Definite Integration

$$\int_a^b f(x)\cdot dx = \left[F(x)\right]_a^b$$

$b \Rightarrow$ upper limit

$a \rightarrow$ Lower limit

$$= \left[F(b) + \cancel{c} - F(a) - \cancel{c}\right]$$

$$= F(b) - F(a)$$

Q.10, 20, 30, 40, 50, 60

$$I = \int_0^1 \frac{dx}{1+x^2} = \left[\tan^{-1} x\right]_0^1$$

$$= \tan^{-1} 1 - \tan^{-1} 0$$

$$= \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

Q.25

$$\int_{\pi/6}^{\pi/4} \operatorname{cosec} x \cdot dx = \left[\log|\operatorname{cosec} x - \cot x|\right]_{\pi/6}^{\pi/4}$$

$$= \boxed{\log}|\sqrt{2} - 1| - \log|2 - \sqrt{3}|$$

$$= \log\left|\frac{\sqrt{2}-1}{2-\sqrt{3}}\right| \quad + \log\left|\frac{1}{2-\sqrt{3}}\right|$$

$$+ \log|2+\sqrt{3}|$$

$$\log\frac{1}{x} = -\log x$$

$$\frac{d}{dx}\sec x = \sec x \cdot \tan x$$

(1) $I = \int_{\pi/6}^{\pi/4} \operatorname{cosec} x \, dx$

(2)

(2) $= \int_{\pi/6}^{\pi/4} \frac{1}{\sin x} \, dx$

(3) $\sin x = t \Rightarrow$ (4) $\cos x \cdot dx = dt$

$\cos x = \sqrt{1-t^2}$

$$dx = \frac{1}{\cos x} \cdot dt = \frac{1}{\sqrt{1-t^2}} dt$$

(5)

$$= \int_{\pi/6}^{\pi/4} \frac{1}{t} \times \frac{1}{\sqrt{1-t^2}} \, dt$$

$$= \int_{\pi/6}^{\pi/4} \frac{t}{t^2\sqrt{1-t^2}} \, dt$$

$t^2 = y$

$2t \cdot dt = dy$

$t\,dt = \frac{1}{2} dy$

$-(\sqrt{2}+1)$

$(\cos x - \sin x)^2$

$\sin x + \cos x$

$$\int_0^{\pi/4} (\sin x - \cos x)\,dx$$

$$\left[-\cos x - \sin x\right]_0^{\pi/4}$$

$$\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) - (-1 - 0)$$

$$-\sqrt{2} + 1$$

$\sqrt{2}+1$

Q.30.

$$\int \cos^3 x \cdot dx$$

$$\int \cos x \cdot \cos^2 x \cdot dx$$

$$\int \cos x (1 - \sin^2 x) \cdot dx$$

$\sin x = t \Rightarrow \cos x\,dx = dt$

$$\int (1 - t^2)\,dt$$

$$t - \frac{t^3}{3}$$

30.

$$I = \int_0^{\pi/6} \cos x \cdot \cos 2x\,dx$$

$$= \int_0^{\pi/6} \cos x \cdot (2\cos^2 x - 1)\,dx$$

$$= 2\int_0^{\pi/6} \cos^3 x - \int_0^{\pi/6} \cos x \cdot dx$$

$$= 2I_1 - I_2 = \cancel{2} \times \frac{11}{\cancel{24}_{12}} - \frac{1}{2}$$

$$= \frac{5}{12}$$

$$I_1 = \int_0^{\pi/6} \cos^3 x = \int_0^{\pi/6} \cos x (1 - \sin^2 x)\,dx$$

$\sin x = t \Rightarrow \cos x \cdot dx = dt$

$$= \int_0^{\pi/6} (1 - t^2) \cdot dt$$

$$= \left[t - \frac{t^3}{3}\right]_0^{\pi/6}$$

$$= \left[\sin x - \frac{\sin^3 x}{3}\right]_0^{\pi/6}$$

$$= \frac{1}{2} - \frac{1}{24} = \frac{11}{24}$$

* Partial frac^n

$$I = \int_1^2 \frac{dx}{(x+1)(x+2)}$$

$$\frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2} = \frac{1}{x+1} - \frac{1}{x+2}$$

$$\frac{1}{\cancel{(x+1)(x+2)}} = \frac{A(x+2)+B(x+1)}{\cancel{(x+1)(x+2)}}$$

$$1 = Ax + 2A + Bx + B$$

$$1 = (A+B)x + (2A+B)$$

$A+B = 0$ $\quad B = -A$ $\quad \boxed{B = -1}$

$2A + B = 1$

$2A - A = 1$

$\boxed{A = 1}$

$$I = \int_1^2 \left(\frac{1}{x+1} - \frac{1}{x+2}\right) dx$$

$$= \Big[\log|x+1| - \log|x+2|\Big]_1^2$$

$$= \left[\log\left|\frac{x+1}{x+2}\right|\right]_1^2$$

$$= \left[\log\left|\frac{3}{4}\right| - \log\left|\frac{2}{3}\right|\right]$$

$$= \log\left|\frac{3/4}{2/3}\right| = \log 9 - \log 8 = \log 3^2 - \log 2^3$$
$$= 2\log 3 - 3\log 2.$$

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

$$= \left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_3^4$$

Integration by Substitution

eg: $I = \int_0^1 \frac{dx}{2x-3}$

$= \left[\cancel{\log|2x-3|}\right]_0^1$

$\boxed{2x-3 = t}$

$2 \cdot dx = dt$

$dx = \frac{1}{2} dt$

when $x = 0$, $t = -3$

$x = 1$, $t = -1$

$I = \int_{-3}^{-1} \frac{1}{t} \cdot \frac{1}{2} dt$

$= \frac{1}{2}\left[\log|t|\right]_{-3}^{-1}$

$= \frac{1}{2}\left[\log|-1| - \log|-3|\right]$

$= \frac{1}{2}\log|1/3|$

5. $I = \int_0^1 \frac{e^x}{1+(e^x)^2} \cdot dx$

$e^x = t$

$e^x \cdot dx = dt$

$x = 0$, $t = 1$

$x = 1$, $t = e$

$I = \int_1^e \frac{dt}{1+t^2} = \left[\tan^{-1} t\right]_1^e$

$= \tan^{-1} e - \tan^{-1} 1$

$= \tan^{-1} e - \frac{\pi}{4}$

2. $\int_0^1 \frac{2x}{1+x^2} \cdot dx$

$1 + x^2 = t$

$2x \cdot dx = dt$

$x = 0$, $t = 1$

$x = 1$, $t = 2$

$I = \int_1^2 \frac{dt}{t}$

$= \left[\log|t|\right]_1^2$

$= \log 2 - \log 1 = \log 2$

10.

$$\int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x}\,dx$$

$x = 0$, $t = 1$

$\cos x = t$

$x = \pi/2$, $t = 0$

$-\sin x \, dx = dt$

$\sin x \, dx = -dt$

$$\int_1^0 \frac{-dt}{1+t^2}$$

$$\left[-\tan^{-1} t\right]_1^0$$

$\pi/4$

(ILATE)

$$I = \int_0^1 \frac{2\tan^{-1} x}{I}\,dx$$

$$= \left[\tan^{-1} x \int 2\,dx - \int\left(\frac{d}{dx}\tan^{-1}x \int 2\,dx\right)dx\right]_0^1$$

$$= \left[\tan^{-1} x \times 2x - \int \frac{1}{1+x^2} \times 2x \, dx\right]_0^1$$

$$= \left[\tan^{-1} x \times 2x - \log|1+x^2|\right]_0^1$$

$$= \left[\left(\frac{\pi}{4} \times 2 - \log 2\right) - (0 - \log 1)\right]$$

$$= \frac{\pi}{2} - \log 2$$

23.

$$I = \int_0^{\pi/2} \frac{dx}{a^2\cos^2 x + b^2 \sin^2 x}$$

$$= \int_0^{\pi/2} \frac{\sec^2 x \cdot dx}{a^2 + b^2 \tan^2 x}$$

$\tan x = t$ $\quad x = 0$

$\sec^2 x \cdot dx = dt$ $\quad t = 0$

$x = \pi/2$

$t = \infty$

$$= \int_0^{\infty} \frac{dt}{a^2 + b^2 t^2} = \frac{1}{b^2} \int_0^{\infty} \frac{dt}{\left(\frac{a}{b}\right)^2 + t^2}$$

$$= \frac{1}{b^2} \left[\frac{1}{a/b} \tan^{-1} \frac{bt}{a} \right]_0^{\infty}$$

$$= \frac{1}{b^2} \left[\frac{b}{a} \times \frac{\pi}{2} - \frac{b}{a} \times 0 \right]$$

$$= \frac{1}{b^{\cancel{2}}} \left[\frac{\pi \cancel{b}}{2a} \right]$$

$$= \pi/2ab.$$

$$I = \int_0^{\pi} \frac{dx}{5 + 4\cos x}$$

$$= \int_0^{\pi} \frac{dx \quad \times 1 + \tan^2 x/2}{5(1 + \tan^2 x/2) + 4(1 - \tan^2 x/2)}$$

$$= \int_0^{\pi} \frac{\sec^2 x/2 \cdot dx}{9 + \tan^2 x/2}$$

$\tan x/2 = t$ $\quad x = 0$

$\frac{1}{2}\sec^2 x/2 \cdot dx = 2dt$ $\quad t = 0$

$x = \pi$

$t = \infty$

$$= I = \int_0^{\infty} \frac{2 \cdot dt}{9 + t^2}$$

$$= 2 \int_0^{\infty} \frac{dt}{3^2 + t^2}$$

$$2\boxed{\left[\frac{1}{3} \tan^{-1} \frac{t}{3}\right]_0^{\infty}}$$

39.

$$I = \int_0^{\pi/2} \frac{\sin 2x}{\sin^4 x + \cos^4 x} \cdot dx$$

$(\cos^2 x)^2$

$\sin^2 x = t$

$x = 0,\ t = 0$

$x = \frac{\pi}{2},\ t = 1$

$2 \cdot \sin x \cdot \cos x \cdot dx = dt$

$$I = \int_0^1 \frac{dt}{t^2 + (1-t)^2}$$

$$= \int_0^1 \frac{dt}{t^2 + 1 + t^2 - 2t}$$

$$= \int_0^1 \frac{dt}{2t^2 - 2t + 1}$$

$\frac{1}{2} - \frac{1}{4}$

$\frac{1}{4}$

$2t^2 - 2t + 1$

$$\boxed{\frac{1}{2}\left[\frac{1}{1/2}\tan^{-1}\left(\frac{t - 1/2}{1/2}\right)\right]_0^1} = \frac{1}{2}\int_0^1 \frac{dt}{t^2 - t + 1/2}$$

$$= \frac{1}{2}\int_0^1 \frac{dt}{t^2 - 2 \times t \times \frac{1}{2} + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + \frac{1}{2}}$$

$$= \frac{1}{2}\int_0^1 \frac{dt}{\left(t - \frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}$$

40.

$$I = \int_0^a \sin^{-1}\sqrt{\frac{x}{a+x}} \cdot dx$$

$x = a\tan^2\theta$

$\sqrt{\frac{x}{a}} = \tan\theta$

$\theta = \tan^{-1}\sqrt{\frac{x}{a}}$

$$= \int_0^a \sin^{-1}\sqrt{\frac{a\tan^2\theta}{a + a\tan^2\theta}} \cdot dx$$

$$= \int_0^a \sin^{-1}\sqrt{\frac{\tan^2\theta}{\sec^2\theta}} \cdot dx$$

$$= \int_0^a \sin^{-1}\sqrt{\frac{\sin^2\theta}{\cancel{\cos^2\theta}} \times \cancel{\cos^2\theta}}\ dx$$

$$= \int_0^a \theta \cdot dx$$

$$= \int_0^a \underset{\text{II}}{1} \times \underset{\text{I}}{\tan^{-1}\sqrt{\frac{x}{a}}} \cdot dx$$

$$= \left[\tan^{-1}\sqrt{\frac{x}{a}} \int 1 \cdot dx - \int\left(\frac{d}{dx}\tan^{-1}\sqrt{\frac{x}{a}} \int 1 \cdot dx\right) dx\right.$$

$$= \left[\tan^{-1}\sqrt{x/a} \cdot x - \left(\frac{1}{1 + \frac{x}{a}} \cdot x \quad dx\right)\right]$$