}=\frac{x+iy}{(x+a)+iy}=\frac{x+iy}{(x+a)+iy}\left(\frac{(x+a)-iy}{(x+a)-iy}\right)$$

$$=\frac{(x+iy)((x+a)-iy)}{(x+a)^2+y^2}=\frac{x(x+a)+y^2}{(x+a)^2+y^2}+i\frac{(x+a)y-xy}{(x+a)^2+y^2}$$

$$w=\underbrace{\frac{x(x+a)+y^2}{(x+a)^2+y^2}}_{u}+i\underbrace{\frac{ay}{(x+a)^2+y^2}}_{v}$$

$$u=\frac{x(x+a)+y^2}{(x+a)^2+y^2};$$

$$u_x=\frac{((x+a)^2+y^2)(2x+a)-(x(x+a)+y^2)(2(x+a))}{((x+a)^2+y^2)^2}$$

$$=\frac{2x(x+a)+2xy^2-2x^2(x+a)-2xy^2-2ax(x+a)-2ay^2}{((x+a)^2+y^2)^2}$$

$$=\frac{(x+a)(2x^2+2ax+ax+a^2-2x^2-2ax)-ay^2}{((x+a)^2+y^2)^2}$$

$$u_x=\frac{a((x+a)^2-y^2)}{((x+a)^2+y^2)^2}\ldots(1)$$

$$u_y=\frac{((x+a)^2+y^2)(2y)-(x(x+a)+y^2)(2y)}{((x+a)^2+y^2)^2}$$

$$=\frac{2y((x+a)^2+y^2-(x(x+a)+y^2))}{((x+a)^2+y^2)^2}$$

$$=\frac{2y(x^2+ax+a^2+y^2-x^2-ax-y^2)}{((x+a)^2+y^2)^2}$$

$$u_y=\frac{2ay(x+a)}{((x+a)^2+y^2)^2}\ldots(2)$$

$$v=\frac{ay}{(x+a)^2+y^2};$$

$$v_x=\frac{((x+a)^2+y^2)(0)-(ay)(2(x+a))}{((x+a)^2+y^2)^2}$$

$$v_x=\frac{-2ay(x+a)}{((x+a)^2+y^2)^2}\ldots(3)$$

$$v_y=\frac{((x+a)^2+y^2)(a)-(ay)(2y)}{((x+a)^2+y^2)^2}$$

$$=\frac{a((x+a)^2+y^2-2y^2)}{((x+a)^2+y^2)^2}$$

$$v_y=\frac{a((x+a)^2-y^2)}{((x+a)^2+y^2)^2}\ldots(4)$$

From (1) and (4), $u_x=v_y$

From (2) and (3), $u_y=-v_x$

Also u_x, u_y, v_x, v_y *are continuous functions in* x *and* y.

Hence $w=\frac{z}{z+a}$ *is analytic.*

$$\text{Now } w=\frac{\bar{z}}{\bar{z}+a}=\frac{x-iy}{x-iy+a}=\frac{x-iy}{(x+a)-iy}=\frac{x-iy}{(x+a)-iy}\left(\frac{(x+a)+iy}{(x+a)+iy}\right)$$

$$=\frac{(x-iy)((x+a)+iy)}{(x+a)^2+y^2}=\frac{x(x+a)+y^2}{(x+a)^2+y^2}+i\frac{(-(x+a)y+xy)}{(x+a)^2+y^2}$$

$$w=\underbrace{\frac{x(x+a)+y^2}{(x+a)^2+y^2}}_{u}+i\underbrace{\frac{-ay}{(x+a)^2+y^2}}_{v}$$

$$u=\frac{x(x+a)+y^2}{(x+a)^2+y^2};$$

$$u_x=\frac{a((x+a)^2-y^2)}{((x+a)^2+y^2)^2}\ldots(5)$$

$$u_y=\frac{2ay(x+a)}{((x+a)^2+y^2)^2}\ldots(6)$$

$$v = \frac{-ay}{(x+a)^2 + y^2};$$

$$v_x = \frac{2ay(x+a)}{((x+a)^2 + y^2)^2} \ldots(7)$$

$$v_y = \frac{-a((x+a)^2 - y^2)}{((x+a)^2 + y^2)^2} \ldots(8)$$

From (5) and (8), $u_x \neq v_y$

From (6) and (7), $u_y \neq -v_x$

Hence $w = \dfrac{\bar{z}}{\bar{z}+a}$ *is not analytic.*

Properties of Analytic function

Property : 1

The function f(z) = u + iv is analytic, show that u = constant and v = constant are orthogonal

Proof:

If $f(z) = u + iv$ is an analytic function of z, then it satisfies C-R equations

$u_x = v_y,\ u_y = -v_x$

Given $u(x, y) = C_1$.............(1)

$v(x, y) = C_2$.............(2)

By total differentiation

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

Differentiate equation (1) & (2) we get $du = 0$, $dv = 0$

$$\therefore \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = 0 \quad \text{and} \quad \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\partial u / \partial x}{\partial u / \partial y} = m_1 (say)$$

$$\frac{dy}{dx} = \frac{-\partial v / \partial x}{\partial v / \partial y} = m_2 (say)$$

$$\therefore m_1 m_2 = -\frac{-\partial u / \partial x}{\partial u / \partial y} \times \frac{-\partial v / \partial x}{\partial v / \partial y} \quad (\because u_x = v_y \quad u_y = -v_x)$$

$$\therefore m_1 m_2 = -1$$

The curves $u(x, y) = C_1$ and $v(x, y) = C_2$ cut orthogonally.

Property : 2

Prove that an analytic function with constant modulus is constant.

Proof:

Let $f(z)=u+iv$ be analytic

By C.R equations satisfied

i.e., $u_x=v_y,\ u_y=-v_x$

$\therefore f(z)=u+iv$

$\Rightarrow |f(z)|=\sqrt{u^2+v^2}=C \Rightarrow |f(z)|^2=u^2+v^2=C^2$

$u^2+v^2=C^2$................(1)

Diff (1) with respect to x

$$2u\frac{\partial u}{\partial x}+2v\frac{\partial v}{\partial x}=0$$

$uu_x+vv_x=0$.............(2)

Diff (1) with respect to y

$$2u\frac{\partial u}{\partial y}+2v\frac{\partial v}{\partial y}=0$$

$-uv_x+vu_x=0$............(3)

$(2)\times u+(3)\times v \Rightarrow (u^2+v^2)u_x=0$

$\Rightarrow u_x=0$

$(2)\times v-(3)\times u \Rightarrow (u^2+v^2)v_x=0$

$\Rightarrow v_x=0$

W.K.T $f'(z)=u_x+iv_x=0$

$f'(z)=0$ Integrate w.r.to z

$f(z)=C$

Property : 3

8. Prove that the real and imaginary parts of an analytic function are harmonic function.

Proof:

Let f (z) = u + iv be an analytic function of z. Then by C- R equations we have,

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\text{...........(1)}\qquad \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}\text{.........(2)}$$

Differentiating (1) partially with respect to x, we get

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 v}{\partial x\partial y}\text{...........(3)}$$

Differentiating (2) partially with respect to y, we get

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial y \partial x} \text{..........} (4)$$

Adding (3) and (4), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} = 0$$

$\therefore$ u satisfies the Laplace equation.

Similarly

Differentiating (1) partially with respect to y, we get

$$\frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u}{\partial y \partial x} \text{..........} (5)$$

Differentiating (2) partially with respect to x, we get

$$\frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x \partial y} \text{..........} (6)$$

Adding (5) and (6), we get

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = -\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y \partial x} = 0$$

$\therefore$ v satisfies the Laplace equation.

Hence the real and imaginary parts of an analytic function are harmonic function.

Property : 4

9. The real part of an analytic function f(z) is constant, prove that f(z) is a constant function.

Proof:

Let f(z) = u + iv

Given u = constant. $\Rightarrow u_x = 0$ and $u_y = 0$

by C-R equations, $u_x = 0 \Rightarrow v_y = 0$ and $u_y = 0 \Rightarrow v_x = 0$

$f'(z) = u_x + iv_x = 0 + i0 = 0$

Integrating, $f(z) = c$ (where c is a constant)

10. If f(z) is an analytic function, prove that $\left\{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right\}|f(z)|^2 = 4|f'(z)|^2$

Proof:

Let f(z) = u + iv be analytic.

Then $u_x = v_y$ and $u_y = -v_x$ (1)

Also $u_{xx} + u_{yy} = 0$ and $v_{xx} + v_{yy} = 0$ (2)

Now $|f(z)|^2 = u^2 + v^2$ and $f'(z) = u_x + iv_x$

$$\therefore \frac{\partial}{\partial x}|f(z)|^2 = 2u.u_x + 2v.v_x$$

and $$\frac{\partial^2}{\partial x^2}|f(z)|^2 = 2\left[u_x^2 + u.u_{xx} + v_x^2 + v.v_{xx}\right] \qquad (3)$$

Similarly $$\frac{\partial^2}{\partial y^2}|f(z)|^2 = 2\left[u_y^2 + u.u_{yy} + v_y^2 + v.v_{yy}\right] \qquad (4)$$

Adding (3) and (4)

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 2\left[u_x^2 + u_y^2 + u(u_{xx} + u_{yy}) + v_x^2 + v_y^2 + v(v_{xx} + v_{yy})\right]$$

$$= 2\left[u_x^2 + v_x^2 + u(0) + v_x^2 + u_x^2 + v(0)\right]$$

$$= 4\left[u_x^2 + v_x^2\right]$$

11. Find the map of the circle (i) $|z| = 3$ under the transformation $w = 2z$

(ii) $|z| = 1$ by the transformation $w = z + 2 + 4i$

Solution (i) : Given $w = 2z$, $|z| = 3$

$|w| = 2|z|$

$|w| = 2(3) = 6$

Hence the image of the circle $|z| = 3$ in the z-plane maps to the circle $|w| = 6$ in the w-plane.

Solution (ii) :

Given: $w = z + 2 + 4i$

$u + iv = x + iy + 2 + 4i = (x + 2) + i(y + 4)$

$u = x + 2, \qquad v = y + 4$

⇨ $x = u - 2, \qquad y = v - 4$

⇨ $|z| = 1$

$x^2 + y^2 = 1$ Hence $(u - 2)^2 + (v - 4)^2 = 1$.

∴ The circle in the z-plane is mapped into the circle in the w -plane with centre (2, 4) and radius 1.

Find the image of the infinite strip $\frac{1}{4} < y < \frac{1}{2}$ under the transformation $w = \frac{1}{z}$

Solution:

$$w = \frac{1}{z} \Rightarrow z = \frac{1}{w}$$

$$z=\frac{1}{u+iv}=\frac{u-iv}{u^2+v^2}\Rightarrow x=\frac{u}{u^2+v^2} \quad \text{\_\_\_\_\_\_(1)} \quad y=-\frac{v}{u^2+v^2} \quad \text{\_\_\_\_\_\_(2)}$$

Given strip is $\frac{1}{4}<y<\frac{1}{2}$ when $y=\frac{1}{4}$

$\frac{1}{4}=-\frac{v}{u^2+v^2}$ (by 2)

$u^2+(v+2)^2=4$..........(3)

which is a circle whose centre is at $(0,-2)$ in the w-plane and radius 2.

When $y=\frac{1}{2}$

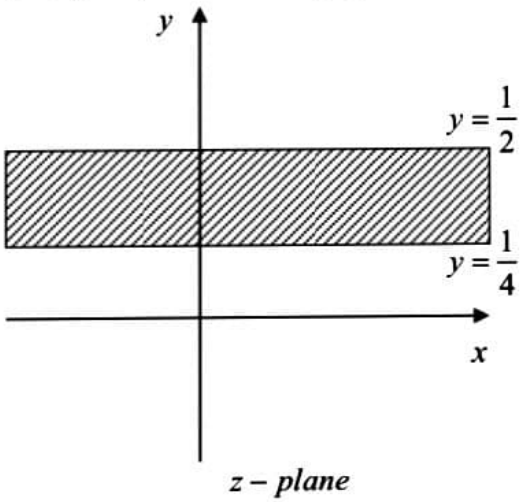
$\frac{1}{2}=\frac{-v}{u^2+v^2}$ (by 2)

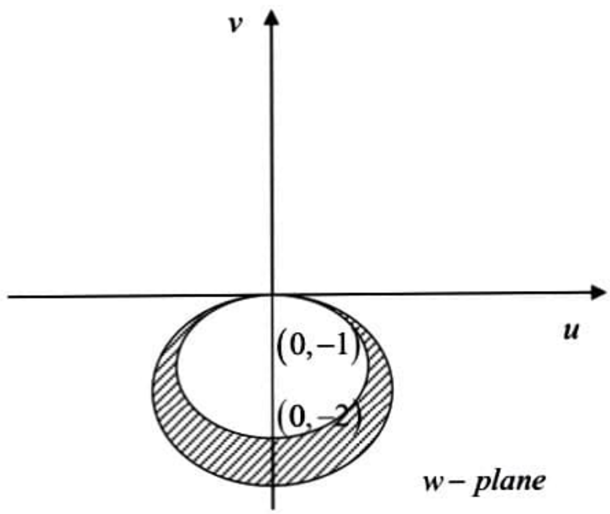
$u^2+v^2+2v=0$

$u^2+(v+1)^2=1$..........(4)

which is a circle whose centre is at $(0,-1)$ and radius is 1 in the w-plane.

Hence the infinite strip $\frac{1}{4}<y<\frac{1}{2}$ is transformed into the region between circles $u^2+(v+1)^2=1$ and $u^2+(v+2)^2=4$ in the w-plane.





Find the image of $|z-2i|=2$ under the transformation $w=\frac{1}{z}$

Solution:

Given $w=\frac{1}{z}\Rightarrow z=\frac{1}{w}$

Now $w=u+iv$

$$z=\frac{1}{w}=\frac{1}{u+iv}=\frac{u-iv}{(u+iv)(u-iv)}=\frac{u-iv}{u^2+v^2}$$

i.e., $x+iy=\frac{u-iv}{u^2+v^2}$ $\therefore$ $x=\frac{u}{u^2+v^2}$..........(1) $y=\frac{-v}{u^2+v^2}$..........(2)

Given $|z-2i|=2$

$|x+iy-2i|=2 \Rightarrow |x+i(y-2)|=2$

$x^2+(y-2)^2=4 \Rightarrow x^2+y^2-4y=0$..................(3)

Sub (1) and (2) in (3)

$$\left(\frac{u}{u^2+v^2}\right)^2+\left(\frac{-v}{u^2+v^2}\right)^2-4\left[\frac{-v}{u^2+v^2}\right]=0$$

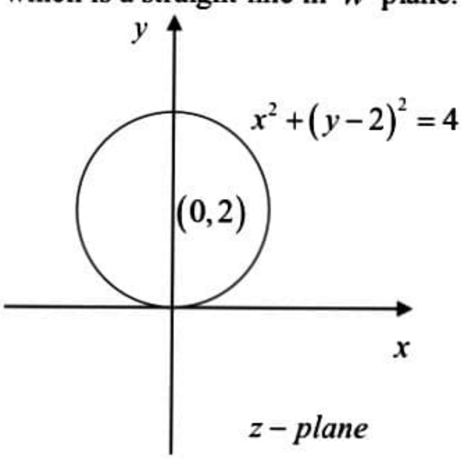
$$\frac{u^2}{(u^2+v^2)^2}+\frac{v^2}{(u^2+v^2)^2}+\left[\frac{4v}{u^2+v^2}\right]=0$$

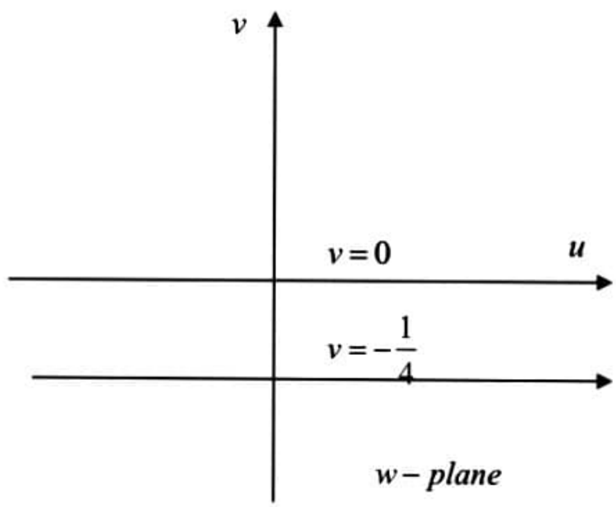
$$\frac{(u^2+v^2)+4v(u^2+v^2)}{(u^2+v^2)^2}=0$$

$$\frac{(1+4v)(u^2+v^2)}{(u^2+v^2)^2}=0$$

$$1+4v=0 \Rightarrow v=-\frac{1}{4} \quad (\therefore u^2+v^2 \neq 0)$$

which is a straight line in w-plane.





Show that the transformation $w=\frac{1}{z}$ transforms in general, circles and straight lines into circles and straight lines.

Solution:

$$w=\frac{1}{z} \Rightarrow z=\frac{1}{w}$$

$$\Rightarrow x+iy=\frac{1}{u+iv}=\frac{u-iv}{u^2+v^2}$$

$$\Rightarrow x=\frac{u}{u^2+v^2} \text{ and } y=\frac{-v}{u^2+v^2}$$

Consider the equation $a(x^2 + y^2) + bx + cy + d = 0$ ------------(1)
This equation represents a circle if $a \neq 0$ and a straight line if $a = 0$

Under the transformation $w = \frac{1}{z}$ equation (1) becomes

$d(u^2 + v^2) + bu - cv + a = 0$ --------------(2)

This equation represents a circle if $d \neq 0$ and a straight line if $d = 0$

Value of a & d	Equation (1) and (2)	Conclusion
$a \neq 0,\ d \neq 0$	Equation (1) and (2) represents a circle, not passing through the origin, in the z–plane and w–plane	The transformation maps a circle not passing through the origin in z–plane into a circle not passing through the origin in w–plane
$a \neq 0,\ d = 0$	Equation (1) represents a circle passing through the origin in the z–plane and equation (2) represents a straight line not passing through the origin in w–plane	The transformation maps a circle passing through the origin in z–plane into a straight line not passing through the origin in w–plane
$a = 0,\ d \neq 0$	Equation (1) represents a straight line not passing through the origin in the z–plane and equation (2) represents a circle passing through the origin in w–plane	The transformation maps a straight line not passing through the origin in the z–plane into a circle passing through the origin in w–plane
$a = 0, d = 0$	Equation (1) and (2) represents a straight line passing through the origin in the z–plane and w–plane	The transformation maps represents a straight line passing through the origin in z–plane into a straight line passing through the origin in w–plane

Thus the transformation $w = \frac{1}{z}$ maps the totality of circles and straight lines as circles or straight lines.

Find the image of the circle $|z-1|=1$ under the transformation $w=z^2$

Solution:

In polar form $z=re^{i\theta}$, $w=Re^{i\phi}$

Given

$$|z-1|=1$$

$$|re^{i\theta}-1|=1$$

$$|r\cos\theta+ir\sin\theta-1|=1$$

$$|(r\cos\theta-1)+ir\sin\theta|=1$$

$$(r\cos\theta-1)^2+(r\sin\theta)^2=1^2$$

$$r^2-2r\cos\theta=0$$

$$r=2\cos\theta----(1)$$

Now, we have

$$w=z^2$$

$$Re^{i\phi}=\left(re^{i\theta}\right)^2$$

$$Re^{i\phi}=r^2e^{i2\theta}$$

$$R=r^2,\qquad \phi=2\theta$$

$$(1)\Rightarrow$$

$$r^2=(2\cos\theta)^2$$

$$=4\cos^2\theta$$

$$=4\left[\frac{1+\cos2\theta}{2}\right]$$

$$r^2=2(1+\cos2\theta)$$

$$R=2(1+\cos\phi)$$

Find the bilinear transformation of the points –1,0,1 in z- plane onto the points 0,*i*,3*i* in w- plane.

Solution:

Given $z_1=-1,\ w_1=0$

$z_2=0,\ w_2=i$

$z_3=i,\ w_3=3i$

Cross-ratio

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}=\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-0)(i-3i)}{(w-3i)(i-0)}=\frac{(z-(-1))(0-1)}{(z-1)(0-(-1))}$$

$$\frac{w(-2i)}{(w-3i)(i)}=\frac{(z+1)(-1)}{(z-1)(1)}$$

$$\frac{2w}{w-3i}=\frac{z+1}{z-1}$$

$$2wz-2w=wz+w-3iz-3i$$

$$w(2z-2-z-1)=-3i(z+1)$$

$$w(z-3)=-3i(z+1)$$

$$\therefore w=-3i\frac{(z+1)}{(z-3)}$$

Find the bilinear transformation which maps the points $z=\infty,i,0$ into $w=0,i,\infty$ respectively.

Solution:

Given $z_1=\infty,\ w_1=0$

$z_2=i,\ w_2=i$

$z_3=0,\ w_3=\infty$

Cross-ratio

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}=\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-w_1)w_3\left(\frac{w_2}{w_3}-1\right)}{w_3\left(\frac{w}{w_3}-1\right)(w_2-w_1)}=\frac{z_1\left(\frac{z}{z_1}-1\right)(z_2-z_3)}{(z-z_3)z_1\left(\frac{z_2}{z_1}-z_1\right)}$$

$$\frac{(w-w_1)\left(\frac{w_2}{w_3}-1\right)}{\left(\frac{w}{w_3}-1\right)(w_2-w_1)}=\frac{\left(\frac{z}{z_1}-1\right)(z_2-z_3)}{(z-z_3)\left(\frac{z_2}{z_1}-1\right)}$$

$$\frac{(w-0)(0-1)}{(0-1)(i-0)}=\frac{(0-1)(i-0)}{(z-0)(0-1)}$$

$$\frac{w}{i}=\frac{i}{z},\qquad w=\frac{i^2}{z},\qquad \therefore w=-\frac{1}{z}$$

Find the bilinear transformation which maps the points z = 1, i, -1 into the points w = 0, 1, ∞.

Solution:

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)}=\frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

Here, $w_3=\infty$