Calculus and Trigonometry Problems

Assignment 1 June 10, 2021

Problem 1:

Evaluate

$$\int_0^{\pi} \frac{\sin(2021x)}{\sin x} dx$$

Solution:

Let us assume,

$$I_n = \int_0^{\pi} \frac{\sin(nx)}{\sin x} dx, \; n \in \mathbb{N}$$

Now let us find I_{n+2}. We have

$$I_{n+2} = \int_0^{\pi} \frac{\sin nx \cos 2x + \cos nx \sin 2x}{\sin x} dx$$

Using the formulae,

$$\cos(2x) = 1 - 2\sin^2 x$$

and

$$\sin(2x) = 2\sin x \cos x$$

we get

$$\Rightarrow I_{n+2} = \int_0^{\pi} \frac{\sin nx \left(1 - 2\sin^2 x\right)}{\sin x} + 2\cos nx \cos x$$

$\Longrightarrow$

$$\Rightarrow I_{n+2} = \int_0^{\pi} 2\cos(n+1)x dx + I_n = I_n$$

Thus we have

$$I_{n+2} = I_n, \; \forall n \in \mathbb{N}$$

Now plugging in $n = 1, 2, 3..$ we get

$$I_1 = I_3 = I_5 = ... = I_{2021}$$

$\Longrightarrow$

$$I_{2021} = I_1 = \int_0^{\pi} \frac{\sin x}{\sin x} dx = \pi$$

Thus

$$\boldsymbol{I_{2021} = \pi}$$

Problem 2:

Given that $\sin\theta + \sin^2\theta = 1$, find the value of

$$\cos^{12}\theta + 3\cos^{10}\theta + 3\cos^8\theta + \cos^6\theta$$

Solution:

When $\sin\theta + \sin^2\theta = 1$ we have

$$\sin\theta = 1 - \sin^2\theta$$

$$\Rightarrow \quad \sin\theta = \cos^2\theta$$

$\Longrightarrow$

$$\cos^{12}\theta = \sin^6\theta$$
$$\cos^{10}\theta = \sin^5\theta$$
$$\cos^8\theta = \sin^4\theta$$
$$\cos^6\theta = \sin^3\theta$$

Then we get

$$\cos^{12}\theta + 3\cos^{10}\theta + 3\cos^8\theta + \cos^6\theta = \sin^6\theta + 3\sin^5\theta + 3\sin^4\theta + \sin^3\theta$$

$$\Rightarrow \quad \cos^{12}\theta + 3\cos^{10}\theta + 3\cos^8\theta + \cos^6\theta = \sin^3\theta(\sin\theta + 1)^3$$

$\Longrightarrow$

$$\cos^{12}\theta + 3\cos^{10}\theta + 3\cos^8\theta + \cos^6\theta = \left(\sin\theta + \sin^2\theta\right)^3 = 1^3 = 1$$

Thus

$$\boldsymbol{\cos^{12}\theta + 3\cos^{10}\theta + 3\cos^8\theta + \cos^6\theta} = 1$$

Problem 3:

If $\tan^2\theta = 1 - a^2$, then find

$$\sec\theta + \tan^3\theta\,\mathrm{cosec}\,\theta$$

in terms of a

Solution:

We have

$$\sec\theta + \tan^3\theta\cos\theta = \frac{1}{\cos\theta} + \frac{\sin^3\theta}{\cos^3\theta} \times \frac{1}{\sin\theta}$$
$$\Rightarrow \sec\theta + \tan^3\theta\,\mathrm{cosec}\,\theta = \frac{1}{\cos\theta} + \frac{\sin^2\theta}{\cos^3\theta}$$
$$\Rightarrow \sec\theta + \tan^3\theta\,\mathrm{cosec}\,\theta = \frac{\sin^2\theta + \cos^2\theta}{\cos^3\theta}$$
$$\Rightarrow \sec\theta + \tan^3\theta\,\mathrm{cosec}\,\theta = \sec^3\theta$$
$$\Rightarrow \sec\theta + \tan^3\theta\,\mathrm{cosec}\,\theta = \left(\sec^2\theta\right)^{\frac{3}{2}}$$
$$\Rightarrow \sec\theta + \tan^3\theta\,\mathrm{cosec}\,\theta = \left(1 + \tan^2\theta\right)^{\frac{3}{2}}$$
$$\Rightarrow \sec\theta + \tan^3\theta\,\mathrm{cosec}\,\theta = \left(1 + 1 - a^2\right)^{\frac{3}{2}}$$

Thus finally

$$\boldsymbol{\sec\theta + \tan^3\theta\,\mathrm{cosec}\,\theta = \left(2 - a^2\right)^{\frac{3}{2}}}$$

Problem 4:

If $\frac{\sin^4\theta}{a} + \frac{\cos^4\theta}{b} = \frac{1}{a+b}$ Then prove that

$$\frac{\sin^8\theta}{a^3} + \frac{\cos^8\theta}{b^3} = \frac{1}{(a+b)^3}$$

Solution:

Dividing both sides by $\cos^4\theta$ we get

$$\frac{\tan^4\theta}{a}+\frac{1}{b}=\frac{1}{a+b}\sec^4\theta$$

$$\Rightarrow\quad \frac{\tan^4\theta}{a}+\frac{1}{b}=\frac{1}{a+b}\left(1+\tan^2\theta\right)^2$$

$$\Rightarrow\quad \frac{\tan^4\theta}{a}+\frac{1}{b}=\frac{1+\tan^4\theta+2\tan^2\theta}{a+b}$$

$$\Rightarrow \tan^4\theta\left(\frac{1}{a}-\frac{1}{a+b}\right)-\frac{2\tan^2\theta}{a+b}+\frac{1}{b}-\frac{1}{a+b}=0$$

$$\Rightarrow \frac{b\tan^4\theta}{a(a+b)}-\frac{2\tan^2\theta}{a+b}+\frac{a}{b(a+b)}=0$$

$$\Rightarrow\quad b^2\tan^4\theta-2ab\tan^2\theta+a^2=0$$

$$\Rightarrow\quad \left(b\tan^2\theta-a\right)^2=0$$

$$\Rightarrow \tan^2\theta=\frac{a}{b}$$

$$\Rightarrow\quad \sec^2\theta=\frac{a+b}{b}$$

$$\Rightarrow \cos^2\theta=\frac{b}{a+b},\quad \sin^2\theta=\frac{a}{a+b}$$

$$\Rightarrow \frac{\sin^8\theta}{a^3}+\frac{\cos^8\theta}{b^3}=\frac{a^4}{a^3(a+b)^4}+\frac{b^4}{b^3(a+b)^4}$$

$$\Rightarrow \frac{\sin^8\theta}{a^3}+\frac{\cos^8\theta}{b^3}=\frac{a}{(a+b)^4}+\frac{b}{(a+b)^4}$$

Thus finally

$$\boldsymbol{\frac{\sin^8\theta}{a^3}+\frac{\cos^8\theta}{b^3}=\frac{1}{(a+b)^3}}$$

Problem 5:

If $\frac{\cos^4\alpha}{\cos^2\beta}+\frac{\sin^4\alpha}{\sin^2\beta}=1$ then find the value of

$$\frac{\sin^4\alpha+\sin^4\beta}{\sin^2\alpha\sin^2\beta}$$

Solution:

Let us assume

$$a = \sin\alpha$$
$$b = \sin\beta$$

Now we have

$$\cos^4\alpha = \left(1 - a^2\right)^2, \cos^2\beta = 1 - b^2$$

So we get

$$\frac{\cos^4\alpha}{\cos^2\beta} + \frac{\sin^4\alpha}{\sin^2\beta} = \frac{a^4}{b^2} + \frac{\left(1-a^2\right)^2}{1-b^2} = 1$$
$$\Rightarrow \frac{a^4 - 2a^2 + 1}{1 - b^2} = \frac{b^2 - a^4}{b^2}$$
$$\Rightarrow a^4b^2 - 2a^2b^2 + b^2 = b^2 - b^4 - a^4 + a^4b^2$$
$$\Rightarrow \quad a^4 + b^4 = 2a^2b^2$$
$$\Rightarrow \quad \frac{a^4 + b^4}{a^2b^2} = 2$$

Hence we get

$$\boldsymbol{\frac{\sin^4\alpha + \sin^4\beta}{\sin^2\alpha\sin^2\beta} = 2}$$

Problem 6:

Find all real values of m such that

$$\sin 2x - 2m(\sin x + \cos x) + 2 = 0$$
$$0 < x < \frac{\pi}{2}$$

has at least one real solution.

Solution:

The given equation can be converted to:

$$m = \frac{\sin 2x + 2}{2(\sin x + \cos x)}$$
$$\Rightarrow m = \frac{2\sin x\cos x + 2}{2(\sin x + \cos x)}$$

Now let us assume that $a = \sin x + \cos x \implies$

$$a = \sqrt{2} \sin\left(x + \frac{\pi}{4}\right)$$

$\implies$

$$a \in (1, \sqrt{2}], \quad \forall x \in \left(0, \frac{\pi}{2}\right)$$

Also we have

$$\sin^2 x + \cos^2 x + 2 \sin x \cos x = a^2$$

$$2 \sin x \cos x = a^2 - 1$$

So we get

$$m = \frac{a^2 + 1}{2a}$$

Now by $AM \geq GM$ we get

$$\frac{a^2 + 1}{2a} > 1$$

Also if

$$f(a) = \frac{a^2 + 1}{2a}$$

We have

$$f'(a) = \frac{1}{2}\left(1 - \frac{1}{a^2}\right) > 0, \forall a > 1$$

So the function $f(a)$ is strictly increasing and since $a \leq \sqrt{2}$, the maximum occurs at $a = \sqrt{2}$. Thus we have

$$m = \frac{a^2 + 1}{2a} \in \left(1, \frac{(\sqrt{2})^2 + 1}{2\sqrt{2}}\right]$$

Thus the given equation has at least one real solution if

$$\boldsymbol{m \in \left(1, \frac{3}{2\sqrt{2}}\right]}$$