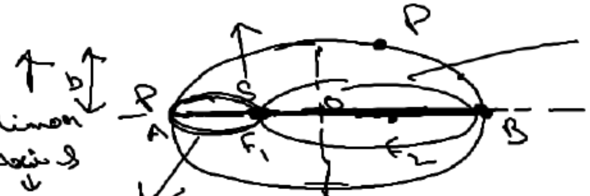
Gravitation

* Kepler's law of Planetary motion —

→ 1st law (law of orbit)

focus → ae



Aphelion

2b = Minor Axis

Perihelion ← Major axis → 2a

$e = 1 \Rightarrow$ Parabola
$e < 1 \rightarrow$ ellipse
$e > 1 \rightarrow$ hyperbola

$AS = AO - SO$

$= a - ae$

$$\boxed{AS = a(1-e)}$$

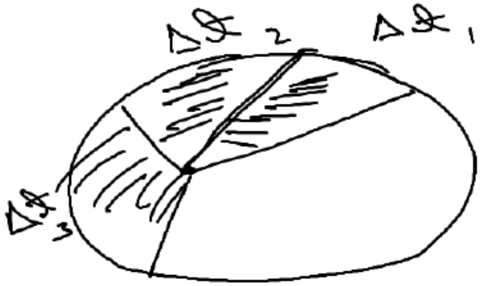
e → eccentricity

$SB = SO + OB$

$= ae + a$

$= a(1+e)$

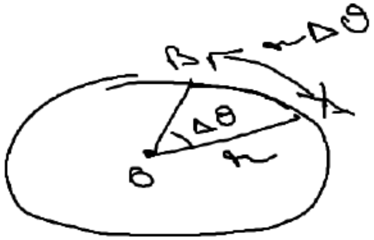
→ Law of Area –



If $\Delta t_1 = \Delta t_2 = \Delta t_3$

then $\Delta A_1 = \Delta A_2 = \Delta A_3$

Derivation



$\theta = \frac{l}{r} \Rightarrow l = r\theta$

Area of $\Delta = \frac{1}{2} r \times r\Delta\theta$

$$\Delta A = \frac{1}{2} r^2 \Delta\theta$$

$$\frac{\Delta A}{\Delta t} = \frac{1}{2} r^2 \frac{\Delta\theta}{\Delta t}$$

$$\lim_{\Delta t \to 0} \frac{\Delta A}{\Delta t} = \frac{1}{2} r^2 \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t}$$

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt}$$

ω

$$m\frac{dA}{dt} = \frac{m}{2} r^2 \omega$$

$I = mr^2$

$$m\frac{dA}{dt} = \frac{1}{2} I\omega$$

$L = I\omega$

$$m\frac{dA}{dt} = \frac{1}{2} L$$

$$\boxed{\frac{dA}{dt} = \frac{L}{2m}}$$

const.

$L_i = L_f$

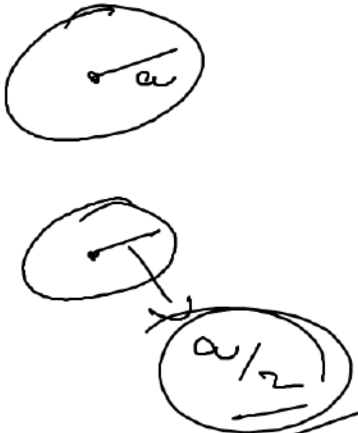
$I_i \omega_i = I_f \omega_f$

→ Keplers 3rd law → Law of period

$$T^2 \propto a^3$$

$$T \propto a^{3/2}$$

$$\boxed{T = K a^{3/2}}$$



Q. Find the no. of days in a year if the distance b/w the earth & the sun is half.

Q. Two satellites of a planet have periods 32 days & 256 days if the radius of the orbit of former is R, find the orbital radius of the latter?

$$T \propto a^{3/2}$$

$$\frac{T_1}{T_2} = \left(\frac{a_1}{a_2}\right)^{3/2}$$

$$\frac{T_2}{T_1} = \left(\frac{a_2}{a_1}\right)^{3/2}$$

$$T_2 = T_1\left(\frac{a_2}{a_1}\right)^{3/2}$$

$$= 365\left(\frac{a/2}{a}\right)^{3/2} = \frac{365}{2^{3/2}} = \frac{365}{\sqrt{8}}$$

$$365 \times \frac{1}{2^{3/2}} = 365 \times \frac{1}{(2^3)^{1/2}}$$

$$\sqrt{8}$$

$T = k a^{3/2}$

$365 = k a^{3/2}$

$$\boxed{k = \frac{365}{a^{3/2}}}$$

$\frac{T_1}{T_2} = \left(\frac{a_1}{a_2}\right)^{3/2}$

$\frac{\cancel{32}}{\cancel{256}}_{8} = \left(\frac{R}{a_2}\right)^{3/2}$

$\left(\frac{1}{8}\right)^{2/3} = \frac{R}{a_2}$

$8^{2/3} = (2^3)^{2/3}$

$= 4$

$\frac{1}{8^{2/3}} = \frac{R}{a_2}$

$\frac{1}{4} = \frac{R}{a_2}$

$a_2 = 4R$

$T = k a^{3/2}$

$\cancel{365}\ T = \frac{365}{a^{3/2}} \times \left(\frac{a}{2}\right)^{3/2}$

$= \frac{365 \times \cancel{a^{3/2}}}{\cancel{a^{3/2}} \times 2^{3/2}}$

$= \frac{365}{2^{3/2}}$

$= \frac{365}{\sqrt{8}}$

* universal law of gravitation –

$F = \frac{m_1 m_2}{R^2} \cdot G$

m_1 m_2

→ $F \propto m_1 m_2$

→ $F \propto \frac{1}{R^2}$

$F \propto \frac{m_1 m_2}{R^2}$

$\boxed{F = G \cdot \frac{m_1 m_2}{r^2}}$

kg^2

Const, constant

$6.67 \times 10^{-11}\ Nm^2/kg^2$

* In vector form —

$\hat{AB} \neq \hat{BA}$ → Direc^n

$|\vec{AB}| = |\vec{BA}|$ → Magn.

$\vec{AB} \neq \vec{BA}$ → vector.

$$\vec{F}_{1,2} = G.\frac{m_1 m_2}{|\vec{AB}|^2}.\hat{AB}$$

$$\vec{F}_{2,1} = G.\frac{m_1 m_2}{|\vec{AB}|^2}.\hat{BA}$$

Acceleration due to gravity

$g \rightarrow 9.8\, m/s^2$

$F = ma$

$m \rightarrow$ mass of object

$M \rightarrow$ mass of earth

$$\frac{GM\cancel{m}}{R^2} = \cancel{m}a$$

$$a = \frac{GM}{R^2} = 9.8\, m/s^2$$

g

Variation in the value of 'g'

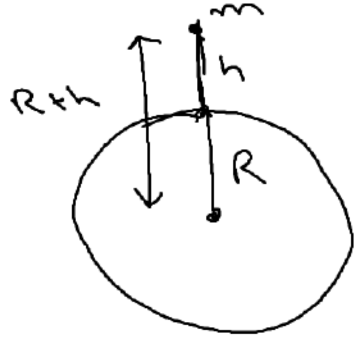
→ Due to shape of Earth → With height

Equator → Polar

by 21 km

$(g)_{min^m}$ $\quad g_{max^m}$

→ ~~...~~ With height



$$F = \frac{GMm}{(R+h)^2}$$

$$g = \frac{GM}{R^2}$$

$$g' = \frac{F}{m} = \frac{GM}{(R+h)^2}$$

$(R+h)^2$

$\left(R\left(1+\frac{h}{R}\right)\right)^2$

$R^2\left(1+\frac{h}{R}\right)^2$

$$g' = \frac{GM}{R^2\left(1+\frac{h}{R}\right)^2}$$

$$= \left(\frac{GM}{R^2}\right)\left(1+\frac{h}{R}\right)^{-2}$$

If $h << R$

Binomial expansion

$$\boxed{g' = g\left(1 - \frac{2h}{R}\right)}$$

$h\uparrow$ $g'\downarrow$

→ With depth —

$$f = \frac{G m' m}{(R-h)^2}$$

$$= \frac{G \frac{M}{R^3}(R-h)^3 \times m}{(R-h)^2}$$

$$= \frac{GMm}{R^3}(R-h)$$

$$g' = \frac{F}{m}$$

$$= \frac{G M m}{R^3 \, m}(R-h)$$

$$= \left(\frac{GM}{R^2}\right) \times \frac{1}{R}(R-h)$$

$$g' = g\left(\frac{R-h}{R}\right)$$

$$g' = g\left(1 - \frac{h}{R}\right)$$

Mass per unit volume of Earth. (ρ)

$$\rho = \frac{M}{V} = \frac{M}{\frac{4}{3}\pi R^3} = \frac{3M}{4\pi R^3}$$

$\rho = \frac{M}{V} \Rightarrow M = \rho V \rightarrow \boxed{m' = \rho V'}$

Mass of inner part of the earth

$$m' = \frac{3M}{4\pi R^3} \times \frac{4}{3}\pi (R-h)^3$$

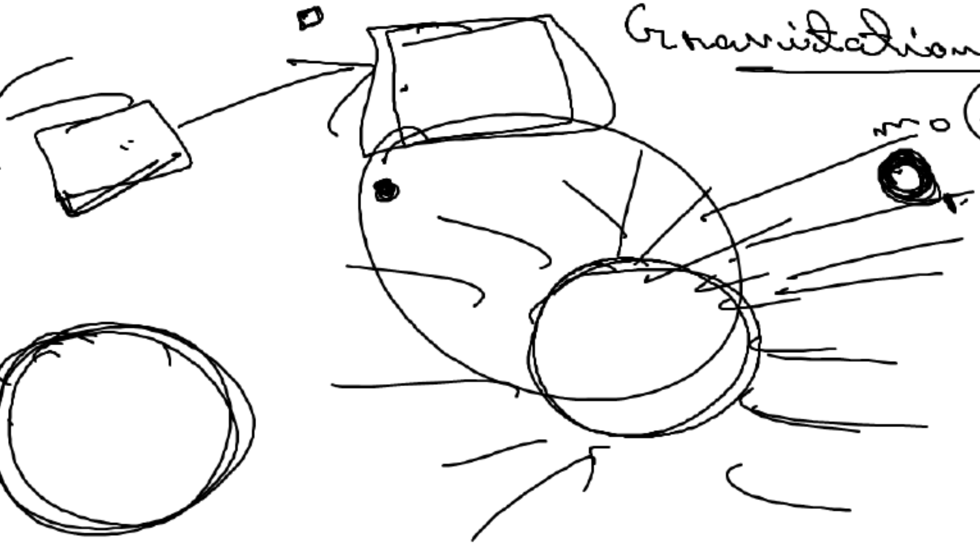
$$m' = \frac{M}{R^3}(R-h)^3$$

$$g' = 9.8\left(1 - \frac{7}{6400}\right)$$

$$=$$

$h \uparrow$

Depth ↑ , g' ↓



Gravitational field → E_f, M_f

E, B

Gravitational fd. → E_g

F_g — Gravitational force

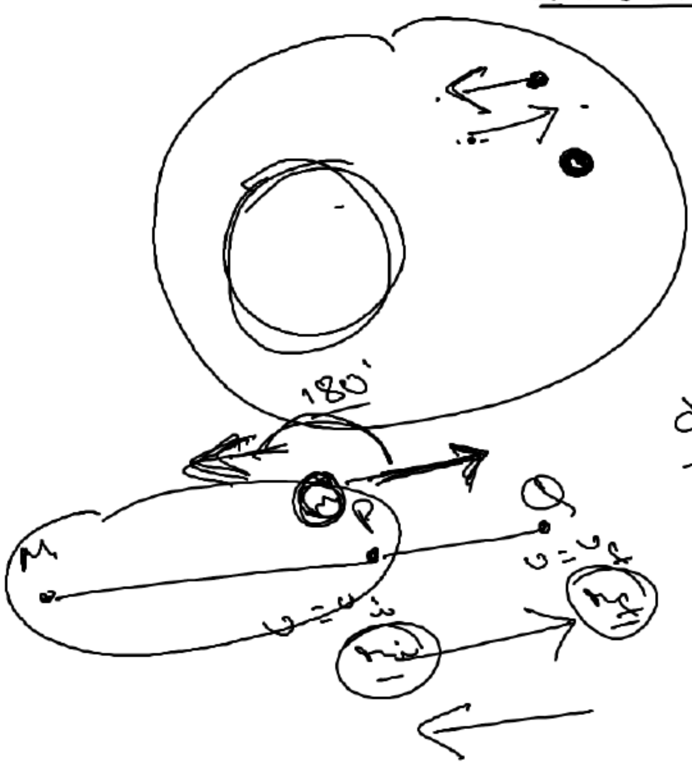
m_0 → test mass

→ very very small

$$E_g = \frac{F_g}{m_0} = \frac{\frac{G M \cdot \cancel{m_0}}{R^2}}{\cancel{m_0}}$$

$$\boxed{E_g = \frac{GM}{R^2}}$$

Gravitational P.E



$\vec{A} \cdot \vec{B} = AB\cos\theta$

$dW = \vec{F_g} \cdot d\vec{r}$

$dW = \frac{GMm}{r^2} \cdot dr \cos 180$

$dW = -\frac{GMm}{r^2} \cdot dr$

$W = \int dW = \int -\frac{GMm}{r^2} \cdot dr$

$= -GMm \int_{r_i}^{r_f} \frac{1}{r^2} \cdot dr$

$= -GMm \int_{r_i}^{r_f} r^{-2} \cdot dr$

$= -GMm \left[\frac{r^{-1}}{-1} \right]_{r_i}^{r_f}$

$\boxed{P.E = -W_{c.f}^{i-f}}$

$W = \vec{F} \cdot \Delta\vec{r}$

$dW = \vec{F} \cdot d\vec{r}$

$W = -GMm \left[-\frac{1}{r} \right]_{r_i}^{r_f}$

$= -GMm \left[\left(-\frac{1}{r_f}\right) - \left(-\frac{1}{r_i}\right) \right]$

$W = -GMm \left[\frac{1}{r_i} - \frac{1}{r_f} \right]$

$U_f - U_i = -W_{cf}^{i-f}$

$= -W$

$= -(-GMm\left[\frac{1}{r_i} - \frac{1}{r_f}\right])$

$U_f - U_i = GMm\left(\frac{1}{r_i} - \frac{1}{r_f}\right)$

* P.E of a point.

ref. pt. $\to \infty$

$r_i = \infty$ $r_f = r$

$U - U_i = GMm\left(\frac{1}{r_i} - \frac{1}{r_f}\right)$

$U - 0 = GMm\left(\frac{1}{\infty} - \frac{1}{r}\right)$

$U = GMm\left(0 - \frac{1}{r}\right)$

$\boxed{U = -\frac{GMm}{r}}$

eg: find P.E of the system.

$$U_{net} = U_1 + U_2 + U_3$$

$$= -\frac{Gm \times m}{L} - \frac{Gmm}{L} - \frac{Gmm}{L}$$

$$\boxed{U_{net} = \left(-\frac{3Gm^2}{L}\right) \text{ Joule}}$$

*

$\text{Force}(f) = \frac{G m_1 m_2}{r^2}$	$E = \frac{Gm}{r^2}$
$U_f - U_i = G m_1 m_2 \left(\frac{1}{r_i} - \frac{1}{r_f}\right)$	$\frac{U_f - U_i}{m_0} = \frac{-W_{i \to f}}{m_0}$
$\boxed{U = -\frac{G m_1 m_2}{r}}$	$V_f - V_i = Gm\left(\frac{1}{r_i} - \frac{1}{r_f}\right)$
	$\boxed{V = -\frac{Gm}{r}}$

* Escape velocity —

$$\boxed{v_e = 11.2\ km/s}$$

T.E of body earth system

$E_i = U_i + K_i$

$$\boxed{E_i = -\frac{GMm}{R} + \frac{1}{2}mv_e^2}$$

$g = \frac{GM}{R^2}$

$GM = gR^2$

$E_f = U_f + K_f$

$$\boxed{E_f = -\frac{GMm}{R+h} + \frac{1}{2}mv^2}$$

$E_i = E_f$

$$-\frac{GMm}{R} + \frac{1}{2}mv_e^2 = -\frac{GMm}{R+h} + \frac{1}{2}mv^2$$

$$\frac{1}{2}mv^2 = \frac{GMm}{R+h} + \left(\underbrace{\frac{1}{2}mv_e^2}_{K.E} - \underbrace{\frac{GMm}{R}}_{P.E}\right)$$

$$\frac{1}{2}mv_e^2 - \frac{GMm}{R} \geq 0$$

$$\frac{1}{2}\cancel{m}v_e^2 \geq \frac{GM\cancel{m}}{R}$$

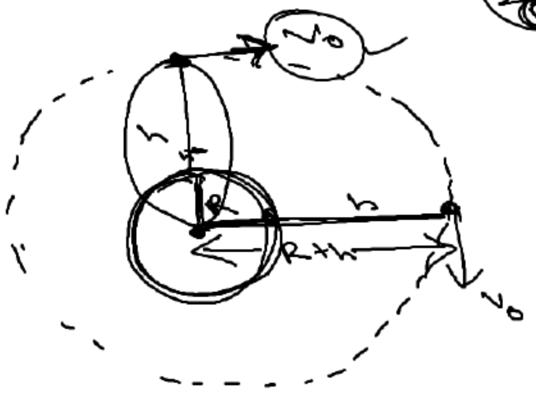
$$\boxed{v_e \geq \sqrt{\frac{2GM}{R}}}$$

$$v_e \geq \sqrt{\frac{2 \times gR^2}{R}}$$

$$\boxed{v_e \geq \sqrt{2gR}}$$

→ Orbital velocity

Sat. Motion (m, h, M, R)



$\frac{mv^2}{R}$

$$F_c = F_g$$

$$\boxed{GM = gR^2}$$

$R \to 6400$ km

$h \to 100$ km

$$\frac{\cancel{m}v_0^2}{\cancel{R+h}} = \frac{GM\cancel{m}}{(R+h)^{\cancel{2}}}$$

If sat. is very close to earth, $R+h \approx R$

$$\boxed{v_0 = \sqrt{\frac{GM}{R+h}} = \sqrt{\frac{gR^2}{R+h}}} = \sqrt{\frac{gR^2}{R}} = \sqrt{gR} = 7.92 \text{ km/s}$$

(velocity req. to put a sat. into its orbit around earth.)

$$g = \frac{GM}{R^2}$$

$$\boxed{GM = gR^2}$$

* Time period of satellite —

$$T = \frac{\text{circumf.}}{\text{orbital speed}}$$

$$T = \frac{2\pi(R+h)}{v_0} = \frac{2\pi(R+h)}{\sqrt{\frac{GM}{R+h}}} = \frac{2\pi(R+h)}{\sqrt{\frac{gR^2}{R+h}}}$$

$$T = \frac{2\pi}{R}\frac{(R+h)^{3/2}}{\sqrt{g}}$$

$$\boxed{T = \frac{2\pi}{R}\left[\frac{(R+h)^3}{g}\right]^{1/2}}$$

when sat. is closer to surface of earth.

$R+h \simeq R$

$$T = \frac{2\pi}{R}\left[\frac{R^3}{g}\right]^{1/2} = 85 \text{ mins.}$$

* Height of sat. from surface of earth ⇒

$$T = \frac{2\pi}{R}\left[\frac{(R+h)^3}{g}\right]^{1/2}$$

$$T^2 = \frac{4\pi^2}{R^2 g}(R+h)^3$$

$$(R+h)^3 = \frac{T^2 R^2 g}{4\pi^2}$$

$$R+h = \left(\frac{T^2 R^2 g}{4\pi^2}\right)^{1/3}$$

$$\boxed{h = \left(\frac{T^2 R^2 g}{4\pi^2}\right)^{1/3} - R}$$

* Energy of satellite –

$r_c = f_c$

$E = U + K$

$\boxed{-27.2\left(\frac{z^2}{n^2}\right)} + 13.6\left(\frac{z^2}{n^2}\right)$

$\boxed{E = -13.6\left(\frac{z^2}{n^2}\right)}$

$\boxed{E = U + K}$

Earth sat. system

because it has some orbital velocity.

$v_0 = \sqrt{\frac{GM}{R+h}} = \sqrt{\frac{gR^2}{R+h}}$

$U = \frac{-GMm}{(R+h)}$

$K = \frac{1}{2}mv_0^2$

$K = \frac{1}{2}m\left(\frac{GM}{R+h}\right)$

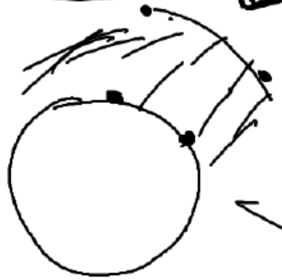
$E = -\frac{GMm}{R+h} + \frac{GMm}{2(R+h)}$

$= \frac{-2GMm + GMm}{2(R+h)} = \frac{-GMm}{2(R+h)}$

$\boxed{E = \frac{-GMm}{2(R+h)}}$ –ve ⇒ sat. is bound to earth.

* GSLV

$T \rightarrow 24\ hr$



$$h = \left(\frac{T^2 R^2 g}{4\pi^2} \right)^{1/3} - R$$

$T = 24\ hr$, $g = 9.8\ m/s^2$, $R = 6400 \times 10^3\ m$

h

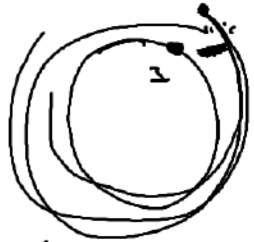
10000

→ PSLV

→ GSLV

PSLV

$T \rightarrow 100\ min.$



400 – 1000

Comb^n → Selec^n → a, b, c (1, 2, 3)

abc

(a, b) → ab, bc, ac → 3

$$ {}^{n}C_{r} = \frac{n!}{(n-r)!\, r!} $$

$n! \to 1 \cdot 2 \cdot 3 \cdots n$

$3! \to 1 \times 2 \times 3$

$5! \to 1 \times 2 \times 3 \times 4 \times 5$

$$ {}^{3}C_{2} = \frac{3!}{(3-2)!\, 2!} = \frac{3 \times \cancel{2} \times 1}{1 \times \cancel{2} \times 1} = 3 $$

$3C3 = 1$

Perm^n → Arrangement

$$ {}^{n}P_{r} = \frac{n!}{(n-r)!} $$

a, b, c → 2 is to be arranged

$$ {}^{3}P_{2} = \frac{3!}{(3-2)!} = \frac{3!}{1!} = 3 \times 2 = 6 $$

ab, ba, bc, cb, ac, ca → 6

$$ \frac{3!}{(3-3)!\, 3!} = \frac{1}{0!} = \frac{1}{1} = 1 $$

(a, b, c) 3

n → Total no. of things

r → No. of things to be selected or arranged

$$ {}^{20}C_{3} = \frac{20!}{(20-3)!\, 3!} $$

$$ = \frac{20!}{17!\, 3!} = \frac{20 \times 19 \times 18 \times \cancel{17!}}{\cancel{17!}\; 3!} = $$