Solution of the 1-Dimensional Schrödinger Equation with the Asymmetric Infinite Square Well Potential

Prepared by [Subrata Sarangi]

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Time-Dependent Schrdinger Equation (TDSE)

- The general Time-Dependent Schrdinger Equation (TDSE) in 3D:

$$i\hbar \frac{\partial \Psi(\mathbf{r}, t)}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \Psi(\mathbf{r}, t) + V(\mathbf{r}) \Psi(\mathbf{r}, t).$$

- $\Psi(\mathbf{r}, t)$: Wavefunction, ∇^2: Laplacian, $V(\mathbf{r})$: Potential energy function.

Separating Time and Spatial Parts

- Assume a separable solution:

$$\Psi(\mathbf{r}, t) = \psi(\mathbf{r}) T(t).$$

- Substituting into TDSE gives:

$$i\hbar \frac{1}{T(t)} \frac{dT(t)}{dt} = -\frac{\hbar^2}{2m} \frac{1}{\psi(\mathbf{r})} \nabla^2 \psi(\mathbf{r}) + V(\mathbf{r}).$$

- Separate variables (time and space dependences must independently satisfy the equation):

$$i\hbar \frac{1}{T(t)} \frac{dT(t)}{dt} = E, \quad -\frac{\hbar^2}{2m} \nabla^2 \psi(\mathbf{r}) + V(\mathbf{r})\psi(\mathbf{r}) = E\psi(\mathbf{r}).$$

- E: Separation constant (interpreted as energy eigenvalue).

Time-Independent Schrödinger Equation (TISE)

- Time part:

$$T(t) = T_0 e^{-iEt/\hbar}.$$

- Spatial part satisfies the TISE:

$$-\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{r}) + V(\mathbf{r})\psi(\mathbf{r}) = E\psi(\mathbf{r}).$$

- For a system with time-independent potential $V(\mathbf{r})$, this governs the spatial behavior of the wavefunction.

Reduction to 1D Schrödinger Equation

- Consider a potential $V(x)$ depending only on x (1D system).
- Wavefunction $\psi(x)$ satisfies:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x).$$

- Boundary conditions depend on the physical situation and $V(x)$.

Potential for Infinite Square Well

- Potential $V(x)$:

$$V(x) = \begin{cases} 0, & 0 < x < a, \\ \infty, & \text{otherwise.} \end{cases}$$

- Physical interpretation:
  - Particle is confined to $0 < x < a$.
  - Wavefunction $\psi(x)$ must vanish outside this region.

Boundary Conditions

- At $x = 0$ and $x = a$: $\psi(x) = 0$ (wavefunction must vanish at the walls of the well).
- Schrödinger Equation inside the well ($V(x) = 0$):

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} = E\psi(x).$$

- Simplified form:

$$\frac{d^2\psi(x)}{dx^2} + k^2\psi(x) = 0, \quad k^2 = \frac{2mE}{\hbar^2}.$$

Solutions to Schrödinger Equation

- General solution inside the well:

$$\psi(x) = A\sin(kx) + B\cos(kx).$$

- Apply boundary conditions:
  - At $x = 0$: $\psi(0) = 0$ implies $B = 0$.
  - At $x = a$: $\psi(a) = 0$ implies $\sin(ka) = 0$.
- Condition for k:

$$ka = n\pi, \quad n = 1, 2, 3, \ldots$$

Eigenvalues and Eigenfunctions

- Wavevector:

$$k_n = \frac{n\pi}{a}.$$

- Energy eigenvalues:

$$E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{n^2\pi^2\hbar^2}{2ma^2}.$$

- Eigenfunctions:

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right), \quad n = 1, 2, 3, \ldots$$

Summary

- Derived the TISE from the TDSE.
- Reduced the 3D case to 1D for the Infinite Square Well potential.
- Obtained eigenvalues:

$$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}.$$

- Found eigenfunctions:

$$\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right).$$