Ch.7 - Integration

Antiderivative

$\frac{d}{dx} f(x) \rightarrow$ Differentiation $\rightarrow$ Symbol to represent Differentiation

$\int f(x)\,\boxed{dx} \rightarrow$ Integration $\rightarrow$ Symbol to represent Integration

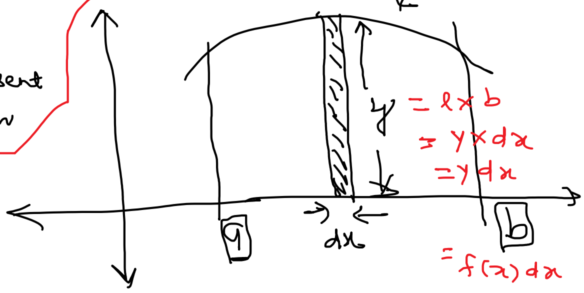
Indefinite integration

$$\int \boxed{\frac{d}{dx} f(x)}\, dx = f(x) + c$$

$$\because \int d f(x) = f(x)$$

Geometrically

$y = f(x)$



Area under curve = Integration

Definite Integration

$$= \int_a^b y\, dx$$

$$\boxed{\text{Area} = \int_a^b f(x)\, dx}$$

#① $\frac{d}{dx} x^n = n x^{n-1}$

Taking Integration both side

$$\int \left(\frac{d}{dx} x^n\right) dx = \int \left(n x^{n-1}\right) dx$$

$$x^n + c_1 = \int \boxed{n}\, x^{n-1}\, dx$$

$$x^n + c_1 = n \int x^{n-1}\, dx$$

replace $\boxed{n}$ by $\boxed{n+1}$

$$x^{n+1} + c_1 = (n+1) \int x^{n+1-1}\, dx$$

$$\frac{x^{n+1}}{n+1} + \underbrace{\frac{c_1}{n+1}}_{c} = \int x^n\, dx \Rightarrow \boxed{\int x^n\, dx = \frac{x^{n+1}}{n+1} + C}$$

#1

Q ① $\int \left(x^3 + 3x^{1/2} + 4x^7 + 8\right) dx$

$\because \frac{d}{dx} 8 = 0$

$\because \int x^0 dx = \frac{x^{0+1}}{0+1} = x$

$\because \int x^n dx = \frac{x^{n+1}}{n+1}$

$\boxed{\text{Sol.}} \rightarrow = \int x^3\, dx + 3\int x^{1/2} dx + 4\int x^7 dx + 8\int dx$

$$= \frac{x^{3+1}}{3+1} + 3\,\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + 4 \times \frac{x^{7+1}}{7+1} + 8\,\frac{x^{0+1}}{0+1} + C$$

$$= \frac{x^4}{4} + \frac{3}{\frac{3}{2}} x^{3/2} + \frac{4}{8} x^8 + 8x + C$$

4

$4 \quad \frac{3}{2} \quad 8$

$= \frac{x^4}{4} + 2x^{3/2} + \frac{1}{2}x^8 + 8x + C$ Ans.

Home-Work

1. Find y given $\frac{dy}{dx}$ in each case.

a) $\frac{dy}{dx} = 2x$

a) $2x$ b) $\int x^7 dx = \frac{x^8}{8}$ c) $3x^2$ d) -15

e) $x - x^3$ f) $10x + 8x^7$ g) $5 - \frac{1}{2}x$ h) $x(x^4 - 6)$ $\int \left(\frac{dy}{dx}\right) dx = 2\int x\,dx$

i) $\sqrt{x}$ j) $(2x - 3)^2$ k) $\frac{2x^5 + 7x}{x} = 2x^4 + 7$ l) $\frac{12}{x^2} - \frac{6}{x^3}$

$y = 2 \times \frac{x^2}{2}$

$\boxed{y = x^2}$

2. Find $f(x)$ given $f'(x)$ in each case.

a) $\frac{9}{8}x^{\frac{1}{2}}$ b) $-2x$ c) $\frac{\sqrt{x}}{3}$ d) $12x^5$

e) $3 - x - 2x^5$ f) $\left(\frac{1}{x^2} + 5\right)^2$ g) $\frac{4}{\sqrt[3]{x}}$ h) $\frac{24x^3 - 8x}{x}$

i) $(x - 4)(x + 7)$ j) $-\frac{5}{6}$ k) $x^2 + x^{-2}$ l) $\frac{14x^8 \quad 3}{x^2}$

3. Find

a) $\int 6x\,dx$ b) $\int 4x + 1\,dx$ c) $\int 4x^{-\frac{1}{2}}\,dx$

d) $\int 7x^{-8}\,dx$ e) $\int (x + 4)^2\,dx$ f) $\int \frac{4x^{-\frac{4}{3}}}{3}\,dx$

g) $\int (9 - 6x)\,dx$ h) $\int \frac{2x + 5x^3}{x}\,dx$ i) $\int 2x(1 - x)^2\,dx$

j) $\int \frac{(2x+1)^2}{\sqrt{x}}\,dx$ k) $\int \left(\frac{3}{\sqrt{x}} - \sqrt{x^3}\right)dx$ l) $\int \sqrt{x} - (\sqrt{x} + 5)^2\,dx$.

#Find these integrals.

a) $\int (2x - 1)^6\,dx$ b) $\int (4 - 3x)^8\,dx$ c) $\int (5x + 2)^5\,dx$

d) $\int \frac{1}{(3x+5)^5}\,dx$ e) $\int \frac{15}{(1-3x)^6}\,dx$ f) $\int \frac{2}{(5+2x)^9}\,dx$

g) $\int \frac{3}{\sqrt{7x+1}}\,dx$ h) $\int \frac{6}{\sqrt{(6x-5)^3}}\,dx$ i) $\int \frac{1}{\sqrt[3]{(7-x)}}\,dx$

$\int \sqrt{7x+1}$ $\int \sqrt{(6x-5)^3}$ $\int \sqrt[3]{(7-x)}$

j) $\int 3\sqrt{(1-x)}\,dx$ **k)** $\int \frac{4}{(1-2x)^7}\,dx$ **l)** $\int \left(\sqrt{(2+3x)}\right)^5 dx$

Sol. (l) $\int (2+3x)^{5/2} \boxed{dx}$

$2+3x = t$

$3 = \frac{dt}{dx}$

$dx = \frac{dt}{3}$

$\int x^n\,dx = \frac{x^{n+1}}{n+1}$

$= \int t^{5/2} \times \frac{dt}{3}$

$= \frac{1}{3}\int \boxed{t^{5/2}\,dt}$

$\because (x^m)^n = x^{m\times n}$

Surd, Index

$\sqrt[3]{x} = x^{1/3}$

$\sqrt[5]{x} = x^{1/5}$

$\sqrt{x} = x^{1/2}$

$(\sqrt{x})^5 = (x^{1/2})^5 = x^{\frac{5}{2}}$

$= \frac{1}{3} \times \frac{t^{5/2+1}}{5/2+1}$

$= \frac{1}{3} \times \frac{t^{7/2}}{7/2} = \boxed{\frac{2}{21}(2+3x)^{7/2}}$ Ans

Methods of Integration

(1) Substitution Method

Put $f^n = t$

find $\boxed{dx}$ in term $\boxed{dt}$

(2) Partial fraction

(3) By part

Formula

(1) $\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$

(2) $\int \sin x\,dx = -\cos x + c$

(3) $\int \cos x\,dx = \sin x + c$

(4) $\int \sec^2 x\,dx = \tan x + c$ $\because \frac{d}{dx}\tan x = \sec^2 x$

(5) $\int \csc^2 x\,dx = -\cot x + c$

(6) $\int \sec x \tan x\,dx = \sec x + c$ $\because \frac{d}{dx}\sec x = \sec x \tan x$

⑥ $\int \sec x \tan x \, dx = \sec x + c$ $\qquad \because \frac{d}{dx}\sec x = \sec x \tan x$

⑦ $\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + C$

⑧ $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$

$= -\cos^{-1} x + c$

⑨ $\int \frac{1}{x^2+1} dx = \boxed{\tan^{-1} x + C}$

$= -\cot^{-1} x + C$

⑩ $\int \frac{1}{x\sqrt{x^2-1}} = \sec^{-1} x + c$

$= -\operatorname{cosec}^{-1} x + C$

⑪ $\int e^x \, dx = e^x + c$

⑫ $\int \frac{1}{x} dx = \log x + C$

⑬ $\int a^x \, dx = \frac{a^x}{\log_e a} + C$

⑭ $\int \tan x \, dx = \log \sec x + C = -\log \cos x + C$

⑮ $\int \cot x \, dx = \log \sin x + C$

⑯ $\int \sec x \, dx = \log|\sec x + \tan x| + c$

⑰ $\int \operatorname{cosec} x \, dx = \log|\operatorname{cosec} x - \cot x| + C$

$\because \sin(A+B) = \sin A \cos B + \cos A \sin B$ ✓

Example 6 Find the following integrals: $\because \sin(x+a) \neq \sin x + \sin a$ ✗

$\sin(A-B) = \sin A \cos B - \cos A \sin B$ ✓

(i) $\int \sin^3 x \cos^2 x \, dx$ (ii) $\int \frac{\sin x}{\sin(x+a)} dx$ (iii) $\int \frac{1}{1+\tan x} dx$

Sol. → $\cos x = t$

Diff. both Side

$\frac{d}{dx}\cos x = \frac{dt}{dx}$

$-\sin x = \frac{dt}{dx}$

$\sin x \, dx = -dt$ — ①

$x + a = t$

$dx = dt$

$\int \frac{\sin(t-a)}{\sin t} dt$

$\int \left(\frac{\cancel{\sin t}\cos a}{\cancel{\sin t}} - \frac{\cos t \sin a}{\sin t} \right) dt$

$\int \frac{1}{1+\frac{\sin x}{\cos x}} dx$

$\int \frac{\cos x}{\cos x + \sin x} dx$

$\frac{1}{2} \int \frac{2\cos x + \cancel{\sin x} - \cancel{\sin x}}{\cos x + \sin x}$

$\int \sin^2 x \cos^2 x \sin x\, dx$

$\int (1-\cos^2 x)\cos^2 x \; (-dt)$

$\int (1-t^2)\, t^2 \; (-dt)$

$-\int (t^2 - t^4)\, dt$

$-\frac{t^3}{3} \quad \frac{}{5} + C$

$-\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + C$ Ans.

$\sin t$ — ①

$\cos a \; t - \boxed{\sin a} \int \cot t \, dt$

$t \cos a - \sin a \log|\sin t| + C$

$\frac{1}{2} \int \frac{2\cos x + \boxed{\sin x - \sin x}}{\cos x + \sin x}$

$\frac{1}{2}\left[\int \frac{\cos x + \sin x}{\cos x + \sin x} dx + \int \frac{\cos x - \sin x}{\cos x + \sin x} dx\right]$

$= \frac{1}{2}\left[\int dx + \int \frac{\cos x - \sin x}{\cos x + \sin x} dx\right]$ —— ①

$\int \frac{\cos x - \sin x}{\cos x + \sin x} dx$

$\cos x + \sin x = t$

$-\sin x + \cos x = \frac{dt}{dx}$

$(\cos x - \sin x)\, dx = dt$

$\int \frac{dt}{t} = \log t + C$

$= \log|\cos x + \sin x| + C$

$= \frac{1}{2}x + \frac{1}{2}\log|\sin x + \cos x| + C$ Ans.

Ex:- 7.2 (Hints)

① $1 + x^2 = t$ ② $\log x = t$ ③ $1 + \log x = t$

④ $\cos x = t$ ⑤ $\sin(ax+b) = t$ ⑥ $\int (ax+b)^{1/2} = \frac{(ax+b)^{3/2}}{3/2 \times a}$ Short cut Ans.

$\boxed{\text{Hint: } ax+b = t}$

⑩ $\int \frac{1}{x - \sqrt{x}} dx = \int \frac{1}{\sqrt{x}(\sqrt{x} - 1)} dx$

$\sqrt{x} - 1 = t$

$x^{1/2} - 1 = t$

$\frac{1}{2\sqrt{x}} dx = dt$

$\frac{1}{\sqrt{x}} dx = 2dt$

$= 2\int \frac{1\, dt}{t}$

$= 2\log t + C$

$= 2\log|\sqrt{x} - 1| + C$ Ans.

Sol. ⑦ $\boxed{\int x\sqrt{(x+2)}\, dx}$

$x + 2 = t$

$dx = dt$

$\int (t-2)\sqrt{t}\, dt$

$= \int t^{3/2} dt - 2\int t^{1/2} dt$

Sol. ⑧ $\int \boxed{x} \sqrt{1+2x^2}\, dx$

$1 + 2x^2 = t$

$4x - dt$

$$\int (t-2)\sqrt{t}\, dt$$

$$\int t^{1/2}(t-2)\, dt$$

$$\int \left(t^{1/2}\, t - 2\, t^{1/2}\right) dt$$

$$= \int t^{3/2}\, dt - 2\int t^{1/2}\, dt$$

$$= \frac{2t^{5/2}}{5} - 2\,\frac{t^{3/2}}{3/2} + c$$

$$= \boxed{\frac{2}{5}(x+2)^{5/2} - \frac{4}{3}(x+2)^{3/2} + c}$$

$$1+2x^2 = t$$

$$4x = \frac{dt}{dx}$$

$$x\,dx = \frac{dt}{4}$$

$$\boxed{dx = \frac{dt}{4x}}$$

$$\frac{1}{4}\int t^{1/2}\, dt = \frac{1}{4}\,\frac{t^{3/2}}{3/2} + c$$

$$\boxed{\frac{1}{6}(1+2x^2)^{3/2} + c}$$ Ans

(9) $$\int (4x+2)\sqrt{x^2+x+1}\; dx$$

$\boxed{\text{Sol.}}$ → $x^2+x+1 = t$

$$2x+1 = \frac{dt}{dx}$$

$$(2x+1)\, dx = dt$$

$$2\int \sqrt{t}\,\boxed{(2x+1)\,dx} = 2\int t^{1/2}\, dt$$

$$= 2\,\frac{t^{3/2}}{3/2} + c$$

$$= \boxed{\frac{4}{3}(x^2+x+1)^{3/2} + c}$$

(11) $$\int \frac{x}{\sqrt{x+4}}\, dx$$

$$x+4 = t$$

$$dx = dt$$

$$\int \frac{t-4}{\sqrt{t}}\, dx$$

$$= \int t^{-1/2}(t-4)\, dt$$

$$= \int t^{1/2}\, dt - 4\, t^{-1/2}\, dt$$

$$= \frac{t^{3/2}}{3/2} - 4\,\frac{t^{1/2}}{1/2} + c$$

$$\boxed{\frac{2}{3}(x+4)^{3/2} - 8(x+4)^{1/2} + c}$$ Ans.

(12) $$\int (x^3-1)^{1/3}\, x^5\, dx$$

$$\boxed{x^3-1 = t}$$

$$3x^2\, dx = dt$$

$$x^2\, dx = \frac{dt}{3}$$

$$\int t^{1/3}\, x^3 \cdot x^2\, dx$$

$$\frac{1}{3}\int t^{1/3}(t+1)\, dt$$

$$\frac{1}{3}\left[\int t^{4/3}\, dt + \int t^{1/3}\, dt\right] = \frac{1}{3}\left[\frac{t^{7/3}}{7/3} + \frac{t^{4/3}}{4/3} + c\right]$$

$$= \frac{1}{7}\, t^{7/3} + \frac{1}{4}\, t^{4/3} + c$$

$$= \frac{1}{7}(x^3-1)^{7/3} + \frac{1}{4}(x^3-1)^{4/3} + c$$ $\boxed{\text{Ans.}}$

(13) $\int \frac{x^2}{\ldots}\, dx$

(13) $\int \frac{x^2}{(2+3x^3)^3} dx$

$2+3x^3 = t$

$9x^2 = \frac{dt}{dx} \Rightarrow x^2 dx = \frac{dt}{9}$

$\frac{1}{9}\int t^{-3} dt = \frac{1}{9} \frac{t^{-2}}{-2} = \frac{-1}{18}(2+3x^3)^{-2} = \frac{-1}{18(2+3x^3)^2}$ Ans

(14) $\int \frac{1}{x(\log x)^m} dx$

$\log x = t$

$\frac{1}{x} = \frac{dt}{dx}$

$\boxed{\frac{1}{x} dx = dt}$

$= \int t^{-m} dt$

$= \frac{t^{-m+1}}{-m+1} + c$

$\frac{(\log x)^{-m+1}}{-m+1} + C$

EXERCISE 7.2

Integrate the functions in Exercises 1 to 37:

1. $\frac{2x}{1+x^2}$ 2. $\frac{(\log x)^2}{x}$ 3. $\frac{1}{x + x\log x}$

4. $\sin x \sin(\cos x)$ 5. $\sin(ax+b)\cos(ax+b)$

6. $\sqrt{ax+b}$ 7. $x\sqrt{x+2}$ 8. $x\sqrt{1+2x^2}$

9. $(4x+2)\sqrt{x^2+x+1}$ 10. $\frac{1}{x-\sqrt{x}}$ 11. $\frac{x}{\sqrt{x+4}}, x>0$

12. $(x^3-1)^{\frac{1}{3}} x^5$ 13. $\frac{x^2}{(2+3x^3)^3}$ 14. $\frac{1}{x(\log x)^m}, x>0, m\neq 1$

15. $\frac{x}{9-4x^2}$ 16. e^{2x+3} 17. $\frac{x}{e^{x^2}}$

Sol. (15) $\int \frac{x\, dt}{9-4x^2}$

$9-4x^2 = t$

$-8x = \frac{dt}{dx}$

$x\,dx = \frac{-dt}{8}$

$= \frac{-1}{8}\boxed{\int \frac{1}{t} dt}$

$= \frac{-1}{8} \log t$

$= \frac{-1}{8} \log|9-4x^2| + c$

Ans.

Sol. (16) $\int e^{2x+3} dx$

$= \boxed{\frac{e^{2x+3}}{2} + c}$

or

$2x+3 = t$

$x\,dx = \frac{-dt}{8}$

8

Ans.

$\because \int e^x dx = e^x$

(17) $\int \frac{x}{e^{x^2}} dx$

$x^2 = t$

$2x = \frac{dt}{dx}$

$\boxed{x\,dx = \frac{dt}{2}}$

$\Rightarrow \frac{1}{2}\int \frac{1}{e^{+t}} dt = \frac{1}{2}\int e^{-t} dt$

$2x+3 = t$

$-t = p$

$-1 = \frac{dp}{dt}$

$dt = -dp$

$= \frac{1}{2} \frac{e^{-t}}{-1} = \frac{-1}{2e^{x^2}} + c$

Ans.

$\frac{d}{dx} f(x) \rightarrow$ Diff. w.r.t. x

$\int f(x)\,dx \rightarrow$ Integration with respect to $\boxed{x}$

Method of Integration $\rightarrow$ Antiderivative

$\int \left[\frac{d}{dx} f(x)\right] dx = f(x)$

(1) Substitution Method — 7.1, 7.2, 7.3, 7.4

(2) Partial fraction Method — 7.5

(3) By Part Method — 7.6

(1) $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

(2) $\int \sin x = -\cos x + c$

$\int \left[\frac{d}{dx} \cos x\right] dx = \int -\sin x \, dx$

(3) $\int \cos x\,dx = \sin x + c$

(4) $\int \sec^2 x\,dx = \tan x + c$

(5) $\int \operatorname{cosec}^2 x\,dx = -\cot x + c$

(6) $\int \sec x \tan x\,dx = \sec x + c$

$\frac{d}{dx}\sec x = \sec x \tan x$

(7) $\int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x + c$

(8) $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + c$

$= -\cos^{-1} x + c$

$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$

$= -\cos^{-1}x + C$

(9) $\int \frac{1}{x^2+1}\,dx = \tan^{-1}x + C$

$= -\cot^{-1}x + C$

(10) $\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1}x + C$

$= -\operatorname{cosec}^{-1}x + C$

(11) $\int e^x\,dx = e^x + C$

(12) $\int \frac{1}{x}\,dx = \log x + C$

(13) $\int a^x\,dx = \frac{a^x}{\log_e a} + C$

(14) $\int \tan x\,dx = \log \sec x + C = -\log \cos x + C$

(15) $\int \cot x\,dx = \log \sin x + C$

(16) $\int \sec x\,dx = \log|\sec x + \tan x| + C$

(17) $\int \operatorname{cosec} x\,dx = \log|\operatorname{cosec} x - \cot x| + C$

$\frac{d}{dx}(ax+b)^n = n(ax+b)^{n-1} \times a$

Substitution Method

$\because \int x^n\,dx = \frac{x^{n+1}}{n+1} + C$

Trick

\# $\int (ax+b)^n \boxed{dx} \rightarrow ax+b = t$

$a = \frac{dt}{dx} \Rightarrow dx = \frac{dt}{a}$

$\int t^n \frac{dt}{a} = \boxed{\frac{1}{a}\frac{t^{n+1}}{n+1}} \Rightarrow \frac{1}{a}\frac{(ax+b)^{n+1}}{n+1}$

Ans.

$= \frac{(ax+b)^{n+1}}{a(n+1)}$

\# (1) $\sin 2x = 2\sin x\cos x = \frac{2\tan x}{1+\tan^2 x}$

(2) $\boxed{\cos 2x} = \boxed{\cos^2 x - \sin^2 x} = \boxed{1 - 2\sin^2 x} = \boxed{2\cos^2 x - 1}$

$= \boxed{\frac{1-\tan^2 x}{1+\tan^2 x}}$

$\boxed{1+\cos 2x = 2\cos^2 x}$

$\boxed{1-\cos 2x = 2\sin^2 x}$

(3) $\tan 2x = \frac{2\tan x}{1-\tan x}$

(4) $\sin 3x = 3\sin x - 4\sin^3 x$.

④ $\sin 3x = 3\sin x - 4\sin^3 x$

⑤ $\cos 3x = 4\cos^3 x - 3\cos x$

$\vdots$

Sol: (24) $\displaystyle\int \frac{2\cos x - 3\sin x}{6\cos x + 4\sin x}\,dx$

Sol. → $\displaystyle\frac{1}{2}\int \frac{2\cos x - 3\sin x}{3\cos x + 2\sin x}\,dx$

$3\cos x + 2\sin x = t$

$-3\sin x + 2\cos x = \frac{dt}{dx}$

$(2\cos x - 3\sin)\,dx = dt$ —— ①

$= \frac{1}{2}\int \frac{1}{t}\,dt$

$= \frac{1}{2}\log|t| + C$

$= \boxed{\frac{1}{2}\log|3\cos x + 2\sin x| + C}$

Ans.

(22) $\displaystyle\int \sec^2(7-4x)\,dx = \boxed{\frac{\tan(7-4x)}{-4} + C}$

$1 + \tan^2\theta = \sec^2\theta$

$\tan^2\theta = \sec^2\theta - 1$

(21) $\displaystyle\int \tan^2(2x-3)\,dx = \int[\sec^2(2x-3) - 1]\,dx$

$= \displaystyle\int \sec^2(2x-3)\,dx - \int dx$

$= \boxed{\frac{\tan(2x-3)}{2} - x + C}$ Ans.

(23) $\displaystyle\int \frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx$

Sol. → $\sin^{-1}x = t$

$\frac{d}{dx}\sin^{-1}x = \frac{dt}{dx}$

$\frac{1}{\sqrt{1-x^2}}\,dx = dt$

$= \displaystyle\int t\,dt$

$= \frac{t^2}{2} + C$

$= \frac{(\sin^{-1}x)^2}{2} + C$ Ans.

(25) $\displaystyle\int \frac{1}{\cos^2 x(1-\tan x)^2}\,dx$

→ $\int \frac{1}{\ }(-dt)$

(25) $\int \frac{1}{\cos^2 x(1-\tan x)^2}\,dx$

$\int \frac{\sec^2 x}{(1-\tan x)^2}\,dx \rightarrow \int \frac{1}{t^2}(-dt)$

$1-\tan x = t$

$-\sec^2 x = \frac{dt}{dx}$

$\sec^2 x\,dx = -dt$

$= -\int t^{-2}(dt)$

$= -\frac{t^{-2+1}}{-2+1} + C$

$= -\frac{t^{-1}}{-1} + C$

$\frac{1}{t} + C$

$\boxed{\frac{1}{1-\tan x} + C}$

Ans.

(26) $\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx \rightarrow 2\int \cos t\,dt$

$\because \int \cos x\,dx = \sin x + C$

$\because \int \sin x\,dx = -\cos x + C$

$\sqrt{x} = t$

$\frac{1}{2\sqrt{x}} = \frac{dt}{dx}$

$\frac{1}{\sqrt{x}}dx = 2dt$

$= +2\sin t$

$= +2\sin\sqrt{x} + C$ Ans.

(27) $\int \sqrt{\sin 2x}\,\cos 2x\,dx$

$\because \sin 2x = t$

$2\cos 2x = \frac{dt}{dx}$

$\cos 2x\,dx = \frac{dt}{2}$

$\rightarrow \frac{1}{2}\int t^{1/2}\,dt$

$= \frac{1}{2}\frac{t^{1/2+1}}{1/2+1} + C$

$\frac{1}{\cancel{2}} \times \frac{\cancel{2}}{3}\, t^{3/2} + C$

$\boxed{\frac{1}{3}(\sin 2x)^{3/2} + C}$

Ans.

(28) $\int \frac{\cos x}{\sqrt{1+\sin x}}\,dx$

Sol. → $1+\sin x = t$

$\cos x\,dx = dt$

$\int t^{-1/2}\,dt$

$\frac{t^{-1/2+1}}{} + C$

$= 2\,t^{1/2} + C$

$= \boxed{2\sqrt{1+\sin x} + C}$

Ans.

$$\frac{t^{-1/2+1}}{-1/2+1}+c$$

(29) $\int \boxed{\cot x} \log \sin x \boxed{dx}$

Sol. → $\log \sin x = t$

$\frac{1}{\sin x} \times \cos x = \frac{dt}{dx}$

$\cot x \, dx = dt$

$= \int t \, dt$

$= \frac{t^2}{2} + c$

$= \frac{(\log \sin x)^2}{2} + c$ Ans.

(30) $\int \frac{\sin x}{1+\cos x} dx$

$1 + \cos x = t$

$-\sin x \, dx = dt$

$\sin x \, dx = -dt$

$- \int \frac{1}{t} dt = -\log|t| + c$

$= \boxed{-\log|1+\cos x| + c}$

(31) $\int \frac{\sin x}{(1+\cos x)^2} dx = -\int t^{-2} dt = -\frac{t^{-2+1}}{-2+1} + c$

$1 + \cos x = t$

$\sin x \, dx = -dt$

$= \frac{1}{t} + c = \boxed{\frac{1}{1+\cos x} + c}$

(32) $\int \frac{1}{1+\cot x} dx = \int \frac{1}{1+\frac{\cos x}{\sin x}} dx = \int \frac{\sin x}{\sin x + \cos x} dx$

$= \frac{1}{2}\int \frac{2\sin x}{\sin x + \cos x} dx = \frac{1}{2}\int \frac{2\sin x + \cos x - \cos x}{\sin x + \cos x} dx$

$= \frac{1}{2}\left[\int \frac{(\sin x + \cos x) + (\sin x - \cos x)}{\sin x + \cos x} dx\right]$

$= \frac{1}{2}\int \frac{\cancel{\sin x + \cos x}}{\cancel{\sin x + \cos x}} dx + \frac{1}{2}\int \frac{\sin x - \cos x}{\sin x + \cos x} dx$

$= \frac{1}{2}x + \frac{1}{2}\int \frac{\sin x - \cos x}{\sin x + \cos x} dx$ —(1)

$$= \frac{x}{2} + \frac{1}{2}\int \frac{\ldots}{\sin x + \cos x}\,dx \quad \text{--- (1)}$$

$\sin x + \cos x = t$

$\cos x - \sin x = \frac{dt}{dx}$

$-(\sin x - \cos x)\,dx = dt$

$(\sin x - \cos x)\,dx = -dt$

$$= \frac{x}{2} + \frac{1}{2}\int \frac{-dt}{t}$$

$$= \boxed{\frac{x}{2} - \frac{1}{2}\log|\sin x + \cos x| + C} \quad \text{Ans.}$$

(33) $\int \frac{1}{1-\tan x}\,dx = \frac{x}{2} - \frac{1}{2}\log|\cos x - \sin x| + C$

(34) $\int \frac{\sqrt{\tan x}}{\sin x \cos x}\,dx$

$$= \int \frac{\sqrt{\tan x}}{\frac{\sin x}{\cos x}\boxed{\cos x \times \cos x}}\,dx$$

$$= \int \frac{\sqrt{\tan x}\,\boxed{\sec^2 x\,dx}}{\tan x}$$

$\tan x = t$

$$\boxed{\frac{d}{dx}\tan x = \sec^2 x}$$

$\sec^2 x = \frac{dt}{dx}$

$\sec^2 x\,dx = dt$

$$\int \frac{\sqrt{t}}{t}\,dt$$

$$\int t^{-1/2}\,dt$$

$$\frac{t^{-1/2+1}}{-1/2+1} + C$$

$$\frac{1}{1/2\sqrt{t}} + C = \boxed{\frac{2}{\sqrt{\tan x}} + C} \quad \text{Ans.}$$

(35) $\int \frac{(1+\log x)^2}{x}\,dx \rightarrow \int t^2\,dt$

$1 + \log x = t$

$\frac{1}{x}\,dx = dt$

$$= \frac{t^3}{3} + C$$

$$= \frac{(1+\log x)^3}{3} + C \quad \text{Ans.}$$

(36) $\int \frac{(x+1)(x+\log x)^2}{x}\,dx$

$$\int \frac{x+1}{x}(x+\log x)^2\,dx$$

$$\int \left(\frac{x}{x} + \frac{1}{x}\right)(x+\log x)^2\,dx$$

$\therefore\ x + \log x = t$

$\therefore\ \left(1 + \frac{1}{x}\right)dx\ \ dt$

$$\therefore \int t^2\,dt$$

$$\frac{t^3}{3} + C$$

$$\int \left(1+\frac{1}{x}\right)(x+\log x)^2\,dx \qquad \frac{t^3}{3}+C$$

$$\frac{(x+\log x)^3}{3}+C$$

37 ★★★ $\displaystyle\int \frac{x^3 \sin(\tan^{-1} x^4)}{1+x^8}\,dx$

$(x^4)^2$

$x^4 = t$ $\qquad \frac{d}{dx}x^4 = 4x^3$

$4x^3 = \frac{dt}{dx}$

$$\boxed{x^3\,dx = \frac{dt}{4}}$$

$\because \frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$

$$\frac{1}{4}\int \frac{\sin(\tan^{-1} t)}{1+t^2}\,dt$$

$\because \tan^{-1} t = p$

$\Rightarrow \frac{1}{1+t^2}\,dt = dp$

$$\frac{1}{4}\int \sin p\;dp$$

$$-\frac{1}{4}\cos p + C = \boxed{-\frac{1}{4}\cos(\tan^{-1} x^4) + C} \rightarrow \text{Ans.}$$

18. $\dfrac{e^{\tan^{-1}x}}{1+x^2}$ **19.** $\dfrac{e^{2x}-1}{e^{2x}+1}$ **20.** $\dfrac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}$

21. $\tan^2(2x-3)$ **22.** $\sec^2(7-4x)$ **23.** $\dfrac{\sin^{-1}x}{\sqrt{1-x^2}}$

24. $\dfrac{2\cos x-3\sin x}{6\cos x+4\sin x}$ **25.** $\dfrac{1}{\cos^2 x(1-\tan x)^2}$ **26.** $\dfrac{\cos\sqrt{x}}{\sqrt{x}}$

27. $\sqrt{\sin 2x}\cos 2x$ **28.** $\dfrac{\cos x}{\sqrt{1+\sin x}}$ **29.** $\cot x \log \sin x$

30. $\dfrac{\sin x}{1+\cos x}$ **31.** $\dfrac{\sin x}{(1+\cos x)^2}$ **32.** $\dfrac{1}{1+\cot x}$

33. $\dfrac{1}{1-\tan x}$ **34.** $\dfrac{\sqrt{\tan x}}{\sin x\cos x}$ **35.** $\dfrac{(1+\log x)^2}{x}$

36. $\dfrac{(x+1)(x+\log x)^2}{x}$ **37.** $\dfrac{x^3\sin(\tan^{-1}x^4)}{1+x^8}$

Choose the correct answer in Exercises 38 and 39.

38. $\displaystyle\int \frac{10x^9+10^x\log_{e^{10}}dx}{x^{10}+10^x}$ equals

(A) $10^x-x^{10}+C$ (B) $10^x+x^{10}+C$

(C) $(10^x-x^{10})^{-1}+C$ (D) $\log(10^x+x^{10})+C$

39. $\displaystyle\int \frac{1\ dx}{\sin^2 x\cos^2 x}$ equals

(A) $\tan x+\cot x+C$ (B) $\tan x-\cot x+C$

(C) $\tan x\cot x+C$ (D) $\tan x-\cot 2x+C$

$\dfrac{a+b}{c}=\dfrac{a}{c}+\dfrac{b}{c}$

$\int f(x)\,g(x)\,dx \neq \int f(x)\,dx \int g(x)\,dx$

$\displaystyle\int \frac{\sin^2 x+\cos^2 x}{\sin^2 x\cdot\cos^2 x}\,dx$

$\displaystyle\int \frac{\cancel{\sin^2 x}}{\cancel{\sin^2 x}\cdot\cos^2 x}\,dx+\int\frac{\cancel{\cos^2 x}}{\sin^2 x\cdot\cancel{\cos^2 x}}\,dx$

$$\int \sec^2 x\,dx + \int \csc^2 x\,dx$$

$$\tan x - \cot x + C \quad \text{Ans.}$$

7.3

Integration using Trigonometric Identities

$\frac{d}{dx}\cos 2x = -2\sin 2x$

$\int \cos 2x\,dx = \frac{\sin 2x}{2} + C$

$$\boxed{\frac{1+\cos 2\theta}{2} = \cos^2\theta}$$

① $\int \cos^2 x\,dx$

$$= \int \frac{1+\cos 2x}{2}$$

$$= \frac{1}{2}\int (1+\cos 2x)\,dx$$

$$= \frac{1}{2}\left[\int dx + \int \cos 2x\,dx\right]$$

$$= \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right] + C$$

$$= \frac{x}{2} + \frac{\sin 2x}{4} + C \quad \text{Ans.}$$

$\because 2\sin A\cos B = \sin(A+B) + \sin(A-B)$

$\because 2\cos A\sin B = \sin(A+B) - \sin(A-B)$

$\because 2\cos A\cos B = \cos(A+B) + \cos(A-B)$

$\because 2\sin A\sin B = \cos(A-B) - \cos(A+B)$

② $\frac{1}{2}\int 2\sin 2x\cos 3x\,dx$

$$= \frac{1}{2}\int [\sin 5x + \sin(-x)]\,dx$$

$\because \sin(-\theta) = -\sin\theta$

$$= \frac{1}{2}\left[\int \sin 5x\,dx - \int \sin x\,dx\right]$$

$\int \sin 5x\,dx$

$5x = t$

$5 = \frac{dt}{dx}$

$5dx = dt$

$dx = \frac{dt}{5}$

$$= \frac{1}{2}\left[-\frac{\cos 5x}{5} + \cos x\right] + C$$

$$= -\frac{1}{10}\cos 5x + \frac{1}{2}\cos x + C$$

Q. ③ $\int \sin^3 x\,dx$

$$\int \sin^2 x \cdot \sin x\,dx$$

$$\int (1-\cos^2 x)\sin x\,dx$$

$\cos x = t$

$-\sin x = \frac{dt}{dx}$

$\sin x\,dx = -dt$

$$\int (1-t^2)(-dt)$$

Alternative Method

$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$\therefore \sin^3\theta = \frac{3\sin\theta - \sin 3\theta}{4}$

$$\int \frac{3}{4}\sin x\,dx - \int \frac{1}{4}\sin 3x\,dx$$

$\frac{d}{dx}\sin\theta = \cos\theta$

$\int \cos x\,dx = \sin x$

$\int \sin x\,dx = -\cos x$

$$= \frac{3}{4}(-\cos x) + \frac{1}{4}\frac{\cos 3x}{3} + C$$

$$= \boxed{-\frac{3}{4}\cos x + \frac{1}{12}\cos 3x + C}$$

$\cos 3x - 4\cos^3 ...$

$\int (1-t^2)(-dt)$

$-\int dt + \int t^2 dt = -t + \frac{t^3}{3} + C$

$= \boxed{\frac{\cos^3 x}{3} - \cos x + C}$

Ans.

$= -\frac{3}{4}\cos x + \frac{1}{12}\cos 3x + C$

Ans.

$(\int \sin x\, dx = -\cos x)$

$\because \cos 3x = 4\cos^3 x - 3\cos x$

Q. $\int \frac{1}{x^2 (x^4+1)^{3/4}} dx = \int \frac{1}{x^2 \times x^{\cancel{4}\times\frac{3}{\cancel{4}}} \times (1+x^{-4})^{3/4}} dx$

$\left(\frac{x^4+1}{x^4} \times x^4\right)^{3/4}$

$\left(\frac{x^4}{x^4} + \frac{1}{x^4} \times x^4\right)^{3/4}$

$(1+x^{-4})^{3/4} \times x^{\cancel{4}\times\frac{3}{\cancel{4}}}$

$= \int \frac{x^{-5}}{(1+x^{-4})^{3/4}} dx$

$1 + x^{-4} = t$

$-4x^{-5} = \frac{dt}{dx}$

$x^{-5} dx = -\frac{dt}{4}$

$= -\frac{1}{4}\int t^{-3/4} dt = -\frac{1}{4} \frac{t^{-3/4+1}}{-3/4+1} + C$

$= \frac{-1}{\cancel{4}} \frac{t^{1/4}}{1/\cancel{4}} + C$

$= -(1+x^{-4})^{1/4} + C$ Ans.

Ex. 7.3

① $\int \sin^2(2x+5) dx = \int \frac{1-\cos 2(2x+5)}{2} dx = \frac{1}{2}\int dx - \frac{1}{2}\int \cos(4x+10) dx$

$= \frac{x}{2} - \frac{1}{2} \frac{\sin(4x+10)}{4} + C$

$= \frac{x}{2} - \frac{\sin(4x+10)}{8} + C$ Ans.

② $\frac{1}{2}\int 2\sin 3x \cos 4x\, dx$

$2\sin A \cos B$

$= \sin(A+B) + \sin(A-B)$

$= \frac{1}{2}\int [\sin 7x + \sin(-x)] dx$

$= \frac{1}{2}\int \sin 7x\, dx - \frac{1}{2}\int \sin x\, dx = -\frac{\cos 7x}{14} + \frac{\cos x}{2} + C$ Ans.

Q. (3) $\frac{1}{2}\int 2\cos 2x \cos 4x \cos 6x\, dx$

$\because 2\cos A \cos B = \cos(A+B) + \cos(A-B)$

Sol → $\frac{1}{2}\int (2\cos 2x \cos 6x) \cos 4x\, dx$

$\frac{1}{2}\int (\cos 8x + \cos 4x) \cos 4x\, dx$

$\because 1 + \cos 2\theta = 2\cos^2\theta$

$\because 1 - \cos 2\theta = 2\sin^2\theta$

$2\times\frac{1}{2}\int 2\cos 8x \cdot \cos 4x + \frac{1}{2}\int \cos^2 4x\, dx$

$\frac{1}{4}\int (\cos 12x + \cos 4x)\, dx + \frac{1}{2}\int \frac{1+\cos 2(4x)}{2}\, dx$

$\frac{1}{4}\int \cos 12x\, dx + \frac{1}{4}\int \cos 4x\, dx + \frac{1}{4}\int dx + \frac{1}{4}\int \cos 8x\, dx$

$\frac{1}{48}\sin 12x + \frac{1}{16}\sin 4x + \frac{1}{4}x + \frac{1}{32}\sin 8x + C$ Ans.

(4) $\int \sin^3(2x+1)\, dx = \int \sin^2(2x+1)\sin(2x+1)\, dx$

$= \int [1 - \cos^2(2x+1)] \sin(2x+1)\, dx$

$= -\frac{1}{2}\int (1-t^2)\, dt$

$\cos(2x+1) = t$

$-2\sin(2x+1) = \frac{dt}{dx}$

$\sin(2x+1)\, dx = -\frac{1}{2}dt$

$= -\frac{1}{2}\int dt + \frac{1}{2}\int t^2\, dt$

$= -\frac{1}{2}t + \frac{1}{2}\frac{t^3}{3} + C$

$= -\frac{1}{2}\cos(2x+1) + \frac{1}{6}\cos^3(2x+1) + C$ Ans.

(5) $\int \sin^3 x \cos^3 x\, dx =$

$\sin x = t$

$\cos x = dt$

$$\int t^3 \cos^2 x \boxed{\cos x \, dx}$$

$\sin x = t$

$\cos x = \frac{dt}{dx}$

$\cos x \, dx = dt$

$$\int t^3 (1 - t^2) \, dt$$

$\cos^2 x = 1 - \sin^2 x = 1 - t^2$

$$\int t^3 \, dt - \int t^5 \, dt$$

$$\frac{t^4}{4} - \frac{t^6}{6} + C = \boxed{\frac{\sin^4 x}{4} - \frac{\sin^6 x}{6} + C} \text{ Ans}$$

$$= \frac{(\sin^2 x)^2}{4} - \frac{(\sin^2 x)^3}{6} + C$$

$$= \frac{(1 - \cos^2 x)^2}{4} - \frac{(1 - \cos^2 x)^3}{6} + C$$

(7) $\frac{1}{2}\int 2 \sin 4x \sin 8x \, dx = \frac{1}{2}\int \left[\cos(4x - 8x) - \cos(4x + 8x)\right] dx$

$\because \cos(-\theta) = \cos\theta$

$$= \frac{1}{2}\int (\cos 4x - \cos 12x) \, dx$$

$\because \sin(-\theta) = -\sin\theta$

$$= \frac{1}{2} \times \frac{\sin 4x}{4} - \frac{\sin 12x}{24} + C \text{ Ans.}$$

EXERCISE 7.3

Find the integrals of the functions in Exercises 1 to 22:

1. $\sin^2 (2x + 5)$ **2.** $\sin 3x \cos 4x$ **3.** $\cos 2x \cos 4x \cos 6x$

4. $\sin^3 (2x + 1)$ **5.** $\sin^3 x \cos^3 x$ **6.** $\sin x \sin 2x \sin 3x$

7. $\sin 4x \sin 8x$ **8.** $\frac{1 - \cos x}{1 + \cos x}$ **9.** $\frac{\cos x}{1 + \cos x}$

10. $\sin^4 x$ **11.** $\cos^4 2x$ **12.** $\frac{\sin^2 x}{1 + \cos x}$

$\because \sin^{-1}\sin x = x$

(21) $\int \sin^{-1}(\cos x) \, dx = \int \sin^{-1} \sin\left(\frac{\pi}{2} - x\right) dx = \int \left(\frac{\pi}{2} - x\right) dx$

$= \frac{\pi}{2}x - \frac{x^2}{2} + c$ Ans.

13. $\frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha}$ **14.** $\frac{\cos x - \sin x}{1 + \sin 2x}$ **15.** $\tan^3 2x \sec 2x$

16. $\tan^4 x$ **17.** $\frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x}$ **18.** $\frac{\cos 2x + 2\sin^2 x}{\cos^2 x}$

19. $\frac{1}{\sin x \cos^3 x}$ **20.** $\frac{\cos 2x}{(\cos x + \sin x)^2}$ **21.** $\sin^{-1}(\cos x)$

Sol. (22) $\int \frac{1}{\cos(x-a)\cos(x-b)}\,dx$

$\cos(a+b) = \cos a \cos b - \sin a \sin b$

$\cos[(x-a)+(x-b)] = \cos(x-a)\cos(x-b) - \sin(x-a)\sin(x-b)$

Sol. →

$\int \frac{\boxed{\sin(b-a)}}{\boxed{\sin(b-a)}\cos(x-a)\cos(x-b)}\,dx$

$\frac{1}{\sin(b-a)}\int \frac{\sin[\boxed{(x-a)} - \boxed{(x-b)}]}{\cos(x-a)\cos(x-b)}\,dx$

$\sin(a+b) = \sin a \cos b + \cos a \sin b$

$\sin[(x-a)+(x-b)] = \sin(x-a)\cos(x-b) + \cos(x-a)\sin(x-b)$

$\frac{1}{\sin(b-a)} \times \left\{\int \left(\frac{\sin(x-a)\cancel{\cos(x-b)}}{\cos(x-a)\cancel{\cos(x-b)}} - \frac{\cancel{\cos(x-a)}\sin(x-b)}{\cancel{\cos(x-a)}\cos(x-b)}\right)dx\right\}$

A B

$\sin[(x-a)-(x-b)] = \sin(x-a)\cos(x-b) - \cos(x-a)\sin(x-b)$

$\boxed{\sin(b-a)}$

$\frac{1}{\sin(b-a)}\left[\int \tan(x-a)\,dx - \int \tan(x-b)\,dx\right]$

$\frac{1}{\sin(b-a)}\left[\log\sec(x-a) - \log\sec(x-b)\right]$

$\frac{1}{\sin(b-a)}\log\frac{\sec(x-a)}{\sec(x-b)} + c$

$\boxed{\frac{1}{\sin(b-a)}\log\frac{\cos(x-b)}{\cos(x-a)} + c}$ Ans.

22. $\dfrac{1}{\cos(x-a)\cos(x-b)}$

Choose the correct answer in Exercises 23 and 24.

23. $\displaystyle\int \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x}\,dx$ is equal to

$\displaystyle\int \left(\frac{\cancel{\sin^2 x}}{\cancel{\sin^2 x}\cos^2 x} - \frac{\cancel{\cos^2 x}}{\sin^2 x \cancel{\cos^2 x}} \right) dx$

(A) $\tan x + \cot x + C$ (B) $\tan x + \operatorname{cosec} x + C$

(C) $-\tan x + \cot x + C$ (D) $\tan x + \sec x + C$

24. $\displaystyle\int \frac{e^x(1+x)}{\cos^2(e^x x)}\,dx$ equals

$\displaystyle\int \frac{dt}{\cos^2 t} = \int \sec^2 t \, dt$

$e^x x = t$

$e^x + x e^x = \dfrac{dt}{dx}$

$e^x(1+x)\,dx = dt$

(A) $-\cot(ex^x) + C$ (B) $\tan(xe^x) + C$

(C) $\tan(e^x) + C$ (D) $\cot(e^x) + C$

Ex: 7.4

(18) $\displaystyle\int \frac{1}{x^2 - a^2}\,dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c$

(19) $\displaystyle\int \frac{1}{a^2 - x^2}\,dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + c$

(20) $\displaystyle\int \frac{1}{x^2 + a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$

(21) $\displaystyle\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \log\left|x + \sqrt{x^2 - a^2}\right| + C$

(22) $\displaystyle\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a} + C$

(23) $\displaystyle\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \log\left|x + \sqrt{x^2 + a^2}\right| + C$

Integration of type

① $\displaystyle\int \frac{1}{ax^2+bx+c}$

② $\displaystyle\int \frac{1}{\sqrt{ax^2+bx+c}}\,dx$

$$ax^2+bx+c = a\left[x^2+\frac{b}{a}x+\frac{c}{a}\right]$$

$$= a\left[x^2+2x\times\frac{b}{2a}+\left(\frac{b}{2a}\right)^2 + \frac{c}{a}-\left(\frac{b}{2a}\right)^2\right]$$

$$= a\left[\left(x+\frac{b}{2a}\right)^2-(d)^2\right]$$

③ $\displaystyle\int \frac{px+q}{ax^2+bx+c}\,dx$

④ $\displaystyle\int \frac{px+q}{\sqrt{ax^2+bx+c}}\,dx$

$$px+q = A\frac{d}{dx}\left[ax^2+bx+c\right]+B$$

$$ax^2+bx+c = \left(x+\frac{b}{2a}\right)^2-d^2$$

Q.① $\displaystyle\int \frac{1}{x^2-25}\,dx = \int \frac{1}{x^2-5^2}\,dx$

$$= \frac{1}{2\times 5}\log\left|\frac{x-5}{x+5}\right| + C$$

$$= \frac{1}{10}\log\left|\frac{x-5}{x+5}\right| + C$$

Q.② $\displaystyle\int \frac{1}{x^2+25} = \int \frac{1}{x^2+5^2}\,dx = \frac{1}{5}\tan^{-1}\frac{x}{5} + C$

Q.③ $\displaystyle\int \frac{1}{25-x^2}\,dx = \int \frac{1}{5^2-x^2}\,dx = \frac{1}{10}\log\left|\frac{5+x}{5-x}\right| + C$

Q.④ $\displaystyle\int \frac{1}{\sqrt{x^2-25}}\,dx = \log\left|x+\sqrt{x^2-25}\right| + C$

q.(4) $\int \frac{1}{\sqrt{x^2-25}}\,dx = \log\left|x + \sqrt{x^2-25}\right| + C$

q.(5) $\int \frac{1}{\sqrt{25-x^2}}\,dx = \sin^{-1}\frac{x}{5} + C$

q.(6) $\int \frac{1}{\sqrt{x^2+25}}\,dx = \log\left|x + \sqrt{x^2+25}\right| + C$

q.(7) $\int \frac{1}{3x^2+13x-10}\,dx$

$f(x) = 3\,\frac{[3x^2+13x-10]}{3}$

$= 3\left[x^2 + \frac{13}{3}x - \frac{10}{3}\right]$

$= 3\left[x^2 + 2\cdot x\cdot\frac{13}{6} + \left(\frac{13}{6}\right)^2 - \frac{10}{3} - \frac{169}{36}\right]$

$= 3\left[\left(x+\frac{13}{6}\right)^2 - \frac{(120+169)}{36}\right]$

$= 3\left[\left(x+\frac{13}{6}\right)^2 - \frac{289}{36}\right]$

$= \boxed{3\left[\left(x+\frac{13}{6}\right)^2 - \left(\frac{17}{6}\right)^2\right]}$

t

$= \frac{1}{3}\int \frac{1}{\left(x+\frac{13}{6}\right)^2 - \left(\frac{17}{6}\right)^2}\,dx$

$x + \frac{13}{6} = t \Rightarrow dx = dt$

$= \frac{1}{3}\int \frac{1}{t^2 - \left(\frac{17}{6}\right)^2}\,dt$

$= \frac{1}{\cancel{3}} \times \frac{1}{\frac{2 \times 17}{6_{\cancel{2}}}} \log\left|\frac{t - \frac{17}{6}}{t + \frac{17}{6}}\right|$

$= \frac{1}{17}\log\left|\frac{x+\frac{13}{6}-\frac{17}{6}}{x+\frac{13}{6}+\frac{17}{6}}\right| + C_1$

$= \frac{1}{17}\log\left|\frac{x-\frac{2}{3}}{x+5}\right| + C_1$

$= \frac{1}{17}\log\left|\frac{3x-2}{3(x+5)}\right| + C_1$

$= \frac{1}{17}\log\left|\frac{3x-2}{x+5}\right| - \log 3 + C_1$

$= \boxed{\frac{1}{17}\log\left|\frac{3x-2}{x+5}\right| + C}$

Ans.

$$= \frac{1}{17}\log\left|\frac{3x-2}{3(x+5)}\right| + c_1$$

Ans.

$$\frac{-\log 3}{17} + c_1 \to c$$

$$\log\left|\frac{f(x)}{3}\right| = \log f(x) - \log 3$$

7.4

Q. 10 $\displaystyle\int \frac{1}{\sqrt{x^2+2x+2}}\,dx$

$\because x^2 + \boxed{2}x + 2 = (x+1)^2 + 1^2$

$$\int \frac{1}{\sqrt{(x+1)^2+1^2}}$$

$x+1 = t \Rightarrow dx = dt$

$$\int \frac{1}{\sqrt{t^2+1^2}}\,dt = \log\left|t+\sqrt{t^2+1}\right| + c$$

$$= \boxed{\log\left|(x+1)+\sqrt{x^2+2x+2}\right| + c}$$

Ex 10

(ii) $\displaystyle\int \frac{x+3}{\sqrt{5-4x-x^2}}\,dx$

$$\boxed{x+3 = A\frac{d}{dx}(5-4x-x^2) + B}$$

$$x+3 = \boxed{A[-4-2x] + B}$$

$1.\ x + 3 = -4A - 2Ax + B$

$1x + 3 = (-2A)x + (B-4A)$

on comparing

$1 = -2A \Rightarrow A = \frac{-1}{2}$

$3 = B - 4 \times \left(\frac{-1}{2}\right)$

$3 = B + 2 \quad \boxed{B = 1}$

$5 - 4x - x^2 = -\{x^2+4x-5\}$

$= -[x^2 + 2\cdot x\cdot 2 + 4 - 5]$

$= -[(x+2)^2 - 3^2]$

$= \boxed{3^2 - (x+2)^2}$ (1)

$3 = B + 2 \quad \boxed{B = 1}$

$$x + 3 = -\frac{1}{2}(-4 - 2x) + 1 \longrightarrow (1)$$

$$\int \frac{x+3}{\sqrt{5-4x-x^2}}\,dx = \underbrace{\int \frac{-\frac{1}{2}(-4-2x)}{\sqrt{5-4x-x^2}}\,dx}_{I_1} + \underbrace{\int \frac{1}{\sqrt{3^2-(x+2)^2}}\,dx}_{I_2}$$

$5-4x-x^2 \to t$

$\boxed{7.4}$

(23) $$\int \frac{5x+3}{\sqrt{x^2+4x+10}}\,dx$$

Linear

Quadratic

$x^2 + 4x + 10 = \boxed{x^2 + 2\cdot x\cdot 2 + 2^2} + 10 - 2^2$

$= (x+2)^2 + 6$

$x^2 + 4x + 10 = (x+2)^2 + (\sqrt{6})^2 \longrightarrow (1)$

$5x + 3 = A\frac{d}{dx}(x^2+4x+10) + B$

$\boxed{5x+3 = A(2x+4) + B}$

$5x + 3 = 2A\,x + (4A + B)$

$2A = 5 \Rightarrow A = 5/2$

$4A + B = 3$

$4\times\frac{5}{2} + B = 3$

$10 + B = 3$

$\boxed{B = -7}$

$5x + 3 = \frac{5}{2}(2x+4) - 7 \longrightarrow (11)$

$$= \int \frac{\frac{5}{2}(2x+4) - 7}{\sqrt{x^2+4x+10}}\,dx$$

$$= \frac{5}{2}\int \frac{2x+4\;dx}{\sqrt{x^2+4x+10}} - 7\int \frac{1}{\sqrt{(x+2)^2 + (\sqrt{6})^2}}\,dx$$

$x^2 + 4x + 10 = t$

$(2x+4)\,dx = dt$

$x + 2 = t_1$

$dx = dt$

$$= \frac{5}{2}\int \frac{dt}{\sqrt{t}} - 7\int \frac{1}{\sqrt{t_1^2 + (\sqrt{6})^2}}\,dt$$

$$= \frac{5}{2}\int t^{-1/2}\,dt - 7 \times \log\left|t_1 + \sqrt{t_1^2 + 6}\right|$$

$\because \int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}$

$\because \int \frac{1}{\sqrt{x^2+a^2}}\,dx = \log\left|x + \sqrt{x^2+a^2}\right| + C$

$$= \frac{5}{2}\,\frac{t^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} - 7\log\left|t_1 + \sqrt{x^2+4x+10}\right|$$

$$= 5\,t^{1/2} - 7\log\left|x + 2 + \sqrt{x^2+4x+10}\right| + C$$

$$= \boxed{5\sqrt{x^2+4x+10} - 7\log\left|x+2+\sqrt{x^2+4x+10}\right| + c}$$ Ans.

Q.(14) $\displaystyle\int \frac{1}{\sqrt{8+3x-x^2}}\,dx$

$$8+3x-x^2 = -(x^2-3x-8)$$

$$= -\left[x^2 - 2\cdot x\cdot\frac{3}{2} + \left(\frac{3}{2}\right)^2 - 8 - \frac{9}{4}\right]$$

$$= -\left[\left(x-\frac{3}{2}\right)^2 - \frac{41}{4}\right]$$

$$= \left(\frac{\sqrt{41}}{2}\right)^2 - \left(x-\frac{3}{2}\right)^2$$

$$\int \frac{1}{\sqrt{\left(\frac{\sqrt{41}}{2}\right)^2 - \left(x-\frac{3}{2}\right)^2}}\,dx$$

$$x - \frac{3}{2} = t$$

$$dx = dt$$

$$= \int \frac{1}{\sqrt{\left(\frac{\sqrt{41}}{2}\right)^2 - t^2}}\,dt$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}$$

$$= \sin^{-1}\frac{t}{\frac{\sqrt{41}}{2}} + c$$

$$= \sin^{-1}\frac{2t}{\sqrt{41}} + c$$

$$= \sin^{-1}\frac{2\left(x-\frac{3}{2}\right)}{\sqrt{41}} + c$$

$$\boxed{\sin^{-1}\left(\frac{2x-3}{\sqrt{41}}\right) + c}$$ Ans.

Q.(22) $\displaystyle\int \frac{x+3}{x^2-2x-5}\,dx$

$$x^2 - 2\cdot x\cdot 1 + 1^2 - 1$$

$$x^2-2x-5 = (x-1)^2 - \sqrt{6}^2 \quad (1)$$

$$\boxed{x+3 = A(2x-2) + B}$$

$$x+3 = 2Ax + (B-2A)$$

$$2A = 1 \Rightarrow A = 1/2$$

$$B - 2A = 3$$

$$B - 1 = 3$$

$$\boxed{B = 4}$$

$$= \int \frac{\frac{1}{2}(2x-2)}{x^2-2x-5}\,dx + 4\int \frac{1}{(x-1)^2 - (\sqrt{6})^2}\,dx$$

$$\boxed{\frac{1}{2}\log\left|x^2-2x-5\right| + \cancel{4}^{\,2} \times \frac{1}{\cancel{2}\times\sqrt{6}}\log\left|\frac{x-1-\sqrt{6}}{x-1+\sqrt{6}}\right| + c}$$

$$\frac{1}{2}\log\left|x^2-2x-5\right| + \cancel{4}^{2}\times\frac{1}{\cancel{2}\times\sqrt{6}}\log\left|\frac{x-1-\sqrt{6}}{x-1+\sqrt{6}}\right| + C$$

Ans.

$B-1=3$

$\boxed{B=4}$

$x+3=\frac{1}{2}(2x-2)+4$ — (1)

(21) $$\int\frac{x+2}{\sqrt{x^2+2x+3}}\,dx$$

$x^2+2x+3=\boxed{x^2+2\cdot x\cdot 1+1}-1+3$

$=(x+1)^2+(\sqrt{2})^2$ — (1)

$x+2=A\frac{d}{dx}\left[x^2+2x+3\right]+B$

$x+2=A(2x+2)+B$ (*)

$x+2=2Ax+(2A+B)$

$2A=1 \Rightarrow A=1/2$

$2A+B=2$ $\boxed{B=1}$

$x+2=\frac{1}{2}(2x+2)+1$ — (1)

Sol. $$\frac{1}{2}\int\frac{2x+2}{\sqrt{x^2+2x+3}}\,dx+\int\frac{1}{\sqrt{(x+1)^2+(\sqrt{2})^2}}\,dx$$

$x^2+2x+3=t$

$(2x+2)\,dx=dt$

$x+1=t$

$dx=dt$

$$\frac{1}{2}\int t^{-1/2}\,dt+\int\frac{1}{\sqrt{t^2+(\sqrt{2})^2}}\,dt$$

$$\frac{1}{2}\,\frac{t^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}+\log\left|t+\sqrt{t^2+2}\right|+C$$

$$=\sqrt{x^2+2x+3}+\log\left|x+1+\sqrt{(x+1)^2+2}\right|+C$$

$$=\boxed{\sqrt{x^2+2x+3}+\log\left|x+1+\sqrt{x^2+2x+3}\right|+C}$$ Ans

Ex 7.5

Partial Fraction

Fraction $=\frac{f(x)}{g(x)}$

Proper Fraction

$\frac{f(x)}{g(x)}$

deg $g(x)$ > deg $f(x)$

$\frac{(x+5)}{(x^2+5x+6)}$ → proper

$=\frac{x+5}{(x+2)(x+3)}$

$4\overline{)7}(1$
$\frac{4}{3}$

$\frac{7}{4}=1\frac{3}{4}=1+\frac{3}{4}$

Improper Fraction

deg $f(x)$ > deg $g(x)$

$\frac{x^{4}+2x}{x^{2}+3x+2}$ → Improper

$=(x^2-3x+7)+\frac{(-13x-14)}{x^2+3x+2}$

$=(x^2-3x+7)-\frac{(13x+14)}{(x^2+3x+2)}$

$x^2+3x+2\overline{)\,x^4+0x^3+0x^2+2x\,}(x^2-3x+7$

$x^4+3x^3+2x^2$

(x^2+5x+6) $(x+2)(x+3)$

$$x^4+3x^3+2x^2$$
$$-3x^3-2x^2+2x$$
$$-3x^3-9x^2-6x$$
$$7x^2+8x$$
$$7x^2+21x+14$$
$$-13x-14$$

$\boxed{\text{Rules}}$ applicable for proper fraction

(1) $$\frac{x+p}{(x+a)(x+b)} = \frac{A}{x+a} + \frac{B}{x+b}$$

(2) $$\frac{x+p}{(x+a)^2(x+b)} = \frac{A}{x+a} + \frac{B}{(x+a)^2} + \frac{C}{x+b}$$

(3) $$\frac{x+p}{(x^2+\boxed{2x}+3)} = \frac{Ax+B}{x^2+2x+3}$$

(4) $$\frac{x+p}{(x^2+2x+3)^2} = \frac{Ax+B}{x^2+2x+3} + \frac{Cx+D}{(x^2+2x+3)^2}$$

Example 11 Find $\int \frac{dx}{(x+1)(x+2)}$

$$\because \frac{1}{(x+a)(x+b)} = \frac{A}{(x+a)} + \frac{B}{(x+b)}$$

Sol- $$\frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2} \quad \text{—— (a)}$$

$$\because \int \frac{1}{x}dx = \log x + C$$

$$\frac{1}{\cancel{(x+1)(x+2)}} = \frac{A(x+2)+B(x+1)}{\cancel{(x+1)(x+2)}}$$

$$1 = A(x+2) + B(x+1) \quad \text{—— (1)}$$

Method $\boxed{1}$ put $\boxed{x=-2}$

$$1 = A(-2+2) + B(-2+1)$$

$$1 = -B$$

$$\boxed{B=-1}$$

put $\boxed{x=-1}$

$$1 = A(-1+2) + B(-1+1)$$

$$\boxed{1 = A}$$

$\boxed{\text{Method } 2^{nd}}$

$1 = Ax + 2A + Bx + B$

$1 = (A+B)x + (2A+B)$

$0x + 1 = (A+B)x + (2A+B)$

$A + B = 0$
$2A + B = 1$

$-A = -1 \Rightarrow \boxed{A = 1}$

$\boxed{B = -1}$

$$\int \frac{1}{(x+1)(x+2)}dx = \int \frac{1}{x+1}dx + \int \frac{-1}{x+2}dx$$

$$= \int \frac{1}{x+1}dx - \int \frac{1}{x+2}dx$$

$$= \log|x+1| - \log|x+2| + C$$

$$\boxed{\int \frac{1}{(x+1)(x+2)}dx = \log\left|\frac{x+1}{x+2}\right| + C \quad \text{Ans.}}$$

Integrate the rational functions in Exercises 1 to 21.

1. $\dfrac{x}{(x+1)(x+2)}$

2. $\dfrac{1}{x^2-9} = \dfrac{1}{(x-3)(x+3)}$

3. $\dfrac{3x-1}{(x-1)(x-2)(x-3)}$

4. $\dfrac{x}{(x-1)(x-2)(x-3)}$

5. $\dfrac{2x}{x^2+3x+2}$ $\quad \dfrac{2x}{(x+1)(x+2)}$

6. $\dfrac{1-x^2}{x(1-2x)}$

Sol ③ $\displaystyle\int \frac{3x-1}{(x-1)(x-2)(x-3)}dx$

$$\frac{3x-1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3} \quad \text{--- (1)}$$

$$\frac{3x-1}{\cancel{(x-1)(x-2)(x-3)}} = \frac{A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)}{\cancel{(x-1)(x-2)(x-3)}}$$

$3x-1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$

put $\boxed{x=1}$

$2 = 2A \Rightarrow \boxed{A=1}$

$\boxed{x=2}$

$5 = -B \Rightarrow \boxed{B=-5}$

put $x=3$

$8 = 2C$

$\boxed{C=4}$

$$\int \frac{3x-1}{(x-1)(x-2)(x-3)}\,dx = \int \frac{1}{(x-1)}\,dx - 5\int \frac{1}{x-2}\,dx + 4\int \frac{1}{x-3}\,dx$$

$$= \underbrace{\log|x-1| - 5\log|x-2| + 4\log|x-3|}$$

$$= \log(x-1) - \log(x-2)^5 + \log(x-3)^4$$

$$= \underbrace{\log \frac{(x-1)(x-3)^4}{(x-2)^5} + C} \quad \underline{\text{Ans}}$$

① $\int \frac{x}{(x+1)(x+2)}\,dx$

Sol. →

$$\frac{x}{(x+1)(x+2)} = \frac{A}{(x+1)} + \frac{B}{x+2}$$

$$\frac{x}{(x+1)(x+2)} = \frac{A(x+2) + B(x+1)}{(x+1)(x+2)}$$

$$\frac{x}{\cancel{(x+1)(x+2)}} = \frac{Ax + 2A + Bx + B}{\cancel{(x+1)(x+2)}}$$

$$1x + 0 = (A+B)x + (2A+B)$$

$A + B = 1$ —— (i)

$2A + B = 0$ —— (ii)

$$A + B = 1$$
$$2A + B = 0$$
$$-A = 1$$

$\boxed{B = 2}$ $\boxed{A = -1}$

$$\int \frac{x}{(x+1)(x+2)}\,dx = \int \frac{A}{x+1}\,dx + \int \frac{B}{x+2}\,dx$$
$$= -1\int \frac{1}{x+1}\,dx + 2\int \frac{1}{x+2}\,dx$$
$$= -\log|x+1| + 2\log|x+2| + C$$
$$= -\log(x+1) + \log(x+2)^2 + C$$
$$= \boxed{\log \frac{(x+2)^2}{x+1} + C} \text{ Ans}$$

(6) $\int \frac{1-x^2}{x(1-2x)}\,dx$

$$\frac{1-x^2}{x-2x^2} = \frac{1}{2} + \left[\frac{\frac{1}{2}x - 1}{2x\left(x-\frac{1}{2}\right)}\right] \quad (1)$$

$$\frac{-(1-x^2)}{-(x-2x^2)} = \frac{x^2-1}{2x^2-x}$$

$4)\,7\,(1$, $\frac{7}{4}$, 3

$$\frac{7}{4} = 1\frac{3}{4} = 1 + \frac{3}{4}$$

$$2x^2 - x \,\big)\, x^2 - 1 \,\big(\, \frac{1}{2}$$
$$\underline{x^2 - \frac{1}{2}x}$$
$$\frac{1}{2}x - 1$$

$$\frac{\frac{1}{2}x - 1}{x\left(x-\frac{1}{2}\right)} = \frac{A}{x} + \frac{B}{x-\frac{1}{2}}$$

$$\frac{1-x^2}{x-2x^2} = \frac{1}{2} + \frac{\frac{1}{2}x-1}{2x^2-x}$$

$$\frac{\frac{1}{2}x - 1}{\cancel{x\left(x-\frac{1}{2}\right)}} = \frac{A\left(x-\frac{1}{2}\right) + Bx}{\cancel{x\left(x-\frac{1}{2}\right)}}$$

$$\frac{1}{2}x + (-1) = x(A+B) + \left(-\frac{1}{2}A\right) \Rightarrow \boxed{A = 2} \quad \boxed{B = -\frac{3}{2}}$$

$$\frac{1}{2}x + (-1) = x(A+B) + \left(-\frac{1}{2}A\right) \Rightarrow \boxed{A = 2} \quad \boxed{B = \frac{-3}{2}}$$

$$\frac{\frac{1}{2}x - 1}{x\left(x-\frac{1}{2}\right)} = \frac{2}{x} + \frac{-3/2}{x-1/2} \quad \text{—— (11)}$$

$$\frac{1-x^2}{x(1-2x)} = \frac{1}{2} + \frac{1}{2}\left[\frac{2}{x} - \frac{3}{2\left(x-\frac{1}{2}\right)}\right]$$

$$\frac{1-x^2}{x(1-2x)} = \frac{1}{2} + \frac{1}{x} - \frac{3}{4} \times \frac{1}{x-\frac{1}{2}}$$

$$\int \frac{1-x^2}{x(1-2x)}dx = \frac{1}{2}x + \log x - \frac{3}{4}\log\left|x-\frac{1}{2}\right| + C$$

$$= \frac{x}{2} + \log x - \frac{3}{4}\log|2x-1| + C \quad \text{Ans.}$$

Q. (7) $\displaystyle\int \frac{x}{(x^2+1)(x-1)}\,dx$

$$\boxed{\frac{x}{(x^2+1)(x-1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1}}$$

$$\frac{x}{\cancel{(x^2+1)(x-1)}} = \frac{(Ax+B)(x-1) + C(x^2+1)}{\cancel{(x^2+1)(x-1)}}$$

$$x = Ax^2 - Ax + Bx - B + Cx^2 + C$$

$$0x^2 + 1x + 0 = (A+C)x^2 + x(B-A) + (C-B)$$

$A + C = 0 \Rightarrow \boxed{A = -C} \Rightarrow C = -A$

$$\left.\begin{aligned} B - A &= 1 \\ C - B &= 0 \end{aligned}\right\}$$

$$\begin{aligned} B - A &= 1 \\ -B - A &= 0 \\ \hline -2A &= 1 \end{aligned}$$

$$\boxed{B = 1 + A = 1 - \frac{1}{2} = \frac{1}{2}}$$

$$\boxed{C = \frac{1}{2}}$$

$$\boxed{A = -\frac{1}{2}}$$

$$-2A = 1 \qquad \boxed{A = -\frac{1}{2}}$$

$$\int \frac{x}{(x^2+1)(x-1)}\,dx = \int \frac{Ax+B}{x^2+1}\,dx + \int \frac{C}{x-1}\,dx$$

$$= \int \frac{-\frac{1}{2}x + \frac{1}{2}}{x^2+1}\,dx + \frac{1}{2}\int \frac{1}{x-1}\,dx$$

$$= \frac{-\frac{1}{2}}{2\times 2}\int \frac{x\times 2}{x^2+1}\,dx + \frac{1}{2}\int \frac{1}{x^2+1}\,dx + \frac{1}{2}\int \frac{1}{x-1}\,dx$$

$$= \frac{-1}{4}\int \frac{2x}{x^2+1}\,dx + \frac{1}{2}\int \frac{1}{x^2+1}\,dx + \frac{1}{2}\int \frac{1}{x-1}\,dx$$

$$x^2+1 = t$$

$$2x\,dx = dt$$

$$= \frac{-1}{4}\int \frac{1}{t}\,dt + \frac{1}{2}\tan^{-1}x + \frac{1}{2}\log|x-1| + C$$

$$= \frac{-1}{4}\log|x^2+1| + \frac{1}{2}\tan^{-1}x + \frac{1}{2}\log|x-1| + C$$

⑧ $\displaystyle\int \frac{x}{(x-1)^2(x+2)}\,dx$

$$\frac{x}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}$$

$$\frac{x}{\cancel{(x-1)^2(x+2)}} = \frac{A(x-1)(x+2) + B(x+2) + C(x-1)^2}{\cancel{(x-1)^2(x+2)}}$$

at $\boxed{x=1}$

$1 = B(3)$ $\qquad \boxed{B = 1/3}$

at $\boxed{x=-2}$

$-2 = 9C$ $\qquad \boxed{C = -2/9}$

at $\boxed{x=0}$

$0 = -2A + 2B + C$

$0 = -2A + \frac{2}{3} - \frac{2}{9}$

$2A = \frac{6-2}{9}$

$\boxed{A = \frac{2}{9}}$

$$\frac{x}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}$$

$$\int \frac{x}{(x-1)^2(x+2)}dx = \frac{2}{9}\int \frac{1}{x-1}dx + \frac{1}{3}\int \frac{1}{(x-1)^2}dx - \frac{2}{9}\int \frac{1}{x+2}dx$$

$$= \frac{2}{9}\log|x-1| - \frac{1}{3}\times\frac{1}{x-1} - \frac{2}{9}\log|x+2| + C$$

$$= \frac{2}{9}\log\frac{|x-1|}{|x+2|} - \frac{1}{3}\times\frac{1}{x-1} + C \quad \text{Ans.}$$

(9) $\int \frac{3x+5}{x^3-x^2-x+1}dx$

$f(x) = x^3 - x^2 - x + 1$

$f(1) = \cancel{1} - \cancel{1} - \cancel{1} + \cancel{1} = 0$

$(x-1)$ is a factor of $f(x)$

$= x^2(x-1) + 0(x-1) - (x-1)$

$= x^2(x-1) - (x-1)$

$= (x-1)(x^2-1)$

$= (x-1)(x-1)(x+1)$

$= (x-1)^2(x+1)$

$$\int \frac{3x+5}{(x-1)^2(x+1)}dx$$

$$\frac{3x+5}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

10. $\frac{2x-3}{(x^2-1)(2x+3)}$ **11.** $\frac{5x}{(x+1)(x^2-4)}$ **12.** $\frac{x^3+x+1}{x^2-1}$

13. $\frac{2}{(1-x)(1+x^2)}$ **14.** $\frac{3x-1}{(x+2)^2}$ **15.** $\frac{1}{x^4-1}$

Sol. $\boxed{15}$ $\int \frac{1}{\ldots}dx$

Sol. 15 $\int \frac{1}{x^4-1}\,dx$

$$\frac{1}{x^4-1} = \frac{1}{(x^2)^2-1^2} = \frac{1}{(x^2+1)\underbrace{(x^2-1)}} = \frac{1}{(x^2+1)(x+1)(x-1)}$$

$$\frac{1}{x^4-1} = \frac{1}{\underbrace{(x^2+1)(x+1)(x-1)}} = \frac{Ax+D}{\underbrace{(x^2+1)}} + \frac{B}{(x+1)} + \frac{C}{x-1} \quad \text{--- (1)}$$

$$\frac{1}{\cancel{x^4-1}} = \frac{(Ax+D)(x+1)(x-1) + B(x^2+1)(x-1) + C(x+1)(x^2+1)}{\cancel{(x^2+1)(x+1)(x-1)}}$$

$$1 = (Ax+D)(x+1)(x-1) + B(x^2+1)(x-1) + C(x+1)(x^2+1)$$

$$\boxed{\begin{array}{l} x=1 \Rightarrow C = 1/4 \\ x=-1 \Rightarrow B = -1/4 \end{array}}$$

at $\boxed{x=0}$

$$1 = D(-1) + B(-1) + C(1)(1)$$

$$1 = -D - B + C$$

$$1 = -D + \frac{1}{4} + \frac{1}{4} \qquad -D = \frac{1}{2} \qquad \boxed{D = -\frac{1}{2}}$$

$$\int \frac{1}{x^4-1}\,dx = -\frac{1}{2}\int \frac{1}{x^2+1}\,dx - \frac{1}{4}\int \frac{1}{x+1}\,dx + \frac{1}{4}\int \frac{1}{x-1}\,dx$$

$$= -\frac{1}{2}\tan^{-1}x - \frac{1}{4}\log|x+1| + \frac{1}{4}\log|x-1| + C$$

$$= \boxed{-\frac{1}{2}\tan^{-1}x + \frac{1}{4}\log\frac{|x-1|}{|x+1|} + C}$$

(14) $$\frac{3x+1}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2} \qquad \boxed{ax^2+bx+c}$$

$$\frac{3x+1}{\cancel{(x+2)^2}} = \frac{A(x+2)+B}{\cancel{(x+2)^2}} \qquad \boxed{A=3}$$

$$2A+B=1$$

$$6+B=1$$

$$3x+1 = Ax + (2A+B) \qquad \boxed{B=-5}$$

$$= \int \frac{1}{\ }\,dx$$

$$\int \frac{3x+1}{(x+2)^2}dx = 3\int \frac{1}{x+2}dx - 5\int \frac{1}{(x+2)^2}dx$$

7.6 → $\boxed{\text{Method of Integration}}$ (3) $\boxed{\text{By Part}}$ method

$$\int f(x)\cdot g(x)\,dx = \int \mathrm{I}\cdot\mathrm{II}\,dx$$

Preference order

ILATE ← Exponential

Inverse, Logarithmic, Algebra, Trigonometric

ILATE, ILATE, E/A

$$\boxed{\int \mathrm{I}\cdot\mathrm{II}\,dx = \mathrm{I}\int \mathrm{II}\,dx - \int\left(\frac{d}{dx}\mathrm{I}\int \mathrm{II}\,dx\right)dx}$$

Q. $\int \underset{\mathrm{I}}{x}\,\underset{\mathrm{II}}{\sin x}\,dx = x\int \sin x\,dx - \int\left\{\frac{d}{dx}x\int \sin x\,dx\right\}dx$

$$= -x\cos x - \int 1\times(-\cos x)\,dx$$

$$= -x\cos x + \int \cos x\,dx$$

$$\boxed{\int x\sin x\,dx = -x\cos x + \sin x + C}$$

Q. $\int \overset{\mathrm{I}}{x^2}\,\overset{\mathrm{II}}{\cos x}\,dx = x^2\int \cos x\,dx - \int\left[\frac{d}{dx}x^2\int \cos x\,dx\right]$

$$= x^2\sin x - 2\boxed{\int x\sin x\,dx}$$

$$= x^2\sin x - 2\left[-x\cos x + \sin x\right] + C$$

$$= x^2\sin x + 2x\cos x - 2\sin x + C \quad \text{Ans.}$$

EXERCISE 7.6

Integrate the functions in Exercises 1 to 22.

1. $x \sin x$
2. $x \sin 3x$
3. $x^2 e^x$
4. $x \log x$
5. $x \log 2x$
6. $x^2 \log x$
7. $x \sin^{-1} x$
8. $x \tan^{-1} x$
9. $x \cos^{-1} x$
10. $(\sin^{-1} x)^2$
11. $\dfrac{x \cos^{-1} x}{\sqrt{1-x^2}}$
12. $x \sec^2 x$
13. $\tan^{-1} x$
14. $x (\log x)^2$
15. $(x^2 + 1) \log x$
16. $e^x (\sin x + \cos x)$
17. $\dfrac{x e^x}{(1+x)^2}$
18. $e^x \left(\dfrac{1+\sin x}{1+\cos x}\right)$
19. $e^x \left(\dfrac{1}{x} - \dfrac{1}{x^2}\right)$
20. $\dfrac{(x-3) e^x}{(x-1)^3}$
21. $e^{2x} \sin x$
22. $\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$

Choose the correct answer in Exercises 23 and 24.

23. $\int x^2 e^{x^3} dx$ equals

(A) $\frac{1}{3} e^{x^3} + C$ (B) $\frac{1}{3} e^{x^2} + C$

(C) $\frac{1}{2} e^{x^3} + C$ (D) $\frac{1}{2} e^{x^2} + C$

24. $\int e^x \sec x (1 + \tan x)\, dx$ equals

(A) $e^x \cos x + C$ (B) $e^x \sec x + C$

(C) $e^x \sin x + C$ (D) $e^x \tan x + C$