Multiple Integration
- Double Integral ⇓ Surface Integration
- Triple Integral ⇓ Volume Integration

1. $$\int_{x=0}^{x=1}\int_{y=0}^{y=x} e^{\frac{y}{x}}\,dx\,dy$$

$$= \int_0^1 \left(\int_0^x e^{\frac{y}{x}}\,dy\right) dx$$

$$= \int_0^1 \left[e^{\frac{y}{x}} \div \frac{\partial}{\partial y}\frac{y}{x}\right]_0^x dx$$

$$\frac{\partial}{\partial y}\frac{y}{x} = \frac{1}{x} \quad \frac{\partial y}{\partial y} = \frac{1}{x} \quad \therefore \int e^x\,dx = e^x$$

$$= \int_0^1 \left[e^{\frac{y}{x}} \div \frac{1}{x}\right]_0^x dx$$

$$= \int_0^1 \left[\frac{e^{\frac{y}{x}}}{\frac{1}{x}}\right]_0^x dx$$

$$= \int_0^1 \left[x\, e^{y/x}\right]_0^x dx$$

$$= \int_0^1 \left[x e^{\frac{x}{x}} - x\right] dx$$

$$= \int_0^1 (ex - x)\,dx$$

Definite Integration

$$= \int_0^1 (ex - x)\,dx$$

$$= \left[e\frac{x^2}{2} - \frac{x^2}{2}\right]_0^1$$

$$= \left\{\frac{e(1)^2}{2} - \frac{1^2}{2}\right\} - \left\{\underbrace{\frac{e(0)^2}{2} - \frac{0^2}{2}}_{0}\right\}$$

$$= \frac{e}{2} - \frac{1}{2} - 0$$

$$= \frac{e-1}{2} \text{ Ans.}$$

$$\frac{d}{dx} e^{\boxed{3x}} = 3e^{3x}$$

$$\int e^{3x}\,dx = \frac{e^{3x}}{3}$$

CBSE Book

1. $\int \frac{1}{x^2+a^2}\,dx$
2. $\int \frac{1}{x^2-a^2}\,dx$
3. $\int \frac{1}{a^2-x^2}\,dx$
4. $\int \frac{1}{\sqrt{x^2+a^2}}$
5. $\int \frac{1}{\sqrt{x^2-a^2}}$
6. $\int \frac{1}{\sqrt{a^2-x^2}}$
7. $\int \sqrt{x^2+a^2}$
8. $\int \sqrt{x^2-a^2}$
9. $\int \sqrt{a^2-x^2}$

$$\int_{x=0}^{x=1} x\,dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1^2}{2} - 0$$

Q. $$\int_0^2 \int_0^{x^2} e^{y/x}\,dx\,dy$$

Q. $$\int_0^a \int_0^{\sqrt{ay}} xy\,dx\,dy$$

Triple Integration

Q. $$\int_0^3 \int_0^2 \left[\int_{x=0}^{x=1} (x + y + z)\,dx\right] dy\,dz$$

$$\underbrace{\frac{\partial}{\partial x} y = 0}, \quad \underbrace{\frac{\partial z}{\partial x} = 0}$$

Sol. → $$\int^3 \int^2 \left[\frac{x^2}{2} + yx + zx\right]^1 dy\,dz$$

$$\int^3 (3 + 2z)\,dz$$

H.W

Sol.→ $\int_0^3\int_0^2 \left[\frac{x^2}{2} + yx + zx\right]_0^1 dy\, dz$

$\int_0^3\left[\int_0^2 \left(\frac{1}{2} + y + z\right) dy\right] dz$

$\int_0^3 \left[\frac{y}{2} + \frac{y^2}{2} + yz\right]_0^2 dz$

$\int_0^3 (3 + 2z)\, dz$

$= \left[3z + \cancel{2}\frac{z^2}{\cancel{2}}\right]_0^3$

$= 9 + 9$

$= 18$

H.W

(1) $\int_{y=0}^{1}\int_{x=y^2}^{1}\int_{z=0}^{1-x} x\, dz\, dx$

(2) $\int_0^9\int_0^x\int_0^{x+y} e^{x+y+z}\, dz\, dy\, dx$

$\int_0^3 (3 + 2x)\, dx$

$\left[3x + \cancel{2}\frac{x^2}{\cancel{2}}\right]_0^3 = 9 + 9 = 18$

$\int_a^b f(x)\, dx = \int_a^b f(t)\, dt$

Q. $\int_0^a \int_0^{\sqrt{ay}} xy\, dx\, dy$

$\int 3x\, dx = 3\int x\, dx = 3\frac{x^2}{2} + C$

Sol.→ $\int_0^a \left(\int_0^{\sqrt{ay}} xy\, dx\right) dy$

$\int_0^a \left[y\, \frac{x^2}{2}\right]_{x=0}^{x=\sqrt{ay}} dy$

$\int_0^a y \frac{(\sqrt{ay})^2}{2}\, dy$

$\int_0^a \frac{y \times a \times y}{2}\, dy$

$= \frac{a}{2}\int_0^a y^2\, dy$

$= \frac{a}{2} \times \left[\frac{y^3}{3}\right]_0^a$

$= \frac{a}{2} \times \frac{a^3}{3}$

$= \frac{a^4}{6}$ Ans.

Q. $\int_0^2 \int_{y=0}^{y=x^2} e^{y/x}\, dx\, dy$

$= \int_0^2 \underset{I}{x}\, \underset{II}{e^x}\, dx \quad - \int_0^2 x\, dx$

ILATE

dy

dz

Sol. $\displaystyle\int_0^2 \left(\int_0^{x^2} e^{\frac{y}{x}}\, dy\right) dx$

$$\int_0^2 \left[x e^{\frac{y}{x}}\right]_{y=0}^{y=x^2} dx$$

$$\int_0^2 \left(x e^{\frac{x^2}{x}} - x e^{\frac{0}{x}}\right) dx$$

$$\int_0^2 \left(x e^{x} - x\right) dx$$

$$= \int_0^2 \underset{I}{x}\, \underset{II}{e^x}\, dx - \int_0^2 \ldots$$

ILATE

$$= \left[x\int e^x dx - \int\left(\frac{dx}{dx}\int e^x dx\right)dx\right]_0^2 - \left[\frac{x^2}{2}\right]_0^2$$

$\therefore \int I \cdot II\, dx = I\int II\, dx - \int\left(\frac{dI}{dx}\int II\, dx\right)dx$

$$= \left[x e^x - \int e^x dx\right]_0^2 - \frac{4}{2} + 0$$

$$= \left[x e^x - e^x\right]_0^2 - 2$$

$$= 2e^2 - e^2 - 0 + e^0 - 2$$

$$= e^2 + 1 - 2$$

$$= e^2 - 1 \quad \text{Ans}$$

Change of order of Integration

$$\int_0^1 \int_{y=x^2}^{y=2-x} xy\, dx\, dy$$

$$\int_0^1 \left(\int_{x^2}^{2-x} xy\, dy\right) dx$$

$$\int_0^1 \left[x \frac{y^2}{2}\right]_{y=x^2}^{2-x} dx$$

$$\int_0^1 \left[x \times \frac{(2-x)^2}{2} - x \times \frac{(x^2)^2}{2}\right] dx$$

$$\int_0^1 \frac{x(4+x^2-4x) - x^5}{2}\, dx$$

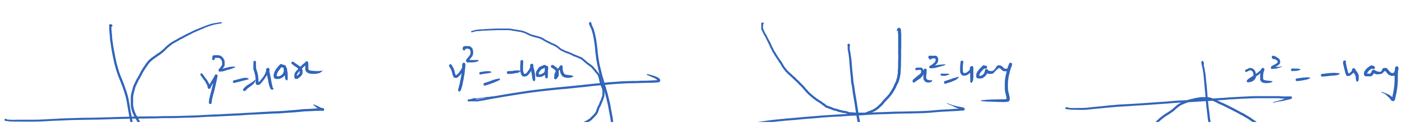
$$\frac{1}{2}\int_0^1 \left(4x + x^3 - 4x^2 - x^5\right) dx$$

$$\frac{1}{2}\left[\frac{4x^2}{2} + \frac{x^4}{4} - \frac{4x^3}{3} - \frac{x^6}{6}\right]_0^1$$

$$\frac{1}{2}\left[\frac{\cancel{4}^2}{\cancel{2}_1} + \frac{1}{4} - \frac{4}{3} - \frac{1}{6}\right]$$

$$\frac{1}{2}\left[\frac{24 + 3 - 16 - 2}{12}\right]$$

$$\frac{1}{2} \times \frac{\cancel{9}^3}{\cancel{12}_4} = \boxed{\frac{3}{8}}$$



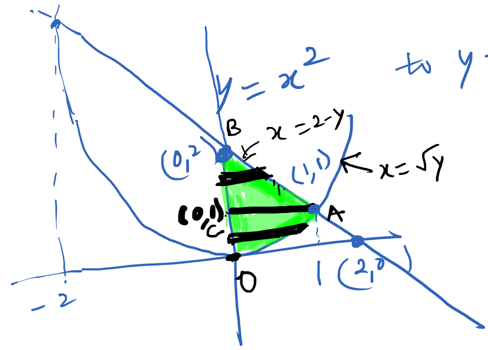

x

<u>change of order</u>

$$\boxed{\int_0^1\int_{x^2}^{2-x} xy\;dx\;dy}$$

$y=x^2$ to $y=2-x$

x	0	2
y	2	0



$x^2=2-x$

$x^2+x-2=0$

$x^2+2x-x-2=0$

$x(x+2)-1(x+2)=0$

$(x+2)(x-1)=0$

$x=-2$ & $x=$

<u>region OAC</u>

$x=0$ to $x=\sqrt{y}$

$y=0$ to $y=1$

<u>region ABC</u>

$x=0$ to $x=2-y$

$y=1$ to $y=2$

$$\text{Area}=\int_0^1\left(\int_0^{\sqrt{y}} xy\;dx\right)dy+\int_1^2\left(\int_0^{2-y} xy\;dx\right)dy$$

$$=\int_0^1\left[y\frac{x^2}{2}\right]_0^{\sqrt{y}}dy+\int_1^2\left[y\frac{x^2}{2}\right]_0^{2-y}dy$$

$$=\int_0^1 y\frac{(\sqrt{y})^2}{2}\,dy+\int_1^2 y\frac{(2-y)^2}{2}\,dy$$

$$=\int_0^1\frac{y^2}{2}\,dy+\int_1^2\frac{y(4+y^2-4y)}{2}\,dy$$

$$=\left[\frac{y^3}{6}\right]_0^1+\frac{1}{2}\int_1^2\left(4y+y^3-4y^2\right)dy$$

$$=\frac{1}{6}+\frac{1}{2}\left[\frac{4y^2}{2}+\frac{y^4}{4}-\frac{4y^3}{3}\right]_1^2$$

$$=\frac{1}{6}+\frac{1}{2}\left[2(2)^2+\frac{2^4}{4}-\frac{4(2)^3}{3}-\frac{4}{2}-\frac{1}{4}+\frac{4}{3}\right]$$

$$=\frac{1}{6}+\frac{1}{2}\left[8+4-\frac{32}{3}-2-\frac{1}{4}+\frac{4}{3}\right]$$

$$=\frac{1}{6}+\frac{1}{2}\left[\frac{10}{1}-\frac{28}{3}-\frac{1}{4}\right]$$

$$=\frac{1}{6}+\frac{5}{1}-\frac{14}{3}-\frac{1}{8}$$

$$=\frac{4+120-112-3}{24}$$

$$=\frac{9}{24}=\boxed{\frac{3}{8}}$$

<u>H.W.</u>

1) $\int_0^{\infty}\int_x^{\infty}\frac{e^{y}}{y}\,dx\,dy$

2) $\int_0^1\int_y^4 e^{x^2}\,dx\,dy$