Q 1.

Find all pairs (x, y) of integers satisfying

$$\begin{cases} x^2 + 11 = xy + y^4 \\ y^2 - 30 = xy. \end{cases}$$

Q 2.

Find the sum of the absolute values of the roots of

$$x^4 - 4x^3 - 4x^2 + 16x - 8.$$

Q 3.

Example 2.10. Prove that for any positive integer n, $(n+1)^5 + n$ is not a prime.

Q 4.

Example 2.14. Let a, b, c be distinct, nonzero real numbers. If two fractions among

$$\frac{a^2 - bc}{a(1 - bc)}; \quad \frac{b^2 - ca}{b(1 - ca)}; \quad \frac{c^2 - ab}{c(1 - ab)}$$

are equal, then prove that all of these are equal, and that their common value equals $a + b + c = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$.

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Q 5.

Example 2.15. Prove that for any positive integers m and n, the number

$$8m^6 + 27m^3n^3 + 27n^6$$

is composite.

Q 6.

Example 3.4. Solve the equation

$$\frac{x^2}{a} + \frac{ab^2}{x^2} = 2\sqrt{2ab}\left(\frac{x}{a} - \frac{b}{x}\right),$$

where a, b are positive real numbers.

Q 7.

Example 3.9. Let $f(x) = ax^2 + bx + c$. Suppose that $f(x) = x$ has no real roots. Prove that the equation $f(f(x)) = x$ also has no real roots.

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Q 8.

Example 3.13. Let a, b, c be real numbers. Prove that the system of equations

$$\begin{cases} ax_1^2 + bx_1 + c = x_2 \\ ax_2^2 + bx_2 + c = x_3 \\ \vdots \\ ax_{n-1}^2 + bx_{n-1} + c = x_n \\ ax_n^2 + bx_n + c = x_1 \end{cases}$$

(a) has no real solutions if $(b-1)^2 - 4ac < 0$;
(b) has a unique real solution if $(b-1)^2 - 4ac = 0$;
(c) has more than one real solution if $(b-1)^2 - 4ac > 0$.

Q 9.

Example 3.14. Do there exist quadratic polynomials $f(x) = ax^2 + bx + c$ and $g(x) = (a+1)x^2 + (b+1)x + (c+1)$ with integer coefficients, both of which have two integer roots?

Q 10.

Example 3.17. Find all real numbers m such that

$$x^2 + my^2 - 4my + 6y - 6x + 2m + 8 \geq 0$$

for all real numbers x and y.

Q 11.

Example 3.18. Find the real values of the parameter m such that all the roots of the equation

$$x(x-1)(x-2)(x-3)=m$$

are real.

Q 12.

Example 3.19. Find all real solutions (x, y) of

$$x^4+4x^2y-11x^2+4xy-8x+8y^2-40y+52=0.$$

Q 13.

Example 3.20. Prove that if $x, y, z \in [0,1]$ then

$$x^2+y^2+z^2 \leq x^2y+y^2z+z^2x+1.$$

Q 14.

Example 3.21. Find the maximum value of

$$\frac{1}{x^2-4x+9}+\frac{1}{y^2-4y+9}+\frac{1}{z^2-4z+9}$$

over all triples (x, y, z) of nonnegative real numbers such that $x+y+z=1$.

Q 15.

Solve in real numbers the system of equations

$$\begin{cases} x - y = 2016 \\ \frac{x+y}{2} - \sqrt{xy} = 72. \end{cases}$$

Q 16.

Solve in real numbers the system of equations

$$\begin{cases} x^2 + 7 = 5y - 6z \\ y^2 + 7 = 10z + 3x \\ z^2 + 7 = -x + 3y. \end{cases}$$

Q 17.

Solve in real numbers the system of equations

$$\begin{cases} x^2 + xy + xz = 20 \\ y^2 + yx + yz = 30 \\ z^2 + xz + zy = 50. \end{cases}$$

Q 18.

Solve over nonzero complex numbers the system

$$\begin{cases} \dfrac{x}{y} + \dfrac{y}{x} = \dfrac{9}{z} \\ \dfrac{y}{z} + \dfrac{z}{y} = \dfrac{16}{x} \\ \dfrac{z}{x} + \dfrac{x}{z} = -\dfrac{25}{y}. \end{cases}$$

Q 19.

Example 4.7. Let a, b, c be nonzero real numbers such that $a^2 + b^2 = c^2$. Solve the system of equations

$$\begin{cases} x^2 + y^2 = z^2 \\ (x+a)^2 + (y+b)^2 = (z+c)^2. \end{cases}$$

Q 20.

Solve in real numbers the system of equations

$$\begin{cases} 3x^2 + 2xy - 2y^2 = 1 \\ 3y^2 + 2yz - 2z^2 = -3 \\ 3z^2 + 2zx - 2x^2 = 2. \end{cases}$$

Q 21.

Solve in integers the system of equations

$$\begin{cases} xy - \dfrac{z}{3} = xyz + 1 \\ yz - \dfrac{x}{3} = xyz - 1 \\ zx - \dfrac{y}{3} = xyz - 9. \end{cases}$$

Q 22.

Solve in real numbers the system of equations

$$\begin{cases} x^2 + xy + y^2 = 3 \\ y^2 + yz + z^2 = 7 \\ z^2 + zx + x^2 = 13. \end{cases}$$

Q 23.

Solve in real numbers the system of equations

$$\begin{cases} \left(x^2 + x + \frac{1}{2}\right)\left(y^2 - y + \frac{1}{2}\right) = z^2 \\ \left(y^2 + y + \frac{1}{2}\right)\left(z^2 - z + \frac{1}{2}\right) = x^2 \\ \left(z^2 + z + \frac{1}{2}\right)\left(x^2 - x + \frac{1}{2}\right) = y^2. \end{cases}$$

Q 24.

Example 5.3. Find all quadratic functions $f(x) = ax^2 + bx + c$ such that a, the discriminant, the product of the roots, and the sum of the roots are consecutive integers in this order.

Q 25.

Solve the system of equations

$$\begin{cases} x + y + z = 4 \\ x^2 + y^2 + z^2 = 14 \\ x^3 + y^3 + z^3 = 34. \end{cases}$$

Q 26.

Let a and b be complex numbers. Solve the equation

$$(x-a)^4+(x-b)^4=(a-b)^4.$$

Q 27.

Solve the system of equations over the real numbers:

$$\begin{cases} x+y+z=6 \\ x^2+y^2+z^2=18 \\ \sqrt{x}+\sqrt{y}+\sqrt{z}=4. \end{cases}$$

Q 28.

Solve in complex numbers the system

$$\begin{cases} x+\frac{1}{2}y+\frac{1}{2}z=1 \\ 2x^2+\frac{1}{2}y^2+\frac{1}{2}z^2=1 \\ 4x^3+\frac{1}{2}y^3+\frac{1}{2}z^3=4. \end{cases}$$

Q 29.

Solve the system of equations

$$\begin{cases} x+y+z=5 \\ \dfrac{x}{zy}+\dfrac{y}{zx}+\dfrac{z}{xy}=\dfrac{9}{4} \\ x^3+y^3+z^3-3xyz=5. \end{cases}$$

Q 30.

Example 5.14. If x, y, z are positive real numbers such that $xyz = 1$ and $xy + yz + zx = 5$, prove that

$$\frac{17}{4} \leq x+y+z \leq 1+4\sqrt{2}.$$

Q 31.

Example 5.15. Find all positive integers a, b, c such that the equations

$$x^2-ax+b=0, \quad x^2-bx+c=0, \quad x^2-cx+a=0$$

have integer roots.

Q 32.

Example 5.16. (a) If $a+b+c=0$, prove that

$$\frac{a^5+b^5+c^5}{5}=\frac{a^3+b^3+c^3}{3}\cdot\frac{a^2+b^2+c^2}{2}.$$

(b) If $a+b+c=0$, prove that

$$\frac{a^7+b^7+c^7}{7}=\frac{a^5+b^5+c^5}{5}\cdot\frac{a^2+b^2+c^2}{2}.$$

(c) Let $P_r=a^r+b^r+c^r$ for real numbers a,b,c. If $a+b+c=0$, find all other pairs of positive integers (m,n) such that

$$\frac{P_{m+n}}{m+n}=\frac{P_m}{m}\cdot\frac{P_n}{n}.$$

Q 33.

Example 7.8. Find all real solutions of the equation

$$\sqrt[4]{97-x}+\sqrt[4]{x}=5.$$

Q 34.

Solve in real numbers the equation

$$\sqrt{2x+1}+\sqrt{6x+1}=\sqrt{12x+1}+1.$$

Q 35.

Example 7.12. Solve the equation

$$\sqrt{x^2+x+1}+\sqrt{x^3-x+1}-\sqrt{\frac{x^5+x^4+1}{7}}=\sqrt{7}.$$

Q 36.

Example 7.13. Find all positive real numbers x, y such that

$$(x+y)\left(1+\frac{1}{xy}\right)+4=2(\sqrt{2x+1}+\sqrt{2y+1}).$$

Q 37.

Prove that for all real numbers $a \geq 1$,

$$\sqrt{a-1}+\sqrt{a^2-1} \leq a\sqrt{a}.$$

Q 38.

Example 7.16. Solve the equation

$$3x+\sqrt{2x^2-x}=\sqrt{3x^2+x}+\sqrt{6x^2-x-1}.$$

Q 39.

Example 8.12. Consider the distinct complex numbers a, b, c, d. Prove that the following are equivalent:
(a) For any $z \in \mathbb{C}$, we have $|z-a|+|z-b| \geq |z-c|+|z-d|$.
(b) There exists $t \in (0,1)$ such that $c = ta + (1-t)b$ and $d = (1-t)a + tb$.

Q 40.

Example 8.13. Let a, b, c be distinct real numbers and let n be a positive integer. Find all nonzero complex numbers z such that

$$az^n + b\bar{z} + \frac{c}{z} = bz^n + c\bar{z} + \frac{a}{z} = cz^n + a\bar{z} + \frac{b}{z}.$$

Q 41.

Evaluate the sum

$$\frac{2}{3+1} + \frac{2^2}{3^2+1} + \frac{2^3}{3^4+1} + \cdots + \frac{2^{n+1}}{3^{2^n}+1}.$$

Q 42.

Example 10.16. Let

$$a_k = \frac{k}{(k-1)^{\frac{4}{3}} + k^{\frac{4}{3}} + (k+1)^{\frac{4}{3}}}.$$

Prove that $a_1 + a_2 + \cdots + a_{999} < 50$.

Q 43.

Example 12.1. Prove that for all $a, b, c > 0$, we have

$$\sqrt{a^2 - ab + b^2} + \sqrt{b^2 - bc + c^2} \geq \sqrt{a^2 + ac + c^2}.$$

Q 44.

Solve the following equation over the real numbers:

$$x^2 + \frac{9x^2}{(x+3)^2} = 16.$$

Q 45.

Example 12.3. Let a, b, c be positive real numbers such that $a^2 \geq b^2 + bc + c^2$. Prove that

$$a > \min(b, c) + \frac{|b^2 - c^2|}{a}.$$

Q 46.

Example 12.4. Let a, b, c be positive real numbers and let $u = 2a^2 + 2ab + b^2$, $v = b^2 + c^2$, and $w = c^2 + 2ca + 2a^2$. Prove that $u + v + w = 2$ if and only if $uv + vw + wu = (ab + bc + ca)^2 + 1$.

Q 47.

Let $x, y, z > 1$ such that $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 2$. Prove that

$$\sqrt{x+y+z} \geq \sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1}.$$

Q 48.

Example 12.6. Let a, b, c be fixed positive real numbers. Find all positive real solutions x, y, z to the system

$$\begin{cases} x + y + z = a + b + c \\ 4xyz - (a^2x + b^2y + c^2z) = abc. \end{cases}$$

Q 49.

Example 12.7. Given that x, y, z are real numbers that satisfy

$$x = \sqrt{y^2 - \frac{1}{16}} + \sqrt{z^2 - \frac{1}{16}}$$

$$y = \sqrt{z^2 - \frac{1}{25}} + \sqrt{x^2 - \frac{1}{25}}$$

$$z = \sqrt{x^2 - \frac{1}{36}} + \sqrt{y^2 - \frac{1}{36}},$$

find $x + y + z$.

Q 50.

Solve in positive real numbers the system of equations

$$\begin{cases} x+y+z=xyz \\ 3\left(x+\frac{1}{x}\right)=5\left(y+\frac{1}{y}\right)=7\left(z+\frac{1}{z}\right). \end{cases}$$

Q 51.

Example 12.10. Let $a_0=\sqrt{2}+\sqrt{3}+\sqrt{6}$, and let $a_{n+1}=\frac{a_n^2-5}{2(a_n+2)}$ for $n\geq 0$. Find a closed-form expression for the general term, a_n.

Q 52.

Example 12.11. Let a,b,c be positive real numbers such that

$$a+b+c+1=4abc.$$

Prove that

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq 3\geq \frac{1}{\sqrt{ab}}+\frac{1}{\sqrt{bc}}+\frac{1}{\sqrt{ca}}.$$

Q 53.

Find all real numbers $a,b,c,d,e\in[-2,2]$ such that

$$a+b+c+d+e=0$$

$$a^3+b^3+c^3+d^3+e^3=0$$

$$a^5+b^5+c^5+d^5+e^5=10.$$

Q 54.

Example 12.13. Let $a, b, c > 0$ be real numbers that satisfy

$$a^2 + b^2 + c^2 + abc = 4.$$

Prove that

$$ab + bc + ca - abc \leq 2.$$

Q 55.

Solve in real numbers the equation

$$13x^5 + x^4 + 2x^3 + 2x^2 + x + \frac{1}{5} = 0.$$

Q 56.

Solve in real numbers the system

$$\begin{cases} x + y^2 = y^3 \\ y + x^2 = x^3. \end{cases}$$

Q 57.

Let a be a positive real number such that $\dfrac{a^2}{a^4 - a^2 + 1} = \dfrac{4}{37}$.

Compute $\dfrac{a^3}{a^6 - a^3 + 1}$.

Q 58.

Solve the equation

$$\sqrt[3]{\frac{x^3-3x+2}{x-2}}+\sqrt[3]{\frac{x^3-3x-2}{x+2}}=2\sqrt[3]{x^2-1}.$$

Q 59.

Solve in real numbers the system

$$\begin{cases} 7(a^5+b^5)=31(a^3+b^3) \\ a^3-b^3=3(a-b). \end{cases}$$

Q 60.

Let z be a non-zero complex number with $z^{23}=1$. Evaluate

$$\sum_{k=0}^{22}\frac{1}{1+z^k+z^{2k}}.$$

Q 61.

Solve in real numbers the system of equations

$$\begin{cases} x^2 = y + 2 \\ y^2 = z + 2 \\ z^2 = x + 2. \end{cases}$$

Q 62.

The polynomial $P(x)$ is defined by

$$P(x) = (x + 2x^2 + \cdots + nx^n)^2 = a_0 + a_1 x + \cdots + a_{2n} x^{2n}.$$

Prove that

$$a_{n+1} + a_{n+2} + \cdots + a_{2n} = \frac{n(n+1)(5n^2+5n+2)}{24}.$$

Q 63.

Solve the following equation in integers:

$$3x^3 - x^2 y - xy^2 + 3y^3 = 2013.$$

Q 64.

Let x and y be real numbers such that

$$x^3 + y^3 + (x+y)^3 + 30xy = 2000.$$

Prove that $x + y = 10$.

Q 65.

Solve in real numbers the system of equations

$$\begin{cases} (xy)^{\log z} + (yz)^{\log x} = 1.001 \\ (yz)^{\log x} + (zx)^{\log y} = 10.001 \\ (zx)^{\log y} + (xy)^{\log z} = 11. \end{cases}$$

Q 66.

Solve in real numbers the system

$$\begin{cases} x^3 = 3x + y \\ y^3 = 3y + z \\ z^3 = 3z + x. \end{cases}$$

Q 67.

Let z_1, z_2, z_3, z_4 be the complex roots of the equation

$$z^4 + az^3 + az + 1 = 0,$$

where a is a real number such that $|a| \leq 1$. Prove that

$$|z_1| = |z_2| = |z_3| = |z_4| = 1.$$

Q 68.

Find the minimum of $2^x - 4^x + 6^x - 8^x - 9^x + 12^x$ over all $x \in \mathbb{R}$.

Q 69.

Find all real numbers x, y greater than 1 such that the numbers

$$\sqrt{x-1} + \sqrt{y-1} \quad \text{and} \quad \sqrt{x+1} + \sqrt{y+1}$$

are nonconsecutive integers.

Q 70.

Solve in nonnegative real numbers the system of equations

$$\begin{cases} (x+1)(y+1)(z+1) = 5 \\ (\sqrt{x} + \sqrt{y} + \sqrt{z})^2 - \min(x, y, z) = 6. \end{cases}$$

Q 71.

Let a, b, c, d be real numbers such that

$$a + b + c + d = a^7 + b^7 + c^7 + d^7 = 0.$$

Prove that $(a + b)(a + c)(a + d) = 0$.

Q 72.

Let k be an integer and let

$$n = \sqrt[3]{k + \sqrt{k^2 - 1}} + \sqrt[3]{k - \sqrt{k^2 - 1}} + 1.$$

Prove that $n^3 - 3n^2$ is an integer.

Q 73.

Let a, b, c, d, e be integers such that

$$a(b + c) + b(c + d) + c(d + e) + d(e + a) + e(a + b) = 0.$$

Prove that $a + b + c + d + e$ divides $a^5 + b^5 + c^5 + d^5 + e^5 - 5abcde$.

Q 74.

Solve the system of equations over the real numbers

$$\begin{cases} \sqrt{xy} - \sqrt{(1-x)(1-y)} = \frac{\sqrt{5}+1}{4} \\ \sqrt{x(1-y)} - \sqrt{y(1-x)} = \frac{\sqrt{5}-1}{4}. \end{cases}$$

Q 75.

Solve the equation

$$x+\sqrt{(x+1)(x+2)}+\sqrt{(x+2)(x+3)}+\sqrt{(x+3)(x+1)}=4$$

over all $x \geq -1$.

Q 76.

Solve in real numbers the system

$$\begin{cases} ab(a+b)+bc(b+c)+ca(c+a)=2 \\ ab+bc+ca=-1 \\ ab(a^2+b^2)+bc(b^2+c^2)+ca(c^2+a^2)=-2. \end{cases}$$

Q 77.

If $P(x), Q(x), R(x)$, and $S(x)$ are all polynomials such that

$$P(x^5)+xQ(x^5)+x^2R(x^5)=(x^4+x^3+x^2+x+1)S(x),$$

prove that $x-1$ is a factor of $P(x)$.

Q 78.

Example 2. Suppose a, b, $c \in R$, $f(x) = ax^2 + bx + c$, and $g(x) = ax + b$. Given that $|f(x)| \leq 1$ when $x \in [-1, 1]$, answer the following questions:

(1) Show that $|c| \leq 1$.
(2) Show that $|g(x)| \leq 2$ for $x \in [-1, 1]$.
(3) Further suppose $a > 0$ and the maximum value of $g(x)$ for $x \in [-1, 1]$ is 2. Find $f(x)$.

Q 79.

Example 3. Suppose the parabola graph of $y = x^2 - (k-1)x - k - 1$ intersects the x-axis at two points A and B, and its vertex is C. Find the minimal area of the triangle ABC.

Q 80.

Example 4. Suppose the quadratic function $f(x) = ax^2 + (2b+1)x - a - 2$ ($a, b \in R$, and $a \neq 0$) has at least one zero in the interval [3,4]. Find the minimum value of $a^2 + b^2$.

Q 81.

Example 7 (2017 China Mathematical Competition). Suppose k and m are real numbers such that the inequality $|x^2 - kx - m| \leq 1$ holds for all $x \in [a, b]$. Show that $b - a \leq 2\sqrt{2}$.

Q 82.

Let $f(x)$ be a quadratic function such that

(1) $f(-1) = 0$;
(2) the inequality $x \leq f(x) \leq \frac{1+x^2}{2}$ holds for all real values of x.

Find the formula of $f(x)$.

Q 83.

Suppose a and b are real numbers such that one root of the equation $x^2 - ax + b = 0$ lies in the interval $[-1, 1]$, while the other lies in $[1, 2]$. Find the value range of $a - 2b$.

Q 84.

(2013 China Mathematical Competition) Find all pairs of positive real numbers (a, b) such that the function $f(x) = ax^2 + b$ has the following property: the inequality

$$f(xy) + f(x + y) \geq f(x)f(y)$$

holds for all real numbers x and y.

Q 85.

Example 2. Suppose the graph of a quadratic function $y = x^2 + bx + c$ has the vertex D, intersects the $x-$axis at A and B (such that A lies on the left), and intersects the $y-$axis at C. If ΔABD and ΔOBC are both isosceles right triangles (here O is the origin), find the value of $b + 2c$.

Q 86.

Example 3. Let a and b be real constants. Suppose for all real numbers k, the graph of

$$y = (k^2 + k + 1)x^2 - 2(a + k)^2 x + (k^2 + 3ak + b)$$

always passes through $A(1, 0)$.

(1) Find the values of a and b;
(2) if B is the other intersection point of the graph of the function and the $x-$axis, find the maximum value of $|AB|$ as k varies.

Q 87.

Example 4. Let a function $f(x) = ax^2 + bx + c$. Suppose $|f(x)| \leq 1$ for all $x \in [-1, 1]$. Show that when $x \in [-1, 1]$, the following inequality always holds:

$$|ax + b| + |c| \leq 3.$$

Q 88.

Example 5. For a function $f(x)$, if $f(x) = x$, then x is called a "fixed point" of $f(x)$. If $f(f(x)) = x$, then x is called a "stable point" of $f(x)$. Let A and B be the set of fixed points and the set of stable points of a function $f(x)$, respectively. (In other words, $A = \{x|f(x) = x\}$ and $B = \{x|f(f(x)) = x\}$.)

(1) Show that $A \subseteq B$;
(2) if $f(x) = ax^2 - 1$ ($a \in \mathbf{R}$ and $x \in \mathbf{R}$), and $A = B \neq \emptyset$, find the value range of a.

Q 89.

Example 6. Let $f(x) = ax^2 + (b+1)x + (b-1)(a \neq 0)$.

(1) If $a = 1$ and $b = -2$, find the fixed points of $f(x)$;
(2) if for all real numbers b, the function $f(x)$ always has two distinct fixed points, find the value range of a;
(3) suppose the condition in (2) holds, and A and B are the points on the graph of $y = f(x)$ that correspond to the fixed points such that A and B are symmetric with respect to the line $y = kx + \frac{1}{2a^2+1}$ (k is a real number). Find the minimum value of b.

Q 90.

Example 7. Suppose $f(x) = ax^2+bx+c$ satisfies $|f(x)| \leq 1$ for all $x \in [0,1]$. Find the maximum value of $|a| + |b| + |c|$.

Q 91.

Example 9. Determine whether there exists a quadratic function $f(x)$ such that for each positive integer k, if $x = \underbrace{55\cdots5}_{k \text{ copies}}$, then $f(x) = \underbrace{55\cdots5}_{2k \text{ copies}}$.

Explain the reason.

Q 92.

Example 10. Suppose there are two points $A(m_1, f(m_1))$ and $B(m_2, f(m_2))$ on the graph of $f(x) = ax^2 + bx + c (a > b > c)$ such that

$$a^2 + [f(m_1) + f(m_2)]a + f(m_1)f(m_2) = 0,$$
$$f(1) = 0.$$

(1) Show that $b \geq 0$;
(2) show that the value range of the distance between the two intersection points of the graph of $f(x)$ and the $x-$axis is $[2, 3)$;
(3) is it true that at least one of $f(m_1+3)$ and $f(m_2+3)$ is positive? Prove your result.

Q 93.

Suppose real numbers $a, b,$ and c satisfy

$$\begin{cases} a^2 - bc - 8a + 7 = 0, \\ b^2 + c^2 + bc - 6a + 6 = 0. \end{cases}$$

Find the value range of a.

Q 94.

Let a and b be real constants. Suppose $f(x) = x^2 + 2bx + 1$ and $g(x) = 2a(x + b)$. We view each pair of real numbers (a, b) as a point in the $a - b$ coordinate plane, and let S be the set of points (a, b) such that the graphs of $y = f(x)$ and $y = g(x)$ have no common point. Find the area of S.

Q 95.

Let $f(x) = ax^2 + bx + c$ be such that $f(x) \in [-1, 1]$ for all $x \in [-1, 1]$. Show that $f(x) \in [-7, 7]$ for all $x \in [-2, 2]$.

Q 96.

In the coordinate plane, the points whose coordinates are both integers are called grid points. Find all the grid points on the graph of $y = \frac{x^2}{10} - \frac{x}{10} + \frac{9}{5}$ such that $y \leq |x|$ and explain why there are no more of them.

Q 97.

Let $f(x)$ be a function defined on $(-\infty, +\infty)$, which is periodic with period 2. For $k \in \mathbf{Z}$, let I_k denote the interval $(2k-1, 2k+1]$. Suppose $f(x) = x^2$ for $x \in I_0$.

(1) Find the formula of $f(x)$ on I_k;
(2) for a positive integer k, find the set (explicitly)

$$M_k = \{a | \text{the equation } f(x) = ax \text{ has two distinct real roots on } I_k\}.$$

Q 98.

Let $f(x) = ax^2 + bx + c$ be such that $|f(x)| \leq 1$ for all $|x| \leq 1$. Show that $|2ax + b| \leq 4$ when $|x| \leq 1$.

Q 99.

Suppose $|ax^2 + bx + c| \leq 1$ for all $x \in [-1, 1]$. Show that $|cx^2 \pm bx + a| \leq 2$ for all $x \in [-1, 1]$.

Q 100.

Let $a, b,$ and c be positive integers, and A and B be the (distinct) intersection points of the graph of $y = ax^2 + bx + c$ with the $x-$axis. If the distances from A and B to the origin are both less than 1, find the minimum value of $a + b + c$.

Q 101.

Problem. Prove that there exist integers a, b, c, d greater than 2007 such that

$$a^2 + b^2 + c^2 + d^2 = abcd + 6.$$

Q 102.

Problem. Let P be a polynomial with real coefficients such that $P(x) > 0$ for all $x > 0$. Prove that there exist polynomials Q, R with nonnegative coefficients such that

$$P(X) = \frac{Q(X)}{R(X)}.$$

Q 103.

Problem. Find all positive rational numbers x, y, z such that

$$x + \frac{1}{y}, \quad y + \frac{1}{z}, \quad z + \frac{1}{x}$$

are all integers.

Q 104.

Let k be a nonzero integer, and define $\lambda = k + \sqrt{k^2 - 1}$. Prove that for every integer n, $\lambda^n + \dfrac{1}{\lambda^n}$ is an even integer.

Q 105.

We say that the equations $x^2 + ax + b = 0$ and $x^2 + cx + d = 0$ are *friends* if they have real and separated roots. More precisely, if we denote by $x_1 < x_2$ and $x_3 < x_4$ the roots of the first, respectively of the second equation, then

$$x_1 < x_3 < x_2 < x_4 \quad \text{or} \quad x_3 < x_1 < x_4 < x_2.$$

Prove that the equation

$$x^2 + \left(\frac{a+c}{2}\right)x + \left(\frac{b+d}{2}\right) = 0$$

has real roots and it is friends with each of the first two equations.

Q 106.

For real numbers α, β denote

$$\mathcal{M}(\alpha, \beta) = \{x \in \mathbb{R} \mid x^2 + \alpha x + \beta = 0\}.$$

Let a, b, c be integers. Prove that if

$$\mathcal{M}(a, b) \cup \mathcal{M}(b, c) \cup \mathcal{M}(c, a) = \emptyset,$$

then $a = b = c$.

Q 107.

For real numbers α, β, γ denote

$$\mathcal{M}(\alpha, \beta, \gamma) = \{x \in \mathbb{R} \mid \alpha x^2 + \beta x + \gamma = 0\}.$$

Let a, b, c be nonzero real numbers. Prove that if

$$\mathcal{M}(a, b, c) \cap \mathcal{M}(b, c, a) \cap \mathcal{M}(c, a, b) \neq \emptyset,$$

then the set

$$\mathcal{M}(a, b, c) \cup \mathcal{M}(b, c, a) \cup \mathcal{M}(c, a, b)$$

has three or four elements.

Q 108.

Let a, b, m, n be real numbers such that

$$m^2 + n^2 - a(m+n) + 2b = 0.$$

Prove that $(m+n+a)^2 \geq 8(mn+b)$.

Q 109.

Let n be a positive integer. Denote by $\mathcal{F}_n$ the set of all quadratic functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the relation

$$f(f(1)) = f(f(2)) = \ldots = f(f(n)).$$

a) Prove that $\mathcal{F}_n = \emptyset$ for all $n \geq 5$.

b) Determine $\mathcal{F}_4$.

Q 110.

Let a, b, c, d be real numbers such that $ad > 0$ and let x_0 be a real root of the third degree equation

$$ax^3 + bx^2 + cx + d = 0.$$

Prove that $x_0 \leq \dfrac{c^2 - 4bd}{4ad}$.

Q 111.

Let a, b, c be real numbers, $a \neq 0$. Prove that if a and $4a + 3b + 2c$ have the same sign, then the quadratic equation $ax^2 + bx + c = 0$ cannot have both roots in the interval $(1, 2)$.

Q 112.

Let a, b, c be real numbers and let $f : \mathbb{R} \to \mathbb{R}$ be a quadratic function with integer coefficients such that

$$|f(k)| < ak^2 + bk + c + 1,$$

for all integers k. Prove that $b^2 - 4ac < 9a^2$.

Q 113.

Let a, b, c, d be real numbers and $f : \mathbb{R} \to \mathbb{R}$ be the function given by

$$f(x) = ax^3 + bx^2 + cx + d.$$

Prove that if

$$f(2) + f(5) < 7 < f(3) + f(4),$$

then there exist two real numbers u and v such that

$$u + v = 7 \quad \text{and} \quad f(u) + f(v) = 7.$$

Q 114.

Let $g, h : \mathbb{R} \to \mathbb{R}$ be two quadratic functions and let $f : \mathbb{R} \to \mathbb{R}$ be such that $f \circ g = h$. Prove that there exist two real numbers m, n and an unbounded interval I such that $f(y) = my + n$, for all $y \in I$.

Q 115.

Let us denote by $\mathcal{Q}$ the set of all quadratic functions. Prove that if a function $f : \mathbb{R} \to \mathbb{R}$ satisfies the implication:

$$g \in \mathcal{Q} \Rightarrow f \circ g \in \mathcal{Q},$$

then $f(x) = mx + n$, for some real numbers m, n.

Q 116.

Let $f, g : \mathbb{R} \to \mathbb{R}$ be two quadratic functions such that if $g(x)$ is an integer, then $f(x)$ is an integer as well. Prove that there exist two integers m and n such that $f(x) = mg(x) + n$, for all real numbers x.

Q 117.

Find all functions $f : [0, \infty) \to [0, \infty)$ which satisfy the relation

$$f(x^2 + x) \leq x \leq f^2(x) + f(x),$$

for all nonnegative real numbers x.

Q 118.

Let f be a quadratic function such that

$$0 \leq f(-1) \leq 1, \quad 0 \leq f(0) \leq 1, \quad 0 \leq f(1) \leq 1.$$

Prove that $f(x) \leq \frac{9}{8}$, for all real numbers $x \in [-1, 1]$.

Q 119.

Let $x \in [0,1]$, $y \in [1,2]$, $z \in [2,3]$ be real numbers. Prove that:

$$\frac{3}{4} \leq x^2 + y^2 + z^2 - xy - yz - zx \leq 7,$$

then find the equality cases.

Q 120.

Let a, b, c be positive integers such that $b > a^2 + c^2$. Prove that the roots of the quadratic equation $ax^2 + bx + c = 0$ are irrational.

Q 121.

Let a, b, x, y be real numbers, $x, y > 0$, $x \neq y$. Prove that if the equality

$$x^n + y^n = a(x^{n-1} + y^{n-1}) + b(x^{n-2} + y^{n-2})$$

holds for $n = k$ and $n = k + 1$, then it holds for all integers n.

Q 122.

Let $a > 2$ be rational. Prove that any integer solution of the equation

$$x^2 - a(a^2 - 3)x + 1 = 0$$

is the cube of an integer.

Q 123.

Solve in the set of positive real numbers the system

$$\begin{cases} x^2 + y^2 + z^2 + xyz = 4 \\ x + y + z = 3 \end{cases}.$$

Q 124.

Let x, y, z be positive reals such that $x^2 + y^2 + z^2 + xyz = 4$. Prove that

$$\sqrt{\frac{(2-x)(2-y)}{(2+x)(2+y)}} + \sqrt{\frac{(2-y)(2-z)}{(2+y)(2+z)}} + \sqrt{\frac{(2-z)(2-x)}{(2+z)(2+x)}} = 1.$$

Q 125.

Let x, y, z be positive real numbers such that $x + y + z = xyz$. Prove that

$$\frac{1}{1+x^2} + \frac{1}{1+y^2} + \frac{1}{1+z^2} + \frac{2}{\sqrt{(1+x^2)(1+y^2)(1+z^2)}} = 1.$$

Q 126.

Let a, b, c be positive reals such that $a + b + c + 2\sqrt{abc} = 1$. Prove that

$$\sqrt{\frac{1-a}{a}} + \sqrt{\frac{1-b}{b}} + \sqrt{\frac{1-c}{c}} = \sqrt{\frac{1-a}{a}} \cdot \sqrt{\frac{1-b}{b}} \cdot \sqrt{\frac{1-c}{c}}.$$

Q 127.

Prove that the equation $x^2 + y^2 + z^2 + xyz = 4$ has infinitely many solutions in integers.

Q 128.

Let a, b, c be nonnegative real numbers such that

$$a^2 + b^2 + c^2 + abc = 4.$$

Prove that $0 \leq ab + bc + ca - abc \leq 2$.

Q 129.

Let a, b, c be nonnegative real numbers such that $x^2+y^2+z^2+2xyz = 1$. Prove that $xyz \leq 1/8$.

Q 130.

Let x, y, z be positive real numbers fulfilling the condition

$$x^2 + y^2 + z^2 = xyz.$$

Prove that the inequality

$$xy + xz + yz \geq 4\left(x + y + z\right) - 9$$

holds.

Q 131.

Let $f(x) = x^2 + ax + b$ be a quadratic function with integer coefficients having the property that the values of f for any two consecutive integers are consecutive squares. Prove that $f(n)$ is a square for any integer n.

Q 132.

Let f, g, and h be three quadratic functions such that each of $f - g$, $g - h$, $h - f$ is still a quadratic function (thus with nonzero coefficient of x^2) having equal zeros. Prove that the functions $f - g$, $g - h$, and $h - f$ have a common zero.

Q 133.

Let $f(x) = x^2 + ax + b$ and $g(x) = x^2 + cx + d$ be two quadratic functions with real coefficients. We say that the corresponding equations $f(x) = 0$ and $g(x) = 0$ are *friends* if they have real and separated roots (see also problem 2 in the chapter *Quadratic Functions and Quadratic Equations*). Prove that the equations $f(x) = 0$ and $g(x) = 0$ are friends if and only if

$$R = (b - d)^2 + (a - c)(ad - bc) < 0.$$

(refer to Prob 105 for more details on “friendly” quadratics)

Q 134.

Find the maximum real number λ that satisfies the following conditions: for any positive real numbers p, q, r, s, there exists a complex number $z = a + bi$ $(a, b \in \mathbb{R})$ such that $|b| \geq \lambda |a|$ and

$$(pz^3 + 2qz^2 + 2rz + s)(qz^3 + 2pz^2 + 2sz + r) = 0.$$

(Contributed by He Yijie)

Q 135.

Find integers a, b, c such that $a \neq 0$ and the quadratic function $f(x) = ax^2 + bx + c$ satisfies

$$f(f(1)) = f(f(2)) = f(f(3)) \ .$$

Q 136.

Let a and b be integers. How many solutions in real pairs (x, y) does the system

$$\lfloor x \rfloor + 2y = a$$

$$\lfloor y \rfloor + 2x = b$$

have?

Q 137.

Find all ordered pairs (x, y) that are solutions of the following system of two equations (where a is a parameter):

$$x - y = 2$$
$$\left(x - \frac{2}{a}\right)\left(y - \frac{2}{a}\right) = a^2 - 1 \; .$$

Find all values of the parameter a for which the solutions of the system are two pairs of nonnegative numbers. Find the minimum value of $x + y$ for these values of a.

Q 138.

Solve the equation:

$$\left(\sqrt{2+\sqrt{2}}\right)^x + \left(\sqrt{2-\sqrt{2}}\right)^x = 2^x \; .$$

Q 139.

Solve the system of equations:

$$\log x + \frac{\log(xy^8)}{\log^2 x + \log^2 y} = 2 \; ,$$
$$\log y + \frac{\log(x^8/y)}{\log^2 x + \log^2 y} = 0 \; .$$

(The logarithms are taken to base 10.)

Q 140.

Find all pairs (a, b) of positive integers with $a \neq b$ for which the system

$$\cos ax + \cos bx = 0$$
$$a \sin ax + b \sin bx = 0$$

has a solution. If so, determine its solutions.

Q 141.

Suppose that

$$\frac{x^2+y^2}{x^2-y^2}+\frac{x^2-y^2}{x^2+y^2}=k\ .$$

Find, in terms of k, the value of the expression

$$\frac{x^8+y^8}{x^8-y^8}+\frac{x^8-y^8}{x^8+y^8}\ .$$

Q 142.

Prove that the equation

$$x^4+5x^3+6x^2-4x-16=0$$

has exactly two real solutions.

Q 143.

Let a be a real number. Solve the equation

$$(a-1)\left(\frac{1}{\sin x}+\frac{1}{\cos x}+\frac{1}{\sin x\cos x}\right)=2\ .$$

Q 144.

Simplify the expression

$$\sqrt[5]{3\sqrt{2}-2\sqrt{5}}\cdot\sqrt[10]{\frac{6\sqrt{10}+19}{2}}\ .$$

Q 145.

Prove that, for any natural number n, the equation

$$x(x+1)(x+2)\cdots(x+2n-1)+(x+2n+1)(x+2n+2)\cdots(x+4n)=0$$

does not have real solutions.

Q 146.

Find all real numbers x that satisfy the equation

$$3^{[(1/2)+\log_3(\cos x+\sin x)]}-2^{\log_2(\cos x-\sin x)}=\sqrt{2}\ .$$

[The logarithms are taken to bases 3 and 2 respectively.]

Q 147.

Prove that if the quadratic equation $x^2+ax+b+1=0$ has nonzero integer solutions, then a^2+b^2 is a composite integer.

Q 148.

Let $f(x)$ be a polynomial with real coefficients for which the equation $f(x)=x$ has no real solution. Prove that the equation $f(f(x))=x$ has no real solution either.

Q 149.

Solve, for real x,

$$x\cdot 2^{1/x}+\frac{1}{x}\cdot 2^{x}=4\ .$$

Q 150.

Suppose that p is a real parameter and that

$$f(x) = x^3 - (p+5)x^2 - 2(p-3)(p-1)x + 4p^2 - 24p + 36 .$$

(a) Check that $f(3-p) = 0$.

(b) Find all values of p for which two of the roots of the equation $f(x) = 0$ (expressed in terms of p) can be the lengths of the two legs in a right-angled triangle with a hypotenuse of $4\sqrt{2}$.

Q 151.

Find all integer values of the parameter a for which the equation

$$|2x+1| + |x-2| = a$$

has exactly one integer among its solutions.

Q 152.

Determine five values of p for which the polynomial $x^2 + 2002x - 1002p$ has integer roots.

Q 153.

Determine all integers x and y that satisfy the equation $x^3 + 9xy + 127 = y^3$.

Q 154.

(a) Solve the following system of equations:

$$(1+4^{2x-y})(5^{1-2x+y}) = 1+2^{2x-y+1} \ ;$$
$$y^2+4x = \log_2(y^2+2x+1) \ .$$

(b) Solve for real values of x:

$$3^x \cdot 8^{x/(x+2)} = 6 \ .$$

Q 155.

Solve the equation

$$5\sin x + \frac{5}{2\sin x} - 5 = 2\sin^2 x + \frac{1}{2\sin^2 x} \ .$$

Q 156.

Solve the equation

$$\tan^2 2x = 2\tan 2x \tan 3x + 1 \ .$$

Q 157.

Let $P(x)$ be the polynomial

$$P(x) = x^{15} - 2004x^{14} + 2204x^{13} - \cdots - 2004x^2 + 2004x \ ,$$

Calculate $P(2003)$.

Q 158.

The real numbers u and v satisfy

$$u^3 - 3u^2 + 5u - 17 = 0$$

and

$$v^3 - 3v^2 + 5v + 11 = 0 .$$

Determine $u + v$.

Q 159.

Solve for positive real values of x, y, t:

$$(x^2 + y^2)^2 + 2tx(x^2 + y^2) = t^2y^2 .$$

Are there infinitely many solutions for which the values of x, y, t are all positive integers?

Optional rider: What is the smallest value of t for a positive integer solution?

Q 160.

Does the equation

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{abc} = \frac{12}{a + b + c}$$

have infinitely many solutions in positive integers a, b, c?

Q 161.

Let a, b, c be integers with $abc \neq 0$, and u, v, w be integers, not all zero, for which

$$au^2 + bv^2 + cw^2 = 0 \ .$$

Let r be any rational number. Prove that the equation

$$ax^2 + by^2 + cz^2 = r$$

is solvable.

Q 162.

Prove that there are infinitely many solutions in positive integers of the system

$$a + b + c = x + y$$
$$a^3 + b^3 + c^3 = x^3 + y^3 \ .$$

Q 163.

Find all integers x which satisfy the equation

$$\cos\left(\frac{\pi}{8}(3x - \sqrt{9x^2 + 160x + 800})\right) = 1 \ .$$

Q 164.

Eliminate θ from the two equations

$$x = \cot\theta + \tan\theta$$

$$y = \sec\theta - \cos\theta \ ,$$

to get a polynomial equation satisfied by x and y.

Q 165.

Suppose that x and y are positive real numbers. Find all real solutions of the equation

$$\frac{2xy}{x+y} + \sqrt{\frac{x^2+y^2}{2}} = \sqrt{xy} + \frac{x+y}{2} .$$

Q 166.

Factor each of the following polynomials as a product of polynomials of lower degree with integer coefficients:

(a) $(x+y+z)^4 - (y+z)^4 - (z+x)^4 - (x+y)^4 + x^4 + y^4 + z^4$;

(b) $x^2(y^3 - z^3) + y^2(z^3 - x^3) + z^2(x^3 - y^3)$;

(c) $x^4 + y^4 - z^4 - 2x^2y^2 + 4xyz^2$;

(d) $(yz + zx + xy)^3 - y^3z^3 - z^3x^3 - x^3y^3$;

(e) $x^3y^3 + y^3z^3 + z^3x^3 - x^4yz - xy^4z - xyz^4$;

(f) $2(x^4 + y^4 + z^4 + w^4) - (x^2 + y^2 + z^2 + w^2)^2 + 8xyzw$;

(g) $6(x^5 + y^5 + z^5) - 5(x^2 + y^2 + z^2)(x^3 + y^3 + z^3)$.

Q 167.

Determine necessary and sufficient conditions on the real parameter a, b, c that

$$\frac{b}{cx+a} + \frac{c}{ax+b} + \frac{a}{bx+c} = 0$$

has exactly one real solution.

Q 168.

Is there a pair of natural numbers, x and y, for which

(a) $x^3 + y^4 = 2^{2003}$?

(b) $x^3 + y^4 = 2^{2005}$?

Provide reasoning for your answers to (a) and (b).

Q 169.

Solve the system of equations

$$x + \frac{1}{y} = y + \frac{1}{z} = z + \frac{1}{x} = t$$

for x, y, z not all equal. Determine xyz.

Q 170.

Find the exact value of:

(a)

$$\sqrt{\frac{1}{6} + \frac{\sqrt{5}}{18}} - \sqrt{\frac{1}{6} - \frac{\sqrt{5}}{18}}\ ;$$

(b)

$$\sqrt{1 + \frac{2}{5}} \cdot \sqrt{1 + \frac{2}{6}} \cdot \sqrt{1 + \frac{2}{7}} \cdot \sqrt{1 + \frac{2}{8}} \cdots \sqrt{1 + \frac{2}{57}} \cdot \sqrt{1 + \frac{2}{58}}\ .$$

Q 171.

Prove that the equation

$$x^2 + 2y^2 + 98z^2 = 77777\ldots777$$

does not have a solution in integers, where the right side has 2006 digits, all equal to 7.

Q 172.

A high school student asked to solve the surd equation

$$\sqrt{3x-2} - \sqrt{2x-3} = 1$$

gave the following answer: *Squaring both sides leads to*

$$3x - 2 - 2x - 3 = 1$$

so $x = 6$. The answer is, in fact, correct.

Show that there are infinitely many real quadruples (a, b, c, d) for which this method leads to a correct solution of the surd equation

$$\sqrt{ax-b} - \sqrt{cx-d} = 1\ .$$

Q 173.

Given two natural numbers x and y for which

$$3x^2 + x = 4y^2 + y\ ,$$

prove that their positive difference is a perfect square. Determine a nontrivial solution of this equation

Q 174.

Solve for t in terms of a, b in the equation

$$\sqrt{\frac{t^3+a^3}{t+a}} + \sqrt{\frac{t^3+b^3}{t+b}} = \sqrt{\frac{a^3-b^3}{a-b}}$$

where $0 < a < b$.

Q 175.

Find all integers x for which

$$(4-x)^{4-x} + (5-x)^{5-x} + 10 = 4^x + 5^x \ .$$

Q 176.

Solve the equation for positive real x:

$$(2^{\log_5 x} + 3)^{\log_5 2} = x - 3 \ .$$

Q 177.

Solve the equation

$$\sqrt{2+\sqrt{2+\sqrt{2+x}}} + \sqrt{3}\sqrt{2-\sqrt{2+\sqrt{2+x}}} = 2x$$

for $x \geq 0$

Q 178.

(a) Find all real numbers x that satisfy the equation

$$(8x-56)\sqrt{3-x} = 30x - x^2 - 97 \ .$$

(b) Find all real numbers x that satisfy the equation

$$\sqrt{x} + \sqrt[3]{x+7} = \sqrt[4]{x+80} \ .$$

Q 179.

Let a be a real parameter. Consider the simultaneous sytem of two equations:

$$\frac{1}{x+y} + x = a - 1 \ ; \tag{1}$$

$$\frac{x}{x+y} = a - 2 \ . \tag{2}$$

(a) For what value of the parameter a does the system have exactly one solution?

(b) Let $2 < a < 3$. Suppose that (x, y) satisfies the sytem. For which value of a in the stated range does $(x/y) + (y/x)$ reach its maximum value?

Q 180.

Solve the equation

$$\sqrt[3]{x^2+2} + \sqrt[3]{4x^2+3x-2} = \sqrt[3]{3x^2+x+5} + \sqrt[3]{2x^2+2x-5} \ .$$

Q 181.

Solve the equation

$$2^{1-2\sin^2 x} = 2 + \log_2(1 - \sin^2 x) \ .$$

Q 182.

Solve the irrational equation

$$\frac{7}{\sqrt{x^2-10x+26} + \sqrt{x^2-10x+29} + \sqrt{x^2-10x+41}} = x^4 - 9x^3 + 16x^2 + 15x + 26 \ .$$

Q 183.

Solve the system of equations

$$\lfloor x \rfloor + 3\{y\} = 3.9 \ ,$$
$$\{x\} + 3\lfloor y \rfloor = 3.4 \ .$$

Q 184.

Determine all real pairs (x, y) that satisfy the system of equations:

$$3\sqrt[3]{x^2y^5} = 4(y^2 - x^2)$$
$$5\sqrt[3]{x^4y} = y^2 + x^2 \ .$$

Q 185.

Solve the equation

$$\left(\frac{1}{10}\right)^{\log_{(1/4)}(\sqrt[4]{x}-1)} - 4^{\log_{10}(\sqrt[4]{x}+5)} = 6 \ ,$$

for $x \geq 1$.

Q 186.

Solve, for real x, y, z the equation

$$\frac{y^2 + z^2 - x^2}{2yz} + \frac{z^2 + x^2 - y^2}{2zx} + \frac{x^2 + y^2 - z^2}{2xy} = 1 \ .$$

Q 187.

Solve the equation

$$\tan 2x \tan\left(2x + \frac{\pi}{3}\right) \tan\left(2x + \frac{2\pi}{3}\right) = \sqrt{3}\ .$$

Q 188.

Find all pairs of natural numbers (x, y) that satisfy the equation

$$2x(xy - 2y - 3) = (x + y)(3x + y)\ .$$

Q 189.

Determine the number of distinct solutions x with $0 \leq x \leq \pi$ for each of the following equations. Where feasible, give an explicit representation of the solution.

(a) $8 \cos x \cos 2x \cos 4x = 1$;

(b) $8 \cos x \cos 4x \cos 5x = 1$.

Q 190.

Solve the equation

$$\sin x \left(1 + \tan x \tan\frac{x}{2}\right) = 4 - \cot x\ .$$

Q 191.

Given the parameters a, b, c, solve the system

$$x + y + z = a + b + c;$$
$$x^2 + y^2 + x^2 = a^2 + b^2 + c^2;$$
$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 3 \ .$$

Q 192.

Solve the following system for real values of x and y:

$$2^{x^2+y} + 2^{x+y^2} = 8$$
$$\sqrt{x} + \sqrt{y} = 2 \ .$$

Q 193.

Let $f(x)$ be a quadratic polynomial. Prove that there exist quadratic polynomials $g(x)$ and $h(x)$ for which

$$f(x)f(x+1) = g(h(x)) \ ,$$

Q 194.

Solve the equation

$$\sqrt{6 + 3\sqrt{2 + \sqrt{2 + x}}} + \sqrt{2 - \sqrt{2 + \sqrt{2 + x}}} = 2x \ .$$

Q 195.

Solve the equation

$$2010^x + 2010^{-x} = 1 + 2x - x^2 \ .$$

Q 196.

Problem 1. Let a, b, c, d be distinct non-zero real numbers satisfying the following two conditions:

$$ac = bd$$
$$\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a} = 4.$$

Determine the largest possible value of the expression $\frac{a}{c} + \frac{c}{a} + \frac{b}{d} + \frac{d}{b}$.

Q 197.

Find all pairs of positive integers m and n such that

$$m! + n! = m^n.$$

Q 198.

For which positive integers m does the equation:

$$(ab)^{2015} = (a^2 + b^2)^m$$

have positive integer solutions?

Q 199.

Find all positive integers n for which the equation

$$(x^2 + y^2)^n = (xy)^{2016}$$

has positive integer solutions.

Q 200.

Let $P(x) = x^3 - 2x + 1$ and let $Q(x) = x^3 - 4x^2 + 4x - 1$. Show that

if $P(r) = 0$ then $Q(r^2) = 0$.