CHAPTER 5 [redacted] (For Board & Competitive Exams)

Complex Numbers-I

Chapter Contents

INTRODUCTION

In order to have solutions to the equations like $x^2 + 1 = 0$, we needed to extend the system of real numbers, such a system came to be known as the system of complex numbers. The study of complex numbers has far reaching bearing in many fields of science and engineering. In the subsequent discussion, some basic concepts will be taken up.

SQUARE ROOT OF NEGATIVE NUMBERS

$\sqrt{-1}$ is defined as i (Pronounced as iota)

i.e., $i = \sqrt{-1} \Rightarrow i^2 = -1 \Rightarrow i^3 = -i \Rightarrow i^4 = 1$

Infact, if a is a positive real number then $\sqrt{-a} = \sqrt{a}\sqrt{-1} = \sqrt{a}i$

(i) If m is an integer, then $i^{4m} = 1$, $i^{4m+1} = i$, $i^{4m+2} = i^2 = -1$, $i^{4m+3} = i^3 = -i$

(ii) The sum of any consecutive four power of i is zero

i.e., $i^n + i^{n+1} + i^{n+2} + i^{n+3} = 0, n \in \mathbb{Z}$

(iii) $\frac{1}{i} = -i$ and $-\frac{1}{i} = i$

EXAMPLE 1 : *Find the value of i^{39}.*

Solution : $i^{39} = i^{4\times 9+3} = i^3 = -i$

EXAMPLE 2 : *Find the value of $y = i + i^2 + i^3 + ... + i^{100} + i^{101}$.*

Solution : We observe that

$$y = (i + i^2 + i^3 + i^4) + (i^5 + i^6 + i^7 + i^8) + ... + (i^{97} + i^{98} + i^{99} + i^{100}) + i^{101}$$

$$= (0 + 0 + ...) + i^{101} = i^{4\times 25+1} = i.$$

EXAMPLE 3 : *If $x = (1 + i)$, then find the value of $(x - 1)^4 + 4$.*

Solution : $x = 1 + i \Rightarrow x - 1 = i$

Now, $(x - 1)^4 = 1$

$\therefore \quad (x - 1)^4 + 4 = 5$

Note : *If a and b are positive real numbers, then $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$. This result also holds true when either $a > 0, b < 0$ or $a < 0, b > 0$.*

But if $a < 0, b < 0$, then this result does not hold true.

$\because$ *If it is true, then*

$i^2 = i.i = \sqrt{-1}\cdot\sqrt{-1} = \sqrt{(-1)(-1)} = \sqrt{1} = 1$

which is a contradiction

Thus if $a < 0, b < 0$, then $\sqrt{a} \times \sqrt{b} \neq \sqrt{ab}$.

Also, if any of a or b is 0, then $\sqrt{a} \times \sqrt{b} = \sqrt{ab} = 0$.

DEFINITION OF A COMPLEX NUMBER

Any number of the form $x + iy$, where $i = \sqrt{-1}$ and x, y are real numbers or the numbers containing i is called complex number. For example $2 + 3i$, $2 - 3i$, 3, 0, $2 + i(2 + i)$ all are complex numbers.

Different Particulars of Complex Numbers

(i) Real and Imaginary Part

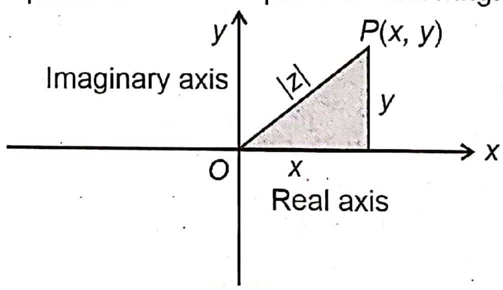
If $z = x + iy$, where $x, y \in R$ and $i = \sqrt{-1}$, is a complex number, then x is called the real part of and y is called imaginary part of z, which can be written as

Re(z) = x and Im(z) = y

- If Re(z) = 0, then z is purely imaginary complex number.
- If Im(z) = 0, then z is purely real complex number.
- $z = 0$ is purely real as well as purely imaginary.

(ii) Representation of Complex Numbers

A complex number $z = x + iy$, $x, y \in R$ can be written as order pair (x, y). It can be represented on Argand plane or Complex plane or Gaussian plane as following.



(iii) Modulus of Complex Numbers

The modulus of a complex number is its distance from (0, 0). If $z = x + iy$, then its modulus represented by $OP = |z| = \sqrt{x^2 + y^2} = \sqrt{(\text{Re}(z))^2 + (\text{Im}(z))^2}$

EXAMPLE 4 : *Find the modulus of $z = x + i\sqrt{1-2x}$, $x \in R$ and $x \le \frac{1}{2}$*

Solution : $|z| = \sqrt{x^2 + (1-2x)} = \sqrt{x^2 + 1 - 2x} = \sqrt{(x-1)^2} = |x - 1|$

(iv) Argument of a Complex Number

The argument of a complex number is the angle which the line joining the number represented in argand plane to the origin makes with positive real axis.

If $z = x + iy$, then its argument is solution of equations $\cos\theta = \frac{x}{\sqrt{x^2+y^2}}$ and $\sin\theta = \frac{y}{\sqrt{x^2+y^2}}$. The argument of z is also called amplitude of z.

The argument of any point lying in the interval $(-\pi, \pi]$ is called principal argument.

Rules to find the principal argument :

Let $z = x + iy$, x, y are non-zero real numbers and $\theta = \tan^{-1}\left|\frac{y}{x}\right|$. Then

(a) $\arg(z) = \theta$, if $x > 0$, $y \ge 0$ (point lies in first quadrant)

(b) $\arg(z) = \pi - \theta$, if $x < 0$, $y \ge 0$ (point lies in second quadrant)

(c) $\arg(z) = \theta - \pi$, if $x < 0$, $y < 0$ (point lies in third quadrant)

(d) $\arg(z) = -\theta$, if $x > 0$, $y < 0$ (point lies in fourth quadrant)

(e) $\arg(0 + i0)$ is not defined.

e.g., For $k > 0$,

(i) $\arg(k) = 0$ (ii) $\arg(-k) = \pi$

(iii) $\arg(ik) = \frac{\pi}{2}$ (iv) $\arg(-ik) = -\frac{\pi}{2}$

EXAMPLE 5 : *Find the principal argument of all complex numbers represented by ± 1 ± i.*

Solution : Here four points exist. $(1 + i)$, $(-1 + i)$, $(-1 - i)$, $(1 - i)$

$$\tan\theta = \left|\frac{y}{x}\right| = 1 \Rightarrow \theta = \frac{\pi}{4}$$

(i) $\arg(1 + i) = \theta = \frac{\pi}{4}$ (1st quadrant)

(ii) $\arg(-1 + i) = \pi - \theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$ (2nd quadrant)

(iii) $\arg(-1 - i) = \theta - \pi = \frac{\pi}{4} - \pi = -\frac{3\pi}{4}$ (3rd quadrant)

(iv) $\arg(1 - i) = -\theta = -\frac{\pi}{4}$ (4th quadrant)

EXAMPLE 6 : *If $z = x + iy$, then find the value of $\arg|z|$.*

Solution : As we know that $|z| \ge 0$. If $|z| > 0$, then $\arg|z| = 0$. If $|z| = 0$, then $\arg|z|$ is not defined.

EXAMPLE 7 : *Find the sum $\arg(1) + \arg(2i) + \arg(-i) + \arg(-1)$*

Solution : We have, $\arg(1) = 0$, $\arg(-1) = \pi$, $\arg(2i) = \frac{\pi}{2}$, $\arg(-i) = -\frac{\pi}{2}$.

Their sum $= 0 + \pi + \frac{\pi}{2} - \frac{\pi}{2} = \pi$

DIFFERENT FORMS OF COMPLEX NUMBERS

A complex number may have three different forms

(i) Cartesian form

$z = x + iy, x \in R, y \in R$ is known as Cartesian form. We have already discussed that $x = \text{Re}(z)$, $y = \text{Im}(z)$

Uses

This form is generally used

(a) To locate any complex number in argand plane

(b) To find the locus of complex number

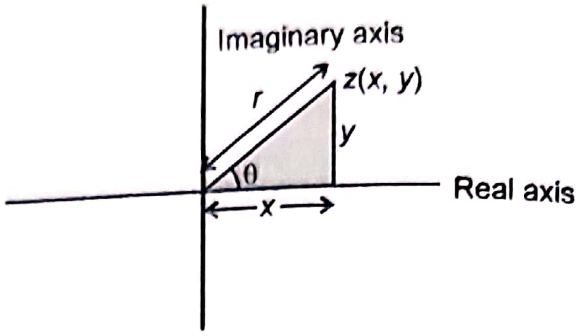
(ii) Polar form (r, θ)

$z = r(\cos\theta + i\sin\theta) = (r, \theta)$ is known as polar form of complex number, where $r = |z|$, $\theta = \arg(z)$

By the given triangle we find that $x = r\cos\theta$, $y = r\sin\theta$

$\Rightarrow$ $z = x + iy = r(\cos\theta + i\sin\theta) = r\,\text{cis}\theta$,

where $\text{cis}\theta = \cos\theta + i\sin\theta = e^{i\theta}$



Uses

This form is generally used

(a) When powers of complex number is very high.

(b) In problems of maximum and minimum.

(c) To find the *n*th roots of unity.

(iii) Euler form

$z = re^{i\theta}$ is known as Euler's form, where $r = |z|$, $\theta = \arg(z)$. Euler form can be converted to p
form by using $e^{i\theta} = \cos\theta + i\sin\theta$

$\Rightarrow \quad z = re^{i\theta} = r(\cos\theta + i\sin\theta)$

EXAMPLE 8 : *If $z = 1 + \sqrt{3}i$, represent this in polar form.*

Solution : $z = 1 + \sqrt{3}i$, $|z| = 2$

$\arg(z) = \frac{\pi}{3}$

$\therefore$ Polar form $= 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

EXAMPLE 9 : *Multiply $z_1 = 1 + i$ and $z_2 = \sqrt{3} + i$ using Euler form.*

Solution : $z_1 = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{2}e^{i\frac{\pi}{4}}$

$z_2 = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = 2e^{i\frac{\pi}{6}}$

$\therefore \quad z_1 z_2 = 2\sqrt{2}e^{i\left(\frac{\pi}{6}+\frac{\pi}{4}\right)} = 2\sqrt{2}e^{i\left(\frac{5\pi}{12}\right)} = 2\sqrt{2}\left(\left(\frac{(\sqrt{3}-1)}{2\sqrt{2}}\right) + i\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)\right) = (\sqrt{3}-1) + i(\sqrt{3}+1)$

ALGEBRA OF COMPLEX NUMBERS

(i) Equality of two Complex Numbers :

Two given complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are said to be equal iff their respec
real and imaginary parts are equal.

i.e., $z_1 = z_2 \Leftrightarrow \quad x_1 + iy_1 = x_2 + iy_2 (x_1, x_2, y_1, y_2 \in R)$

$\Leftrightarrow \quad x_1 = x_2$ and $y_1 = y_2$

Note : *The Inequality does not hold good in case of imaginary numbers i.e. $z_1 \ngtr z_2$ or z_1 ≮*

For example, $2 + 6i > 1 - 8i$

or $6 - 7i < 8 + 3i$

make no sense.

EXAMPLE 10 : *Find the values of x and y, if $2x + 3yi = 2 + 12i$, where $x, y \in R$.*

Solution : $2x + (3y)i = 2 + 12i$

$\Rightarrow$ $2x = 2$ and $3y = 12$ (Equating real and imaginary parts separately)

$\Rightarrow$ $x = 1$ and $y = 4$

(ii) Addition of Two Complex Numbers :

Let $z_1 = a + ib$ and $z_2 = c + id$ be any two complex numbers.

Then, the sum $z_1 + z_2$ is defined as follows :

$z_1 + z_2 = (a + c) + i(b + d)$, which is again a complex number.

e.g., If $z_1 = 4 + i5$ and $z_2 = 3 + i(-4)$, then $z_1 + z_2 = (4 + 3) + i(5 - 4) = 7 + i$.

In polar form or cis form

$$z_1 + z_2 = r_1(\cos\theta_1 + i\sin\theta_1) + r_2(\cos\theta_2 + i\sin\theta_2)$$
$$= (r_1\cos\theta_1 + r_2\cos\theta_2) + i(r_1\sin\theta_1 + r_2\sin\theta_2)$$

Properties of addition of complex numbers :

The addition of complex numbers satisfy the following properties

(a) Closure law : The sum of two complex numbers is a complex number.

i.e., if z_1 and z_2 are any two complex numbers, then $z_1 + z_2$ is also a complex number

Proof : Let $z_1 = a + ib$ and $z_2 = c + id$ be two complex numbers, then $z_1 + z_2 = (a + ib) + (c + id) = (a + c) + i(b + d)$, which is also a complex number.

(b) Commutative law : For any two complex numbers z_1 and z_2,

$$z_1 + z_2 = z_2 + z_1$$

Proof : Let $z_1 = a + ib$ and $z_2 = c + id$ be two complex numbers, then $(a, b, c, d \in R)$

$$z_1 + z_2 = (a + ib) + (c + id) = (a + c) + i(b + d)$$

and $$z_2 + z_1 = (c + id) + (a + ib) = (c + a) + i(d + b)$$

But we know that addition of real numbers is commutative, *i.e.*, $a + c = c + a$ and $b + d = d + b$

Thus, $z_1 + z_2 = z_2 + z_1$

(c) Associative law : For any three complex numbers z_1, z_2 and z_3

$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$

Proof : Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ and $z_3 = a_3 + ib_3$ be three complex numbers, then $(a_1, a_2, a_3, b_1, b_2, b_3 \in R)$

$$(z_1 + z_2) + z_3 = [(a_1 + ib_1) + (a_2 + ib_2)] + (a_3 + ib_3)$$
$$= [(a_1 + a_2) + i(b_1 + b_2)] + (a_3 + ib_3)$$
$$= [(a_1 + a_2) + a_3] + i[(b_1 + b_2) + b_3]$$
$$= [a_1 + (a_2 + a_3)] + i[b_1 + (b_2 + b_3)] \text{ [addition of real numbers is associative]}$$
$$= (a_1 + ib_1) + [(a_2 + a_3) + i(b_2 + b_3)]$$
$$= (a_1 + ib_1) + [(a_2 + ib_2) + (a_3 + ib_3)]$$
$$= z_1 + (z_2 + z_3)$$

Thus, $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$

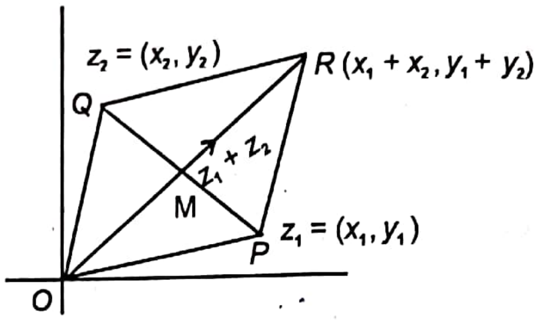
(d) **Additive identity** : For every complex number $x + iy$, there exists a complex number $0 + i0$ such that $(x + iy) + (0 + i0) = (x + 0) + i(y + 0) = x + iy$.

Here, $0 + i0$ (denoted as 0), called the additive identity.

(e) **Additive inverse** : For every complex number $z = a + ib$, we have the complex number $-a + i(-b)$ [denoted as $-z$], called the additive inverse or negative of z.

(iii) Geometrical Representation of $z_1 + z_2$:

Let $P(z_1)$ and $Q(z_2)$ be two given points with Argand plane as shown in the figure. Let us complete the parallelogram $OPRQ$ and let the diagonals intersect at M. We know that the diagonals in the parallelogram bisect each other. Hence M is the mid-point of the diagonal PQ and so the ordered pair $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ represents M, and consequently $R(x_1 + x_2, y_1 + y_2)$.



Thus the sum of two complex numbers (z_1) and (z_2) is represented by the diagonal OR of the parallelogram $OPRQ$.

Example : $(2 + 6i) + (8 + 7i) = (2 + 8) + i(6 + 7) = 10 + 13i$

(iv) Subtraction :

$z_1 - z_2 = (x_1 + iy_1) - (x_2 + iy_2)$ (Here $x_1, x_2, y_1, y_2 \in R$)

$= (x_1 - x_2) + i(y_1 - y_2)$

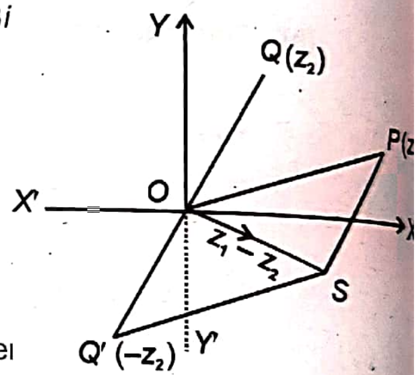
$= (\text{Re}(z_1) - \text{Re}(z_2)) + i(\text{Im}(z_1) - \text{Im}(z_2))$

In polar form

$z_1 - z_2 = r_1(\cos\theta_1 + i\sin\theta_1) - r_2(\cos\theta_2 + i\sin\theta_2)$

$= (r_1\cos\theta_1 - r_2\cos\theta_2) + i(r_1\sin\theta_1 - r_2\sin\theta_2)$

In parallelogram $OPSQ'$, S represents the complex number



(v) Multiplication :

We have $z_1.z_2 = (x_1 + iy_1)(x_2 + iy_2) = x_1x_2 + ix_1y_2 + ix_2y_1 + i^2y_1y_2$ (Here $x_1, x_2, y_1, y_2 \in R$)

$\Rightarrow \quad z_1z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$

Which shows that

$\text{Re}(z_1z_2) = \text{Re}(z_1)\,\text{Re}(z_2) - \text{Im}(z_1)\,\text{Im}(z_2)$

$\text{Im}(z_1z_2) = \text{Re}(z_1)\,\text{Im}(z_2) + \text{Re}(z_2)\,\text{Im}(z_1)$

In polar form $z_1z_2 = r_1(\cos\theta_1 + i\sin\theta_1)\,r_2(\cos\theta_2 + i\sin\theta_2)$

$= r_1r_2e^{i(\theta_1 + \theta_2)}$

$\Rightarrow \quad z_1z_2 = r_1r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$

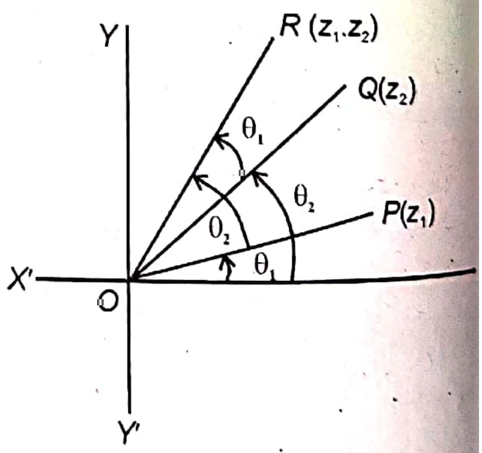
Let $P(z_1 = r_1e^{i\theta_1})$ and $Q(z_2 = r_2e^{i\theta_2})$ be two given points in the Argand plane so that

$OP = r_1 = |z_1|$

$OQ = r_2 = |z_2|$

$\angle XOP = \theta_1 = \arg(z_1)$

$\angle XOQ = \theta_2 = \arg(z_2)$



Then, $z_1z_2 = r_1e^{i\theta_1}r_2e^{i\theta_2} = r_1r_2e^{i(\theta_1+\theta_2)} = r_1r_2[\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)]$

which shows that

$|z_1z_2| = r_1r_2 = |z_1|\,|z_2|$

$\arg(z_1z_2) = \theta_1 + \theta_2 = \arg(z_1) + \arg(z_2)$

But principal $\text{amp}(z_1z_2) = \text{amp}(z_1) + \text{amp}(z_2) + 2k\pi$, where $k \in \{1, 0, -1\}$

Here we observe that when we multiply two complex numbers z_1 and z_2, the resulting number is also a complex number which can be obtained by rotating one complex number through the argument of the other in anticlockwise direction about origin of reference.

Properties of multiplication of complex numbers :

(a) **Closure law :** The product of two complex numbers is a complex number, *i.e.*, if z_1 and z_2 are two complex numbers, then z_1z_2 is also a complex number.

Proof : Let $z_1 = a + ib$ and $z_2 = c + id$ be two complex numbers ($a, b, c, d \in R$)

then, $z_1z_2 = (a + ib)(c + id) = (ac - bd) + i(ad + bc)$,

which is also a complex number.

(b) **Commutative law :** For any two complex numbers z_1 and z_2, we have

$z_1z_2 = z_2z_1$.

Proof : Let $z_1 = a + ib$ and $z_2 = c + id$ be two complex numbers, then ($a, b, c, d \in R$)

$z_1z_2 = (a + ib)(c + id) = (ac - bd) + i(ad + bc)$

and $z_2z_1 = (c + id)(a + ib) = (ca - db) + i(cb + da)$

Since addition and multiplication of real numbers is commutative

$\therefore$ $ac - bd = ca - db$ and $ad + bc = cb + da$

$\Rightarrow$ $z_1z_2 = z_2z_1$

(c) **Associative law :** For any three complex numbers z_1, z_2, z_3, we have $(z_1z_2)z_3 = z_1(z_2z_3)$.

Proof : Let $z_1 = a + ib$, $z_2 = c + id$ and $z_3 = e + if$ be three complex numbers, then ($a, b, c, d, e, f \in R$)

$$\begin{aligned}(z_1z_2)z_3 &= [(a + ib)(c + id)](e + if)\\ &= [(ac - bd) + i(ad + bc)](e + if)\\ &= [(ac - bd)e - (ad + bc)f] + i[(ac - bd)f \pm (ad + bc)e]\\ &= [ace - bde - adf - bcf] + i[acf - bdf + ade + bce]\end{aligned}$$

and
$$\begin{aligned}z_1(z_2z_3) &= (a + ib)[(c + id)(e + if)]\\ &= (a + ib)[(ce - df) + i(cf + de)]\\ &= [a(ce - df) - b(cf + de)] + i[a(cf + de) + b(ce - df)]\\ &= (ace - adf - bcf - bde) + i(acf + ade + bce - bdf)\end{aligned}$$

Since addition of real numbers is commutative and associative

$\therefore$ $ace - bde - adf - bcf = ace - adf - bcf - bde$

and $acf - bdf + ade + bce = acf + ade + bce - bdf$

Thus, $(z_1z_2)z_3 = z_1(z_2z_3)$

(d) **Multiplicative identity :** For any complex number $x + iy$, there exists complex number $1 + i0$ such that, $(x + iy)(1 + i0) = (x\cdot1 - y\cdot0) + i(x\cdot0 + y\cdot1)$

$= x + iy$

$1 + i0$ is called the multiplicative identity.

(e) **Multiplicative inverse :** For every non-zero complex number $z = x + iy$, $(x \neq 0, y \neq 0)$, we have the complex number $\frac{x - iy}{x^2 + y^2}$ such that $(x + iy)\frac{x - iy}{x^2 + y^2} = 1 + i0 = 1$ (multiplicative identity) and $\frac{x - iy}{x^2 + y^2} = \frac{x}{x^2 + y^2} - \frac{iy}{x^2 + y^2}$ is called the multiplicative inverse of $x + iy$.

Note : *The multiplicative inverse (M.I) of a non-zero complex number is calculated as follows*

Multiplicative inverse of $x + iy = \frac{1}{x + iy} = \frac{1}{x + iy} \cdot \frac{x - iy}{x - iy}$

$$= \frac{x - iy}{x^2 - i^2 y^2} = \frac{x - iy}{x^2 + y^2} \quad (x, y \in R)$$

$$= \frac{x}{x^2 + y^2} - \frac{iy}{x^2 + y^2}$$

(vi) Distributive law : For any three complex numbers, z_1, z_2, z_3, we have

$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$ and $(z_1 + z_2)z_3 = z_1 z_3 + z_2 z_3$

Proof : Let $z_1 = a + ib$, $z_2 = c + id$ and $z_3 = e + if$ be three complex numbers, then $(a, b, c, d, e, f \in R)$

$$\begin{aligned} z_1(z_2 + z_3) &= (a + ib)[(c + id) + (e + if)] \\ &= (a + ib)[(c + e) + i(d + f)] \\ &= [(a(c + e) - b(d + f)) + i(a(d + f) + b(c + e))] \\ &= (ac + ae - bd - bf) + i(ad + af + bc + be) \end{aligned}$$

and

$$\begin{aligned} z_1 z_2 + z_1 z_3 &= [(a + ib)(c + id)] + [(a + ib)(e + if)] \\ &= [(ac - bd) + i(ad + bc)] + [(ae - bf) + i(af + be)] \\ &= (ac - bd + ae - bf) + i(ad + bc + af + be) \end{aligned}$$

Since addition of real numbers is commutative and associative

$\therefore \quad ac + ae - bd - bf = ac - bd + ae - bf$

and $ad + af + bc + be = ad + bc + af + be$

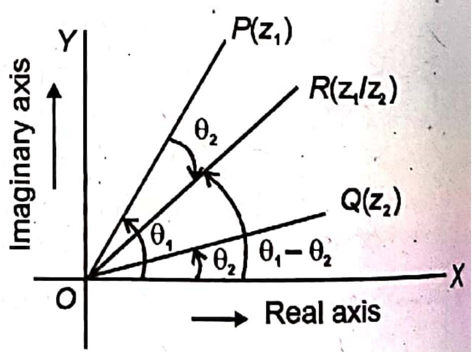
Thus, $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$

Similarly, $(z_1 + z_2)z_3 = z_1 z_3 + z_2 z_3$

(vii) Division :

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} \times \frac{x_2 - iy_2}{x_2 - iy_2} = \frac{(x_1 x_2 + y_1 y_2) + i(x_2 y_1 - x_1 y_2)}{x_2^2 + y_2^2}$$; (Provided $z_2 \neq 0$) $(x_1, x_2, y_1, y_2 \in R)$

$$= \left(\frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2}\right) + i\left(\frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}\right)$$



In polar form

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)} = \frac{|z_1|}{|z_2|} e^{i(\arg(z_1) - \arg(z_2))}$$

which shows that

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \text{ and } \arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$$

But principal amp $\left(\frac{z_1}{z_2}\right) = \text{amp}(z_1) - \text{amp}(z_2) + 2k\pi$, where $k \in \{-1, 0, 1\}$

Here we observe that when we divide two complex numbers, the resulting number is also a complex number which can be obtained by rotating the complex number on numerator through the amplitude of the complex number on denominator in clockwise direction about origin of reference.

(viii) Square root of a complex number :

Let $z = x + iy$ be a complex number then $(x, y \in R)$

$$\sqrt{z} = \sqrt{x+iy} = \pm(a+ib)\,(say)\ \ (a, b \in R)$$

Squaring both sides:

$x + iy = a^2 - b^2 + i2ab.$

$\therefore \quad a^2 - b^2 = x$ ...(i)

and $2ab = y$ ...(ii)

$\therefore \quad a^2 + b^2 = \sqrt{x^2+y^2}$ ...(iii)

from eq. (i) and (iii)

$$\therefore \quad a^2 = \frac{\sqrt{x^2+y^2}+x}{2} \text{ and } b^2 = \frac{\sqrt{x^2+y^2}-x}{2}$$

$$\therefore \quad \sqrt{x+iy} = \pm\left(\sqrt{\frac{\sqrt{x^2+y^2}+x}{2}} + i\sqrt{\frac{\sqrt{x^2+y^2}-x}{2}}\right), y \ge 0$$

$$\text{and} \quad \sqrt{x-iy} = \pm\left(\frac{\sqrt{\sqrt{x^2+y^2}+x}}{2} - i\frac{\sqrt{\sqrt{x^2+y^2}-x}}{2}\right), y < 0$$

EXAMPLE 11 : *If $z_1 = 2 + 3i$ and $z_2 = -1 + 2i$, then find*

(i) $z_1 + z_2$ *(ii)* $z_1 - z_2$ *(iii)* $z_1 \cdot z_2$ *(iv)* $\frac{z_1}{z_2}$

Solution : We have, $z_1 = 2 + 3i$ and $z_2 = -1 + 2i$

(i) $z_1 + z_2 = (2 + 3i) + (-1 + 2i) = (2 - 1) + (3 + 2)i = 1 + 5i$

(ii) $z_1 - z_2 = (2 + 3i) - (-1 + 2i) = (2 + 1) + (3 - 2)i = 3 + i$

(iii) $z_1z_2 = (2 + 3i)(-1 + 2i) = (-2 - 6) + (4 - 3)i = -8 + i$

(iv) $\frac{z_1}{z_2} = \frac{2+3i}{-1+2i} = (2+3i)\cdot\frac{1}{-1+2i}$

$$= (2+3i)\cdot\frac{1}{-1+2i}\cdot\frac{-1-2i}{-1-2i} = \frac{(2+3i)(-1-2i)}{(-1)^2-(2i)^2}$$

$$= \frac{1}{5}[(-2+6)+(-4-3)i] = \frac{1}{5}(4-7i)$$

EXAMPLE 12 : *If $z_1 = 3 - 2i$, $z_2 = 2 - i$ and $z_3 = 2 + 5i$, then find $z_1 + z_2 - 2z_3$.*

Solution : $z_1 + z_2 - 2z_3 = (3 - 2i) + (2 - i) - 2(2 + 5i)$

$= [(3 - 2i) + (2 - i)] - (4 + 10i)$

$= [(3 + 2) + i(-2 - 1)] - (4 + 10i)$

$= (5 - 3i) - (4 + 10i)$

$= (5 - 4) + i(-3 - 10)$

$= 1 - 13i$

EXAMPLE 13 : *If $z = 4 + 7i$ be a complex number, then find*

(i) Additive inverse of z *(ii) Multiplicative inverse of z*

Solution :

(i) Additive inverse of $(4 + 7i) = (-4 - 7i)$

(ii) Multiplicative inverse of $(4 + 7i)$ $= \frac{1}{4+7i} = \frac{1}{4+7i}\cdot\frac{4-7i}{4-7i}$

$$= \frac{(4-7i)}{(4)^2-(7i)^2} = \frac{4-7i}{65}$$

$$= \frac{4}{65} - \frac{7}{65}i$$

EXAMPLE 14 : *Express the following in the form of a + ib.*

(i) $\left(\frac{1}{4}+4i\right)^3$ (ii) $(1-2i)^{-3}$

Solution : (i) $\left(\frac{1}{4}+4i\right)^3=\left(\frac{1}{4}\right)^3+3\left(\frac{1}{4}\right)(4i)^2+3\left(\frac{1}{4}\right)^2(4i)+(4i)^3$

$$=\frac{1}{64}+12i^2+\frac{3}{4}i+64i^3$$

$$=\frac{1}{64}-12+\frac{3}{4}i-64i \qquad [\because i^2=-1 \text{ and } i^3=-i]$$

$$=-\frac{767}{64}+\left(\frac{-253}{4}\right)i$$

(ii) $(1-2i)^{-3}=\frac{1}{(1-2i)^3}$

$$=\frac{1}{(1)^3-3(1)^2(2i)+3(1)(-2i)^2-(2i)^3}$$

$$=\frac{1}{1-6i-12+8i}$$

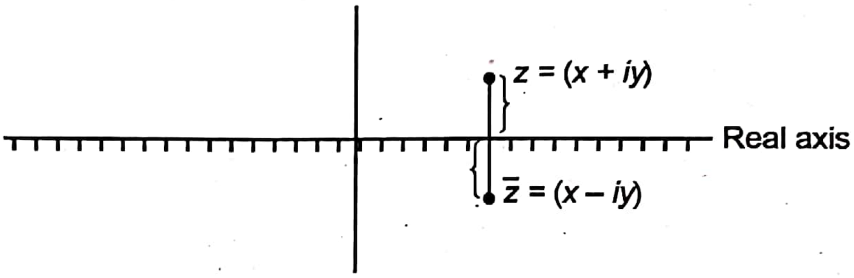
$$=\frac{1}{-11+2i}=\frac{1}{-11+2i}\left(\frac{-11-2i}{-11-2i}\right)$$

$$=\frac{-11-2i}{(-11)^2-(2i)^2}=\frac{-11-2i}{125}$$

$$=-\frac{11}{125}+\left(\frac{-2}{125}\right)i$$

Conjugate of a Complex Number

The reflection of a complex number $z = x + iy$ $(x, y \in R)$ in the real axis mirror is called the conjugate of z. This point is denoted by $\bar{z} = x - iy$.



Properties of Conjugate of a Complex Number

(i) $\overline{(\bar{z})}=z$

(ii) $|z| = |\bar{z}| = |-z| = |-\bar{z}| = |iz| = |i\bar{z}|$

(iii) $z\bar{z} = |z|^2 = (\text{Re}(z))^2 + (\text{Im}(z))^2$

(iv) $\text{Re}(z) = \frac{z+\bar{z}}{2}$, $Im(z)=\frac{z-\bar{z}}{2i}$

If z is purely real number, then Im(z) = 0 $\Rightarrow$ $z = \bar{z}$

If z is purely imaginary, then Re(z) = 0 $\Rightarrow$ $z + \bar{z} = 0$ or $z = -\bar{z}$.

(v) $\overline{(z_1+z_2)}=\bar{z}_1+\bar{z}_2$

In general $\overline{z_1+z_2+\ldots\ldots+z_n}=\bar{z}_1+\bar{z}_2+\bar{z}_3+\ldots\ldots+\bar{z}_n$

(vi) $\overline{z_1\cdot z_2}=\bar{z}_1\cdot\bar{z}_2$ or in general $\overline{z_1\cdot z_2\ldots\ldots z_n}=\bar{z}_1\cdot\bar{z}_2\ldots\ldots\bar{z}_n$

And also we may write that $\overline{(z^n)}=(\bar{z})^n$

(vii) If $z_2 \neq 0$, then $\overline{\left(\frac{z_1}{z_2}\right)}=\frac{\bar{z}_1}{\bar{z}_2}$

(viii) $\arg(z) + \arg(\bar{z}) = 2k\pi$, $k \in Z$

EXAMPLE 15 : *If $z_1 = \bar{z}_1$ and $z_2 = -\bar{z}_2$ and z_1 and z_2 both are non-zero complex numbers, then find the sum of all possible principal values of $\arg(z_1)$ and $\arg(z_2)$.*

Solution : If $z_1 = \bar{z}_1$ then z_1 is purely real number and in this case $\arg(z_1) = 0$ or π. Similarly if $z_2 = -\bar{z}_2$, then z_2 is purely imaginary number and in this case $\arg(z_2) = \frac{\pi}{2}, -\frac{\pi}{2}$

Hence sum $= 0 + \pi + \frac{\pi}{2} - \frac{\pi}{2} = \pi$.

EXAMPLE 16 : *Find real values of x and y for which the complex numbers $-5 + ix^2y$ and $x^2 + y + 36i$ are conjugate of each other.*

Solution : Since $-5 + ix^2y$ and $x^2 + y + 36i$ are conjugate of each other

$\therefore \quad -5 + ix^2y = x^2 + y - 36i$

Equate real and imaginary parts

$\Rightarrow \quad -5 = x^2 + y \quad ...(i)$

and $x^2y = -36 \quad ...(ii)$

From (i) and (ii), we get

$$x^2 - \frac{36}{x^2} = -5$$

$\Rightarrow \quad x^4 + 5x^2 - 36 = 0$

$\Rightarrow \quad (x^2 + 9)(x^2 - 4) = 0$

$\Rightarrow \quad x^2 - 4 = 0$ [$\because x^2 + 9 \neq 0$ for any real x]

$\Rightarrow \quad x = \pm 2$

From (i) we get

$x = 2, y = -9$ or $x = -2, y = -9$

any fraction must have real numbers denominator

Cannot compare complex nos

EXERCISE

[Basics for Complex Numbers]

1. $\sqrt{-5}\sqrt{-\frac{16}{5}} =$ √5 i

   (1) 4 (2) –4 (3) 3 (4) 2

2. The real values of x and y for which the equation is $(x + iy)(2 - 3i) = 4 + i$ is satisfied are

   (1) $x = \frac{5}{13}, y = \frac{8}{13}$ (2) $x = \frac{8}{13}, y = \frac{5}{13}$ (3) $x = \frac{5}{13}, y = \frac{14}{13}$ (4) $x = -\frac{5}{13}, y = \frac{8}{13}$

3. If $(x + iy)(p + iq) = (x^2 + y^2)i$, then, where $x, y, p, q \in R$

   (1) $p = x, q = y$ (2) $p = x^2, q = y^2$ (3) $x = q, y = p$ (4) $x = -q, y = p$

4. $\left|(1+i)\frac{(2+i)}{(3+i)}\right|$ equals Property of modulus

   (1) $-\frac{1}{2}$ (2) $\frac{1}{2}$ (3) 1 (4) –1

5. $a + ib > c + id$ is meaningful only when $(a, b, c, d \in R)$

   (1) $b = 0, c = 0$ (2) $b = 0, d = 0$ (3) $a = 0, c = 0$ (4) $a = 0, d = 0$

6. The real values of x and y for which the equation $(x^4 + 2xi) - (3x^2 + yi) = (3 - 5i) + (1 + 2yi)$ is satisfied, is

   (1) $x = -2$ and $y = \frac{1}{3}$ (2) $x = 3, \frac{1}{3}, y = \pm 2$

   $x = 2$ and $y = 3$

   (3) $x = \pm 1, y = 2$ (4) $x = \pm 2, y = 1$

   $x^4 - 3x^2 - 4 = 0$

   $x^2 = a$

7. If $a^2 + b^2 = 1$, then $\frac{1+b+ia}{1+b-ia}$ is equal to, (where $a, b \in R$).

(1) 1 (2) 2 (3) $b + ia$ (4) $a + ib$

8. If $z = 3 - 4i$, then $z^4 - 3z^3 + 3z^2 + 99z - 95$ is equal to

(1) 5 (2) 6 (3) –5 (4) –4

9. If $z = x - iy$ and $z^{1/3} = p + iq$, then $\frac{\left(\frac{x}{p}+\frac{y}{q}\right)}{p^2+q^2}$ is equal to, where $x, y, p, q \in R$

(1) –2 (2) –1 (3) 2 (4) 1

PROPERTIES OF MODULUS OF A COMPLEX NUMBER

The following properties of modulus of complex numbers hold good:

(i) $|z| = 0$ iff $z = 0$ and $|z| > 0$ iff $z \neq 0$.

(ii) $-|z| \le \text{Re}(z) \le |z|$ and

$-|z| \le \text{Im}(z) \le |z|$

(iii) $|z_1 z_2| = |z_1|\,|z_2|$

In general $|z_1 z_2 z_3 \ldots z_n| = |z_1|\,|z_2|\,|z_3| \ldots |z_n|$

(iv) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, where $z_2 \neq 0$

(v) $|z_1 \pm z_2|^2 = |z_2|^2 + |z_1|^2 \pm 2\text{Re}(z_1 \bar{z}_2) = |z_1|^2 + |z_2|^2 \pm 2\text{Re}(\bar{z}_1 . z_2)$

We have,

$|z_1 \pm z_2|^2 = (z_1 \pm z_2)(\overline{z_1 \pm z_2})$

$= (z_1 \pm z_2)(\bar{z}_1 \pm \bar{z}_2)$

$= z_1\bar{z}_1 \pm z_1\bar{z}_2 \pm \bar{z}_1 z_2 + z_2\bar{z}_2$

$= |z_1|^2 + |z_2|^2 \pm (z_1\bar{z}_2 + \overline{z_1\bar{z}_2})$

$= |z_1|^2 + |z_2|^2 \pm 2\text{Re}(z_1 . \bar{z}_2)$

(vi) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$

(vii) $|z_1 + z_2| \le |z_1| + |z_2|$

(viii) $|z_1 - z_2| \ge \big|\,|z_1| - |z_2|\,\big|$

(ix) $\big|\,|z_1| - |z_2|\,\big| \le |z_1| + |z_2|$

(x) $|z_1 + z_2| = |z_1 - z_2| \Leftrightarrow \arg(z_1) - \arg(z_2) = \frac{\pi}{2}$

(xi) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 \Leftrightarrow \frac{z_1}{z_2}$ is purely imaginary number.

(xii) If $|z_1| \le 1$ and $|z_2| \le 1$, then

$|z_1 - z_2|^2 \le (|z_1| - |z_2|)^2 + (\arg z_1 - \arg z_2)^2$ and $|z_1 + z_2|^2 \le (|z_1| + |z_2|)^2 - (\arg z_1 - \arg z_2)^2$

(xiii) $|z_1 + z_2 + \ldots + z_n| = |\bar{z}_1 + \bar{z}_2 + \ldots + \bar{z}_n|$

(xiv) The minimum value of $|z - z_1| + |z - z_2| = |z_1 - z_2|$

(xv) $|z|^2 = z\bar{z}$

PROPERTIES OF ARGUMENT OF A COMPLEX NUMBER

The following properties regarding argument of complex numbers hold good:

(i) $\arg(\bar{z}) = -\arg(z)$, provided z is not negative purely real number.

(ii) $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) + 2k\pi \quad (k = 0, 1 \text{ or } -1)$

In general $\arg(z_1 z_2 z_3 \ldots z_n) = \arg(z_1) + \arg(z_2) + \arg(z_3) + \ldots\ldots + \arg(z_n) + 2k\pi,\ k \in I$.

(iii) $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) + 2k\pi;\ k = 0, 1 \text{ or } -1$

(iv) $\arg\left(\frac{z}{\bar{z}}\right) = 2\arg(z) + 2k\pi;\ (k = 0, 1 \text{ or } -1)$

(v) $\arg(z^n) = n\arg(z) + 2k\pi$; k is integer depending on 'n'

(vi) If $\arg(z) = 0$ or π, then z is purely real number.

(vii) If $\arg(z) = \frac{\pi}{2}$ or $-\frac{\pi}{2}$, then z is purely imaginary number.

Remarks : The proper value of k must be chosen so that argument lies in $(-\pi, \pi]$.

EXAMPLE 17 : *If $\arg(z_1) = \frac{2\pi}{3}$ and $\arg(z_2) = \frac{\pi}{2}$ then find the principal value of $\arg(z_1z_2)$ and also find the quadrant of z_1z_2.*

Solution :

$$\arg(z_1z_2) = \arg(z_1) + \arg(z_2) + 2k\pi$$
$$= \frac{2\pi}{3} + \frac{\pi}{2} + 2k\pi$$
$$= \frac{7\pi}{6} + 2k\pi$$

To bring the argument in the principal range $k = -1$

$\arg(z_1z_2) = \frac{7\pi}{6} - 2\pi = -\frac{5\pi}{6}$ which shows that z_1z_2 lies in III[rd] quadrant.

EXAMPLE 18 : *If $z = \frac{(1+i)(1-i)}{(2-i)}$, find the principal argument of z and also write the polar form of z.*

Solution :

$$z = \frac{(1+i)(1-i)}{(2-i)} = \frac{2(2+i)}{(2^2-i^2)} = \frac{2}{5}(2+i) = \frac{4}{5} + \frac{i2}{5}$$

Principal argument of $z = \tan^{-1}\left(\frac{1}{2}\right)$

$$|z| = \sqrt{\frac{16}{25} + \frac{4}{25}} = \sqrt{\frac{20}{25}} = \sqrt{\frac{4}{5}}$$

Polar form $z = \sqrt{\frac{4}{5}}(\cos\theta + i\sin\theta)$, where $\theta = \tan^{-1}\frac{1}{2}$.

EXAMPLE 19 : *Find the complex number z if $z\bar{z} = 2$ and $z + \bar{z} = 2$.*

Solution :

Let, $z = x + iy$

$z\bar{z} = 2 \Rightarrow (x+iy)(x-iy) = 2 \Rightarrow x^2 + y^2 = 2$ ...(i)

Also, $z + \bar{z} = x + iy + x - iy = 2x = 2 \Rightarrow x = 1$ ...(ii)

By (i) and (ii), we have

$x = 1, y = \pm 1$

Hence, $z = 1 + i$ or $1 - i$

SOLVED EXAMPLES

EXAMPLE 20 : *Find the modulus and principal argument of* $\frac{2+i}{4i+(1+i)^2}$.

Solution : We have,

$$\frac{2+i}{4i+(i+1)^2} = \frac{2+i}{4i+i^2+1+2i} = \frac{2+i}{6i} = \frac{1}{6} - i\frac{1}{3}$$

$$\Rightarrow \left|\frac{2+i}{4i+(1+i)^2}\right| = \sqrt{\frac{1}{36}+\frac{1}{9}} = \frac{\sqrt{5}}{6} \text{ and } \arg\left(\frac{2+i}{4i+(1+i)^2}\right) = -\tan^{-1}\left(\frac{\frac{1}{3}}{\frac{1}{6}}\right) = -\tan^{-1}(2)$$

EXAMPLE 21 : *If the roots of the equation* $z^2 + (p + iq)z + r + is = 0$ *are real where* $p, q, r, s, \in R$, *then determine* $s^2 + q^2r$

Solution : Let $x \in R$ be a root of the given equation

$\Rightarrow \quad x^2 + (p + iq)x + r + is = 0$

$\Rightarrow \quad (x^2 + px + r) + i(qx + s) = 0 = 0 + i.0$

Equating real and imaginary parts from respective sides we get $x^2 + px + r = 0$ and $qx + s = 0$

$\Rightarrow \quad x = -\frac{s}{q}$

Substituting for x in the first relation, we have

$$\frac{s^2}{q^2} - \frac{ps}{q} + r = 0$$

$\Rightarrow \quad (s^2 + q^2r) = pqs$

EXAMPLE 22 : *Find the greatest and least values of* $|z_1 + z_2|$ *if* $z_1 = 24 + 7i$ *and* $|z_2| = 7$.

Solution : We have, $|z_1| = |24 + 7i| = 25$ and $|z_2| = 7$

$\therefore \quad$ Greatest $|z_1 + z_2| = |z_1| + |z_2| = 25 + 7 = 32$

Least $|z_1 + z_2| = ||z_1| - |z_2|| = 25 - 7 = 18$

EXAMPLE 23 : *If* $|z-2| = z + 2(1-i)$, *then z is equal to*

(1) $\frac{1}{2} - 4i$ (2) $\frac{1}{2} + 2i$

(3) $1 + 2i$ (4) $\frac{3}{2} - 2i$

Solution : Let $z = x + iy$

$\Rightarrow \quad |x - 2 + iy| = x + iy + 2(1-i)$

$\Rightarrow \quad \sqrt{(x-2)^2 + y^2} = x + 2 + i(y-2)$

$\therefore \quad (x-2)^2 + y^2 = (x+2)^2 \quad \text{and} \quad 0 = y - 2$

$\therefore \quad y = 2, \quad \text{and} \quad x = \frac{1}{2}$

$\therefore \quad z = x + iy = \frac{1}{2} + 2i$

Hence, correct answer is (2).

EXAMPLE 24 : *If $iz^3 + z^2 - z + i = 0$, then show that $|z| = 1$.*

Solution : We have,

$iz^3 + z^2 - z + i = 0$

$\Rightarrow \quad i(z)(z^2 + i) + (z^2 + i) = 0$

$\Rightarrow \quad (z^2 + i)(iz + 1) = 0$

$\Rightarrow \quad$ Either $z^2 + i = 0$ or $iz + 1 = 0$

when $z^2 + i = 0$

$\Rightarrow \quad z^2 = -i$

$\Rightarrow \quad |z^2| = |-i| = 1$

$\Rightarrow \quad |z| = 1$, as $|z|$ can't be negative.

when $iz + 1 = 0$

$\Rightarrow \quad z = -\dfrac{1}{i} = i$

$\Rightarrow \quad |z| = |i| = 1.$

In either case, $|z| = 1$.

EXAMPLE 25 : *If $z_1, z_2, z_3, \ldots, z_n$ are complex numbers such that $|z_1| = |z_2| = \ldots = |z_n|$ $= \left|\dfrac{1}{z_1} + \dfrac{1}{z_2} + \dfrac{1}{z_3} + \ldots + \dfrac{1}{z_n}\right| = 1$, then show that $|z_1 + z_2 + z_3 + \ldots + z_n| = 1$.*

Solution : $\because \quad |z| = 1$

$\Rightarrow \quad z\bar{z} = 1$

$\Rightarrow \quad \dfrac{1}{z} = \bar{z}$

Now, $\left|\dfrac{1}{z_1} + \dfrac{1}{z_2} + \dfrac{1}{z_3} + \ldots + \dfrac{1}{z_n}\right| = 1$

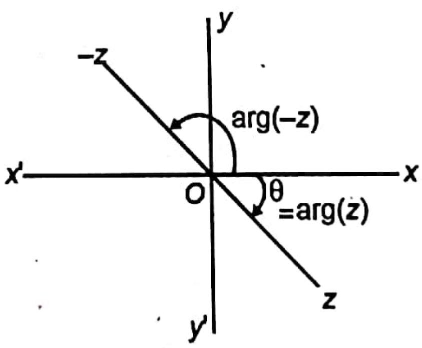
$\Rightarrow \quad |\bar{z}_1 + \bar{z}_2 + \bar{z}_3 + \ldots + \bar{z}_n| = 1$

$\Rightarrow \quad |\overline{z_1 + z_2 + z_3 + \ldots z_n}| = 1$

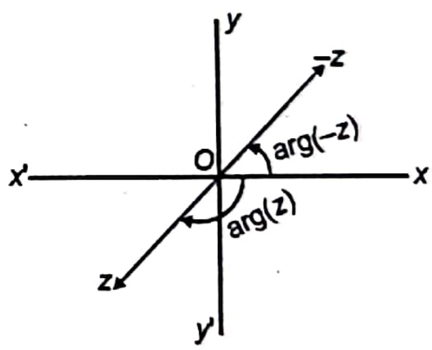
$\Rightarrow \quad |z_1 + z_2 + z_3 + \ldots + z_n| = 1$; as $|z| = |\bar{z}|$.

EXAMPLE 26 : *If $\arg(z) < 0$, then find $\arg(-z) - \arg(z)$.*

Solution : The following figures illustrates the answer



(i)



(ii)

Clearly $\arg(-z) - \arg(z) = \pi$

Remarks : (i) If the same question be asked in objective type question with suitable options, then students may verify the option by taking particular z whose $\arg(z) < 0$.

(ii) If $\arg(z) > 0$, then $\arg(-z) - \arg(z) = -\pi$

$\arg(z) - \arg(-z) = \pi$

and $\arg(z) < 0$, then $\arg(z) - \arg(-z) = -\pi$

Quick Recap

1. A number of the form $z = x + iy$, $x, y \in R$ where $i = \sqrt{-1}$ is called complex number.

   (i) Modulus of $z = x + iy$ is given by $\sqrt{x^2 + y^2}$

   (ii) Argument θ, of $z = x + iy$ is given by $\cos\theta = \frac{x}{\sqrt{x^2+y^2}}$ and $\sin\theta = \frac{y}{\sqrt{x^2+y^2}}$

2. $i^{4n} = 1$, $i^{4n+1} = i$, $i^{4n+2} = -1$, $i^{4n+3} = -i$. In general $i^{4n+r} = i^r$.

3. If $z_1 = a + ib$ and $z_2 = c + id$, then ($a, b, c, d \in R$)

   (i) $z_1 = z_2 \Rightarrow a = c, b = d$

   (ii) $z_1 + z_2 = (a + c) + i(b + d)$

   (iii) $z_1z_2 = (ac - bd) + i(ad + bc)$

   (iv) $z_1^{-1} = \left(\frac{a}{a^2+b^2}\right) + i\left(\frac{-b}{a^2+b^2}\right)$ (called multiplicative inverse)

4. **Conjugate :** If $z = a + ib$ ($a, b \in R$), then conjugate of z is denoted $\bar{z}$ and is given by $\bar{z} = a - ib$. Let z, z_1 and z_2 be complex numbers, then

   (i) $\bar{\bar{z}} = z$

   (ii) $z + \bar{z} = 2\text{Re}(z)$

   (iii) $z - \bar{z} = 2i\ \text{Im}(z)$

   (iv) $z = \bar{z} \Leftrightarrow z$ is purely real

   (v) $z + \bar{z} = 0 \Leftrightarrow z$ is purely imaginary

   (vi) $z\bar{z} = |z|^2$

   (vii) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

   (viii) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

   (ix) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$

5. **Modulus :** If $z = a + ib$ ($a, b \in R$), then modulus of z is denoted by $|z|$ and is given by $|z| = \sqrt{a^2 + b^2}$. Let z, z_1, z_2 be complex numbers, then

   (i) $|z| = 0 \Leftrightarrow z = 0$

(ii) $|z| = |-z| = |\bar{z}|$

(iii) $-|z| \le \text{Re}(z) \le |z|$; $-|z| \le \text{Im}(z) \le |z|$

(iv) $|z_1 z_2| = |z_1||z_2|$

(v) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

(vi) $|az_1 + bz_1|^2 = a^2|z_1|^2 + b^2|z_2|^2 + 2ab\,\text{Re}(z_1\bar{z_2})$

(vii) $|z_1 + z_2| \le \underbrace{|z_1| + |z_2|}_{max}$

(viii) $|z_1 - z_2| \ge \underbrace{||z_1| - |z_2||}_{min}$

(ix) If $\left|z + \frac{1}{z}\right| = k$, then $|z| \in \left[\frac{-k+\sqrt{k^2+4}}{2}, \frac{k+\sqrt{k^2+4}}{2}\right]$

6. **Argument or Amplitude :** Let $z = a + ib$, $\alpha = \tan^{-1}\left|\frac{b}{a}\right|$ $(a, b \in R)$.

(i) If θ is the argument of z, then

(a) $\theta = \alpha$, if z lies in first quadrant.

(b) $\theta = \pi - \alpha$, if z lies in second quadrant.

(c) $\theta = \alpha - \pi$, if z lies in third quadrant.

(d) $\theta = -\alpha$, if z lies in fourth quadrant.

(ii) $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) + 2k\pi$, $k = -1$ if $\arg z_1 + \arg z_2 > \pi$

$= 1$ if $\arg z_1 + \arg z_2 \le -\pi$

$= 0$ if $-\pi < \arg z_1 + \arg z_2 \le \pi$

(iii) $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) + 2k\pi$ $k = -1$ if $\arg z_1 - \arg z_2 > \pi$

$k = 1$ if $\arg z_1 - \arg z_2 \le -\pi$

$k = 0$ if $-\pi < \arg z_1 - \arg z_2 \le \pi$

(iv) $\arg(z_1) = \arg(z_2) \Leftrightarrow |z_1 + z_2| = |z_1| + |z_2|$

(v) $\arg(\bar{z}) = -\arg(z)$ (where $\arg(z) \ne \pi$)

(vi) $\arg z = 0 \Rightarrow z$ is purely real and $\arg(z) = \frac{\pi}{2} \Rightarrow z$ is purely imaginary.

(vii) $\arg(z) - \arg(-z) = \pm\pi$, according as amp(z) is positive or negative.

7. **Polar form :** If modulus and argument of a complex number $z = a + ib$ are r, θ respectively, then $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$ (Euler form).

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$\cos\theta + i\sin\theta = e^{i\theta}$

STEP-1

For School & Board Exams

Assignment

SECTION - A

Multiple Choice Questions (MCQs) Type and Assertion-Reason Based Questions :

1. The value of $\frac{(i^{11}+i^{12}+i^{13}+i^{14}+i^{15})}{(1+i)}$ is

   (1) $\frac{-(1+i)}{2}$ (2) $\frac{(1-i)}{2}$ (3) $\frac{(1+i)}{2}$ (4) $\frac{1}{2}$

2. If $\left(\frac{1+i}{1-i}\right)^3-\left(\frac{1-i}{1+i}\right)^3=a+ib$, then values of a and b respectively are $(a, b \in R)$

   (1) 0 and 2 (2) 0 and –2 (3) 2 and 0 (4) 2 and 2

3. The conjugate of $\frac{(1+2i)^2}{3-i}$ is

   (1) $\frac{-13}{10}+\frac{9}{10}i$ (2) $\frac{-13}{10}-\frac{9}{10}i$ (3) $\frac{13}{10}+\frac{9}{10}i$ (4) $\frac{13}{10}-\frac{9}{10}i$

4. If $z = 3 + i + 9i^2 - 6i^3$, then $(\overline{z^{-1}})$ is

   (1) $2 + i$ (2) $-\frac{3}{79}+\frac{4}{79}i$ (3) $1 - i$ (4) $-\frac{6}{85}+\frac{7}{85}i$

5. The modulus of $i^{25}+(i+2)^3$ is

   (1) $\sqrt{47}$ (2) $4\sqrt{15}$ (3) $\sqrt{35}$ (4) $2\sqrt{37}$

6. The polar form of $(i^{41})^3$ is

   (1) $\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}$ (2) $\cos\pi + i\sin\pi$

   (3) $-\cos\pi - i\sin\pi$ (4) $\cos\left(\frac{-\pi}{2}\right)+i\sin\left(\frac{-\pi}{2}\right)$

7. The argument of $-2+2\sqrt{3}i$ is

   (1) $-\frac{\pi}{3}$ (2) $\frac{\pi}{3}$ (3) $\frac{2\pi}{3}$ (4) $-\frac{2\pi}{3}$

8. The amplitude of $\frac{1}{i}$ is equal to

   (1) 0 (2) $\frac{\pi}{2}$ (3) $-\frac{\pi}{2}$ (4) π

9. The additive inverse of $5 + 7i$ is

   (1) $5 - 7i$ (2) $-5 + 7i$ (3) $5 + 7i$ (4) $-5 - 7i$

10. The complex number $\frac{1+2i}{1-i}$ lies in

    (1) First quadrant (2) Second quadrant (3) Third quadrant (4) Fourth quadrant

11. The value of $(1 + i)^5 \times (1 - i)^5$ is

    (1) –8 (2) $8i$ (3) 8 (4) 32

12. If $z = 3 - 2i$, then the value of $\text{Re}(z)\,(\text{Im}(z))^2$ is

(1) 6 (2) 12 (3) –6 (4) –12

13. If $z_1 = 1 + i$ and $z_2 = -3 + 2i$, then $\text{Im}\left(\frac{z_1 z_2}{\bar{z}_1}\right)$ is

(1) 2 (2) –3 (3) 3 (4) –2

14. The value of $(1 + i)(1 - i^2)(1 + i^4)(1 - i^5)$ is

(1) $2i$ (2) 8 (3) –8 (4) $8i$

15. If $z = 4 - 9i$, then $z\bar{z}$ is

(1) –92 (2) –97 (3) 92 (4) 97

16. The modulus of $\overline{6+i^3} + \overline{9-i} + \overline{9+i^2}$ is

(1) 36 (2) $\sqrt{533}$ (3) $\sqrt{465}$ (4) 49

17. The value of $\left|\frac{1}{2+i} - \frac{1}{2-i}\right|$ is

(1) $-\frac{2}{5}$ (2) $\frac{4}{25}$ (3) $\frac{2}{5}$ (4) 0

Direction : In question number 18 to 20, two statements are given one labelled as Assertion (A) and the other labelled as Reason (R). Select the correct answer from the following options.

(1) Both Assertion (A) and Reason (R) are true and the Reason (R) is the correct explanation of the Assertion (A)

(2) Both Assertion (A) and Reason (R) are true and Reason (R) is not the correct explanation of the Assertion (A)

(3) Assertion (A) is true, but Reason (R) is false

(4) Assertion (A) is false, but Reason (R) is true

18. Assertion (A) : Let z_1 and z_2 be two complex numbers such that $\arg(z_1) = \frac{\pi}{3}$ and $\arg(z_2) = \frac{\pi}{6}$ then $\arg(z_1 z_2) = \frac{\pi}{2}$.

and

Reason (R) : $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) + 2k\pi, k \in \{0, 1, -1\}$

19. Assertion (A) : The expression $\left(\frac{2i}{1+i}\right)^n$ is a positive integer for all the values of n.

and

Reason (R) : Here $n = 8$ is the least positive for which the above expression is a positive integer.

20. Assertion (A) : The argument of $z = 0 + 0i$ is $\frac{\pi}{2}$.

and

Reason (R) : Principal argument of a purely real number is 0 or π.

SECTION - B

Very Short Answer (VSA) Type Questions :

21. Find the value of the expression $1 + i^2 + i^4 + i^6 + i^8 + \ldots\ldots + i^{20}$.

22. Find the multiplicative inverse of $3 + 2i$.

23. Express $\left(\frac{1}{3} + 7i\right) + (9 - 2i) - \left(-\frac{4}{3} + 2i\right)$ in the form $a + ib$.

24. Find the conjugate of $\frac{1}{2 - 7i}$.

25. Find the principal argument of $2\sqrt{3} - 2i$.

SECTION - C

Short Answer (SA) Type Questions :

26. Evaluate $\left(-2 - \frac{1}{4}i\right)^3$.

27. Find the conjugate of $\frac{(2 - 3i)^2}{2 + i}$.

28. Find the multiplicative inverse of $\frac{2 + 4i}{(1 + i)^2}$.

29. If z_1, z_2, z_3 are $2 - 8i$, $7 - 8i$ and $1 - i$ respectively, then find the value of $\text{Im}\left(\frac{z_1 \bar{z}_2}{\bar{z}_3}\right)$.

SECTION - D

Long Answer (LA) Type Questions :

30. If $z = x + iy$ satisfies $\arg(z - 1) = \arg(z + 3i)$, then find the value of $(x - 1) : y$.

31. If $z_1 = 3 + 2i$ and $z_2 = 5 - 3i$, then show that $z_1 \bar{z}_2$ and $\bar{z}_1 z_2$ are conjugates of each other.

SECTION - E

Case Study Based Questions :

A software ultron is analyzing four complex number $z_1 = 1 + i$, $z_2 = 2 - 2i$, $z_3 = -3 - 3i$ and $z_4 = -3 + 3i$.

32. The complex number with least modulus is

    (1) z_1 (2) z_2 (3) z_3 (4) z_4

33. The complex number with greatest value of principal argument is

    (1) z_1 (2) z_2 (3) z_3 (4) z_4

34. Which of the following is purely real?

    (1) $z_1 + z_2$ (2) $z_2 + z_3$ (3) $z_3 + z_4$ (4) z_4.

35. Which of the following is purely imaginary?

    (1) $2z_1 - z_2$ (2) $z_1 + 2z_2$ (3) $z_3 + z_4$ (4) $z_1 + z_2$

36. Conjugate of $z_1 + z_2 + z_3 + z_4$ is

    (1) $-3 + i$ (2) $3 - i$ (3) $3 + i$ (4) $-3 - i$

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STEP-2

For JEE (Main) & State Engineering Entrance Exams

Assignment

bjective Type Questions (One option is correct)

[Basics of Complex Numbers]

If $z = \frac{1+i}{\sqrt{2}}$, then the value of z^{1929} is

(1) $1+i$ (2) -1

(3) $\frac{1+i}{2}$ (4) $\frac{1+i}{\sqrt{2}}$

. If the multiplicative inverse of a complex number is $\frac{\sqrt{2}+5i}{17}$, then the complex number is

(1) $\frac{\sqrt{2}-5i}{17}$ (2) $\frac{\sqrt{2}+5i}{29}$

(3) $\frac{17}{27}(\sqrt{2}-5i)$ (4) $\frac{17}{27}(\sqrt{2}+5i)$

3. $\left[i^{17}+\frac{1}{i^{315}}\right]^9$ is equal to

(1) $32i$ (2) -512

(3) 512 (4) $512i$

4. $\frac{(1+i)^3}{2+i}$ is equal to

(1) $\frac{2}{5}-\frac{6}{5}i$ (2) 0

(3) $-\frac{1}{5}+\frac{6}{5}i$ (4) $-\frac{2}{5}+\frac{6}{5}i$

5. $\left(\frac{2}{1-i}+\frac{3}{1+i}\right)\left(\frac{2+3i}{4+5i}\right)$ is equal to

(1) $-\frac{117}{82}-\frac{13}{82}i$ (2) $-\frac{117}{82}+\frac{13}{82}i$

(3) $\frac{117}{82}-\frac{13i}{82}$ (4) $\frac{117}{82}+\frac{13i}{82}$

6. $\frac{3-\sqrt{-16}}{1-\sqrt{-25}}$ is equal to

(1) $\frac{-1}{24}$ (2) 0

(3) $\frac{23}{26}+\frac{11}{26}i$ (4) $23+5i$

7. If z is a complex number, then

(1) $\text{Re}(z) = z+\bar{z}$ (2) $\text{Re}(z) = \frac{z\cdot\bar{z}}{2}$

(3) $\text{Re}(z) = \frac{z-\bar{z}}{2}$ (4) $\text{Re}(z) = \frac{z+\bar{z}}{2}$

8. The multiplicative inverse of $(3+\sqrt{5}i)^2$ is

(1) $\frac{1}{49}-\frac{3\sqrt{5}}{98}i$ (2) $\frac{1}{49}+\frac{3\sqrt{5}}{98}i$

(3) $4+6\sqrt{5}i$ (4) $4-6\sqrt{5}i$

9. $\frac{3+2i\sin\theta}{1-2i\sin\theta}$ will be purely imaginary, then $\sin^2\theta$ is equal to

(1) $\frac{1}{4}$ (2) $\frac{1}{2}$

(3) $\frac{3}{4}$ (4) 1

10. If $x+iy = \frac{3}{2+\cos\theta+i\sin\theta}$, then the value of x^2+y^2 is

(1) $\frac{9}{4+5\sin\theta}$ (2) $\frac{9}{4+5\cos\theta}$

(3) $\frac{9}{5+4\cos\theta}$ (4) $\frac{9}{5+4\sin\theta}$

[Modulus of Complex Numbers]

11. If $z = \frac{1}{(1+i)(1-2i)}$, then $|z|$ is

(1) $\frac{2}{10}$ (2) $\frac{\sqrt{7}}{10}$

(3) $\frac{9}{\sqrt{10}}$ (4) $\frac{1}{\sqrt{10}}$

12. If $z_1 = 3+i$ and $z_2 = 2-i$, then $\left|\frac{z_1+z_2-1}{z_1-z_2+i}\right|$ is

(1) $\sqrt{\frac{8}{5}}$ (2) $\frac{\sqrt{8}}{5}$

(3) $\frac{8}{5}$ (4) $\frac{8}{\sqrt{5}}$

13. For any complex number z, the minimum value of $|z| + |z - 3i|$ is

(1) 2 (2) 3

(3) 9 (4) $\sqrt{3}$

14. If $|z - 4 + 3i| \le 2$, then the least and the greatest values of $|z|$ are

(1) 3, 7 (2) 4, 7

(3) 3, 9 (4) 4, 5

15. The maximum value of $|z|$ when z satisfies the condition $\left|z + \frac{2}{z}\right| = 2$ is

(1) $\sqrt{3} - 1$ (2) $\sqrt{3} + 1$

(3) $\sqrt{3}$ (4) $\sqrt{2} + \sqrt{3}$

16. If $|z_1| = 2$, $|z_2| = 3$, $|z_3| = 4$ and $|2z_1 + 3z_2 + 4z_3| = 4$, then the expression $|8z_2z_3 + 27z_3z_1 + 64z_1z_2|$ equals

(1) 72 (2) 24

(3) 96 (4) 92

[Arguments of Complex Numbers]

17. The argument of the complex number $(1 + i)^4$ is

(1) 135° (2) 180°

(3) 90° (4) 45°

18. If $z = \frac{-4 + 2\sqrt{3}i}{5 + \sqrt{3}i}$, then the value of arg(z) is

(1) π (2) $\frac{\pi}{3}$

(3) $\frac{2\pi}{3}$ (4) $\frac{\pi}{4}$

19. If $z = \cos\frac{\pi}{4} + i\sin\frac{\pi}{6}$, then

(1) $|z| = 1$, $\arg(z) = \frac{\pi}{4}$

(2) $|z| = 1$, $\arg(z) = \frac{\pi}{6}$

(3) $|z| = \frac{\sqrt{3}}{2}, \arg(z) = \frac{5\pi}{24}$

(4) $|z| = \frac{\sqrt{3}}{2}, \arg(z) = \tan^{-1}\frac{1}{\sqrt{2}}$

20. The sum of principal arguments of complex numbers $1 + i$, $-1 + i\sqrt{3}$, $-\sqrt{3} - i$, $\sqrt{3} - i$, i, $-3i$, 2, –1 is

(1) $\frac{11\pi}{12}$ (2) $\frac{13\pi}{12}$

(3) $\frac{12\pi}{13}$ (4) $\frac{\pi}{15}$

21. If z_1, z_2 and z_3, z_4 are two pairs of conjugate complex numbers, then $\arg\left(\frac{z_1}{z_3}\right) + \arg\left(\frac{z_2}{z_4}\right)$ is

(1) 0 (2) $\frac{\pi}{2}$

(3) $\frac{3\pi}{4}$ (4) π

22. If $\arg z = \frac{\pi}{4}$, then

(1) $\text{Re}(z^2) = 9\text{Im}(z^2)$ (2) $\text{Im}(z^2) = 0$

(3) $\text{Re}(z^2) = 0$ (4) $\text{Re}(z) = 0$

23. The polar form of complex number $\sqrt{3} + i$ is

(1) $2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ (2) $2\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)$

(3) $2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$ (4) $2\left(-\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

Numerical Value Based Questions

24. If $x = 3 + 4i$ then the value of $204x - 70x^2 + 12x^3 - x^4$ is

25. Non-zero complex number z is such that $z + \frac{1}{z}$ is purely real then sum of possible values of arg(z) is k (in degrees), then k equals

26. Let $|z| = 2$ and $w = \frac{z+1}{z-1}$, where $z, w \in C$ then $2|w - 1| - |w + 1| + 1$ equals

27. The number of solutions of the equation $z^2 + \bar{z} = 0$ is ________.

PREVIOUS YEARS QUESTIONS

[Basics of Complex Numbers]

28. Let $\left(-2 - \frac{1}{3}i\right)^3 = \frac{x + iy}{27}$ $(i = \sqrt{-1})$, where x and y are real numbers, then $y - x$ equals

[JEE (Main)-2019]

(1) –85 (2) –91

(3) 85 (4) 91

29. Let $z \in C$ with $Im(z) = 10$ and it satisfies $\frac{2z-n}{2z+n} = 2i-1$ for some natural number n. Then **[JEE (Main)-2019]**

(1) $n = 20$ and $Re(z) = 10$

(2) $n = 20$ and $Re(z) = -10$

(3) $n = 40$ and $Re(z) = -10$

(4) $n = 40$ and $Re(z) = 10$

30. If the equation, $x^2 + bx + 45 = 0$ $(b \in R)$ has conjugate complex roots and they satisfy $|z+1| = 2\sqrt{10}$, then **[JEE (Main)-2020]**

(1) $b^2 - b = 42$ (2) $b^2 - b = 30$

(3) $b^2 + b = 12$ (4) $b^2 + b = 72$

31. The imaginary part of $(3+2\sqrt{-54})^{1/2} - (3-2\sqrt{-54})^{1/2}$ can be **[JEE (Main)-2020]**

(1) $\sqrt{6}$ (2) $-\sqrt{6}$

(3) $-2\sqrt{6}$ (4) 6

32. If $\left(\frac{1+i}{1-i}\right)^{m/2} = \left(\frac{1+i}{i-1}\right)^{n/3} = 1$, $(m, n \in N)$ then the greatest common divisor of the least values of m and n is ________. **[JEE (Main)-2020]**

33. The least positive integers n such that $\frac{(2i)^n}{(1-i)^{n-2}}, i = \sqrt{-1}$, is a positive integer, is ________. **[JEE (Main)-2021]**

34. Let for some real numbers α and β, $a = \alpha - i\beta$. If the system of equations $4ix + (1+i)y = 0$ and $8\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)x + \bar{a}y = 0$ has more than one solution, then $\frac{\alpha}{\beta}$ is equal to **[JEE (Main)-2022]**

(1) $-2+\sqrt{3}$ (2) $2-\sqrt{3}$

(3) $2+\sqrt{3}$ (4) $-2-\sqrt{3}$

35. Let α and β be the roots of the equation $x^2 + (2i-1) = 0$. Then, the value of $|\alpha^8 + \beta^8|$ is equal to **[JEE (Main)-2022]**

(1) 50   (2) 250

(3) 1250   (4) 1500

36. For two non-zero complex numbers z_1 and z_2, if $Re(z_1z_2) = 0$ and $Re(z_1 + z_2) = 0$, then which of the following are possible? **[JEE (Main)-2023]**

A. $Im(z_1) > 0$ and $Im(z_2) > 0$

B. $Im(z_1) < 0$ and $Im(z_2) > 0$

C. $Im(z_1) > 0$ and $Im(z_2) < 0$

D. $Im(z_1) < 0$ and $Im(z_2) < 0$

Choose the correct answer from the options given below

(1) B and C   (2) B and D

(3) A and B   (4) A and C

37. Let the complex number $z = x + iy$ be such that $\frac{2z-3i}{2z+i}$ is purely imaginary. If $x + y^2 = 0$, then $y^4 + y^2 - y$ is equal to **[JEE (Main)-2023]**

(1) $\frac{2}{3}$ (2) $\frac{3}{2}$

(3) $\frac{3}{4}$ (4) $\frac{4}{3}$

38. Let $S = \left\{z \in \mathbb{C} - \{i, 2i\} : \frac{z^2+8iz-15}{z^2-3iz-2} \in \mathbb{R}\right\}$. If $\alpha - \frac{13}{11}i \in S, \alpha \in \mathbb{R} - \{0\}$, then $242\alpha^2$ is equal to ________. **[JEE (Main)-2023]**

39. If the set $\left\{Re\left(\frac{z-\bar{z}+z\bar{z}}{2-3z+5\bar{z}}\right) : z \in \mathbb{C}, Re(z) = 3\right\}$ is equal to the interval $(\alpha, \beta]$, then $24(\beta - \alpha)$ is equal to **[JEE (Main)-2023]**

(1) 36   (2) 27

(3) 30   (4) 42

[Modulus of Complex Numbers]

40. Let z_1 and z_2 be any two non-zero complex numbers such that $3|z_1| = 4|z_2|$. If $z = \frac{3z_1}{2z_2} + \frac{2z_2}{3z_1}$ then **[JEE (Main)-2019]**

(1) $\text{Im}(z) = 0$ (2) $\frac{3}{2} \le |z| \le \frac{5}{2}$

(3) $|z| = \frac{1}{2}\sqrt{\frac{17}{2}}$ (4) $\text{Re}(z) = 0$

41. Let z be a complex number such that $|z| + z = 3 + i$ (where $i = \sqrt{-1}$). Then $|z|$ is equal to **[JEE (Main)-2019]**

(1) $\frac{\sqrt{41}}{4}$ (2) $\frac{5}{4}$

(3) $\frac{5}{3}$ (4) $\frac{\sqrt{34}}{3}$

42. If $\frac{z-\alpha}{z+\alpha}$ $(\alpha \in R)$ is a purely imaginary number and $|z| = 2$, then the value of α is **[JEE (Main)-2019]**

(1) $\sqrt{2}$ (2) 2

(3) $\frac{1}{2}$ (4) 1

43. Let $z \in C$ be such that $|z| < 1$. If $\omega = \frac{5+3z}{5(1-z)}$, then **[JEE (Main)-2019]**

(1) $5\,\text{Re}(\omega) > 4$ (2) $5\,\text{Re}(\omega) > 1$

(3) $4\,\text{Im}(\omega) > 5$ (4) $5\,\text{Im}(\omega) < 1$

44. If z be a complex number satisfying $|\text{Re}(z)| + |\text{Im}(z)| = 4$, then $|z|$ cannot be **[JEE (Main)-2020]**

(1) $\sqrt{10}$ (2) $\sqrt{8}$

(3) $\sqrt{\frac{17}{2}}$ (4) $\sqrt{7}$

45. The region represented by $\{z = x + iy \in C : |z| - \text{Re}(z) \le 1\}$ is also given by the inequality **[JEE (Main)-2020]**

(1) $y^2 \le x + \frac{1}{2}$ (2) $y^2 \le 2\left(x + \frac{1}{2}\right)$

(3) $y^2 \ge x + 1$ (4) $y^2 \ge 2(x+1)$

46. Let $z = x + iy$ be a non-zero complex number such that $z^2 = i|z|^2$, where $i = \sqrt{-1}$, then z lies on the **[JEE (Main)-2020]**

(1) Line, $y = x$

(2) Imaginary axis

(3) Real axis

(4) Line, $y = -x$

47. Let $S = \{z \in C : z^2 + \bar{z} = 0\}$. Then $\sum_{z \in S}(\text{Re}(z) + \text{Im}(z))$ is equal to ______. **[JEE (Main)-2022]**

48. Let S be the set of all (α, β), $\pi < \alpha, \beta < 2\pi$, for which the complex number $\frac{1 - i\sin\alpha}{1 + 2i\sin\alpha}$ is purely imaginary and $\frac{1 + i\cos\beta}{1 - 2i\cos\beta}$ is purely real. Let $Z_{\alpha\beta} = \sin 2\alpha + i\cos 2\beta$, $(\alpha, \beta) \in S$. Then $\sum_{(\alpha,\beta)\in S}\left(iZ_{\alpha\beta} + \frac{1}{i\bar{Z}_{\alpha\beta}}\right)$ is equal to **[JEE (Main)-2022]**

(1) 3 (2) $3i$

(3) 1 (4) $2 - i$

49. Let $z = a + ib$, $b \ne 0$ be complex numbers satisfying $z^2 = \bar{z} \cdot 2^{1-|z|}$. Then the least value of $n \in N$, such that $z^n = (z+1)^n$, is equal to ______. **[JEE (Main)-2022]**

50. If $z \ne 0$ be a complex number such that $z - \frac{1}{z} = 2$, then the maximum value of $|z|$ is **[JEE (Main)-2022]**

(1) $\sqrt{2}$ (2) 1

(3) $\sqrt{2} - 1$ (4) $\sqrt{2} + 1$

51. If $z = x + iy$ satisfies $|z| - 2 = 0$ and $|z - i| - |z + 5i| = 0$, then **[JEE (Main)-2022]**

(1) $x + 2y - 4 = 0$ (2) $x^2 + y - 4 = 0$

(3) $x + 2y + 4 = 0$ (4) $x^2 - y + 3 = 0$

52. Sum of squares of modulus of all the complex numbers z satisfying $\bar{z} = iz^2 + z^2 - z$ is equal to ________. **[JEE (Main)-2022]**

53. Let $\alpha_1, \alpha_2, ..., \alpha_7$ be the roots of the equation $x^7 + 3x^5 - 13x^3 - 15x = 0$ and $|\alpha_1| \geq |\alpha_2| \geq ... \geq |\alpha_7|$. Then $\alpha_1\alpha_2 - \alpha_3\alpha_4 + \alpha_5\alpha_6$ is equal to ________. **[JEE (Main)-2023]**

54. If for $z = \alpha + i\beta$, $|z + 2| = z + 4(1 + i)$, then $\alpha + \beta$ and $\alpha\beta$ are the roots of the equation **[JEE (Main)-2023]**

(1) $x^2 + 3x - 4 = 0$

(2) $x^2 + 7x + 12 = 0$

(3) $x^2 + x - 12 = 0$

(4) $x^2 + 2x - 3 = 0$

55. Let $S = \{z \in \mathbb{C} : \bar{z} = i(z^2 + \text{Re}(\bar{z}))\}$. Then $\sum_{z \in S} |z|^2$ is equal to **[JEE (Main)-2023]**

(1) $\frac{5}{2}$ (2) 4

(3) $\frac{7}{2}$ (4) 3

[Arguments of Complex Numbers]

56. Let $z = \left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^5 + \left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)^5$. If $R(z)$ and $I(z)$ respectively denote the real and imaginary parts of z, then **[JEE (Main)-2019]**

(1) $I(z) = 0$

(2) $R(z) > 0$ and $I(z) > 0$

(3) $R(z) < 0$ and $I(z) > 0$

(4) $R(z) = -3$

57. If z and w are two complex numbers such that $|zw| = 1$ and $\arg(z) - \arg(w) = \frac{\pi}{2}$, then **[JEE (Main)-2019]**

(1) $z\bar{w} = \frac{1-i}{\sqrt{2}}$ (2) $\bar{z}w = i$

(3) $z\bar{w} = \frac{-1+i}{\sqrt{2}}$ (4) $\bar{z}w = -i$

58. If $\frac{3 + i\sin\theta}{4 - i\cos\theta}$, $\theta \in [0, 2\pi]$, is a real number, then an argument of $\sin\theta + i\cos\theta$ is **[JEE (Main)-2020]**

(1) $\pi - \tan^{-1}\left(\frac{3}{4}\right)$

(2) $\pi - \tan^{-1}\left(\frac{4}{3}\right)$

(3) $-\tan^{-1}\left(\frac{3}{4}\right)$

(4) $\tan^{-1}\left(\frac{4}{3}\right)$

59. The value of $\left(\frac{1 + \sin\frac{2\pi}{9} + i\cos\frac{2\pi}{9}}{1 + \sin\frac{2\pi}{9} - i\cos\frac{2\pi}{9}}\right)^3$ is **[JEE (Main)-2020]**

(1) $-\frac{1}{2}(1 - i\sqrt{3})$ (2) $\frac{1}{2}(1 - i\sqrt{3})$

(3) $\frac{1}{2}(\sqrt{3} - i)$ (4) $-\frac{1}{2}(\sqrt{3} - i)$

60. If α and β are the distinct roots of the equation $x^2 + (3)^{1/4}x + 3^{1/2} = 0$, then the value of $\alpha^{96}(\alpha^{12} - 1) + \beta^{96}(\beta^{12} - 1)$ is equal to **[JEE (Main)-2021]**

(1) 28×3^{25}

(2) 56×3^{24}

(3) 52×3^{24}

(4) 56×3^{25}

61. Let $i = \sqrt{-1}$. If $\frac{(-1 + i\sqrt{3})^{21}}{(1 - i)^{24}} + \frac{(1 + i\sqrt{3})^{21}}{(1 + i)^{24}} = k$, and $n = [|k|]$ be the greatest integral part of $|k|$. Then $\sum_{j=0}^{n+5}(j+5)^2 - \sum_{j=0}^{n+5}(j+5)$ is equal to ________. **[JEE (Main)-2021]**

62. If $z = 2 + 3i$, then $z^5 + (\bar{z})^5$ is equal to **[JEE (Main)-2022]**

(1) 244 (2) 224

(3) 245 (4) 265

63. Let z_1 and z_2 be two complex numbers such that $\overline{z_1} = i\overline{z_2}$ and $\arg\left(\frac{z_1}{\overline{z_2}}\right) = \pi$. Then **[JEE (Main)-2022]**

(1) $\arg z_2 = \left(\frac{\pi}{4}\right)$ (2) $\arg z_2 = -\frac{3\pi}{4}$

(3) $\arg z_1 = \frac{\pi}{4}$ (4) $\arg z_1 = -\frac{3\pi}{4}$

64. The area of the polygon, whose vertices are the non-real roots of the equation $\bar{z} = iz^2$ is **[JEE (Main)-2022]**

(1) $\frac{3\sqrt{3}}{4}$ (2) $\frac{3\sqrt{3}}{2}$

(3) $\frac{3}{2}$ (4) $\frac{3}{4}$

65. Let $z = 1 + i$ and $z_1 = \frac{1+i\bar{z}}{\bar{z}(1-z)+\frac{1}{z}}$. Then $\frac{12}{\pi}\arg(z_1)$ is equal to ________. **[JEE (Main)-2023]**

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STEP-3

For JEE (Advanced)

SOLVED EXAMPLES

EXAMPLE 1 : *Let z be a complex number such that $z \neq 1$ and $|z| = 1$, then prove that $\frac{z-1}{z+1}$ is a purely imaginary number?*

Solution : $\because \quad |z| = 1 \Rightarrow z \cdot \bar{z} = 1 \Rightarrow \bar{z} = \frac{1}{z}$ and $\text{Re}(z) = \frac{1}{2}(z+\bar{z})$

$$\therefore \quad \text{Re}\left(\frac{z-1}{z+1}\right) = \frac{1}{2}\left(\frac{z-1}{z+1} + \left(\overline{\frac{z-1}{z+1}}\right)\right)$$

$$= \frac{1}{2}\left(\frac{z-1}{z+1} + \frac{\bar{z}-1}{\bar{z}+1}\right) = \frac{1}{2}\left[\frac{z-1}{z+1} + \frac{\frac{1}{z}-1}{\frac{1}{z}+1}\right] = \frac{1}{2}\left(\frac{z-1}{z+1} + \frac{1-z}{z+1}\right) = 0$$

$\therefore \quad \frac{z-1}{z+1}$ is a purely imaginary number.

EXAMPLE 2 : *If $|z - 2i| \leq \sqrt{2}$, then the maximum value of $|3 + i(z-1)|$ is equal to*

(1) $\sqrt{2}$ (2) $2\sqrt{2}$

(3) $3\sqrt{2}$ (4) $3 - 2\sqrt{2}$

Solution : $|3 + i(z-1)| = |i(z-1-3i)| = |z-1-3i|$

$$= |z - 2i + (-1-i)|$$

$$\leq |z-2i| + |1+i|$$

$$\leq \sqrt{2} + \sqrt{2}$$

$$= 2\sqrt{2}$$

$\therefore$ Maximum value $= 2\sqrt{2}$

Hence, correct answer is (2).

EXAMPLE 3 : *STATEMENT-1 : The number 2 + 3i is a complex number as 2, 3, $\in$ R.*

and

STATEMENT-2 : z = x + iy is a complex number, iff $x \in R$, $y \in R$.

(1) Statement-1 is True, Statement-2 is True; Statement-2 **is** *a correct explanation for Statement-1*

(2) Statement-1 is True, Statement-2 is True; Statement-2 **is NOT** *a correct explanation for Statement-1*

(3) Statement-1 is True, Statement-2 is False

(4) Statement-1 is False, Statement-2 is True

Solution : Clearly statement 1 is true. Statement 2 is not always true.

For example, $z = (2 + 3i) + i(4 - 6i)$ is a complex number but $2 + 3i \notin R$ and $4 - 6i \notin R$.

Hence answer is (3).

EXAMPLE 4 : *Find the complex number z such that* $\frac{z}{z+1}=-1+i$

Solution : Let $z=x+iy \quad (x, y \in R)$

$$\frac{x+iy}{x+iy+1}=-1+i$$

$\Rightarrow \quad x+iy=((x+1)+iy)(-1+i)$

$\Rightarrow \quad x+iy=(-x-1-y)+i(x-y+1)$

$\therefore \quad x=-x-1-y \text{ and } y=x-y+1$

$\therefore \quad 2x=-1-y \text{ and } x=2y-1$

$\therefore \quad x=-\frac{3}{5} \text{ and } y=\frac{1}{5}$

$\therefore$ The complex number $z=-\frac{3}{5}+\frac{1}{5}i$

EXAMPLE 5 : *STATEMENT-1 : The complex number* $z=i\sin\left(i\frac{\pi}{6}\right)$ *will be a pure real number.*

and

STATEMENT-2 : The complex number z = x + iy, x, y ∈R is called pure real if y = 0.

(1) *Statement-1 is True, Statement-2 is True; Statement-2* **is** *a correct explanation for Statement-1*

(2) *Statement-1 is True, Statement-2 is True; Statement-2* **is NOT** *a correct explanation for Statement-1*

(3) *Statement-1 is True, Statement-2 is False*

(4) *Statement-1 is False, Statement-2 is True*

Solution : From Euler's formula, we know that

$$\sin\theta=\frac{e^{i\theta}-e^{-i\theta}}{2i}$$

$\Rightarrow \quad i\sin\left(i\frac{\pi}{6}\right)=\frac{e^{i\left(i\frac{\pi}{6}\right)}-e^{-i\left(i\frac{\pi}{6}\right)}}{2}=\frac{e^{-\frac{\pi}{6}}-e^{\frac{\pi}{6}}}{2}=a$ (pure real number)

$\Rightarrow$ Statement 1 is true.

Statement 2 is correct and explains the statement 1.

Hence answer is (1).

EXAMPLE 6 : *Compute the product for* $n\geq 2$,

$$\left[1+\left(\frac{1+i}{2}\right)\right]\cdot\left[1+\left(\frac{1+i}{2}\right)^2\right]\left[1+\left(\frac{1+i}{2}\right)^4\right]\ldots\left[1+\left(\frac{1+i}{2}\right)^{2n}\right]$$

Solution : Let $x=\frac{1+i}{2}$

$\therefore$ The product becomes

$(1+x)(1+x^2)(1+x^4)\ldots(1+x^{2^n})$

$=\frac{1}{1-x}(1-x^2)(1+x^2)(1+x^4)\ \ldots\ (1+x^{2^n})$

$=\frac{1-x^{2^{n+1}}}{1-x}=\frac{1-\left(\frac{1+i}{2}\right)^{2^{n+1}}}{1-\left(\frac{1+i}{2}\right)}$

$=\frac{2}{(1-i)}\left[1-\left(\left(\frac{1+i}{2}\right)^2\right)^{2^n}\right]=(1+i)\left[1-\left(\frac{i}{2}\right)^{2^n}\right]$

EXAMPLE 7 : *Express $(1^2 + 2^2)(3^2 + 4^2)(5^2 + 7^2)$ as sum of the squares in four different ways.*

Solution : $(1 + 2i)(3 + 4i)(5 + 7i) = (-5 + 10i)(5 + 7i) = -95 + 15i$

$(1 + 2i)(3 + 4i)(5 - 7i) = (-5 + 10i)(5 - 7i) = 45 + 85i$

$(1 + 2i)(3 - 4i)(5 + 7i) = (11 + 2i)(5 + 7i) = 41 + 87i$

and $(1 - 2i)(3 + 4i)(5 + 7i) = (11 - 2i)(5 + 7i) = 69 + 67i$

Taking modulus of both sides of above four arrangements, we get:

$(1^2 + 2^2)(3^2 + 4^2)(5^2 + 7^2) = 95^2 + 15^2 = 45^2 + 85^2 = 41^2 + 87^2 = 69^2 + 67^2$

EXAMPLE 8 : *Determine the largest real number k such that $|z_1z_2 + z_2z_3 + z_3z_1| \geq k\,|z_1 + z_2 + z_3|$ for all complex numbers z_1, z_2, z_3 with unit absolute value.*

Solution : From the given condition,

$|z_1| = |z_2| = |z_3| = 1 \Rightarrow \frac{1}{z_1} = \bar{z}_1, \frac{1}{z_2} = \bar{z}_2$ and $\bar{z}_3 = \frac{1}{z_3}$

Now $|z_1z_2 + z_2z_3 + z_3z_1| = \left|z_1z_2z_3\left(\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right)\right|$

$= |z_1||z_2||z_3||\bar{z}_1 + \bar{z}_2 + \bar{z}_3|$

$= |\overline{z_1 + z_2 + z_3}| = |z_1 + z_2 + z_3| \geq k\,|z_1 + z_2 + z_3|$ (given)

Hence the largest possible $k = 1$.

EXAMPLE 9 : *If z_1 and z_2 are two complex numbers such that $|z_1| < 1 < |z_2|$, then prove that $\left|\frac{1 - z_1\bar{z}_2}{z_1 - z_2}\right| < 1$.*

Solution : Let us assume the result

i.e., $\left|\frac{1 - z_1\bar{z}_2}{z_1 - z_2}\right| < 1$

$\Rightarrow \frac{|1 - z_1\bar{z}_2|}{|z_1 - z_2|} < 1$

$\Rightarrow |1 - z_1\bar{z}_2|^2 < |z_1 - z_2|^2$

$\Rightarrow (1 - z_1\bar{z}_2)(1 - \bar{z}_1 z_2) < (z_1 - z_2)(\bar{z}_1 - \bar{z}_2)$

$\Rightarrow 1 - \bar{z}_1 z_2 - z_1\bar{z}_2 + z_1\bar{z}_1 z_2\bar{z}_2 < z_1\bar{z}_1 - z_1\bar{z}_2 - \bar{z}_1 z_2 + z_2\bar{z}_2$

$\Rightarrow 1 - |z_1|^2 - |z_2|^2 + |z_1|^2|z_2|^2 < 0$

$\Rightarrow (1 - |z_1|^2)(1 - |z_2|^2) < 0$

$\Rightarrow (|z_1|^2 - 1)(|z_2|^2 - 1) < 0$

$\Rightarrow |z_1|^2 < 1 < |z_2|$

$\Rightarrow |z_1| < 1 < |z_2|$ as $|z|$ can't be negative which is the case.

or $\boxed{|z_2| < 1 < |z_1|}$

EXAMPLE 10 : *Prove that there doesn't exist complex number z such that $|z| < \frac{1}{3}$ and $\sum_{r=1}^{n} a_r z^r = 1$ where $|a_r| < 2$*

Solution : On contrary let us assume the given conditions hold good.

i.e., $|z| < \frac{1}{3}$ and $\sum_{r=1}^{n} a_r z^r = 1$, where $|a_r| < 2$

Now, $\sum_{r=1}^{n} a_r z^r = 1$

$\Rightarrow$ $a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \ldots\ldots + a_n z^n = 1$

$\Rightarrow$ $1 = |1| = |a_1 z + a_2 z^2 + a_3 z^3 + \ldots\ldots + a_n z^n|$

$$\le |a_1 z| + |a_2 z^2| + |a_3 z^3| + \ldots + |a_n z^n|$$

$$= |a_1|\,|z| + |a_2||z|^2 + |a_3||z|^3 + \ldots\ldots + |a_n||z|^n$$

$$< 2\frac{1}{3} + 2.\left(\frac{1}{3}\right)^2 + 2.\left(\frac{1}{3}\right)^3 + \ldots\ldots 2\left(\frac{1}{3}\right)^n$$

$$< 2\left(\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \ldots + \frac{1}{3^n} + \frac{1}{3^{n+1}} + \ldots\ldots to\ \infty\right)$$

$$= 2.\frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{\frac{2}{3}}{\frac{2}{3}} = 1$$

$\Rightarrow$ $1 < 1$, a contradiction.

This contradiction shows that our assumption is wrong. Consequently there exists no complex number z satisfying the given condition.

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Assignment

☞ QR codes are provided for Video solutions of the questions. Please scan QR code to watch the solution.

• Questions to be discussed in Class.

SECTION - A

Objective Type Questions (One option is correct)

Consider three complex numbers z_1, z_2, z_3 such that $|z_1| = 1, |z_2| = 2, |z_3| = 3$. The value of

$\left|\dfrac{36z_1+9z_2+4z_3}{\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}}\right|$ is

(1) 9 (2) 18

(3) 36 (4) 54

If $\text{Im}(z) = 2$, and $z_1 = 1 + i$, then the minimum value of $|z - z_1|$ is (where $i=\sqrt{-1}$)

(1) 2 (2) $\sqrt{2}$

(3) 1 (4) $\frac{1}{\sqrt{2}}$

*. If α, β, γ are three distinct non-zero complex numbers such that $|\beta|=\frac{1}{|\gamma|}$, then the value of $\left|\dfrac{\beta\gamma-\alpha}{1-\bar{\alpha}\beta\gamma}\right|$ will be

(1) $\frac{3}{2}$ (2) $\frac{1}{2}$

(3) 1 (4) 2

The argument of complex number 'z' satisfying the equation $z+\bar{z}=6$, can not be

(1) $\frac{\pi}{6}$ (2) $\frac{3\pi}{4}$

(3) $\frac{\pi}{4}$ (4) $\frac{\pi}{3}$

5. If a complex number z moves such that $|\text{Re}(\bar{z})|+|\text{Im}(z)|=\sqrt{3}$, then the area bounded by all such positions of z is

(1) 4 square units

(2) 6 square units

(3) 8 square units

(4) 12 square units

6. If both the roots of the quadratic equation $ax^2 + bx + c = 0$; $a, b, c \in R - \{0\}$ are non-real and if 'θ' denotes the principle argument of the complex number $Z = a + ic$, then which of the following options is necessarily true?

(1) $\theta \notin \left(-\pi, -\frac{\pi}{2}\right)$

(2) $\theta \in \left(\frac{\pi}{2}, \pi\right)$

(3) $\theta \notin \left(-\frac{\pi}{2}, 0\right)$

(4) $\theta \notin \left(0, \frac{\pi}{2}\right)$

7. Consider two complex numbers z_1 and z_2 such that $|z_1 + z_2| = |z_1| - |z_2|$, then the complex number $z=\frac{z_1}{z_2}$ will always lie on a

(1) Ray (2) Square

(3) Pair of rays (4) Circle

8. The sum of the real roots of polynomial $72 - 36z - 26z^2 + 13z^3 + 2z^4 - z^5 = 0$ will be

(1) –2 (2) 2

(3) 3 (4) –3

9. If $f(x) = 7x^5 + 3x^4 + a_1x^3 + a_2x^2 + a_3x + 1$ such that for any $z = x + iy$ where $xy \neq 0$ and $x, y \in R$, $f(\bar{z})=\overline{f(z)}$ then the product of $a_1\ a_2\ a_3$

(1) Must be purely real

(2) May be purely real

(3) May be imaginary

(4) Must be imaginary

10*. a, b, c are real numbers in the polynomial $p(z) = 2z^4 + az^3 + bz^2 + cz + 5$. If two roots of the equation $p(z) = 0$ are 2 and $i\,(=\sqrt{-1})$, then the value of a is

(1) $\frac{3}{2}$ (2) $-\frac{3}{4}$

(3) $\frac{-13}{2}$ (4) $\frac{-3}{2}$

11. Given that the equation $z^2 + (a + ib)z + c + id = 0$, where a, b, c, d are non-zero real numbers, has a real root then

(1) $abc = c^2 + a^2d$

(2) $acd = b^2 + c^2a$

(3) $bcd = a^2 + d^2b$

(4) $abd = d^2 + b^2c$

12. Which of the given polynomial is divisible by $x^3 + x^2 + x + 1$?

(1) $x^{52} + x^{45} + x^{38} + x^{15}$

(2) $x^{51} + x^{45} + x^{36} + x^{15}$

(3) $x^{51} + x^{46} + x^{39} + x^{14}$

(4) $x + x^{44} + x^{56} + x^{61}$

13. If $x = -5 + 2\sqrt{-4}$, then the value of the expression $x^4 + 9x^3 + 35x^2 - x + 4$ is equal to

(1) 158 (2) –164

(3) –160 (4) 164

14*. If the equation $az^2 + bz + c = 0$ where $a, b, c \in C$ has one purely imaginary root, then

(1) $(b\bar{c}+c\bar{b})(a\bar{b}+\bar{a}b)+(c\bar{a}-a\bar{c})^2=0$

(2) $(b\bar{c}+\bar{b}c)(c\bar{a}+\bar{a}c)+(a\bar{b}-\bar{a}b)^2=0$

(3) $(a\bar{b}+\bar{a}b)(c\bar{a}+a\bar{c})+(b\bar{c}-\bar{b}c)^2=0$

(4) $(a+b+c)(\bar{a}+\bar{b}+\bar{c})=0$

15. It is given that the equation $|z|^2 - 2iz + 2\alpha(\lambda + i) = 0$ possesses solution for all $\alpha \in R$, then the number of integral value(s) of 'λ' for which it is true is

(1) Zero (2) One

(3) Two (4) Infinite

SECTION - B

Objective Type Questions

(One or more than one option(s) is/are correct)

1. If a, b, c are real numbers and z is a complex number such that, $a^2 + b^2 + c^2 = 1$ and $b + ic = (1 + a)z$, then $\frac{1+iz}{1-iz}$ equals.

(1) $\frac{b-ic}{1-ia}$ (2) $\frac{a+ib}{1+c}$

(3) $\frac{1-c}{a-ib}$ (4) $\frac{1+a}{b+ic}$

2. If z satisfies $|z - 1| < |z + 3|$, then $\omega = 2z + 3 - i$ satisfies

(1) $|\omega - 5 - i| < |\omega + 3 + i|$

(2) $|\omega - 5| < |\omega + 3|$

(3) $\text{Im}(i\omega) > 1$

(4) $|\arg(\omega - 1)| < \frac{\pi}{2}$

3. If $|z+\omega|^2=|z|^2+|\omega|^2$, where z and ω are complex numbers, then

(1) $\frac{z}{\omega}$ is purely real

(2) $\frac{z}{\omega}$ is purely imaginary

(3) $z\bar{\omega} + \bar{z}\omega = 0$

(4) $\text{amp}\left(\frac{z}{\omega}\right) = \frac{\pi}{2}$

4. Let z_1, z_2 be two complex numbers such that $|z_1| = 1$ and $|z_2| = 2$ respectively then

(1) $\min |z_1 - z_2| = 1$

(2) $\max |2z_1 + z_2| = 4$

(3) $\left|z_2 + \frac{1}{z_1}\right| \le 3$

(4) $\min |z_1 - z_2| = 2$

5. Let complex number z satisfy $\left|z - \frac{2}{z}\right| = 1$, then $|z|$ can take all values except

(1) 1 (2) 2

(3) 3 (4) 4

6*. If $z_1 = a + ib$ and $z_2 = c + id$ are complex numbers such that $|z_1| = |z_2| = 1$ and $\text{Re}(z_1\bar{z}_2) = 0$, then the pair of complex numbers $\omega_1 = a + ic$ and $\omega_2 = b + id$ satisfy $(a, b, c, d \in R)$

(1) $|\omega_1| = 1$

(2) $|\omega_2| = 1$

(3) $\text{Re}(\omega_1\bar{\omega}_2) = 0$

(4) $|\omega_1| = 2|\omega_2|$

7. If z_1, z_2 be two complex numbers satisfying the equation $\left|\frac{z_1 + z_2}{z_1 - z_2}\right| = 1$, then

(1) $z_1\bar{z}_2 + z_2\bar{z}_1 = 1$

(2) $\overline{\left(\frac{z_1}{z_2}\right)} = -\frac{z_1}{z_2}$

(3) $z_1\bar{z}_2 + z_2\bar{z}_1 = 0$

(4) $\text{Re}(z_1\bar{z}_2) = 0$

8*. If $Z_1, Z_2, Z_3 \in C$ such that $|Z_1| = |Z_2| = |Z_3| = 1$ and $\frac{Z_1^2}{Z_2Z_3} + \frac{Z_2^2}{Z_1Z_3} + \frac{Z_3^2}{Z_1Z_2} + 1 = 0$, then the integral values that $|Z_1 + Z_2 + Z_3|$ can attain, is

(1) 1 (2) 2

(3) 3 (4) 4

9. If $\left|\frac{2iz_1 - z_1 - z_2}{2iz_1 + z_1 + z_2}\right| = \left|\frac{\cos 2\phi + i\sin 2\phi}{\cos 2\phi - i\sin 2\phi}\right|$ and z_1, z_2 are non-zero complex numbers, then

(1) $\frac{z_1 + z_2}{z_1}$ is purely real

(2) $\frac{z_1}{z_2}$ is purely real

(3) $\frac{z_1 + z_2}{z_1 - z_2}$ is purely real

(4) $\frac{z_1^2}{z_2^2}$ is purely real

10*. If $z \in C$ and $|z|^2 - 2|z| - 3 < 0$, then the possible value(s) of $|z^2 + 2z\cos\theta|$ for all values of θ is(are)

(1) 12 (2) 14

(3) 15 (4) 16

11. If $|z - 5 + 12i| \le 1$ and α, β are the greatest and the least values of $|z|$ respectively, then

(1) $|\alpha + i\beta|^2 = 340$

(2) $|\alpha - i\beta|^2 = 240$

(3) HCF $(\alpha, \beta) = 4$

(4) $\frac{\alpha}{2} + \frac{\beta}{4} = 10$

12. If z is a complex number and the minimum value of $|z| + |z - 1| + |2z - 3|$ is k if $\beta = 2[\alpha] + 3 = 3[\alpha - k]$ then (where [.] denotes the greatest integer function)

(1) $[\alpha + k] = 11$

(2) $[\beta - k] = 19$

(3) $[\alpha + \beta] = 29$

(4) $\frac{\beta}{[\alpha - 2]} = 3$

13*. If $z_r = p_r + iq_r, p_r, q_r \in I$ such that (where $i = \sqrt{-1}$) $3p_r^2 - |p_rq_r| - 2q_r^2 + 7 = 0$ for $r = 1, 2, 3, \ldots, n$. Then

(1) $n = 4$

(2) $n = 2$

(3) $\sum_{r=1}^{n} |z_r|^2 = 20$

(4) $|z_r| = 2$ for $1 \le r \le n$

14. Which of the following statements is(are) correct.

(1) If the smallest value of $|z|^2 + |z - 3|^2 + |6i - z|^2$ occurs at $z = z_0$, then $|z_0| = \sqrt{5}$ and $\text{Re}(z_0) < \text{Im}(z_0)$

(2) If the smallest value of $|z|^2 + |z - 3|^2 + |z - 6i|^2$ occurs at $z = z_0$, then $|z_0| = \sqrt{5}$ and $\text{Re}(z_0) > \text{Im}(z)$

(3) If $|z_1| = |z_2| = |z_3| = 2$, then the minimum value of $|z_1 + z_2|^2 + |z_2 + z_3|^2 + |z_3 + z_1|^2$ is 6

(4) If $|z_1| = |z_2| = |z_3| = 2$, then the minimum value of $|z_1 + z_2|^2 + |z_2 + z_3|^2 + |z_3 + z_1|^2$ is 12

SECTION - C

Linked Comprehension Type Questions

*This section contains 1 paragraph. Based upon each paragraph, some multiple choice questions have to be answered. Each question has 4 choices (1), (2), (3) and (4), out of which **ONLY ONE** is correct.*

Comprehension

Consider the complex number $z = \frac{5 + 12i}{-3 - 4i}$ then

Choose the correct answer :

1. The additive inverse of z is

(1) $\frac{5 - 12i}{-3 - 4i}$

(2) $\frac{5 - 12i}{3 + 4i}$

(3) $\frac{5 + 12i}{3 + 4i}$

(4) $\frac{5 - 12i}{3 - 4i}$

2. The multiplicative inverse of z is

(1) $\frac{5-12i}{3+4i}$ (2) $\frac{5+12i}{3+4i}$

(3) $\frac{3+4i}{5+12i}$ (4) $\frac{-3-4i}{5+12i}$

3. The conjugate of z is

(1) $\frac{5+12i}{3-4i}$ (2) $\frac{-5+12i}{3-4i}$

(3) $\frac{5+12i}{3+4i}$ (4) $\frac{-5+12i}{-3-4i}$

SECTION - D

Matrix-Match Type Questions

This section contains 2 questions. Each question contains statements given in two columns which have to be matched. The statements in Column-I are labelled A, B, C and D, while the statements in Column-II are labelled p, q, r and s. Any given statement in Column-I can have correct matching with One OR More statement(s) in Column-II.

1. Let $z_1 = -1 + i$ and $z_2 = -2$, then match the following.

Column-I (Complex Number)	Column-II (Principal Argument)
(A) z_1z_2	(p) $\frac{3\pi}{4}$
(B) $\frac{z_1}{z_2}$	(q) $\frac{\pi}{4}$
(C) $z_1 - z_2$	(r) $-\frac{\pi}{4}$
(D) $z_2 - z_1$	(s) $-\frac{3\pi}{4}$

2. Match the entries of Column-I with those of Column-II

Column-I	Column-II
(A) If $\lvert z \rvert = 12$, then the greatest value of $\lvert z + 3 + 4i \rvert$ is	(p) 17
(B) If $\lvert z \rvert = 5$, then the least value of $\lvert z - 5 - 12i \rvert$ is	(q) 2
(C) If $\lvert z - 3 + 4i \rvert = 7$, then the least value of $\lvert z \rvert$ is	(r) 6
(D) If $\lvert z - i \rvert = 5$, then the greatest value of $\lvert z \rvert$ is	(s) 8

SECTION - E

Assertion-Reason Type Questions

This section contains 2 questions. Each question contains STATEMENT-1 (Assertion) and STATEMENT-2 (Reason). Each question has 4 choices (1), (2), (3) and (4) out of which **ONLY ONE** is correct.

(1) Statement-1 is True, Statement-2 is True; Statement-2 **is** a correct explanation for Statement-1

(2) Statement-1 is True, Statement-2 is True; Statement-2 **is NOT** a correct explanation for Statement-1

(3) Statement-1 is True, Statement-2 is False

(4) Statement-1 is False, Statement-2 is True

1. STATEMENT-1 : $3 + i x^2y$ and $x^2 + y + 4i$ are conjugate complex numbers, then $x^2 + y^2 = 5$

and

STATEMENT-2 : If the sum and product of two complex numbers is real, they are conjugates of each other.

2. STATEMENT-1 : If $\left|z+\frac{1}{z}\right|=a$ where z is a complex number and a is a real number, the least and greatest values of |z| are $\frac{\sqrt{a^2+4}-a}{2}$ and $\frac{\sqrt{a^2+4}+a}{2}$.

and

STATEMENT-2 : For a equal to zero the greatest and the least values of |z| are equal.

SECTION - F

Integer Answer Type Questions

1. The minimum value of $|z-3|+|z-4i|$ is

2. If $z=4+3\sqrt{20}\,i$, $i=\sqrt{-1}$, then the positive value of $(\sqrt{z}+\sqrt{\bar{z}})^2$ is equal to

3*. If a, b, c, p, q, r are six complex numbers such that $\frac{p}{a}+\frac{q}{b}+\frac{r}{c}=1+i$ and $\frac{a}{p}+\frac{b}{q}+\frac{c}{r}=0$, where $i=\sqrt{-1}$ then the value of $\frac{p^2}{a^2}+\frac{q^2}{b^2}+\frac{r^2}{c^2}=\lambda$, then value of λ^4 is

4. If $x=\frac{1+i}{2}$, where $i=\sqrt{-1}$ then the real part of expression $2x^4-2x^2+x+3$ equals

5*. If z_1, z_2, z_3 are three distinct complex numbers such that $\frac{3}{|z_2-z_3|}=\frac{5}{|z_3-z_1|}=\frac{7}{|z_1-z_2|}$ then $\frac{9}{z_2-z_3}+\frac{25}{z_3-z_1}+\frac{49}{z_1-z_2}$ is equal to

6. $8iz^3+12z^2-18z+27i=0$, where $i=\sqrt{-1}$ then $2|z|$ is

7. If $|z-i|\le 2$ and $w=5+3i$, where $i=\sqrt{-1}$ then the maximum value of $|iz+w|$ is

8. If z_1 and z_2 are complex numbers such that $|5z_1-3z_2|^2+|3z_1+5z_2|^2=\lambda(|z_1|^2+|z_2|^2)$ then λ is equal to

9. If $z + \sqrt{2}\,|z + 1| + i = 0$, where $i = \sqrt{-1}$, then $|\text{Re}(z) + \text{Im}(z)|$ is

10. If $\frac{5z_2}{11z_1}$ is purely imaginary then the value of $\left|\frac{2z_1 + 3z_2}{2z_1 - 3z_2}\right|$ is equal to

SECTION - G (PREVIOUS YEARS QUESTIONS)

Only One Option Correct Type Questions

1. Let $z = x + iy$ be a complex number where x and y are integers. Then the area of the rectangle whose vertices are the roots of the equation $z\bar{z}^3 + \bar{z}z^3 = 350$ is **[IIT-JEE-2009 (Paper-1)]**

(A) 48 (B) 32 (C) 40 (D) 80

2. Let z be a complex number such that the imaginary part of z is non-zero and $a = z^2 + z + 1$ is real. Then a **cannot** take the value **[IIT-JEE-2012 (Paper-1)]**

(A) –1 (B) $\frac{1}{3}$ (C) $\frac{1}{2}$ (D) $\frac{3}{4}$

One or More Option(s) Correct Type Questions

3. Let a, b, x and y be real numbers such that $a - b = 1$ and $y \neq 0$. If the complex number $z = x + iy$ satisfies $\text{Im}\left(\frac{az+b}{z+1}\right) = y$, then which of the following is(are) possible value(s) of x? **[JEE (Adv)-2017 (Paper-1)]**

(A) $-1+\sqrt{1-y^2}$ (B) $1-\sqrt{1+y^2}$ (C) $1+\sqrt{1+y^2}$ (D) $-1-\sqrt{1-y^2}$

4. Let $\bar{z}$ denote the complex conjugate of a complex number z. If z is a non-zero complex number for which both real and imaginary parts of $(\bar{z})^2 + \frac{1}{z^2}$ are integers, then which of the following is/are possible value(s) of $|z|$? **[JEE (Adv)-2022 (Paper-2)]**

(A) $\left(\frac{43+3\sqrt{205}}{2}\right)^{1/4}$ (B) $\left(\frac{7+\sqrt{33}}{4}\right)^{1/4}$ (C) $\left(\frac{9+\sqrt{65}}{4}\right)^{1/4}$ (D) $\left(\frac{7+\sqrt{13}}{6}\right)^{1/4}$

Integer / Numerical Value Type Questions

5. If z is any complex number satisfying $|z - 3 - 2i| \leq 2$, then the minimum value of $|2z - 6 + 5i|$ is **[IIT-JEE-2011 (Paper-1)]**

6. Let z be a complex number with non-zero imaginary part. If $\frac{2+3z+4z^2}{2-3z+4z^2}$ is a real number, then the value of $|z|^2$ is ________. **[JEE (Adv)-2022 (Paper-1)]**

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