Algebra and Pre-Calculus Problems

Assignment 1 June 10, 2021

Problem 1:

Find
$$\frac{1}{r^2}+\frac{1}{s^2}+\frac{1}{t^2}$$
given that r, s, t are the roots of $x^3-6x^2+5x-7=0$

Solution:

Equation whose roots are squares of the roots of

$$f(x)=x^3-6x^2+5x-7=0$$

is

$$f(\sqrt{x})=0$$

i.e.,
$$x\sqrt{x}+5\sqrt{x}=6x+7$$
squaring both sides and re arranging we get

$$x^3-26x^2-59x-49=0$$

and equation whose roots are reciprocals to that of above is obtained by replacing x with $\frac{1}{x}$ in the above polynomial equation. So that results in

$$49x^3+59x^2+26x-1=0$$

and whose sum of the roots is $\frac{-59}{49}$ by Vieta's Formulae. Thus

$$\boldsymbol{\frac{1}{r^2}+\frac{1}{s^2}+\frac{1}{t^2}}=\frac{-59}{49}$$

Problem 2:

Evaluate

$$\int \frac{dx}{x^3 - 25x}$$

Solution:

We can re-write the given integral as

$$I = \int \frac{dx}{x^3 - 25x} = \frac{1}{50} \int \frac{\frac{50dx}{x^3}}{1 - \frac{25}{x^2}}$$

Now put

$$1 - \frac{25}{x^2} = t$$

Then we have

$$\frac{50}{x^3} dx = dt$$

Thus we have

$$I = \frac{1}{50} \int \frac{dt}{t}$$

$\implies$

$$I = \frac{1}{50} \ln |t| = \frac{1}{50} \ln \left| 1 - \frac{25}{x^2} \right| + C$$

Problem 3:

Find the value of

$$\int_{-\infty}^{\infty} \frac{x^2}{x^4 + 1}$$

Solution:

By using the Even function property,

$$\int_{-a}^{a} f(x) dx = 2 \int_{0}^{a} f(x) dx$$

we have

$$I = 2 \int_{0}^{\infty} \frac{x^2 \mathrm{d}x}{x^4 + 1}$$

now by $x^2 = \tan\theta$ we get

$$I = \int_0^{\frac{\pi}{2}} \sqrt{\tan\theta}\mathrm{d}\theta \to (1)$$

now to evaluate I consider its complement

$$I = \int_0^{\frac{\pi}{2}} \sqrt{\cot\theta}\mathrm{d}\theta \to (2)$$

adding both

$$2I = \int_0^{\frac{\pi}{2}} \left(\sqrt{\tan\theta} + \sqrt{\cot\theta}\right) \mathrm{d}\theta = 2\int_0^{\frac{\pi}{4}} \left(\sqrt{\tan\theta} + \sqrt{\cot\theta}\right) \mathrm{d}\theta =$$

$\Longrightarrow$

$$2I = 2\int_0^{\frac{\pi}{4}} \left(\frac{\sin\theta + \cos\theta}{\sqrt{\sin\theta\cos\theta}}\right)$$

and use $\sin\theta\cos\theta = \frac{1}{\sqrt{2}}\sqrt{1 - (\sin\theta - \cos\theta)^2}$ and use again the substitution $\sin\theta - \cos\theta = t$. With the above substitution we get

$$I = \sqrt{2}\int_{-1}^{0} \frac{dt}{\sqrt{1-t^2}}$$

$\Longrightarrow$

$$I = \sqrt{2}\sin^{-1}(t)\Big|_{-1}^{0}$$

Plugging the upper and lower bounds we get

$$\boldsymbol{I = \frac{\pi}{\sqrt{2}}}$$

Problem 4:

Find the value of

$$\int_0^{\pi/2} \sqrt{\sin 2x} \cdot \sin x \, dx$$

Solution:

Given that

$$I = \int_0^{\pi/2} \sqrt{\sin 2x} \cdot \sin x \, dx \to (1)$$

Using the fact that

$$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$$

We get

$$I = \int_0^{\pi/2} \sqrt{\sin 2x} \cos x \, dx \to (2)$$

Adding $(1), (2)$ we get

$$2I = \int_0^{\pi/2} \sqrt{\sin 2x}(\sin x + \cos x)dx$$

Now recall that

$$\sin 2x = 1 - (\sin x - \cos x)^2$$

So we get

$$2I = \int_0^{\frac{\pi}{2}} \sqrt{1 - (\sin x - \cos x)^2} \, (\sin x + \cos x)dx$$

Now use the substitution $\sin x - \cos x = t$, then we get

$$2I = \int_{-1}^{1} \sqrt{1-t^2} \, dt = 2\left(\frac{t}{2}\sqrt{1-t^2} + \frac{1}{2}\sin^{-1}(t)\right)\Bigg|_0^1$$

Plugging in the bounds we get

$$\boldsymbol{I = \frac{\pi}{4}}$$

Problem 5:

Solve the differential equation

$$\left(xy^3 + x^2y^7\right)\frac{dy}{dx} = 1$$

Solution:

The equation is evidently a non linear differential equation. Let us apply a trick to convert the equation to the Linear form. We can re-write the given equation as:

$$\frac{1}{x^2}\frac{dx}{dy} = \frac{y^3}{x} + y^7$$

Now use the substitution $\frac{1}{x} = u$ and differentiate both sides with respect to y, we get

$$\frac{1}{x^2}\frac{dx}{dy} = -\frac{du}{dy}$$

So we get

$$-\frac{du}{dy} = y^3 u + y^7$$

$$\Rightarrow \quad \frac{du}{dy} + uy^3 = -y^7$$

which is the Linear first order differential equation of the form

$$\frac{du}{dy} + P(y)u = Q(y)$$

So using the method of Integrating Factor, we can solve it and i am leaving it to you as an exercise. The answer is:

$$\boldsymbol{\frac{1}{x} = 4 - y^4 + Ce^{-\frac{y^4}{4}}}$$

Problem 6:

If $I = \int_0^1 x^{1004} \cdot (1-x)^{1004} dx$ and $J = \int_0^1 x^{1004} \cdot \left(1 - x^{2010}\right)^{1004} dx$, Find the value of $\frac{I}{J}$

Solution:

Let us start with J first. We use substitution $x^{2010} = t^2$

We get

$$J = \frac{1}{1005}\int_0^1 (1-t)^{1004}(1+t)^{1004} dt = \frac{1}{1005}\int_0^1 t^{1004}(2-t)^{1004} dt$$

Now let use another clever substitution $t = 2u$. We get

$$J = \frac{1}{1005}\int_0^{\frac{1}{2}} 2^{2009} y^{1004}(1-y)^{1004} dy$$

Hence

$$J=\frac{2^{2009}}{1005}\int_0^{\frac{1}{2}} y^{1004}(1-y)^{1004}dy$$

Also using

$$\int_a^b f(x)dx=\int_a^b f(a+b-x)dx$$

we get

$$I=2\int_0^{\frac{1}{2}} y^{1004}(1-y)^{1004}dy$$

Finally we get

$$\boldsymbol{\frac{I}{J}=\frac{2010}{2^{2009}}}$$

Problem 7:

Given that $a, b \in \mathbb{R}$ such that

$$a^4+b^4-6a^2b^2=15$$
$$a^3b-ab^3=4$$

Then find the value of a^2+b^2

Solution:

The first equation can be written as:

$$(a^2-b^2)^2-4a^2b^2=15$$

The second equation can be written as:

$$ab(a^2-b^2)=4$$

Let $\alpha=a^2-b^2$ and $\beta=ab$ Then we have

$$\alpha^2-4\beta^2=15 \rightarrow (1)$$

and

$$\alpha\beta=4 \rightarrow (2)$$

Now using the fact that

$$(a^2+b^2)^2=(a^2-b^2)^2+4a^2b^2$$

we get

$$(a^2+b^2)^2 = \alpha^2 + 4\beta^2 = 2\alpha^2 - 15$$

Now from $(1),(2)$ we have

$$\alpha^4 - 15\alpha^2 = 64$$

$\implies$

$$4\alpha^4 - 60\alpha^2 = 256$$

$\implies$

$$(2\alpha^2 - 15)^2 = 481$$

$\implies$

$$2\alpha^2 - 15 = \sqrt{481}$$

$\implies$

$$(a^2+b^2)^2 = \sqrt{481}$$

Finally

$$\boldsymbol{a^2 + b^2 = \sqrt[4]{481}}$$