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Mathematics

Class XI

Detailed Explanation to all
Objective & **Subjective Problems**

A Highly Useful **Question-Solution Bank**
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Abhishek Chauhan

ARIHANT PRAKASHAN
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卐 **ISBN** : 978-93-5176-469-4

卐

Published by Arihant Publications (India) Ltd.

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PREFACE

The Department of Education in Science & Mathematics (DESM) & National Council of Educational Research & Training (NCERT) developed **Exemplar Problems** in Science and Mathematics for Secondary and Senior Secondary Classes with the objective to provide the students a large number of quality problems in various forms and format *viz*. Multiple Choice Questions, Short Answer Questions, Long Answer Questions etc., with varying levels of difficulty.

NCERT Exemplar Problems are very important for both; School & Board Examinations as well as competitive examinations like Engineering Entrances. The questions given in exemplar book are mainly of higher difficulty order by practicing these problems, you will able to manage with the margin between a good score and a very good or an excellent score.

Approx 20% problems asked in any Board Examination or Entrance Examinations are of higher difficulty order, exemplar problems will make you ready to solve these difficult problems.

This book **NCERT Exemplar Problems-Solutions Mathematics XI** contains Explanatory & Accurate Solutions to all the questions given in NCERT Exemplar Mathematics book.

For the overall benefit of the students we have made unique this book in such a way that it presents not only hints and solutions but also detailed and authentic explanations. Through these detailed explanations, students can learn the concepts which will enhance their thinking and learning abilities.

We have introduced some additional features with the solutions which are as follows

- **Thinking Process** Along with the solutions to questions we have given thinking process that tell how to approach to solve a problem. Here, we have tried to cover all the loopholes which may lead to onfusion. All formulae and hints are discussed in detail.
- **Note** We have provided notes also to solutions in which special points are mentioned which are of great value for the students.

For the completion of this book, I would like to thank Priyanshi Garg who helped me at project management level.

With the hope that this book will be of great help to the students,
I wish great success to my readers.

Author

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1

Sets

Short Answer Type Questions

Q.1 Write the following sets in the roaster form.

(i) $A = \{x : x \in R, 2x + 11 = 15\}$

(ii) $B = \{x \mid x^2 = x, x \in R\}$

(iii) $C = \{x \mid x$ is a positive factor of a prime number $p\}$

Thinking Process

Solve the equation and get the value of x.

Sol. **(i)** We have, $A = \{x : x \in R, 2x + 11 = 15\}$

$\therefore \quad 2x + 11 = 15$

$\Rightarrow \quad 2x = 15 - 11 \Rightarrow 2x = 4$

$\Rightarrow \quad x = 2$

$\therefore \quad A = \{2\}$

(ii) We have, $B = \{x \mid x^2 = x, x \in R\}$

$\therefore \quad x^2 = x$

$\Rightarrow \quad x^2 - x = 0 \Rightarrow x(x - 1) = 0$

$\Rightarrow \quad x = 0, 1$

$\therefore \quad B = \{0, 1\}$

(iii) We have, $C = \{x \mid x$ is a positive factor of prime number $p\}$.

Since, positive factors of a prime number are 1 and the number itself.

$\therefore \quad C = \{1, p\}$

Q. 2 Write the following sets in the roaster form.

(i) $D = \{t \mid t^3 = t, t \in R\}$

(ii) $E = \{w \mid \frac{w - 2}{w + 3} = 3, w \in R\}$

(iii) $F = \{x \mid x^4 - 5x^2 + 6 = 0, x \in R\}$

Thinking Process

Solve the given equation and get the value of respective variable.

Sol. **(i)** We have, $D = \{t \mid t^3 = t, t \in R\}$

$\therefore \quad t^3 = t$

$\Rightarrow \quad t^3 - t = 0 \Rightarrow t(t^2 - 1) = 0$

$\Rightarrow \quad t(t-1)(t+1) = 0 \Rightarrow t = 0, 1, -1$

$\therefore \quad D = \{-1, 0, 1\}$

(ii) We have, $E = \{w \mid \frac{w-2}{w+3} = 3, w \in R\}$

$\therefore \quad \frac{w-2}{w+3} = 3$

$\Rightarrow \quad w - 2 = 3w + 9 \Rightarrow w - 3w = 9 + 2$

$\Rightarrow \quad -2w = 11 \Rightarrow w = \frac{-11}{2}$

$\therefore \quad E = \left\{\frac{-11}{2}\right\}$

(iii) We have, $F = \{x \mid x^4 - 5x^2 + 6 = 0, x \in R\}$

$\therefore \quad x^4 - 5x^2 + 6 = 0$

$\Rightarrow \quad x^4 - 3x^2 - 2x^2 + 6 = 0$

$\Rightarrow \quad x^2(x^2 - 3) - 2(x^2 - 3) = 0$

$\Rightarrow \quad (x^2 - 3)(x^2 - 2) = 0$

$\Rightarrow \quad x = \pm\sqrt{3}, \pm\sqrt{2}$

$\therefore \quad F = \{-\sqrt{3}, -\sqrt{2}, \sqrt{2}, \sqrt{3}\}$

Note *In roaster form, the order in which elements are listed is immaterial. Thus, we can also write* $F = \{-\sqrt{3}, \sqrt{2}, -\sqrt{2}, \sqrt{3}\}$.

Q. 3 If $Y = \{x \mid x$ is a positive factor of the number $2^{p-1}(2^p - 1)$, where $2^p - 1$ is a prime number$\}$. Write Y in the roaster form.

Thinking Process

First, write all the factors of 2^{p-1}, *where* $p = 1, 2, 3, \ldots, p$ *and then get y.*

Sol. $Y = \{x \mid x$ is a positive factor of the number $2^{p-1}(2^p - 1)$, where $2^p - 1$ is a prime number$\}$.

So, the factor of 2^{p-1} are $1, 2, 2^2, 2^3, \ldots, 2^{p-1}$.

$\therefore \quad Y = \{1, 2, 2^2, 2^3, \ldots, 2^{p-1}, 2^p - 1\}$

Q. 4 State which of the following statements are true and which are false. Justify your answer.

(i) $35 \in \{x \mid x$ has exactly four positive factors$\}$.

(ii) $128 \in \{y \mid$ the sum of all the positive factors of y is $2y\}$.

(iii) $3 \notin \{x \mid x^4 - 5x^3 + 2x^2 - 112x + 6 = 0\}$.

(iv) $496 \notin \{y \mid$ the sum of all the positive factors of y is $2y\}$.

Sol. **(i)** Since, the factors of 35 are 1, 5, 7 and 35. So, statement (i) is true.

(ii) Since, the factors of 128 are 1, 2, 4, 8, 16, 32, 64 and 128.

$\therefore$ Sum of factors $= 1 + 2 + 4 + 8 + 16 + 32 + 64 + 128$

$= 255 \neq 2 \times 128$

So, statement (ii) is false.

(iii) We have, $x^4 - 5x^3 + 2x^2 - 112x + 6 = 0$

$\therefore$ For $x = 3$,

$$(3)^4 - 5(3)^3 + 2(3)^2 - 112(3) + 6 = 0$$

$\Rightarrow$ $81 - 135 + 18 - 336 + 6 = 0$

$\Rightarrow$ $-346 = 0$

which is not true.

Hence, statement (iii) is true.

(iv) $\because$ $496 = 2^4 \times 31$

So, the factors of 496 are 1, 2, 4, 8, 16, 31, 62, 124, 248 and 496.

$\therefore$ Sum of factors $= 1 + 2 + 4 + 8 + 16 + 31 + 62 + 124 + 248 + 496$

$= 992 = 2\,(496)$

So, $496 \in \{y \mid$ the sum of all the positive factor of y is $2y\}$.

Hence, statement (iv) is false.

Q. 5 If $L = \{1, 2, 3, 4\}$, $M = \{3, 4, 5, 6\}$ and $N = \{1, 3, 5\}$, then verify that $L - (M \cup N) = (L - M) \cap (L - N)$.

Sol. Given, $L = \{1, 2, 3, 4\}$, $M = \{3, 4, 5, 6\}$ and $N = \{1, 3, 5\}$

$\therefore$ $M \cup N = \{1, 3, 4, 5, 6\}$

$L - (M \cup N) = \{2\}$

Now, $L - M = \{1, 2\}$, $L - N = \{2, 4\}$

$\therefore$ $(L - M) \cap (L - N) = \{2\}$

Hence, $L - (M \cup N) = (L - M) \cap (L - N)$.

Q. 6 If A and B are subsets of the universal set U, then show that

(i) $A \subset A \cup B$ (ii) $A \subset B \Leftrightarrow A \cup B = B$

(iii) $(A \cap B) \subset A$

Sol. **(i)** Let $x \in A$

$\Rightarrow$ $x \in A$ or $x \in B \Rightarrow x \in A \cup B$

Hence, $\subset A \cup B$

(ii) If $A \subset B$

Let $x \in A \cup B$

$\Rightarrow$ $x \in A$ or $x \in B \Rightarrow x \in B$ $[\because A \subset B]$

$\Rightarrow$ $A \cup B \subset B$ ... (i)

But $B \subset A \cup B$ ... (ii)

From Eqs. (i) and (ii),

$A \cup B = B$

If $A \cup B = B$

Let $y \in A$

$\Rightarrow$ $y \in A \cup B \Rightarrow y \in B$ $[\because A \cup B = B]$

$\Rightarrow$ $A \subset B$

Hence, $A \subset B \Leftrightarrow A \cup B = B$

(iii) Let $x \in A \cap B$

$\Rightarrow$ $x \in A$ and $x \in B \Rightarrow x \in A$

Hence, $A \cap B \subset A$

Q. 7 Given that $N = \{1, 2, 3, \ldots, 100\}$. Then, write

(i) the subset of N whose elements are even numbers.

(ii) the subset of N whose elements are perfect square numbers.

Sol. We have, $N = \{1, 2, 3, 4, \ldots, 100\}$

(i) Required subset $= \{2, 4, 6, 8, \ldots, 100\}$

(ii) Required subset $= \{1, 4, 9, 16, 25, 36, 49, 64, 81, 100\}$

Q. 8 If $X = \{1, 2, 3\}$, if n represents any member of X, write the following sets containing all numbers represented by

(i) $4n$ (ii) $n + 6$ (iii) $\frac{n}{2}$ (iv) $n - 1$

Sol. Given, $X = \{1, 2, 3\}$

(i) $\{4n \mid n \in X\} = \{4, 8, 12\}$

(ii) $\{n + 6 \mid n \in X\} = \{7, 8, 9\}$

(iii) $\left\{\frac{n}{2} \mid n \in X\right\} = \left\{\frac{1}{2}, 1, \frac{3}{2}\right\}$

(iv) $\{n - 1 \mid n \in X\} = \{0, 1, 2\}$

Q. 9 If $Y = \{1, 2, 3, \ldots, 10\}$ and a represents any element of Y, write the following sets, containing all the elements satisfying the given conditions.

(i) $a \in Y$ but $a^2 \notin Y$

(ii) $a + 1 = 6, a \in Y$

(iii) a is less than 6 and $a \in Y$

Sol. Given, $Y = \{1, 2, 3, \ldots, 10\}$

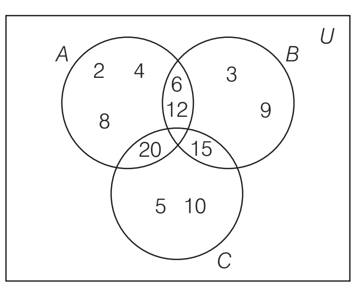
(i) $\{a : a \in Y \text{ and } a^2 \notin Y\} = \{4, 5, 6, 7, 8, 9, 10\}$

(ii) $\{a : a + 1 = 6, a \in Y\} = \{5\}$

(iii) is less than 6 and $a \in Y\} = \{1, 2, 3, 4, 5,\}$

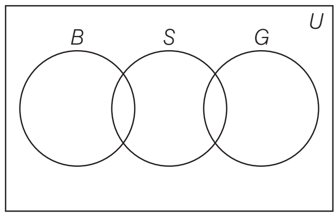
Q. 10 A, B and C are subsets of universal set U. If $A = \{2, 4, 6, 8, 12, 20\}$, $B = \{3, 6, 9, 12, 15\}$, $C = \{5, 10, 15, 20\}$ and U is the set of all whole numbers, draw a Venn diagram showing the relation of U, A, B and C.

Sol.



Q. 11 Let U be the set of all boys and girls in a school, G be the set of all girls in the school, B be the set of all boys in the school and S be the set of all students in the school who take swimming. Some but not all, students in the school take swimming. Draw a Venn diagram showing one of the possible interrelationship among sets U, G, B and S.

Sol.



Q. 12 For all sets A, B and C, show that $(A - B) \cap (A - C) = A - (B \cup C)$.

Thinking Process

To prove this we have to show that $(A-B) \cap (A-C) \subseteq A-(B \cup C)$ and $A-(B \cup C) \subseteq (A-B) \cap (A-C)$.

Sol. Let $x \in (A - B) \cap (A - C)$

$\Rightarrow x \in (A - B)$ and $x \in (A - C)$

$\Rightarrow (x \in A$ and $x \notin B)$ and $(x \in A$ and $x \notin C)$

$\Rightarrow x \in A$ and $(x \notin B$ and $x \notin C)$

$\Rightarrow x \in A$ and $x \notin (B \cup C)$

$\Rightarrow x \in A - (B \cup C)$

$\Rightarrow (A - B) \cap (A - C) \subset A - (B \cup C)$ ... (i)

Now, let $y \in A - (B \cup C)$

$\Rightarrow y \in A$ and $y \notin (B \cup C)$

$\Rightarrow y \in A$ and $(y \notin B$ and $y \notin C)$

$\Rightarrow (y \in A$ and $y \notin B)$ and $(y \in A$ and $y \notin C)$

$\Rightarrow y \in (A - B)$ and $y \in (A - C)$

$\Rightarrow y \in (A - B) \cap (A - C)$

$\Rightarrow A - (B \cup C) \subset (A - B) \cap (A - C)$ ...(ii)

From Eqs. (i) and (ii),

$$A - (B \cup C) = (A - B) \cap (A - C)$$

Q. 13 For all sets A and B, $(A - B) \cup (A \cap B) = A$.

Thinking Process

To solve the above problem, use distributive law on sets
i.e., $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Sol. LHS $= (A - B) \cup (A \cap B)$

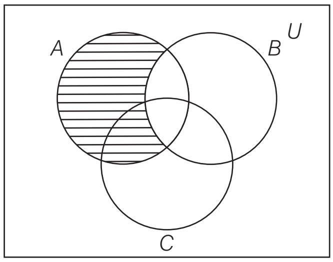
$= [(A - B) \cup A] \cap [(A - B) \cup B]$

$= A \cap (A \cup B) = A =$ RHS

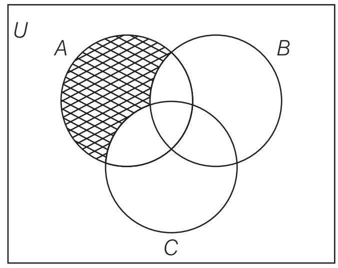
Hence, given statement is true.

Q. 14 For all sets A, B and C, $A-(B-C)=(A-B)-C$.

Sol. See the Venn diagrams given below, where shaded portions are representing $A-(B-C)$ and $(A-B)-C$ respectively.



$A-(B-C)$



$(A-B)-C$

Clearly, $A-(B-C)\neq(A-B)-C$.

Hence, given statement is false.

Q. 15 For all sets A, B and C, if $A\subset B$, then $A\cap C\subset B\cap C$.

Sol. Let $x\in A\cap C$

$\Rightarrow$ $x\in A$ and $x\in C$

$\Rightarrow$ $x\in B$ and $x\in C$ $[\because A\subset B]$

$\Rightarrow$ $x\in(B\cap C)$ $\Rightarrow$ $(A\cap C)\subset(B\cap C)$

Hence, given statement is true.

Q. 16 For all sets A, B and C, if $A\subset B$, then $A\cup C\subset B\cup C$.

Sol. Let $x\in A\cup C$

$\Rightarrow$ $x\in A$ and $x\in C$

$\Rightarrow$ $x\in B$ and $x\in C$ $[\because A\subset B]$

$\Rightarrow$ $x\in B\cup C$ $\Rightarrow$ $A\cup C\subset B\cup C$

Hence, given statement is true.

Q. 17 For all sets A, B and C, if $A\subset C$ and $B\subset C$, then $A\cup B\subset C$.

Sol. Let $x\in A\cup B$

$\Rightarrow$ $x\in A$ and $x\in B$

$\Rightarrow$ $x\in C$ and $x\in C$ $[\because A\subset C$ and $B\subset C]$

$\Rightarrow$ $x\in C$ $\Rightarrow$ $A\cup B\subset C$

Hence, given statement is true.

Q. 18 For all sets A and B, $A\cup(B-A)=A\cup B$.

Thinking Process

To solve the above problem, use distributive law i.e., $A\cup(B\cap C)=(A\cup B)\cap(A\cup C)$.

Sol. $\because$ LHS $=A\cup(B-A)=A\cup(B\cap A')$ $[\because A-B=A\cap B']$

$=(A\cup B)\cap(A\cup A')=(A\cup B)\cap U$ $[\because A\cup A'=U]$

$=A\cup B=$ RHS $[\because A\cap U=A]$

Q. 19 For all sets A and B, $A-(A-B)=A\cap B$.

Sol.
$$\begin{aligned}\text{LHS} &= A-(A-B)=A-(A\cap B') && [\because A-B=A\cap B']\\ &= A\cap(A\cap B')'=A\cap[A'\cup(B')'] && [\because (A\cap B)'=A'\cup B']\\ &= A\cap(A'\cup B) && [\because (A')'=A]\\ &= (A\cap A')\cup(A\cap B)=\phi\cup(A\cap B)\\ &= A\cap B=\text{RHS}\end{aligned}$$

Q. 20 For all sets A and B, $A-(A\cap B)=A-B$.

Sol.
$$\begin{aligned}\text{LHS} &= A-(A\cap B)=A\cap(A\cap B)' && [\because A-B=A\cap B']\\ &= A\cap(A'\cup B') && [\because (A\cap B)'=A'\cup B']\\ &= (A\cap A')\cup(A\cap B')=\phi\cup(A\cap B')\\ &= A\cap B' && [\because \phi\cup A=A]\\ &= A-B=\text{RHS}\end{aligned}$$

Q. 21 For all sets A and B, $(A\cup B)-B=A-B$.

Sol.
$$\begin{aligned}\text{LHS} &= (A\cup B)-B=(A\cup B)\cap B' && [\because A-B=A\cap B']\\ &= (A\cap B')\cup(B\cap B')=(A\cap B')\cup\phi && [\because B\cap B'=\phi]\\ &= A\cap B' && [\because A\cup\phi=A]\\ &= A-B=\text{RHS}\end{aligned}$$

Q. 22 Let $T=\left\{x \mid \dfrac{x+5}{x-7}-5=\dfrac{4x-40}{13-x}\right\}$. Is T an empty set? Justify your answer.

Thinking Process

First of all solve the given equation and get the value of x.

Sol. Since,
$$T=\left\{x \mid \frac{x+5}{x-7}-5=\frac{4x-40}{13-x}\right\}$$

$$\begin{aligned}\because\quad & \frac{x+5}{x-7}-5=\frac{4x-40}{13-x}\\ \Rightarrow\quad & \frac{x+5-5(x-7)}{x-7}=\frac{4x-40}{13-x}\\ \Rightarrow\quad & \frac{x+5-5x+35}{x-7}=\frac{4x-40}{13-x}\\ \Rightarrow\quad & \frac{-4x+40}{x-7}=\frac{4x-40}{13-x}\\ \Rightarrow\quad & -(4x-40)(13-x)=(4x-40)(x-7)\\ \Rightarrow\quad & (4x-40)(x-7)+(4x-40)(13-x)=0\\ \Rightarrow\quad & (4x-40)(x-7+13-x)=0\\ \Rightarrow\quad & 4(x-10)6=0\\ \Rightarrow\quad & 24(x-10)=0\\ \Rightarrow\quad & x=10\\ \therefore\quad & T=\{10\}\end{aligned}$$

Hence, T is not an empty set.

Long Answer Type Questions

Q. 23 If A, B and C be sets. Then, show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Sol. Let $x \in A \cap (B \cup C)$

$\Rightarrow$ $x \in A$ and $x \in (B \cup C)$

$\Rightarrow$ $x \in A$ and ($x \in B$ or $x \in C$)

$\Rightarrow$ ($x \in A$ and $x \in B$) or ($x \in A$ and $x \in C$)

$\Rightarrow$ $x \in A \cap B$ or $x \in A \cap C$

$x \in (A \cap B) \cup (A \cap C)$

$\Rightarrow$ $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$ ...(i)

Again, let $y \in (A \cap B) \cup (A \cap C)$

$\Rightarrow$ $y \in (A \cap B)$ or $y \in (A \cap C)$

$\Rightarrow$ ($y \in A$ and $y \in B$) or ($y \in A$ and $y \in C$)

$\Rightarrow$ $y \in A$ and ($y \in B$ or $y \in C$)

$\Rightarrow$ $y \in A$ and $y \in B \cup C$

$\Rightarrow$ $y \in A \cap (B \cup C)$

$\Rightarrow$ $(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$ ...(ii)

From Eqs.(i) and(ii),

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Q. 24 Out of 100 students; 15 passed in English, 12 passed in Mathematics, 8 in Science, 6 in English and Mathematics, 7 in Mathematics and Science, 4 in English and Science, 4 in all the three. Find how many passed

(i) in English and Mathematics but not in Science.

(ii) in Mathematics and Science but not in English.

(iii) in Mathematics only.

(iv) in more than one subject only.

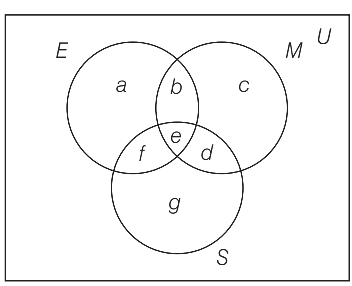
Sol. Let M be the set of students who passed in Mathematics, E be the set of students who passed in English and S be the set of students who passed in Science.

Then, $n(u) = 100$,

$n(E) = 15$, $n(M) = 12$, $n(S) = 8$, $n(E \cap M) = 6$, $n(M \cap S) = 7$,

$n(E \cap S) = 4$, and $n(E \cap M \cap S) = 4$,

$\because$ $n(E) = 15$



$\Rightarrow$ $a + b + e + f = 15$ ...(i)

and $n(M) = 12$

$\Rightarrow$ $b + c + e + d = 12$ ...(ii)

Also, $n(S) = 8$

$\Rightarrow \quad d + e + f + g = 8$ ...(iii)

$n(E \cap M) = 6$

$\Rightarrow \quad b + e = 6$ ...(iv)

$n(M \cap S) = 7$

$\Rightarrow \quad e + d = 7$ ... (v)

$n(E \cap S) = 4$

$\Rightarrow \quad e + f = 4$ ... (vi)

$n(E \cap M \cap S) = 4$

$\Rightarrow \quad e = 4$ ... (vii)

From Eqs. (vi) and (vii), $f = 0$

From Eqs. (v) and (vii), $d = 3$

From Eqs. (iv) and (vii), $b = 2$

On substituting the values of d, e and f in Eq. (iii), we get

$$3 + 4 + 0 + g = 8$$

$$\Rightarrow \quad g = 1$$

On substituting the value of b, e and d in Eq. (ii), we get

$$2 + c + 4 + 3 = 12$$

$$\Rightarrow \quad c = 3$$

On substituting b, e, and f in Eq. (i), we get

$$a + 2 + 4 + 0 = 15$$

$$\Rightarrow \quad a = 9$$

(i) Number of students who passed in English and Mathematics but not in Science

$$= b = 2$$

(ii) Number of students who passed in Mathematics and Science but not in English

$$= d = 3$$

(iii) Number of students who passed in Mathematics only $= c = 3$

(iv) Number of students who passed in more than one subject

$$= b + e + d + f$$

$$= 2 + 4 + 3 + 0 = 9$$

Alternate Method

Let E denotes the set of student who passed in English. M denotes the set of students who passed in Mathematics. S denotes the set of students who passed in Science.

Now, $n(U) = 100,\ n(E) = 15,\ n(m) = 12,\ n(S) = 8,$

$n(E \cap M) = 6,\ n(M \cap S) = 7,$

$n(E \cap S) = 4,\ n(E \cap M \cap S) = 4$

(i) Number of students passed in English and Mathematics but not in Science

i.e., $n(E \cap M \cap S') = n(E \cap M) - n(E \cap M \cap S)$ $[\because A \cap B' = A - (A \cap B)]$

$$= 6 - 4 = 2$$

(ii) Number of students passed in Mathematics and Science but not in English.

i.e., $n(M \cap S \cap E') = n(M \cap S) - n(M \cap S \cap E)$

$$= 7 - 4 = 3$$

(iii) Number of students passed in mathematics only

i.e., $n(M \cap S' \cap E') = n(M) - n(M \cap S) - n(M \cap E) + n(M \cap S \cap E)$

$$= 12 - 7 - 6 + 4 = 3$$

(iv) Number of students passed in more than one subject only

i.e., $n(E \cap M) + n(M \cap S) + n(E \cap S) - 3\,n(E \cap M \cap S) + n(E \cap M \cap S)$

$$= 6 + 7 + 4 - 4 \times 3 + 4$$

$$= 17 - 12 + 4 = 5 + 4 = 9$$

Q. 25 In a class of 60 students, 25 students play cricket and 20 students play tennis and 10 students play both the games. Find the number of students who play neither.

Sol. Let C be the set of students who play cricket and T be the set of students who play tennis.

Then, $n(U) = 60,\ n(C) = 25,\ n(T) = 20,$ and $n(C \cap T) = 10$

$\therefore$ $n(C \cup T) = n(C) + n(T) - n(C \cap T)$

$= 25 + 20 - 10 = 35$

$\therefore$ Number of students who play neither $= n(U) - n(C \cup T)$

$= 60 - 35 = 25$

Q. 26 In a survey of 200 students of a school, it was found that 120 study Mathematics, 90 study Physics and 70 study Chemistry, 40 study Mathematics and Physics, 30 study Physics and Chemistry, 50 study Chemistry and Mathematics and 20 none of these subjects. Find the number of students who study all the three subjects.

Thinking Process

To solve this problem, use the formula for all the three subjects
$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C).$

Sol. Let M be the set of students who study Mathematics, P be the set of students who study Physics and C be the set of students who study Chemistry.

Then, $n(U) = 200,\ n(M) = 120,\ n(P) = 90,$

$n(C) = 70,\ n(M \cap P) = 40,\ n(P \cap C) = 30,$

$n(C \cap M) = 50,\ n(M' \cap P' \cap C') = 20,$

$n(U) - n(M \cup P \cup C) = 20,$

$n(M \cup P \cup C) = 200 - 20 = 180$

$\because$ $n(M \cup P \cup C) = n(M) + n(P) + n(C) - n(C \cap M) + n(M \cap P \cap C)$

$\Rightarrow$ $180 = 120 + 90 + 70 - 40 - 30 - 50 + n(M \cap P \cap C)$

$\Rightarrow$ $180 = 160 + n(M \cap P \cap C)$

$\Rightarrow$ $n(M \cap P \cap C) = 180 - 160 = 20$

So, the number of students who study all the three subjects is 20.

Q. 27 In a town of 10000 families, it was found that 40% families buy newspaper A, 20% families buy newspaper B, 10% families buy newspaper C, 5% families buy A and B, 3% buy B and C and 4% buy A and C. If 2% families buy all the three newspaper. Find

(i) the number of families which buy newspaper A only.

(ii) the number of families which buy none of A, B and C.

Sol. Let A be the set of families which buy newspaper A, B be the set of families which buy newspaper B and C be the set of families which buy newspaper C.

Then, $n(U) = 10000,\ n(A) = 40\%\ n(B) = 20\%$ and $n(C) = 10\%$

$n(A \cap B) = 5\%,$

$n(B \cap C) = 3\%$

$n(A \cap C) = 4\%$

$n(A \cap B \cap C) = 2\%$

(i) Number of families which buy newspaper A only

$$= n(A) - n(A \cap B) - n(A \cap C) + n(A \cap B \cap C)$$
$$= (40 - 5 - 4 + 2)\% = 33\%$$

$$10000 \times 33/100 = 3300$$

(ii) Number of families which buy none of A, B and C

$$= n(U) - n(A \cup B \cup C)$$
$$= n(U) - [n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)]$$
$$= 100 - [40 + 20 + 10 - 5 - 3 - 4 + 2]$$
$$= 100 - 60\% = 40\%$$
$$= 10000 \times \frac{40}{100} = 4000$$

Q. 28 In a group of 50 students, the number of students studying French, English, Sanskrit were found to be as follows French = 17, English = 13, Sanskrit = 15 French and English = 09, English and Sanskrit = 4, French and Sanskrit = 5, English, French and Sanskrit = 3. Find the number of students who study

(i) only French.
(ii) only English.
(iii) only Sanskrit.
(iv) English and Sanskrit but not French.
(v) French and Sanskrit but not English.
(vi) French and English but not Sanskrit.
(vii) atleast one of the three languages.
(viii) none of the three languages.

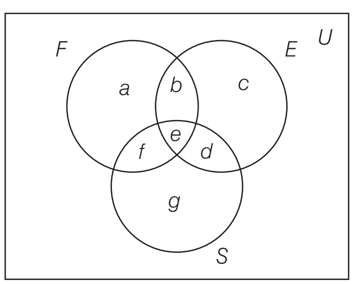
Sol. Let F be the set of students who study French, E be the set of students who study English and S be the set of students who study Sanskrit.

Then, $n(U) = 50$, $n(F) = 17$, $n(E) = 13$, and $n(S) = 15$,

$n(F \cap E) = 9$, $n(E \cap S) = 4$, $n(F \cap S) = 5$,

$n(F \cap E \cap S) = 3$,

$\because \quad n(F) = 17$



$\Rightarrow \quad a + b + e + f = 17 \quad$ ...(i)

$n(E) = 13$

$\Rightarrow \quad b + c + d + e = 13 \quad$ ... (ii)

$n(S) = 15$

$\Rightarrow \quad d + e + f + g = 15 \quad$ ... (iii)

$n(F \cap E) = 9$

$\Rightarrow \quad b + e = 9 \quad$ ... (iv)

$n(E \cap S) = 4$

$\Rightarrow \quad e + d = 4$ ... (v)

$n(F \cap S) = 5$

$\Rightarrow \quad f + e = 5$ ... (vi)

$n(F \cap E \cap S) = 3$

$\Rightarrow \quad e = 3$ ... (vii)

From Eqs. (vi) and (vii), $f = 2$

From Eqs. (v)and (vii), $d = 1$

From Eqs. (iv) and (vii), $b = 6$

On substituting the values of e, f and d in Eq. (iii), we get

$$1 + 3 + 2 + g = 15$$

$$\Rightarrow \quad g = 9$$

On substituting the values of b, d and e in Eq. (ii), we get

$$6 + c + 1 + 3 = 13$$

$$\Rightarrow \quad c = 3$$

On substituting the values of b, e and f in Eq. (i), we get

$$a + 6 + 3 + 2 = 17$$

$$\Rightarrow \quad a = 6$$

(i) Number of students who study French only, $a = 6$

(ii) Number of students who study English only, $c = 3$

(iii) Number of students who study Sanskrit only, $g = 9$

(iv) Number of students who study English and Sanskrit but not French, $d = 1$

(v) Number of students who study French and Sanskrit but not English, $f = 2$

(vi) Number of students who study French and English but not Sanskrit, $b = 6$

(vii) Number of students who study atleast one of the three languages

$$= a + b + c + d + e + f + g$$
$$= 6 + 6 + 3 + 1 + 3 + 2 + 9 = 30$$

(viii) Number of students who study none of three languages = Total students − Students who study atleast one of the three languages

$$= 50 - 30 = 20$$

Objective Type Questions

Q. 29 Suppose, $A_1, A_2, \ldots, A_{30}$ are thirty sets each having 5 elements and B_1, B_2, B_n are n sets each with 3 elements, let $\bigcup_{i=1}^{30} A_i = \bigcup_{j=1}^{n} B_j = S$ and each element of S belongs to exactly 10 of the A_i's and exactly 9 of the B_j's. Then, n is equal to

(a) 15 (b) 3

(c) 45 (d) 35

Thinking Process

First find the total number of elements for the both sets, then compare them.

Sol. *(c)* If elements are not repeated, then number of elements in $A_1 \cup A_2 \cup A_3, \ldots \cup A_{30}$ is 30×5.

But each element is used 10 times, so

$$S = \frac{30 \times 5}{10} = 15$$

If elements in $B_1, B_2, \ldots, B_n$ are not repeated, then total number of elements is $3n$ but each element is repeated 9 times, so

$$S = \frac{3n}{9} \Rightarrow 15 = \frac{3n}{9}$$

$$\therefore \quad n = 45$$

Q. 30 Two finite sets have m and n elements. The number of subsets of the first set is 112 more than that of the second set. The values of m and n are, respectively

(a) 4, 7 (b) 7, 4 (c) 4, 4 (d) 7, 7

Thinking Process

We know that, if a set A contains n elements, then the number of subsets of A is equal to 2^n.

Sol. *(a)* Since, number of subsets of a set containing m elements is 112 more than the subsets of the set containing n elements.

$$\because \quad 2^m - 2^n = 112$$

$$\Rightarrow \quad 2^n \cdot (2^{m-n} - 1) = 2^4 \cdot 7$$

$$\Rightarrow \quad 2^n = 2^4 \text{ and } 2^{m-n} - 1 = 7$$

$$\Rightarrow \quad n = 4 \text{ and } 2^{m-n} = 8$$

$$\Rightarrow \quad 2^{m-n} = 2^3 \Rightarrow m - n = 3$$

$$\Rightarrow \quad m - 4 = 3 \Rightarrow m = 4 + 3$$

$$\therefore \quad m = 7$$

Q. 31 The set $(A \cap B')' \cup (B \cap C)$ is equal to

(a) $A' \cup B \cup C$ (b) $A' \cup B$ (c) $A' \cup C'$ (d) $A' \cap B$

Sol. *(b)* We know that, $(A \cap B)' = (A' \cup B')$ and $(A')' = A$

$$\therefore \quad = (A \cap B')' \cup (B \cap C)$$

$$= [A' \cup (B')'] \cup (B \cap C)$$

$$= (A' \cup B) \cup (B \cap C) = A' \cup B$$

Q. 32 Let F_1 be the set of parallelograms, F_2 the set of rectangles, F_3 the set of rhombuses, F_4 the set of squares and F_5 the set of trapeziums in a plane. Then, F_1 may be equal to

(a) $F_2 \cap F_3$ (b) $F_3 \cap F_4$

(c) $F_2 \cup F_5$ (d) $F_2 \cup F_3 \cup F_4 \cup F_1$

Sol. *(d)* Every rectangle, rhombus, square in a plane is a parallelogram but every trapezium is not a parallelogram.

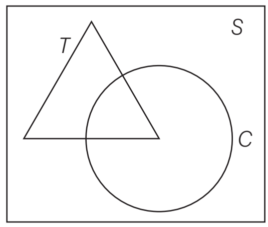
So, F_1 is either of F_1, F_2, F_3 and F_4.

$$\therefore \quad F_1 = F_2 \cup F_3 \cup F_4 \cup F_1$$

Q. 33 Let S = set of points inside the square, T = set of points inside the triangle and C = set of points inside the circle. If the triangle and circle intersect each other and are contained in a square. Then,

(a) $S \cap T \cap C = \phi$ (b) $S \cup T \cup C = C$
(c) $S \cup T \cup C = S$ (d) $S \cup T = S \cap C$

Sol. ***(c)*** The given sets can be represented in Venn diagram as shown below



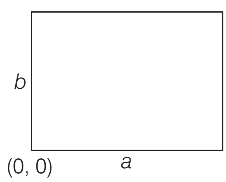
It is clear from the diagram that, $S \cup T \cup C = S$.

Q. 34 If R be the set of points inside a rectangle of sides a and b ($a, b > 1$) with two sides along the positive direction of X-axis and Y-axis. Then,

(a) $R = \{(x, y) : 0 \le x \le a, 0 \le y \le b\}$
(b) $R = \{(x, y) : 0 \le x < a, 0 \le y \le b\}$
(c) $R = \{(x, y) : 0 \le x \le a, 0 < y < b\}$
(d) $R = \{(x, y) : 0 < x < a, 0 < y < b\}$

Sol. ***(d)*** Since, R be the set of points inside the rectangle.

$\therefore \quad R = \{(x, y) : 0 < x < a$ and $0 < y < b\}$



Q. 35 In a town of 840 persons, 450 persons read Hindi, 300 read English and 200 read both. Then, the number of persons who read neither, is

(a) 210 (b) 290 (c) 180 (d) 260

Sol. ***(b)*** Let H be the set of persons who read Hindi and E be the set of persons who read English.

Then, $n(U) = 840$, $n(H) = 450$, $n(E) = 300$, $n(H \cap E) = 200$

Number of persons who read neither $= n(H' \cap F')$

$= n(H \cup E)'$

$= n(U) - n(H \cup E)$

$= 840 - [n(H) + n(E) - n(H \cap E)]$

$= 840 - (450 + 300 - 200)$

$= 840 - 550 = 290$

Q. 36 If $X = \{8^n - 7n - 1 \mid n \in N\}$ and $y = \{49n - 49 \mid n \in N\}$. Then,

(a) $X \subset Y$ (b) $Y \subset X$ (c) $X = Y$ (d) $X \cap Y = \phi$

Thinking Process

If every element of A is an elements of B, then $A \subseteq B$.

Sol. *(a)* $X = \{8^n - 7n - 1 \mid n \in N\} = \{0, 49, 490, \ldots\}$

$Y = \{49n - 49 \mid n \in N\} = \{0, 49, 98, 147, \ldots\}$

Clearly, every elements of X is in Y but every element of Y is not in X.

$\therefore \quad X \subset Y$

Q. 37 A survey shows that 63% of the people watch a news channel whereas 76%watch another channel. If x% of the people watch both channel, then

(a) $x = 35$ (b) $x = 63$ (c) $39 \le x \le 63$ (d) $x = 39$

Sol. *(c)* Let A be the set of percentage of those people who watch a news channel and B be the set of percentage of those people who watch another channel.

$n(A) = 63, \; n(B) = 76, \text{ and } n(A \cap B) = x$

$\because \quad n(A \cup B) \le 100$

$\Rightarrow \quad n(A) + n(B) - n(A \cap B) \le 100$

$\Rightarrow \quad 63 + 76 - x \le 100 \quad \Rightarrow \quad 139 - x \le 100$

$\Rightarrow \quad 139 - 100 \le x \quad \Rightarrow \quad 39 \le x$

$\because \quad n(A) = 63$

$\Rightarrow \quad x(A \cap B) \le n(A) \quad \Rightarrow \quad x \le 63$

$\therefore \quad 39 \le x \le 63$

Q. 38 If sets A and B are defined as

$$A = \{(x, y) \mid y = \frac{1}{x}, 0 \ne x \in R\}, B = \{(x, y) \mid y = -x, x \in R,\}. \text{ Then,}$$

(a) $A \cap B = A$ (b) $A \cap B = B$ (c) $A \cap B = \phi$ (d) $A \cup B = A$

Sol. *(c)* Let $x \in R$

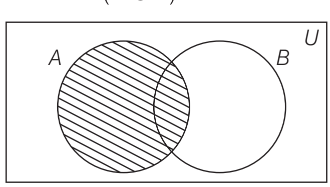
We know that, $\quad -x \ne \frac{1}{x}$

$\therefore \quad A \cap B = \phi$

Q. 39 If A and B are two sets, then $A \cap (A \cup B)$ equals to

(a) A (b) B (c) ϕ (d) $A \cap B$

Sol. *(a)* $\because \quad A \cap (A \cup B) = A$



$A \cap (A \cup B)$

Q. 40 If $A = \{1, 3, 5, 7, 9, 11, 13, 15, 17\}$, $B = \{2, 4, \ldots, 18\}$ and N the set of natural numbers is the universal set, then $(A' \cup (A \cup B) \cap B')$ is

(a) ϕ (b) N (c) A (d) B

Thinking Process

To solve this problem, use the distributive law i.e., $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Sol. ***(b)***

$$A' \cup [(A \cup B) \cap B] \quad [\because A \cap (B \cup C) = (A \cap B) \cup (A \cap C)]$$
$$= A' \cup [(A \cap B') \cup (B \cap B')]$$
$$= A' \cup [(A \cap B') \cup \phi] = A' \cup (A \cap B')$$
$$= (A' \cup A) \cap (A' \cup B')$$
$$= A' \cup B' = N \cap (A' \cup B')$$
$$= A' \cup B' = (A \cap B)' \quad [\because A \cap B = \phi]$$
$$= \phi = N$$

Q. 41 If $S = \{x \mid x$ is a positive multiple of 3 less than 100$\}$ and $P = \{x \mid x$ is a prime number less than 20$\}$. Then, $n(S) + n(P)$ is equal to

(a) 34 (b) 31 (c) 33 (d) 41

Sol. ***(d)*** $\because$ $S = \{x \mid x$ is a positive multiple of 3 less than 100$\}$

$\therefore$ $n(S) = 33$

and $P = \{x \mid x$ is a prime number less than 20$\}$

$\therefore$ $n(P) = 8$

$$n(S) + n(P) = 33 + 8 = 41$$

Q. 42 If X and Y are two sets and X' denotes the complement of X, then $X \cap (X \cup Y)'$ is equal to

(a) X (b) Y (c) ϕ (d) $X \cap Y$

Sol. ***(c)***

$$X \cap (X \cup Y)' = X \cap (X' \cap Y') \quad [\because (A \cup B)' = A' \cap B']$$
$$= (X \cap X') \cap (X \cap Y')$$
$$= \phi \cap (X \cap Y') = \phi \quad [\because \phi \cap A = \phi]$$

Fillers

Q. 43 The set $\{x \in R : 1 \le x < 2\}$ can be written as ....

Sol. The set $\{x \in R : 1 \le x < 2\}$ can be written as $(1, 2)$.

Q. 44 When $A = \phi$, then number of elements in $P(A)$ is ....

Sol. $\therefore$ $A = \phi \Rightarrow n(A) = 0$

$$n\{P(A)\} = 2^{n(A)} = 2^0 = 1$$

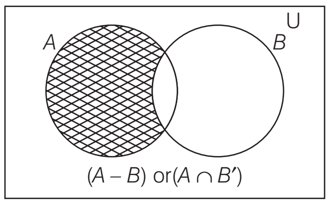
So, number of element in *P(A)* is 1.

Q. 45 If A and B are finite sets ,such that $A \subset B$, then $n(A \cup B)$ is equal to ....

Sol. If A and B are two finite sets such that $A \subset B$, then $n(A \cup B) = n(B)$.

Q. 46 If A and B are any two sets, then $A - B$ is equal to ....

Sol. If A and B are any two sets, then $A - B = A \cap B'$



$\therefore \quad A - B = A \cap B'$

Q. 47 Power set of the set $A = \{1, 2\}$ is ....

Thinking Process

We know that, the power set is a collection of all the subset of a set. To solve this problem, write the all subset of the given set.

Sol. $\therefore$ $A = \{1, 2\}$

So, the subsets of A are ϕ, $\{1\}$, $\{2\}$ and $\{1, 2\}$.

$\therefore \quad P(A) = \{\phi, \{1\}, \{2\}, \{1, 2\}\}$

Q. 48 If the sets $A = \{1, 3, 5\}$, $B = \{2, 4, 6\}$ and $C = \{0, 2, 4, 6, 8\}$. Then, the universal set of all the three sets A, B and C can be ....

Sol. Universal set for A, B and C is given by $U = \{0, 1, 2, 3, 4, 5, 6, 8\}$

Q. 49 If $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{1, 2, 3, 5\}$, $B = \{2, 4, 6, 7\}$ and $C = \{2, 3, 4, 8\}$. Then,

(i) $(B \cup C)'$ is .... (ii) $(C - A)'$ is ....

Sol. If $U = \{1, 2, 3, 4, 5, \dots, 10\}$,

$A = \{1, 2, 3, 5\}$, $B = \{2, 4, 6, 7\}$ and $C = \{2, 3, 4, 8\}$

$\therefore \quad B \cup C = \{2, 3, 4, 6, 7, 8\}$

(i) $(B \cup C)' = U - (B \cup C) = \{1, 5, 9, 10\}$

(ii) $C - A = \{4, 8\}$

$\therefore \quad (C - A)' = U - (C - A) = \{1, 2, 3, 5, 6, 7, 9, 10\}$

Q. 50 For all sets A and B, $A - (A \cap B)$ is equal to ....

Sol. $A - (A \cap B) = A - B = A \cap B'$

Q. 51 Match the following sets for all sets A, B and C

	Column I		Column II
(i)	$((A' \cup B') - A)'$	(a)	$A - B$
(ii)	$[(B' \cup (B' - A)]'$	(b)	A
(iii)	$(A - B) - (B - C)$	(c)	B
(iv)	$(A - B) \cap (C - B)$	(d)	$(A \times B) \cap (A \times C)$
(v)	$A \times (B \cap C)$	(e)	$(A \times B) \cup (A \times C)$
(vi)	$A \times (B \cup C)$	(f)	$(A \cap C) - B$

Sol. (i) $[(A' \cup B') - A]' = [(A' \cup B') \cap A']'$ $\quad [\because A - B = A \cap B']$

$= [(A \cap B)' \cap A']'$ $\quad [\because (A \cap B)' = A' \cup B']$

$= [(A \cap B)']' \cup (A')' = (A \cap B) \cup A$ $\quad [\because (A')' = A]$

$= A$

(ii) $[B' \cup (B' - A)]' = [B' \cup (B' \cap A')]'$ $\quad [\because A - B = A \cap B']$

$= [B' \cup (B \cup A)']'$ $\quad [\because A' \cap B' = (A \cup B)']$

$= (B')' \cap [(B \cup A)']'$ $\quad [\because (A \cup B)' = A' \cap B']$

$= B \cap (B \cup A)$ $\quad [\because (A')' = A]$

$= B$

(iii) $(A - B) - (B - C) = (A \cap B') - (B \cap C')$ $\quad [\because A - B = A \cap B']$

$= (A \cap B') \cap (B \cap C')'$

$= (A \cap B') \cap [B' \cup (C')']$

$= (A \cap B') \cap (B' \cup C)$ $\quad [\because (A')' = A]$

$= [A \cap (B' \cup C)] \cap [B' \cap (B' \cup C)]$

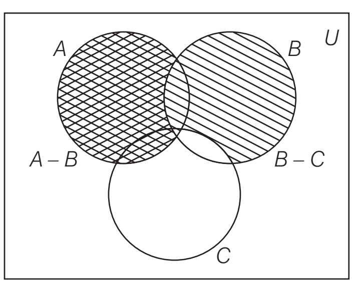
$= [A \cap (B' \cup C)] \cap B'$

$= (A \cap B') \cap [(B' \cup C) \cap B']$

$= (A \cap B') \cap B' = A \cap B' = A - B$

Alternate Method

It is clear from the diagram, $(A - B) - (B - C) = A - B$.



(iv) $(A - B) \cap (C - B)$

$\Rightarrow \quad (A \cap B') \cap (C \cap B')$ $\quad [\because A - B = A \cap B']$

$\Rightarrow \quad (A \cap C) \cap B'$

$\Rightarrow \quad (A \cap C) - B$ $\quad [\because A \cap B' = A - B]$

(v) $A \times B \cap C = (A \times B) \cap (A \times C)$

(vi) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

Hence, the correct matches are

(i) $\leftrightarrow$ (b), (ii) $\leftrightarrow$ (c), (iii) $\leftrightarrow$ (a), (iv) $\leftrightarrow$ (f), (v) $\leftrightarrow$ (d), (vi) $\leftrightarrow$ (e)

True/False

Q. 52 If A is any set, then $A \subset A$.

Sol. ***True***

Since, every set is the subset of itself.
Therefore, for any set A, $A \subset A$.

Q. 53 If $M = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $B = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, then $B \not\subset M$.

Sol. ***False***

$$M = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$
$$B = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

Since, every elements of B is also in M.

$\therefore$ $B \subset M$

Q. 54 The sets $\{1, 2, 3, 4\}$ and $\{3, 4, 5, 6\}$ are equal

Sol. ***False***

Since, $2 \in \{1, 2, 3, 4\}$

But $2 \notin \{3, 4, 5, 6\}$

$\therefore$ $\{1, 2, 3, 4\} \neq \{3, 4, 5, 6\}$

Q. 55 $Q \cup Z = Q$, where Q is the set of rational numbers and Z is the set of integers.

Sol. ***True***

Since, every integer is also a rational number, then $Z \subset Q$
where, Z is the set of integer and Q is the set of rational number.

$\therefore$ $Q \cup Z = Q$

Q. 56 Let sets R and T be defined as

$$R = \{x \in Z \mid x \text{ is divisible by } 2\}$$
$$T = \{x \in Z \mid x \text{ is divisible by } 6\}. \text{ Then, } T \subset R$$

Sol. ***True***

$$R = \{x \in Z \mid x \text{ is divisible by } 2\} = \{\ldots -6, -4, -2, 0, 2, 4, 6, \ldots\}$$
$$T = \{x \in Z \mid x \text{ is divisible by } 6\} = \{\ldots, -12, -6, 0, 6, 12, \ldots\}$$

Thus, this every elements of T is also in R.

$\therefore$ $T \subset R$

Q. 57 Given $A = \{0, 1, 2\}$, $B = \{x \in R \mid 0 \le x \le 2\}$. Then, $A = B$.

Sol. ***False***

$$A = \{0, 1, 2\}, \text{ and } B = \{x \in R \mid 0 \le x \le 2\}$$

$\Rightarrow$ $n(A) = 3$

So, A is finite. Since, there are infinite real numbers from 0 to 2. So, B is infinite.

$\therefore$ $A \neq B$

2

Relations and Functions

Short Answer Type Questions

Q. 1 If $A = \{-1, 2, 3\}$ and $B = \{1, 3\}$, then determine

(i) $A \times B$ (ii) $B \times A$ (iii) $B \times B$ (iv) $A \times A$

Sol. $A = \{-1, 2, 3\}$ and $B = \{1, 3\}$

(i) $A \times B = \{(-1, 1), (-1, 3), (2, 1), (2, 3), (3, 1), (3, 3)\}$

(ii) $B \times A = \{(1, -1), (1, 2), (1, 3), (3, -1), (3, 2), (3, 3)\}$

(iii) $B \times B = \{(1, 1), (1, 3), (3, 1), (3, 3)\}$

(iv) $A \times A = \{(-1, -1), (-1, 2), (-1, 3), (2, -1), (2, 2), (2, 3), (3, -1), (3, 2), (3, 3)\}$

Q. 2 If $P = \{x : x < 3, x \in N\}$, $Q = \{x : x \le 2, x \in W\}$, then find $(P \cup Q) \times (P \cap Q)$, where W is the set of whole numbers.

Sol. We have, $P = \{x : x < 3, x \in N\} = \{1, 2\}$

and $Q = \{x : x \le 2, x \in W\} = \{0, 1, 2\}$

$\therefore$ $P \cup Q = \{0, 1, 2\}$ and $P \cap Q = \{1, 2\}$

$$(P \cup Q) \times (P \cap Q) = \{0, 1, 2\} \times \{1, 2\}$$
$$= \{(0, 1), (0, 2), (1, 1), (1, 2), (2, 1), (2, 2)\}$$

Q. 3 If $A = \{x : x \in W, x < 2\}$, $B = \{x : x \in N, 1 < x < 5\}$ and $C = \{3, 5\}$, then find

(i) $A \times (B \cap C)$ (ii) $A \times (B \cup C)$

Sol. We have, $A = \{x : x \in W, x < 2\} = \{0, 1\}$

and $B = \{x : x \in N, 1 < x < 5\}$

$= \{2, 3, 4\}$ and $C = \{3, 5\}$

(i) $\because$ $B \cap C = \{3\}$

$\therefore$ $A \times (B \cap C) = \{0, 1\} \times \{3\} = \{(0, 3), (1, 3)\}$

(ii) $\because$ $(B \cup C) = \{2, 3, 4, 5\}$

$\therefore$ $A \times (B \cup C) = \{0, 1\} \times \{2, 3, 4, 5\}$

$= \{(0, 2), (0, 3), (0, 4), (0, 5), (1, 2), (1, 3), (1, 4), (1, 5)\}$

Q. 4 In each of the following cases, find a and b.

(i) $(2a + b, a - b) = (8, 3)$

(ii) $\left(\frac{a}{4}, a - 2b\right) = (0, 6 + b)$

Sol. **(i)** We have, $(2a + b, a - b) = (8, 3)$

$\Rightarrow \quad 2a + b = 8$ and $a - b = 3$

[since, two ordered pairs are equal, if their corresponding first and second elements are equal]

On substituting, $b = a - 3$ in $2a + b = 8$, we get

$$2a + a - 3 = 8 \Rightarrow 3a - 3 = 8$$

$$\Rightarrow \quad 3a = 11 \Rightarrow a = \frac{11}{3}$$

Again, substituting $a = \frac{11}{3}$ in $b = a - 3$, we get

$$b = \frac{11}{3} - 3 = \frac{11 - 9}{3} = \frac{2}{3}$$

$$\therefore \quad a = \frac{11}{3} \text{ and } b = \frac{2}{3}$$

(ii) We have, $\left(\frac{a}{4}, a - 2b\right) = (0, 6 + b)$

$$\Rightarrow \quad \frac{a}{4} = 0 \Rightarrow a = 0$$

and $\quad a - 2b = 6 + b$

$\Rightarrow \quad 0 - 2b = 6 + b$

$\Rightarrow \quad -3b = 6$

$\therefore \quad b = -2$

$\therefore \quad a = 0, b = -2$

Q. 5 $A = \{1, 2, 3, 4, 5\}, S = \{(x, y) : x \in A, y \in A\}$, then find the ordered which satisfy the conditions given below.

(i) $x + y = 5$ (ii) $x + y < 5$

(iii) $x + y > 8$

Sol. We have, $A = \{1, 2, 3, 4, 5\}$ and $S = \{(x, y) : x \in A, y \in A\}$

(i) The set of ordered pairs satisfying $x + y = 5$ is,

$$\{(1, 4), (2, 3), (3, 2), (4, 1)\}.$$

(ii) The set of ordered pairs satisfying $x + y < 5$ is $\{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (3, 1)$.

(iii) The set of ordered pairs satisfying $x + y > 8$ is $\{(4, 5), (5, 4), (5, 5)\}$.

Q. 6 If $R = \{(x, y) : x, y \in W, x^2 + y^2 = 25\}$, then find the domain and range of R.

Thinking Process

First, write the relation in Roaster form, then find the domain and range of R.

Sol. We have, $R = \{(x, y) : x, y \in W, x^2 + y^2 = 25\}$

$= \{(0, 5), (3, 4), (4, 3), (5, 0)\}$

Domain of R = Set of first element of ordered pairs in R

$= \{0, 3, 4, 5\}$

Range of R = Set of second element of ordered pairs in R

$= \{5, 4, 3, 0,\}$

Q. 7 If $R_1 = \{(x, y) | y = 2x + 7$, where $x \in R$ and $-5 \le x \le 5\}$ is a relation. Then, find the domain and range of R_1.

Sol. We have, $R_1 = \{x, y) | y = 2x + 7$, where $x \in R$ and $-5 \le x \le 5\}$

Domain of $R_1 = \{-5 \le x \le 5, x \in R\}$

$= [-5, 5]$

$\because \quad y = 2x + 7$

When $x = -5$, then $y = 2(-5) + 7 = -3$

When $x = 5$, then $y = 2(5) + 7 = 17$

$\therefore$ Range of $R_1 = \{-3 \le y \le 17, y \in R\}$

$= [-3, 17]$

Q. 8 If $R_2 = \{x, y) | x$ and y are integers and $x^2 + y^2 = 64\}$ is a relation, then find the value of R_2.

Sol. We have, $R_2 = \{(x, y)\}$ x and y are integers and $x^2 + y^2 = 64\}$

Since, 64 is the sum of squares of 0 and $\pm$ 8.

When $x = 0$, then $y^2 = 64 \Rightarrow y = \pm 8$

$x = 8$, then $y^2 = 64 - 8^2 \Rightarrow 64 - 64 = 0$

$x = -8$, then $y^2 = 64 - (-8)^2 = 64 - 64 = 0$

$\therefore \quad R_2 = \{(0, 8), (0, -8), (8, 0), (-8, 0)\}$

Q. 9 If $R_3 = \{(x, |x|) | x$ is a real number$\}$ is a relation, then find domain and range of R_3.

Sol. We have, $R_3 = \{(x, |x|) | x$ is real number$\}$

Clearly, domain of $R_3 = R$

Since, image of any real number under R_3 is positive real number or zero.

$\therefore$ Range of $R_3 = R^+ \cup \{0\}$ or $(0, \infty)$

Q. 10 Is the given relation a function? Give reason for your answer.

(i) $h = \{(4, 6), (3, 9), (-11, 6), (3, 11)\}$

(ii) $f = \{(x, x) | x$ is a real number$\}$

(iii) $g = \left\{\left(x, \frac{1}{x}\right) \middle| x \text{ is a positive integer}\right\}$

(iv) $s = \{(x, x^2) | x$ is a positive integer$\}$

(v) $t = \{(x, 3) | x$ is a real number$\}$

Sol. **(i)** We have, $h = \{(4, 6), (3, 9), (-11, 6), (3, 11)\}$.

Since, 3 has two images 9 and 11. So, it is not a function.

(ii) We have, $f = \{(x, x) | x$ is a real number.

We observe that, every element in the domain has unique image. So, it is a function.

(iii) We have, $g = \left\{\left(x, \frac{1}{x}\right) \middle| x \text{ is a positive integer}\right\}$

For every x, it is a positive integer and $\frac{1}{x}$ is unique and distinct. Therefore, every element in the domain has unique image. So, it is a function.

(iv) We have, $s = \{(x, x^2) | x$ is a positive integer$\}$

Since, the square of any positive integer is unique. So, every element in the domain has unique image. Hence, it is a function.

(v) We have, $t = \{(x, 3) | x$ is a real number$\}$.

Since, every element in the domain has the image 3. So, it is a constant function.

Q. 11 If f and g are real functions defined by $f(x) = x^2 + 7$ and $g(x) = 3x + 5$. Then, find each of the following.

(i) $f(3) + g(-5)$

(ii) $f\left(\frac{1}{2}\right) \times g(14)$

(iii) $f(-2) + g(-1)$

(iv) $f(t) - f(-2)$

(v) $\frac{f(t) - f(5)}{t - 5}$, if $t \neq 5$

Sol. Given, f and g are real functions defined by $f(x) = x^2 + 7$ and $g(x) = 3x + 5$.

(i) $f(3) = (3)^2 + 7 = 9 + 7 = 16$ and $g(-5) = 3(-5) + 5 = -15 + 5 = -10$

$\therefore f(3) + g(-5) = 16 - 10 = 6$

(ii) $f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 + 7 = \frac{1}{4} + 7 = \frac{29}{4}$

and $g(14) = 3(14) + 5 = 42 + 5 = 47$

$\therefore \quad f\left(\frac{1}{2}\right) \times g(14) = \frac{29}{4} \times 47 = \frac{1363}{4}$

(iii) $f(-2) = (-2)^2 + 7 = 4 + 7 = 11$ and $g(-1) = 3(-1) + 5 = -3 + 5 = 2$

$\therefore \quad f(-2) + g(-1) = 11 + 2 = 13$

(iv) $f(t) = t^2 + 7$ and $f(-2) = (-2)^2 + 7 = 4 + 7 = 11$

$\therefore \quad f(t) - f(-2) = t^2 + 7 - 11 = t^2 - 4$

(v) $f(t) = t^2 + 7$ and $f(5) = 5^2 + 7 = 25 + 7 = 32$

$\therefore \quad \dfrac{f(t) - f(5)}{t - 5}$, if $t \neq 5$

$$= \frac{t^2 + 7 - 32}{t - 5}$$

$$= \frac{t^2 - 25}{t - 5} = \frac{(t - 5)(t + 5)}{(t - 5)}$$

$$= t + 5 \qquad [\because \ t \neq 5]$$

Q. 12 Let f and g be real functions defined by $f(x) = 2x + 1$ and $g(x) = 4x - 7$.

(i) For what real numbers x, $f(x) = g(x)$?

(ii) For what real numbers x, $f(x) < g(x)$?

Sol. We have, $f(x) = 2x + 1$ and $g(x) = 4x - 7$

(i) $\because \quad f(x) = g(x)$

$\Rightarrow \quad 2x + 1 = 4x - 7 \Rightarrow 2x = 8$

$\therefore \quad x = 4$

(ii) $\because \quad f(x) < g(x)$

$\Rightarrow \quad 2x + 1 < 4x - 7$

$\Rightarrow \quad 2x - 4x + 1 < 4x - 7 - 4x$

$\Rightarrow \quad -2x + 1 < -7$

$\Rightarrow \quad -2x < -7 - 1$

$\Rightarrow \quad -2x < -8$

$\Rightarrow \quad \dfrac{-2x}{-2} > \dfrac{-8}{-2}$

$\therefore \quad x > 4$

Q. 13 If f and g are two real valued functions defined as $f(x) = 2x + 1$ and $g(x) = x^2 + 1$, then find

(i) $f + g$ (ii) $f - g$ (iii) fg (iv) $\dfrac{f}{g}$

Sol. We have, $f(x) = 2x + 1$ and $g(x) = x^2 + 1$

(i) $(f + g)(x) = f(x) + g(x)$

$= 2x + 1 + x^2 + 1 = x^2 + 2x + 2$

(ii) $(f - g)(x) = f(x) - g(x) = (2x + 1) - (x^2 + 1)$

$= 2x + 1 - x^2 - 1 = 2x - x^2 = x(2 - x)$

(iii) $(fg)(x) = f(x) \cdot g(x) = (2x + 1)(x^2 + 1)$

$= 2x^3 + 2x + x^2 + 1 = 2x^3 + x^2 + 2x + 1$

(iv) $\dfrac{f}{g}(x) = \dfrac{f(x)}{g(x)} = \dfrac{2x + 1}{x^2 + 1}$

Q. 14 Express the following functions as set of ordered pairs and determine their range.

$$f : x \to R,\ f(x) = x^3 + 1, \text{ where } x = \{-1, 0, 3, 9, 7\}$$

Sol. We have, $f : X \to R,\ f(x) = x^3 + 1.$

Where $X = \{-1, 0, 3, 9, 7\},$

When $x = -1$, then $f(-1) = (-1)^3 + 1 = -1 + 1 = 0$

$x = 0$, then $f(0) = (0)^3 + 1 = 0 + 1 = 1$

$x = 3$, then $f(3) = (3)^3 + 1 = 27 + 1 = 28$

$x = 9$, then $f(9) = (9)^3 + 1 = 729 + 1 = 730$

$x = 7$, then $f(7) = (7)^3 + 1 = 343 + 1 = 344$

$f = \{(-1, 0), (0, 1), (3, 28), (9, 730), (7, 344)\}$

$\therefore$ Range of $f = \{0, 1, 28, 730, 344\}$

Q. 15 Find the values of x for which the functions $f(x) = 3x^2 - 1$ and $g(x) = 3 + x$ are equal.

Sol. $\because$ $f(x) = g(x)$

$\Rightarrow$ $3x^2 - 1 = 3 + x$

$\Rightarrow$ $3x^2 - x - 4 = 0$

$\Rightarrow$ $3x^2 - 4x + 3x - 4 = 0$

$\Rightarrow$ $x(3x - 4) + 1(3x - 4) = 0$

$\Rightarrow$ $(3x - 4)(x + 1) = 0$

$\therefore$ $x = -1, \frac{4}{3}$

Long Answer Type Questions

Q. 16 Is $g = \{(1, 1), (2, 3), (3, 5), (4, 7),\}$ a function, justify. If this is described by the relation, $g(x) = \alpha x + \beta$, then what values should be assigned to α and β?

Thinking Process

First, find the two equation by substitutions different values of x and $g(x)$.

Sol. We have, $g = \{(1, 1), (2, 3), (3, 5), (4, 7)\}$

Since, every element has unique image under g. So, g is a function.

Now, $g(x) = \alpha x + \beta$

When $x = 1$, then $g(1) = \alpha(1) + \beta$ ...(i)

$\Rightarrow$ $1 = \alpha + \beta$

When $x = 2$, then $g(2) = \alpha(2) + \beta$

$\Rightarrow$ $3 = 2\alpha + \beta$ ...(ii)

On solving Eqs. (i) and (ii), we get

$\alpha = 2, \beta = -1$

Q. 17 Find the domain of each of the following functions given by

(i) $f(x) = \dfrac{1}{\sqrt{1 - \cos x}}$ (ii) $f(x) = \dfrac{1}{\sqrt{x + |x|}}$

(iii) $f(x) = x|x|$ (iv) $f(x) = \dfrac{x^3 - x + 3}{x^2 - 1}$

(v) $f(x) = \dfrac{3x}{28 - x}$

Sol. (i) We have, $f(x) = \dfrac{1}{\sqrt{1 - \cos x}}$

$\because \quad -1 \le \cos x \le 1$

$\Rightarrow \quad -1 \le -\cos x \le 1$

$\Rightarrow \quad 0 \le 1 - \cos x \le 2$

So, $f(x)$ is defined, if $1 - \cos x \ne 0$

$\cos x \ne 1$

$x \ne 2n\pi - \forall n \in Z$

$\therefore$ Domain of $f = R - \{2n\pi : n \in Z\}$

(ii) We have, $f(x) = \dfrac{1}{\sqrt{x + |x|}}$

$\because \quad x + |x| = x - x = 0, x < 0$

$= x + x = 2x, x \ge 0$

Hence, $f(x)$ is defined, if $x > 0$.

$\therefore$ Domain of $f = R^+$

(iii) We have, $f(x) = x|x|$

Clearly, $f(x)$ is defined for any $x \in R$.

$\therefore$ Domain of $f = R$

(iv) We have, $f(x) = \dfrac{x^3 - x + 3}{x^2 - 1}$

$f(x)$ is not defined, if $x^2 - 1 = 0$

$\Rightarrow \quad (x - 1)(x + 1) = 0$

$\Rightarrow \quad x = -1, 1$

$\therefore$ Domain of $f = R - \{-1, 1\}$

(v) We have, $f(x) = \dfrac{3x}{28 - x}$

Clearly, $f(x)$ is defined, if $28 - x \ne 0$

$\Rightarrow \quad x \ne 28$

$\therefore$ Domain of $f = R - \{28\}$

Q. 18 Find the range of the following functions given by

(i) $f(x) = \dfrac{3}{2 - x^2}$ (ii) $f(x) = 1 - |x - 2|$

(iii) $f(x) = |x - 3|$ (iv) $f(x) = 1 + 3\cos 2x$

Thinking Process

First, find the value of x in terms of y, where $y = f(x)$. Then, find the values of y for which x attain real values.

Sol. (i) We have, $f(x) = \dfrac{3}{2 - x^2}$

Let $y = f(x)$

Then, $y = \dfrac{3}{2 - x^2} \Rightarrow 2 - x^2 = \dfrac{3}{y}$

$\Rightarrow$ $x^2 = 2 - \dfrac{3}{y} \Rightarrow x = \sqrt{\dfrac{2y - 3}{y}}$

x assums real values, if $2y - 3 \geq 0$ and $y > 0 \Rightarrow y \geq \dfrac{3}{2}$

$\therefore$ Range of $f = \left[\dfrac{3}{2}, \infty\right)$

(ii) We know that, $|x - 2| \geq 0 \Rightarrow -|x - 2| \leq 0$

$\Rightarrow$ $1 - |x - 2| \leq 1 \Rightarrow f(x) \leq 1$

$\therefore$ Range of $f = (-\infty, 1]$

(iii) We know that, $|x - 3| \geq 0 \Rightarrow f(x) \geq 0$

$\therefore$ Range of $f = [0, \infty)$

(iv) We know that, $-1 \leq \cos 2x \leq 1 \Rightarrow -3 \leq 3\cos 2x \leq 3$

$\Rightarrow$ $1 - 3 \leq 1 + 3\cos 2x \leq 1 + 3 \Rightarrow -2 \leq 1 + 3\cos 2x \leq 1 + 3$

$\Rightarrow$ $-2 \leq f(x) \leq 4$

$\therefore$ Range of $f = [-2, 4]$

Q. 19 Redefine the function

$$f(x) = |x - 2| + |2 + x|, -3 \leq x \leq 3$$

Thinking Process

First find the interval in which $|x-2|$ and $|2 + x|$ is defined, then find the value of $f(x)$ in that interval.

Sol. Since, $|x - 2| = -(x - 2), x < 2$

$x - 2, x \geq 2$

and $|2 + x| = -(2 + x), x < -2$

$(2 + x), x \geq -2$

$\therefore$ $f(x) = |x - 2| + |2 + x|, -3 \leq x \leq 3$

$$= \begin{cases} -(x - 2) - (2 + x), & -3 \leq x < -2 \\ -(x - 2) + 2 + x, & -2 \leq x < 2 \\ x - 2 + 2 + x, & 2 \leq x \leq 3 \end{cases}$$

$$= \begin{cases} -2x, & -3 \leq x < -2 \\ 4, & -2 \leq x < 2 \\ 2, & 2 \leq x \leq 3 \end{cases}$$

Q. 20 If $f(x) = \dfrac{x-1}{x+1}$, then show that

(i) $f\left(\dfrac{1}{x}\right) = -f(x)$ (ii) $f\left(-\dfrac{1}{x}\right) = \dfrac{-1}{f(x)}$

Sol. We have, $f(x) = \dfrac{x-1}{x+1}$

(i) $f\left(\dfrac{1}{x}\right) = \dfrac{\dfrac{1}{x} - 1}{\dfrac{1}{x} + 1} = \dfrac{(1-x)/x}{(1+x)/x} = \dfrac{1-x}{1+x} = \dfrac{-(x-1)}{x+1} = -f(x)$

(ii) $f\left(-\dfrac{1}{x}\right) = \dfrac{-\dfrac{1}{x} - 1}{-\dfrac{1}{x} + 1} = \dfrac{(-1-x)/x}{(-1+x)/x} \Rightarrow f\left(-\dfrac{1}{x}\right) = \dfrac{-(x+1)}{x-1}$

Now, $\dfrac{-1}{f(x)} = \dfrac{-1}{\dfrac{x-1}{x+1}} = \dfrac{-(x+1)}{x-1}$

$\therefore$ $f\left(-\dfrac{1}{x}\right) = -\dfrac{1}{f(x)}$

Q. 21 If $f(x) = \sqrt{x}$ and $g(x) = x$ be two functions defined in the domain $R^+ \cup \{0\}$, then find the value of

(i) $(f+g)(x)$ (ii) $(f-g)(x)$

(iii) $(fg)(x)$ (iv) $\left(\dfrac{f}{g}\right)(x)$

Sol. We have, $f(x) = \sqrt{x}$ and $g(x) = x$ be two function defined in the domain $R^+ \cup \{0\}$.

(i) $(f+g)(x) = f(x) + g(x) = \sqrt{x} + x$ **(ii)** $(f-g)(x) = f(x) - g(x) = \sqrt{x} - x$

(ii) $(fg)(x) = f(x) \cdot g(x) = \sqrt{x} \cdot x = x^{\frac{3}{2}}$ **(iv)** $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)} = \dfrac{\sqrt{x}}{x} = \dfrac{1}{\sqrt{x}}$

Q. 22 Find the domain and range of the function $f(x) = \dfrac{1}{\sqrt{x-5}}$.

Sol. We have, $f(x) = \dfrac{1}{\sqrt{x-5}}$

$f(x)$ is defined, if $x - 5 > 0 \Rightarrow x > 5$

$\therefore$ Domain of $f = (5, \infty)$

Let $f(x) = y$

$\therefore$ $y = \dfrac{1}{\sqrt{x-5}} \Rightarrow \sqrt{x-5} = \dfrac{1}{y}$

$\Rightarrow$ $x - 5 = \dfrac{1}{y^2}$

$\therefore$ $x = \dfrac{1}{y^2} + 5$

$\because$ $x \in (5, \infty) \Rightarrow y \in R^+$

Hence, range of $f = R^+$

Q. 23 If $f(x) = y = \frac{ax - b}{cx - a}$, then prove that $f(y) = x$.

Sol. We have, $f(x) = y = \frac{ax - b}{cx - a}$

$$\therefore \quad f(y) = \frac{ay - b}{cy - a} = \frac{a\left(\frac{ax - b}{cx - a}\right) - b}{c\left(\frac{ax - b}{cx - a}\right) - a}$$

$$= \frac{a(ax - b) - b(cx - a)}{c(ax - b) - a(cx - a)} = \frac{a^2x - ab - bcx + ab}{acx - bc - acx + a^2}$$

$$= \frac{a^2x - bcx}{a^2 - bc} = \frac{x(a^2 - bc)}{(a^2 - bc)} = x$$

$\therefore \quad f(y) = x$ **Hence proved.**

Objective Type Questions

Q. 24 Let $n(A) = m$ and $n(B) = n$. Then, the total number of non-empty relations that can be defined from A to B is

(a) m^n (b) $n^m - 1$ (c) $mn - 1$ (d) $2^{mn} - 1$

Thinking Process

First find the number of element in $A \times B$ and then find the number of relation by using $2^{m(A \times B)} - 1$.

Sol. ***(d)*** We have, $n(A) = m$ and $n(B) = n$

$$n(A \times B) = n(A) \cdot n(B) = mn$$

Total number of relation from A to $B = 2^{mn} - 1 = 2^{n(A \times B) - 1} - 1$

Q. 25 If $[x]^2 - 5[x] + 6 = 0$, where $[\cdot]$ denote the greatest integer function, then

(a) $x \in [3, 4]$ (b) $x \in (2, 3]$ (c) $x \in [2, 3]$ (d) $x \in [2, 4)$

Thinking Process

If a and b are two successive positive integer and $[x] = a, b$, then $x \in [a, b]$

Sol. ***(c)*** We have, $[x]^2 - 5[x] + 6 = 0$

$\Rightarrow \quad [x]^2 - 3[x] - 2[x] + 6 = 0$

$\Rightarrow \quad [x]([x] - 3) - 2([x] - 3) = 0$

$\Rightarrow \quad ([x] - 3)([x] - 2) = 0$

$\Rightarrow \quad [x] = 2, 3$

$\therefore \quad x \in [2, 3]$

Q. 26 Range of $f(x)=\dfrac{1}{1-2\cos x}$ is

(a) $\left[\frac{1}{3}, 1\right]$ (b) $\left[-1, \frac{1}{3}\right]$

(c) $(-\infty, -1] \cup \left[\frac{1}{3}, \infty\right)$ (d) $\left[-\frac{1}{3}, 1\right]$

Sol. ***(b)*** We know that, $-1 \le -\cos x \le 1$

$$\Rightarrow \quad -2 \le -2\cos x \le 2$$
$$\Rightarrow \quad 1-2 \le 1-2\cos x \le 1+2$$
$$\Rightarrow \quad -1 \le 1-2\cos x \le 3$$
$$\Rightarrow \quad -1 \le \frac{1}{1-2\cos x} \le \frac{1}{3}$$
$$\Rightarrow \quad -1 \le f(x) \le \frac{1}{3}$$
$$\therefore \quad \text{Range of } f = \left[-1, \frac{1}{3}\right]$$

Q. 27 Let $f(x)=\sqrt{1+x^2}$, then

(a) $f(xy)=f(x)\cdot f(y)$ (b) $f(xy)\ge f(x)\cdot f(y)$

(c) $f(xy)\le f(x)\cdot f(y)$ (d) None of these

Sol. ***(c)*** We have, $f(x)=\sqrt{1+x^2}$

$$f(xy)=\sqrt{1+x^2y^2}$$
$$f(x)\cdot f(y)=\sqrt{1+x^2}\cdot\sqrt{1+y^2}$$
$$=\sqrt{(1+x^2)(1+y^2)}$$
$$=\sqrt{1+x^2+y^2+x^2y^2}$$
$$\because \quad \sqrt{1+x^2y^2}\le\sqrt{1+x^2+y^2+x^2y^2}$$
$$\Rightarrow \quad f(xy)\le f(x)\cdot f(y)$$

Q. 28 Domain of $\sqrt{a^2-x^2}$ $(a>0)$ is

(a) $(-a, a)$ (b) $[-a, a]$

(c) $[0, a]$ (d) $(-a, 0]$

Sol. ***(b)*** Let $f(x)=\sqrt{a^2-x^2}$

$f(x)$ is defined, if $a^2-x^2\ge 0$

$$\Rightarrow \quad x^2-a^2\le 0$$
$$\Rightarrow \quad (x-a)(x+a)\le 0$$
$$\Rightarrow \quad -a\le x\le a \qquad [\because a>0]$$
$$\therefore \quad \text{Domain of } f=[-a, a]$$

Q. 29 If $f(x) = ax + b$, where a and b are integers, $f(-1) = -5$ and $f(3) = 3$, then a and b are equal to

(a) $a = -3, b = -1$
(b) $a = 2, b = -3$
(c) $a = 0, b = 2$
(d) $a = 2, b = 3$

Sol. ***(b)*** We have,
$$f(x) = ax + b$$
$$f(-1) = a(-1) + b$$
$$-5 = -a + b \qquad \text{...(i)}$$
and,
$$f(3) = a(3) + b$$
$$3 = 3a + b \qquad \text{...(ii)}$$
On solving Eqs. (i) and (ii), we get
$$a = 2 \text{ and } b = -3$$

Q. 30 The domain of the function f defined by
$$f(x) = \sqrt{4 - x} + \frac{1}{\sqrt{x^2 - 1}} \text{ is equal to}$$

(a) $(-\infty, -1) \cup (1, 4]$
(b) $(-\infty, -1] \cup (1, 4]$
(c) $(-\infty, -1) \cup [1, 4]$
(d) $(-\infty, -1) \cup [1, 4)$

Sol. ***(a)*** We have,
$$f(x) = \sqrt{4 - x} + \frac{1}{\sqrt{x^2 - 1}}$$
$f(x)$ is defined, if
$$4 - x \ge 0 \text{ or } x^2 - 1 > 0$$
$$x - 4 \le 0 \text{ or } (x + 1)(x - 1) > 0$$
$$x \le 4 \text{ or } x < -1 \text{ and } x > 1$$
$\therefore$ Domain of $f = (-\infty, -1) \cup (1, 4]$

Q. 31 The domain and range of the real function f defined by $f(x) = \dfrac{4 - x}{x - 4}$ is given by

(a) Domain $= R$, Range $= \{-1, 1\}$
(b) Domain $= R - \{1\}$, Range $= R$
(c) Domain $= R - \{4\}$, Range $= \{-1\}$
(d) Domain $= R - \{-4\}$, Range $= \{-1, 1\}$

Thinking Process

A function $\dfrac{f(x)}{g(x)}$ *is defined, if* $g(x) \ne 0$.

Sol. ***(c)*** We have,
$$f(x) = \frac{4 - x}{x - 4}$$
$f(x)$ is defined, if $x - 4 \ne 0$ *i.e.*, $x \ne 4$

$\therefore$ Domain of $f = R - \{4\}$

Let $f(x) = y$
$$\therefore \quad y = \frac{4 - x}{x - 4} \Rightarrow xy - 4y = 4 - x$$
$$\Rightarrow \quad xy + x = 4 + 4y \Rightarrow x(y + 1) = 4(1 + y)$$
$$\therefore \quad x = \frac{4(1 + y)}{y + 1}$$
x assumes real values, if $y + 1 \ne 0$ *i.e.*, $y = -1$.

$\therefore$ Range of $f = R - \{-1\}$

Q. 32 The domain and range of real function f defined by

$$f(x) = \sqrt{x-1} \text{ is given by}$$

(a) Domain $= (1, \infty)$, Range $= (0, \infty)$ (b) Domain $= [1, \infty)$, Range $= (0, \infty)$
(c) Domain $= (1, \infty)$, Range $= [0, \infty)$ (d) Domain $= [1, \infty)$, Range $= [0, \infty)$

Thinking Process

A function is defined $f(x) = \sqrt{x}$ is defined $x \geq 0$.

Sol. *(d)* We have, $f(x) = \sqrt{x-1}$

$f(x)$ is defined, if $x - 1 \geq 0$.

$\Rightarrow x \geq 1$

$\therefore$ Domain of $f = [1, \infty)$

Let $f(x) = y$

$\therefore y = \sqrt{x-1}$

$\Rightarrow y^2 = x - 1$

$\therefore x = y^2 + 1$

x assumes real values for $y \in R$.

but $y \geq 0$

$\therefore$ Range of $f = [0, \infty)$

Q. 33 The domain of the function f given by $f(x) = \dfrac{x^2 + 2x + 1}{x^2 - x - 6}$.

(a) $R - \{3, -2\}$ (b) $R - \{-3, 2\}$ (c) $R - [3, -2]$ (d) $R - (3, -2)$

Sol. *(a)* We have, $f(x) = \dfrac{x^2 + 2x + 1}{x^2 - x - 6}$

$f(x)$ is defined, if $x^2 - x - 6 = 0$

$\Rightarrow x^2 - 3x + 2x - 6 = 0$

$\Rightarrow x(x - 3) + 2(x - 3) = 0$

$\Rightarrow (x - 3)(x + 2) = 0$

$\therefore x = -3, -2$

$\therefore$ Domain of $f = R - \{3, -2\}$

Q. 34 The domain and range of the function f given by $f(x) = 2 - |x - 5|$ is

(a) Domain $= R^+$, Range $= (-\infty, 1]$
(b) Domain $= R$, Range $= [-\infty, 2]$
(c) Domain $= R$, Range $= (-\infty, 2)$
(d) Domain $= R^+$, Range $= (-\infty, 2]$

Sol. *(b)* We have, $f(x) = 2 - |x - 5|$

$f(x)$ is defined for all $x \in R$

$\therefore$ Domain of $f = R$

We know that, $|x - 5| \geq 0 \Rightarrow -|x - 5| \leq 0$

$\Rightarrow 2 - |x - 5| \leq 2$

$\therefore f(x) \leq 2$

$\therefore$ Range of $f = [-\infty, 2]$

Q. 35 The domain for which the functions defined by $f(x) = 3x^2 - 1$ and $g(x) = 3 + x$ are equal to

(a) $\left[-1, \frac{4}{3}\right]$ (b) $\left[1, \frac{4}{3}\right]$

(c) $\left[-1, -\frac{4}{3}\right]$ (d) $\left[-2, -\frac{4}{3}\right]$

Sol. *(a)* We have, $f(x) = 3x^2 - 1$ and $g(x) = 3 + x$

$$f(x) = g(x)$$

$\Rightarrow$ $3x^2 - 1 = 3 + x$

$\Rightarrow$ $3x^2 - x - 4 = 0$

$\Rightarrow$ $3x^2 - 4x + 3x - 4 = 0$

$\Rightarrow$ $x(3x - 4) + 1(3x - 4) = 0$

$\Rightarrow$ $(3x - 4)(x + 1) = 0$

$\therefore$ $x = -1, \frac{4}{3}$

So, domain for which $f(x)$ and $g(x)$ are equal to $\left[-1, \frac{4}{3}\right]$.

Fillers

Q. 36 Let f and g be two real functions given by

$$f = \{(0, 1), (2, 0), (3, -4), (4, 2), (5, 1)\}$$

and $g = \{(1, 0), (2, 2), (3, -1), (4, 4), (5, 3)\}$,

then the domain of $f \cdot g$ is given by............

Thinking Process

First find the domain of f and domain of g. Then,

domain of $f \cdot g$ = domain of $f \cap$ domain of g.

Sol. We have, $f = \{(0, 1), (2, 0), (3, -4), (4, 2), (5, 1)\}$

and $g = \{(1, 0), (2, 2), (3, -1), (4, 4), (5, 3)\}$

$\therefore$ Domain of $f = \{0, 2, 3, 4, 5\}$,

and Domain of $g = \{1, 2, 3, 4, 5\}$

$\therefore$ Domain of $(f \cdot g)$ = Domain of $f \cap$ Domain of $g = \{2, 3, 4, 5\}$

Q. 37 Let $f = \{(2, 4), (5, 6), (8, -1), (10, -3)\}$
and $g = \{(2, 5), (7, 1), (8, 4), (10, 13), (11, 5)\}$
be two real functions. Then, match the following.

Column I		Column II	
(i)	$f - g$	(a)	$\left\{\left(2, \frac{4}{5}\right), \left(8, \frac{-1}{4}\right), \left(10, \frac{-3}{13}\right)\right\}$
(ii)	$f + g$	(b)	$\{(2, 20), (8, -4), (10, -39)\}$
(c)	$f \cdot g$	(c)	$\{(2, -1), (8, -5), (10, -16)\}$
(d)	$\frac{f}{g}$	(d)	$\{(2, 9), (8, 3), (10, -10)\}$

The domain of $f - g, f + g, f \cdot g, \dfrac{f}{g}$ is domain of $f \cap$ domain of g. Then, find their images.

Sol. We have,

$$f = \{(2, 4), (5, 6), (8, 1), (10, -3)\}$$

and $$g = \{(2, 5), (7, 1), (8, 4), (10, 13), (11, 5)\}$$

So, $f - g, f + g, f \,.\, g, \dfrac{f}{g}$ are defined in the domain (domain of $f \cap$ domain of g)

i.e., $\{2, 5, 8, 10\} \cap \{2, 7, 8, 10, 11\} \Rightarrow \{2, 8, 10\}$

(i) $(f - g)(2) = f(2) - g(2) = 4 - 5 = -1$

$$(f - g)(8) = f(8) - g(8) = -1 - 4 = -5$$
$$(f - g)(10) = f(10) - g(10) = -3 - 13 = -16$$
$$\therefore \quad f - g = \{(2, -1), (8, -5), (10, -16)\}$$

(ii) $(f + g)(2) = f(2) + g(2) = 4 + 5 = 9$

$$(f + g)(8) = f(8) + g(8) = -1 + 4 = 3$$
$$(f + g)(10) = f(10) + g(10) = -3 + 13 = 10$$
$$\therefore \quad f + g = \{(2, 9), (8, 3), (10, 10)\}$$

(iii) $(f \cdot g)(2) = f(2) \cdot g(2) = 4 \times 5 = 20$

$$(f \cdot g)(8) = f(8) \cdot g(8) = -1 \times 4 = -4$$
$$(f \cdot g)(10) = f(10) \cdot g(10) = -3 \times 13 = -39$$
$$\therefore \quad fg = \{(2, 20), (8, -4), (10, -39)\}$$

(iv) $\left(\dfrac{f}{g}\right)(2) = \dfrac{f(2)}{g(2)} = \dfrac{4}{5}$

$$\left(\frac{f}{g}\right)(8) = \frac{f(8)}{g(8)} = \frac{-1}{4}$$
$$\left(\frac{f}{g}\right)(10) = \frac{f(10)}{g(10)} = \frac{-3}{13}$$
$$\therefore \quad \frac{f}{g} = \left\{\left(2, \frac{4}{5}\right), \left(8, -\frac{1}{4}\right), \left(10, \frac{-3}{13}\right)\right\}$$

Hence, the correct matches are (i) $\rightarrow$ (c), (ii) $\rightarrow$ (d), (iii) $\rightarrow$ (b), (iv) $\rightarrow$ (a).

True/False

Q. 38 The ordered pair (5, 2) belongs to the relation

$$R = \{(x, y) : y = x - 5,\ x, y \in Z\}$$

Sol. ***False***

We have, $R = \{(x, y) : y = x - 5,\ x, y \in Z\}$

If $x = 5$, then $y = 5 - 5 = 0$

Hence, (5, 2) does not belong to R.

Q. 39 If $P = \{1, 2\}$, then $P \times P \times P = \{(1, 1, 1), (2, 2, 2), (1, 2, 2), (2, 1, 1)\}$

Sol. ***False***

We have, $P = \{1, 2\}$ and $n(P) = 2$

$\therefore$ $n(P \times P \times P) = n(P) \times n(P) \times n(P) = 2 \times 2 \times 2 = 8$

But given $P \times P \times P$ has 4 elements.

Q. 40 If $A = \{1, 2, 3\}$, $B = \{3, 4\}$ and $C = \{4, 5, 6\}$, then $(A \times B) \cup (A \times C)$
$= \{(1, 3), (1, 4), (1, 5), (1, 6), (2, 3), (2, 4), (2, 5), (2, 6), (3, 3), (3, 4), (3, 5), (3, 6)\}$.

Thinking Process

First, we find $A \times B$ and $A \times C$, then we will find $(A \times B) \cup (A \times C)$.

Sol. ***True***

We have, $A = \{1, 2, 3\}$, $B = \{3, 4\}$ and $C = \{4, 5, 6\}$

$A \times B = \{(1, 3), (1, 4), (2, 3), (2, 4), (3, 3), (3, 4)\}$

$A \times C = \{(1, 4), (1, 5), (1, 6), (2, 4), (2, 5), (2, 6), (3, 4), (3, 5), (3, 6)\}$

$(A \times B) \cup (A \times C) = \{(1, 3), (1, 4), (1, 5), (1, 6), (2, 3), (2, 4), (2, 5), (2, 6), (3, 3), (3, 4), (3, 5), (3, 6)\}$

Q. 41 If $(x - 2, y + 5) = \left(-2, \frac{1}{3}\right)$ are two equal ordered pairs, then $x = 4$, $y = \frac{-14}{3}$

Sol. ***False***

We have, $(x - 2, y + 5) = \left(-2, \frac{1}{3}\right)$

$\Rightarrow$ $x - 2 = -2,\ y + 5 = \frac{1}{3} \Rightarrow x = -2 + 2,\ y = \frac{1}{3} - 5$

$\therefore$ $x = 0,\ y = \frac{-14}{3}$

Q. 42 If $A \times B = \{(a, x), (a, y), (b, x), (b, y)\}$, then $A = \{a, b\}$ and $B = \{x, y\}$.

Sol. ***True***

We have, $A \times B = \{(a, x), (a, y), (b, x), (b, y)\}$

A = Set of first element of ordered pairs in $A \times B = \{a, b\}$

B = Set of second element of ordered pairs in $A \times B = \{x, y\}$

3

Trigonometric Functions

Short Answer Type Questions

Q. 1 Prove that $\dfrac{\tan A + \sec A - 1}{\tan A - \sec A + 1} = \dfrac{1 + \sin A}{\cos A}$.

Thinking Process

Here, use the formulae i.e., $\sec^2 A - \tan^2 A = 1$ and $a^2 - b^2 = (a+b)(a-b)$ to solve the above problem.

Sol.

$$\text{LHS} = \frac{\tan A + \sec A - 1}{\tan A - \sec A + 1}$$

$$= \frac{\tan A + \sec A - (\sec^2 A - \tan^2 A)}{(\tan A - \sec A + 1)} \qquad [\because \sec^2 A - \tan^2 A = 1]$$

$$= \frac{(\tan A + \sec A) - (\sec A + \tan A)(\sec A - \tan A)}{(1 - \sec A + \tan A)}$$

$$= \frac{(\sec A + \tan A)(1 - \sec A + \tan A)}{1 - \sec A + \tan A}$$

$$= \sec A + \tan A = \frac{1}{\cos A} + \frac{\sin A}{\cos A}$$

$$= \frac{1 + \sin A}{\cos A} = \text{RHS}$$

Hence proved.

Q. 2 If $\dfrac{2\sin\alpha}{1 + \cos\alpha + \sin\alpha} = y$, then prove that $\dfrac{1 - \cos\alpha + \sin\alpha}{1 + \sin\alpha}$ is also equal to y.

Sol. Given that, $\dfrac{2\sin\alpha}{1 + \cos\alpha + \sin\alpha} = y$

Now, $\dfrac{1 - \cos\alpha + \sin\alpha}{1 + \sin\alpha} = \dfrac{(1 - \cos\alpha + \sin\alpha)}{(1 + \sin\alpha)} \cdot \dfrac{(1 + \cos\alpha + \sin\alpha)}{(1 + \cos\alpha + \sin\alpha)}$

$$= \frac{\{(1 + \sin\alpha) - \cos\alpha\}}{(1 + \sin\alpha)} \cdot \frac{\{(1 + \sin\alpha) + \cos\alpha\}}{(1 + \cos\alpha + \sin\alpha)}$$

$$= \frac{(1 + \sin\alpha)^2 - \cos^2\alpha}{(1 + \sin\alpha)(1 + \sin\alpha + \cos\alpha)}$$

$$= \frac{(1 + \sin^2\alpha + 2\sin\alpha) - \cos^2\alpha}{(1 + \sin\alpha)(1 + \sin\alpha + \cos\alpha)}$$

$$= \frac{1 + \sin^2 \alpha + 2\sin\alpha - 1 + \sin^2 \alpha}{(1 + \sin\alpha)(1 + \sin\alpha + \cos\alpha)}$$

$$= \frac{2\sin^2 \alpha + 2\sin\alpha}{(1 + \sin\alpha)(1 + \sin\alpha + \cos\alpha)}$$

$$= \frac{2\sin\alpha(1 + \sin\alpha)}{(1 + \sin\alpha)(1 + \sin\alpha + \cos\alpha)}$$

$$= \frac{2\sin\alpha}{1 + \sin\alpha + \cos\alpha} = y$$

Hence proved.

Q. 3 If $m\sin\theta = n\sin(\theta + 2\alpha)$, then prove that $\tan(\theta + \alpha)\cot\alpha = \dfrac{m+n}{m-n}$.

Sol. Given that, $m\sin\theta = n\sin(\theta + 2\alpha)$

$\therefore$ $\dfrac{\sin(\theta + 2\alpha)}{\sin\theta} = \dfrac{m}{n}$

Using componendo and dividendo, we get

$$\frac{\sin(\theta + 2\alpha) + \sin\theta}{\sin(\theta + 2\alpha) - \sin\theta} = \frac{m+n}{m-n}$$

$$\Rightarrow \frac{2\sin\left(\frac{\theta + 2\alpha + \theta}{2}\right)\cdot\cos\left(\frac{\theta + 2\alpha - \theta}{2}\right)}{2\cos\left(\frac{\theta + 2\alpha + \theta}{2}\right)\cdot\sin\left(\frac{\theta + 2\alpha - \theta}{2}\right)} = \frac{m+n}{m-n}$$

$\left[\because \sin x + \sin y = 2\sin\frac{x+y}{2}\cdot\cos\frac{x-y}{2} \text{ and } \sin x - \sin y = 2\cos\frac{x+y}{2}\sin\cdot\frac{x-y}{2}\right]$

$$\Rightarrow \frac{\sin(\theta + \alpha)\cdot\cos\alpha}{\cos(\theta + \alpha)\cdot\sin\alpha} = \frac{m+n}{m-n}$$

$$\Rightarrow \tan(\theta + \alpha)\cdot\cot\alpha = \frac{m+n}{m-n}$$

Hence proved.

Q. 4 If $\cos(\alpha + \beta) = \dfrac{4}{5}$ and $\sin(\alpha - \beta) = \dfrac{5}{13}$, where α lie between 0 and $\dfrac{\pi}{4}$, then find that value of $\tan 2\alpha$.

Sol. Given that, $\cos(\alpha + \beta) = \dfrac{4}{5}$ and $\sin(\alpha - \beta) = \dfrac{5}{13}$

$$\Rightarrow \sin(\alpha + \beta) = \sqrt{1 - \frac{16}{25}} = \sqrt{\frac{9}{25}} = \pm\frac{3}{5}$$

$$\therefore \sin(\alpha + \beta) = \frac{3}{5}$$

and $$\cos(\alpha - \beta) = \sqrt{1 - \frac{25}{169}} = \sqrt{\frac{144}{169}} = \pm\frac{12}{13}$$

$$\therefore \cos(\alpha - \beta) = \frac{12}{13}$$

$\left[\text{since, } \alpha \text{ lies between 0 and } \frac{\pi}{4}\right]$

Now, $$\tan(\alpha + \beta) = \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)}$$

$$= \frac{\frac{3}{5}}{\frac{4}{5}} = \frac{3}{4}$$

and $$\tan(\alpha-\beta)=\frac{\sin(\alpha-\beta)}{\cos(\alpha-\beta)}=\frac{\frac{5}{13}}{\frac{12}{13}}=\frac{5}{12}$$

$\therefore$ $$\tan 2\alpha=\tan(\alpha+\beta+\alpha-\beta)$$

$$=\frac{\tan(\alpha+\beta)+\tan(\alpha-\beta)}{1-\tan(\alpha+\beta)\cdot\tan(\alpha-\beta)}\qquad\left[\because \tan(x\pm y)=\frac{\tan x\pm\tan y}{1\mp\tan x\cdot\tan y}\right]$$

$$=\frac{\frac{3}{4}+\frac{5}{12}}{1-\frac{3}{4}\cdot\frac{5}{12}}=\frac{\frac{9+5}{12}}{\frac{16-5}{16}}=\frac{14\times16}{12\times11}=\frac{56}{33}$$

Q. 5 If $\tan x=\frac{b}{a}$, then find the value of $\sqrt{\frac{a+b}{a-b}}+\sqrt{\frac{a-b}{a+b}}$.

Thinking Process

First of all rationalise the given expression and used the formula, i.e., $\cos 2x=\cos^2 x-\sin^2 x$.

Sol. Given that, $\tan x=\frac{b}{a}$

$$\therefore\quad \sqrt{\frac{a+b}{a-b}}+\sqrt{\frac{a-b}{a+b}}=\frac{\sqrt{(a+b)^2}+\sqrt{(a-b)^2}}{\sqrt{(a-b)(a+b)}}$$

$$=\frac{(a+b)+(a-b)}{\sqrt{a^2-b^2}}=\frac{2a}{\sqrt{a^2-b^2}}=\frac{2a}{a\sqrt{1-\left(\frac{b}{a}\right)^2}}\qquad\left[\because \frac{b}{a}=\tan x\right]$$

$$=\frac{2}{\sqrt{1-\tan^2 x}}=\frac{2\cos x}{\sqrt{\cos^2 x-\sin^2 x}}\qquad[\because \cos 2x=\cos^2 x-\sin^2 x]$$

$$=\frac{2\cos x}{\sqrt{\cos 2x}}$$

Q. 6 Prove that $\cos\theta\cos\frac{\theta}{2}-\cos 3\theta\cos\frac{9\theta}{2}=\sin 7\theta\sin 8\theta$.

Sol. $$\text{LHS}=\cos\theta\cos\frac{\theta}{2}-\cos 3\theta\cos\frac{9\theta}{2}$$

$$=\frac{1}{2}\left[2\cos\theta\cdot\cos\frac{\theta}{2}-2\cos 3\theta\cdot\cos\frac{9\theta}{2}\right]$$

$$=\frac{1}{2}\left[\cos\left(\theta+\frac{\theta}{2}\right)+\cos\left(\theta-\frac{\theta}{2}\right)-\cos\left(3\theta+\frac{9\theta}{2}\right)-\cos\left(3\theta-\frac{9\theta}{2}\right)\right]$$

$$=\frac{1}{2}\left(\cos\frac{3\theta}{2}+\cos\frac{\theta}{2}-\cos\frac{15\theta}{2}-\cos\frac{3\theta}{2}\right.$$

$$=\frac{1}{2}\left[\cos\frac{\theta}{2}-\cos\frac{15\theta}{2}\right]$$

$$=-\frac{1}{2}\left[2\sin\left(\frac{\theta+15\theta}{2}\right)\cdot\sin\left(\frac{\theta-15\theta}{2}\right)\right]\qquad\left[\because \cos x-\cos y=-2\sin\frac{x+y}{2}\cdot\sin\frac{x-y}{2}\right]$$

$$=+(\sin 8\theta\cdot\sin 7\theta)=\text{RHS}$$

$\therefore$ LHS = RHS **Hence proved.**

Q. 7 If $a\cos\theta + b\sin\theta = m$ and $a\sin\theta - b\cos\theta = n$, then show that $a^2 + b^2 = m^2 + n^2$.

Sol. Given that, $a\cos\theta + b\sin\theta = m$ ...(i)

and $a\sin\theta - b\cos\theta = n$ ...(ii)

On squaring and adding of Eqs. (i) and (ii), we get

$$m^2 + n^2 = (a\cos\theta + b\sin\theta)^2 + (a\sin\theta - b\cos\theta)^2$$
$$= a^2\cos^2\theta + b^2\sin^2\theta + 2ab\sin\theta\cdot\cos\theta + a^2\sin^2\theta + b^2\cos^2\theta - 2ab\sin\theta\cdot\cos\theta$$

$\Rightarrow$ $m^2 + n^2 = a^2(\cos^2\theta + \sin^2\theta) + b^2(\sin^2\theta + \cos^2\theta)$

$\Rightarrow$ $m^2 + n^2 = a^2 + b^2$ **Hence proved.**

Q. 8 Find the value of $\tan 22°30'$.

Sol. Let $\theta = 45°$

We know that, $\tan\frac{\theta}{2} = \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = \frac{2\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} \Rightarrow \tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta}$

$\therefore$ $\tan 22°30' = \frac{\sin 45°}{1+\cos 45°}$ $[\because \theta = 45°]$

$$= \frac{\frac{1}{\sqrt{2}}}{1+\frac{1}{\sqrt{2}}} = \frac{1}{\sqrt{2}+1}$$

Q. 9 Prove that $\sin 4A = 4\sin A\cos^3 A - 4\cos A\sin^3 A$.

Thinking Process

Here, apply the formula i.e., $\sin 2x = 2\sin x\cos x$ *and* $\cos 2x = \cos^2 x - \sin^2 x$

Sol.
$$\text{LHS} = \sin 4A$$
$$= 2\sin 2A\cdot\cos 2A$$
$$= 2(2\sin A\cdot\cos A)(\cos^2 A - \sin^2 A)$$
$$= 4\sin A\cdot\cos^3 A - 4\cos A\sin^3 A$$
$\left[\because \cos 2A = \cos^2 A - \sin^2 A \text{ and } \sin 2A = 2\sin A\cdot\cos A\right]$

$\therefore$ LHS = RHS **Hence proved.**

Q. 10 If $\tan\theta + \sin\theta = m$ and $\tan\theta - \sin\theta = n$, then prove that $m^2 - n^2 = 4\sin\theta\tan\theta$.

Sol. Given that, $\tan\theta + \sin\theta = m$ ...(i)

and $\tan\theta - \sin\theta = n$ ...(ii)

Now, $m + n = \tan\theta + \sin\theta + \tan\theta - \sin\theta$

$m + n = 2\tan\theta$ ...(iii)

Also, $m - n = \tan\theta + \sin\theta - \tan\theta + \sin\theta$

$m - n = 2\sin\theta$ ...(iv)

From Eqs. (iii) and (iv),

$$(m+n)(m-n) = 4\sin\theta\cdot\tan\theta$$
$$m^2 - n^2 = 4\sin\theta\cdot\tan\theta$$ **Hence proved.**

Q. 11 If $\tan(A+B)=p$ and $\tan(A-B)=q$, then show that $\tan 2A=\dfrac{p+q}{1-pq}$.

Sol. Given that $\tan(A+B)=p$ ...(i)

and $\tan(A-B)=q$ ...(ii)

$\therefore$ $\tan 2A=\tan(A+B+A-B)$

$$=\frac{\tan(A+B)+\tan(A-B)}{1-\tan(A+B)\tan(A-B)} \quad \left[\because \tan(x+y)=\frac{\tan x+\tan y}{1-\tan x\tan y}\right]$$

$$=\frac{p+q}{1-pq}$$ [from Eqs. (i) and (ii)]

Q. 12 If $\cos\alpha+\cos\beta=0=\sin\alpha+\sin\beta$, then prove that $\cos 2\alpha+\cos 2\beta=-2\cos(\alpha+\beta)$.

Sol. Given that, $\cos\alpha+\cos\beta=0=\sin\alpha+\sin\beta$

$\Rightarrow$ $(\cos\alpha+\cos\beta)^2-(\sin\alpha+\sin\beta)^2=0$

$\Rightarrow$ $\cos^2\alpha+\cos^2\beta+2\cos\alpha\cos\beta-\sin^2\alpha-\sin^2\beta-2\sin\alpha\sin\beta=0$

$\Rightarrow$ $\cos^2\alpha-\sin^2\alpha+\cos^2\beta-\sin^2\beta=2(\sin\alpha\sin\beta-\cos\alpha\cos\beta)$

$\Rightarrow$ $\cos 2\alpha+\cos 2\beta=-2\cos(\alpha+\beta)$ **Hence proved.**

Q. 13 If $\dfrac{\sin(x+y)}{\sin(x-y)}=\dfrac{a+b}{a-b}$, then show that $\dfrac{\tan x}{\tan y}=\dfrac{a}{b}$.

Sol. Given that, $\dfrac{\sin(x+y)}{\sin(x-y)}=\dfrac{a+b}{a-b}$

Using componendo and dividendo,

$$\Rightarrow \quad =\frac{\sin(x+y)+[\sin(x-y)]}{\sin(x+y)-\sin(x-y)}=\frac{a+b+a-b}{a+b-a+b}$$

$$\Rightarrow \quad =\frac{2\sin\left(\frac{x+y+x-y}{2}\right)\cdot\cos\left(\frac{x+y-x+y}{2}\right)}{2\cos\left(\frac{x+y+x-y}{2}\right)\cdot\sin\left(\frac{x+y-x+y}{2}\right)}=\frac{2a}{2b}$$

$$\left[\because \sin x+\sin y=2\sin\frac{x+y}{2}\cdot\cos\frac{x-y}{2} \text{ and } \sin x-\sin y=2\cos\frac{x+y}{2}\cdot\sin\frac{x-y}{2}\right]$$

$$\Rightarrow \quad =\frac{\sin x\cdot\cos y}{\cos x\cdot\sin y}=\frac{a}{b}$$

$$\Rightarrow \quad \frac{\tan x}{\tan y}=\frac{a}{b}$$

Q. 14 If $\tan\theta=\dfrac{\sin\alpha-\cos\alpha}{\sin\alpha+\cos\alpha}$, then show that $\sin\alpha+\cos\alpha=\sqrt{2}\cos\theta$.

Sol. Given that, $\tan\theta=\dfrac{\sin\alpha-\cos\alpha}{\sin\alpha+\cos\alpha}$

$$\Rightarrow \quad \tan\theta=\frac{\cos\alpha(\tan\alpha-1)}{\cos\alpha(\tan\alpha+1)}$$

$$\Rightarrow \quad \tan\theta=\frac{\tan\alpha-\tan\frac{\pi}{4}}{1+\tan\frac{\pi}{4}\cdot\tan\alpha} \quad \left[\because \tan\frac{\pi}{4}=1\right]$$

$$\Rightarrow \quad \tan\theta = \tan\left(\alpha - \frac{\pi}{4}\right)$$

$$\Rightarrow \quad \theta = \alpha - \frac{\pi}{4} \Rightarrow \alpha = \theta + \frac{\pi}{4}$$

$$\therefore \quad \sin\alpha + \cos\alpha = \sin\left(\theta + \frac{\pi}{4}\right) + \cos\left(\theta + \frac{\pi}{4}\right)$$

$$= \sin\theta\cdot\cos\frac{\pi}{4} + \cos\theta\cdot\sin\frac{\pi}{4} + \cos\theta\cdot\cos\frac{\pi}{4} - \sin\theta\cdot\sin\frac{\pi}{4}$$

$$= \frac{1}{\sqrt{2}}\sin\theta + \frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}\cos\theta - \frac{1}{\sqrt{2}}\sin\theta \quad \left[\because \sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}\right]$$

$$= \frac{2}{\sqrt{2}}\cdot\cos\theta = \sqrt{2}\cos\theta$$

Q. 15 If $\sin\theta + \cos\theta = 1$, then find the general value of θ.

Thinking Process

If $\sin\theta = \sin\alpha$, then $\theta = n\pi + (-1)^n\cdot\alpha$, gives general solution of the given equation.

Sol. Given that, $\sin\theta + \cos\theta = 1$

On squaring both sides, we get

$$\sin^2\theta + \cos^2\theta + 2\sin\theta\cdot\cos\theta = 1$$

$$\Rightarrow \quad 1 + 2\sin\theta\cdot\cos = 1 \quad [\because \sin 2x = 2\sin x\cos x]$$

$$\Rightarrow \quad \sin 2\theta = 0 \quad \Rightarrow \quad 2\theta = n\pi + (-1)^n\cdot 0$$

$$\therefore \quad \theta = \frac{n\pi}{2}$$

Alternate Method

$$\sin\theta + \cos\theta = 1$$

$$\Rightarrow \quad \frac{1}{\sqrt{2}}\cdot\sin\theta + \frac{1}{\sqrt{2}}\cdot\cos\theta = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \sin\theta\cdot\cos\frac{\pi}{4} + \cos\theta\cdot\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \quad \left[\because \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \cos\frac{\pi}{4}\right]$$

$$\Rightarrow \quad \sin\left(\theta + \frac{\pi}{4}\right) = \sin\frac{\pi}{4} \quad [\because \sin(x+y) = \sin x\cdot\cos y + \cos x\cdot\sin y]$$

$$\Rightarrow \quad \theta + \frac{\pi}{4} = n\pi + (-1)^n\frac{\pi}{4}$$

$$\therefore \quad \theta = n\pi + (-1)^n\frac{\pi}{4} - \frac{\pi}{4}$$

Q. 16 Find the most general value of θ satisfying the equation $\tan\theta = -1$ and $\cos\theta = \frac{1}{\sqrt{2}}$.

Sol. The given equations are

$$\tan\theta = -1 \quad \text{...(i)}$$

and $$\cos\theta = \frac{1}{\sqrt{2}} \quad \text{...(ii)}$$

From Eq. (i), $$\tan\theta = -\tan\frac{\pi}{4}$$

$$\Rightarrow \quad \tan\theta = \tan\left(2\pi - \frac{\pi}{4}\right) \quad \Rightarrow \quad \tan\theta = \tan\frac{7\pi}{4}$$

$$\therefore \quad \theta = \frac{7\pi}{4}$$

From Eq. (ii), $\cos\theta = \frac{1}{\sqrt{2}} \Rightarrow \cos\theta = \cos\frac{\pi}{4}$

$\Rightarrow \cos\theta = \cos\left(2\pi - \frac{\pi}{4}\right) \Rightarrow \cos\theta = \cos\frac{7\pi}{4}$

$\therefore \theta = \frac{7\pi}{4}$

Hence, the most general value of θ *i.e.*, $\theta = 2n\pi + \frac{7\pi}{4}$.

Q. 17 If $\cot\theta + \tan\theta = 2\operatorname{cosec}\theta$, then find the general value of θ.

Sol. Given that, $\cot\theta + \tan\theta = 2\operatorname{cosec}\theta$

$\Rightarrow \frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta} = \frac{2}{\sin\theta}$

$\Rightarrow \frac{\cos^2 + \sin^2\theta}{\sin\theta\cdot\cos\theta} = \frac{2}{\sin\theta}$

$\Rightarrow \frac{1}{\cos\theta} = 2$ $[\because \sin^2\theta + \cos^2\theta = 1]$

$\Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \cos\theta = \cos\frac{\pi}{3}$

$\therefore \theta = 2n\pi \pm \frac{\pi}{3}$

Q. 18 If $2\sin^2\theta = 3\cos\theta$, where $0 \le \theta \le 2\pi$, then find the value of θ.

Sol. Given that, $2\sin^2\theta = 3\cos\theta$

$\Rightarrow 2 - 2\cos^2\theta = 3\cos\theta$

$\Rightarrow 2\cos^2\theta + 3\cos\theta - = 0$

$\Rightarrow 2\cos^2\theta + 4\cos\theta - \cos\theta - 2 = 0$

$\Rightarrow 2\cos\theta(\cos\theta + 2) - 1(\cos\theta + 2) = 0$

$\Rightarrow (\cos\theta + 2)(2\cos\theta - 1) = 0$

$\Rightarrow \cos\theta = -2$ not possible $[\because -1 \le \cos\theta \le 1]$

$2\cos\theta = 1$

$\Rightarrow \cos\theta = \frac{1}{2}$

$\Rightarrow \cos\theta = \cos\frac{\pi}{3}$

$\therefore \theta = \frac{\pi}{3}$

Also, $\cos\theta = \cos\left(2\pi - \frac{\pi}{3}\right)$

$\Rightarrow \cos\theta = \cos\frac{5\pi}{6}$

$\therefore \theta = \frac{5\pi}{6}$

So, the values of θ are $\frac{\pi}{3}$ and $\frac{5\pi}{6}$.

Q. 19 If $\sec x \cos 5x + 1 = 0$, where $0 < x \le \frac{\pi}{2}$, then find the value of x.

Sol. Given that, $\sec x \cos 5x + 1 = 0$

$$\frac{\cos 5x}{\cos x} + 1 = 0 \Rightarrow \cos 5x + \cos x = 0$$

$$\Rightarrow 2\cos\left(\frac{5x + x}{2}\right)\cdot\cos\left(\frac{5x - x}{2}\right) = 0 \qquad \left[\because \cos x + \cos y = 2\cos\frac{x+y}{2}\cdot\cos\frac{x-y}{2}\right]$$

$$\Rightarrow 2\cos 3x \cdot \cos 2x = 0$$

$$\Rightarrow \cos 3x = 0 \text{ or } \cos 2x = 0$$

$$\Rightarrow \cos 3x = \cos\frac{\pi}{2} \text{ or } \cos 2x = \cos\frac{\pi}{2}$$

$$\therefore 3x = \frac{\pi}{2} \Rightarrow 2x = \frac{\pi}{2}$$

and $$x = \frac{\pi}{6} \Rightarrow x = \frac{\pi}{4}$$

Hence, the solutions are $\frac{\pi}{2}, \frac{\pi}{4}$ and $\frac{\pi}{6}$.

Long Answer Type Questions

Q. 20 If $\sin(\theta + \alpha) = a$ and $\sin(\theta + \beta) = b$, then prove that $\cos(\alpha + \beta) - 4ab\cos(\alpha - \beta) = 1 - 2a^2 - 2b^2$.

Thinking Process

Express $\cos(\alpha - \beta) = \cos(\theta + \alpha) - (\theta + \beta)$.

Sol. Given that, $\sin(\theta + \alpha) = a$ ...(i)

and $\sin(\theta + \beta) = b$ ...(ii)

$\therefore$ $\cos(\theta + \alpha) = \sqrt{1 - a^2}$ and $\cos(\theta + \beta) = \sqrt{1 - b^2}$

$\therefore$ $\cos(\alpha - \beta) = \cos\{\theta + \alpha - (\theta + \beta)\}$

$$= \cos(\theta + \beta)\cos(\theta + \alpha) + \sin(\theta + \alpha)\sin(\theta + \beta)$$

$$= \sqrt{1 - a^2}\sqrt{1 - b^2} + a \cdot b = ab + \sqrt{(1 - a^2)(1 - b^2)}$$

$$= ab + \sqrt{1 - a^2 - b^2 + a^2b^2}$$

and $\cos(\alpha - \beta) = ab + \sqrt{1 - a^2 - b^2 + a^2b^2}$

$$= \cos 2(\alpha - \beta) - 4ab\cos(\alpha - \beta)$$

$$= 2\cos^2(\alpha - \beta) - 1 - 4ab\cos(\alpha - \beta)$$

$$= 2\cos(\alpha - \beta)(\cos\alpha - \beta - 2ab) - 1$$

$$= 2(ab + \sqrt{1 - a^2 - b^2 + a^2b^2})(ab + \sqrt{1 - a^2 - b^2 + a^2b^2} - 2ab) - 1$$

$$= 2[(\sqrt{1 - a^2 - b^2 + a^2b^2} + ab)(\sqrt{1 - a^2 - b^2 + a^2b^2} - ab)] - 1$$

$$= 2[1 - a^2 - b^2 + a^2b^2 - a^2b^2] - 1$$

$$= 2 - 2a^2 - 2b^2 - 1$$

$$= 1 - 2a^2 - 2b^2$$

Hence proved.

Q. 21 If $\cos(\theta + \phi) = m\cos(\theta - \phi)$, then prove that $\tan\theta = \dfrac{1-m}{1+m}\cot\phi$.

Sol. Given that,
$$\cos(\theta + \phi) = m\cos(\theta - \phi)$$
$$\Rightarrow \quad \frac{\cos(\theta + \phi)}{\cos(\theta - \phi)} = \frac{m}{1}$$

Using componendo and dividendo rule,
$$\frac{\cos(\theta - \phi) - \cos(\theta + \phi)}{\cos(\theta - \phi) + \cos(\theta + \phi)} = \frac{1-m}{1+m}$$
$$\Rightarrow \quad \frac{-2\sin\left(\frac{\theta - \phi + \theta + \phi}{2}\right)\cdot\sin\left(\frac{\theta - \phi - \theta - \phi}{2}\right)}{2\cos\left(\frac{\theta - \phi + \theta + \phi}{2}\right)\cdot\cos\left(\frac{\theta - \phi - \theta - \phi}{2}\right)} = \frac{1-m}{1+m}$$
$$\Rightarrow \quad \frac{\sin\theta\cdot\sin\phi}{\cos\theta\cdot\cos\phi} = \frac{1-m}{1+m} \qquad \left[\begin{array}{l}\because \sin(-\theta) = -\sin\theta \\ \text{and} \cos(-\theta) = \cos\theta\end{array}\right]$$
$$\Rightarrow \quad \tan\theta\cdot\tan\phi = \frac{1-m}{1+m}$$
$$\Rightarrow \quad \tan\theta = \left(\frac{1-m}{1+m}\right)\cot\phi$$

Q. 22 Find the value of the expression
$$3\left[\sin^4\left(\frac{3\pi}{2} - \alpha\right) + \sin^4(3\pi + \alpha)\right] - 2\left[\sin^6\left(\frac{\pi}{2} + \alpha\right) + \sin^6(5\pi - \alpha)\right].$$

Sol. Given expression,
$$3\left[\sin^4\left(\frac{3\pi}{2} - \alpha\right) + \sin^4(3\pi + \alpha)\right] - 2\left[\sin^6\left(\frac{\pi}{2} + \alpha\right) + \sin^6(5\pi - \alpha)\right]$$
$$= 3[\cos^4\alpha + \sin^4(\pi + \alpha)] - 2[\cos^6\alpha + \sin^6(\pi - \alpha)]$$
$$= 3[\cos^4\alpha + \sin^4\alpha] - 2[\cos^6\alpha + \sin^6\alpha] = 3 - 2 = 1$$

Q. 23 If $a\cos 2\theta + b\sin 2\theta = c$ has α and β as its roots, then prove that
$$\tan\alpha + \tan\beta = \frac{2b}{a+c}.$$

Sol. Given that, $a\cos 2\theta + b\sin 2\theta = c$
$$\Rightarrow \quad a\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) + b\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = c \qquad \left[\because \sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta} \text{ and } \cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}\right]$$
$$\Rightarrow \quad a(1 - \tan^2\theta) + 2b\tan\theta = c(1 + \tan^2\theta)$$
$$\Rightarrow \quad a - a - \tan^2\theta + 2b\tan\theta = c + c\tan^2\theta$$
$$\Rightarrow \quad (a + c)\tan^2\theta - 2b\tan\theta + c - a = 0$$

Since, this equation has $\tan\alpha$ and $\tan\beta$ as its roots.
$$\because \quad \tan\alpha + \tan\beta = \frac{-(-2b)}{a+c} = \frac{2b}{a+c}$$

Q. 24 If $x = \sec\phi - \tan\phi$ and $y = \text{cosec}\,\phi + \cot\phi$, then show that $xy + x - y + 1 = 0$.

Sol. Given that, $x = \sec\phi - \tan\phi$ ...(i)

and $y = \text{cosec}\,\phi + \cot\phi$ ...(ii)

Now, $1\cdot xy = (\sec\phi - \tan\phi)(\text{cosec}\,\phi + \cot\phi)$

$\Rightarrow xy = \sec\phi\cdot\text{cosec}\,\phi - \text{cosec}\,\phi\cdot\tan\phi + \sec\phi\cdot\cot\phi - \tan\phi\cdot\cot\phi$

$\Rightarrow xy = \sec\phi\cdot\text{cosec}\,\phi - \dfrac{1}{\cos\phi} + \dfrac{1}{\sin\phi} - 1$

$\Rightarrow 1 + xy = \sec\phi\,\text{cosec}\,\phi - \sec\phi + \text{cosec}\,\phi$ ...(iii)

From Eqs. (i) and (ii), we get

$x - y = \sec\phi - \tan\phi - \text{cosec}\,\phi - \cot\phi$

$\Rightarrow x - y = \sec\phi - \text{cosec}\,\phi - \dfrac{\sin\phi}{\cos\phi} - \dfrac{\cos\phi}{\sin\phi}$

$\Rightarrow x - y = \sec\phi - \text{cosec}\,\phi - \left(\dfrac{\sin^2\phi + \cos^2\phi}{\sin\phi\cdot\cos\phi}\right)$

$\Rightarrow x - y = \sec\phi - \text{cosec}\,\phi - \dfrac{1}{\sin\phi\cdot\cos\phi}$

$\Rightarrow x - y = \sec\phi - \text{cosec}\,\phi - \text{cosec}\,\phi\cdot\sec\phi$

$\Rightarrow x - y = -(\sec\phi\cdot\text{cosec}\,\phi - \sec\phi + \text{cosec}\,\phi)$

$\Rightarrow x - y = -(xy + 1)$ [from Eq. (iii)]

$\Rightarrow xy + x - y + 1 = 0$ **Hence proved.**

Q. 25 If θ lies in the first quadrant and $\cos\theta = \dfrac{8}{17}$, then find the value of $\cos(30° + \theta) + \cos(45° - \theta) + \cos(120° - \theta)$.

Sol. Given that, $\cos 3\theta = \dfrac{8}{17} \Rightarrow \sin\theta = \sqrt{1 - \dfrac{64}{289}}$

$\Rightarrow \sin\theta = \sqrt{\dfrac{289 - 64}{289}} \Rightarrow \sin\theta = \pm\dfrac{15}{17}$

$\Rightarrow \sin\theta = \dfrac{15}{17}$ [since, θ lies in first quadrant]

Now, $\cos(30° + \theta) + \cos(45° - \theta) + \cos(120° - \theta)$

$= \cos(30° + \theta) + \cos(45° - \theta) + \cos(90° + 30° - \theta)$

$= \cos(30° + \theta) + \cos(45° - \theta) - \sin(30° - \theta)$

$= \cos 30°\cos\theta - \sin 30°\sin\theta + \cos 45°\cos\theta + \sin 45°\sin\theta - \sin 30°\cos\theta + \cos 30°\sin\theta$

$= \dfrac{\sqrt{3}}{2}\cos\theta - \dfrac{1}{2}\sin\theta + \dfrac{1}{\sqrt{2}}\cos\theta + \dfrac{1}{\sqrt{2}}\sin\theta - \dfrac{1}{2}\cos\theta\,\dfrac{\sqrt{3}}{2}\sin\theta$

$= \left(\dfrac{\sqrt{3}}{2} + \dfrac{1}{\sqrt{2}} - \dfrac{1}{2}\right)\cos\theta + \left(\dfrac{1}{\sqrt{2}} - \dfrac{1}{2} + \dfrac{\sqrt{3}}{2}\right)\sin\theta$

$= \left(\dfrac{\sqrt{6} + 2 - \sqrt{2}}{2\sqrt{2}}\right)\cos\theta + \left(\dfrac{2 - \sqrt{2} + \sqrt{6}}{2\sqrt{2}}\right)\sin\theta$

$= \left(\dfrac{\sqrt{6} + 2 - \sqrt{2}}{2\sqrt{2}}\right)\dfrac{8}{17} + \left(\dfrac{2 - \sqrt{2} + \sqrt{6}}{2\sqrt{2}}\right)\dfrac{15}{17}$

$$= \frac{1}{17(2\sqrt{2})}(8\sqrt{6} + 16 - 8\sqrt{2} + 30 - 15\sqrt{2} + 15\sqrt{6})$$

$$= \frac{1}{17(2\sqrt{2})}(23\sqrt{6} - 23\sqrt{2} + 46)$$

$$= \frac{23\sqrt{6}}{17(2\sqrt{2})} - \frac{23\sqrt{2}}{17(2\sqrt{2})} + \frac{46}{17(2\sqrt{2})}$$

$$= \frac{23\sqrt{3}}{17(2)} - \frac{23}{17(2)} + \frac{23}{17\sqrt{2}}$$

$$= \frac{23}{17}\left(\frac{\sqrt{3}-1}{2} + \frac{1}{\sqrt{2}}\right)$$

Q. 26 Find the value of $\cos^4\frac{\pi}{8} + \cos^4\frac{3\pi}{8} + \cos^4\frac{5\pi}{8} + \cos^4\frac{7\pi}{8}$.

Sol. Given expression, $\cos^4\frac{\pi}{8} + \cos^4\frac{3\pi}{8} + \cos^4\frac{5\pi}{8} + \cos^4\frac{7\pi}{8}$

$$= \cos^4\frac{\pi}{8} + \cos^4\frac{3\pi}{8} + \cos^4\left(\pi - \frac{3\pi}{8}\right) + \cos^4\left(\pi - \frac{\pi}{8}\right)$$

$$= \cos^4\frac{\pi}{8} + \cos^4\frac{3\pi}{8} + \cos^4\frac{3\pi}{8} + \cos^4\frac{\pi}{8}$$

$$= 2\left[\cos^4\frac{\pi}{8} + \cos^4\frac{3\pi}{8}\right] = 2\left[\cos^4\frac{\pi}{8} + \cos^4\left(\frac{\pi}{2} - \frac{\pi}{8}\right)\right]$$

$$= 2\left[\cos^4\frac{\pi}{8} + \sin^4\frac{\pi}{8}\right]$$

$$= 2\left[\left(\cos^2\frac{\pi}{8} + \sin^2\frac{\pi}{8}\right)^2 - 2\cos^2\frac{\pi}{8}\cdot\sin^2\frac{\pi}{8}\right]$$

$$= 2\left[1 - 2\cos^2\frac{\pi}{8}\cdot\sin^2\frac{\pi}{8}\right] = 2 - \left(2\sin\frac{\pi}{8}\cdot\cos\frac{\pi}{8}\right)^2$$

$$= 2 - \left(\sin\frac{2\pi}{8}\right)^2 = 2 - \left(\frac{1}{\sqrt{2}}\right)^2$$

$$= 2 - \frac{1}{2} = \frac{3}{2}$$

Q. 27 Find the general solution of the equation $5\cos^2\theta + 7\sin^2\theta - 6 = 0$.

Sol. Given equation, $5\cos^2\theta + 7\sin^2\theta - 6 = 0$

$\Rightarrow \quad 5\cos^2 + 7(1 - \cos^2\theta) - 6 = 0$

$\Rightarrow \quad 5\cos^2\theta + 7 - 7\cos^2\theta - 6 = 0$

$\Rightarrow \quad 5\cos^2\theta + 7 - 7\cos^2\theta - 6 = 0 \quad \Rightarrow \quad -2\cos^2\theta + 1 = 0$

$\Rightarrow \quad 2\cos^2\theta - 1 = 0 \quad \left[\because \cos^2\theta = \cos^2\alpha \;\; \therefore \theta = n\pi \pm \alpha\right]$

$\Rightarrow \quad \cos^2\theta = \frac{1}{2}$

$\Rightarrow \quad \cos^2\theta = \cos^2\frac{\pi}{4}$

$\therefore \quad \theta = n\pi \pm \frac{\pi}{4}$

Q. 28 Find the general of the equation $\sin x - 3\sin 2x + \sin 3x = \cos x - 3\cos 2x + \cos 3x$.

Sol. Given equation, $\sin x - 3\sin 2x + \sin 3x = \cos x - 3\cos 2x + \cos 3x$

$$\Rightarrow \quad 2\sin\left(\frac{x+3x}{2}\right)\cdot\cos\left(\frac{3x-x}{2}\right) - 3\sin 2x = 2\cos\left(\frac{3x+x}{2}\right)\cdot\cos\left(\frac{3x-x}{2}\right) - 3\cos 2x$$

$$\Rightarrow \quad 2\sin 2x\cos x - 3\sin 2x = 2\cos 2x\cdot\cos x - 3\cos 2x$$

$$\Rightarrow \quad \sin 2x(2\cos x - 3) = \cos 2x(2\cos x - 3)$$

$$\Rightarrow \quad \frac{\sin 2x}{\cos 2x} = 1$$

$$\Rightarrow \quad \tan 2x = 1$$

$$\Rightarrow \quad \tan 2x = \tan\frac{\pi}{4}$$

$$\Rightarrow \quad 2x = n\pi + \frac{\pi}{4}$$

$$\therefore \quad x = \frac{n\pi}{2} + \frac{\pi}{8}$$

Q. 29 Find the general solution of the equation

$$(\sqrt{3}-1)\cos\theta + (\sqrt{3}+1)\sin\theta = 2$$

Sol. Given equation is,

$$(\sqrt{3}-1)\cos\theta + (\sqrt{3}+1)\sin\theta = 2 \quad \ldots(i)$$

Put $\sqrt{3}-1 = r\sin\alpha$ and $\sqrt{3}+1 = r\cos\alpha$

$$\therefore \quad r^2 = (\sqrt{3}-1)^2 + (\sqrt{3}+1)^2$$

$$\Rightarrow \quad = 3 + 1 - 2\sqrt{3} + 3 + 1 + 2\sqrt{3}$$

$$\Rightarrow \quad r^2 = 8$$

$$\therefore \quad r = 2\sqrt{2}$$

now, $$\tan\alpha = \frac{\sqrt{3}-1}{\sqrt{3}+1} = \frac{\tan\frac{\pi}{3} - \tan\frac{\pi}{4}}{1 + \tan\frac{\pi}{3}\cdot\frac{\pi}{4}}$$

$$\Rightarrow \quad \tan\alpha = \tan\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$$

$$\Rightarrow \quad \tan\alpha = \tan\frac{\pi}{12}$$

$$\therefore \quad \alpha = \frac{\pi}{12}$$

From Eq. (i), $r\sin\alpha\cos\theta + r\cos\alpha\sin\theta = 2$

$$r[\sin(\theta+\alpha)] = 2$$

$$\Rightarrow \quad \sin(\theta+\alpha) = \frac{2}{2\sqrt{2}}$$

$$\Rightarrow \quad \sin(\theta+\alpha) = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \sin(\theta+\alpha) = \sin\frac{\pi}{4} \quad \theta + \alpha = n\pi + (-1)^n\frac{\pi}{4}$$

$$\theta = n\pi + (-1)^n\cdot\frac{\pi}{4} - \frac{\pi}{12}$$

Alternate Method

$$(\sqrt{3}-1)\cos\theta + (\sqrt{3}+1)\sin\theta = 2 \quad \ldots\text{(i)}$$

Put $\sqrt{3}-1 = r\cos\alpha$ and $\sqrt{3}+1 = r\sin\alpha$

$\therefore$ $r = 2\sqrt{2}$

Now, $$\tan\alpha = \frac{\sqrt{3}+1}{\sqrt{3}-1} = \frac{1+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}}$$

$\Rightarrow$ $$\tan\alpha = \frac{\tan\frac{\pi}{4}+\tan\frac{\pi}{6}}{1-\tan\frac{\pi}{4}\cdot\tan\frac{\pi}{6}}$$

$\Rightarrow$ $$\tan\alpha = \tan\left(\frac{\pi}{4}+\frac{\pi}{6}\right) \Rightarrow \tan\alpha = \tan\frac{5\pi}{12}$$

$\therefore$ $$\alpha = \frac{5\pi}{12}$$

From Eq. (i), $r\cos\alpha\cos\theta + r\sin\alpha\sin\theta = 2$

$$r[\cos(\theta-\alpha)] = 2$$

$\Rightarrow$ $$\cos(\theta-\alpha) = \frac{2}{2\sqrt{2}}$$

$\Rightarrow$ $$\cos(\theta-\alpha) = \frac{1}{\sqrt{2}}$$

$\Rightarrow$ $$\cos(\theta-\alpha) = \cos\frac{\pi}{4}$$

$\Rightarrow$ $$\theta-\alpha = 2n\pi \pm \frac{\pi}{4}$$

$\therefore$ $$\theta = 2n\pi \pm \frac{\pi}{4} + \frac{5\pi}{12}$$

Objective Type Questions

Q. 30 If $\sin\theta + \operatorname{cosec}\theta = 2$, then $\sin^2\theta + \operatorname{cosec}^2\theta$ is equal to

(a) 1 (b) 4 (c) 2 (d) None of these

Sol. *(c)* Given that, $\sin\theta + \operatorname{cosec}\theta = 2$

$\Rightarrow$ $$\sin^2\theta + \operatorname{cosec}^2\theta + 2\sin\theta\cdot\operatorname{cosec}\theta = 4$$

$\Rightarrow$ $$\sin^2\theta + \operatorname{cosec}^2\theta = 4-2$$

$\Rightarrow$ $$\sin^2\theta + \operatorname{cosec}^2\theta = 2$$

Q. 31 If $f(x) = \cos^2 x + \sec^2 x$, then

(a) $f(x) < 1$ (b) $f(x) = 1$ (c) $2 < f(x) < 1$ (d) $f(x) \geq 2$

Sol. *(d)* Given that, $f(x) = \cos^2 x + \sec^2 x$

We know that, AM $\geq$ GM

$$\frac{\cos^2 x + \sec^2 x}{2} \geq \sqrt{\cos^2 x\cdot\sec^2 x}$$

$\Rightarrow$ $$\cos^2 x + \sec^2 x \geq 2 \quad [\because \cos x\cdot\sec x = 1]$$

$\Rightarrow$ $$f(x) \geq 2$$

Q. 32 If $\tan\theta = \frac{1}{2}$ and $\tan\phi = \frac{1}{3}$, then the value of $\theta + \phi$ is

(a) $\frac{\pi}{6}$ (b) π (c) 0 (d) $\frac{\pi}{4}$

Sol. *(d)* Given that, $\tan\theta = \frac{1}{2}$ and $\tan\phi = \frac{1}{3}$

Now, $\tan(\theta + \phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta \cdot \tan\phi}$

$\Rightarrow$ $\tan(\theta + \phi) = \frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{2} \cdot \frac{1}{3}} \Rightarrow \tan(\theta + \phi) = \frac{\frac{3+2}{6}}{\frac{6-1}{6}} = \frac{5}{5} = 1$

$\Rightarrow$ $\tan(\theta + \phi) = \tan\frac{\pi}{4}$

$\therefore$ $\theta + \phi = \frac{\pi}{4}$

Q. 33 Which of the following is not correct?

(a) $\sin\theta = -\frac{1}{5}$ (b) $\cos\theta = 1$ (c) $\sec\theta - \frac{1}{2}$ (d) $\tan\theta = 20$

Sol. *(c)* We know that, the range of $\sec\theta$ is $R - (-1, 1)$.

Hence, $\sec\theta$ cannot be equal to $\frac{1}{2}$.

Q. 34 The value of $\tan 1° \tan 2° \tan 3° \dots \tan 89°$ is

(a) 0 (b) 1 (c) $\frac{1}{2}$ (d) Not defined

Sol. *(b)* Given expression, $\tan 1° \tan 2° \tan 3° \dots \tan 89°$

$= \tan 1° \tan 2° \dots \tan 45° \cdot \tan(90° - 44°) \tan(90° - 43°) \dots \tan(90° - 1°)$

$= \tan 1° \cdot \cot 1° \cdot \tan 2° \cdot \cot 2° \dots \tan 89° \cdot \cot 89°$

$= 1 \cdot 1 \dots 1 \cdot 1 = 1$

Q. 35 The value of $\frac{1 - \tan^2 15°}{1 + \tan^2 15°}$ is

(a) 1 (b) $\sqrt{3}$ (c) $\frac{\sqrt{3}}{2}$ (d) 2

Sol. *(c)* Given expression, $\frac{1 - \tan^2 15°}{1 + \tan^2 15°}$

Let $\theta = 15°$

We know that, $\cos 2\theta = \frac{1 - \tan^2\theta}{1 + \tan^2\theta}$

$\therefore$ $\cos 30° = \frac{1 - \tan^2 15°}{1 + \tan^2 15°}$

$\Rightarrow$ $\frac{1 - \tan^2 15°}{1 + \tan^2 15°} = \frac{\sqrt{3}}{2}$ $\left[\because \cos 30° = \frac{\sqrt{3}}{2}\right]$

Q. 36 The value of $\cos 1° \cos 2° \cos 3° \dots \cos 179°$ is

(a) $\frac{1}{\sqrt{2}}$ (b) 0 (c) 1 (d) -1

Sol. ***(b)*** Given expression, $\cos 1° \cos 2° \cos 3° \dots \cos 179°$

$$= \cos 1° \cos 2° \dots \cos 90° \dots \cos 179° \quad [\because \cos 90° = 0]$$
$$= 0$$

Q. 37 If $\tan\theta = 3$ and θ lies in third quadrant, then the value of $\sin\theta$ is

(a) $\frac{1}{\sqrt{10}}$ (b) $-\frac{1}{\sqrt{10}}$ (c) $\frac{-3}{\sqrt{10}}$ (d) $\frac{3}{\sqrt{10}}$

Sol. ***(c)*** Given that, $\tan\theta = 3$

$\Rightarrow$ $\sec^2\theta = 1 + \tan^2\theta$

$\Rightarrow$ $\sec\theta = \sqrt{1 + 9} = \pm\sqrt{10}$

$\Rightarrow$ $\sec\theta = -\sqrt{10}$

$\Rightarrow$ $\cos\theta = -\frac{1}{\sqrt{10}}$

$\Rightarrow$ $\sin\theta = \pm\sqrt{1 - \frac{1}{10}} = \pm\sqrt{\frac{9}{10}} = \pm\frac{3}{\sqrt{10}}$ [since, θ lies in third quadrant]

$\therefore$ $\sin\theta = -\frac{3}{\sqrt{10}}$

Q. 38 The value of $\tan 75° - \cot 75°$ is

(a) $2\sqrt{3}$ (b) $2 + \sqrt{3}$ (c) $2 - \sqrt{3}$ (d) 1

Sol. ***(a)*** Given expression, $\tan 75° - \cot 75°$

$$= \frac{\sin 75°}{\cos 75°} - \frac{\cos 75°}{\sin 75°}$$
$$= \frac{\sin^2 75° - \cos^2 75°}{\sin 75° \cdot \cos 75°}$$
$$= \frac{-2\cos 150°}{\sin 150°}$$
$$= \frac{-2\cos(90° + 60°)}{\sin(90° + 60°)}$$
$$= \frac{+2\sin 60°}{\cos 60°}$$
$$= \frac{2 \cdot \frac{\sqrt{3}}{2}}{\frac{1}{2}} = 2\sqrt{3}$$

Q. 39 Which of the following is correct?

(a) $\sin 1° > \sin 1$ (b) $\sin 1° < \sin 1$

(c) $\sin 1° = \sin 1$ (d) $\sin 1° = \frac{\pi}{18°} \sin 1$

Sol. ***(b)*** We know that, if θ is increasing, then $\sin\theta$ is also increasing.

$\therefore$ $\sin 1° < \sin 1$ $[\because 1 \text{ rad} = 57° 30']$

Q. 40 If $\tan\alpha = \dfrac{m}{m+1}$ and $\tan\beta = \dfrac{1}{2m+1}$, then $\alpha + \beta$ is equal to

(a) $\dfrac{\pi}{2}$ (b) $\dfrac{\pi}{3}$ (c) $\dfrac{\pi}{6}$ (d) $\dfrac{\pi}{4}$

Sol. *(d)* Given that, $\tan\alpha = \dfrac{m}{m+1}$ and $\tan\beta = \dfrac{1}{2m+1}$

Now, $$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha \cdot \tan\beta}$$

$$\Rightarrow \quad \tan(\alpha+\beta) = \frac{\dfrac{m}{m+1} + \dfrac{1}{2m+1}}{1 - \left(\dfrac{m}{m+1}\right)\left(\dfrac{1}{2m+1}\right)}$$

$$\Rightarrow \quad \tan(\alpha+\beta) = \frac{m(2m+1) + m + 1}{(m+1)(2m+1) - m}$$

$$\Rightarrow \quad \tan(\alpha+\beta) = \frac{2m^2 + m + m + 1}{2m^2 + 2m + m + 1 - m}$$

$$\Rightarrow \quad \tan(\alpha+\beta) = \frac{2m^2 + 2m + 1}{2m^2 + 2m + 1} \Rightarrow \tan(\alpha+\beta) = 1$$

$$\therefore \quad \alpha + \beta = \frac{\pi}{4}$$

Q. 41 The minimum value of $3\cos x + 4\sin x + 8$ is

(a) 5 (b) 9 (c) 7 (d) 3

Thinking Process

For the expression $A\cos\theta + B\sin\theta$, then the minimum value is $-\sqrt{A^2 + B^2}$.

Sol. *(d)* Given expression, $3\cos x + 4\sin x + 8$

Let $y = 3\cos x + 4\sin x + 8$

$\Rightarrow$ $y - 8 = 3\cos x + 4\sin x$

$\therefore$ Minimum value of $y - 8 = -\sqrt{9+16}$

$\Rightarrow$ $y - 8 = -5 \Rightarrow y = -5 + 8$

$\therefore$ $y = 3$

Hence, the minimum value of $3\cos x + 4\sin x + 8$ is 3.

Q. 42 The value of $\tan 3A - \tan 2A - \tan A$ is

(a) $\tan 3A \tan 2A \tan A$

(b) $-\tan 3A \tan 2A \tan A$

(c) $\tan A \tan 2A - \tan 2A \tan 3A - \tan 3A \tan A$

(d) None of the above

Sol. *(a)* Let $3A = A + 2A$

$$\tan 3A = \tan(A + 2A)$$

$$\Rightarrow \quad \tan 3A = \frac{\tan A + \tan 2A}{1 - \tan A \cdot \tan 2A}$$

$$\Rightarrow \quad \tan A + \tan 2A = \tan 3A - \tan 3A \cdot \tan 2A \cdot \tan A$$

$$\Rightarrow \quad \tan 3A - \tan 2A - \tan A = \tan 3A \cdot \tan 2A \cdot \tan A$$

Q. 43 The value of $\sin(45° + \theta) - \cos(45° - \theta)$ is

(a) $2\cos\theta$ (b) $2\sin\theta$ (c) 1 (d) 0

Thinking Process

Use formula i.e., $\sin(A+B) = \sin A\cos B + \cos A\sin B$

and $\cos(A-B) = \cos A\cos B + \sin A\sin B$.

Sol. *(d)* Given expression,

$\sin(45° + \theta) - \cos(45° - \theta)$

$$= \sin 45° \cdot \cos\theta + \cos 45° \cdot \sin\theta - \cos 45° \cdot \cos\theta - \sin 45° \cdot \sin\theta$$

$$= \frac{1}{\sqrt{2}} \cdot \cos\theta + \frac{1}{\sqrt{2}} \cdot \sin\theta - \frac{1}{\sqrt{2}} \cdot \cos\theta - \frac{1}{\sqrt{2}}\sin\theta$$

$$= 0$$

Q. 44 The value of $\cot\left(\frac{\pi}{4} + \theta\right)\cot\left(\frac{\pi}{4} - \theta\right)$ is

(a) –1 (b) 0 (c) 1 (d) Not defined

Thinking Process

Use formula i.e., $\cot(A+B) = \left(\frac{\cot A\cot B - 1}{\cot A + \cot B}\right)$ *and* $\cot(A-B) = \left(\frac{\cot A\cot B + 1}{\cot A - \cot B}\right)$.

Sol. *(c)* Given expression, $\cot\left(\frac{\pi}{4} + \theta\right) - \cot\left(\frac{\pi}{4} - \theta\right)$

$$= \left(\frac{\cot\frac{\pi}{4}\cot\theta - 1}{\cot\frac{\pi}{4} + \cot\theta}\right) \cdot \left(\frac{\cot\frac{\pi}{4}\cot\theta + 1}{\cot\theta - \cot\frac{\pi}{4}}\right)$$

$$= \left(\frac{\cot\theta - 1}{\cot\theta + 1}\right) \cdot \left(\frac{\cot\theta + 1}{\cot\theta - 1}\right)$$

$$= 1$$

Q. 45 $\cos 2\theta \cos 2\phi + \sin^2(\theta - \phi) - \sin^2(\theta + \phi)$ is equal to

(a) $\sin 2(\theta + \phi)$ (b) $\cos 2(\theta + \phi)$ (c) $\sin 2(\theta - \phi)$ (d) $\cos 2(\theta - \phi)$

Sol. *(b)* Given expression, $\cos 2\theta \cos 2\phi + \sin^2(\theta - \phi) - \sin^2(\theta + \phi)$

$$= \cos 2\theta \cdot \cos 2\phi + \sin(\theta - \phi + \theta + \phi) \cdot \sin(\theta - \phi - \theta - \phi)$$

$$= \cos 2\theta \cdot \cos 2\phi - \sin 2\theta \cdot \sin 2\phi$$

$$= \cos(2\theta + 2\phi) = \cos 2(\theta + \phi)$$

Q. 46 The value of $\cos 12° + \cos 84° + \cos 156° + \cos 132°$ is

(a) $\frac{1}{2}$ (b) 1 (c) $-\frac{1}{2}$ (d) $\frac{1}{8}$

Thinking Process

Use the formula $\cos A + \cos B = 2\cos\frac{A+B}{2} \cdot \cos\frac{A-B}{2}$ *and*

$\cos A - \cos B = -2\sin\frac{A+B}{2} \cdot \sin\frac{A-B}{2}$ *to solve this problem.*

Sol. *(c)* Given expression, $\cos 12° + \cos 84° + \cos 150° + \cos 132°$

$$= \cos 12° + \cos 150° + \cos 84° + \cos 132°$$

$$= 2\cos\left(\frac{12° + 150°}{2}\right)\cdot\cos\left(\frac{12° - 150°}{2}\right) + 2\cos\left(\frac{84° + 132°}{2}\right)\cdot\cos\left(\frac{84° - 132°}{2}\right)$$

$$= 2\cos 84° \cos 72° + 2\cos 108° \cdot \cos 24°$$

$$= 2\cos 84° \cos(90° - 18°) + 2\cos(90° + 18°)\cdot\cos 24°$$

$$= 2\cos 84° \sin 18° - 2\sin 18° \cdot \cos 24°$$

$$= 2\sin 18° (\cos 84° - \cos 24°)$$

$$= 2\sin 18° \cdot 2\sin\left(\frac{84° + 24°}{2}\right)\cdot\sin\left(\frac{84° - 24°}{2}\right)$$

$$= -4\sin 18° \cdot \sin 54° \sin 30°$$

$$= -4\left(\frac{\sqrt{5} - 1}{4}\right)\cdot\cos 36° \cdot \frac{1}{2}$$

$$= -(\sqrt{5} - 1)\left(\frac{\sqrt{5} + 1}{4}\right)\cdot\frac{1}{2} = -\left(\frac{5 - 1}{8}\right) = \frac{-4}{8} = \frac{-1}{2}$$

Q. 47 If $\tan A = \frac{1}{2}$ and $\tan B = \frac{1}{3}$, then $\tan(2A + B)$ is equal to

(a) 1 (b) 2 (c) 3 (d) 4

Sol. *(c)* Given that, $\tan A = \frac{1}{2}$ and $\tan B = \frac{1}{3}$

Now, $$\tan(2A + B) = \frac{\tan 2A + \tan B}{1 - \tan 2A \cdot \tan B} \quad \dots(i)$$

Also, $$\tan 2A = \frac{2\tan A}{1 - \tan^2 A} = \frac{2 \cdot \frac{1}{2}}{1 - \frac{1}{4}} = \frac{4}{3}$$

From Eq. (i), $$\tan(2A + B) = \frac{\frac{4}{3} + \frac{1}{3}}{1 - \frac{4}{3}\cdot\frac{1}{3}} = \frac{\frac{4}{3} + \frac{1}{3}}{\frac{9 - 4}{9}} = \frac{\frac{5}{3}}{\frac{5}{9}} = 3$$

Q. 48 The value of $\sin\frac{\pi}{10}\sin\frac{13\pi}{10}$ is

(a) $\frac{1}{2}$ (b) $-\frac{1}{2}$ (c) $-\frac{1}{4}$ (d) 1

Sol. *(c)* Given expression, $\sin\frac{\pi}{10}\sin\frac{13\pi}{10} = \sin\frac{\pi}{10}\sin\left(\pi + \frac{3\pi}{10}\right)$

$$= -\sin\frac{\pi}{10}\sin\frac{3\pi}{10} = -\sin 18° \cdot \sin 54°$$

$$= -\sin 18° \cdot \cos 36°$$

$$= -\left(\frac{\sqrt{5} - 1}{4}\right)\left(\frac{\sqrt{5} + 1}{4}\right) \quad \text{[since, put this value here]}$$

$$= -\left(\frac{5 - 1}{16}\right) = -\frac{1}{4}$$

Q. 49 The value of $\sin 50° - \sin 70° + \sin 10°$ is

(a) 1 (b) 0 (c) $\frac{1}{2}$ (d) 2

Thinking Process

Here, use the formula i.e., $\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\cdot \sin\left(\frac{A-B}{2}\right)$ *also* $\sin(-\theta) = -\sin\theta$

.

Sol. *(b)* Given expression, $\sin 50° - \sin 70° + \sin 10°$

$$= 2\cos\left(\frac{50° + 70°}{2}\right)\cdot \sin\left(\frac{50° - 70°}{2}\right) + \sin 10°$$

$$= -2\cos 60° \sin 10° + \sin 10°$$

$$= -2\cdot\frac{1}{2}\sin 10° + \sin 10° = 0$$

Q. 50 If $\sin\theta + \cos\theta = 1$, then the value of $\sin 2\theta$ is

(a) 1 (b) $\frac{1}{2}$ (c) 0 (d) -1

Sol. *(c)* Given that, $\sin\theta + \cos\theta = 1$

On squaring both sides, we get

$$\sin^2\theta + \cos^2\theta + 2\sin\theta\cdot\cos\theta = 1$$

$$\Rightarrow \quad 1 + \sin 2\theta = 1$$

$$\therefore \quad \sin 2\theta = 0$$

Q. 51 If $\alpha + \beta = \frac{\pi}{4}$, then the value of $(1 + \tan\alpha)(1 + \tan\beta)$ is

(a) 1 (b) 2 (c) -2 (d) Not defined

Thinking Process

Formula i.e., $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\cdot\tan B}$ *to solve this problem.*

Sol. *(b)* Given that, $\alpha + \beta = \frac{\pi}{4}$

Now, $(1 + \tan\alpha)(1 + \tan\beta) = 1 + \tan\alpha + \tan\beta + \tan\alpha\tan\beta$ ...(i)

We know that, $\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\cdot\tan\beta}$

$$\Rightarrow \quad 1 = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\cdot\tan\beta}$$

$$\Rightarrow \quad \tan\alpha + \tan\beta = 1 - \tan\alpha\tan\beta$$

From Eq. (i),

$$(1 + \tan\alpha)(1 + \tan\beta) = 1 + 1 - \tan\alpha\cdot\tan\beta + \tan\alpha\cdot\tan\beta$$

$$= 2$$

Q. 52 If $\sin\theta = \frac{-4}{5}$ and θ lies in third quadrant, then the value of $\cos\frac{\theta}{2}$ is

(a) $\frac{1}{5}$ (b) $-\frac{1}{\sqrt{10}}$ (c) $-\frac{1}{\sqrt{5}}$ (d) $\frac{1}{\sqrt{10}}$

Thinking Process

Use $\cos\theta = \sqrt{1-\sin^2\theta}$ *and* $\cos\theta = 2\cos^2\frac{\theta}{2} - 1.$

Sol. *(c)* Given that, $\sin\theta = \frac{-4}{5}$

$$\cos\theta = \sqrt{1-\frac{16}{25}} = \sqrt{\frac{25-16}{25}} = \pm\frac{3}{5}$$

$$\cos\theta = \frac{-3}{5} \quad \text{[since, } \theta \text{ lies in third quadrant]}$$

$$\Rightarrow 2\cos^2\frac{\theta}{2} - 1 = \frac{-3}{5}$$

$$\Rightarrow 2\cos^2\frac{\theta}{2} = 1 - \frac{3}{5}$$

$$\Rightarrow 2\cos^2\frac{\theta}{2} = \frac{2}{5}$$

$$\therefore \cos\frac{\theta}{2} = \pm\frac{1}{\sqrt{5}}$$

$$\Rightarrow \cos\frac{\theta}{2} = -\frac{1}{\sqrt{5}} \quad \text{[since, } \theta \text{ lies in third quadrant]}$$

Q. 53 The number of solutions of equation $\tan x + \sec x = 2\cos x$ lying in the interval $[0, 2\pi]$ is

(a) 0 (b) 1 (c) 2 (d) 3

Sol. *(c)* Given equation, $\tan + \sec x = 2\cos x$

$$\Rightarrow \frac{\sin x}{\cos x} + \frac{1}{\cos x} = 2\cos x$$

$$\Rightarrow 1 + \sin x = 2\cos^2 x$$

$$\Rightarrow 1 + \sin x = 2(1 - \sin^2 x)$$

$$\Rightarrow 1 + \sin x = 2 - 2\sin^2 x$$

$$\Rightarrow 2\sin^2 x + \sin x - 1 = 0$$

$$\Rightarrow 2\sin^2 x + 2\sin x - \sin x - 1 = 0$$

$$\Rightarrow 2\sin x(\sin x + 1) - 1(\sin x + 1) = 0$$

$$\Rightarrow (\sin x + 1)(2\sin x - 1) = 0$$

$$\Rightarrow \sin x + 1 = 0 \text{ or } (2\sin x - 1) = 0$$

$$\Rightarrow \sin x = -1, \sin x = \frac{1}{2}$$

$$\therefore x = \frac{3\pi}{2}, x = \frac{\pi}{6}$$

Hence, only two solutions possible.

Q. 54 The value of $\sin\frac{\pi}{18}+\sin\frac{\pi}{9}+\sin\frac{2\pi}{9}+\sin\frac{5\pi}{18}$ is

(a) $\sin\frac{7\pi}{18}+\sin\frac{4\pi}{9}$
(b) 1
(c) $\cos\frac{\pi}{6}+\cos\frac{3\pi}{7}$
(d) $\cos\frac{\pi}{9}+\sin\frac{\pi}{9}$

Thinking Process

Here, apply the formulae i.e., $\sin A+\sin B=2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$.

Sol. ***(a)*** Given expression, $\sin\frac{\pi}{18}+\sin\frac{\pi}{9}+\sin\frac{2\pi}{9}+\sin\frac{5\pi}{18}$

$$= \sin 10^\circ + \sin 20^\circ + \sin 40^\circ + \sin 50^\circ$$
$$= \sin 50^\circ + \sin 10^\circ + \sin 40^\circ + \sin 20^\circ$$
$$= \sin 130^\circ + \sin 10^\circ + \sin 140^\circ + \sin 20^\circ$$
$$= 2\sin 70^\circ\cos 60^\circ + 2\sin 80^\circ\cdot\cos 60^\circ \qquad \left[\because \sin x+\sin y = 2\sin\frac{x+y}{2}\cdot\cos\frac{x-y}{2}\right]$$
$$= 2\cdot\frac{1}{2}\sin 70^\circ + 2\cdot\frac{1}{2}\sin 80^\circ \qquad \left[\because \cos 60^\circ = \frac{1}{2}\right]$$
$$= \sin 70^\circ + \sin 80^\circ = \sin\frac{7\pi}{18}+\sin\frac{4\pi}{9}$$

Q. 55 If A lies in the second quadrant and $3\tan A+4=0$, then the value of $2\cot A-5\cos A+\sin A$ is

(a) $\frac{-53}{10}$
(b) $\frac{23}{10}$
(c) $\frac{37}{10}$
(d) $\frac{7}{10}$

Thinking Process

Use the formulae i.e., $\sec A=\sqrt{1+\tan^2 A}$ *and* $\sin A=\sqrt{1-\cos^2 A}$, $\sec A=\frac{1}{\cos A}$ and $\tan A=\frac{1}{\cot A}$.

Sol. ***(b)*** Given equation,

$$3\tan A+4=0$$
$$\Rightarrow \quad 3\tan A = -4$$
$$\Rightarrow \quad \tan A = \frac{-4}{3}$$
$$\Rightarrow \quad \cot A = \frac{-3}{4}$$
$$\Rightarrow \quad \sec A = \sqrt{1+\frac{16}{9}} = \sqrt{\frac{25}{9}} = \pm\frac{5}{3}$$
$$\Rightarrow \quad \sec A = \frac{-5}{3} \qquad \text{[since, } A \text{ lies in second quadrant]}$$
$$\cos A = \frac{-3}{5}$$
$$\sin A = \sqrt{1-\frac{9}{25}} = \frac{\sqrt{25-9}}{25} = \pm\frac{4}{5}$$
$$\sin A = \frac{4}{5} \qquad \text{[since, } A \text{ lies in second quadrant]}$$

$$\therefore \quad 2\cot A - 5\cos A + \sin A = 2\left(\frac{-3}{4}\right) - 5\left(\frac{-3}{5}\right) + \frac{4}{5}$$
$$= \frac{-6}{4} + 3 + \frac{4}{5}$$
$$= \frac{-30 + 60 + 16}{20} = \frac{46}{20}$$
$$= \frac{23}{10}$$

Q. 56 The value of $\cos^2 48° - \sin^2 12°$ is

(a) $\frac{\sqrt{5}+1}{8}$ (b) $\frac{\sqrt{5}-1}{8}$ (c) $\frac{\sqrt{5}+1}{5}$ (d) $\frac{\sqrt{5}+1}{2\sqrt{2}}$

Sol. *(a)* Given expression, $\cos^2 48° - \sin^2 12°$
$$= \cos(48° + 12°) - \cos(48° - 12°)$$
$$= \cos 60° \cdot \cos 36°$$
$$= \frac{1}{2} \cdot \frac{\sqrt{5}+1}{4}$$
$$= \frac{\sqrt{5}+1}{8}$$

Q. 57 If $\tan\alpha = \frac{1}{7}$ and $\tan\beta = \frac{1}{3}$, then $\cos 2\alpha$ is equal to

(a) $\sin 2\beta$ (b) $\sin 4\beta$ (c) $\sin 3\beta$ (d) $\cos 2\beta$

Thinking Process

Use $\cos 2\alpha = \frac{1-\tan^2\alpha}{1+\tan^2\alpha}$ *and* $\sin 2\alpha = \frac{2\tan\alpha}{1+\tan^2\alpha}$

Sol. *(b)* Given that, $\tan = \frac{1}{7}$ and $\tan\beta = \frac{1}{3}$
$$\cos 2\alpha = \frac{1 - \frac{1}{49}}{1 + \frac{1}{49}} = \frac{\frac{48}{49}}{\frac{50}{49}}$$
$$= \frac{48}{50} = \frac{24}{25}$$
$$\Rightarrow \quad \cos 2\alpha = \frac{24}{25} \quad \text{...(i)}$$

We know that, $$\sin 4\beta = \frac{2\tan 2\beta}{1 + \tan^2 2\beta} \quad \text{...(ii)}$$

and $$\tan 2\beta = \frac{2\tan\beta}{1-\tan^2\beta} = \frac{2 \times \frac{1}{3}}{1 - \frac{1}{9}}$$
$$= \frac{\frac{2}{3}}{\frac{8}{9}} = \frac{2 \times 9}{3 \times 8} = \frac{3}{4}$$

From Eq, (ii),

$$\sin 4\beta = \frac{2 \times \frac{3}{4}}{1 + \frac{9}{16}} = \frac{\frac{6}{4}}{\frac{25}{16}} = \frac{6 \times 16}{4 \times 25}$$

$\Rightarrow$ $$\sin 4\beta = \frac{24}{25}$$

$\Rightarrow$ $$\sin 4\beta = \cos 2\alpha$$

$\therefore$ $$\cos 2\alpha = \sin 4\beta$$ [from Eq. (i)]

Q. 58 If $\tan\theta = \frac{a}{b}$, then $b\cos 2\theta + a\sin 2\theta$ is equal to

(a) a (b) b (c) $\frac{a}{b}$ (d) None of these

Sol. ***(b)*** Given that, $\tan\theta = \frac{a}{b}$

$$\therefore \quad b\cos 2\theta + a\sin 2\theta = b\left(\frac{1 - \tan^2\theta}{1 + \tan^2\theta}\right) + a\left(\frac{2\tan\theta}{1 + \tan^2\theta}\right)$$

$$= b\left(\frac{1 - \frac{a^2}{b^2}}{1 + \frac{a^2}{b^2}}\right) + a\left(\frac{\frac{2a}{b}}{1 + \frac{a^2}{b^2}}\right)$$

$$= b\left(\frac{b^2 - a^2}{b^2 + a^2}\right) + \frac{2a^2 b}{a^2 + b^2}$$

$$= \frac{b}{a^2 + b^2}[b^2 - a^2 + 2a^2] = \frac{(a^2 + b^2)b}{(a^2 + b^2)}$$

$$= b$$

Q. 59 If for real values of x, $\cos\theta = x + \frac{1}{x}$, then

(a) θ is an acute angle (b) θ is right angle
(c) θ is an obtuse angle (d) No value of θ is possible

Thinking Process

The quadratic equation $ax^2 + bx + c = 0$ has real roots, then $b^2 - 4ac = 0$, use this condition to solve the above problem.

Sol. ***(d)*** Here, $$\cos\theta = x + \frac{1}{x}$$

$\Rightarrow$ $$\cos\theta = \frac{x^2 + 1}{x}$$

$$x^2 - x\cos\theta + 1 = 0$$

For real value of x, $(-\cos\theta)^2 - 4 \times 1 \times 1 = 0$

$$\cos^2\theta = 4$$

$$\cos\theta = \pm 2$$

which is not possible. $[\because -1 \le \cos\theta \le 1]$

Fillers

Q. 60 The value of $\dfrac{\sin 50^\circ}{\sin 130^\circ}$ is ....

Sol. Here,
$$\frac{\sin 50^\circ}{\sin 130^\circ} = \frac{\sin(180^\circ - 130^\circ)}{\sin 130^\circ}$$
$$= \frac{\sin 130^\circ}{\sin 130^\circ} = 1$$

Q. 61 If $k = \sin\left(\dfrac{\pi}{18}\right)\sin\left(\dfrac{5\pi}{18}\right)\sin\left(\dfrac{7\pi}{18}\right)$, then the numerical value of k is ....

Sol. Here,
$$k = \sin\left(\frac{\pi}{18}\right)\sin\left(\frac{5\pi}{18}\right)\sin\left(\frac{7\pi}{18}\right)$$
$$= \sin 10^\circ \sin 50^\circ \sin 70^\circ$$
$$= \sin 10^\circ \cos 40^\circ \cdot \cos 20^\circ$$
$$= \frac{1}{2}\sin 10^\circ [2\cos 40^\circ \cdot \cos 20^\circ]$$
$$= \frac{1}{2}\sin 10^\circ [\cos 60^\circ + \cos 20^\circ] \quad [\because 2\cos x \cdot \cos y = \cos(x+y) + \cos(x-y)]$$
$$= \frac{1}{2}\sin 10^\circ \cdot \frac{1}{2} + \frac{1}{2}\sin 10^\circ \cos 20^\circ$$
$$= \frac{1}{4}\sin 10^\circ + \frac{1}{4}[\sin 30^\circ - \sin 10^\circ]$$
$$= \frac{1}{8}$$

Q. 62 If $\tan A = \dfrac{1-\cos\theta}{\sin B}$, then $\tan 2A =$ ....

Thinking Process

Use $\cos\theta = 1 - 2\sin^2\dfrac{\theta}{2}$ *and* $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$.

Sol. Given that,
$$\tan A = \frac{1-\cos B}{\sin B}$$
$$= \frac{1 - 1 + 2\sin^2\frac{B}{2}}{2\sin\frac{B}{2}\cdot\cos\frac{B}{2}} = \tan\frac{B}{2}$$

Now,
$$\tan 2A = \frac{2\tan A}{1-\tan^2 A}$$
$$\Rightarrow \quad \tan 2A = \frac{2\cdot\tan\frac{B}{2}}{1-\tan^2\frac{B}{2}}$$
$$\Rightarrow \quad \tan 2A = \tan B$$

Q. 63 If $\sin x + \cos x = a$, then

(i) $\sin^6 x + \cos^6 x =$ ....

(ii) $|\sin x - \cos x| =$ ....

Sol. Given that, $\sin x + \cos x = a$

On squaring both sides, we get

$$(\sin x + \cos x)^2 = (a)^2$$

$$\Rightarrow \quad \sin^2 x + \cos^2 x + 2\sin x \cos x = a^2$$

$$\Rightarrow \quad \sin x \cdot \cos x = \frac{1}{2}(a^2 - 1)$$

(i) $\sin^6 x + \cos^6 x = (\sin^2 x)^3 + (\cos^2 x)^3$

$$= (\sin^2 x + \cos^2 x)(\sin^4 x - \sin^2 x \cos^2 x + \cos^4 x)$$

$$= \sin^4 x + \cos^4 x - \frac{1}{4}(a^2 - 1)^2$$

$$= (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x - \frac{1}{4}(a^2 - 1)^2$$

$$= 1 - 2 \cdot \frac{1}{4}(a^2 - 1)^2 - \frac{1}{4}(a^2 - 1)^2 = \frac{1}{4}[4 - 3(a^2 - 1)^2]$$

(ii) $|\sin x - \cos x| = \sqrt{(\sin x - \cos x)^2}$

$$= \sqrt{\sin^2 x + \cos^2 x - 2\sin x \cos x}$$

$$= \sqrt{1 - 2\frac{1}{2}(a^2 - 1)} = \sqrt{1 - a^2 + 1} = \sqrt{2 - a^2}$$

Q. 64 In right angled ΔABC with $\angle C = 90°$ the equation whose roots are $\tan A$ and $\tan B$ is ....

Sol. In right angled $\Delta\, ABC$, $\angle C = 90°$

$$\therefore \quad \tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\Rightarrow \quad \frac{1}{0} = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\Rightarrow \quad \tan A \tan B = 1 \quad \ldots\text{(i)}$$

$$\tan A + \tan B = \frac{\sin A}{\cos A} + \frac{\sin B}{\cos B}$$

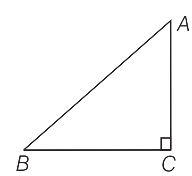
$$= \frac{\sin A}{\cos A} + \frac{\sin(90° - A)}{\cos(90° - A)} \quad [\because \angle C = 90°, \angle B = 90° - A]$$

$$= \frac{\sin A}{\cos A} + \frac{\cos A}{\sin A}$$

$$= \frac{\sin^2 A + \cos^2 A}{\sin A \cdot \cos A}$$

$$= \frac{1}{\sin A \cdot \cos A} = \frac{2}{2 \cdot \sin A \cdot \cos A}$$

$$= \frac{2}{\sin 2A} \quad [\because \sin 2x = 2\sin x \cos x]$$



So, the required equation is $x^2 - \left(\frac{2}{\sin A}\right)x + 1$.

Q. 65 $3(\sin x - \cos x)^4 + 6(\sin x + \cos x)^2 + 4(\sin^6 x + \cos^6 x) = \text{.........}$.

Thinking Process

Use formulae i.e., $(a^3 + b^3) = (a+b)(a^2 - ab + b^2)$ *and* $a^2 + b^2 = (a+b)^2 - 2ab$.

Sol. Given expression, $3(\sin x - \cos x)^4 + 6(\sin x + \cos x)^2 + 4(\sin^6 x + \cos^6 x)$

$$= 3[\sin^2 x + \cos^2 x - 2\sin x\cos x]^2 + 6[\sin^2 x + \cos^2 x + 2\cdot\sin x\cdot\cos x] + 4[(\sin^2 x)^3 + (\cos^2 x)^3]$$

$$= 3(1 - \sin 2x)^2 + 6(1 + \sin 2x) + 4[(\sin^2 + \cos^2 x)(\sin^4 x - \sin x\cos^2 x + \cos^4 x)$$

$$= 3(1 + \sin^2 23x - 2\sin 2x) + 6 + 6\sin 2x + 4[(\sin^2 x + \cos^2 x)^2 3\sin x\cos^2 x]$$

$$= 3 + 3\sin^2 2x - 6\sin 2x + 6 + 6\sin 2x$$

$$= 4 - 3\sin^2 2x = 13$$

Q. 66 Given $x > 0$, the value of $f(x) = -3\cos\sqrt{3 + x + x^2}$ lie in the interval ....

Sol. Given function, $f(x) = -3\cos\sqrt{3 + x + x^2}$

We know that, $-1 \le \cos x \le 1$

$\Rightarrow$ $-3 \le 3\cos x \le 3$

$\Rightarrow$ $3 \ge -3\cos x \ge -3$

$\Rightarrow$ $-3 \le -3\cos x \le 3$

So, the value of $f(x)$ lies in $[-3, 3]$.

Q. 67 The maximum distance of a point on the graph of the function $y = \sqrt{3}\sin x + \cos x$ from X-axis is ....

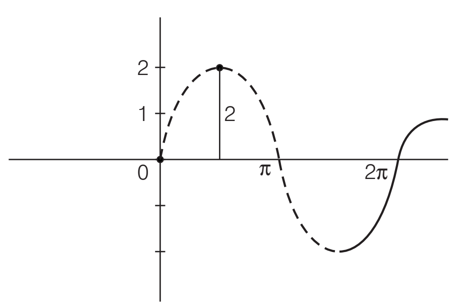
Sol. Given that, $y = \sqrt{3}\sin x + \cos x$

$$y = 2\left[\frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x\right]$$

$$= 2\left[\sin x\cdot\cos\frac{\pi}{6} + \cos x\sin\frac{\pi}{6}\right]$$

$$= 2\sin(x + \pi/6)$$

Graph of $y = 2\sin x$



Hence, the maximum distance is 2 units.

True/False

Q. 68 In each of the questions 68 to 75, state whether the statements is True or False? Also, give justification.

Thinking Process

If $\tan A = \dfrac{1-\cos B}{\sin B}$, *then* $\tan 2A = \tan B$

Sol. ***True***

Given that, $$\tan A = \frac{1-\cos B}{\sin B} = \frac{1-1+2\sin^2 \frac{B}{2}}{2\sin\frac{B}{2}\cdot\cos\frac{B}{2}} = \tan\frac{B}{2}$$

Now, $$\tan 2A = \frac{2\tan A}{1-\tan^2 A} = \frac{2\cdot\tan\frac{B}{2}}{1-\tan^2\frac{B}{2}} = \tan B$$

Q. 69 The equality $\sin A + \sin 2A + \sin 3A = 3$ holds for some real value of A.

Sol. ***False***

Given that, $\sin A + \sin 2A + \sin 3A = 3$

It is possible only if $\sin A$, $\sin 2A$, $\sin 3A$ each has a value one because maximum value of $\sin A$ is a certain angle is 1. Which is not possible because angle are different.

Q. 70 $\sin 10°$ is greater than $\cos 10°$.

Sol. ***False***

$$\sin 10° = \sin(90° - 80°)$$
$$\sin 10° = \cos 80°$$

$\therefore$ $$\cos 80° < \cos 10°$$

Hence, $$\sin 10° < \cos 10°$$

Q. 71 $\cos\dfrac{2\pi}{15}\cos\dfrac{4\pi}{15}\cos\dfrac{8\pi}{15}\cos\dfrac{16\pi}{15} = \dfrac{1}{16}$

Sol. ***True***

$$\text{LHS} = \cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\cos\frac{16\pi}{15}$$
$$= \cos 24°\cos 48°\cos 96°\cos 192°$$
$$= \frac{1}{16\sin 24°}[(2\sin 24°\cos 24°)(2\cos 48°)(2\cos 96°)(2\cos 192°)]$$
$$= \frac{1}{16\sin 24°}[2\sin 48°\cos 48°(2\cos 96°)(2\cos 192°)]$$
$$= \frac{1}{16\sin 24°}[(2\sin 96°\cos 96°)(2\cos 192°)]$$
$$= \frac{1}{16\sin 24°}[2\sin 192°\cos 192°)$$
$$= \frac{1}{16\sin 24°}\sin 384° = \frac{\sin(360°+24°)}{16\sin 24°}$$
$$= \frac{1}{16} = \text{RHS}$$

Hence proved.

Q. 72 One value of θ which satisfies the equation $\sin^4\theta - 2\sin^2\theta - 1$ lies between 0 and 2π.

Sol. ***False***

Given equation, $\sin^4\theta - 2\sin^2\theta - 1 = 0$

$\Rightarrow \sin^2\theta = \dfrac{2 \pm \sqrt{4+4}}{2}$

$\Rightarrow \sin^2\theta = \dfrac{2 \pm 2\sqrt{2}}{2}$

$\Rightarrow \sin^2\theta = (1+\sqrt{2}) \text{ or } (1-\sqrt{2}) \Rightarrow -1 \le \sin\theta \le 1$

$\Rightarrow \sin^2\theta \le 1$

$\because \sin^2\theta = \sqrt{2+1} \text{ or } (1-\sqrt{2})$

which is not possible.

Q. 73 If $\operatorname{cosec} x = 1 + \cot x$,then $x = 2n\pi, 2n\pi + \dfrac{\pi}{2}$

Sol. ***True***

Given that, $\operatorname{cosec} x = 1 + \cot x$

$\Rightarrow \dfrac{1}{\sin x} = 1 + \dfrac{\cos x}{\sin x} \Rightarrow \dfrac{1}{\sin x} = \dfrac{\sin x + \cos x}{\sin x}$

$\Rightarrow \sin x + \cos x = 1$

$\Rightarrow \dfrac{1}{\sqrt{2}} \cdot \sin x + \dfrac{1}{\sqrt{2}} \cdot \cos x = \dfrac{1}{\sqrt{2}}$

$\Rightarrow \sin\dfrac{\pi}{4}\sin x + \cos x \cos\dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}}$

$\Rightarrow \cos\left(x - \dfrac{\pi}{4}\right) = \cos\dfrac{\pi}{4}$

$\therefore x - \dfrac{\pi}{4} = 2n\pi \pm \dfrac{\pi}{4}$

For positive sign, $x = 2n\pi + \dfrac{\pi}{4} + \dfrac{\pi}{4} = 2n\pi + \dfrac{\pi}{2}$

For negative sign, $x = 2n\pi - \dfrac{\pi}{4} + \dfrac{\pi}{4} = 2n\pi$

Q. 74 If $\tan\theta + \tan 2\theta + \sqrt{3}\tan\theta\tan 2\theta = \sqrt{3}$, then $\theta = \dfrac{n\pi}{3} + \dfrac{\pi}{9}$.

Sol. ***True***

$\tan\theta + \tan 2\theta + \sqrt{3}\tan\theta\tan 2\theta = \sqrt{3}$

$\Rightarrow \tan\theta + \tan 2\theta = \sqrt{3} - \sqrt{3}\tan\theta\tan 2\theta$

$\Rightarrow \tan\theta + \tan 2\theta = \sqrt{3}(1 - \tan\theta\tan 2\theta)$

$\Rightarrow \dfrac{\tan\theta + \tan 2\theta}{1 - \tan\theta\tan 2\theta} = \sqrt{3}$

$\Rightarrow \tan(\theta + 2\theta) = \tan\dfrac{\pi}{3} \Rightarrow \tan 3\theta = \tan\dfrac{\pi}{3}$

$\therefore 3\theta = n\pi + \dfrac{\pi}{3}$

$\theta = \dfrac{n\pi}{3} + \dfrac{\pi}{9}$

Q. 75 If $\tan(\pi\cos\theta) = \cot(\pi\sin\theta)$, then $\cos\left(\theta - \frac{\pi}{4}\right) = \pm\frac{1}{2\sqrt{2}}$.

Thinking Process

Use the formulae i.e., $\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta$ *and* $\cos(A - B) = \cos A\cos B + \sin A\sin B$.

Sol. ***True***

We have, $\tan(\pi\cos\theta) = \cot(\pi\sin\theta)$

$\Rightarrow \tan(\pi\cos\theta) = \tan\left[\frac{\pi}{2} - (\pi\sin\theta)\right]$

$\Rightarrow \pi\cos = \frac{\pi}{2} - \pi\sin\theta$

$\Rightarrow \pi(\sin\theta + \cos\theta) = \frac{\pi}{2}$

$\Rightarrow \sin\theta + \cos\theta = \frac{1}{2}$

$\Rightarrow \frac{1}{\sqrt{2}}\cdot\sin\theta + \frac{1}{\sqrt{2}}\cdot\cos\theta = \frac{1}{2\sqrt{2}}$

$\Rightarrow \sin\theta\cdot\sin\frac{\pi}{4} + \cos\theta\cdot\cos\frac{\pi}{4} = \frac{1}{2\sqrt{2}}$

$\therefore \cos\left(\theta - \frac{\pi}{4}\right) = \frac{1}{2\sqrt{2}}$

Q. 76 In the following match each item given under the Column I to its correct answer given under the Column II.

	Column I		Column II
(i)	$\sin(x + y)\sin(x - y)$	(a)	$\cos^2 x - \sin^2 y$
(ii)	$\cos(x + y)\cos(x - y)$	(b)	$1 - \tan\theta / 1 + \tan\theta$
(iii)	$\cot\left(\frac{\pi}{4} + \theta\right)$	(c)	$1 + \tan\theta / 1 - \tan\theta$
(iv)	$\tan\left(\frac{\pi}{4} + \theta\right)$	(d)	$\sin^2 x - \sin^2 y$

Sol.

(i) $\sin(x + y)\sin(x - y) = \sin^2 x - \sin^2 y$

(ii) $\cos(x + y)\cos(x - y) = \cos^2 x - \sin^2 y$

(iii) $\cot\left(\frac{\pi}{4} + \theta\right) = \frac{\cot\frac{\pi}{4}\cot\theta - 1}{\cot\frac{\pi}{4} + \cot\theta}$

$= \frac{-1 + \cot\theta}{1 + \cot\theta} = \frac{1 - \tan\theta}{1 + \tan\theta}$

(iv) $\tan\left(\frac{\pi}{4} + \theta\right) = \frac{\tan\frac{\pi}{4} + \tan\theta}{1 - \tan\frac{\pi}{4}\tan\theta} = \frac{1 + \tan\theta}{1 - \tan\theta}$

Hence, the correct mathes are **(i)** → (d), **(ii)** → (a), **(iii)** → (b), **(iv)** → (c).

4

Principle of Mathematical Induction

Short Answer Type Questions

Q. 1 Give an example of a statement $P(n)$ which is true for all $n \geq 4$ but $P(1)$, $P(2)$ and $P(3)$ are not true. Justify your answer.

Sol. Let the statement $P(n): 3n < n!$

For $n = 1$, $3 \times 1 < 1!$ [false]

For $n = 2$, $3 \times 2 < 2! \Rightarrow 6 < 2$ [false]

For $n = 3$, $3 \times 3 < 3! \Rightarrow 9 < 6$ [false]

For $n = 4$, $3 \times 4 < 4! \Rightarrow 12 < 24$ [true]

For $n = 5$, $3 \times 5 < 5! \Rightarrow 15 < 5 \times 4 \times 3 \times 2 \times 1 \Rightarrow 15 < 120$ [true]

Q. 2 Give an example of a statement $P(n)$ which is true for all n. Justify your answer.

Sol. Consider the statement

$$P(n): 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

For $n = 1$, $1 = \frac{1(1+1)(2 \times 1 + 1)}{6}$

$\Rightarrow$ $1 = \frac{2(3)}{6}$

$\Rightarrow$ $1 = 1$

For $n = 2$, $1 + 2^2 = \frac{2(2+1)(4+1)}{6}$

$\Rightarrow$ $5 = \frac{30}{6} \Rightarrow 5 = 5$

For $n = 3$, $1 + 2^2 + 3^2 = \frac{3(3+1)(7)}{6}$

$\Rightarrow$ $1 + 4 + 9 = \frac{3 \times 4 \times 7}{6}$

$\Rightarrow$ $14 = 14$

Hence, the given statement is true for all n.

Prove each of the statements in the following questions from by the Principle of Mathematical Induction.

Q. 3 $4^n - 1$ is divisible by 3, for each natural number n.

Thinking Process

In step I put $n=1$, the obtained result should be a divisible by 3. In step II put $n=k$ and take $P(k)$ equal to multiple of 3 with non-zero constant say q. In step III put $n=k+1$, in the statement and solve till it becomes a multiple of 3.

Sol. Let $P(n)$: $4^n - 1$ is divisible by 3 for each natural number n.

Step **I** Now, we observe that $P(1)$ is true.

$$P(1) = 4^1 - 1 = 3$$

It is clear that 3 is divisible by 3.

Hence, $P(1)$ is true.

Step **II** Assume that, $P(n)$ is true for $n = k$

$P(k)$: $4^k - 1$ is divisible by 3

$$x4^k - 1 = 3q$$

Step **III** Now, to prove that $P(k + 1)$ is true.

$$\begin{aligned} P(k+1) &: 4^{k+1} - 1 \\ &= 4^k \cdot 4 - 1 \\ &= 4^k \cdot 3 + 4^k - 1 \\ &= 3 \cdot 4^k + 3q && [\because 4^k - 1 = 3q] \\ &= 3(4^k + q) \end{aligned}$$

Thus, $P(k + 1)$ is true whenever $P(k)$ is true.

Hence, by the principle of mathematical induction $P(n)$ is true for all natural number n.

Q. 4 $2^{3n} - 1$ is divisible by 7, for all natural numbers n.

Sol. Let $P(n)$: $2^{3n} - 1$ is divisible by 7

Step **I** We observe that $P(1)$ is true.

$$P(1) : 2^{3\times1} - 1 = 2^3 - 1 = 8 - 1 = 7$$

It is clear that $P(1)$ is true.

Step **II** Now, assume that $P(n)$ is true for $n = k$,

$P(k)$: $2^{3k} - 1$ is divisible by 7.

$$\Rightarrow \quad 2^{3k} - 1 = 7q$$

Step **III** Now, to prove $P(k + 1)$ is true.

$$\begin{aligned} P(k+1) &: 2^{3(k+1)} - 1 \\ &= 2^{3k} \cdot 2^3 - 1 \\ &= 2^{3k}(7 + 1) - 1 \\ &= 7 \cdot 2^{3k} + 2^{3k} - 1 \\ &= 7 \cdot 2^{3k} + 7q && \text{[from step II]} \\ &= 7(2^{3k} + q) \end{aligned}$$

Hence, $P(k + 1)$: is true whenever $P(k)$ is true.

So, by the principle of mathematical induction $P(n)$ is true for all natural number n.

Q. 5 $n^3 - 7n + 3$ is divisible by 3, for all natural numbers n.

Sol. Let $P(n): n^3 - 7n + 3$ is divisible by 3, for all natural number n.

Step **I** We observe that $P(1)$ is true.

$$P(1) = (1)^3 - 7(1) + 3$$
$$= 1 - 7 + 3$$
$$= -3, \text{ which is divisible by 3.}$$

Hence, $P(1)$ is true.

Step **II** Now, assume that $P(n)$ is true for $n = k$.

$$\therefore \quad P(k) = k^3 - 7k + 3 = 3q$$

Step **III** To prove $P(k + 1)$ is true

$$P(k+1) : (k+1)^3 - 7(k+1) + 3$$
$$= k^3 + 1 + 3k(k+1) - 7k - 7 + 3$$
$$= k^3 - 7k + 3 + 3k(k+1) - 6$$
$$= 3q + 3[k(k+1) - 2]$$

Hence, $P(k + 1)$ is true whenever $P(k)$ is true. [from step II]

So, by the principle of mathematical induction $P(n)$: is true for all natural number n.

Q. 6 $3^{2n} - 1$ is divisible by 8, for all natural numbers n.

Sol. Let $P(n): 3^{2n} - 1$ is divisible by 8, for all natural numbers.

Step **I** We observe that $P(1)$ is true.

$$P(1) : 3^{2(1)} - 1 = 3^2 - 1$$
$$= 9 - 1 = 8, \text{which is divisible by 8.}$$

Step **II** Now, assume that $P(n)$ is true for $n = k$.

$$P(k) : 3^{2k} - 1 = 8q$$

Step **III** Now, to prove $P(k + 1)$ is true.

$$P(k+1) : 3^{2(k+1)} - 1$$
$$= 3^{2k} \cdot 3^2 - 1$$
$$= 3^{2k} \cdot (8 + 1) - 1$$
$$= 8 \cdot 3^{2k} + 3^{2k} - 1$$
$$= 8 \cdot 3^{2k} + 8q$$
$$= 8(3^{2k} + q) \quad \text{[from step II]}$$

Hence, $P(k + 1)$ is true whenever $P(k)$ is true.

So, by the principle of mathematical induction $P(n)$ is true for all natural numbers n.

Q. 7 For any natural numbers n, $7^n - 2^n$ is divisible by 5.

Sol. Consider the given statement is

$P(n) : 7^n - 2^n$ is divisible by 5, for any natural number n.

Step **I** We observe that $P(1)$ is true.

$$P(1) = 7^1 - 2^1 = 5, \text{ which is divisible by 5.}$$

Step **II** Now, assume that $P(n)$ is true for $n = k$.

$$P(k) = 7^k - 2^k = 5q$$

Step **III** Now, to prove $P(k + 1)$ is true,

$$P(k+1) : 7^{k+1} - 2^{k+1}.$$
$$= 7^k \cdot 7 - 2^k \cdot 2$$

$$= 7^k \cdot (5+2) - 2^k \cdot 2$$
$$= 7^k \cdot 5 + 2 \cdot 7^k - 2^k \cdot 2$$
$$= 5 \cdot 7^k + 2(7^k - 2^k)$$
$$= 5 \cdot 7^k + 2(5q)$$
$$= 5(7^k + 2q), \text{ which is divisible by 5.} \quad \text{[from step II]}$$

So, $P(k+1)$ is true whenever $P(k)$ is true.
Hence, by the principle of mathematical induction $P(n)$ is true for any natural number n.

Q. 8 For any natural numbers n, $x^n - y^n$ is divisible by $x - y$, where x and y are any integers with $x \neq y$.

Sol. Let $P(n): x^n - y^n$ is divisible by $x - y$, where x and y are any integers with $x \neq y$.

Step **I** We observe that $P(1)$ is true.

$$P(1): x^1 - y^1 = x - y$$

Step **II** Now, assume that $P(n)$ is true for $n = k$.

$$P(k): x^k - y^k \text{ is divisible by } (x - y).$$

$$\therefore \quad x^k - y^k = q(x - y)$$

Step **III** Now, to prove $P(k+1)$ is true.

$$P(k+1): x^{k+1} - y^{k+1}$$
$$= x^k \cdot x - y^k \cdot y$$
$$= x^k \cdot x - x^k \cdot y + x^k \cdot y - y^k \cdot y$$
$$= x^k(x - y) + y(x^k - y^k)$$
$$= x^k(x - y) + yq(x - y)$$
$$= (x - y)[x^k + yq], \text{ which is divisible by } (x - y). \quad \text{[from step II]}$$

Hence, $P(k+1)$ is true whenever $P(k)$ is true. So, by the principle of mathematical induction $P(n)$ is true for any natural number n.

Q. 9 $n^3 - n$ is divisible by 6, for each natural number $n \geq 2$.

Thinking Process

In step I put $n=2$, the obtained result should be divisible by 6. Then, follow the same process as in question no. 4.

Sol. Let $P(n): n^3 - n$ is divisible by 6, for each natural number $n \geq 2$.

Step **I** We observe that $P(2)$ is true. $P(2): (2)^3 - 2$

$$\Rightarrow \quad 8 - 2 = 6, \text{ which is divisible by 6.}$$

Step **II** Now, assume that $P(n)$ is true for $n = k$.

$$P(k): k^3 - k \text{ is divisible by 6.}$$

$$\therefore \quad k^3 - k = 6q$$

Step **III** To prove $P(k+1)$ is true

$$P(k+1): (k+1)^3 - (k+1).$$
$$= k^3 + 1 + 3k(k+1) - (k+1)$$
$$= k^3 + 1 + 3k^2 + 3k - k - 1$$
$$= k^3 - k + 3k^2 + 3k$$
$$= 6q + 3k(k+1) \quad \text{[from step II]}$$

We know that, $3k(k+1)$ is divisible by 6 for each natural number $n = k$.
So, $P(k+1)$ is true. Hence, by the principle of mathematical induction $P(n)$ is true.

Q. 10 $n(n^2 + 5)$ is divisible by 6, for each natural number n.

Sol. Let $P(n): n(n^2 + 5)$ is divisible by 6, for each natural number n.

Step **I** We observe that $P(1)$ is true.

$$P(1): 1(1^2 + 5) = 6, \text{ which is divisible by 6.}$$

Step **II** Now, assume that $P(n)$ is true for $n = k$.

$$P(k): k(k^2 + 5) \text{ is divisible by 6.}$$

$$\therefore \quad k(k^2 + 5) = 6q$$

Step **III** Now, to prove $P(k + 1)$ is true, we have

$$P(k+1): (k+1)[(k+1)^2 + 5]$$
$$= (k+1)[k^2 + 2k + 1 + 5]$$
$$= (k+1)[k^2 + 2k + 6]$$
$$= k^3 + 2k^2 + 6k + k^2 + 2k + 6$$
$$= k^3 + 3k^2 + 8k + 6$$
$$= k^3 + 5k + 3k^2 + 3k + 6$$
$$= k(k^2 + 5) + 3(k^2 + k + 2)$$
$$= (6q) + 3(k^2 + k + 2)$$

We know that, $k^2 + k + 2$ is divisible by 2, where, k is even or odd.

Since, $P(k + 1): 6q + 3(k^2 + k + 2)$ is divisible by 6. So, $P(k + 1)$ is true whenever $P(k)$ is true.

Hence, by the principle of mathematical induction $P(n)$ is true.

Q. 11 $n^2 < 2^n$, for all natural numbers $n \geq 5$.

Sol. Consider the given statement

$P(n): n^2 < 2^n$ for all natural numbers $n \geq 5$.

Step **I** We observe that $P(5)$ is true

$$P(5): 5^2 < 2^5$$
$$= 25 < 32$$

Hence, $P(5)$ is true.

Step **II** Now, assume that $P(n)$ is true for $n = k$.

$$P(k) = k^2 < 2^k \text{ is true.}$$

Step **III** Now, to prove $P(k + 1)$ is true, we have to show that

$$P(k+1): (k+1)^2 < 2^{k+1}$$

Now, $\quad k^2 < 2^k = k^2 + 2k + 1 < 2^k + 2k + 1$

$$= (k+1)^2 < 2^k + 2k + 1 \quad \ldots \text{(i)}$$

Now, $(2k + 1) < 2^k$ $\quad = 2^k + 2k + 1 < 2^k + 2^k$

$$= 2^k + 2k + 1 < 2 \cdot 2^k$$
$$= 2^k + 2k + 1 < 2^{k+1} \quad \ldots \text{(ii)}$$

From Eqs. (i) and (ii), we get $(k + 1)^2 < 2^{k+1}$

So, $P(k + 1)$ is true, whenever $P(k)$ is true. Hence, by the principle of mathematical induction $P(n)$ is true for all natural numbers $n \geq 5$.

Q. 12 $2n < (n+2)!$ for all natural numbers n.

Sol. Consider the statement

$P(n): 2n < (n+2)!$ for all natural number n.

Step **I** We observe that, $P(1)$ is true. $P(1): 2(1) < (1+2)!$

$\Rightarrow \quad 2 < 3! \Rightarrow 2 < 3\times 2\times 1 \Rightarrow 2 < 6$

Hence, $P(1)$ is true.

Step **II** Now, assume that $P(n)$ is true for $n = k$,

$$P(k): 2k < (k+2)! \text{ is true.}$$

Step **III** To prove $P(k+1)$ is true, we have to show that

$$P(k+1): 2(k+1) < (k+1+2)!$$

Now,
$$2k < (k+2)!$$
$$2k + 2 < (k+2)! + 2$$
$$2(k+1) < (k+2)! + 2 \qquad \ldots\text{(i)}$$

Also,
$$(k+2)! + 2 < (k+3)! \qquad \ldots\text{(ii)}$$

From Eqs. (i) and (ii),

$$2(k+1) < (k+1+2)!$$

So, $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, by principle of mathematical induction $P(n)$ is true.

Q. 13 $\sqrt{n} < \dfrac{1}{\sqrt{1}} + \dfrac{1}{\sqrt{2}} + \ldots + \dfrac{1}{\sqrt{n}}$, for all natural numbers $n \geq 2$.

Sol. Consider the statement

$P(n): \sqrt{n} < \dfrac{1}{\sqrt{1}} + \dfrac{1}{\sqrt{2}} + \ldots + \dfrac{1}{\sqrt{n}}$, for all natural numbers $n \geq 2$.

Step **I** We observe that $P(2)$ is true.

$$P(2): \sqrt{2} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}}, \text{ which is true.}$$

Step **II** Now, assume that $P(n)$ is true for $n = k$.

$$P(k): \sqrt{k} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{k}} \text{ is true.}$$

Step **III** To prove $P(k+1)$ is true, we have to show that

$$P(k+1): \sqrt{k+1} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{k+1}} \text{ is true.}$$

Given that,
$$\sqrt{k} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{k}}$$

$$\Rightarrow \quad \sqrt{k} + \frac{1}{\sqrt{k+1}} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}}$$

$$\Rightarrow \quad \frac{(\sqrt{k})(\sqrt{k+1}) + 1}{\sqrt{k+1}} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} \qquad \ldots\text{(i)}$$

If
$$\sqrt{k+1} < \frac{\sqrt{k}\sqrt{k+1} + 1}{\sqrt{k+1}}$$

$$\Rightarrow \quad k + 1 < \sqrt{k}\,\sqrt{k+1} + 1$$

$$\Rightarrow \quad k < \sqrt{k(k+1)} \Rightarrow \sqrt{k} < \sqrt{k+1} \qquad \ldots\text{(ii)}$$

From Eqs. (i) and (ii),

$$\sqrt{k+1} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{k+1}}$$

So, $P(k+1)$ is true, whenever $P(k)$ is true. Hence, $P(n)$ is true.

Q. 14 $2+4+6+\ldots+2n=n^2+n$, **for all natural numbers** n.

Sol. Let $P(n): 2+4+6+\ldots+2n=n^2+n$

For all natural numbers n.

Step **I** We observe that $P(1)$ is true.

$$P(1): 2=1^2+1$$
$$2=2, \text{ which is true.}$$

Step **II** Now, assume that $P(n)$ is true for $n=k$.

$\therefore$ $$P(k): 2+4+6+\ldots+2k=k^2+k$$

Step **III** To prove that $P(k+1)$ is true.

$$P(k+1): 2+4+6+8+\ldots+2k+2(k+1)$$
$$=k^2+k+2(k+1)$$
$$=k^2+k+2k+2$$
$$=k^2+2k+1+k+1$$
$$=(k+1)^2+k+1$$

So, $P(k+1)$ is true, whenever $P(k)$ is true.
Hence, $P(n)$ is true.

Q. 15 $1+2+2^2+\ldots+2^n=2^{n+1}-1$ **for all natural numbers** n.

Sol. Consider the given statement

$P(n): 1+2+2^2+\ldots+2^n=2^{n+1}-1$, for all natural numbers n

Step **I** We observe that $P(0)$ is true.

$$P(1): 1=2^{0+1}-1$$
$$1=2^1-1$$
$$1=2-1$$
$$1=1, \text{ which is true.}$$

Step **II** Now, assume that $P(n)$ is true for $n=k$.

So, $P(k): 1+2+2^2+\ldots+2^k=2^{k+1}-1$ is true.

Step **III** Now, to prove $P(k+1)$ is true.

$$P(k+1): 1+2+2^2+\ldots+2^k+2^{k+1}$$
$$=2^{k+1}-1+2^{k+1}$$
$$=2\cdot 2^{k+1}-1$$
$$=2^{k+2}-1$$
$$=2^{(k+1)+1}-1$$

So, $P(k+1)$ is true, whenever $P(k)$ is true.
Hence, $P(n)$ is true.

Q. 16 $1+5+9+\ldots+(4n-3)=n(2n-1)$, **for all natural numbers** n.

Sol. Let $P(n): 1+5+9+\ldots+(4n-3)=n(2n-1)$, for all natural numbers n.

Step **I** We observe that $P(1)$ is true.

$$P(1): 1=1(2\times 1-1),\ 1=2-1 \text{ and } 1=1, \text{ which is true.}$$

Step **II** Now, assume that $P(n)$ is true for $n=k$.

So, $P(k): 1+5+9+\ldots+(4k-3)=k(2k-1)$ is true.

Step **III** Now, to prove $P(k+1)$ is true.

$$\begin{aligned} P(k+1): 1+5+9+\ldots+(4k-3)+4(k+1)-3 &= k(2k-1)+4(k+1)-3 \\ &= 2k^2-k+4k+4-3 \\ &= 2k^2+3k+1 \\ &= 2k^2+2k+k+1 \\ &= 2k(k+1)+1(k+1) \\ &= (k+1)(2k+1) \\ &= (k+1)[2k+1+1-1] \\ &= (k+1)[2(k+1)-1] \end{aligned}$$

So, $P(k+1)$ is true, whenever $p(k)$ is true, hence $P(n)$ is true.

Long Answer Type Questions

Use the Principle of Mathematical Induction in the following questions.

Q. 17 A sequence $a_1, a_2, a_3, \ldots$ is defined by letting $a_1 = 3$ and $a_k = 7a_{k-1}$, for all natural numbers $k \geq 2$. Show that $a_n = 3 \cdot 7^{n-1}$ for all natural numbers.

Sol. A sequence $a_1, a_2, a_3, \ldots$ is defined by letting $a_1 = 3$ and $a_k = 7a_{k-1}$, for all natural numbers $k \geq 2$.

Let $P(n): a_n = 3 \cdot 7^{n-1}$ for all natural numbers.

Step **I** We observe $P(2)$ is true.

For $n = 2$, $a_2 = 3 \cdot 7^{2-1} = 3 \cdot 7^1 = 21$ is true.

As $a_1 = 3, a_k = 7a_{k-1}$

$\Rightarrow$ $a_2 = 7 \cdot a_{2-1} = 7 \cdot a_1$

$\Rightarrow$ $a_2 = 7 \times 3 = 21$ $[\because a_1 = 3]$

Step **II** Now, assume that $P(n)$ is true for $n = k$.

$$P(k): a_k = 3 \cdot 7^{k-1}$$

Step **III** Now, to prove $P(k+1)$ is true, we have to show that

$$P(k+1): a_{k+1} = 3 \cdot 7^{k+1-1}$$

$$\begin{aligned} a_{k+1} &= 7 \cdot a_{k+1-1} = 7 \cdot a_k \\ &= 7 \cdot 3 \cdot 7^{k-1} = 3 \cdot 7^{k-1+1} \end{aligned}$$

So, $P(k+1)$ is true, whenever $p(k)$ is true. Hence, $P(n)$ is true.

Q. 18 A sequence $b_0, b_1, b_2, \ldots$ is defined by letting $b_0 = 5$ and $b_k = 4 + b_{k-1}$, for all natural numbers k. Show that $b_n = 5 + 4n$, for all natural number n using mathematical induction.

Sol. Consider the given statement,

$P(n): b_n = 5 + 4n$, for all natural numbers given that $b_0 = 5$ and $b_k = 4 + b_{k-1}$

Step **I** $P(1)$ is true.

$$P(1): b_1 = 5 + 4 \times 1 = 9$$

As $b_0 = 5, b_1 = 4 + b_0 = 4 + 5 = 9$

Hence, $P(1)$ is true.

Step II Now, assume that $P(n)$ is true for $n = k$.

$$P(k): b_k = 5 + 4k$$

Step III Now, to prove $P(k+1)$ is true, we have to show that

$$\therefore \quad P(k+1): b_{k+1} = 5 + 4(k+1)$$

$$b_{k+1} = 4 + b_{k+1-1}$$
$$= 4 + b_k$$
$$= 4 + 5 + 4k = 5 + 4(k+1)$$

So, by the mathematical induction $P(k+1)$ is true whenever $P(k)$ is true, hence $P(n)$ is true.

Q. 19 A sequence $d_1, d_2, d_3, \ldots$ is defined by letting $d_1 = 2$ and $d_k = \frac{d_{k-1}}{k}$, for all natural numbers, $k \geq 2$. Show that $d_n = \frac{2}{n!}$, for all $n \in N$.

Sol. Let $P(n): d_n = \frac{2}{n!}$, $\forall n \in N$, to prove $P(2)$ is true.

Step I $\quad P(2): d_2 = \frac{2}{2!} = \frac{2}{2 \times 1} = 1$

As, given $\quad d_1 = 2$

$$\Rightarrow \quad d_k = \frac{d_{k-1}}{k}$$

$$\Rightarrow \quad d_2 = \frac{d_1}{2} = \frac{2}{2} = 1$$

Hence, $P(2)$ is true.

Step II Now, assume that $P(k)$ is true.

$$P(k): d_k = \frac{2}{k!}$$

Step III Now, to prove that $P(k+1)$ is true, we have to show that $P(k+1): d_{k+1} = \frac{2}{(k+1)!}$

$$d_{k+1} = \frac{d_{k+1-1}}{k} = \frac{d_k}{k}$$
$$= \frac{2}{k!k} = \frac{2}{(k+1)!}$$

So, $P(k+1)$ is true. Hence, $P(n)$ is true.

Q. 20 Prove that for all $n \in N$

$$\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots + \cos[\alpha+(n-1)\beta]$$
$$= \frac{\cos\left[\alpha + \left(\frac{n-1}{2}\right)\beta\right]\sin\left(\frac{n\beta}{2}\right)}{\sin\frac{\beta}{2}}$$

Thinking Process

To prove this, use the formula $2\cos A \sin B = \sin(A+B) - \sin(A-B)$ *and*

$$\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\cdot\sin\left(\frac{A-B}{2}\right)$$

Sol. Let $P(n): \cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots + \cos[\alpha+(n-1)\beta]$

$$= \frac{\cos\left[\alpha + \left(\frac{n-1}{2}\right)\beta\right]\sin\left(\frac{n\beta}{2}\right)}{\sin\frac{\beta}{2}}$$

Step **I** We observe that $P(1)$

$$P(1): \cos\alpha = \frac{\cos\left[\alpha+\left(\frac{1-1}{2}\right)\right]\beta\sin\frac{\beta}{2}}{\sin\frac{\beta}{2}} = \frac{\cos(\alpha+0)\sin\frac{\beta}{2}}{\sin\frac{\beta}{2}}$$

$$\cos\alpha = \cos\alpha$$

Hence, $P(1)$ is true.

Step **II** Now, assume that $P(n)$ is true for $n = k$.

$$P(k): \cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots + \cos[\alpha+(k-1)\beta]$$

$$= \frac{\cos\left[\alpha+\left(\frac{k-1}{2}\right)\right]\beta\sin\frac{k\beta}{2}}{\sin\frac{\beta}{2}}$$

Step **III** Now, to prove $P(k+1)$ is true, we have to show that

$$P(k+1): \cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots + \cos[\alpha+(k-1)\beta]$$

$$+ \cos[\alpha+(k+1-1)\beta] = \frac{\cos\left(\alpha+\frac{k\beta}{2}\right)\sin(k+1)\frac{\beta}{2}}{\sin\frac{\beta}{2}}$$

$$\text{LHS} = \cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots + \cos[\alpha+(k-1)\beta] + \cos(\alpha+k\beta)$$

$$= \frac{\cos\left[\alpha+\left(\frac{k-1}{2}\right)\beta\right]\sin\frac{k\beta}{2}}{\sin\frac{\beta}{2}} + \cos(\alpha+k\beta)$$

$$= \frac{\cos\left[\alpha+\left(\frac{k-1}{2}\right)\beta\right]\sin\frac{k\beta}{2} + \cos(\alpha+k\beta)\sin\frac{\beta}{2}}{\sin\frac{\beta}{2}}$$

$$= \frac{\sin\left(\alpha+\frac{k\beta}{2}-\frac{\beta}{2}+\frac{k\beta}{2}\right) - \sin\left(\alpha+\frac{k\beta}{2}-\frac{\beta}{2}-\frac{k\beta}{2}\right) + \sin\left(\alpha+k\beta+\frac{\beta}{2}\right) - \sin\left(\alpha+k\beta-\frac{\beta}{2}\right)}{2\sin\frac{\beta}{2}}$$

$$= \frac{\sin\left(\alpha+k\beta+\frac{\beta}{2}\right) - \sin\left(\alpha-\frac{\beta}{2}\right)}{2\sin\frac{\beta}{2}}$$

$$= \frac{2\cos\frac{1}{2}\left(\alpha+\frac{\beta}{2}+k\beta+\alpha-\frac{\beta}{2}\right)\sin\frac{1}{2}\left(\alpha+\frac{\beta}{2}+k\beta-\alpha+\frac{\beta}{2}\right)}{2\sin\frac{\beta}{2}}$$

$$= \frac{\cos\left(\frac{2\alpha+k\beta}{2}\right)\sin\left(\frac{k\beta+\beta}{2}\right)}{\sin\frac{\beta}{2}} = \frac{\cos\left(\alpha+\frac{k\beta}{2}\right)\sin(k+1)\frac{\beta}{2}}{\sin\frac{\beta}{2}} = \text{RHS}$$

So, $P(k+1)$ is true. Hence, $P(n)$ is true.

Q. 21 Prove that $\cos\theta \cos 2\theta \cos 2^2\theta \ldots \cos 2^{n-1}\theta = \dfrac{\sin 2^n \theta}{2^n \sin\theta}, \forall n \in N.$

Sol. Let $P(n) : \cos\theta \cos 2\theta \ldots \cos 2^{n-1}\theta = \dfrac{\sin 2^n\theta}{2^n \sin\theta}$

Step **I** For $n = 1$, $P(1) : \cos\theta = \dfrac{\sin 2^1\theta}{2^1 \sin\theta}$

$$= \frac{\sin 2\theta}{2\sin\theta} = \frac{2\sin\theta\cos\theta}{2\sin\theta} = \cos\theta$$

which is true.

Step **II** Assume that $P(n)$ is true, for $n = k$.

$$P(k) : \cos\theta \cdot \cos 2\theta \cdot \cos 2^2\theta \ldots \cos 2^{k-1}\theta = \frac{\sin 2^k\theta}{2^k \sin\theta} \text{ is true.}$$

Step **III** To prove $P(k+1)$ is true.

$$P(k+1) : \cos\theta \cdot \cos 2\theta \cdot \cos 2^2\theta \ldots \cos 2^{k-1}\theta \cdot \cos 2^k\theta$$

$$= \frac{\sin 2^k\theta}{2^k \sin\theta} \cdot \cos 2^k\theta$$

$$= \frac{2\sin 2^k\theta \cdot \cos 2^k\theta}{2 \cdot 2^k \sin\theta}$$

$$= \frac{\sin 2 \cdot 2^k\theta}{2^{k+1} \sin\theta} = \frac{\sin 2^{(k+1)}\theta}{2^{k+1}\sin\theta}$$

which is true.

So, $P(k+1)$ is true. Hence, $P(n)$ is true.

Q. 22 Prove that, $\sin\theta + \sin 2\theta + \sin 3\theta + \ldots + \sin n\theta = \dfrac{\sin\dfrac{n\theta}{2}\sin\dfrac{(n+1)}{2}\theta}{\sin\dfrac{\theta}{2}}$,

for all $n \in N$.

Thinking Process

To use the formula of $2\sin A \sin B = \cos(A-B) - \cos(A+B)$ *and*

$\cos A - \cos B = 2\sin\dfrac{A+B}{2} \cdot \sin\dfrac{B-A}{2}$ *also* $\cos(-\theta) = \cos\theta$.

Sol. Consider the given statement

$P(n) : \sin\theta + \sin 2\theta + \sin 3\theta + \ldots + \sin n\theta$

$$= \frac{\sin\dfrac{n\theta}{2}\sin\dfrac{(n+1)\theta}{2}}{\sin\dfrac{\theta}{2}}, \text{ for all } n \in N$$

Step **I** We observe that $P(1)$ is

$$P(1) : \sin\theta = \frac{\sin\dfrac{\theta}{2} \cdot \sin\dfrac{(1+1)}{2}\theta}{\sin\dfrac{\theta}{2}} = \frac{\sin\dfrac{\theta}{2} \cdot \sin\theta}{\sin\dfrac{\theta}{2}}$$

$$\sin\theta = \sin\theta$$

Hence, $P(1)$ is true.

Step **II** Assume that $P(n)$ is true, for $n = k$.

$$P(k): \sin\theta + \sin 2\theta + \sin 3\theta + \dots + \sin k\theta = \frac{\sin\frac{k\theta}{2}\sin\left(\frac{k+1}{2}\right)\theta}{\sin\frac{\theta}{2}} \text{ is true.}$$

Step **III** Now, to prove $P(k+1)$ is true.

$$P(k+1): \sin\theta + \sin 2\theta + \sin 3\theta + \dots + \sin k\theta + \sin(k+1)\theta = \frac{\sin\frac{(k+1)\theta}{2}\sin\left(\frac{k+1+1}{2}\right)\theta}{\sin\frac{\theta}{2}}$$

$$\text{LHS} = \sin\theta + \sin 2\theta + \sin 3\theta + \dots + \sin k\theta + \sin(k+1)\theta$$

$$= \frac{\sin\frac{k\theta}{2}\sin\left(\frac{k+1}{2}\right)\theta}{\sin\frac{\theta}{2}} + \sin(k+1)\theta = \frac{\sin\frac{k\theta}{2}\sin\left(\frac{k+1}{2}\right)\theta + \sin(k+1)\theta\cdot\sin\frac{\theta}{2}}{\sin\frac{\theta}{2}}$$

$$= \frac{\cos\left[\frac{k\theta}{2} - \left(\frac{k+1}{2}\right)\theta\right] - \cos\left[\frac{k\theta}{2} + \left(\frac{k+1}{2}\right)\theta\right] + \cos\left[(k+1)\theta - \frac{\theta}{2}\right] - \cos\left[(k+1)\theta + \frac{\theta}{2}\right]}{2\sin\frac{\theta}{2}}$$

$$= \frac{\cos\frac{\theta}{2} - \cos\left(k\theta + \frac{\theta}{2}\right) + \cos\left(k\theta + \frac{\theta}{2}\right) - \cos\left(k\theta + \frac{3\theta}{2}\right)}{2\sin\frac{\theta}{2}}$$

$$= \frac{\cos\frac{\theta}{2} - \cos\left(k\theta + \frac{3\theta}{2}\right)}{2\sin\frac{\theta}{2}} = \frac{2\sin\frac{1}{2}\left(\frac{\theta}{2} + k\theta + \frac{3\theta}{2}\right)\cdot\sin\frac{1}{2}\left(k\theta + \frac{3\theta}{2} - \frac{\theta}{2}\right)}{2\sin\frac{\theta}{2}}$$

$$= \frac{\sin\left(\frac{k\theta + 2\theta}{2}\right)\cdot\sin\left(\frac{k\theta + \theta}{2}\right)}{\sin\frac{\theta}{2}} = \frac{\sin(k+1)\frac{\theta}{2}\cdot\sin(k+1+1)\frac{\theta}{2}}{\sin\frac{\theta}{2}}$$

So, $P(k+1)$ is true, whenever $P(k)$ is true. Hence, $P(n)$ is true.

Q. 23 Show that $\frac{n^5}{5} + \frac{n^3}{3} + \frac{7n}{15}$ is a natural number, for all $n \in N$.

Thinking Process

Here, use the formula $(a+b)^5 = a^5 + 5ab^4 + 10a^2b^3 + 10a^3b^2 + 5a^4b + b^5$

and $(a+b)^3 = a^3 + b^3 + 3ab(a+b)$

Sol. Consider the given statement

$P(n): \frac{n^5}{5} + \frac{n^3}{3} + \frac{7n}{15}$ is a natural number, for all $n \in N$.

Step **I** We observe that $P(1)$ is true.

$P(1): \frac{(1)^5}{5} + \frac{1^3}{3} + \frac{7(1)}{15} = \frac{3+5+7}{15} = \frac{15}{15} = 1$, which is a natural number. Hence, $P(1)$ is true.

Step **II** Assume that $P(n)$ is true, for $n = k$.

$P(k): \frac{k^5}{5} + \frac{k^3}{3} + \frac{7k}{15}$ is natural number.

Step **III** Now, to prove $P(k+1)$ is true.

$$\frac{(k+1)^5}{5}+\frac{(k+1)^3}{3}+\frac{7(k+1)}{15}$$

$$=\frac{k^5+5k^4+10k^3+10k^2+5k+1}{5}+\frac{k^{3.}+1+3k(k+1)}{3}+\frac{7k+7}{15}$$

$$=\frac{k^5+5k^4+10k^3+10k^2+5k+1}{5}+\frac{k^3+1+3k^2+3k}{3}+\frac{7k+7}{15}$$

$$=\frac{k^5}{5}+\frac{k^3}{3}+\frac{7k}{15}+\frac{5k^4+10k^3+10k^2+5k+1}{5}+\frac{3k^2+3k+1}{3}+\frac{7k+7}{15}$$

$$=\frac{k^5}{5}+\frac{k^3}{3}+\frac{7k}{15}+k^4+2k^3+2k^2+k+k^2+k+\frac{1}{5}+\frac{1}{3}+\frac{7}{15}$$

$$=\frac{k^5}{5}+\frac{k^3}{3}+\frac{7k}{15}+k^4+2k^3+3k^2+2k+1, \text{ which is a natural number}$$

So, $P(k+1)$ is true, whenever $P(k)$ is true. Hence, $P(n)$ is true.

Q. 24 Prove that $\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{2n}>\frac{13}{24}$, for all natural numbers $n>1$.

Sol. Consider the given statement

$$P(n):\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{2n}>\frac{13}{24}, \text{ for all natural numbers } n>1.$$

Step **I** We observe that, $P(2)$ is true,

$$P(2):\frac{1}{2+1}+\frac{1}{2+2}>\frac{13}{24}.$$

$$\frac{1}{3}+\frac{1}{4}>\frac{13}{24}$$

$$\frac{4+3}{12}>\frac{13}{24}$$

$$\frac{7}{12}>\frac{13}{24}, \text{ which is true.}$$

Hence, $P(2)$ is true.

Step **II** Now, we assume that $P(n)$ is true,

For $n=k$,

$$P(k):\frac{1}{k+1}+\frac{1}{k+2}+\ldots+\frac{1}{2k}>\frac{13}{24}.$$

Step **III** Now, to prove $P(k+1)$ is true, we have to show that

$$P(k+1):\frac{1}{k+1}+\frac{1}{k+2}+\ldots+\frac{1}{2k}+\frac{1}{2(k+1)}>\frac{13}{24}$$

Given,
$$\frac{1}{k+1}+\frac{1}{k+2}+\ldots+\frac{1}{2k}>\frac{13}{24}$$

$$\frac{1}{k+1}+\frac{1}{k+2}+\frac{1}{2k}+\frac{1}{2(k+1)}>\frac{13}{24}+\frac{1}{2(k+1)}$$

$$\frac{13}{24}+\frac{1}{2(k+1)}>\frac{13}{24}$$

$$\because \quad \frac{1}{k+1}+\frac{1}{k+2}+\ldots+\frac{1}{2k}+\frac{1}{2(k+1)}>\frac{13}{24}$$

So, $P(k+1)$ is true, whenever $P(k)$ is true. Hence, $P(n)$ is true.

Q. 25 Prove that number of subsets of a set containing n distinct elements is 2^n, for all $n \in N$.

Sol. Let $P(n)$: Number of subset of a set containing n distinct elements is 2^n, for all $n \in N$.

Step **I** We observe that $P(1)$ is true, for $n = 1$.

Number of subsets of a set contain 1 element is $2^1 = 2$, which is true.

Step **II** Assume that $P(n)$ is true for $n = k$.

$P(k)$: Number of subsets of a set containing k distinct elements is 2^k, which is true.

Step **III** To prove $P(k+1)$ is true, we have to show that

$P(k+1)$: Number of subsets of a set containing $(k+1)$ distinct elements is 2^{k+1}.

We know that, with the addition of one element in the set, the number of subsets become double.

$\therefore$ Number of subsets of a set containing $(k+1)$ distinct elements $= 2 \times 2^k = 2^{k+1}$.

So, $P(k+1)$ is true. Hence, $P(n)$ is true.

Objective Type Questions

Q. 26 If $10^n + 3 \cdot 4^{n+2} + k$ is divisible by 9, for all $n \in N$, then the least positive integral value of k is

(a) 5 (b) 3 (c) 7 (d) 1

Sol. ***(a)*** Let $P(n): 10^n + 3 \cdot 4^{n+2} + k$ is divisible by 9, for all $n \in N$.

For $n = 1$, the given statement is also true $10^1 + 3 \cdot 4^{1+2} + k$ is divisible by 9.

$\because$ $$= 10 + 3 \cdot 64 + k = 10 + 192 + k$$
$$= 202 + k$$

If $(202 + k)$ is divisible by 9, then the least value of k must be 5.

$\because$ $202 + 5 = 207$ is divisible by 9

$\Rightarrow$ $$\frac{207}{9} = 23$$

Hence, the least value of k is 5.

Q. 27 For all $n \in N$, $3 \cdot 5^{2n+1} + 2^{3n+1}$ is divisible by

(a) 19 (b) 17 (c) 23 (d) 25

Sol. ***(b, c)***

Given that, $3 \cdot 5^{2n+1} + 2^{3n+1}$

For $n = 1$,

$$3 \cdot 5^{2(1)+1} + 2^{3(1)+1}$$
$$= 3 \cdot 5^3 + 2^4$$
$$= 3 \times 125 + 16 = 375 + 16 = 391$$

Now, $391 = 17 \times 23$

which is divisible by both 17 and 23.

Q. 28 If $x^n - 1$ is divisible by $x - k$, then the least positive integral value of k is

(a) 1 (b) 2 (c) 3 (d) 4

Sol. Let $P(n): x^n - 1$ is divisible by $(x - k)$.

For $n = 1$, $x^1 - 1$ is divisible by $(x - k)$.

Since, if $x - 1$ is divisible by $x - k$. Then, the least possible integral value of k is 1.

Fillers

Q. 29 If $P(n): 2n < n!$, $n \in N$, then $P(n)$ is true for all $n \geq$ ....

Sol. Given that, $P(n): 2n < n!$, $n \in N$

For $n = 1$, $2 < !$ [false]

For $n = 2$, $2 \times 2 < 2! 4 < 2$ [false]

For $n = 3$, $2 \times 3 < 3!$

$6 < 3!$

$6 < 3 \times 2 \times 1$

$(6 < 6)$ [false]

For $n = 4$, $2 \times 4 < 4!$

$8 < 4 \times 3 \times 2 \times 1$

$(8 < 24)$ [true]

For $n = 5$, $2 \times 5 < 5!$

$10 < 5 \times 4 \times 3 \times 2 \times 1$

$(10 < 120)$ [true]

Hence, $P(n)$ is for all $n \geq 4$.

True/False

Q. 30 Let $P(n)$ be a statement and let $P(k) \Rightarrow P(k + 1)$, for some natural number k, then $P(n)$ is true for all $n \in N$.

Sol. ***False***

The given statement is false because $P(1)$ is true has not been proved.

5

Complex Numbers and Quadratic Equations

Short Answer Type Questions

Q. 1 For a positive integer n, find the value of $(1-i)^n\left(1-\frac{1}{i}\right)^n$.

Sol. Given expression $= (1-i)^n\left(1-\frac{1}{i}\right)^n$

$$= (1-i)^n (i-1)^n \cdot i^{-n} = (1-i)^n (1-i)^n (-1)^n \cdot i^{-n}$$

$$= [(1-i)^2]^n (-1)^n \cdot i^{-n} = (1+i^2-2i)^n (-1)^n i^{-n} \qquad [\because i^2 = -1]$$

$$= (1-1-2i)^n (-1)^n i^{-n} = (-2)^n \cdot i^n (-1)^n i^{-n}$$

$$= (-1)^{2n} \cdot 2^n = 2^n$$

Q. 2 Evaluate $\sum_{n=1}^{13} (i^n + i^{n+1})$, where $n \in N$.

Thinking Process

Use $i^2 = -1, i^4 = (-1)^2 = 1, i^3 = -i$, and $i^5 = i$ to solve it

Sol. Given that, $\sum_{n=1}^{13} (i^n + i^{n+1}), n \in N$

$$= (i + i^2 + i^3 + i^4 + i^5 + i^6 + i^7 + i^8 + i^9 + i^{10} + i^{11} + i^{12} + i^{13}) + (i^2 + i^3 + i^4 + i^5 + i^6 + i^7 + i^8 + i^9 + i^{10} + i^{11} + i^{12} + i^{13} + i^{14})$$

$$= (i + 2i^2 + 2i^3 + 2i^4 + 2i^5 + 2i^6 + 2i^7 + 2i^8 + 2i^9 + 2i^{10} + 2i^{11} + 2i^{12} + 2i^{13} + i^{14})$$

$$= i - 2 - 2i + 2 + 2i + 2(i^4)i^2 + 2(i)^4 i^3 + 2(i^2)^4 + 2(i^2)^4 i + 2(i^2)^5 + 2(i^2)^5 \cdot i + 2(i^2)^6 + 2(i^2)^6 \cdot i + (i^2)^7$$

$$= i - 2 - 2i + 2 + 2i - 2 - 2i + 2 + 2i - 2 - 2i + 2 + 2i - 1 - 1 + i$$

Alternate Method

$$\sum_{n=1}^{13}(i^n+i^{n+1}),\ n\in N=\sum_{h=1}^{13} i^n(1+i)$$

$$=(1+i)[i+i^2+i^3+i^4+i^5+i^6+i^7+i^8+i^9+i^{10}+i^{11}+i^{12}+i^{13}]$$

$$=(1+i)[i^{13}]\quad [\because i^n+i^{n+1}+i^{n+2}+i^{n+3}=0,\text{ where } n\in N \text{ i.e., } \sum_{n=1}^{12} i^n=0$$

$$=(1+i)i$$

$[\because (i^4)^3\cdot i=i]$

$$=(i^2+i)=i-1$$

Q. 3 If $\left(\frac{1+i}{1-i}\right)^3-\left(\frac{1-i}{1+i}\right)^3=x+iy$, then find (x, y).

Thinking Process

If two complex numbers $z_1=x_1+iy^1$ and $z_2=x_2+iy_2$ are equal

i.e., $z_1=z_2 \Rightarrow x_1+iy_1=x_2+iy_2$, then $x_1=x_2$ and $y_1=y_2$.

Sol. Given that, $\left(\frac{1+i}{1-i}\right)^3-\left(\frac{1-i}{1+i}\right)^3=x+iy$ ...(i)

$\therefore$ $$\left(\frac{1+i}{1-i}\right)^3=\frac{1+i^3+3i(1+i)}{1-i^3-3i(1-i)}=\frac{1-i+3i+3i^2}{1+i-3i+3i^2}$$

$$=\frac{2i-2}{-2i-2}=\frac{i-1}{-i-1}=\frac{1-i}{1+i}$$

$$=\frac{(1-i)}{(1+i)}\frac{(1-i)}{(1-i)}=\frac{1+i^2-2i}{1+1}=\frac{1-1-2i}{2}$$

$\Rightarrow$ $$\left(\frac{1+i}{1-i}\right)^3=-i \quad \text{...(ii)}$$

Similarly, $$\left(\frac{1-i}{1+i}\right)^3=\frac{-1}{i}=\frac{i^2}{i}=i \quad \text{...(iii)}$$

Using Eqs. (ii) and (iii) in Eq. (i), we get

$$-i-i=x+iy$$

$\Rightarrow$ $$-2i=x+iy$$

On comparing real and imaginary part of complex number, we get

$$x=0 \text{ and } y=-2$$

So, $$(x, y)=(0,-2)$$

Q. 4 If $\frac{(1+i)^2}{2-i}=x+iy$, then find the value of $x+y$.

Sol. Given that, $$\frac{(1+i)^2}{2-i}=x+iy$$

$\Rightarrow$ $$\frac{(1+i^2+2i)}{2-i}=x+iy \Rightarrow \frac{2i}{2-i}=x+iy$$

$\Rightarrow$ $$\frac{2i(2+i)}{(2-i)(2+i)}=x+iy \Rightarrow \frac{4i+2i^2}{4-i^2}=x+iy$$

$$\Rightarrow \quad \frac{4i-2}{4+1} = x+iy \quad \Rightarrow \quad \frac{-2}{5} + \frac{4i}{5} = x+iy$$

On comparing both sides, we get

$$x = -2/5 \quad \Rightarrow \quad y = 4/5$$

$$\Rightarrow \quad x + y = \frac{-2}{5} + \frac{4}{5} = 2/5$$

Q. 5 If $\left(\frac{1-i}{1+i}\right)^{100} = a+ib$, then find (a, b).

Sol. Given that, $\left(\frac{1-i}{1+i}\right)^{100} = a+ib$

$$\Rightarrow \quad \left[\frac{(1-i)}{(1+i)}\cdot\frac{(1-i)}{(1-i)}\right]^{100} = a+ib \quad \Rightarrow \quad \left(\frac{1+i^2-2i}{1-i^2}\right)^{100} = a+ib$$

$$\Rightarrow \quad \left(\frac{-2i}{2}\right)^{100} = a+ib \qquad [\because i^2 = -1]$$

$$\Rightarrow \quad (i^4)^{25} = a+ib \quad \Rightarrow \quad 1 = a+ib$$

Then, $a = 1$ and $b = 0$ $\qquad [\because i^4 = 1]$

$\therefore \quad (a, b) = (1, 0)$

Q. 6 If $a = \cos\theta + i\sin\theta$, then find the value of $\frac{1+a}{1-a}$.

Thinking Process

To solve the above problem use the trigonometric formula $\cos\theta = 2\cos^2\theta/2 - 1 = 1 - 2\sin^2\theta/2$ *and* $\sin\theta = 2\sin\theta/2\cdot\cos\theta/2$.

Sol. Given that, $a = \cos\theta + i\sin\theta$

$$\therefore \quad \frac{1+a}{1-a} = \frac{1+\cos\theta + i\sin\theta}{1-\cos\theta - i\sin\theta}$$

$$= \frac{1+2\cos^2\theta/2 - 1 + 2i\sin\theta/2\cdot\cos\theta/2}{1-1+2\sin^2\theta/2 - 2i\sin\theta/2\cdot\cos\theta/2} = \frac{2\cos\theta/2(\cos\theta/2 + i\sin\theta/2)}{2\sin\theta/2(\sin\theta/2 - i\cos\theta/2)}$$

$$= -\frac{2\cos\theta/2(\cos\theta/2 + i\sin\theta/2)}{2i\sin\theta/2(\cos\theta/2 + i\sin\theta/2)} = -\frac{1}{i}\cot\theta/2$$

$$= \frac{+i^2}{i}\cot\theta/2 = i\cot\theta/2 \qquad \left[\because \frac{-1}{i} = \frac{i^2}{i}\right]$$

Q. 7 If $(1+i)z = (1-i)\bar{z}$, then show that $z = -i\bar{z}$.

Sol. We have, $(1+i)z = (1-i)\bar{z} \quad \Rightarrow \quad \frac{z}{\bar{z}} = \frac{(1-i)}{(1+i)}$

$$\Rightarrow \quad \frac{z}{\bar{z}} = \frac{(1-i)}{(1+i)}\frac{(1-i)}{(1-i)} \quad \Rightarrow \quad \frac{z}{\bar{z}} = \frac{1+i^2-2i}{1-i^2} \qquad [\because i^2 = -1]$$

$$\Rightarrow \quad \frac{z}{\bar{z}} = \frac{1-1-2i}{2} \quad \Rightarrow \quad \frac{z}{\bar{z}} = -i$$

$\therefore \quad z = -i\bar{z}$ **Hence proved.**

Q. 8 If $z = x + iy$, then show that $z\bar{z} + 2(z + \bar{z}) + b = 0$, where $b \in R$, represents a circle.

Sol. Given that, $z = x + iy$

Then, $\bar{z} = x - iy$

Now, $z\bar{z} + 2(z + \bar{z}) + b = 0$

$\Rightarrow (x + iy)(x - iy) + 2(x + iy + x - iy) + b = 0$

$\Rightarrow x^2 + y^2 + 4x + b = 0$, which is the equation of a circle.

Q. 9 If the real part of $\dfrac{\bar{z}+2}{\bar{z}-1}$ is 4, then show that the locus of the point representing z in the complex plane is a circle.

Sol. Let $z = x + iy$

Now,
$$\frac{\bar{z}+2}{\bar{z}-1} = \frac{x - iy + 2}{x - iy - 1}$$
$$= \frac{[(x+2) - iy][(x-1) + iy]}{[(x-1) - iy][(x-1) + iy]}$$
$$= \frac{(x-1)(x+2) - iy(x-1) + iy(x+2) + y^2}{(x-1)^2 + y^2}$$
$$= \frac{(x-1)(x+2) + y^2 + i[(x+2)y - (x-1)y]}{(x-1)^2 + y^2} \quad [\because -i^2 = 1]$$

Taking real part, $\dfrac{(x-1)(x+2) + y^2}{(x-1)^2 + y^2} = 4$

$\Rightarrow x^2 - x + 2x - 2 + y^2 = 4(x^2 - 2x + 1 + y^2)$

$\Rightarrow 3x^2 + 3y^2 - 9x + 6 = 0$, which represents a circle.

Hence, z lies on the circle.

Q. 10 Show that the complex number z, satisfying the condition $\arg\left(\dfrac{z-1}{z+1}\right) = \dfrac{\pi}{4}$ lies on a circle.

Thinking Process

First use, $\arg\left(\dfrac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$. *Also apply* $\arg(z) = \theta = \tan^{-1}\dfrac{y}{x}$, *where* $z = x + iy$ *and then use the property* $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\dfrac{x-y}{1+xy}\right)$

Sol. Let $z = x + iy$

Given that, $\arg\left(\dfrac{z-1}{z+1}\right) = \pi/4$

$\Rightarrow \arg(z-1) - \arg(z+1) = \pi/4$

$\Rightarrow \arg(x + iy - 1) - \arg(x + iy + 1) = \pi/4$

$\Rightarrow \arg(x - 1 + iy) - \arg(x + 1 + iy) = \pi/4$

$$\Rightarrow \quad \tan^{-1}\frac{y}{x-1} - \tan^{-1}\frac{y}{x+1} = \pi/4$$

$$\Rightarrow \quad \tan^{-1}\left[\frac{\frac{y}{x-1}-\frac{y}{x+1}}{1+\left(\frac{y}{x-1}\right)\left(\frac{y}{x+1}\right)}\right] = \pi/4$$

$$\Rightarrow \quad \frac{y\left[\frac{x+1-x+1}{x^2-1}\right]}{\frac{x^2-1+y^2}{x^2-1}} = \tan \pi/4$$

$$\Rightarrow \quad \frac{2y}{x^2+y^2-1} = 1$$

$$\Rightarrow \quad x^2+y^2-1 = 2y$$

$\Rightarrow \quad x^2+y^2-2y-1 = 0$, which represents a circle.

Q. 11 Solve the equation $|z| = z + 1 + 2i$.

Sol. The given equation is $|z| = z + 1 + 2i$ ...(i)

Let $z = x + iy$

From Eq. (i), $|x + iy| = x + iy + 1 + 2i$

$\Rightarrow \quad \sqrt{x^2+y^2} = x + iy + 1 + 2i$ $\left[\because |z| + \sqrt{x^2 = y^2}\right]$

$\Rightarrow \quad \sqrt{x^2+y^2} = (x+1) + i\,(y+2)$

On squaring both sides, we get

$$x^2 + y^2 = (x+1)^2 + i^2(y+2)^2 + 2i\,(x+1)(y+2)$$

$$\Rightarrow \quad x^2 + y^2 = x^2 + 2x + 1 - y^2 - 4y - 4 + 2i\,(x+1)(y+2)$$

On comparing real and imaginary parts,

$$x^2 + y^2 = x^2 + 2x + 1 - y^2 - 4y - 4$$

i.e., $2y^2 = 2x - 4y - 3$ ...(ii)

and $2\,(x+1)(y+2) = 0$

$(x+1) = 0$ or $(y+2) = 0$

$\Rightarrow \quad x = -1$ or $y = -2$

For $x = -1$, $2y^2 = -2 - 4y - 3$

$2y^2 + 4y + 5 = 0$ [using Eq. (ii)]

$$\Rightarrow \quad y = \frac{-4 \pm \sqrt{16 - 2\times 4\times 5}}{4}$$

$$\Rightarrow \quad y = \frac{-4 \pm \sqrt{-24}}{4} \notin R$$

For $y = -2$, $2(-2)^2 = 2x - 4(-2) - 3$ [usign Eq. (ii)]

$\Rightarrow \quad 8 = 2x + 8 - 3$

$\Rightarrow \quad 2x = 3 \Rightarrow x = 3/2$

$\therefore \quad z = x + iy = 3/2 - 2i$

Long Answer Type Questions

Q. 12 If $|z+1| = z + 2(1+i)$, then find the value of z.

Sol. Given that, $|z+1| = z + 2(1+i)$ ...(i)

$$z = x + iy$$

Then, $|x+iy+1| = x + iy + 2(1+i)$

$\Rightarrow$ $|x+1+iy| = (x+2) + i(y+2)$

$\Rightarrow$ $\sqrt{(x+1)^2 + y^2} = (x+2) + i(y+2)$

On squaring both sides, we get

$$(x+1)^2 + y^2 = (x+2)^2 + i^2(y+2)^2 + 2i(x+2)(y+2)$$

$\Rightarrow$ $x^2 + 2x + 1 + y^2 = x^2 + 4x + 4 - y^2 - 4y - 4 + 2i(x+2)(y+2)$

$\Rightarrow$ $x^2 + y^2 + 2x + 1 = x^2 - y^2 + 4x - 4y + 2i(x+2)(y+2)$

On comparing real and imaginary parts, we get

$$x^2 + y^2 + 2x + 1 = x^2 - y^2 + 4x - 4y$$

$\Rightarrow$ $2y^2 - 2x + 4y + 1 = 0$ ...(ii)

and $2(x+2)(y+2) = 0$

$\Rightarrow$ $x + 2 = 0$ or $y + 2 = 0$

$x = -2$ or $y = -2$ ...(iii)

For $x = -2$, $2y^2 + 4 + 4y + 1 = 0$ [using Eq. (ii)]

$\Rightarrow$ $2y^2 + 4y + 5 = 0$

$\Rightarrow$ $16 - 4 \times 2 \times 5 < 0$

$\therefore$ Discriminant, $D = b^2 - 4ac < 0$

$\Rightarrow$ $2y^2 + 4y + 5$ has no real roots.

For $y = -2$, $2(-2)^2 - 2x + 4(-2) + 1 = 0$ [using Eq. (ii)]

$\Rightarrow$ $8 - 2x - 8 + 1 = 0$

$\Rightarrow$ $x = 1/2$

$\therefore$ $z = x + iy = \frac{1}{2} - 2i$

Q. 13 If $\arg(z-1) = \arg(z+3i)$, then find $x - 1 : y$, where $z = x + iy$.

Sol. Given that, $\arg(z-1) = \arg(z+3i)$

and let $z = x + iy$

Now, $\arg(z-1) = \arg(z+3i)$

$\Rightarrow$ $\arg(x + iy - 1) = \arg(x + iy + 3i)$

$\Rightarrow$ $\arg(x - 1 + iy) = \arg[x + i(y+3)]$

$\Rightarrow$ $\tan^{-1}\frac{y}{x-1} = \tan^{-1}\frac{y+3}{x}$

$\Rightarrow$ $\frac{y}{x-1} = \frac{y+3}{x} \Rightarrow xy = (x-1)(y+3)$

$\Rightarrow$ $xy = xy - y + 3x - 3 \Rightarrow 3x - 3 = y$

$$\Rightarrow \quad \frac{3(x-1)}{y} = 1 \Rightarrow \frac{x-1}{y} = \frac{1}{3}$$

$$\therefore \quad (x-1) : y = 1 : 3$$

Q. 14 Show that $\left|\frac{z-2}{z-3}\right| = 2$ represents a circle. Find its centre and radius.

Thinking Process

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are two complex numbers, then $\left|\frac{z_1}{z_2}\right| = \left|\frac{z_1}{z_2}\right|, (z_2 \neq 0)$, use this concept to solve the above problem. Also, we know that general equation of a circle is $x^2 + y^2 + 2gx + 2fy + c = 0$, with centre $(-g, -f)$ and radius $= \sqrt{g^2 + f^2 - c}$.

Sol. Let $z = x + iy$

Given, equation is $\left|\frac{z-2}{z-3}\right| = 2 \Rightarrow \left|\frac{z-2}{z-3}\right| = 2$

$$\Rightarrow \quad \left|\frac{x+iy-2}{x+iy-3}\right| = 2$$

$$\Rightarrow \quad |x-2+iy| = 2|x-3+iy|$$

$$\Rightarrow \quad \sqrt{(x-2)^2 + y^2} = 2\sqrt{(x-3)^2 + y^2} \qquad \left[\because |x+iy| = \sqrt{x^2+y^2}\right]$$

On squaring both sides, we get

$$x^2 - 4x + 4 + y^2 = 4(x^2 - 6x + 9 + y^2)$$

$$\Rightarrow \quad 3x^2 + 3y^2 - 20x + 32 = 0$$

$$\Rightarrow \quad x^2 + y^2 - \frac{20}{3}x + \frac{32}{3} = 0 \qquad \ldots\text{(i)}$$

On comparing the above equation with $x^2 + y^2 + 2gx + 2fy + c = 0$, we get

$$\Rightarrow \quad 2g = \frac{-20}{3} \Rightarrow g = \frac{-10}{3}$$

and $\quad 2f = 0 \Rightarrow f = 0$ and $c = \frac{32}{3}$

$$\therefore \quad \text{Centre} = (-g, -f) = (10/3, 0)$$

Also, $\quad$ radius $(r) = \sqrt{(10/3)^2 + 0 - 32/3} \qquad [\because r = \sqrt{g^2 + f^2 - c}]$

$$= \frac{1}{3}\sqrt{(100-96)} = 2/3$$

Q. 15 If $\frac{z-1}{z+1}$ is a purely imaginary number $(z \neq -1)$, then find the value of $|z|$.

Thinking Process

If $z = x + iy$ is a purely imaginary number, then its real part must be equal to zero i.e., $x = 0$,

Sol. Let $\quad z = x + iy$

$$\frac{z-1}{z+1} = \frac{x+iy-1}{x+iy+1}, \ z \neq -1$$

$$= \frac{x-1+iy}{x+1+iy} = \frac{(x-1+iy)(x+1-iy)}{(x+1+iy)(x+1-iy)}$$

$$= \frac{(x^2 - 1) + iy(x+1) - iy(x-1) - i^2y^2}{(x+1)^2 - (iy)^2}$$

$$\Rightarrow \quad \frac{z-1}{z+1} = \frac{(x^2-1) + y^2 + i[y(x+1) - y(x-1)]}{(x+1)^2 + y^2}$$

Given that, $\frac{z-1}{z+1}$ is a purely imaginary numbers.

Then, $$\frac{(x^2-1)+y^2}{(x+1)^2+y^2} = 0$$

$$\Rightarrow \quad x^2 - 1 + y^2 = 0 \Rightarrow x^2 + y^2 = 1$$

$$\Rightarrow \quad \sqrt{x^2+y^2} = \sqrt{1} \Rightarrow |z| = 1 \qquad [\because |z| = \sqrt{x^2+y^2}]$$

Q. 16 z_1 and z_2 are two complex numbers such that $|z_1| = |z_2|$ and $\arg(z_1) + \arg(z_2) = \pi$, then show that $z_1 = -\bar{z}_2$.

Sol. Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ are two complex numbers.

Given that, $|z_1| = |z_2|$

and $\arg(z_1) + \arg(z_2) = \pi$

If $|z_1| = |z_2|$

$\Rightarrow r_1 = r_2$ ...(i)

and if $\arg(z_1) + \arg(z_2) = \pi$

$\Rightarrow \theta_1 + \theta_2 = \pi$

$\Rightarrow \theta_1 = \pi - \theta_2$

Now, $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

$\Rightarrow z_1 = r_2[\cos(\pi - \theta_2) + i\sin(\pi - \theta_2)]$ $[\because r_1 = r_2$ and $\theta_1 = (\pi - \theta_2)]$

$\Rightarrow z_1 = r_2(-\cos\theta_2 + i\sin\theta_2)$

$\Rightarrow z_1 = -r_2(\cos\theta_2 - i\sin\theta_2)$

$\Rightarrow z_1 = -[r_2(\cos\theta_2 - i\sin\theta_2)]$

$\Rightarrow z_1 = -\bar{z}_2$ $[\because \bar{z}_2 = r_2(\cos\theta_2 - i\sin\theta_2)]$

Q. 17 If $|z_1| = 1\ (z_1 \neq -1)$ and $z_2 = \frac{z_1 - 1}{z_1 + 1}$, then show that the real part of z_2 is zero.

Sol. Let $z_1 = x + iy$

$\Rightarrow |z_1| = \sqrt{x^2+y^2} = 1$ $[\because |z_1| = 1$, given$]$ ...(i)

Now, $$z_2 = \frac{z_1 - 1}{z_1 + 1} = \frac{x + iy - 1}{x + iy + 1}$$

$$= \frac{x - 1 + iy}{x + 1 + iy} = \frac{(x - 1 + iy)(x + 1 - iy)}{(x + 1 + iy)(x + 1 - iy)}$$

$$= \frac{x^2 - 1 + iy(x+1) - iy(x-1) - i^2y^2}{(x+1)^2 - i^2y^2}$$

$$= \frac{x^2 - 1 + ixy + iy - ixy + iy + y^2}{(x+1)^2 + y^2}$$

$$= \frac{x^2 + y^2 - 1 + 2iy}{(x+1)^2 + y^2} = \frac{1 - 1 + 2iy}{(x+1)^2 + y^2} \quad [\because x^2 + y^2 = 1]$$

$$= 0 + \frac{2yi}{(x+1)^2 + y^2}$$

Hence, the real part of z_2 is zero.

Q. 18 If z_1, z_2 and z_3, z_4 are two pairs of conjugate complex numbers, then find $\arg\left(\frac{z_1}{z_4}\right) + \arg\left(\frac{z_2}{z_3}\right)$.

Thinking Process

First let, $z = r(\cos\theta + i\sin\theta)$, then conjugate of z i.e., $\bar{z} = r(\cos\theta - i\sin\theta)$. Use the property $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$.

Sol. Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$,

Then, $z_2 = \bar{z}_1 = r_1(\cos\theta_1 - i\sin\theta_1) = r_1[\cos(-\theta_1) + \sin(-\theta_1)]$

Also, let $z_3 = r_2(\cos\theta_2 + i\sin\theta_2)$,

Then, $z_4 = \bar{z}_3 = r_2(\cos\theta_2 - i\sin\theta_2)$

$$\arg\left(\frac{z_1}{z_4}\right) + \arg\left(\frac{z_2}{z_3}\right) = \arg(z_1) - \arg(z_4) + \arg(z_2) - \arg(z_3)$$

$$= \theta_1 - (-\theta_2) + (-\theta_1) - \theta_2 \quad [\because \arg(z) = \theta]$$

$$= \theta_1 + \theta_2 - \theta_1 - \theta_2 = 0$$

Q. 19 If $|z_1| = |z_2| = \cdots = |z_n| = 1$, then show that

$$|z_1 + z_2 + z_3 + \cdots + z_n| = \left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} + \cdots + \frac{1}{z_n}\right|.$$

Sol. Given that, $|z_1| = |z_2| = \cdots = |z_n| = 1$

$\Rightarrow$ $|z_1|^2 = |z_2|^2 = \cdots = |z_n|^2 = 1$

$\Rightarrow$ $z_1\bar{z}_1 = z_2\bar{z}_2 = z_3\bar{z}_3 = \cdots = z_n\bar{z}_n = 1$

$\Rightarrow$ $z_1 = \frac{1}{\bar{z}_1}, z_2 = \frac{1}{\bar{z}_2} = \cdots = z_n = \frac{1}{\bar{z}_n}$

Now, $|z_1 + z_2 + z_3 + z_4 + \cdots + z_n|$

$$= \left|\frac{z_1\bar{z}_1}{\bar{z}_1} + \frac{z_2\bar{z}_2}{\bar{z}_2} + \frac{z_3\bar{z}_3}{\bar{z}_3} + \cdots + \frac{z_n\bar{z}_n}{\bar{z}_n}\right| \quad \left[\because z_1\bar{z} = 1, \text{ where } z_1 = \frac{1}{z}, z_1 = \frac{\bar{z}}{\bar{z} - \bar{z}}, z_1 = \bar{z}\right]$$

$$= \left|\frac{|z_1|^2}{\bar{z}_1} + \frac{|z_2|^2}{\bar{z}_2} + \frac{|z_3|^2}{\bar{z}_3} + \cdots + \frac{|z_n|^2}{\bar{z}_n}\right|$$

$$= \left|\frac{1}{\bar{z}_1} + \frac{1}{\bar{z}_2} + \frac{1}{\bar{z}_3} + \cdots + \frac{1}{\bar{z}_n}\right| = \sqrt{\frac{1}{\bar{z}_1} + \frac{1}{\bar{z}_2} + \frac{1}{\bar{z}_3}}$$

$$= \left|\frac{1}{z_1} + \frac{1}{z_2} + \cdots + \frac{1}{z_n}\right|$$

Hence proved.

Q. 20 If the complex numbers z_1 and z_2, $\arg(z_1) - \arg(z_2) = 0$, then show that $|z_1 - z_2| = |z_1| - |z_2|$.

Sol. Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$\Rightarrow$ $\arg(z_1) = \theta_1$ and $\arg(z_2) = \theta_2$

Given that, $\arg(z_1) - \arg(z_2) = 0$

$$\theta_1 - \theta_2 = 0 \Rightarrow \theta_1 = \theta_2$$

$$z_2 = r_2(\cos\theta_1 + i\sin\theta_1) \qquad [\because \theta_1 = \theta_2]$$

$$z_1 - z_2 = (r_1\cos\theta_1 - r_2\cos\theta_1) + i(r_1\sin\theta_1 - r_2\sin\theta_1)$$

$$|z_1 - z_2| = \sqrt{(r_1\cos\theta_1 - r_2\cos\theta_1)^2 + (r_1\sin\theta_1 - r_2\sin\theta_1)^2}$$

$$= \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos^2\theta_1 - 2r_1r_2\sin^2\theta_1}$$

$$= \sqrt{r_1^2 + r_2^2 - 2r_1r_2(\sin^2\theta_1 + \cos^2\theta_1)}$$

$$= \sqrt{r_1^2 + r_2^2 - 2r_1r_2} = \sqrt{(r_1 - r_2)^2}$$

$\Rightarrow$ $|z_1 - z_2| = r_1 - r_2$ $\qquad [\because r = |z|]$

$= |z_1| - |z_2|$ **Hence proved.**

Q. 21 Solve the system of equations $\text{Re}(z^2) = 0$, $|z| = 2$.

Sol. Given that, $\text{Re}(z^2) = 0, |z| = 2$

Let $z = x + iy$

$$|z| = \sqrt{x^2 + y^2}$$

$\because$ $\sqrt{x^2 + y^2} = 2$

$\Rightarrow$ $x^2 + y^2 = 4$ ...(i)

and $\text{Re}(z) = x$

Also, $z = x + iy$

$\Rightarrow$ $z^2 = x^2 + 2ixy - y^2$

$\Rightarrow$ $z^2 = (x^2 - y^2) + 2ixy$

$\Rightarrow$ $\text{Re}(z^2) = x^2 - y^2$ $\qquad [\because \text{Re}(z^2) = 0]$

$\Rightarrow$ $x^2 - y^2 = 0$ ...(ii)

From Eqs. (i) and (ii),

$$x^2 + x^2 = 4$$

$\Rightarrow$ $2x^2 = 4 \Rightarrow x^2 = 2$

$\Rightarrow$ $x = \pm\sqrt{2}$

$\therefore$ $y = \pm\sqrt{2}$

$\because$ $z = x + iy$

$\Rightarrow$ $z = \sqrt{2} \pm i\sqrt{2}, -\sqrt{2} \pm i\sqrt{2}$

Q. 22 Find the complex number satisfying the equation $z + \sqrt{2}\,|(z+1)| + i = 0$.

Sol. Given equation is $z + \sqrt{2}\,|(z+1)| + i = 0$ ...(i)

Let $z = x + iy$

$\Rightarrow x + iy + \sqrt{2}\,|x + iy + 1| + i = 0$

$\Rightarrow x + i(1+y) + \sqrt{2}\left[\sqrt{(x+1)^2 + y^2}\right] = 0$

$\Rightarrow x + i(1+y) + \sqrt{2}\sqrt{(x^2 + 2x + 1 + y^2)} = 0$

$\Rightarrow x + \sqrt{2}\sqrt{x^2 + 2x + 1 + y^2} = 0$

$\Rightarrow x^2 = 2(x^2 + 2x + 1 + y^2$

$\Rightarrow x^2 + 4x + 2y^2 + 2 = 0$ ...(ii)

$1 + y = 0$

$\Rightarrow y = -1$

For $y = -1$, $x^2 + 4x + 2 + 2 = 0$ [using Eq. (ii)]

$\Rightarrow x^2 + 4x + 4 = 0 \Rightarrow (x+2)^2 = 0$

$\Rightarrow x + 2 = 0 \Rightarrow x = -2$

$\therefore z = x + iy = -2 - i$

Q. 23 Write the complex number $z = \dfrac{1-i}{\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}}$ in polar form.

Sol. Given that, $z = \dfrac{1-i}{\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}} = \dfrac{-\sqrt{2}\left[\frac{-1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right]}{\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}}$

$= \dfrac{-\sqrt{2}[\cos(\pi - \pi/4) + i\sin(\pi - \pi/4)]}{\cos\pi/3 + i\sin\pi/3}$

$= \dfrac{-\sqrt{2}[\cos 3\pi/4 + i\sin 3\pi/4]}{\cos\pi/3 + i\sin\pi/3}$

$= -\sqrt{2}\left[\cos\left(\frac{3\pi}{4} - \frac{\pi}{3}\right) + i\sin\left(\frac{3\pi}{4} - \frac{\pi}{3}\right)\right]$

$= -\sqrt{2}\left[\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right]$

Q. 24 If z and w are two complex numbers such that $|zw| = 1$ and $\arg(z) - \arg(w) = \dfrac{\pi}{2}$, then show that $\bar{z}\,w = -i$.

Sol. Let $z = r_1(\cos\theta_1 + i\sin\theta_1)$ and $w = r_2(\cos\theta_2 + i\sin\theta_2)$

Also, $|zw| = |z|\,|w| = r_1\,r_2 = 1$

$\therefore r_1 r_2 = 1$

Further, $\arg(z) = \theta_1$ and $\arg(w) = \theta_2$

But $\quad \arg(z) - \arg(w) = \dfrac{\pi}{2}$

$\Rightarrow \quad \theta_1 - \theta_2 = \dfrac{\pi}{2}$

$\Rightarrow \quad \arg\left(\dfrac{z}{w}\right) = \dfrac{\pi}{2}$

Now, to prove $\bar{z}w = -i$

$$\text{LHS} = \bar{z}w$$
$$= r_1(\cos\theta_1 - i\sin\theta_1)\, r_2(\cos\theta_2 + i\sin\theta_2)$$
$$= r_1 r_2[\cos(\theta_2 - \theta_1) + i\sin(\theta_2 - \theta_1)]$$
$$= r_1 r_2[\cos(-\pi/2) + i\sin(-\pi/2)]$$
$$= 1[0 - i]$$
$$= -i = \text{RHS}$$

Hence proved.

Q. 25 Fill in the blanks of the following.

(i) For any two complex numbers z_1, z_2 and any real numbers a, b, $|az_1 - bz_2|^2 + |bz_1 + az_2|^2 = \cdots$

(ii) The value of $\sqrt{-25} \times \sqrt{-9}$ is ...

(iii) The number $\dfrac{(1-i)^3}{1-i^3}$ is equal to ...

(iv) The sum of the series $i + i^2 + i^3 + \cdots$ upto 1000 terms is ...

(v) Multiplicative inverse of $1 + i$ is ...

(vi) If z_1 and z_2 are complex numbers such that $z_1 + z_2$ is a real number, then $z_1 = \cdots$

(vii) $\arg(z) + \arg \bar{z}$ where, $(\bar{z} \neq 0)$ is ...

(viii) If $|z + 4| \leq 3$, then the greatest and least values of $|z + 1|$ are ... and ...

(ix) If $\left|\dfrac{z-2}{z+2}\right| = \dfrac{\pi}{6}$, then the locus of z is ...

(x) If $|z| = 4$ and $\arg(z) = \dfrac{5\pi}{6}$, then $z = \cdots$

Sol. **(i)** $|az_1 - bz_2|^2 + |bz_1 + az_2|^2$

$$= |az_1|^2 + |bz_2|^2 - 2\text{Re}(az_1 \cdot b\bar{z}_2) + |bz_1|^2 + |az_2|^2 + 2\text{Re}(az_1 \cdot b\bar{z}_2)$$
$$= (a^2 + b^2)|z_1|^2 + (a^2 + b^2)|z_2|^2$$
$$= (a^2 + b^2)(|z_1|^2 + |z_2|^2)$$

(ii) $\sqrt{-25} \times \sqrt{-9} = i\sqrt{25} \times i\sqrt{9} = i^2(5 \times 3) = -15$

(iii) $\dfrac{(1-i)^3}{1-i^3} = \dfrac{(1-i)^3}{(1-i)(1+i+i^2)}$

$$= \frac{(1-i)^2}{i} = \frac{1+i^2-2i}{i} = \frac{-2i}{i} = -2$$

(iv) $i + i^2 + i^3 + \ldots$ upto 1000 terms $= i + i^2 + i^3 + i^4 + \ldots i^{1000} = 0$

$$\left[\because i^n + i^{n+1} + i^{n+2} + i^{n+3} = 0, \text{ where } n \in N \text{ i.e., } \sum_{n=1}^{1000} i^n = 0\right]$$

(v) Multiplicative inverse of $1 + i = \dfrac{1}{1+i} = \dfrac{1-i}{1-i^2} = \dfrac{1}{2}(1-i)$

(vi) Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$, which is real.

If $z_1 + z_2$ is real, then $y_1 + y_2 = 0$

$\Rightarrow \quad y_1 = -y_2$

$\because \quad z_2 = x_2 - iy_1$

$\Rightarrow \quad z_2 = \bar{z}_1$ [when $x_1 = x_2$]

(vii) $\arg(z) + \arg(\bar{z}), (\bar{z} \neq 0)$

$\Rightarrow \quad \theta + (-\theta) = 0$

(viii) Given that, $|z + 4| \leq 3$

For the greatest value of $|z + 1|$.

$$\Rightarrow \quad |z+1| = |z + 4 - 3| \leq |z + 4| + |-3|$$
$$= |z + 4 - 3| \leq 3 + 3$$
$$= |z + 4 - 3| \leq 6$$

So, greatest value of $|z+1| = 6$

For, now, least value of $|z+1|$, we know that the least value of the modulus of a complex number is zero. So, the least value of $|z+1|$ is zero.

(ix) Given that, $\left|\dfrac{z-2}{z+2}\right| = \dfrac{\pi}{6}$

$$\Rightarrow \quad \frac{|x + iy - 2|}{|x + iy + 2|} = \frac{\pi}{6} \quad \Rightarrow \quad \frac{|x - 2 + iy|}{|x + 2 + iy|} = \frac{\pi}{6}$$

$$\Rightarrow \quad 6|x - 2 + iy| = \pi\,|x + 2 + iy|$$

$$\Rightarrow \quad 6\sqrt{(x-2)^2 + y^2} = \pi\sqrt{(x+2)^2 + y^2}$$

$$\Rightarrow \quad 36[x^2 + 4 - 4x + y^2] = \pi^2\,[x^2 + 4x + 4 + y^2]$$

$\Rightarrow (36 - \pi^2)\,x^2 + (36 - \pi^2)\,y^2 - (144 + 4\pi^2)\,x + 144 + 4\pi^2 = 0$, which is a circle.

(x) Given that, $|z| = 4$ and $\arg(z) = \dfrac{5\pi}{6}$

Let $\quad z = x + iy = r(\cos\theta + i\sin\theta)$

$\Rightarrow \quad |z| = r = 4$ and $\arg(z) = \theta$

$\because \quad \tan\theta = \dfrac{5\pi}{6}$

$$\Rightarrow \quad z = 4\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = 4\,[\cos(\pi - \pi/6) + i\sin(\pi - \pi/6)]$$

$$= 4\left[-\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right] = 4\left[\frac{-\sqrt{3}}{2} + i\,\frac{1}{2}\right] = -2\sqrt{3} + 2i$$

True/False

Q. 26 State true or false for the following.

(i) The order relation is defined on the set of complex numbers.

(ii) Multiplication of a non-zero complex number by $-i$ rotates the point about origin through a right angle in the anti-clockwise direction.

(iii) For any complex number z, the minimum value of $|z| + |z-1|$ is 1.

(iv) The locus represented by $|z-1| = |z-i|$ is a line perpendicular to the join of the points (1, 0) and (0, 1).

(v) If z is a complex number such that $z \neq 0$ and $\text{Re}(z) = 0$, then, $\text{Im}(z^2) = 0$.

(vi) The inequality $|z-4| < |z-2|$ represents the region given by $x > 3$.

(vii) Let z_1 and z_2 be two complex numbers such that $|z_1 + z_2| = |z_1| + |z_2|$, then $\arg(z_1 - z_2) = 0$.

(viii) 2 is not a complex number.

Sol. (i) ***False***

We can compare two complex numbers when they are purely real. Otherwise comparison of complex number is not possible.

(ii) ***False***

$$(x, y)\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -y \end{bmatrix}, \text{ which is false.}$$

(iii) ***True***

Let $$z = x + iy$$

$$|z| + |z-1| = \sqrt{x^2 + y^2} + \sqrt{(x-1)^2 + y^2}$$

If $x = 0, y = 0$, then the value of $|z| + |z-1| = 1$.

(iv) ***True***

Let $$z = x + iy$$

$$|z-1| = |z-i|$$

Then, $$|x - 1 + iy| = |x - i(1-y)|$$

$$(x-1)^2 + y^2 = x^2 + (1-y)^2$$

$$x^2 - 2x + 1 + y^2 = x^2 + 1 + y^2 - 2y$$

$$-2x + 1 = 1 - 2y$$

$$-2x + 2y = 0$$

$$x - y = 0 \quad \text{...(i)}$$

Equation of a line through the points (1, 0) and (0, 1),

$$y - 0 = \frac{1-0}{0-1}(x-1)$$

$$\Rightarrow \quad y = -(x-1) \quad \Rightarrow \quad x + y = 1 \quad \text{...(ii)}$$

which is perpendicular to the line $x - y = 0$.

(v) ***False***

Let $z = x + iy$, $z \neq 0$ and Re $(z) = 0$

i.e., $x = 0$

$\therefore$ $z = iy$

$\text{Im}(z^2) = i^2 y^2 = -y^2$ which is real.

(vi) ***True***

Given inequality, $|z-4| < |z-2|$

Let $z = x + iy$

$\therefore$ $|x - 4 + iy| < |x - 2 + iy|$

$\Rightarrow$ $\sqrt{(x-4)^2 + y^2} < \sqrt{(x-2)^2 + y^2}$

$\Rightarrow$ $(x-4)^2 + y^2 < (x-2)^2 + y^2$

$\Rightarrow$ $x^2 - 8x + 16 + y^2 < x^2 - 4x + 4 + y^2$

$\Rightarrow$ $-8x + 16 < -4x + 4$

$\Rightarrow$ $-8x < -4x - 12$

$\Rightarrow$ $-4x < -12$

$\Rightarrow$ $4x > 12$

$\Rightarrow$ $x > 3$

(vii) ***False***

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

Given that, $|z_1 + z_2| = |z_1| + |z_2|$

$|x_1 + iy_1 + x_2 + iy_2| = |x_1 + iy_1| + |x_2 + iy_2|$

$\Rightarrow$ $\sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2} = \sqrt{(x_1^2 + y_1^2)} + \sqrt{(x_2^2 + y_2^2)}$

On squaring both sides, we get

$$(x_1 + x_2)^2 + (y_1 + y_2)^2 = (x_1^2 + y_1^2) + (x_2^2 + y_2^2) + 2\sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)}$$

$\Rightarrow$ $x_1^2 + x_2^2 + 2x_1x_2 + y_1^2 + y_2^2 + 2y_1y_2 = x_1^2 + y_1^2 + x_2^2 + y_2^2 + 2\sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)}$

$\Rightarrow$ $2x_1x_2 + 2y_1y_2 = 2\sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)}$

$\Rightarrow$ $x_1x_2 + y_1y_2 = \sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)}$

On squaring both sides, we get

$$x_1^2x_2^2 + y_1^2y_2^2 + 2x_1x_2y_1y_2 = x_1^2x_2^2 + y_1^2x_2^2 + x_1^2y_2^2 + y_1^2y_2^2$$

$\Rightarrow$ $(x_1y_2 - x_2y_1)^2 = 0$

$\Rightarrow$ $x_1y_2 = x_2y_1$

$\Rightarrow$ $\dfrac{y_1}{x_1} = \dfrac{y_2}{x_2}$

$\Rightarrow$ $\left(\dfrac{y_1}{x_1}\right) - \left(\dfrac{y_2}{x_2}\right) = 0$

$\Rightarrow$ $\arg(z_1) - \arg(z_2) = 0$

(viii) ***True***

We know that, 2 is a real number.

Since, 2 is not a complex number.

Q. 27 Match the statements of Column A and Column B.

	Column A		Column B
(i)	The polar form of $i+\sqrt{3}$ is	(a)	Perpendicular bisector of segment joining (–2, 0) and (2, 0).
(ii)	The amplitude of $-1+\sqrt{-3}$ is	(b)	On or outside the circle having centre at (0, – 4) and radius 3.
(iii)	It $\lvert z+2\rvert=\lvert z-2\rvert$, then locus of z is	(c)	$\frac{2\pi}{3}$
(iv)	It $\lvert z+2i\rvert=\lvert z-2i\rvert$, then locus of z is	(d)	Perpendiculor bisectar of segment joining (0, – 2) and (0, 2).
(v)	Region represented by	(e)	$2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$
(vi)	Region represented by $\lvert z+4\rvert\le 3$ is	(f)	On or inside the circle having centre (– 4, 0) and radius 3 units.
(vii)	Conjugate of $\frac{1+2i}{1-i}$ lies in	(g)	First quadrant
(viii)	Reciprocal of $1-i$ lies in	(h)	Third quadrant

Sol. **(i)** Given, $$z=i+\sqrt{3}=r(\cos\theta+\sin\theta)$$

$\because$ $$r\cos\theta=\sqrt{3},\ r\sin\theta=1$$

$\Rightarrow$ $$r^2=1+3=4 \Rightarrow r=2 \qquad [\because r>0]$$

$\Rightarrow$ $$\tan\alpha=\left|\frac{r\sin\theta}{r\cos\theta}\right|=\frac{1}{\sqrt{3}}$$

$\Rightarrow$ $$\tan\alpha=\frac{1}{\sqrt{3}} \Rightarrow \alpha=\frac{\pi}{6}$$

$\because$ $$x>0,\ y>0$$

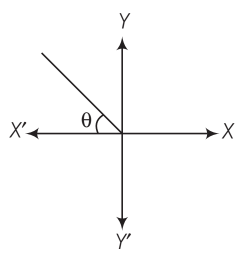
and $$\arg(z)=\theta=\frac{\pi}{6}$$

So the polar form of z is $2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$.

(ii) Given that, $$z=-1+\sqrt{-3}=-1+i\sqrt{3}$$

$\therefore$ $$\tan\alpha=\left|\frac{\sqrt{3}}{-1}\right|=\sqrt{3}$$

$\Rightarrow$ $$\tan\alpha=\tan\frac{\pi}{3} \Rightarrow \alpha=\frac{\pi}{3}$$



$\because$ $$x<0,\ y>0$$

$\Rightarrow$ $$\theta=\pi-\alpha=\pi-\frac{\pi}{3}=\frac{2\pi}{3}$$

(iii) Given that, $|z + 2| = |z - 2|$

$\Rightarrow$ $|x + 2 + iy| = |x - 2 + iy|$

$\Rightarrow$ $(x + 2)^2 + y^2 = (x - 2)^2 + y^2$

$\Rightarrow$ $x^2 + 4x + 4 = x^2 - 4x + 4 \quad \Rightarrow \quad 8x = 0$

$\therefore$ $x = 0$

It is a straight line which is a perpendicular bisector of segment joining the points (–2, 0) and (2, 0).

(iv) Given that, $|z + 2i| = |z - 2i|$

$\Rightarrow$ $|x + i(y + 2)| = |x + i(y - 2)|$

$\Rightarrow$ $x^2 + (y + 2)^2 = x^2 + (y - 2)^2$

$\Rightarrow$ $4y = 0 \quad \Rightarrow \quad y = 0$

It is a straight line, which is a perpendicular bisector of segment joining (0, –2) and (0, 2).

(v) Given that, $|z + 4i| \geq 3 = |x + iy + 4i| \geq 3$

$\Rightarrow$ $= |x + i\,(y + 4)| \geq 3$

$\Rightarrow$ $= \sqrt{x^2 + (y + 4)^2} \geq 3$

$\Rightarrow$ $= x^2 + (y + 4)^2 \geq 9$

$\Rightarrow$ $= x^2 + y^2 + 8y + 16 \geq 9$

$= x^2 + y^2 + 8y + 7 \geq 0$

Which represent a circle. On or outside having centre (0, – 4) and radius 3.

(vi) Given that, $|z + 4| \leq 3$

$\Rightarrow$ $|x + iy + 4| \leq 3$

$\Rightarrow$ $|x + 4 + iy| \leq 3$

$\Rightarrow$ $\sqrt{(x + 4)^2 + y^2} \leq 3$

$\Rightarrow$ $(x + 4)^2 + y^2 \leq 9$

$\Rightarrow$ $x^2 + 8x + 16 + y^2 \leq 9$

$\Rightarrow$ $x^2 + 8x + y^2 + 7 \leq 0$

It represent the region which is on or inside the circle having the centre (– 4, 0) and radius 3.

(vii) Given that, $z = \dfrac{1 + 2i}{1 - i} = \dfrac{(1 + 2i)(1 + i)}{(1 - i)(1 + i)}$

$= \dfrac{1 + 2i + i + 2i^2}{1 - i^2} = \dfrac{1 - 2 + 3i}{1 + 1} = \dfrac{-1 + 3i}{2}$

$\therefore$ $\bar{z} = \dfrac{-1}{2} - \dfrac{3i}{2}$

Hence, $\left(\dfrac{-1}{2}, \dfrac{-3}{2}\right)$ lies in third quadrant.

(viii) Given that, $z = 1 - i$

$\therefore$ $\dfrac{1}{z} = \dfrac{1}{1 - i} = \dfrac{1 + i}{(1 - i)(1 + i)} = \dfrac{1 + i}{1 - i^2} = \dfrac{1}{2}(1 + i)$

Hence, $\left(\dfrac{1}{2}, \dfrac{1}{2}\right)$ lies in first quadrant.

Hence, the correct matches are **(a)** → (v), **(b)** → (iii), **(c)** → (i), **(d)** → (iv), **(e)** → (ii), **(f)** → (vi), **(g)** → (viii), **(h)** → (vii)

Q. 28 What is the conjugate of $\dfrac{2-i}{(1-2i)^2}$?

Sol. Given that,

$$z = \frac{2-i}{(1-2i)^2} = \frac{2-i}{1+4i^2-4i}$$
$$= \frac{2-i}{1-4-4i} = \frac{2-i}{-3-4i}$$
$$= \frac{(2-i)}{-(3+4i)} = -\left[\frac{(2-i)(3-4i)}{(3+4i)(3-4i)}\right]$$
$$= -\left(\frac{6-8i-3i+4i^2}{9+16}\right) = -\frac{(-11i+2)}{25}$$
$$= \frac{-1}{25}(2-11i) \Rightarrow z = \frac{1}{25}(-2+11i)$$

$\therefore$ $$\bar{z} = \frac{1}{25}(-2-11i) = \frac{-2}{25} - \frac{11}{25}i$$

Q. 29 If $|z_1| = |z_2|$, is it necessary that $z_1 = z_2$.

Sol. Given that, $$|z_1| = |z_2|$$
Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

$\Rightarrow$ $$|x_1 + iy_1| = |x_2 + iy_2|$$
$\Rightarrow$ $$x_1^2 + y_1^2 = x_2^2 + y_2^2$$
$\Rightarrow$ $$x_1^2 = x_2^2,\ y_1^2 = y_2^2$$
$\Rightarrow$ $$x_1 = \pm x_2,\ y_1 = \pm y_2$$
$\Rightarrow$ $$z_1 = x_1 + iy_1 \text{ or } z_1 = \pm x_2 \pm iy_2$$

Hence, it is not neccessary that $z_1 = z_2$.

Q. 30 If $\dfrac{(a^2+1)^2}{2a-i} = x+iy$, then what is the value of x^2+y^2?

Sol. Given that, $$\frac{(a^2+1)^2}{2a-i} = x+iy \Rightarrow \frac{(a^2+1)^2}{(2a-i)} = x+iy$$

$\Rightarrow$ $$\frac{(a^2+1)^2(2a+i)}{(2a-i)(2a+i)} = x+iy$$
$\Rightarrow$ $$\frac{(a^2+1)^2(2a+i)}{4a^2+1} = x+iy$$
$\Rightarrow$ $$x = \frac{2a(a^2+1)^2}{4a^2+1} \quad \text{and} \quad y = \frac{(a^2+1)^2}{4a^2+1}$$

$\therefore$ $$x^2+y^2 = 4a^2\left[\frac{(a^2+1)^2}{4a^2+1}\right]^2 + \left[\frac{(a^2+1)^2}{4a^2+1}\right]^2$$
$$= \frac{(4a^2+1)(a^2+1)^4}{(4a^2+1)^2} = \frac{(a^2+1)^4}{(4a^2+1)}$$

Q. 31 Find the value of z, if $|z| = 4$ and $\arg(z) = \dfrac{5\pi}{6}$.

Sol. Let $z = r(\cos\theta + i\sin\theta)$

Also, $|z| = r = 4$ and $\theta = \dfrac{5\pi}{6}$ $[\because \arg(z) = \theta]$

$\therefore$ $z = 4\left[\cos\dfrac{5\pi}{6} + i\sin\dfrac{5\pi}{6}\right]$ $[\because z = r(\cos\theta + i\sin\theta)]$

$$= 4\left[\cos\left(\pi - \frac{\pi}{6}\right) + i\sin\left(\pi - \frac{\pi}{6}\right)\right]$$

$$= 4\left[-\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right]$$

$$= 4\left[-\frac{\sqrt{3}}{2} + i\frac{1}{2}\right] = -2\sqrt{3} + 2i$$

Q. 32 Find the value of $\left|(1+i)\dfrac{(2+i)}{(3+i)}\right|$.

Thinking Process

First, convert the given expression in the formed $a + ib$, then use $|a + ib| = \sqrt{a^2 + b^2}$.

Sol. Given that, $\left|(1+i)\dfrac{(2+i)}{(3+i)}\right| = \left|\dfrac{(2+i+2i+i^2)}{(3+i)}\right| = \left|\dfrac{2+3i-1}{3+i}\right|$

$$= \left|\frac{1+3i}{3+i}\right| = \left|\frac{(1+3i)(3-i)}{(3+i)(3-i)}\right|$$

$$= \left|\frac{3+9i-i-3i^2}{9-i^2}\right| = \left|\frac{3+8i+3}{9+1}\right| = \left|\frac{6+8i}{10}\right|$$

$$= \sqrt{\frac{6^2}{100} + \frac{8^2}{100}} = \sqrt{\frac{36+64}{100}} = \sqrt{\frac{100}{100}} = 1$$

Q. 33 Find the principal argument of $(1 + i\sqrt{3})^2$.

Thinking Process

Let $z = a + ib$, then the polar form of z is $r(\cos\theta + i\sin\theta)$, where $r = |z| = \sqrt{a^2 + b^2}$ and $\tan\theta = \dfrac{b}{a}\cdot$ Here, θ is argument or amplitude of z i.e., arg $(z) = \theta$. The principal argument is a unique value of θ such that $-\pi \le \theta \le \pi$.

Sol. Given that, $z = (1 + i\sqrt{3})^2$

$\Rightarrow$ $z = 1 - 3 + 2i\sqrt{3} \Rightarrow z = -2 + i2\sqrt{3}$

$\Rightarrow$ $\tan\alpha = \left|\dfrac{2\sqrt{3}}{-2}\right| = |-\sqrt{3}| = \sqrt{3}$ $\left[\because \tan\alpha = \left|\dfrac{\text{Im}(z)}{\text{Re}(z)}\right|\right]$

$\Rightarrow$ $\tan\alpha = \tan\dfrac{\pi}{3} \Rightarrow \alpha = \dfrac{\pi}{3}$

$\because$ Re $(z) < 0$ and Im $(z) > 0$

$\Rightarrow$ $\arg(z) = \pi - \dfrac{\pi}{3} \Rightarrow = \dfrac{2\pi}{3}$

Q. 34 Where does z lie, if $\left|\frac{z-5i}{z+5i}\right|=1$?

Thinking Process

If $z_1=x_1+iy_1$ and $z_2=x_2+iy_2$, then $|z_1|=\sqrt{x_1^2+y_1^2}$ and $|z_2|=\sqrt{x_2^2+y_2^2}$.

Also, use the modulus property i.e., $\left|\frac{z_1}{z_2}\right|=\frac{|z_1|}{|z_2|}$,

Sol. Let $z=x+iy$

Given that, $\left|\frac{z-5i}{z+5i}\right|=\left|\frac{x+iy-5i}{x+iy+5i}\right|$

$\Rightarrow \left|\frac{z-5i}{z+5i}\right|=\frac{|x+i(y-5)|}{|x+i(y+5)|}$ $\quad\left[\because \left|\frac{z-5i}{z+5i}\right|=1\right]$

$\Rightarrow \left|\frac{z-5i}{z+5i}\right|=\frac{\sqrt{x^2+(y-5)^2}}{\sqrt{x^2+(y+5)^2}}$

On squaring both sides, we get

$$x^2+(y-5)^2=x^2+(y+5)^2$$

$\Rightarrow -10y=+10y$

$\Rightarrow 20y=0$

$\therefore y=0$

So, z lies on real axis.

Objective Type Questions

Q. 35 $\sin x+i\cos 2x$ and $\cos x-i\sin 2x$ are conjugate to each other for

(a) $x=n\pi$

(b) $x=\left(n+\frac{1}{2}\right)\frac{\pi}{2}$

(c) $x=0$

(d) No value of x

Sol. *(d)* Let $z=\sin x+i\cos 2x$

and $\bar{z}=\sin x-i\cos 2x$ ...(i)

Given that, $\bar{z}=\cos x-i\sin 2x$ ...(ii)

$\therefore \sin x-i\cos 2x=\cos x-i\sin 2x$

$\Rightarrow \sin x=\cos x$ and $\cos 2x=\sin 2x$

$\Rightarrow \tan x=1$ and $\tan 2x=1$

$\Rightarrow \tan x=\tan\frac{\pi}{4}$ and $\tan 2x=\tan\frac{\pi}{4}$

$\Rightarrow x=n\pi+\frac{\pi}{4}$ and $2x=n\pi+\frac{\pi}{4}$

$\Rightarrow 2x-x=0 \Rightarrow x=0$

Q. 36 The real value of α for which the expression $\dfrac{1 - i\sin\alpha}{1 + 2i\sin\alpha}$ is purely real is

(a) $(n+1)\dfrac{\pi}{2}$ (b) $(2n+1)\dfrac{\pi}{2}$ (c) $n\pi$ (d) None of these

where, $n \in N$

Thinking Process

First, convert the given expansion into $a + ib$ form and then check whether the complex number $a + ib$ is purely real.

Sol. ***(c)*** Given expression, $z = \dfrac{1 - i\sin\alpha}{1 + 2i\sin\alpha}$ [let]

$$= \frac{(1 - i\sin\alpha)(1 - 2i\sin\alpha)}{(1 + 2i\sin\alpha)(1 - 2i\sin\alpha)}$$

$$= \frac{1 - i\sin\alpha - 2i\sin\alpha + 2i^2\sin^2\alpha}{1 - 4i^2\sin^2\alpha}$$

$$= \frac{1 - 3i\sin\alpha - 2\sin^2\alpha}{1 + 4\sin^2\alpha}$$

$$= \frac{1 - 2\sin^2\alpha}{1 + 4\sin^2\alpha} - \frac{3i\sin\alpha}{1 + 4\sin^2\alpha}$$

It is given that z is a purely real.

$$\therefore \quad \frac{-3\sin\alpha}{1 + 4\sin^2\alpha} = 0$$

$$\Rightarrow \quad -3\sin\alpha = 0 \Rightarrow \sin\alpha = 0$$

$$\alpha = n\pi$$

Q. 37 If $z = x + iy$ lies in the third quadrant, then $\dfrac{\bar{z}}{z}$ also lies in the third quadrant, if

(a) $x > y > 0$ (b) $x < y < 0$ (c) $y < x < 0$ (d) $y > x > 0$

Sol. ***(b)*** Given that, $z = x + iy$ lies in third quadrant.

$$x < 0 \text{ and } y < 0.$$

Now, $$\frac{\bar{z}}{z} = \frac{x - iy}{x + iy} = \frac{(x - iy)(x - iy)}{(x + iy)(x - iy)} = \frac{x^2 - y^2 - 2ixy}{x^2 + y^2}$$

$$\frac{\bar{z}}{z} = \frac{x^2 - y^2}{x^2 + y^2} - \frac{2ixy}{x^2 + y^2}$$

Since, $\dfrac{\bar{z}}{z}$ also lies in third quadrant.

$$\therefore \quad \frac{x^2 - y^2}{x^2 + y^2} < 0 \text{ and } \frac{-2xy}{x^2 + y^2} < 0$$

$$x^2 - y^2 < 0 \text{ and } -2xy < 0$$

$$\Rightarrow \quad x^2 < y^2 \text{ and } xy > 0$$

So, $$x < y < 0$$

Q. 38 The value of $(z+3)(\bar{z}+3)$ is equivalent to

(a) $|z+3|^2$ (b) $|z-3|$ (c) z^2+3 (d) None of these

Sol. *(a)* Given that, $(z+3)(\bar{z}+3)$

Let $z = x + iy$

$$\Rightarrow \quad (z+3)(\bar{z}+3) = (x+iy+3)(x+3-iy)$$
$$= (x+3)^2 - (iy)^2 = (x+3)^2 + y^2$$
$$= |x+3+iy|^2 = |z+3|^2$$

Q. 39 If $\left(\frac{1+i}{1-i}\right)^x = 1$, then

(a) $x = 2n+1$ (b) $x = 4n$ (c) $x = 2n$ (d) $x = 4n+1$

where, $n \in N$

Sol. *(b)* Given that, $\left(\frac{1+i}{1-i}\right)^x = 1$

$$\Rightarrow \quad \left[\frac{(1+i)(1+i)}{(1-i)(1+i)}\right]^x = 1 \Rightarrow \left[\frac{1+2i+i^2}{1-i^2}\right]^x = 1$$

$$\Rightarrow \quad \left[\frac{2i}{1+1}\right]^x = 1 \Rightarrow \left[\frac{2i}{2}\right]^x = 1$$

$$\Rightarrow \quad i^x = 1 \Rightarrow i^x = (i^{4n}) \qquad [\because i^{4n} = 1, n \in N]$$

$$\Rightarrow \quad x = 4n$$

Q. 40 A real value of x satisfies the equation $\left(\frac{3-4ix}{3+4ix}\right) = \alpha - i\beta \ (\alpha, \beta \in R)$, if $\alpha^2 + \beta^2$ is equal to

(a) 1 (b) −1 (c) 2 (d) −2

Sol. *(a)* Given equation, $\left(\frac{3-4ix}{3+4ix}\right) = \alpha - i\beta \ (\alpha, \beta \in R)$

$$\Rightarrow \quad \left[\frac{3-4ix}{3+4ix}\right] = \alpha - i\beta$$

Now, $$(\alpha - i\beta) = \frac{(3-4ix)(3-4ix)}{(3+4ix)(3-4ix)} = \frac{9+16i^2x^2-24ix}{9-16i^2x^2}$$

$$\Rightarrow \quad \alpha - i\beta = \frac{9-16x^2-24ix}{9+16x^2}$$

$$\Rightarrow \quad \alpha - i\beta = \frac{9-16x^2}{9+16x^2} - \frac{i24x}{9+16x^2} \qquad \text{...(i)}$$

$$\therefore \quad \alpha + i\beta = \frac{9-16x^2}{9+16x^2} + \frac{i24x}{9+16x^2} \qquad \text{...(ii)}$$

So, $$(\alpha - i\beta)(\alpha + i\beta) = \left(\frac{9-16x^2}{9+16x^2}\right)^2 - \left(\frac{i24x}{9+16x^2}\right)^2$$

$$\therefore \quad \alpha^2 + \beta^2 = \frac{81+256x^4-288x^2+576x^2}{(9+16x^2)^2}$$

$$= \frac{81+256x^4+288x^2}{(9+16x^2)^2}$$

$$= \frac{(9+16x^2)^2}{(9+16x^2)^2} = 1$$

Q. 41 Which of the following is correct for any two complex numbers z_1 and z_2?

(a) $|z_1z_2| = |z_1||z_2|$ (b) $\arg(z_1z_2) = \arg(z_1)\cdot\arg(z_2)$

(c) $|z_1+z_2| = |z_1|+|z_2|$ (d) $|z_1+z_2| \ge |z_1|-|z_2|$

Sol. *(a)* Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

$\Rightarrow \quad |z_1| = r_1 \quad$ ...(i)

and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$\Rightarrow \quad |z_2| = r_2 \quad$ ...(ii)

Now, $z_1z_2 = r_1r_2[\cos\theta_1\cos\theta_2 + i\sin\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + i^2\sin\theta_1\sin\theta_2]$

$= r_1r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$

$\Rightarrow \quad |z_1z_2| = r_1r_2$

$\therefore \quad |z_1z_2| = |z_1||z_2| \quad$ [using Eqs. (i) and (ii)]

Q. 42 The point represented by the complex number $(2-i)$ is rotated about origin through an angle $\frac{\pi}{2}$ in the clockwise direction, the new position of point is

(a) $1+2i$ (b) $-1-2i$ (c) $2+i$ (d) $-1+2i$

Thinking Process

Here, $z < i\alpha$ is a complex number, where modulus is r and argument $(\theta+\alpha)$. If P (z) rotates in clockwise sense through an angle α, then its new position will be $z(\theta - i\alpha)$.

Sol. *(b)* Given that, $z = 2 - i$

It is rotated about origin through an angle $\frac{\pi}{2}$ in the clockwise direction

$\therefore$ New position $= ze^{-i\pi/2} = (2-i)e^{-i\pi/2}$

$$= (2-i)\left[\cos\left(\frac{-\pi}{2}\right) + i\sin\left(\frac{-\pi}{2}\right)\right] = (2-i)[0-i]$$

$= -2i - 1 = -1 - 2i$

Q. 43 If $x, y \in R$, then $x + iy$ is a non-real complex number, if

(a) $x = 0$ (b) $y = 0$ (c) $x \ne 0$ (d) $y \ne 0$

Sol. *(d)* Given that, $x, y \in R$

Then, $x + iy$ is non-real complex number if and only if $y \ne 0$.

Q. 44 If $a + ib = c + id$, then

(a) $a^2 + c^2 = 0$
(b) $b^2 + c^2 = 0$
(c) $b^2 + d^2 = 0$
(d) $a^2 + b^2 = c^2 + d^2$

Thinking Process

If two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are equal, then

$$|z_1| = |z_2| \Rightarrow \sqrt{x_1^2 + y_1^2} = \sqrt{x_2^2 + y_2^2}$$

Sol. ***(d)*** Given that,
$$a + ib = c + id$$
$$\Rightarrow \quad |a + ib| = |c + id|$$
$$\Rightarrow \quad \sqrt{a^2 + b^2} = \sqrt{c^2 + d^2}$$
On squaring both sides, we get
$$a^2 + b^2 = c^2 + d^2$$

Q. 45 The complex number z which satisfies the condition $\left|\dfrac{i + z}{i - z}\right| = 1$ lies on

(a) circle $x^2 + y^2 = 1$
(b) the X-axis
(c) the Y-axis
(d) the line $x + y = 1$

Sol. ***(b)*** Given that,
$$\left|\frac{i + z}{i - z}\right| = 1$$
Let
$$z = x + iy$$
$$\therefore \quad \left|\frac{x + i(y + 1)}{-x - i(y - 1)}\right| = 1 \Rightarrow \frac{x^2 + (y + 1)^2}{x^2 + (y - 1)^2} = 1$$
$$\Rightarrow \quad x^2 + (y + 1)^2 = x^2 + (y - 1)^2$$
$$\Rightarrow \quad 4y = 0 \Rightarrow y = 0$$
So, z lies on X-axis (real axis).

Q. 46 If z is a complex number, then

(a) $|z^2| > |z|$
(b) $|z^2| = |z|^2$
(c) $|z^2| < |z|^2$
(d) $|z^2| \geq |z|^2$

Sol. ***(b)*** If z is a complex number, then $z = x + iy$
$$|z| = |x + iy| \quad \text{and} \quad |z|^2 = |x + iy|^2$$
$$\Rightarrow \quad |z|^2 = x^2 + y^2 \quad \text{...(i)}$$
and
$$z^2 = (x + iy)^2 = x^2 + i^2y^2 + i2xy$$
$$z^2 = x^2 - y^2 + i2xy$$
$$\Rightarrow \quad |z^2| = \sqrt{(x^2 - y^2)^2 + (2xy)^2}$$
$$\Rightarrow \quad |z^2| = \sqrt{x^4 + y^4 - 2x^2y^2 + 4x^2y^2}$$
$$\Rightarrow \quad |z^2| = \sqrt{x^4 + y^4 + 2x^2y^2} = \sqrt{(x^2 + y^2)^2}$$
$$\Rightarrow \quad |z^2| = x^2 + y^2 \quad \text{...(ii)}$$
From Eqs. (i) and (ii),
$$|z|^2 = |z^2|$$

Q. 47 $|z_1 + z_2| = |z_1| + |z_2|$ is possible, if

(a) $z_2 = \bar{z}_1$ (b) $z_2 = \dfrac{1}{z_1}$

(c) $\arg(z_1) = \arg(z_2)$ (d) $|z_1| = |z_2|$

Sol. *(c)* Given that, $|z_1 + z_2| = |z_1| + |z_2|$

$$\Rightarrow |r_1(\cos\theta_1 + i\sin\theta_1) + r_2(\cos\theta_2 + i\sin\theta_2)| = |r_1(\cos\theta_1 + i\sin\theta_1)| + |r_2(\cos\theta_2 + i\sin\theta_2)|$$

$$\Rightarrow |(r_1\cos\theta_1 + r_2\cos\theta) + i(r_1\sin\theta_1 + r_2\sin\theta_2)| = r_1 + r_2$$

$$\Rightarrow \sqrt{r_1^2\cos^2\theta_1 + r_2^2\cos^2\theta_2 + 2r_1r_2\cos\theta_1\cos\theta_2 + r_1^2\sin^2\theta_1 + r_2^2\sin^2\theta_2 + 2r_1r_2\sin\theta_1\sin\theta_2} = r_1 + r_2$$

$$\Rightarrow \sqrt{r_1^2 + r_2^2 + 2r_1r_2[\cos(\theta_1 - \theta_2)]} = r_1 + r_2$$

On squaring both sides, we get

$$r_1^2 + r_2^2 + 2r_1r_2\cos(\theta_1 - \theta_2) = r_1^2 + r_2^2 + 2r_1r_2$$

$$\Rightarrow 2r_1r_2[1 - \cos(\theta_1 - \theta_2)] = 0$$

$$\Rightarrow 1 - \cos(\theta_1 - \theta_2) = 0$$

$$\Rightarrow \cos(\theta_1 - \theta_2) = 1$$

$$\Rightarrow \cos(\theta_1 - \theta_2) = \cos 0°$$

$$\Rightarrow \theta_1 - \theta_2 = 0°$$

$$\Rightarrow \theta_1 = \theta_2$$

$$\therefore \arg(z_1) = \arg(z_2)$$

Q. 48 The real value of θ for which the expression $\dfrac{1 + i\cos\theta}{1 - 2i\cos\theta}$ is a real number is

(a) $n\pi + \dfrac{\pi}{4}$ (b) $n\pi + (-1)^n\dfrac{\pi}{4}$

(c) $2n\pi \pm \dfrac{\pi}{2}$ (d) None of these

Sol. *(c)* Given expression $= \dfrac{1 + i\cos\theta}{1 - 2i\cos\theta} = \dfrac{(1 + i\cos\theta)(1 + 2i\cos\theta)}{(1 - 2i\cos\theta)(1 + 2i\cos\theta)}$

$$= \frac{1 + i\cos\theta + 2i\cos\theta + 2i^2\cos^2\theta}{1 - 4i^2\cos^2\theta}$$

$$= \frac{1 + 3i\cos\theta - 2\cos^2\theta}{1 + 4\cos^2\theta}$$

For real value of θ, $\dfrac{3\cos\theta}{1 + 4\cos^2\theta} = 0$

$$\Rightarrow 3\cos\theta = 0$$

$$\Rightarrow \cos\theta = \cos\frac{\pi}{2}$$

$$\Rightarrow \theta = 2n\pi \pm \frac{\pi}{2}$$

Q. 49 The value of arg (x), when $x < 0$ is

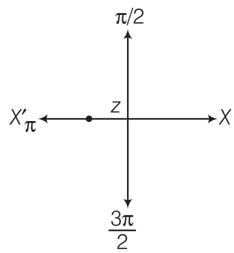
(a) 0 (b) $\frac{\pi}{2}$ (c) π (d) None of these

Sol. *(c)* Let $z = x + 0i$ and $x < 0$

$$|z| = \sqrt{(-1)^2 + (0^2)} = 1$$

Since, the point $(x, 0)$ represent $z = x + 0i$ lies on the negative side of real axis.

$\therefore$ Principal arg $(z) = \pi$



Q. 50 If $f(z) = \frac{7 - z}{1 - z^2}$, where $z = 1 + 2i$, then $|f(z)|$ is equal to

(a) $\frac{|z|}{2}$ (b) $|z|$

(c) $2|z|$ (d) None of these

Sol. *(a)* Let $z = 1 + 2i$

$\Rightarrow$ $|z| = \sqrt{1 + 4} = \sqrt{5}$

Now,

$$f(z) = \frac{7 - z}{1 - z^2} = \frac{7 - 1 - 2i}{1 - (1 + 2i)^2}$$

$$= \frac{6 - 2i}{1 - 1 - 4i^2 - 4i} = \frac{6 - 2i}{4 - 4i}$$

$$= \frac{(3 - i)(2 + 2i)}{(2 - 2i)(2 + 2i)}$$

$$= \frac{6 - 2i + 6i - 2i^2}{4 - 4i^2} = \frac{6 + 4i + 2}{4 + 4}$$

$$= \frac{8 + 4i}{8} = 1 + \frac{1}{2}i$$

$$f(z) = 1 + \frac{1}{2}i$$

$\therefore$ $|f(z)| = \sqrt{1 + \frac{1}{4}} = \sqrt{\frac{4 + 1}{4}} = \frac{\sqrt{5}}{2} = \frac{|z|}{2}$

6

Linear Inequalities

Short Answer Type Questions

Solve for x, the inequalities in following questions.

Q. 1 $\dfrac{4}{x-1} \leq 3 \leq \dfrac{6}{x+1} (x > 0)$

Thinking Process

First solve the first two inequalities, then solve the last two inequality to get range of x.

Sol. Consider first two inequalities,

$$\frac{4}{x+1} \leq 3$$

$\Rightarrow$ $4 \leq 3(x+1)$

$\Rightarrow$ $4 \leq 3x + 3$

$\Rightarrow$ $4 - 3 \leq 3x$ [subtracting 3 on both sides]

$\Rightarrow$ $1 \leq 3x$

$\therefore$ $x \geq \dfrac{1}{3}$ ...(i)

and consider last two inequalities,

$$3 \leq \frac{6}{x+1}$$

$\Rightarrow$ $3(x+1) \leq 6$

$\Rightarrow$ $3x + 3 \leq 6$

$\Rightarrow$ $3x \leq 6 - 3$ [subtracting 3 to both sides]

$\Rightarrow$ $3x \leq 3$ [dividing by 3]

$\therefore$ $x \leq 1$ ...(ii)

From Eqs. (i) and (ii),

$$x \in \left[\frac{1}{3}, 1\right]$$

$$\frac{1}{3} \leq x \leq 1$$

Q. 2 $\dfrac{|x-2|-1}{|x-2|-2} \le 0$

Thinking Process

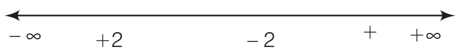
First, let $y = |x-2|$ and then for the obtained values of y use the property $|x-a| \ge k \Leftrightarrow x \le a-k$ or $x \ge a+k$ to get the range of x.

Sol. Let $|x-2| = y$

$$\frac{y-1}{y-2} \le 0$$

$\Rightarrow$ $y-1=0$ and $y-2=0$

$\Rightarrow$ $y=1$ and $y=2$



$\Rightarrow$ $1 \le y < 2$

$\Rightarrow$ $1 \le |x-2| < 2$

$\Rightarrow$ $1 \le |x-2|$ and $|x-2| < 2$

$\Rightarrow$ $x-2 \le -1$

$\Rightarrow$ $x-2 \ge 1$

and $-2 < x-2 < 2$

$\Rightarrow$ $x \le 1$ or $x \ge 3$ and $0 < x < 4$

$\Rightarrow$ $x \in (0, 1] \cup [3, 4)$

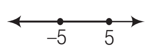
Q. 3 $\dfrac{1}{|x|-3} \le \dfrac{1}{2}$

Sol. Given, $\dfrac{1}{|x|-3} \le \dfrac{1}{2}$

$\Rightarrow$ $|x| - 3 \ge 2$ $\left[\because \dfrac{1}{a} < \dfrac{1}{b} \Rightarrow a > b\right]$

$\Rightarrow$ $|x| \ge 5$ [adding 3 to both sides]

$\Rightarrow$ $x \le -5$ or $x \ge 5$ [$\because$ $|x| \ge a \Rightarrow |x| \le -a \Rightarrow |x| \ge a$]



$\Rightarrow$ $x \in (-\infty, -5] \cup [5, \infty)$ ...(i)

But $|x| - 3 \ne 0$

Either $|x| - 3 < 0$ or $|x| - 3 > 0$

$\Rightarrow$ $|x| < 3$ or $|x| > 3$

$\Rightarrow$ $-3 < x < 3$ or $x < -3$ or $x > 3$ ...(ii)

[$\because$ $|x| < a \Rightarrow -a < x < a$ and $|x| > a \Rightarrow x < -a$ $x > a$]

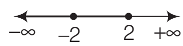
On combining results of Eqs. (i) and (ii), we get

$$x \in (-\infty, -5] \cup (-3, 3) \cup [5, \infty)$$

Q. 4 $|x-1| \le 5, |x| \ge 2$

Sol.
$$|x-1| \le 5$$
$$-5 \le x-1 \le 5$$
$$\Rightarrow \quad -4 \le x \le 6$$
$$\Rightarrow \quad x \in [-4, 6] \quad \text{...(i)}$$

and
$$|x| \ge 2$$
$$\Rightarrow \quad x \le -2 \text{ or } x \ge 2$$
$$\Rightarrow \quad x \in (-\infty, -2] \cup [2, \infty) \quad \text{...(ii)}$$



On combining Eqs. (i) and (ii), we get
$$x \in (-4, -2] \cup [2, 6]$$

Q. 5 $-5 \le \dfrac{2-3x}{4} \le 9$

Sol. We have,
$$-5 \le \frac{2-3x}{4}$$
$$\Rightarrow \quad -20 \le 2-3x \quad \text{[multiplying by 4 on both sides]}$$
$$\Rightarrow \quad 3x \le 2+20$$
$$\Rightarrow \quad 3x \le 22$$
$$\Rightarrow \quad x \le \frac{22}{3}$$
and
$$\frac{2-3x}{4} \le 9$$
$$\Rightarrow \quad 2-3x \le 36$$
$$\Rightarrow \quad -3x \le 36-2$$
$$\Rightarrow \quad -3x \le 34$$
$$\Rightarrow \quad 3x \ge -34$$
$$\Rightarrow \quad x \ge -\frac{34}{3}$$
$$\Rightarrow \quad -\frac{34}{3} \le x \le \frac{22}{3}$$
$$\Rightarrow \quad x \in \left[\frac{-34}{3}, \frac{22}{3}\right]$$

Q. 6 $4x + 3 \ge 2x + 17,\ 3x - 5 < -2$

Sol. We have,
$$4x + 3 \ge 2x + 17$$
$$\Rightarrow \quad 4x - 2x \ge 17 - 3 \Rightarrow 2x \ge 14$$
$$\Rightarrow \quad x \ge \frac{14}{2}$$
$$\Rightarrow \quad x \ge 7 \quad \text{...(i)}$$
Also, we have
$$3x - 5 < -2$$
$$\Rightarrow \quad 3x < -2 + 5 \Rightarrow 3x < 3$$
$$\Rightarrow \quad x < 1 \quad \text{...(ii)}$$
On combining Eqs. (i) and (ii), we see that solution is not possible because nothing is common between these two solutions. (*i.e.*, $x < 1, x \ge 7$).

Q. 7 A company manufactures cassettes. Its cost and revenue functions are $C(x) = 26000 + 30x$ and $R(x) = 43x$, respectively, where x is the number of cassettes produced and sold in a week. How many cassettes must be sold by the company to realise some profit?

Sol. Cost function, $C(x) = 26000 + 30x$
and revenue function, $R(x) = 43x$
For profit, $R(x) > C(x)$

$\Rightarrow \quad 26000 + 30x < 43x$
$\Rightarrow \quad 30x - 43x < -26000$
$\Rightarrow \quad -13x < -26000$
$\Rightarrow \quad 13x > 26000$
$\Rightarrow \quad x > \frac{26000}{13}$
$\therefore \quad x > 2000$

Hence, more than 2000 cassettes must be produced to get profit.

Q. 8 The water acidity in a pool is considered normal when the average pH reading of three daily measurements is between 8.2 and 8.5. If the first two pH readings are 8.48 and 8.35, then find the range of pH value for the third reading that will result in the acidity level being normal.

Sol. Given, first pH value = 8.48
and second pH value = 8.35
Let third pH value be x.
Since, it is given that average pH value lies between 8.2 and 8.5.

$\therefore \quad 8.2 < \frac{8.48 + 8.35 + x}{3} < 8.5$
$\Rightarrow \quad 8.2 < \frac{16.83 + x}{3} < 8.5$
$\Rightarrow \quad 3 \times 8.2 < 16.83 + x < 8.5 \times 3$
$\Rightarrow \quad 24.6 < 16.83 + x < 25.5$
$\Rightarrow \quad 24.6 - 16.83 < x < 25.5 - 16.83$
$\Rightarrow \quad 7.77 < x < 8.67$

Thus, third pH value lies between 7.77 and 8.67.

Q. 9 A solution of 9% acid is to be diluted by adding 3% acid solution to it. The resulting mixture is to be more than 5% but less than 7% acid. If there is 460 L of the 9% solution, how many litres of 3% solution will have to be added?

Sol. Let x L of 3% solution be added to 460 L of 9% solution of acid.
Then, total quantity of mixture = $(460 + x)$ L
Total acid content in the $(460 + x)$ L of mixture

$$= \left(460 \times \frac{9}{100} + x \times \frac{3}{100}\right)$$

It is given that acid content in the resulting mixture must be more than 5% but less than 7% acid.

Therefore, $5\% \text{ of } (460 + x) < 460 \times \frac{9}{100} + \frac{3x}{100} < 7\% \text{ of } (460 + x)$

$\Rightarrow \frac{5}{100} \times (460 + x) < 460 \times \frac{9}{100} + \frac{3}{100} x < \frac{7}{100} \times (460 + x)$

$\Rightarrow 5 \times (460 + x) < 460 \times 9 + 3x < 7 \times (460 + x)$ [multiplying by 100]

$\Rightarrow 2300 + 5x < 4140 + 3x < 3220 + 7x$

Taking first two inequalities, $2300 + 5x < 4140 + 3x$

$\Rightarrow 5x - 3x < 4140 - 2300$

$\Rightarrow 2x < 1840$

$\Rightarrow x < \frac{1840}{2}$

$\Rightarrow x < 920$ ...(i)

Taking last two inequalities, $4140 + 3x < 3220 + 7x$

$\Rightarrow 3x - 7x < 3220 - 4140$

$\Rightarrow -4x < -920$

$\Rightarrow 4x > 920$

$\Rightarrow x > \frac{920}{4}$

$\Rightarrow x > 230$ ...(ii)

Hence, the number of litres of the 3% solution of acid must be more than 230 L and less than 920 L.

Q. 10 A solution is to be kept between 40°C and 45°C. What is the range of temperature in degree fahrenheit, if the conversion formula is $F = \frac{9}{5} C + 32$?

Sol. Let the required temperature be x°F.

Given that, $F = \frac{9}{5} C + 32$

$\Rightarrow 5F = 9C + 32 \times 5$

$\Rightarrow 9C = 5F - 32 \times 5$

$\therefore C = \frac{5F - 160}{9}$

Since, temperature in degree calcius lies between 40°C to 45°C.

Therefore, $40 < \frac{5F - 160}{9} < 45$

$\Rightarrow 40 < \frac{5x - 160}{9} < 45$

$\Rightarrow 40 \times 9 < 5x - 160 < 45 \times 9$ [multiplying throughout by 9]

$\Rightarrow 360 < 5x - 160 < 405$ [adding 160 throughout]

$\Rightarrow 360 + 160 < 5x < 405 + 160$

$\Rightarrow 520 < 5x < 565$

$\Rightarrow \frac{520}{5} < x < \frac{565}{5}$ [divide throughout by 5]

$\Rightarrow 104 < x < 113$

Hence, the range of temperature in degree fahrenheit is 104 °F to 113°F.

Q. 11 The longest side of a triangle is twice the shortest side and the third side is 2 cm longer than the shortest side. If the perimeter of the triangle is more than 166 cm, then find the minimum length of the shortest side.

Sol. Let the length of shortest side be x cm.

According to the given information,

Longest side $= 2 \times$ Shortest side

$= 2x$ cm

and third side $= 2 +$ Shortest side

$= (2 + x)$ cm

Perimeter of triangle $= x + 2x + (x + 2) = 4x + 2$

According to the question,

Perimeter > 166 cm

$\Rightarrow \quad 4x + 2 > 166$

$\Rightarrow \quad 4x > 166 - 2$

$\Rightarrow \quad 4x > 164$

$\therefore \quad x > \frac{164}{4} = 41\text{cm}$

Hence, the minimum length of shortest side be 41cm.

Q. 12 In drilling world's deepest hole it was found that the temperature T in degree celcius, x km below the earth's surface was given by $T = 30 + 25(x - 3)$, $3 \le x \le 15$. At what depth will the temperature be between 155°C and 205°C?

Sol. Given that, $T = 30 + 25(x - 3)$, $3 \le x \le 15$

According to the question,

$155 < T < 205$

$\Rightarrow \quad 155 < 30 + 25(x - 3) < 205$

$\Rightarrow \quad 155 - 30 < 25(x - 3) < 205 - 30$ [subtracting 30 in whole]

$\Rightarrow \quad 125 < 25(x - 3) < 175$

$\Rightarrow \quad \frac{125}{25} < x - 3 < \frac{175}{25}$ [dividing by 25 in whole]

$\Rightarrow \quad 5 < x - 3 < 7$

$\Rightarrow \quad 5 + 3 < x < 7 + 3$ [adding 3 in whole]

$\Rightarrow \quad 8 < x < 10$

Hence, at the depth 8 to10 km temperature lies between 155° to 205°C.

Long Answer Type Questions

Q.13 Solve the following system of inequalities $\frac{2x+1}{7x-1} > 5, \frac{x+7}{x-8} > 2.$

Sol. The given system of inequations is

$$\frac{2x+1}{7x-1} > 5 \quad \text{...(i)}$$

and $$\frac{x+7}{x-8} > 2 \quad \text{...(ii)}$$

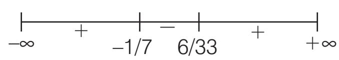
Now, $$\frac{2x+1}{7x-1} - 5 > 0$$

$$\Rightarrow \frac{(2x+1)-5(7x-1)}{7x-1} > 0$$

$$\Rightarrow \frac{2x+1-35x+5}{7x-1} > 0$$

$$\Rightarrow \frac{-33x+6}{7x-1} > 0 \Rightarrow \frac{33x-6}{7x-1} < 0$$

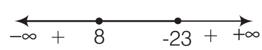
$$\Rightarrow x \in \left(\frac{1}{7}, \frac{6}{33}\right) \quad \text{...(iii)}$$



and $$\frac{x+7}{x-8} > 2 \Rightarrow \frac{x+7}{x-8} - 2 > 0$$

$$\Rightarrow \frac{x+7-2(x-8)}{x-8} > 0 \Rightarrow \frac{x+7-2x+16}{x-8} > 0$$

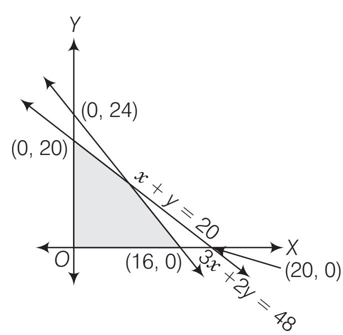
$$\Rightarrow \frac{-x+23}{x-8} > 0 \Rightarrow \frac{x-23}{x-8} < 0$$



$$\Rightarrow x \in (8, 23) \quad \text{...(iv)}$$

Since, the intersection of Eqs. (iii) and (iv) is the null set. Hence, the given system of equation has no solution.

Q.14 Find the linear inequalities for which the shaded region in the given figure is the solution set.



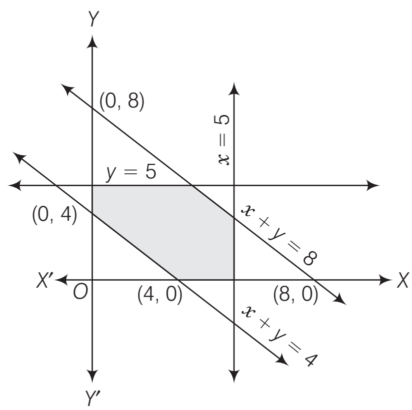
Sol. Consider the line $3x + 2y = 48$, we observe that the shaded region and the origin are on the same side of the line $3x + 2y = 48$ and (0, 0) satisfy the linear constraint $3x + 2y \le 48$. So, we must have one inequation as $3x + 2y \le 48$.

Now, consider the line $x + y = 20$. We find that the shaded region and the origin are on the same side of the line $x + y = 20$ and (0, 0) satisfy the constraints $x + y \le 20$. So, the second inequation is $x + y \le 20$.

We also notice that the shaded region is above *X*-axis and is on the right side of *Y*-axis, so we must have $x \ge 0$, $y \ge 0$.

Thus, the linear inequations corresponding to the given solution set are $3x + 2y \le 48$, $x + y \le 20$ and $x \ge 0$, $y \ge 0$.

Q. 15 Find the linear inequalities for which the shaded region in the given figure is the solution set.



Sol. Consider the line $x + y = 4$.

We observe that the shaded region and the origin lie on the opposite side of this line and (0, 0) satisfies $x + y \le 4$. Therefore, we must have $x + y \ge 4$ as the linear inequation corresponding to the line $x + y = 4$.

Consider the line $x + y = 8$, clearly the shaded region and origin lie on the same side of this line and (0, 0) satisfies the constraints $x + y \le 8$. Therefore, we must have $x + y \le 8$, as the linear inequation corresponding to the line $x + y = 8$.

Consider the line $x = 5$. It is clear from the graph that the shaded region and origin are on the left of this line and (0, 0) satisfy the constraint $x \le 5$.

Hence, $x \le 5$ is the linear inequation corresponding to $x = 5$.

Consider the line $y = 5$, clearly the shaded region and origin are on the same side (below) of the line and (0, 0) satisfy the constrain $y \le 5$.

Therefore, $y \le 5$ is an inequation corresponding to the line $y = 5$.

We also notice that the shaded region is above the *X*-axis and on the right of the *Y*-axis *i.e.*, shaded region is in first quadrant. So, we must have $x \ge 0$, $y \ge 0$.

Thus, the linear inequations comprising the given solution set are

$$x + y \ge 4;\ x + y \le 8;\ x \le 5;\ y \le 5;\ x \ge 0 \text{ and } y \ge 0.$$

Q.16 Show that the following system of linear inequalities has no solution $x+2y\le 3, 3x+4y\ge 12, x\ge 0, y\ge 1$.

Sol. Consider the inequation $x + 2y \le 3$ as an equation, we have

$$x + 2y = 3$$

$\Rightarrow$ $x = 3 - 2y$

$\Rightarrow$ $2y = 3 - x$

x	3	1	0
y	0	1	1.5

Now, (0, 0) satisfy the inequation $x + 2y \le 3$.

So, half plane contains (0, 0) as the solution and the line $x + 2y = 3$ intersect the coordinate axis at (3, 0) and (0, 3/2).

Consider the inequation $3x + 4y \ge 12$ as an equation, we have $3x + 4y = 12$

$\Rightarrow$ $4y = 12 - 3x$

x	0	4	2
y	3	0	3/2

Thus, coordinate axis intersected by the line $3x + 4y = 12$ at points (4, 0) and (0, 3).

Now, (0, 0) does not satisfy the inequation $3x + 4y = 12$.

Therefore, half plane of the solution does not contained (0, 0).

Consider the inequation $y \ge 1$ as an equation, we have

$$y = 1.$$

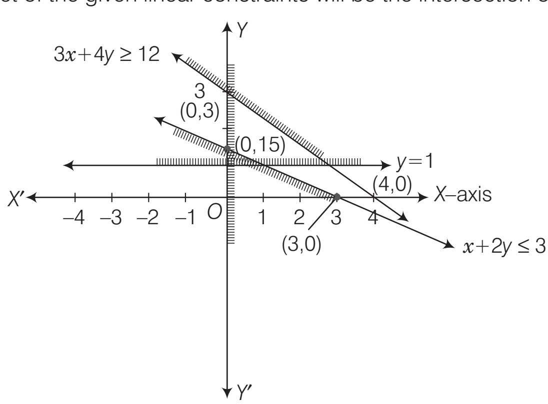
It represents a straight line parallel to X-axis passing through point (0, 1).

Now, (0, 0) does not satisfy the inequation $y \ge 1$.

Therefore, half plane of the solution does not contains (0, 0).

Clearly $x \ge 0$ represents the region lying on the right side of Y-axis.

The solution set of the given linear constraints will be the intersection of the above region.



It is clear from the graph the shaded portions do not have common region.

So, solution set is null set.

Q.17 Solve the following system of linear inequalities
$3x + 2y \geq 24,\ 3x + y \leq 15,\ x \geq 4.$

Sol. Consider the inequation $3x + 2y \geq 24$ as an equation, we have $3x + 2y = 24$.

$\Rightarrow$ $$2y = 24 - 3x$$

x	0	8	4
y	12	0	6

Hence, line $3x + y = 24$ intersect coordinate axes at points $(8, 0)$ and $(0,12)$.
Now, $(0,0)$ does not satisfy the inequation $3x + 2y \geq 24$.
Therefore, half plane of the solution set does not contains $(0,0)$.
Consider the inequation $3x + y \leq 15$ as an equation, we have

$$3x + y = 15$$

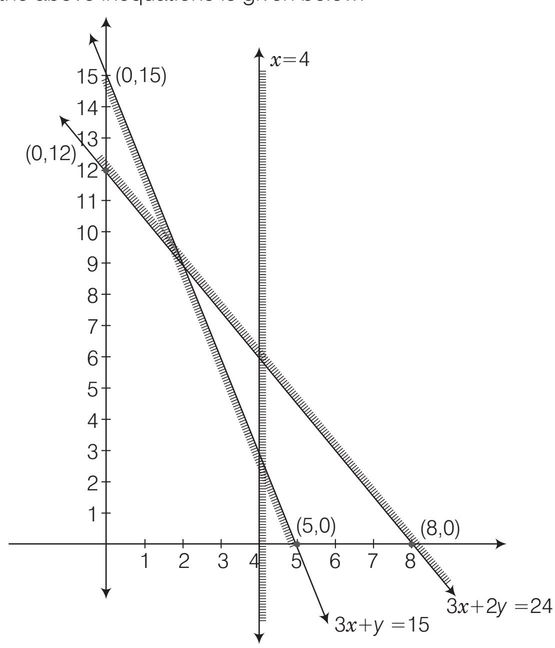
$\Rightarrow$ $$y = 15 - 3x$$

x	0	5	3
y	15	0	6

Line $3x + y = 15$ intersects coordinate axes at points $(5,0)$ and $(0,15)$.
Now, point $(0,0)$ satisfy the inequation $3x + y \leq 15$.
Therefore, the half plane of the solution contain origin.
Consider the inequality $x \geq 4$ as an equation, we have

$$x = 4$$

It represents a straight line parallel to Y-axis passing through $(4,0)$. Now, point $(0,0)$ does not satisfy the inequation $x \geq 4$.
Therefore, half plane does not contains $(0,0)$,
The graph of the above inequations is given below.



It is clear from the graph that there is no common region corresponding to these inequality.
Hence, the given system of inequalities have no solution.

Q.18 Show that the solution set of the following system of linear inequalities is an unbounded region $2x + y \geq 8$, $x + 2y \geq 10$, $x \geq 0$, $y \geq 0$.

Sol. Consider the inequation $2x + y \geq 8$ as an equation, we have

$$2x + y = 8$$

$$\Rightarrow \quad y = 8 - 2x$$

x	0	4	3
y	8	0	2

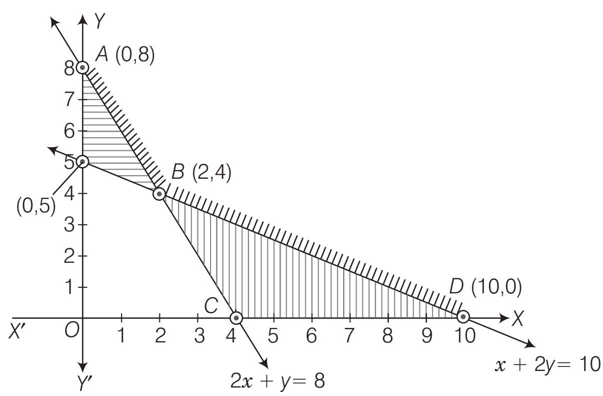
The line $2x + y = 8$ intersects coordinate axes at (4, 0) and (0, 8).
Now, point (0, 0) does not satisfy the inequation $2x + y \geq 8$.
Therefore, half plane does not contain origin.
Consider the inequation $x + 2y \geq 10$, as an equation, we have

$$x + 2y = 10$$

$$\Rightarrow \quad 2y = 10 - x$$

x	10	0	8
y	0	5	1

The line $2x + y = 8$ intersects the coordinate axes at (10, 0) and (0, 5).
Now, point (0, 0) does not satisfy the inequation $x + 2y \geq 10$.
Therefore, half plane does not contain (0, 0).
Consider the inequation $x \geq 0$ and $y \geq 0$ clearly, it represents the region in first quadrant.
The graph of the above inequations is given below



It is clear from the graph that common shaded portion is unbounded.

Objective Type Questions

Q. 19 If $x < 5$, then

(a) $-x < -5$ (b) $-x \le -5$ (c) $-x > -5$ (d) $-x \ge -5$

Sol. *(c)* If $x < 5$, then $-x > -5$

[if we multiply by negative numbers, then inequality get reversed]

Q. 20 If x, y and b are real numbers and $x < y$, $b < 0$, then

(a) $\frac{x}{b} < \frac{y}{b}$ (b) $\frac{x}{b} \le \frac{y}{b}$ (c) $\frac{x}{b} > \frac{y}{b}$ (d) $\frac{x}{b} \ge \frac{y}{b}$

Sol. *(c)* It is given that, $x < y$, $b < 0$

$\Rightarrow$ $\frac{x}{b} > \frac{y}{b}$ [$\because b < 0$]

Q. 21 If $-3x + 17 < -13$, then

(a) $x \in (10, \infty)$ (b) $x \in [10, \infty)$ (c) $x \in (-\infty, 10]$ (d) $x \in [-10, 10)$

Sol. *(a)* Given that, $-3x + 17 < -13$

$\Rightarrow$ $3x - 17 > 13$ [multiplying by -1 on both sides]

$\Rightarrow$ $3x > 13 + 17$ [adding 17 on both sides]

$\Rightarrow$ $3x > 30$

$\therefore$ $x > 10$

Q. 22 If x is a real number and $|x| < 3$, then

(a) $x \ge 3$ (b) $-3 < x < 3$ (c) $x \le -3$ (d) $-3 \le x \le 3$

Sol. *(b)* Given, $|x| < 3$

$\Rightarrow$ $-3 < x < 3$ [$\because |x| < a \Rightarrow -a < x < a$]

Q. 23 Let x and b are real numbers. If $b > 0$ and $|x| > b$, then

(a) $x \in (-b, \infty)$ (b) $x \in [-\infty, b)$

(c) $x \in (-b, b)$ (d) $x \in (-\infty, -b) \cup (b, \infty)$

Sol. *(d)* Given, $|x| > b$ and $b > 0$

$\Rightarrow$ $x < -b$ or $x > b$

$\Rightarrow$ $x \in (-\infty, -b) \cup (b, \infty)$

Q. 24 If $|x - 1| > 5$, then

(a) $x \in (-4, 6)$ (b) $x \in [-4, 6]$

(c) $x \in (-\infty, -4) \cup (6, \infty)$ (d) $x \in [-\infty, -4) \cup [6, \infty)$

Sol. *(c)* Given, $|x - 1| > 5$

$\Rightarrow$ $(x - 1) < -5$ or $(x - 1) > 5$ [$\because |x| > a \Rightarrow x < a$ or $x > a$]

$\Rightarrow$ $x < -4$ or $x > 6$

$\Rightarrow$ $x \in (-\infty, -4) \cup (6, \infty)$

Q. 25 If $|x+2| \leq 9$, then

(a) $x \in (-7, 11)$ (b) $x \in [-11, 7]$

(c) $x \in (-\infty, -7) \cup (11, \infty)$ (d) $x \in (-\infty, -7) \cup [11, \infty)$

Sol. ***(b)*** Given, $|x+2| \leq 9$,

$\Rightarrow$ $-9 \leq x + 2 \leq 9$ [$\because |x| \leq a \Rightarrow -a \leq x \leq a$]

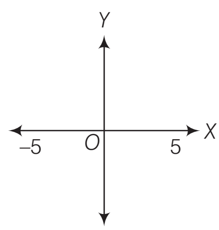
$\Rightarrow$ $-9 - 2 \leq x \leq 9 - 2$ [subtracting 2 througout]

$\Rightarrow$ $-11 \leq x \leq 7$

$\Rightarrow$ $x \in [-11, 7]$

The inequality representing the following graphs is

Q. 26



(a) $|x| < 5$ (b) $|x| \leq 5$

(c) $|x| > 5$ (d) $|x| \geq 5$

Sol. ***(a)*** The given graph represent $x > -5$ and $x < 5$.

On combining these two result, we get

$$|x| < 5.$$

Solution of a linear inequality in variable x is represented on number line in following questions.

Q. 27

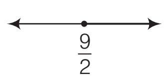
(a) $x \in (-\infty, 5)$ (b) $x \in (-\infty, 5]$

(c) $x \in [5, \infty)$ (d) $x \in (5, \infty)$

Sol. ***(d)*** The given graph represents all the values greater than 5 except $x = 5$ on the real line

So, $x \in (5, \infty)$.

Q. 28



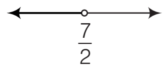
(a) $x \in \left(\frac{9}{2}, \infty\right)$ (b) $x \in \left[\frac{9}{2}, \infty\right)$

(c) $x \in -\left[\infty, \frac{9}{2}\right)$ (d) $x \in \left(-\infty, \frac{9}{2}\right]$

Sol. ***(b)*** The given graph represents all the values greater than $\frac{9}{2}$ including $\frac{9}{2}$ as the real line.

$$x \in \left[\frac{9}{2}, \infty\right)$$

Q. 29

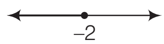


(a) $x \in \left(-\infty, \frac{7}{2}\right)$
(b) $x \in \left(-\infty, \frac{7}{2}\right]$
(c) $x \in \left[\frac{7}{2}, -\infty\right)$
(d) $x \in \left(\frac{7}{2}, \infty\right)$

Sol. ***(a)*** The given graph represents all the values less than $\frac{7}{2}$ on the real line.

$$\Rightarrow \quad x \in \left(-\infty, \frac{7}{2}\right)$$

Q. 30



(a) $x \in (-\infty, -2)$
(b) $x \in (-\infty, -2]$
(c) $x \in (-2, \infty]$
(d) $x \in [-2, \infty)$

Sol. ***(b)*** The given graph represents all values less than –2 including –2.

$$\Rightarrow \quad x \in (-\infty, -2]$$

True/False

Q. 31 State which of the following statements is true of false.

(i) If $x < y$ and $b < 0$, then $\frac{x}{b} < \frac{y}{b}$.

(ii) If $xy > 0$, then $x > 0$ and $y < 0$

(iii) If $xy > 0$, then $x < 0$ and $y < 0$

(iv) If $xy < 0$, then $x < 0$ and $y < 0$

(v) If $x < -5$ and $x < -2$, then $x \in (-\infty, -5)$

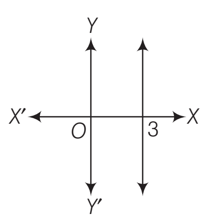
(vi) If $x < -5$ and $x > 2$, then $x \in (-5, 2)$

(vii) If $x > -2$ and $x < 9$, then $x \in (-2, 9)$

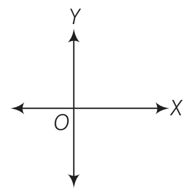
(viii) If $|x| > 5$, then $x \in (-\infty, -5) \cup [5, \infty)$

(ix) If $|x| \le 4$, then $x \in [-4, 4]$

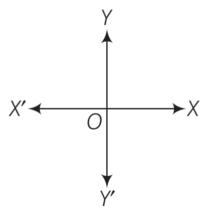
(x) Graph of $x < 3$ is



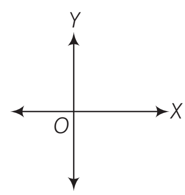
(xi) Graph of $x \geq 0$ is



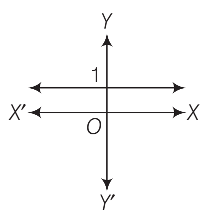
(xii) Graph of $y \leq 0$ is



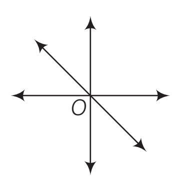
(xiii) Solution set of $x \geq 0$ and and $y \leq 0$ is



(xiv) Solution set of $x \geq 0$ and $y \leq 1$ is



(xv) Solution set of $x + y \geq 0$ is



Sol. **(i)** If $x < y$ and $b < 0$

$\Rightarrow$ $$\frac{x}{b} > \frac{y}{b}$$

Hence, statement (i) is false.

(ii) If $xy > 0$, then, $x > 0$, $y > 0$ or $x < 0$, $y < 0$.
Hence, statement (ii) is true.

(iii) If $xy > 0$, then $x < 0$ and $y < 0$.
Hence, statement (iii) is true.

(iv) If $xy<0 \Rightarrow x<0$, $y>0$ or $x>0$, $y<0$.
Hence, statement (iv) is false.

(v) If $x<-5$ and $x<-2$, then
$$x \in (-\infty, -5)$$
Hence, statement (v) is true.

(vi) If $x<-5$ and $x>2$, then x have no value.
Hence, statement (vi) is false.

(vii) If $x>-2$ and $x<9$, then $x \in (-2, 9)$.
Hence, statement (vii)is true.

(viii) If $|x| > 5$, then either $x<-5$ or $x>5$.
$$\Rightarrow \quad x \in (-\infty, -5) \cup (5, \infty)$$
Hence, statement (viii) is false.

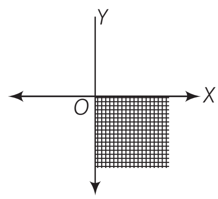
(ix) If $|x| \leq 4$, then
$$-4 \leq x \leq 4$$
$$\Rightarrow \quad x \in [-4, 4]$$
Hence, statement (ix) is true.

(x) The given graph represents $x \leq 3$.
Hence, statement (x) is false.

(xi) The given graph represents $x \geq 0$.
Hence, statement (xi) is true.

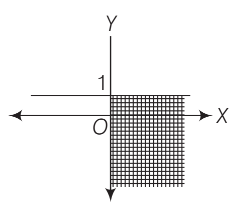
(xii) The given graph represent $y \geq 0$.
Hence, statement (xii) is false.

(xiii) Solution set of $x \geq 0$ and $y \leq 0$ is



Hence, statement (xiii) is false.

(xiv) Solution set of $x \geq 0$ and $y \leq 1$ is



Hence, statement (xiv) is false.

(xv) The given graph represents $x + y \geq 0$.
Hence, statement (xv) is correct.

Fillers

Q. 32 Fill in the blanks of the following

(i) If $-4x \geq 12$, then $x \ldots -3$.

(ii) If $\frac{-3}{4}x \leq -3$, then $x \ldots 4$.

(iii) If $\frac{2}{x+2} > 0$, then $x \ldots -2$.

(iv) If $x > -5$, then $4x \ldots -20$.

(v) If $x > y$ and $z < 0$, then $-xz \ldots -yz$.

(vi) If $p > 0$ and $q < 0$, then $p - q \ldots p$.

(vii) If $|x + 2| > 5$, then $x \ldots -7$ or $x \ldots 3$.

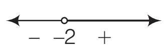
(viii) If $-2x + 1 \geq 9$, then $x \ldots -4$.

Sol. **(i)** If $-4x \geq 12 \Rightarrow x \leq -3$

(ii) If $\frac{-3}{4}x \leq -3$

$\Rightarrow$ $x \geq (-3) \times \frac{4}{-3} \Rightarrow x \geq 4$

(iii) If $\frac{2}{x+2} > 0$



$\Rightarrow$ $x > -2$

(iv) If $x > -5 \Rightarrow 4x > -20$

(v) If $x > y$ and $z < 0$, then

$xz < yz$ [since, $z < 0$]

$\Rightarrow$ $-xz > -yz$

(vi) If $p > 0$ and $q < 0$,

then $p - q > p$

e.g., consider $2 > 0$ and $-3 < 0$.

$\Rightarrow$ $2 - (-3) = 2 + 3 = 5 > 2$

(vii) If $|x + 2| > 5$, then

$x + 2 < -5$ or $x + 2 > 5$

$\Rightarrow$ $x < -5 - 2$ or $x > 5 - 2$

$\Rightarrow$ $x < -7$ or $x > 3$

$\Rightarrow$ $x \in (-\infty, -7) \cup (3, \infty)$

(viii) If $-2x + 1 \geq 9$, then

$-2x \geq 9 - 1 \Rightarrow -2x \geq 8$

$\Rightarrow$ $2x \leq -8 \Rightarrow x \leq -4$

7

Permutations and Combinations

Short Answer Type Questions

Q. 1 Eight chairs are numbered 1 to 8. Two women and 3 men wish to occupy one chair each. First the women choose the chairs from amongst the chairs 1 to 4 and then men select from the remaining chairs. Find the total number of possible arrangements.

Sol. First women choose the chairs from among 1 to 4 chairs. *i.e.,* total number of chairs is 4.
Since, there are two women, so number of arrangements = ${}^4P_2$ ways.

Now, men have to choose chairs from remaining 6 chairs.
Since, there are 3 men, so number can be arranged in ${}^6P_3$ ways.

$\therefore$ Total number of possible arrangements = ${}^4P_2 \times {}^6P_3$

$$= \frac{4!}{4-2!} \times \frac{6!}{6-3!}$$

$$= \frac{4!}{2!} \times \frac{6!}{3!}$$

$$= \frac{4 \times 3 \times 2!}{2!} \times \frac{6 \times 5 \times 4 \times 3!}{3!}$$

$$= 4 \times 3 \times 6 \times 5 \times 4 = 1440$$

Q. 2 If the letters of the word 'RACHIT' are arranged in all possible ways as listed in dictionary. Then, what is the rank of the word 'RACHIT'?

Sol. The letters of the word 'RACHIT' in alphabetical order are A, C, H, I, R and T.

Now, words beginning with $A = 5!$
words beginning with $C = 5!$
words beginning with $H = 5!$
words beginning with $I = 5!$

Word beginning with R *i.e.*, RACHIT = 1

$\therefore$ Rank of the word 'RACHIT' in dictionary = $4 \times 5! + 1 = 4 \times 120 + 1$
$= 480 + 1 = 481$

Q. 3 A candidate is required to answer 7 questions out of 12 questions, which are divided into two groups, each containing 6 questions. He is not permitted to attempt more than 5 questions from either group. Find the number of different ways of doing questions.

Sol. Since, candidate cannot attempt more than 5 questions from either group.
Thus, he is able to attempt minimum two questions from either group.
The number of questions attempted from each group is given in following table

Group I	5	4	3	2
Group II	2	3	4	5

Since, each group have 6 questions and total attempted 7 questions.

$\therefore$ Total number of possible ways $= {}^6C_5 \times {}^6C_2 + {}^6C_4 \times {}^6C_3 + {}^6C_3 \times {}^6C_4 + {}^6C_2 \times {}^6C_5$
$= 2\,[{}^6C_5 \times {}^6C_2 + {}^6C_4 \times {}^6C_3]$
$= 2\,[6 \times 15 + 15 \times 20]$
$= 2\,[90 + 300]$
$= 2 \times 390 = 780$

Q. 4 Out of 18 points in a plane, no three are in the same line except five points which are collinear. Find the number of lines that can be formed joining the point.

Sol. Total number of points = 18
Out of which 5 points are collinear, we get a straight line by joining any two points.
$\therefore$ Number of straight line formed by joining the 18 points taking 2 at a time = ${}^{18}C_2$
and number of straight line formed by joining 5 points taking 2 at a time = ${}^5C_2$
But 5 collinear points, when joined pairwise give only one line.
$\therefore$ Required number of straight line $= {}^{18}C_2 - {}^5C_2 + 1$
$= 153 - 10 + 1 = 144$

Q. 5 We wish to select 6 person from 8 but, if the person A is chosen, then B must be chosen. In how many ways can selections be made?

Sol. Total number of person = 8
Number of person to be selected = 6
It is given that, if A is chosen then, B must be chosen.
Therefore, following cases arise.
Case **I** When A is chosen, B must be chosen.
Number of ways $= {}^{8-2}C_{6-2} = {}^6C_4$
Case **II** When A is not chosen.
Then, B may be chosen.
$\therefore$ Number of ways $= {}^{8-1}C_6 = {}^7C_6$
Hence, required number of ways $= {}^6C_4 + {}^7C_6$
$= 15 + 7 = 22$

Q. 6 How many committee of five person with a chairperson can be selected from 12 persons?

Sol. $\because$ Total number of persons = 12

and number of persons to be selected = 5

Out of 12 persons a chairperson is selected = ${}^{12}C_1$ = 12 ways

Now, remaining 4 persons are selected out of 11 persons.

$\therefore$ Number of ways = ${}^{11}C_4 = 330$

$\therefore$ Total number of ways to form a committee of 5 persons = $12 \times 330 = 3960$

Q. 7 How many automobile license plates can be made, if each plate contains two different letters followed by three different digits?

Sol. There are 26 English alphabets and 10 digits (0 to 9).

Since, it is given that each plate contains two different letters followed by three different digits.

$\therefore$ Arrangement of 26 letters, taken 2 at a time = ${}^{26}P_2 = \dfrac{26!}{24!} = 26 \times 25 = 650$

and three-digit number can be formed out of the 10 digits = ${}^{10}P_3 = 10 \times 9 \times 8 = 720$ ways

$\therefore$ Total number of licence plates = $650 \times 720 = 468000$

Q. 8 A bag contains 5 black and 6 red balls, determine the number of ways in which 2 black and 3 red balls can be selected from the lot.

Sol. It is given that bag contains 5 black and 6 red balls.

So, 2 black balls is selected from 5 black balls in ${}^5C_2$ ways.

and 3 red balls are selected from 6 red balls in ${}^6C_3$ ways.

$\therefore$ Total number of ways in which 2 black and 3 red balls are selected = ${}^5C_2 \times {}^6C_3$

$= 10 \times 20 = 200$ ways

Q. 9 Find the number of permutations of *n* distinct things taken *r* together, in which 3 particular things must occur together.

Sol. Total number of things = n

We have to arrange r things out of n in which three things must occur together.

Therefore, combination of n things taken r at a time in which 3 things always occurs

$= {}^{n-3}C_{r-3}$

If three things taken together, then it is considered as 1 group.

Arrangement of these three things = 3!

Now, we have to arrange = $r - 3 + 1 = (r - 2)$ objects

$\therefore$ Arranged of $(r-2)$ objects = $r - 2!$

$\therefore$ Total number of arrangements = ${}^{n-3}C_{r-3} \times r - 2! \times 3!$

Q. 10 Find the number of different words that can be formed from the letters of the word 'TRIANGLE', so that no vowels are together.

Sol. Number of letters in the word 'TRIANGLE' = 8, out of which 5 are consonants and 3 are vowels.

If vowels are not together, then we have following arrangement.

V	C	V	C	V	C	V	C	V	C	V

Consonants can be arranged in $= 5! = 120$ ways and vowels can occupy at 6 places.

The 3 vowels can be arranged at 6 place in ${}^6P_3$ ways $= \frac{6!}{6-3!} = \frac{6!}{3!}$

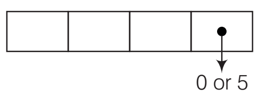
$$= \frac{6 \times 5 \times 4 \times 3!}{3!} = 120$$

Total number of arrangement $= 120 \times 120 = 14400$

Q. 11 Find the number of positive integers greater than 6000 and less than 7000 which are divisible by 5, provided that no digit is to be repeated.

Sol. We know that a number is divisible by 5, If at the units place of the number is 0 or 5.

We have to form 4 -digit number which is greater than 6000 and less than 7000. So, unit digit can be filled in 2 ways.



Since, repeatition is not allowed. Therefore, tens place can be filled in 7 ways, similarily hundreds place can be filled in 8 ways.

But we have to form a number greater than 6000 and less than 7000.

Hence, thousand place can be filled in only 1 ways.

6	8	7	2

Total number of integers $= 1 \times 8 \times 7 \times 2$

$= 14 \times 8 = 112$

Q. 12 There are 10 persons named $P_1, P_2, P_3, \ldots, P_{10}$. Out of 10 persons, 5 persons are to be arranged in a line such that in each arrangement P_1 must occur whereas P_4 and P_5 do not occur. Find the number of such possible arrangements.

Sol. Given that, $P_1, P_2, \ldots, P_{10}$, are 10 persons, out of which 5 persons are to be arranged but P_1 must occur whereas P_4 and P_5 never occur.

$\therefore$ Selection depends on only $10 - 3 = 7$ persons

As, we have already occur P_1, Therefore, we have to select only 4 persons out of 7.

Number of selection $= {}^7C_4 = \frac{7!}{4!\,(7-4)!} = \frac{7!}{4!\,3!} = \frac{5040}{24 \times 6} = 35$

$\therefore$ Required number of arrangement of 5 persons $= 35 \times 5! = 35 \times 120 = 4200$

Q. 13 There are 10 lamps in a hall. Each one of them can be switched on independently. Find the number of ways in which the hall can be illuminated.

Thinking Process

The number of ways in which the hall can be illuminated is equivalent to the number of selections of one or more things out of n different things is

$$^nC_1 + {^nC_2} + {^nC_3} + \ldots + {^nC_n} = 2^n - 1$$

Sol. Total number of ways $= {^{10}C_1} + {^{10}C_2} + {^{10}C_3} + {^{10}C_4} + {^{10}C_5} + {^{10}C_6} + \ldots + {^{10}C_{10}}$

$= 2^{10} - 1$ $\quad [\because {^nC_0} + {^nC_1} + {^nC_2} + \ldots = 2^n]$

$= 1024 - 1 = 1023$

Q. 14 A box contains two white, three black and four red balls. In how many ways can three balls be drawn from the box, if atleast one black ball is to be included in the draw?

Sol. There are 2 white, three black and four red balls.
We have to draw 3 balls, out of these 9 balls in which atleast one black ball is included.
Hence, we can select the balls in the following ways.

Black balls	1	2	3
Other than black	2	1	0

$\therefore$ Required number of selections $= {^3C_1} \times {^6C_2} + {^3C_2} \times {^6C_1} + {^3C_3} \times {^6C_0}$

$= 3 \times 15 + 3 \times 6 + 1$

$= 45 + 18 + 1 = 64$

Q. 15 If ${^nC_{r-1}} = 36$, ${^nC_r} = 84$ and ${^nC_{r+1}} = 126$, then find the value of ${^rC_2}$.

Sol. Given, ${^nC_{r-1}} = 36$ ...(i)

$\Rightarrow$ ${^nC_r} = 84$ ...(ii)

$\Rightarrow$ ${^nC_{r+1}} = 126$ ...(iii)

On dividing Eq. (i) by Eq. (ii), we get

$$\frac{^nC_{r-1}}{^nC_r} = \frac{36}{84} \quad \left[\because {^nC_r} = \frac{n!}{(n-r!\,r!)} \text{ and } n! = n\,(n-1)!\right]$$

$$\Rightarrow \frac{n!}{(r-1)!\{n-(r-1)\}!} \cdot \frac{r!\,(n-r)!}{n!} = \frac{3}{7}$$

$$\Rightarrow \frac{1}{(r-1)!(n-r+1)!} \cdot \frac{r(r-1)!\,(n-r)!}{1} = \frac{3}{7}$$

$$\Rightarrow \frac{1 \cdot r}{(n-r+1)\,(n-r)!} \cdot (n-r)! = \frac{3}{7} \Rightarrow \frac{r}{n-r+1} = \frac{3}{7}$$

$\Rightarrow$ $7r = 3n - 3r + 3$

$\Rightarrow$ $10r - 3n = 3$ ...(iv)

On dividing Eq. (ii) by Eq. (iii), we get

$$\frac{{}^nC_r}{{}^nC_{r+1}} = \frac{84}{126}$$

$$\Rightarrow \quad \frac{n!}{r!\,(n-r)!} \cdot \frac{(r+1)!\,(n-r-1)!}{n!} = \frac{14}{21}$$

$$\Rightarrow \quad \frac{1}{r!\,(n-r)!(n-r-1)!} \cdot \frac{(r+1)\,r!\,(n-r-1)!}{r} = \frac{2}{3} \quad \Rightarrow \quad \frac{r+1}{n-r} = \frac{2}{3}$$

$$\Rightarrow \quad 3r + 3 = 2n - 2r \quad \Rightarrow \quad 2n - 5r = 3 \qquad \text{...(v)}$$

On multiplying Eq. (iv) by 2 and Eq. (v) by 3, we get

$$20r - 6n = 6 \qquad \text{...(vi)}$$

$$6n - 15r = 9 \qquad \text{...(vii)}$$

On adding Eqs. (vi) and (vii),

$$5r = 15 \quad \Rightarrow \quad r = 3$$

From Eq. (v), $2n = 3 + 15$

$$\Rightarrow \quad 2n = 18 \quad \Rightarrow \quad n = 9$$

$$\therefore \quad {}^rC_2 = {}^3C_2 = \frac{3!}{2!1!} = \frac{3 \times 2!}{2!} = 3$$

Q. 16 Find the number of integers greater than 7000 that can be formed with the digits 3, 5, 7, 8 and 9 where no digits are repeated.

Sol. Here, we have to find the number of integers greater than 7000 with the digits 3, 5, 7, 8 and 9. So, with these digits we can make maximum five-digit number because repeaition is not allowed.

Now, all the five-digit numbers are greater than 7000.

Number of ways of forming 5-digit number $= 5 \times 4 \times 3 \times 2 \times 1 = 120$

and all the four-digit numbers greater than 7000 can be formed in following manner.

Thousand place can be filled in 3 ways. Hundred place can be filled in 4 ways. Tenth place can be filled in 3 ways. Units place can be filled in 2 ways.

Thus, we have total number of 4-digit number $= 3 \times 4 \times 3 \times 2 = 72$

$\therefore$ Total number of integers $= 120 + 72 = 192$

Q. 17 If 20 lines are drawn in a plane such that no two of them are parallel and no three are concurrent, in how many points will they intersect each other?

Sol. It is given that no two lines are parallel means all line are intersecting and no three lines are concurrent means three lines intersect at a point.

Since, we know that for one point of intersection, we required two lines.

$$\therefore \quad \text{Number of point of intersection} = {}^{20}C_2 = \frac{20!}{2!18!} = \frac{20 \times 19 \times 18!}{2 \times 1 \times 18!}$$

$$= \frac{20 \times 19}{2} = 19 \times 10 = 190$$

Q. 18 In a certain city, all telephone numbers have six digits, the first two digits always being 41 or 42 or 46 or 62 or 64. How many telephone numbers have all six digits distinct?

Sol. If first two digit is 41, the remaining 4 digits can be arranged in

$$= {}^8P_4 = \frac{8!}{8-4!} = \frac{8!}{4!}$$

$$= \frac{8 \times 7 \times 6 \times 5 \times 4!}{4!}$$

$$= 8 \times 7 \times 6 \times 5 = 1680$$

Similarly, if first two digit is 42, 46, 62, or 64, the remaining 4 digits can be arranged in ${}^8P_4$ ways *i.e.*, 1680 ways.

$\therefore$ Total number of telephone numbers have all six digits distinct $= 5 \times 1680 = 8400$

Q. 19 In an examination, a student has to answer 4 questions out of 5 questions, questions 1 and 2 are however compulsory. Determine the number of ways in which the student can make the choice.

Sol. It is given that 2 questions are compulsory out of 5 questions.

So, these two questions are always included in the selection.

We know that, the selection of n distinct objects taken r at a time in which p objects are always included in ${}^{n-p}C_{r-p}$ ways.

$\therefore$ Total number of ways $= {}^{5-2}C_{4-2} = {}^3C_2$

$$= \frac{3!}{2!1!} = \frac{3 \times 2!}{2!} = 3$$

Q. 20 If a convex polygon has 44 diagonals, then find the number of its sides.

Sol. Let the convex polygon has n sides.

$\therefore$ Number of diagonals $= {}^nC_2 - n$

According to the question,

$${}^nC_2 - n = 44$$

$$\frac{n!}{2!(n-2)!} - n = 44$$

$$\Rightarrow \quad \frac{n(n-1)}{2} - n = 44$$

$$\Rightarrow \quad n\left[\frac{(n-1)}{2} - 1\right] = 44 \quad \Rightarrow \quad n\left(\frac{n-1-2}{2}\right) = 44$$

$$\Rightarrow \quad n(n-3) = 44 \times 2 \quad \Rightarrow \quad n(n-3) = 88$$

$$\Rightarrow \quad n^2 - 3n - 88 = 0 \quad \Rightarrow \quad (n-11)(n+8) = 0$$

$$\Rightarrow \quad n = 11, -8 \qquad [\because n \neq -8]$$

$$\therefore \quad n = 11$$

Long Answer Type Questions

Q. 21 18 mice were placed in two experimental groups and one control group with all groups equally large. In how many ways can the mice be placed into three groups?

Sol. It is given that 18 mice were placed equally in two experimental groups and one control group *i.e.*, three groups.

$\therefore$ Required arrangements $= \dfrac{\text{Total arrangement}}{\text{Equally likely arrangement}} = \dfrac{18!}{6!6!6!}$

Q. 22 A bag contains six white marbles and five red marbles. Find the number of ways in which four marbles can be drawn from the bag, if (i) they can be of any colour. (ii) two must be white and two red. (iii) they must all be of the same colour.

Sol. Total number of marbles = 6 white + 5 red = 11 marbles

(i) If they can be of any colour means we have to select 4 marbles out of 11.

$\therefore$ Required number of ways = ${}^{11}C_4$

(ii) If two must be white, then selection will be ${}^6C_2$ and two must be red, then selection will be ${}^5C_2$.

$\therefore$ Required number of ways = ${}^6C_2 \times {}^5C_2$

(iii) If they all must be of same colour, then selection of 4 white marbles out of 6 = ${}^6C_4$

and selection of 4 red marble out of 5 = ${}^5C_4$

$\therefore$ Required number of ways = ${}^6C_4 + {}^5C_4$

Q. 23 In how many ways can a football team of 11 players be selected from 16 players? How many of them will

(i) include 2 particular players?

(ii) exclude 2 particular players?

Sol. Total number of players = 16

We have to select a team of 11 players

(i) include 2 particular players = ${}^{16-2}C_{11-2} = {}^{14}C_9$

[since, selection of n objects taken r at a time in which p objects are always included is ${}^{n-p}C_{r-p}$]

(ii) Exclude 2 particular players = ${}^{16-2}C_{11} = {}^{14}C_{11}$

[since, selection of n objects taken r at a time in which p objects are never included is ${}^{n-p}C_r$]

Q. 24 A sports team of 11 students is to be constituted, choosing atleast 5 from class XI and atleast 5 from class XII. If there are 20 students in each of these classes, in how many ways can the team be constituted?

Sol. Total students in each class = 20

We have to selects atleast 5 students from each class.

Hence, selection of sport team of 11 students from each class is given in following table

Class XI	5	6
Class XII	6	5

$\therefore$ Total number of ways of selecting a team of 11 players $= {}^{20}C_5 \times {}^{20}C_6 + {}^{20}C_6 \times {}^{20}C_5$

$= 2 \times {}^{20}C_5 \times {}^{20}C_6$

Q. 25 A group consists of 4 girls and 7 boys. In how many ways can a team of 5 members be selected, if the team has

(i) no girls.

(ii) atleast one boy and one girl.

(iii) atleast three girls.

Sol. Number of girls = 4 and Number of boys = 7

We have to select a team of 5 members provided that

(i) team having no girls.

$\therefore$ Required selection $= {}^7C_5 = \frac{7!}{5!2!} = \frac{7 \times 6}{2} = 21$

(ii) atleast one boy and one girl

$\therefore$ Required selection $= {}^7C_1 \times {}^4C_4 + {}^7C_2 \times {}^4C_3 + {}^7C_3 \times {}^4C_2 + {}^7C_4 \times {}^4C_1$

$= 7 \times 1 + 21 \times 4 + 35 \times 6 + 35 \times 4$

$= 7 + 84 + 210 + 140 = 441$

(iii) when atleast three girls are included $= {}^4C_3 \times {}^7C_2 + {}^4C_4 \times {}^7C_1$

$= 4 \times 21 + 7 = 84 + 7 = 91$

Q. 26 A committee of 6 is to be chosen from 10 men and 7 women, so as to contain atleast 3 men and 2 women. In how many different ways can this be done, if two particular women refuse to serve on the same committee?

Sol. $\because$ Total number of men = 10

and total number of women = 7

We have to form a committee containing atleast 3 men and 2 women.

Number of ways $= {}^{10}C_3 \times {}^7C_3 + {}^{10}C_4 \times {}^7C_2$

If two particular women to be always there .

$\therefore$ Number of ways $= {}^{10}C_4 \times {}^5C_0 + {}^{10}C_3 \times {}^5C_1$

Total number of committee when two particular women are never together

= Total – Together

$= ({}^{10}C_3 \times {}^7C_3 + {}^{10}C_4 \times {}^7C_2) - ({}^{10}C_4 \times {}^5C_0 + {}^{10}C_3 \times {}^5C_1)$

$= (120 \times 35 + 210 \times 21) - (210 + 120 \times 5)$

$= 4200 + 4410 - (210 + 600)$

$= 8610 - 810 = 7800$

Objective Type Questions

Q. 27 If $^nC_{12} = {}^nC_8$, then n is equal to

(a) 20 (b) 12 (c) 6 (d) 30

Sol. ***(a)*** Given that, $^nC_{12} = {}^nC_8$

$\Rightarrow \quad {}^nC_{n-12} = {}^nC_8 \qquad [\because {}^nC_r = {}^nC_{n-r}]$

$\Rightarrow \quad n - 12 = 8$

$\Rightarrow \quad n = 12 + 8 = 20$

Q. 28 The number of possible outcomes when a coin is tossed 6 times is

(a) 36 (b) 64 (c) 12 (d) 32

Sol. ***(b)*** Number of outcomes when tossing a coin 1 times = 2 (head or tail)

$\therefore$ Total possible outcomes when a coin tossed 6 times = $2^6 = 64$

[$\because 2^n$ for n time tossed coin]

Q. 29 The number of different four-digit numbers that can be formed with the digits 2, 3, 4, 7 and using each digit only once is

(a) 120 (b) 96 (c) 24 (d) 100

Sol. ***(c)*** Given, digits 2, 3, 4 and 7, we have to form four-digit numbers using these digits.

$\therefore$ Required number of ways = $^4P_4 = 4! = 4 \times 3 \times 2! = 24$

Q. 30 The sum of the digits in unit place of all the numbers formed with the help of 3, 4, 5 and 6 taken all at a time is

(a) 432 (b) 108 (c) 36 (d) 18

Sol. ***(b)*** If we fixed 3 at units place.

Total possible number is 3! *i.e.*,6.

Sum of the digits in unit place of all these numbers = $3! \times 3$

Similarly, if we fixed 4, 5 and 6 at units place, in each case total possible numbers are 3!.

Required sum of unit digits of all such numbers = $(3 + 4 + 5 + 6) \times 3!$

$= 18 \times 3! = 18 \times 6 = 108$

Q. 31 The total number of words formed by 2 vowels and 3 consonants taken from 4 vowels and 5 consonants is

(a) 60 (b) 120 (c) 7200 (d) 720

Sol. ***(c)*** Given that, total number of vowels = 4

and total number of consonants = 5

Total number of words formed by 2 vowels and 3 consonants

$$= {}^4C_2 \times {}^5C_3 = \frac{4!}{2!2!} \times \frac{5!}{3!2!}$$

$$= \frac{4 \times 3!}{2!2!} \times \frac{5 \times 4 \times 3 \times 2!}{3! \times 2!} = \frac{4 \times 5 \times 4 \times 3}{4}$$

$$= 5 \times 4 \times 3 = 60$$

Choose what order they appear in 5! *i.e.*, 120.

So, total number of words = $60 \times 120 = 7200$

Q. 32 If a five-digit number divisible by 3 is to be formed using the numbers 0, 1, 2, 3, 4 and 5 without repetitions, then the total number of ways this can be done is

(a) 216 (b) 600 (c) 240 (d) 3125

Sol. ***(a)*** We know that, a number is divisible by 3, when sum of digits in the number must be divisible by 3.

So, if we consider the digits 0, 1, 2, 4, 5, then $(0+1+2+4+5)=12)$

We see that, sum is divisible by 3. Therefore, five-digit numbers using the digit

$$0, 1, 2, 4, 5 = 4\times4\times3\times2\times1 = 96$$

4	4	3	2	1

and if we consider the digit 1, 2, 3, 4, 5, then $(1+2+3+4+5=15)$

This sum is also divisible by 3.

So, five-digit number can be formed using the digit 1, 2, 3, 4, 5 in 5! ways.

Total number of ways $= 96 + 5! = 96 + 120 = 216$

Q. 33 Everybody in a room shakes hands with everybody else. If the total number of hand shakes is 66, then the total number of persons in the room is

(a) 11 (b) 12 (c) 13 (d) 14

Sol. ***(b)*** Let the total number of person in the room is n.

We know that, two person form 1 hand shaken.

$\therefore$ Required number of hand shakes $= {}^nC_2 = \dfrac{n!}{2!\,(n-2)!} = \dfrac{n\,(n-1)}{2}$

According to the question, $\dfrac{n\,(n-1)}{2} = 66$

$\Rightarrow \quad n\,(n-1) = 132$

$\Rightarrow \quad n^2 - n - 132 = 0$

$\Rightarrow \quad (n-12)\,(n+11) = 0$

$\Rightarrow \quad n = 12, -11$ [inadmissible]

$\therefore \quad n = 12$

Q. 34 The number of triangles that are formed by choosing the vertices from a set of 12 points, seven of which lie on the same line is

(a) 105 (b) 15 (c) 175 (d) 185

Sol. ***(d)*** Total number of triangles formed from 12 points taking 3 at a time $= {}^{12}C_3$

But out of 12 points 7 are collinear. So, these 7 points constitute a straight line mean no triangle is formed by joining these 7 points.

$\therefore$ Required number of triangles $= {}^{12}C_3 - {}^7C_3 = 220 - 35 = 185$

Q. 35 The number of parallelograms that can be formed from a set of four parallel lines intersecting another set of three parallel lines is

(a) 6 (b) 18 (c) 12 (d) 9

Sol. ***(b)*** To form parallelogram we required a pair of line from a set of 4 lines and another pair of line from another set of 3 lines.

$\therefore$ Required number of parallelograms $= {}^4C_2 \times {}^3C_2 = 6\times3 = 18$

Q. 36 The number of ways in which a team of eleven players can be selected from 22 players always including 2 of them and excluding 4 of them is

(a) $^{16}C_{11}$ (b) $^{16}C_5$ (c) $^{16}C_9$ (d) $^{20}C_9$

Sol. *(c)* Total number of players = 22

We have to select a team of 11 players. Selection of 11 players when 2 of them is always included and 4 are never included.

Total number of players = 22 − 2 − 4 = 16

$\therefore$ Required number of selections = $^{16}C_9$

Q. 37 The number of 5-digit telephone numbers having atleast one of their digits repeated is

(a) 90000 (b) 10000 (c) 30240 (d) 69760

Sol. *(d)* If all the digits repeated, then number of 5 digit telephone numbers can be formed in 10^5 ways and if no digit repeated, then 5-digit telephone numbers can be formed in $^{10}P_5$ ways.

$$\therefore \text{ Required number of ways} = 10^5 - {}^{10}P_5 = 100000 - \frac{10!}{5!}$$

$$= 100000 - 10 \times 9 \times 8 \times 7 \times 6$$

$$= 100000 - 30240 = 69760$$

Q. 38 The number of ways in which we can choose a committee from four men and six women, so that the committee includes atleast two men and exactly twice as many women as men is

(a) 94 (b) 126 (c) 128 (d) None of these

Sol. *(a)* $\because$ Number of men = 4

and number of women = 6

It is given that committee includes two men and exactly twice as many women as men.

Thus, possible selection is given in following table

Men	Women
2	4
3	6

$$\text{Required number of committee formed} = {}^4C_2 \times {}^6C_4 + {}^4C_3 \times {}^6C_6$$

$$= 6 \times 15 + 4 \times 1 = 94$$

Q. 39 The total number of 9-digit numbers which have all different digits is

(a) 10! (b) 9! (c) $9 \times 9!$ (d) $10 \times 10!$

Sol. *(c)* We have to form 9-digit numbers with the digit 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 0 cannot be placed at the first place from left. So, first place from left can be filled in 9 ways.

Since, repetition is not allowed, so remaining 8 places can be filled in 9! ways.

$\therefore$ Required number of ways = $9 \times 9!$

Q. 40 The number of words which can be formed out of the letters of the word 'ARTICLE', so that vowels occupy the even place is

(a) 1440 (b) 144 (c) 7! (d) $^4C_4 \times {}^3C_3$

Sol. *(b)* Total number of letters in the word article is 7, out of which A, E, I are vowels and R, T, C, L are consonants.

Since, it is given that vowels occupy even place, therefore the arrangement of vowel, consonant can be understand with the help of following diagram.

1	2	3	4	5	6	7

Now, vowels can be placed at 2, 4 and 6th position.

Therefore, number of arrangement = $^3P_3 = 3! = 6$ ways

and consonants can be placed at 1, 3, 5 and 7th position.

Therefore, number of arrangement = $^4P_4 = 4! = 24$

$\therefore$ Total number of words = $6 \times 24 = 144$

Q. 41 Given 5 different green dyes, four different blue dyes and three different red dyes, the number of combinations of dyes which can be chosen taking atleast one green and one blue dye is

(a) 3600 (b) 3720 (c) 3800 (d) 3600

Sol. *(b)* Possible number of choosing green dyes = 2^5

Possible number of choosing blue dyes = 2^4

Possible number of choosing red dyes = 2^3

If atleast one blue and one green dyes are selected.

Then, total number of selection = $(2^5 - 1)(2^4 - 1) \times 2^3 = 3720$

Fillers

Q. 42 If $^nP_r = 840$ and $^nC_r = 35$, then r is equal to ....

Sol. Given that, $^nP_r = 840$ and $^nC_r = 35$

$\because \quad {}^nP_r = {}^nC_r \cdot r!$

$\Rightarrow \quad 840 = 35 \times r!$

$\Rightarrow \quad r! = \dfrac{840}{35} = 24$

$\Rightarrow \quad r! = 4 \times 3 \times 2 \times 1$

$\Rightarrow \quad r! = 4!$

$\therefore \quad r = 4$

Q. 43 $^{15}C_8 + {}^{15}C_9 - {}^{15}C_6 - {}^{15}C_7$ is equal to ....

Sol. $^{15}C_8 + {}^{15}C_9 - {}^{15}C_6 - {}^{15}C_7 = {}^{15}C_{15-8} + {}^{15}C_{15-9} - {}^{15}C_6 - {}^{15}C_7$ $[\because {}^nC_r = {}^nC_{n-r}]$

$= {}^{15}C_7 + {}^{15}C_6 - {}^{15}C_6 - {}^{15}C_7$

$= 0$

Q. 44 The number of permutations of n different objects, taken r at a line, when repetitions are allowed, is ....

Sol. Number of permutations of n different things taken r at a time when repetition is allowed $= n^r$

Q. 45 The number of different words that can be formed from the letters of the word 'INTERMEDIATE' such that two vowels never come together is ....

Sol. Total number of letters in the word 'INTERMEDIATE' = 12
out of which 6 are consonants and 6 are vowels. The arrangement of these 12 alphabets in which two vowels never come together can be understand with the help of follow manner.

V	C	V	C	V	C	V	C	V	C	V	C	V

6 consonants out of which 2 are alike can be placed in $\frac{6!}{2!}$ ways and 6 vowels, out of which 3 E's alike and 2 I's are alike can be arranged at seven place in ${}^7P_6 \times \frac{1}{3!} \times \frac{1}{2!}$ ways.

$\therefore$ Total number of words $= \frac{6!}{2!} \times {}^7P_6 \times \frac{1}{3!} \times \frac{1}{2!} = 151200$

Q. 46 Three balls are drawn from a bag containing 5 red, 4 white and 3 black balls. The number of ways in which this can be done, if atleast 2 are red, is.

Sol. Required number of ways $= {}^5C_2 \times {}^7C_1 + {}^5C_3$ [since, at least two red]

$= 10 \times 7 + 10$

$= 70 + 10 = 80$

Q. 47 The number of six-digit numbers all digits of which are odd, is ....

Sol. Among the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, clearly 1, 3, 5, 7 and 9 are odd.

$\therefore$ Number of six-digit numbers $= 5 \times 5 \times 5 \times 5 \times 5 \times 5 = 5^6$

Q. 48 In a football championship, 153 matches were played. Every two teams played one match with each other. The number of teams, participating in the championship is ....

Sol. Let the number of team participating in championship be n.
Since, it is given that every two teams played one match with each other.

$\therefore$ Total match played $= {}^nC_2$

According to the question,

$${}^nC_2 = 153$$

$$\Rightarrow \frac{n(n-1)}{2} = 153$$

$$\Rightarrow n^2 - n = 306$$

$$\Rightarrow n^2 - n - 306 = 0$$

$$\Rightarrow (n - 18)(n + 17) = 0$$

$$\Rightarrow n = 18, -17 \quad \text{[inadmissible]}$$

$$\therefore n = 18$$

Q. 49 The total number of ways in which six '+' and four '–' signs can be arranged in a line such that no two signs '–' occur together, is ....

Sol. The arrangement can be understand with the help of following figure.

–	+	–	+	–	+	–	+	–	+	–	+	–

Thus, '+' sign can be arranged in 1 way because all are identical.
and 4 negative signs can be arranged at 7 places in ${}^7C_4$ ways.

$\therefore$ total number of ways $= {}^7C_4 \times 1$

$$= \frac{7!}{4!3!} = \frac{7 \times 6 \times 5 \times 4!}{3! \times 4!}$$

$$= \frac{7 \times 6 \times 5}{3 \times 2 \times 1} = 35 \text{ ways}$$

Q. 50 A box contains 2 white balls, 3 black balls and 4 red balls. The number of ways three balls be drawn from the box, if atleast one black ball is to be included in the draw is ....

Sol. Since, there are 2 white, 3 black and 4 red balls. It is given that atleast one black ball is to be included in the draw.

$\therefore$ Required number of ways $= {}^3C_1 \times {}^6C_2 + {}^3C_2 \times {}^6C_1 + {}^3C_3$

$$= 3 \times 15 + 3 \times 6 + 1$$
$$= 45 + 18 + 1 = 64$$

True/False

Q. 51 There are 12 points in a plane of which 5 points are collinear, then the number of lines obtained by joining these points in pairs is ${}^{12}C_2 - {}^5C_2$.

Sol. ***False***

Required number of lines $= {}^{12}C_2 - {}^5C_2 + 1$

Q. 52 Three letters can be posted in five letter boxes in 3^5 ways.

Sol. ***False***

Required number of ways $= 5^3 = 125$

Q. 53 In the permutations of n things r, taken together, the number of permutations in which m particular things occur together is ${}^{n-m}P_{r-m} \times {}^rP_m$.

Sol. ***False***

Arrangement of n things, taken r at a time in which m things occur together, we considered these m things as 1 group.

Number of object excluding those m objects $= (r - m)$

Now, first we have to arrange $(r - m + 1)$ objects.

Number of arrangements $= (r - m + 1)!$ and m objects which we consider as 1 group, can be arranged in $m!$ ways.

$\therefore$ Required number of arrangements $= (r - m + 1)! \times m!$

Q. 54 In a steamer there are stalls for 12 animals and there are horses, cows and calves (not less than 12 each) ready to be shipped. They can be loaded in 3^{12} ways.

Sol. ***True***

There are three types of animals and stalls available for 12 animals.

Number of ways of loading = 3^{12}

Q. 55 If some or all of n objects are taken at a time, then the number of combinations is $2^n - 1$.

Sol. ***True***

If some or all objects taken at a time, then number of selection would be

$$^nC_1 + {^nC_2} + {^nC_3} + \dots + {^nC_n} = 2^n - 1 \qquad [\because \ {^nC_0} + {^nC_1} + {^nC_2} + \dots + {^nC_n} = 2^n]$$

Q. 56 There will be only 24 selections containing atleast one red ball out of a bag containing 4 red and 5 black balls. It is being given that the balls of the same colour are identical.

Sol. Total number of selection $= [(4 + 1)(5 + 1) - 1] - 5$

$= (5 \times 6 - 1) - 5$

$= (30 - 1) - 5 = 24$

Q. 57 Eighteen guests are to be seated, half on each side of a long table. Four particular guests desire to sit on one particular side and three others on other side of the table. The number of ways in which the seating arrangements can be made is $\frac{11!}{5!6!}(9!)(9!)$.

Sol. ***True***

After seating 4 on one side and 3 on the other side, we have to select out of 11;5 on one side and 6 on the other side.

Now, remaining selecting of one half side $= {^{(18-4-3)}C_5} = {^{11}C_5}$

and the other half side $= {^{(11-5)}C_6} = {^6C_6}$

Total arrangements $= {^{11}C_5} \times 9! \times {^6C_6} \times 9!$

$= \frac{11!}{5!6!} \times 9! \times 1 \times 9!$

$= \frac{11!}{5!6!} \times 9! \times 9!$

Q. 58 A candidate is required to answer 7 questions, out of 12 questions which are divided into two groups, each containing 6 questions. He is not permitted to attempt more than 5 questions from either group. He can choose the seven questions in 650 ways.

Sol. ***False***

He can attempt questions in following manner

Group (*A*)	2	3	4	5
Group (*B*)	5	4	3	2

Number of ways of attempting 7 questions

$$= {}^6C_2 \times {}^6C_5 + {}^6C_3 \times {}^6C_4 + {}^6C_4 \times {}^6C_3 + {}^6C_5 \times {}^6C_2$$
$$= 2\,({}^6C_2 \times {}^6C_5 + {}^6C_3 \times {}^6C_4)$$
$$= 2\,(15 \times 6 + 20 \times 15)$$
$$= 2\,(90 + 300)$$
$$= 2 \times 390 = 780$$

Q. 59 To fill 12 vacancies there are 25 candidates of which 5 are from scheduled castes. If 3 of the vacancies are reserved for scheduled caste candidates while the rest are open to all, the number of ways in which the selection can be made is ${}^5C_3 \times {}^{20}C_9$.

Sol. ***False***

We have to select 3 scheduled caste candidate out of 5 in ${}^5C_3$ ways.

and we have to select 9 other candidates out of 22 in ${}^{22}C_9$ ways.

$\therefore$ Total number of selections = ${}^5C_3 \times {}^{22}C_9$

Matching The Columns

Q. 60 There are 3 books on Mathematics, 4 on Physics and 5 on English. How many different collections can be made such that each collection consists?

	Column I		Column II
(i)	One book of each subject	(a)	3968
(ii)	Atleast one book of each subject	(b)	60
(iii)	Atleast one book of English	(c)	3255

Sol. There are three books of Mathematics 4 of Physics and 5 on English.

(i) One book of each subject = ${}^3C_1 \times {}^4C_1 \times {}^5C_1$

$$= 3 \times 4 \times 5 = 60$$

(ii) Atleast one book of each subject = $(2^3 - 1) \times (2^4 - 1) \times (2^5 - 1)$

$$= 7 \times 15 \times 31 = 3255$$

(iii) Atleast one book of English = Selection based on following manner

English book	1	2	3	4	5
Others	11	10	9	8	7

$$= (2^5 - 1) \times 2^7$$
$$= 128 \times 31 = 3968$$

Q. 61 Five boys and five girls form a line. Find the number of ways of making the seating arrangement under the following condition.

	Column I		Column II
(i)	Boys and girls alternate	(a)	$5! \times 6!$
(ii)	No two girls sit together	(b)	$10! - 5!\ 6!$
(iii)	All the girls sit together	(c)	$(5!)^2 + (5!)^2$
(iv)	All the girls are never together	(d)	$2!\ 5!\ 5!$

Sol. **(i)** Boys and girls alternate

Total arrangements $= (5!)^2 + (5!)^2$

(ii) No two girls sit together $= 5!6!$

(iii) All the girls sit together $= 2!\ 5!\ 5!$

(iv) All the girls are never together $= 10! - 5!\ 6!$

Q. 62 There are 10 professors and 20 lecturers, out of whom a committee of 2 professors and 3 lecturers is to be formed. Find

	Column I		Column II
(i)	in how many ways committee can be formed?	(a)	${}^{10}C_2 \times {}^{19}C_3$
(ii)	in how many ways a particular professor is included?	(b)	${}^{10}C_2 \times {}^{19}C_2$
(iii)	in how many ways a particular lecturer is included?	(c)	${}^{9}C_1 \times {}^{20}C_3$
(iv)	in how many ways a particular lecturer is excluded?	(d)	${}^{10}C_2 \times {}^{20}C_3$

Sol. **(i)** We have to select 2 professors out of 10 and 3 lecturers out of 20 $= {}^{10}C_2 \times {}^{20}C_3$

(ii) When a particular professor included $= {}^{10-1}C_1 \times {}^{20}C_3 = {}^{9}C_1 \times {}^{20}C_3$

(iii) When a particular lecturer included $= {}^{10}C_2 \times {}^{19}C_2$

(iv) When a particular lecturer excluded $= {}^{10}C_2 \times {}^{19}C_3$

Q. 63 Using the digits 1, 2, 3, 4, 5, 6, 7, a number of 4 different digits is formed. Find

	Column I		Column II
(i)	how many numbers are formed?	(a)	840
(ii)	how many numbers are exactly divisible by 2?	(b)	200
(iii)	how many numbers are exactly divisible by 25?	(c)	360
(iv)	how many of these are exactly divisible by 4?	(d)	40

Sol. **(i)** Total numbers of 4 digit formed with digits 1, 2, 3, 4, 5, 6, 7

$$= 7 \times 6 \times 5 \times 4 = 840$$

(ii) When a number is divisible by 2. At its unit place only even numbers occurs.

Total numbers $= 4 \times 5 \times 6 \times 3 = 360$

(iii) Total numbers which are divisible by 25 = 40

(iv) A number is divisible by 4, If its last two digit is divisible by 4.

$\therefore$ Total such numbers = 200

Q. 64 How many words (with or without dictionary meaning) can be made from the letters of the word MONDAY, assuming that no letter is repeated, if

Column I		Column II	
(i)	4 letters are used at a time.	(a)	720
(ii)	All letters are used at a time	(b)	240
(iii)	All letters are used but the first is a vowel.	(c)	360

Sol. **(i)** 4 letters are used at a time $= {}^6P_4 = \frac{6!}{2!} = 6 \times 5 \times 4 \times 3 = 360$

(ii) All letters used at a time $= 6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$

(iii) All letters used but first is vowel $= 2 \times 5! = 2 \times 5 \times 4 \times 3 \times 2 \times 1 = 240$

8

Binomial Theorem

Short Answer Type Questions

Q. 1 Find the term independent of x, where $x \neq 0$, in the expansion of $\left(\frac{3x^2}{2} - \frac{1}{3x}\right)^{15}$.

Thinking Process

The general term in the expansion of $(x-a)^n$ i.e., $T_{r+1} = {}^nC_r (x)^{n-r} (-a)^r$. For the term independent of x, put $n - r = 0$, then we get the value of r.

Sol. Given expansion is $\left(\frac{3x^2}{2} - \frac{1}{3x}\right)^{15}$.

Let T_{r+1} term is the general term.

Then,
$$T_{r+1} = {}^{15}C_r \left(\frac{3x^2}{2}\right)^{15-r} \left(-\frac{1}{3x}\right)^r$$
$$= {}^{15}C_r \ 3^{15-r} \ x^{30-2r} \ 2^{r-15} \ (-1)^r \cdot 3^{-r} \cdot x^{-r}$$
$$= {}^{15}C_r (-1)^r \ 3^{15-2r} 2^{r-15} x^{30-3r}$$

For independent of x,
$$30 - 3r = 0$$
$$3r = 30 \Rightarrow r = 10$$

$\because$ $T_{r+1} = T_{10+1} = 11\text{th term is independent of } x.$

$\therefore$
$$T_{10+1} = {}^{15}C_{10} (-1)^{10} \ 3^{15-20} \ 2^{10-15}$$
$$= {}^{15}C_{10} \ 3^{-5} \ 2^{-5}$$
$$= {}^{15}C_{10} (6)^{-5}$$
$$= {}^{15}C_{10} \left(\frac{1}{6}\right)^5$$

Q. 2 If the term free from x in the expansion of $\left(\sqrt{x} - \frac{k}{x^2}\right)^{10}$ is 405, then find the value of k.

Sol. Given expansion is $\left(\sqrt{x} - \frac{k}{x^2}\right)^{10}$.

Let T_{r+1} is the general term.

Then, $T_{r+1} = {}^{10}C_r (\sqrt{x})^{10-r} \left(\frac{-k}{x^2}\right)^r$

$= {}^{10}C_r (x)^{\frac{1}{2}(10-r)} (-k)^r \cdot x^{-2r}$

$= {}^{10}C_r x^{5-\frac{r}{2}} (-k)^r \cdot x^{-2r}$

$= {}^{10}C_r \ x^{5-\frac{r}{2}-2r} (-k)^r$

$= {}^{10}C_r \ x^{\frac{10-5r}{2}} (-k)^r$

For free from x, $\frac{10-5r}{2} = 0$

$\Rightarrow$ $10 - 5r = 0 \Rightarrow r = 2$

Since, $T_{2+1} = T_3$ is free from x.

$\therefore$ $T_{2+1} = {}^{10}C_2 (-k)^2 = 405$

$\Rightarrow$ $\frac{10 \times 9 \times 8!}{2! \times 8!} (-k)^2 = 405$

$\Rightarrow$ $45k^2 = 405 \Rightarrow k^2 = \frac{405}{45} = 9$

$\therefore$ $k = \pm 3$

Q. 3 Find the coefficient of x in the expansion of $(1 - 3x + 7x^2)(1 - x)^{16}$.

Sol. Given, expansion $= (1 - 3x + 7x^2)(1 - x)^{16}$.

$= (1 - 3x + 7x^2)({}^{16}C_0 1^{16} - {}^{16}C_1 1^{15} x^1 + {}^{16}C_2 1^{14} x^2 + \ldots + {}^{16}C_{16} x^{16})$

$= (1 - 3x + 7x^2)(1 - 16x + 120x^2 + \ldots)$

$\therefore$ Coefficient of $x = -3 - 16 = -19$

Q. 4 Find the term independent of x in the expansion of $\left(3x - \frac{2}{x^2}\right)^{15}$.

Thinking Process

The general term in the expansion of $(x - a)^n$ i.e., $T_{r+1} = {}^nC_r (x)^{n-r} (-a)^r$.

Sol. Given expansion is $\left(3x - \frac{2}{x^2}\right)^{15}$.

Let T_{r+1} is the general term.

$\therefore$ $T_{r+1} = {}^{15}C_r (3x)^{15-r} \left(\frac{-2}{x^2}\right)^r = {}^{15}C_r (3x)^{15-r} (-2)^r x^{-2r}$

$= {}^{15}C_r 3^{15-r} x^{15-3r} (-2)^r$

For independent of x, $15 - 3r = 0 \Rightarrow r = 5$

Since, $T_{5+1} = T_6$ is independent of x.

$$\therefore \quad T_{5+1} = {}^{15}C_5\, 3^{15-5}(-2)^5$$

$$= -\frac{15 \times 14 \times 13 \times 12 \times 11 \times 10!}{5 \times 4 \times 3 \times 2 \times 1 \times 10!} \cdot 3^{10} \cdot 2^5$$

$$= -3003 \cdot 3^{10} \cdot 2^5$$

Q. 5 Find the middle term (terms) in the expansion of

(i) $\left(\frac{x}{a} - \frac{a}{x}\right)^{10}$ (ii) $\left(3x - \frac{x^3}{6}\right)^9$

Thinking Process

In the expansion of $(a+b)^n$, if n is even, then this expansion has only one middle term i.e., $\left(\frac{n}{2}+1\right)$th term is the middle term and if n is odd, then this expansion has two middle terms i.e., $\left(\frac{n+1}{2}\right)$th and $\left(\frac{n+1}{2}+1\right)$th are two middle terms.

Sol. **(i)** Given expansion is $\left(\frac{x}{a} - \frac{a}{x}\right)^{10}$.

Here, the power of Binomial *i.e.*, $n = 10$ is even.

Since, it has one middle term $\left(\frac{10}{2} + 1\right)$th term *i.e.*, 6th term.

$$\therefore \quad T_6 = T_{5+1} = {}^{10}C_5 \left(\frac{x}{a}\right)^{10-5} \left(\frac{-a}{x}\right)^5$$

$$= -{}^{10}C_5 \left(\frac{x}{a}\right)^5 \left(\frac{a}{x}\right)^5$$

$$= -\frac{10 \times 9 \times 8 \times 7 \times 6 \times 5!}{5! \times 5 \times 4 \times 3 \times 2 \times 1} \left(\frac{x}{a}\right)^5 \left(\frac{x}{a}\right)^{-5}$$

$$= -9 \times 4 \times 7 = -252$$

(ii) Given expansion is $\left(3x - \frac{x^3}{6}\right)^9$.

Here, $n = 9$ [odd]

Since, the Binomial expansion has two middle terms *i.e.*, $\left(\frac{9+1}{2}\right)$th and $\left(\frac{9+1}{2} + 1\right)$th *i.e.*, 5th term and 6th term.

$$\therefore \quad T_5 = T_{(4+1)} = {}^9C_4 (3x)^{9-4} \left(-\frac{x^3}{6}\right)^4$$

$$= \frac{9 \times 8 \times 7 \times 6 \times 5!}{4 \times 3 \times 2 \times 1 \times 5!}\, 3^5\, x^5\, x^{12}\, 6^{-4}$$

$$= \frac{7 \times 6 \times 3 \times 3^1}{2^4}\, x^{17} = \frac{189}{8}\, x^{17}$$

$$\therefore \quad T_6 = T_{5+1} = {}^9C_5 (3x)^{9-5} \left(-\frac{x^3}{6}\right)^5$$

$$= -\frac{9 \times 8 \times 7 \times 6 \times 5!}{5! \times 4 \times 3 \times 2 \times 1} \cdot 3^4 \cdot x^4 \cdot x^{15} \cdot 6^{-5}$$

$$= \frac{-21 \times 6}{3 \times 2^5} x^{19} = \frac{-21}{16} x^{19}$$

Q. 6 **Find the coefficient of x^{15} in the expansion of $(x - x^2)^{10}$.**

Sol. Given expansion is $(x - x^2)^{10}$.

Let the term T_{r+1} is the general term.

$$\therefore \quad T_{r+1} = {}^{10}C_r x^{10-r} (-x^2)^r$$

$$= (-1)^r \cdot {}^{10}C_r \cdot x^{10-r} \cdot x^{2r}$$

$$= (-1)^r \, {}^{10}C_r \, x^{10+r}$$

For the coefficient of x^{15},

$$10 + r = 15 \Rightarrow r = 5$$

$$T_{5+1} = (-1)^5 \, {}^{10}C_5 \, x^{15}$$

$$\therefore \quad \text{Coefficient of } x^{15} = -1 \, \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5!}{5 \times 4 \times 3 \times 2 \times 1 \times 5!}$$

$$= -3 \times 2 \times 7 \times 6 = -252$$

Q. 7 **Find the coefficient of $\frac{1}{x^{17}}$ in the expansion of $\left(x^4 - \frac{1}{x^3}\right)^{15}$.**

Thinking Process

In this type of questions, first of all find the general terms, in the expansion $(x - y)^n$ using the formula $T_{r+1} = {}^nC_r \, x^{n-r} (-y)^r$ and then put $n - r$ equal to the required power of x of which coefficient is to be find out.

Sol. Given expansion is $\left(x^4 - \frac{1}{x^3}\right)^{15}$.

Let the term T_{r+1} contains the coefficient of $\frac{1}{x^{17}}$ *i.e.*, x^{-17}.

$$\therefore \quad T_{r+1} = {}^{15}C_r (x^4)^{15-r} \left(-\frac{1}{x^3}\right)^r$$

$$= {}^{15}C_r \, x^{60-4r} (-1)^r x^{-3r}$$

$$= {}^{15}C_r \, x^{60-7r} (-1)^r$$

For the coefficient x^{-17},

$$60 - 7r = -17$$

$$\Rightarrow \quad 7r = 77 \Rightarrow r = 11$$

$$\Rightarrow \quad T_{11+1} = {}^{15}C_{11} \, x^{60-77} (-1)^{11}$$

$$\therefore \quad \text{Coefficient of } x^{-17} = \frac{-15 \times 14 \times 13 \times 12 \times 11!}{11! \times 4 \times 3 \times 2 \times 1}$$

$$= -15 \times 7 \times 13 = -1365$$

Q. 8 Find the sixth term of the expansion $(y^{1/2}+x^{1/3})^n$, if the Binomial coefficient of the third term from the end is 45.

Sol. Given expansion is $(y^{1/2}+x^{1/3})^n$.

The sixth term of this expansion is

$$T_6 = T_{5+1} = {}^nC_5 (y^{1/2})^{n-5} (x^{1/3})^5 \quad \text{...(i)}$$

Now, given that the Binomial coefficient of the third term from the end is 45.

We know that, Binomial coefficient of third term from the end = Binomial coefficient of third term from the begining = ${}^nC_2$

$\because \quad {}^nC_2 = 45$

$\Rightarrow \quad \dfrac{n(n-1)(n-2)!}{2!(n-2)!} = 45$

$\Rightarrow \quad n(n-1) = 90$

$\Rightarrow \quad n^2 - n - 90 = 0$

$\Rightarrow \quad n^2 - 10n + 9n - 90 = 0$

$\Rightarrow \quad n(n-10) + 9(n-10) = 0$

$\Rightarrow \quad (n-10)(n+9) = 0$

$\Rightarrow \quad (n+9) = 0$ or $(n-10) = 0$

$\therefore \quad n = 10 \qquad [\because n \neq -9]$

From Eq. (i),

$$T_6 = {}^{10}C_5 \, y^{5/2} \, x^{5/3} = 252 y^{5/2} \cdot x^{5/3}$$

Q. 9 Find the value of r, if the coefficients of $(2r+4)$th and $(r-2)$th terms in the expansion of $(1+x)^{18}$ are equal.

Thinking Process

Coefficient of $(r+1)$th term in the expansion of $(1+x)^n$ is ${}^nC_r$. Use this formula to solve the above problem.

Sol. Given expansion is $(1+x)^{18}$.

Now, $(2r+4)$th term *i.e.*, T_{2r+3+1}.

$\therefore \quad T_{2r+3+1} = {}^{18}C_{2r+3}(1)^{18-2r-3}(x)^{2r+3}$

$= {}^{18}C_{2r+3}\, x^{2r+3}$

Now, $(r-2)$th term *i.e.*, T_{r-3+1}.

$\therefore \quad T_{r-3+1} = {}^{18}C_{r-3}\, x^{r-3}$

As, $\quad {}^{18}C_{2r+3} = {}^{18}C_{r-3} \qquad [\because {}^nC_x = {}^nC_y \Rightarrow x + y = n]$

$\Rightarrow \quad 2r + 3 + r - 3 = 18$

$\Rightarrow \quad 3r = 18$

$\therefore \quad r = 6$

Q. 10 If the coefficient of second, third and fourth terms in the expansion of $(1+x)^{2n}$ are in AP, then show that $2n^2 - 9n + 7 = 0$.

Thinking Process

In the expansion of $(x+y)^n$, the coefficient of $(r+1)$th term is ${}^nC_r$. Use this formula to get the required coefficient. If a, b and c are in AP, then $2b = a + c$.

Sol. Given expansion is $(1+x)^{2n}$.

Now, coefficient of 2nd term = ${}^{2n}C_1$

Coefficient of 3rd term = ${}^{2n}C_2$

Coefficient of 4th term = ${}^{2n}C_3$

Given that, ${}^{2n}C_1$, ${}^{2n}C_2$ and ${}^{2n}C_3$ are in AP.

Then, $$2\cdot{}^{2n}C_2 = {}^{2n}C_1 + {}^{2n}C_3$$

$$\Rightarrow \quad 2\left[\frac{2n(2n-1)(2n-2)!}{2\times1\times(2n-2)!}\right] = \frac{2n(2n-1)!}{(2n-1)!} + \frac{2n(2n-1)(2n-2)(2n-3)!}{3!(2n-3)!}$$

$$\Rightarrow \quad n(2n-1) = n + \frac{n(2n-1)(2n-2)}{6}$$

$$\Rightarrow \quad n(12n-6) = n(6+4n^2-4n-2n+2)$$

$$\Rightarrow \quad 12n-6 = (4n^2-6n+8)$$

$$\Rightarrow \quad 6(2n-1) = 2(2n^2-3n+4)$$

$$\Rightarrow \quad 3(2n-1) = 2n^2-3n+4$$

$$\Rightarrow \quad 2n^2-3n+4-6n+3 = 0$$

$$\Rightarrow \quad 2n^2-9n+7 = 0$$

Q. 11 Find the coefficient of x^4 in the expansion of $(1+x+x^2+x^3)^{11}$.

Sol. Given, expansion $= (1+x+x^2+x^3)^{11} = [(1+x)+x^2(1+x)]^{11}$

$$= [(1+x)(1+x^2)]^{11} = (1+x)^{11}\cdot(1+x^2)^{11}$$

Now, above expansion becomes

$$= ({}^{11}C_0 + {}^{11}C_1x + {}^{11}C_2x^2 + {}^{11}C_3x^3 + {}^{11}C_4x^4 + \ldots)({}^{11}C_0 + {}^{11}C_1x^2 + {}^{11}C_2x^4 + \ldots)$$

$$= (1+11x+55x^2+165x^3+330x^4+\ldots)(1+11x^2+55x^4+\ldots)$$

$\therefore$ Coefficient of $x^4 = 55 + 605 + 330 = 990$

Long Answer Type Questions

Q. 12 If p is a real number and the middle term in the expansion of $\left(\frac{p}{2}+2\right)^8$ is 1120, then find the value of p.

Sol. Given expansion is $\left(\frac{p}{2}+2\right)^8$.

Here, $n = 8$ [even]

Since, this Binomial expansion has only one middle term *i.e.*, $\left(\frac{8}{2}+1\right)$th = 5th term

$$T_5 = T_{4+1} = {}^8C_4\left(\frac{p}{2}\right)^{8-4}\cdot 2^4$$

$$\Rightarrow \quad 1120 = {}^8C_4\, p^4\cdot 2^{-4}\, 2^4$$

$$\Rightarrow \quad 1120 = \frac{8\times7\times6\times5\times4!}{4!\times4\times3\times2\times1}p^4$$

$$\Rightarrow \quad 1120 = 7 \times 2 \times 5 \times p^4$$
$$\Rightarrow \quad p^4 = \frac{1120}{70} = 16 \Rightarrow p^4 = 2^4$$
$$\Rightarrow \quad p^2 = 4 \Rightarrow p = \pm 2$$

Q. 13 Show that the middle term in the expansion of $\left(x - \frac{1}{x}\right)^{2n}$ is
$$\frac{1 \times 3 \times 5 \times \ldots \times (2n-1)}{n!} \times (-2)^n.$$

Sol. Given, expansion is $\left(x - \frac{1}{x}\right)^{2n}$. This Binomial expansion has even power. So, this has one middle term.

i.e., $\left(\frac{2n}{2} + 1\right)$th term $= (n+1)$th term

$$T_{n+1} = {}^{2n}C_n (x)^{2n-n}\left(-\frac{1}{x}\right)^n = {}^{2n}C_n\, x^n (-1)^n x^{-n}$$
$$= {}^{2n}C_n (-1)^n = (-1)^n \frac{(2n)!}{n!n!} = \frac{1\cdot 2\cdot 3\cdot 4\cdot 5 \ldots (2n-1)(2n)}{n!n!}(-1)^n$$
$$= \frac{1\cdot 3\cdot 5\ldots (2n-1)\cdot 2\cdot 4\cdot 6\ldots (2n)}{1 2\cdot 3\cdot \ldots n(n!)}(-1)^n$$
$$= \frac{1\cdot 3\cdot 5\ldots(2n-1)\,.\,2^n(1\cdot 2\cdot 3\ldots n)(-1)^n}{(1\cdot 2\cdot 3\ldots n)(n!)}$$
$$= \frac{[1\cdot 3\cdot 5\ldots(2n-1)]}{n!}(-2)^n$$

Hence proved.

Q. 14 Find n in the Binomial $\left(\sqrt[3]{2} + \frac{1}{\sqrt[3]{3}}\right)^n$, if the ratio of 7th term from the beginning to the 7th term from the end is $\frac{1}{6}$.

Sol. Here, the Binomial expansion is $\left(\sqrt[3]{2} + \frac{1}{\sqrt[3]{3}}\right)^n$.

Now, 7th term from beginning $T_7 = T_{6+1} = {}^nC_6(\sqrt[3]{2})^{n-6}\left(\frac{1}{\sqrt[3]{3}}\right)^6$ ...(i)

and 7th term from end *i.e.,* T_7 from the beginning of $\left(\frac{1}{\sqrt[3]{3}} + \sqrt[3]{2}\right)^n$

i.e., $$T_7 = {}^nC_6\left(\frac{1}{\sqrt[3]{3}}\right)^{n-6}(\sqrt[3]{2})^6 \quad \ldots(ii)$$

Given that, $$\frac{{}^nC_6(\sqrt[3]{2})^{n-6}\left(\frac{1}{\sqrt[3]{3}}\right)^6}{{}^nC_6\left(\frac{1}{\sqrt[3]{3}}\right)^{n-6}(\sqrt[3]{2})^6} = \frac{1}{6} \Rightarrow \frac{2^{\frac{n-6}{3}}\cdot 3^{-6/3}}{3^{-\left(\frac{n-6}{3}\right)}\cdot 2^{6/3}} = \frac{1}{6}$$

$$\Rightarrow \left(2^{\frac{n-6}{3}}\cdot 2^{\frac{-6}{3}}\right)\left(3^{\frac{-6}{3}}\cdot 3^{\frac{(n-6)}{3}}\right) = 6^{-1}$$

$$\Rightarrow \quad \left(2^{\frac{n-6}{3}-\frac{6}{3}}\right)\cdot\left(3^{\frac{n-6}{3}-\frac{6}{3}}\right)=6^{-1} \Rightarrow (2\cdot 3)^{\frac{n}{3}-4}=6^{-1}$$

$$\Rightarrow \quad \frac{n}{3}-4=-1 \Rightarrow \frac{n}{3}=3$$

$$\therefore \quad n=9$$

Q. 15 In the expansion of $(x+a)^n$, if the sum of odd terms is denoted by O and the sum of even term by E. Then, prove that

(i) $O^2 - E^2 = (x^2 - a^2)^n$.

(ii) $4OE = (x+a)^{2n} - (x-a)^{2n}$.

Sol. **(i)** Given expansion is $(x+a)^n$.

$$\therefore \quad (x+a)^n = {}^nC_0\, x^n a^0 + {}^nC_1 x^{n-1}a^1 + {}^nC_2 x^{n-2}a^2 + {}^nC_3 x^{n-3}a^3 + \ldots + {}^nC_n a^n$$

Now, sum of odd terms

i.e., $$O = {}^nC_0 x^n + {}^nC_2 x^{n-2}a^2 + \ldots$$

and sum of even terms

i.e., $$E = {}^nC_1 x^{n-1}a + {}^nC_3 x^{n-3}a^3 + \ldots$$

$$\because \quad (x+a)^n = O + E \quad \ldots(i)$$

Similarly, $$(x-a)^n = O - E \quad \ldots(ii)$$

$$\therefore \quad (O+E)(O-E) = (x+a)^n (x-a)^n \quad \text{[on multiplying Eqs. (i) and (ii)]}$$

$$\Rightarrow \quad O^2 - E^2 = (x^2-a^2)^n$$

(ii) $4OE = (O+E)^2 - (O-E)^2 = [(x+a)^n]^2 - [(x-a)^n]^2$ [from Eqs. (i) and (ii)]

$$= (x+a)^{2n} - (x-a)^{2n}$$ **Hence proved**.

Q. 16 If x^p occurs in the expansion of $\left(x^2+\frac{1}{x}\right)^{2n}$, then prove that its coefficient is $\dfrac{2n!}{\left(\dfrac{4n-p}{3}\right)!\left(\dfrac{2n+p}{3}\right)!}$.

Sol. Given expansion is $\left(x^2+\frac{1}{x}\right)^{2n}$.

Let x^p occur in the expansion of $\left(x^2+\frac{1}{x}\right)^{2n}$.

$$T_{r+1} = {}^{2n}C_r (x^2)^{2n-r}\left(\frac{1}{x}\right)^r$$

$$= {}^{2n}C_r x^{4n-2r} x^{-r} = {}^{2n}C_r x^{4n-3r}$$

Let $$4n - 3r = p$$

$$\Rightarrow \quad 3r = 4n - p \Rightarrow r = \frac{4n-p}{3}$$

$$\therefore \quad \text{Coefficient of } x^p = {}^{2n}C_r = \frac{(2n)!}{r!\,(2n-r)!} = \frac{(2n)!}{\left(\frac{4n-p}{3}\right)!\left(2n-\frac{4n-p}{3}\right)!}$$

$$= \frac{(2n)!}{\left(\frac{4n-p}{3}\right)!\left(\frac{6n-4n+p}{3}\right)!} = \frac{(2n)!}{\left(\frac{4n-p}{3}\right)!\left(\frac{2n+p}{3}\right)!}$$

Q. 17 Find the term independent of x in the expansion of

$$(1+x+2x^3)\left(\frac{3}{2}x^2-\frac{1}{3x}\right)^9.$$

Sol. Given expansion is $(1+x+2x^3)\left(\frac{3}{2}x^2-\frac{1}{3x}\right)^9$.

Now, consider $\left(\frac{3}{2}x^2-\frac{1}{3x}\right)^9$

$$T_{r+1} = {}^9C_r\left(\frac{3}{2}x^2\right)^{9-r}\left(-\frac{1}{3x}\right)^r$$

$$= {}^9C_r\left(\frac{3}{2}\right)^{9-r}x^{18-2r}\left(-\frac{1}{3}\right)^r x^{-r} = {}^9C_r\left(\frac{3}{2}\right)^{9-r}\left(-\frac{1}{3}\right)^r x^{18-3r}$$

Hence, the general term in the expansion of $(1+x+2x^3)\left(\frac{3}{2}x^2-\frac{1}{3x}\right)^9$

$$= {}^9C_r\left(\frac{3}{2}\right)^{9-r}\left(-\frac{1}{3}\right)^r x^{18-3r} + {}^9C_r\left(\frac{3}{2}\right)^{9-r}\left(-\frac{1}{3}\right)^r x^{19-3r} + 2\cdot{}^9C_r\left(\frac{3}{2}\right)^{9-r}\left(-\frac{1}{3}\right)^r x^{21-3r}$$

For term independent of x, putting $18-3r=0$, $19-3r=0$ and $21-3r=0$, we get

$$r=6,\ r=19/3,\ r=7$$

Since, the possible value of r are 6 and 7.
Hence, second term is not independent of x.

$\therefore$ The term independent of x is ${}^9C_6\frac{3^{9-6}}{2}\left(-\frac{1}{3}\right)^6 + 2\cdot{}^9C_7\frac{3^{9-7}}{2}\left(-\frac{1}{3}\right)^7$

$$= \frac{9\times8\times7\times6!}{6!\times3\times2}\cdot\frac{3^3}{2^3}\cdot\frac{1}{3^6} - 2\cdot\frac{9\times8\times7!}{7!\times2\times1}\cdot\frac{3^2}{2^2}\cdot\frac{1}{3^7}$$

$$= \frac{84}{8}\cdot\frac{1}{3^3} - \frac{36}{4}\cdot\frac{2}{3^5} = \frac{7}{18} - \frac{2}{27} = \frac{21-4}{54} = \frac{17}{54}$$

Objective Type Questions

Q. 18 The total number of terms in the expansion of $(x+a)^{100}+(x-a)^{100}$ after simplification is

(a) 50 (b) 202 (c) 51 (d) None of these

Sol. ***(c)*** Here, $(x+a)^{100}+(x-a)^{100}$

Total number of terms is 102 in the expansion of $(x+a)^{100}+(x-a)^{100}$

50 terms of $(x+a)^{100}$ cancel out 50 terms of $(x-a)^{100}$. 51 terms of $(x+a)^{100}$ get added to the 51 terms of $(x-a)^{100}$.

Alternate Method

$$(x+a)^{100}+(x-a)^{100} = {}^{100}C_0\,x^{100} + {}^{100}C_1\,x^{99}a + \dots + {}^{100}C_{100}\,a^{100}$$
$$+ {}^{100}C_0\,x^{100} - {}^{100}C_1\,x^{99}a + \dots + {}^{100}C_{100}\,a^{100}$$
$$= 2\,\underbrace{[{}^{100}C_0\,x^{100} + {}^{100}C_2\,x^{98}a^2 + \dots + {}^{100}C_{100}\,a^{100}]}_{51 \text{ terms}}$$

Q. 19 If the integers $r > 1$, $n > 2$ and coefficients of $(3r)$th and $(r+2)$nd terms in the Binomial expansion of $(1+x)^{2n}$ are equal, then

(a) $n = 2r$ (b) $n = 3r$

(c) $n = 2r + 1$ (d) None of these

Thinking Process

In the expansion of $(x+y)^n$, the coefficient of $(r+1)$th term is ${}^nC_r$.

Sol. *(a)* Given that, $r > 1$, $n > 2$ and the coefficients of $(3r)$th and $(r+2)$th term are equal in the expansion of $(1+x)^{2n}$.

Then, $T_{3r} = T_{3r-1+1} = {}^{2n}C_{3r-1}\, x^{3r-1}$

and $T_{r+2} = T_{r+1+1} = {}^{2n}C_{r+1}\, x^{r+1}$

Given, ${}^{2n}C_{3r-1} = {}^{2n}C_{r+1}$ $[\because {}^nC_x = {}^nC_y \Rightarrow x + y = n]$

$\Rightarrow 3r - 1 + r + 1 = 2n$

$\Rightarrow 4r = 2n \Rightarrow n = \dfrac{4r}{2}$

$\therefore n = 2r$

Q. 20 The two successive terms in the expansion of $(1+x)^{24}$ whose coefficients are in the ratio 1 : 4 are

(a) 3rd and 4th (b) 4th and 5th

(c) 5th and 6th (d) 6th and 7th

Sol. *(c)* Let two successive terms in the expansion of $(1+x)^{24}$ are $(r+1)$th and $(r+2)$th terms.

$\therefore T_{r+1} = {}^{24}C_r\, x^r$

and $T_{r+2} = {}^{24}C_{r+1}\, x^{r+1}$

Given that, $\dfrac{{}^{24}C_r}{{}^{24}C_{r+1}} = \dfrac{1}{4}$

$$\Rightarrow \frac{\dfrac{(24)!}{r!(24-r)!}}{\dfrac{(24)!}{(r+1)!(24-r-1)!}} = \frac{1}{4}$$

$$\Rightarrow \frac{(r+1)\, r!\, (23-r)!}{r!(24-r)\,(23-r)!} = \frac{1}{4}$$

$$\Rightarrow \frac{r+1}{24-r} = \frac{1}{4} \Rightarrow 4r + 4 = 24 - r$$

$\Rightarrow 5r = 20 \Rightarrow r = 4$

$\therefore T_{4+1} = T_5$ and $T_{4+2} = T_6$

Hence, 5th and 6th terms.

Q. 21 The coefficient of x^n in the expansion of $(1+x)^{2n}$ and $(1+x)^{2n-1}$ are in the ratio

(a) 1 : 2 (b) 1 : 3

(c) 3 : 1 (d) 2 : 1

Sol. *(d)* $\because$ Coefficient of x^n in the expansion of $(1+x)^{2n} = {}^{2n}C_n$

and coefficient of x^n in the expansion of $(1+x)^{2n-1} = {}^{2n-1}C_n$

$$\because \quad \frac{{}^{2n}C_n}{{}^{2n-1}C_n} = \frac{\frac{(2n)!}{n!n!}}{\frac{(2n-1)!}{n!(n-1)!}}$$

$$= \frac{(2n)!\,n!\,(n-1)!}{n!\,n!\,(2n-1)!}$$

$$= \frac{2n\,(2n-1)!\,n!\,(n-1)!}{n!\,n(n-1)!\,(2n-1)!}$$

$$= \frac{2n}{n} = \frac{2}{1} = 2:1$$

Q. 22 If the coefficients of 2nd, 3rd and the 4th terms in the expansion of $(1+x)^n$ are in AP, then the value of n is

(a) 2 (b) 7

(c) 11 (d) 14

Sol. *(b)* The expansion of $(1+x)^n$ is ${}^nC_0 + {}^nC_1x + {}^nC_2x^2 + {}^nC_3x^3 + \ldots + {}^nC_nx^n$

$\therefore$ Coefficient of 2nd term $= {}^nC_1$,

Coefficient of 3rd term $= {}^nC_2$,

and coefficient of 4th term $= {}^nC_3$.

Given that, ${}^nC_1$, ${}^nC_2$ and ${}^nC_3$ are in AP.

$$\therefore \quad 2\,{}^nC_2 = {}^nC_1 + {}^nC_3$$

$$\Rightarrow \quad 2\left[\frac{(n)!}{(n-2)!\,2!}\right] = \frac{(n)!}{(n-1)!} + \frac{(n)!}{3!(n-3)!}$$

$$\Rightarrow \quad \frac{2\cdot n\,(n-1)\,(n-2)!}{(n-2)!\,2!} = \frac{n\,(n-1)!}{(n-1)!} + \frac{n(n-1)\,(n-2)\,(n-3)!}{3\cdot 2\cdot 1\,(n-3)!}$$

$$\Rightarrow \quad n(n-1) = n + \frac{n(n-1)\,(n-2)}{6}$$

$$\Rightarrow \quad 6n - 6 = 6 + n^2 - 3n + 2$$

$$\Rightarrow \quad n^2 - 9n + 14 = 0$$

$$\Rightarrow \quad n^2 - 7n - 2n + 14 = 0$$

$$\Rightarrow \quad n(n-7) - 2\,(n-7) = 0$$

$$\Rightarrow \quad (n-7)\,(n-2) = 0$$

$$\therefore \quad n = 2 \text{ or } n = 7$$

Since, $n = 2$ is not possible.

$$\therefore \quad n = 7$$

Q. 23 If A and B are coefficient of x^n in the expansions of $(1+x)^{2n}$ and $(1+x)^{2n-1}$ respectively, then $\dfrac{A}{B}$ equals to

(a) 1 (b) 2

(c) $\dfrac{1}{2}$ (d) $\dfrac{1}{n}$

Sol. *(b)* Since, the coefficient of x^n in the expansion of $(1+x)^{2n}$ is ${}^{2n}C_n$.

$\therefore \quad A = {}^{2n}C_n$

Now, the coefficient of x^n in the expansion of $(1+x)^{2n-1}$ is ${}^{2n-1}C_n$.

$\therefore \quad B = {}^{2n-1}C_n$

Now, $\quad \dfrac{A}{B} = \dfrac{{}^{2n}C_n}{{}^{2n-1}C_n} = \dfrac{2}{1} = 2$

Same as solution No. 21.

Q. 24 If the middle term of $\left(\dfrac{1}{x} + x\sin x\right)^{10}$ is equal to $7\dfrac{7}{8}$, then the value of x is

(a) $2n\pi + \dfrac{\pi}{6}$ (b) $n\pi + \dfrac{\pi}{6}$

(c) $n\pi + (-1)^n \dfrac{\pi}{6}$ (d) $n\pi + (-1)^n \dfrac{\pi}{3}$

Sol. *(c)* Given expansion is $\left(\dfrac{1}{x} + x\sin x\right)^{10}$.

Since, $n = 10$ is even, so this expansion has only one middle term *i.e.,* 6th term.

$$\therefore \quad T_6 = T_{5+1} = {}^{10}C_5 \left(\frac{1}{x}\right)^{10-5} (x\sin x)^5$$

$$\Rightarrow \quad \frac{63}{8} = {}^{10}C_5\, x^{-5} x^5 \sin^5 x$$

$$\Rightarrow \quad \frac{63}{8} = \frac{10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5!}{5\cdot 4\cdot 3\cdot 2\cdot 1\cdot 5!}\sin^5 x$$

$$\Rightarrow \quad \frac{63}{8} = 2\cdot 9\cdot 2\cdot 7\cdot \sin^5 x$$

$$\Rightarrow \quad \sin^5 x = \frac{1}{32}$$

$$\Rightarrow \quad \sin^5 x = \left(\frac{1}{2}\right)^5$$

$$\Rightarrow \quad \sin x = \frac{1}{2}$$

$$\therefore \quad x = n\pi + (-1)^n \ \pi/6$$

Fillers

Q. 25 The largest coefficient in the expansion of $(1+x)^{30}$ is ....

Thinking Process

In the expansion of $(1+x)^n$, the largest coefficient is ${}^nC_{n/2}$ (when n is even).

Sol. Largest coefficient in the expansion of $(1+x)^{30} = {}^{30}C_{30/2} = {}^{30}C_{15}$

Q. 26 The number of terms in the expansion of $(x+y+z)^n$ ....

Sol. Given expansion is $(x+y+z)^n = [x+(y+z)]^n$.

$$[x+(y+z)]^n = {}^nC_0x^n + {}^nC_1x^{n-1}(y+z) + {}^nC_2x^{n-2}(y+z)^2 + \dots + {}^nC_n(y+z)^n$$

$\therefore$ Number of terms $= 1+2+3+\dots+n+(n+1)$

$$= \frac{(n+1)(n+2)}{2}$$

Q. 27 In the expansion of $\left(x^2 - \frac{1}{x^2}\right)^{16}$, the value of constant term is ....

Sol. Let constant be T_{r+1}.

$$\therefore \quad T_{r+1} = {}^{16}C_r(x^2)^{16-r}\left(-\frac{1}{x^2}\right)^r$$

$$= {}^{16}C_r\, x^{32-2r}(-1)^r x^{-2r}$$

$$= {}^{16}C_r\, x^{32-4r}(-1)^r$$

For constant term, $32 - 4r = 0 \Rightarrow r = 8$

$$\therefore \quad T_{8+1} = {}^{16}C_8$$

Q. 28 If the seventh term from the beginning and the end in the expansion of $\left(\sqrt[3]{2} + \frac{1}{\sqrt[3]{3}}\right)^n$ are equal, then n equals to ....

Sol. Given expansions is $\left(\sqrt[3]{2} + \frac{1}{\sqrt[3]{3}}\right)^n$.

$$\therefore \quad T_7 = T_{6+1} = {}^nC_6(\sqrt[3]{2})^{n-6}\left(\frac{1}{\sqrt[3]{3}}\right)^6 \quad \dots\text{(i)}$$

Since, T_7 from end is same as the T_7 from beginning of $\left(\frac{1}{\sqrt[3]{3}} + \sqrt[3]{2}\right)^n$.

Then,
$$T_7 = {}^nC_6\left(\frac{1}{\sqrt[3]{3}}\right)^{n-6}(\sqrt[3]{2})^6 \quad \dots\text{(ii)}$$

Given that,
$${}^nC_6(2)^{\frac{n-6}{3}}(3)^{-6/3} = {}^nC_6(3)^{-\frac{(n-6)}{3}}2^{6/3}$$

$$\Rightarrow \quad (2)^{\frac{n-12}{3}} = \left(\frac{1}{3^{1/3}}\right)^{n-12}$$

which is true, when $\frac{n-12}{3} = 0$.

$$\Rightarrow \quad n - 12 = 0 \Rightarrow n = 12$$

Q. 29 The coefficient of $a^{-6}b^4$ in the expansion of $\left(\frac{1}{a}-\frac{2b}{3}\right)^{10}$ is ....

Thinking Process

In the expansion of $(x-a)^n, T_{r+1} = {}^nC_r\, x^{n-r}(-a)^r$

Sol. Given expansion is $\left(\frac{1}{a}-\frac{2b}{3}\right)^{10}$.

Let T_{r+1} has the coefficient of $a^{-6}b^4$.

$$\therefore \quad T_{r+1} = {}^{10}C_r \left(\frac{1}{a}\right)^{10-r}\left(-\frac{2b}{3}\right)^r$$

For coefficient of $a^{-6}b^4$, $\quad 10-r=6 \Rightarrow r=4$

$$\text{Coefficient of } a^{-6}b^4 = {}^{10}C_4(-2/3)^4$$

$$\therefore \quad = \frac{10\cdot 9\cdot 8\cdot 7\cdot 6!}{6!\cdot 4\cdot 3\cdot 2\cdot 1}\cdot\frac{2^4}{3^4} = \frac{1120}{27}$$

Q. 30 Middle term in the expansion of $(a^3+ba)^{28}$ is ....

Sol. Given expansion is $(a^3+ba)^{28}$.

$\because \quad n = 28$ [even]

$\therefore$ Middle term $= \left(\frac{28}{2}+1\right)$th term $=$ 15th term

$$\begin{aligned}\therefore \quad T_{15} &= T_{14+1}\\ &= {}^{28}C_{14}(a^3)^{28-14}(ba)^{14}\\ &= {}^{28}C_{14}\, a^{42}b^{14}a^{14}\\ &= {}^{28}C_{14}\, a^{56}b^{14}\end{aligned}$$

Q. 31 The ratio of the coefficients of x^p and x^q in the expansion of $(1+x)^{p+q}$ is ....

Sol. Given expansion is $(1+x)^{p+q}$.

$\therefore$ Coefficient of $x^p = {}^{p+q}C_p$

and coefficient of $x^q = {}^{p+q}C_q$

$$\therefore \quad \frac{{}^{p+q}C_p}{{}^{p+q}C_q} = \frac{{}^{p+q}C_p}{{}^{p+q}C_p} = 1:1$$

Q. 32 The position of the term independent of x in the expansion of $\left(\sqrt{\frac{x}{3}}+\frac{3}{2x^2}\right)^{10}$ is ....

Sol. Given expansion is $\left(\sqrt{\frac{x}{3}}+\frac{3}{2x^2}\right)^{10}$.

Let the constant term be T_{r+1}.

Then, $T_{r+1} = {}^{10}C_r \left(\sqrt{\dfrac{x}{3}}\right)^{10-r} \left(\dfrac{3}{2x^2}\right)^r$

$= {}^{10}C_r \cdot x^{\frac{10-r}{2}} \cdot 3^{\frac{-10+r}{2}} \cdot 3^r \cdot 2^{-r} \cdot x^{-2r}$

$= {}^{10}C_r \; x^{\frac{10-5r}{2}} \; 3^{\frac{-10+3r}{2}} \; 2^{-r}$

For constant term, $10 - 5r = 0 \Rightarrow r = 2$

Hence, third term is independent of x.

Q. 33 If 25^{15} is divided by 13, then the remainder is ....

Sol. Let $25^{15} = (26-1)^{15}$

$= {}^{15}C_0 \, 26^{15} - {}^{15}C_1 26^{14} + \ldots - {}^{15}C_{15}$

$= {}^{15}C_0 26^{15} - {}^{15}C_1 26^{14} + \ldots - 1 - 13 + 13$

$= {}^{15}C_0 \, 26^{15} - {}^{15}C_1 \, 26^{14} + \ldots - 13 + 12$

It is clear that, when 25^{15} is divided by 13, then remainder will be 12.

True/False

Q. 34 The sum of the series $\sum_{r=0}^{10} {}^{20}C_r$ is $2^{19} + \dfrac{{}^{20}C_{10}}{2}$.

Sol. ***False***

Given series $= \sum_{r=0}^{10} {}^{20}C_r = {}^{20}C_0 + {}^{20}C_1 + {}^{20}C_2 + \ldots + {}^{20}C_{10}$

$= {}^{20}C_0 + {}^{20}C_1 + \ldots + {}^{20}C_{10} + {}^{20}C_{11} + \ldots \; {}^{20}C_{20} - ({}^{20}C_{11} + \ldots + {}^{20}C_{20})$

$= 2^{20} - ({}^{20}C_{11} + \ldots + {}^{20}C_{20})$

Hence, the given statement is false.

Q. 35 The expression $7^9 + 9^7$ is divisible by 64.

Sol. ***True***

Given expression $= 7^9 + 9^7 = (1+8)^7 - (1-8)^9$

$= ({}^7C_0 + {}^7C_1 8 + {}^7C_2 8^2 + \ldots + {}^7C_7 8^7) - ({}^9C_0 - {}^9C_1 8 + {}^9C_2 8^2 \ldots - {}^9C_9 8^9)$

$= (1 + 7 \times 8 + 21 \times 8^2 + \ldots) - (1 - 9 \times 8 + 36 \times 8^2 + \ldots - 8^9)$

$= (7 \times 8 + 9 \times 8) + (21 \times 8^2 - 36 \times 8^2) + \ldots$

$= 2 \times 64 + (21 - 36)64 + \ldots$

which is divisible by 64.

Hence, the statement is true.

Q. 36 The number of terms in the expansion of $[(2x + y^3)^4]^7$ is 8.

Sol. ***False***

Given expansion is $[(2x + y^3)^4]^7 = (2x + y^3)^{28}$.

Since, this expansion has 29 terms.

So, the given statement is false.

Q. 37 The sum of coefficients of the two middle terms in the expansion of $(1+x)^{2n-1}$ is equal to ${}^{2n-1}C_n$.

Sol. ***False***

Here, the Binomial expansion is $(1+x)^{2n-1}$.

Since, this expansion has two middle term *i.e.*, $\left(\frac{2n-1+1}{2}\right)$th term and $\left(\frac{2n-1+1}{2}+1\right)$th term *i.e.*, nth term and $(n+1)$th term.

$\therefore$ Coefficient of nth term $= {}^{2n-1}C_{n-1}$

Coefficient of $(n+1)$th term $= {}^{2n-1}C_n$

Sum of coefficients $= {}^{2n-1}C_{n-1} + {}^{2n-1}C_n$

$= {}^{2n-1+1}C_n = {}^{2n}C_n$ $\quad [\because {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r]$

Q. 38 The last two digits of the numbers 3^{400} are 01.

Sol. ***True***

Given that, $3^{400} = 9^{200} = (10-1)^{200}$

$\Rightarrow (10-1)^{200} = {}^{200}C_0 10^{200} - {}^{200}C_1 10^{199} + \ldots - {}^{200}C_{199} 10^1 + {}^{200}C_{200} 1^{200}$

$\Rightarrow (10-1)^{200} = 10^{200} - 200 \times 10^{199} + \ldots - 10 \times 200 + 1$

So, it is clear that the last two digits are 01.

Q. 39 If the expansion of $\left(x - \frac{1}{x^2}\right)^{2n}$ contains a term independent of x, then n is a multiple of 2.

Sol. ***False***

Given Binomial expansion is $\left(x - \frac{1}{x^2}\right)^{2n}$.

Let T_{r+1} term is independent of x.

Then, $T_{r+1} = {}^{2n}C_r (x)^{2n-r} \left(-\frac{1}{x^2}\right)^r$

$= {}^{2n}C_r \, x^{2n-r} (-1)^r x^{-2r} = {}^{2n}C_r \, x^{2n-3r} (-1)^r$

For independent of x,

$2n - 3r = 0$

$\therefore r = \frac{2n}{3}$,

which is not a integer.

So, the given expansion is not possible.

Q. 40 The number of terms in the expansion of $(a+b)^n$, where $n \in N$, is one less than the power n.

Sol. ***False***

We know that, the number of terms in the expansion of $(a+b)^n$, where $n \in N$, is one more than the power n.

9

Sequence and Series

Short Answer Type Questions

Q. 1 The first term of an AP is a and the sum of the first p terms is zero, show that the sum of its next q terms is $\dfrac{-a(p+q)q}{p-1}$.

Sol. Let the common difference of an AP is d.
According to the question,

$$S_p = 0$$

$$\Rightarrow \quad \frac{p}{2}[2a+(p-1)d] = 0 \qquad \left[\because S_n = \frac{n}{2}\{2a+(n-1)d\}\right]$$

$$\Rightarrow \quad 2a+(p-1)d = 0$$

$$\therefore \quad d = \frac{-2a}{p-1}$$

Now, sum of next q terms $= S_{p+q} - S_p = S_{p+q} - 0$

$$= \frac{p+q}{2}[2a+(p+q-1)d]$$

$$= \frac{p+q}{2}[2a+(p-1)d+qd]$$

$$= \frac{p+q}{2}\left[2a+(p-1)\cdot\frac{-2a}{p-1}+\frac{q(-2a)}{p-1}\right]$$

$$= \frac{p+q}{2}\left[2a+(-2a)-\frac{2aq}{p-1}\right]$$

$$= \frac{p+q}{2}\left[\frac{-2aq}{p-1}\right]$$

$$= \frac{-a(p+q)q}{(p-1)}$$

Q. 2 A man saved ₹ 66000 in 20 yr. In each succeeding year after the first year, he saved ₹ 200 more than what he saved in the previous year. How much did he save in the first year?

Sol. Let saved in first year ₹ a. Since, each succeeding year an increment ₹ 200 has made. So,it forms an AP whose

First term = a, common difference $(d) = 200$ and $n = 20$ yr

$\therefore \quad S_{20} = \frac{20}{2}[2a + (20 - 1)d] \quad \left[\because S_n = \frac{n}{2}\{2a + (n - 1)d\}\right]$

$\Rightarrow \quad 66000 = 10\,[2a + 19d]$

$\Rightarrow \quad 66000 = 20a + 190d$

$\Rightarrow \quad 66000 = 20a + 190 \times 200$

$\Rightarrow \quad 20a = 66000 - 38000$

$\Rightarrow \quad 20a = 28000$

$\therefore \quad a = \frac{28000}{20} = 1400$

Hence, he saved ₹ 1400 in the first year.

Q. 3 A man accepts a position with an initial salary of ₹ 5200 per month. It is understood that he will receive an automatic increase of ₹ 320 in the very next month and each month thereafter.

(i) Find his salary for the tenth month.

(ii) What is his total earnings during the first year?

Sol. Since, the man get a fixed increment of ₹ 320 each month.

Therefore, this forms an AP whose First term = 5200 and Common difference $(d) = 320$

(i) Salary for tenth month *i.e.*, for $n = 10$,

$a_{10} = a + (n - 1)d$

$\Rightarrow \quad a_{10} = 5200 + (10 - 1) \times 320$

$\Rightarrow \quad a_{10} = 5200 + 9 \times 320$

$\therefore \quad a_{10} = 5200 + 2880$

$\therefore \quad a_{10} = 8080$

(ii) Total earning during the first year.

In a year there are 12 month *i.e.*, $n = 12$,

$$S_{12} = \frac{12}{2}[2 \times 5200 + (12 - 1)320]$$

$$= 6\,[10400 + 11 \times 320]$$

$$= 6[10400 + 3520] = 6 \times 13920 = 83520$$

Q. 4 If the p th and qth terms of a GP are q and p respectively, then show that its $(p + q)$th term is $\left(\frac{q^p}{p^q}\right)^{\frac{1}{p-q}}$.

Sol. Let the first term and common ratio of GP be a and r, respectively.

According to the question, $\quad p$th term $= q$

$\Rightarrow \quad a \cdot r^{p-1} = q \quad \quad$ ...(i)

and $\quad q$ th term $= p$

$\Rightarrow \quad ar^{q-1} = p \quad \quad$ ...(ii)

On dividing Eq. (i) by Eq. (ii), we get

$$\frac{ar^{p-1}}{ar^{q-1}} = \frac{q}{p}$$

$$\Rightarrow \quad r^{p-1-q+1} = \frac{q}{p}$$

$$\Rightarrow \quad r^{p-q} = \frac{q}{p} \quad \Rightarrow \quad r = \left(\frac{q}{p}\right)^{\frac{1}{p-q}}$$

On substituting the value of r in Eq. (i), we get

$$a\left(\frac{q}{p}\right)^{\frac{p-1}{p-q}} = q \quad \Rightarrow \quad a = \frac{q}{\left(\frac{q}{p}\right)^{\frac{p-1}{p-q}}} = q \cdot \left(\frac{p}{q}\right)^{\frac{p-1}{p-q}}$$

$$\therefore \quad (p+q)\text{th term, } T_{p+q} = a \cdot r^{p+q-1} = q \cdot \left(\frac{p}{q}\right)^{\frac{p-1}{p-q}} \cdot (r)^{p+q-1}$$

$$= q \cdot \left(\frac{p}{q}\right)^{\frac{p-1}{p-q}} \left[\left(\frac{q}{p}\right)^{\frac{1}{p-q}}\right]^{p+q-1} = q \cdot \left(\frac{p}{q}\right)^{\frac{p-1}{p-q}} \left(\frac{q}{p}\right)^{\frac{p+q-1}{p-q}}$$

$$= q \cdot \left(\frac{p}{q}\right)^{\frac{p-1}{p-q}} \left(\frac{p}{q}\right)^{\frac{-(p+q-1)}{p-q}} = q \cdot \left(\frac{p}{q}\right)^{\frac{p-1}{p-q} - \frac{(p+q-1)}{p-q}}$$

$$= q \cdot \left(\frac{p}{q}\right)^{\frac{p-1-p-q+1}{p-q}} = q \cdot \left(\frac{p}{q}\right)^{\frac{-q}{p-q}}$$

$$a = q \cdot \left(\frac{p}{q}\right)^{\frac{p-1}{p-q}}$$

Now, $(p+q)$th term *i.e.*, $a_{p+q} = ar^{p+q-1}$

$$= q \cdot \left(\frac{p}{q}\right)^{\frac{p-1}{p-q}} \cdot \left(\frac{q}{p}\right)^{\frac{p+q-1}{p-q}}$$

$$= q \cdot \frac{q^{\frac{p+q-1-p+1}{p-q}}}{p^{\frac{p+q-1-p+1}{p-q}}} = q \cdot \left(\frac{q^{\frac{q}{p-q}}}{p^{\frac{q}{p-q}}}\right)$$

Q. 5 A carpenter was hired to build 192 window frames. The first day he made five frames and each day, thereafter he made two more frames than he made the day before. How many days did it take him to finish the job?

Sol. Here, $a = 5$ and $d = 2$

Let he finished the job in n days.

Then, $\quad S_n = 192$

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

$$\Rightarrow \quad 192 = \frac{n}{2}[2 \times 5 + (n-1)2]$$

$$\Rightarrow \quad 192 = \frac{n}{2}[10 + 2n - 2]$$

$$\Rightarrow \quad 192 = \frac{n}{2}[8 + 2n]$$
$$\Rightarrow \quad 192 = 4n + n^2$$
$$\Rightarrow \quad n^2 + 4n - 192 = 0$$
$$\Rightarrow \quad (n - 12)(n + 16) = 0$$
$$\Rightarrow \quad n = 12, -16 \qquad [\because n \neq -16]$$
$$\therefore \quad n = 12$$

Q. 6 **The sum of interior angles of a triangle is 180°. Show that the sum of the interior angles of polygons with 3, 4, 5, 6, ... sides form an arithmetic progression. Find the sum of the interior angles for a 21 sided polygon.**

Sol. We know that, sum of interior angles of a polygon of side $n = (2n - 4) \times 90° = (n - 2) \times 180°$

Sum of interior angles of a polygon with sides 3 is 180.

Sum of interior angles of polygon with side $4 = (4 - 2) \times 180° = 360°$

Similarly, sum of interior angles of polygon with side 5, 6, 7... are 540°, 720°, 900°,...

The series will be 180°, 360° 540°, 720°, 900°,...

Here, $a = 180°$

and $d = 360° - 180° = 180°$

Since, common difference is same between two consecutive terms of the series.

So, it form an AP.

We have to find the sum of interior angles of a 21 sides polygon.

It means, we have to find the 19th term of the above series.

$$\therefore \quad a_{19} = a + (19 - 1)d$$
$$= 180 + 18 \times 180 = 3420$$

Q. 7 **A side of an equilateral triangle is 20 cm long. A second equilateral triangle is inscribed in it by joining the mid-points of the sides of the first triangle. The process is continued as shown in the accompanying diagram. Find the perimeter of the sixth inscribed equilateral triangle.**

Sol. Side of equilateral $\Delta ABC = 20$ cm. By joining the mid-points of this triangle, we get another equilateral triangle of side equal to half of the length of side of ΔABC.

Continuing in this way, we get a set of equilateral triangles with side equal to half of the side of the previous triangle.

$\therefore$ Perimeter of first triangle $= 20 \times 3 = 60$ cm

Perimeter of second triangle $= 10 \times 3 = 30$ cm

Perimeter of third triangle $= 5 \times 3 = 15$ cm

Now, the series will be 60, 30, 15,...

Here, $a = 60$

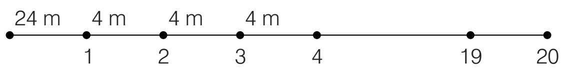
$$\therefore \quad r = \frac{30}{60} = \frac{1}{2} \qquad \left[\because \frac{\text{second term}}{\text{first term}} = r\right]$$

We have, to find perimeter of sixth inscribed triangle. It is the sixth term of the series.

$$\therefore \quad a_6 = ar^{6-1} \qquad [\because a_n = ar^{n-1}]$$
$$= 60 \times \left(\frac{1}{2}\right)^5 = \frac{60}{32} = \frac{15}{8} \text{ cm}$$

Q. 8 In a potato race 20 potatoes are placed in a line at intervals of 4 m with the first potato 24 m from the starting point. A contestant is required to bring the potatoes back to the starting place one at a time. How far would he run in bringing back all the potatoes?

Sol. According to the given information, we have following diagram.



Distance travelled to bring first potato $= 24 + 24 = 2 \times 24 = 48$ m

Distance travelled to bring second potato $= 2(24 + 4) = 2 \times 28 = 56$ m

Distance travelled to bring third potato $= 2(24 + 4 + 4) = 2 \times 32 = 64$ m

Then, the series of distances are 48, 56, 64,...

Here, $a = 48,$

$d = 56 - 48 = 8$

and $n = 20$

To find the total distance that he run in bringing back all potatoes, we have to find the sum of 20 terms of the above series.

$$\therefore \quad S_{20} = \frac{20}{2}[2 \times 48 + 19 \times 8] \qquad \left[\because S_n = \frac{n}{2}\{2a + (n-1)d\}\right]$$

$$= 10[96 + 152]$$

$$= 10 \times 248 = 2480 \text{ m}$$

Q. 9 In a cricket tournament 16 school teams participated. A sum of ₹ 8000 is to be awarded among themselves as prize money. If the last placed team is awarded ₹ 275 in prize money and the award increases by the same amount for successive finishing places, how much amount will the first place team receive?

Sol. Let the first place team got ₹ a.

Since, award money increases by the same amount for successive finishing places. Therefore series is an AP.

Let the constant amount be d.

Here, $l = 275, n = 16$ and $S_{16} = 8000$

$\therefore \quad l = a + (n -)d$

$\Rightarrow \quad l = a + (16 - 1)(-d)$

[we take common difference (–ve) because series is decreasing]

$\Rightarrow \quad 275 = a - 15d \qquad$ ...(i)

and $\quad S_{16} = \frac{16}{2}[2a + (n-1)\cdot(-d)]$

$\Rightarrow \quad 8000 = 8[2a + (16 - 1)(-d)]$

$\Rightarrow \quad 8000 = 8[2a - 15d]$

$\Rightarrow \quad 1000 = 2a - 15d \qquad$ ...(ii)

On subtracting Eq. (i) from Eq. (ii), we get

$$(2a - 15d) - (a - 15d) = 1000 - 275$$

$\Rightarrow \quad 2a - 15d - a + 15d = 725$

$\therefore \quad a = 725$

Hence, first place team receive ₹ 725.

Q. 10 If $a_1, a_2, a_3, \ldots, a_n$ are in AP, where $a_i > 0$ for all i, show that

$$\frac{1}{\sqrt{a_1}+\sqrt{a_2}}+\frac{1}{\sqrt{a_2}+\sqrt{a_3}}+\ldots+\frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}}=\frac{n-1}{\sqrt{a_1}+\sqrt{a_n}}$$

Sol. Since, $a_1, a_2, a_3, \ldots, a_n$ are in AP.

$\Rightarrow$ $a_2 - a_1 = a_3 - a_2 = \ldots = a_n - a_{n-1} = d$ [common difference]

If $a_2 - a_1 = d$, then $(\sqrt{a_2})^2 - (\sqrt{a_1})^2 = d$

$\Rightarrow$ $(\sqrt{a_2}-\sqrt{a_1})(\sqrt{a_2}+\sqrt{a_1}) = d$

$\Rightarrow$ $$\frac{1}{\sqrt{a_1}+\sqrt{a_2}}=\frac{\sqrt{a_2}-\sqrt{a_1}}{d}$$

Similarly, $$\frac{1}{\sqrt{a_2}+\sqrt{a_3}}=\frac{\sqrt{a_3}-\sqrt{a_2}}{d}$$

... ...

... ...

... ...

$$\frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}}=\frac{\sqrt{a_n}-\sqrt{a_{n-1}}}{d}$$

On adding these terms, we get

$$\frac{1}{\sqrt{a_1}+\sqrt{a_2}}+\frac{1}{\sqrt{a_2}+\sqrt{a_3}}+\ldots+\frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}}$$

$$=\frac{1}{d}[\sqrt{a_2}-\sqrt{a_1}+\sqrt{a_3}-\sqrt{a_2}+\ldots+\sqrt{a_n}-\sqrt{a_{n-1}}]$$ [using above relations]

$$=\frac{1}{d}[\sqrt{a_n}-\sqrt{a_1}]$$ ...(i)

Again, $a_n = a_1 + (n-1)d$ [$\because T_n = a + (n-1)d$]

$\Rightarrow$ $a_n - a_1 = (n-1)d$

$\Rightarrow$ $(\sqrt{a_n})^2 - (\sqrt{a_1})^2 = (n-1)d$

$\Rightarrow$ $(\sqrt{a_n}-\sqrt{a_1})(\sqrt{a_n}+\sqrt{a_1}) = (n-1)d$ $\Rightarrow$ $\sqrt{a_n}-\sqrt{a_1} = \dfrac{(n-1)d}{\sqrt{a_n}+\sqrt{a_1}}$

On putting this value in Eq. (i), we get

$$\frac{1}{\sqrt{a_1}+\sqrt{a_2}}+\frac{1}{\sqrt{a_2}+\sqrt{a_3}}+\ldots+\frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}}$$

$$=\frac{(n-1)d}{d(\sqrt{a_n}+\sqrt{a_1})}=\frac{n-1}{\sqrt{a_n}+\sqrt{a_1}}$$ **Hence proved.**

Q. 11 Find the sum of the series

$(3^3-2^3)+(5^3-4^3)+(7^3-6^3)+\ldots$ to (i) n terms. (ii) 10 terms.

Sol. Given series, $(3^3-2^3)+(5^3-4^3)+(7^3-6^3)+$ ...(i)

$= (3^3+5^3+7^3+\ldots)-(2^3+4^3+6^3+\ldots)$

Let T_n be the nth term of the series (i),

then T_n = (nth term of $3^3, 5^3, 7^3, \ldots$) − (nth term of $2^3, 4^3, 6^3, \ldots$) $= (2n+1)^3-(2n)^3$

$= (2n+1-2n)[(2n+1)^2+(2n+1)2n+(2n)^2]$ [$\because a^3-b^3=(a-b)(a^2+ab+b^2)$]

$= [4n^2+1+4n+4n^2+2n+4n^2] = [12n^2+6n]+1$

(i) Let S_n denote the sum of n term of series (i).

Then,
$$S_n = \Sigma T_n = \Sigma(12n^2 + 6n)$$
$$= 12\Sigma n^2 + 6\Sigma n + \Sigma n$$
$$= 12 \cdot \frac{n(n+1)(2n+1)}{6} + \frac{6n(n+1)}{2} + n$$
$$= 2n(n+1)(2n+1) + 3n(n+1) + n$$
$$= 2n(n+1)(2n+1) + 3n(n+1) + n$$
$$= (2n^2 + 2n)(2n+1) + 3n^2 + 3n + n$$
$$= 4n^3 + 2n^2 + 4n^2 + 2n + 3n^2 + 3n + n$$
$$= 4n^3 + 9n^2 + 6n$$

(ii) Sum of 10 terms,
$$S_{10} = 4 \times (10)^3 + 9 \times (10)^2 + 6 \times 10$$
$$= 4 \times 1000 + 9 \times 100 + 60$$
$$= 4000 + 900 + 60 = 4960$$

Q. 12 Find the rth term of an AP sum of whose first n terms is $2n + 3n^2$.

Sol. Given that, sum of n terms of an AP,
$$S_n = 2n + 3n^2$$
$$T_n = S_n - S_{n-1}$$
$$= (2n + 3n^2) - [2(n-1) + 3(n-1)^2]$$
$$= (2n + 3n^2) - [2n - 2 + 3(n^2 + 1 - 2n)]$$
$$= (2n + 3n^2) - (2n - 2 + 3n^2 + 3 - 6n)$$
$$= 2n + 3n^2 - 2n + 2 - 3n^2 - 3 + 6n$$
$$= 6n - 1$$
$\therefore$ rth term $T_r = 6r - 1$

Long Answer Type Questions

Q. 13 If A is the arithmetic mean and G_1, G_2 be two geometric mean between any two numbers, then prove that $2A = \frac{G_1^2}{G_2} + \frac{G_2^2}{G_1}$.

Sol. Let the numbers be a and b.

Then,
$$A = \frac{a+b}{2}$$
$\Rightarrow$ $2A = a + b$ ... (i)

and G_1, G_2 be geometric mean between a and b, then a, G_1, G_2, b are in GP.

Let r be the common ratio.

Then, $b = ar^{4-1}$ $[\because a_n = ar^{n-1}]$

$\Rightarrow$ $b = ar^3 \Rightarrow \frac{b}{a} = r^3$

$\therefore$ $r = \left(\frac{b}{a}\right)^{1/3}$

Now, $G_1 = ar = a\left(\frac{b}{a}\right)^{1/3}$ $\left[\because r = \left(\frac{b}{a}\right)^{1/3}\right]$

and $G_2 = ar^2 = a\left(\frac{b}{a}\right)^{2/3}$

$$\text{RHS} = \frac{G_1^2}{G_2} + \frac{G_2^2}{G_1} = \frac{\left[a\left(\frac{b}{a}\right)^{1/3}\right]^2}{a\left(\frac{b}{a}\right)^{2/3}} + \frac{\left[a\left(\frac{b}{a}\right)^{2/3}\right]^2}{a\left(\frac{b}{a}\right)^{1/3}}$$

$$= \frac{a^2\left(\frac{b}{a}\right)^{2/3}}{a\left(\frac{b}{a}\right)^{2/3}} + \frac{a^2\left(\frac{b}{a}\right)^{4/3}}{a\left(\frac{b}{a}\right)^{1/3}}$$

$$= a + a\left(\frac{b}{a}\right) = a + b = 2A \quad \text{[using Eq. (i)]}$$

$= \text{LHS}$

Q. 14 If $\theta_1, \theta_2, \theta_3, \ldots, \theta_n$ are in AP whose common difference is d, show that

$$\sec\theta_1 \sec\theta_2 + \sec\theta_2 \sec\theta_3 + \ldots + \sec\theta_{n-1} \sec\theta_n = \frac{\tan\theta_n - \tan\theta_1}{\sin d}.$$

Sol. Since, $\theta_1, \theta_2, \theta_3, \ldots, \theta_n$ are in AP.

$\Rightarrow$ $\theta_2 - \theta_1 = \theta_3 - \theta_2 = \cdots = \theta_n - \theta_{n-1} = d$ ...(i)

Now, we have to prove

$$\sec q_1 \sec q_2 + \sec q_2 \sec q_3 + \cdots + \sec q_{n-1} \sec\theta_n = \frac{\tan\theta_n - \tan\theta_1}{\sin d}$$

or it can be written as

$$\sin d\,[\sec\theta_1 \sec\theta_2 + \sec\theta_2 \sec\theta_3 + \cdots + \sec\theta_{n-1} \sec\theta_n] = \tan\theta_n - \tan\theta_1$$

Now, taking only first term of LHS

$$\sin d \sec\theta_1 \sec\theta_2 = \frac{\sin d}{\cos\theta_1 \cos\theta_2} = \frac{\sin(\theta_2 - \theta_1)}{\cos\theta_1 \cos\theta_2} \quad \text{[from Eq. (i)]}$$

$$= \frac{\sin\theta_2 \cos\theta_1 - \cos\theta_2 \sin\theta_1}{\cos\theta_1 \cos\theta_2}$$

$[\because \sin(A - B) = \sin A \cdot \cos B - \cos A \cdot \sin B]$

$$= \frac{\sin\theta_2 \cos\theta_1}{\cos\theta_1 \cos\theta_2} - \frac{\cos\theta_2 \sin\theta_1}{\cos\theta_1 \cos\theta_2} = \tan\theta_2 - \tan\theta_1$$

Similarly, we can solve other terms which will be $\tan\theta_3 - \tan\theta_2, \tan\theta_4 - \tan\theta_3, \cdots$

$\therefore$ $\text{LHS} = \tan\theta_2 - \tan\theta_1 + \tan\theta_3 - \tan\theta_2 + \cdots + \tan\theta_n - \tan\theta_{n-1}$

$= -\tan\theta_1 + \tan\theta_n = \tan\theta_n - \tan\theta_1$

$= \text{RHS}$ **Hence proved.**

Q. 15 If the sum of p terms of an AP is q and the sum of q terms is p, then show that the sum of $p+q$ terms is $-(p+q)$. Also, find the sum of first $p-q$ terms (where, $p>q$).

Sol. Let first term and common difference of the AP be a and d, respectively.

Then, $$S_p = q$$

$$\Rightarrow \quad \frac{p}{2}[2a+(p-1)d] = q$$

$$2a+(p-1)d = \frac{2q}{p} \quad \text{...(i)}$$

and $$S_q = p$$

$$\Rightarrow \quad \frac{q}{2}[2a+(q-1)d] = p$$

$$\Rightarrow \quad 2a+(q-1)d = \frac{2p}{q} \quad \text{...(ii)}$$

On subtracting Eq. (ii) from Eq. (i), we get

$$2a+(p-1)d-2a-(q-1)d = \frac{2q}{p}-\frac{2p}{q}$$

$$\Rightarrow \quad [(p-1)-(q-1)]\,d = \frac{2q^2-2p^2}{pq}$$

$$\Rightarrow \quad [p-1-q+1]\,d = \frac{2(q^2-p^2)}{pq}$$

$$\Rightarrow \quad (p-q)d = \frac{2(q^2-p^2)}{pq}$$

$$\therefore \quad d = \frac{-2(p+q)}{pq} \quad \text{...(iii)}$$

On substituting the value of d in Eq. (i), we get

$$2a+(p-1)\left(\frac{-2(p+q)}{pq}\right) = \frac{2q}{p}$$

$$\Rightarrow \quad 2a = \frac{2q}{p}+\frac{2(p+q)(p-1)}{pq}$$

$$\Rightarrow \quad a = \left[\frac{q}{p}+\frac{(p+q)(p-1)}{pq}\right] \quad \text{...(iv)}$$

Now, $$S_{p+q} = \frac{p+q}{2}[2a+(p+q-1)d]$$

$$= \frac{p+q}{2}\left[\frac{2q}{p}+\frac{2(p+q)(p-1)}{pq}-\frac{(p+q-1)2(p+q)}{pq}\right]$$

$$= (p+q)\left[\frac{q}{p}+\frac{(p+q)(p-1)-(p+q-1)(p+q)}{pq}\right]$$

$$= (p+q)\left[\frac{q}{p}+\frac{(p+q)(p-1-p-q+1)}{pq}\right]$$

$$= p+q\left[\frac{q}{p}-\frac{p+q}{p}\right] = (p+q)\left[\frac{q-p-q}{p}\right]$$

$$S_{p+q} = -(p+q)$$

$$S_{p-q} = \frac{p-q}{2}[2a+(p-q-1)d]$$

$$= \frac{p-q}{2}\left[\frac{2q}{p} + \frac{2(p+q)(p-1)}{pq} - \frac{(p-q-1)2\,(p+q)}{pq}\right]$$

$$= (p-q)\left[\frac{q}{p} + \frac{p+q(p-1-p+q+1)}{pq}\right]$$

$$= (p-q)\left[\frac{q}{p} + \frac{(p+q)q}{pq}\right]$$

$$= (p-q)\left[\frac{q}{p} + \frac{p+q}{p}\right] = (p-q)\frac{(p+2q)}{p}$$

Q. 16 If pth, qth and rth terms of an AP and GP are both and c respectively, then show that $a^{b-c}\cdot b^{c-a}\cdot c^{a-b} = 1$.

Sol. Let A, d are the first term and common difference of AP and x, R are the first term and common ratio of GP, respectively.
According to the given condition,

$$A + (p-1)d = a \quad \text{...(i)}$$
$$A + (q-1)d = b \quad \text{...(ii)}$$
$$A + (r-1)d = c \quad \text{...(iii)}$$

and

$$a = xR^{p-1} \quad \text{...(iv)}$$
$$b = xR^{q-1} \quad \text{...(v)}$$
$$c = xR^{r-1} \quad \text{...(vi)}$$

On subtracting Eq. (ii) from Eq. (i), we get

$$d(p-1-q+1) = a-b$$

$$\Rightarrow \quad a-b = d(p-q) \quad \text{...(vii)}$$

On subtracting Eq. (iii) from Eq. (ii), we get

$$d(q-1-r+1) = b-c$$

$$\Rightarrow \quad b-c = d\,(q-r) \quad \text{...(viii)}$$

On subtracting Eq. (i) from Eq. (iii), we get

$$d(r-1-p+1) = c-a$$

$$\Rightarrow \quad c-a = d\,(r-p) \quad \text{...(ix)}$$

Now, we have to prove $a^{b-c}\ b^{c-a}\ c^{a-b} = 1$

$$\text{Taking LHS} = a^{b-c}\ b^{c-a}\ c^{a-b}$$

Using Eqs. (iv), (v), (vi) and (vii), (viii), (ix),

$$\text{LHS} = (xR^{p-1})^{d\,(q-r)}\,(xR^{q-1})^{d\,(r-p)}\,(xR^{r-1})^{d\,(p-q)}$$

$$= x^{d(q-r)+d(r-p)+d(p-q)}\ R^{(p-1)d(q-r)+(q-1)d\,(r-p)+(r-1)d(p-q)}$$

$$= x^{d(q-r+r-p+p-q)}$$

$$R^{d\,(pq-pr-q+r+qr-pq-r+p+rp-rq-p+q)} = x^0 R^0 = 1$$

$$= \text{RHS}$$

Hence proved.

Objective Type Questions

Q. 17 If the sum of n terms of an AP is given by $S_n = 3n + 2n^2$, then the common difference of the AP is

(a) 3 (b) 2 (c) 6 (d) 4

Sol. *(d)* Given, $S_n = 3n + 2n^2$

First term of the AP,

$\therefore \quad T_1 = 3 \times 1 + 2\,(1)^2 = 3 + 2 = 5$

and $\quad T_2 = S_2 - S_1$

$= [3 \times 2 + 2 \times (2)^2] - [3 \times 1 + 2 \times (1)^2]$

$= 14 - 5 = 9$

$\therefore$ Common difference $(d) = T_2 - T_1 = 9 - 5 = 4$

Q. 18 If the third term of GP is 4, then the product of its first 5 terms is

(a) 4^3 (b) 4^4 (c) 4^5 (d) None of these

Sol. *(c)* It is given that, $T_3 = 4$

Let a and r the first term and common ratio, respectively.

Then, $\quad ar^2 = 4 \quad$ ...(i)

Product of first 5 terms $= a \cdot ar \cdot ar^2 \cdot ar^3 \cdot ar^4$

$= a^5\, r^{10} = (ar^2)^5 = (4)^5 \quad$ [using Eq. (i)]

Q. 19 If 9 times the 9th term of an AP is equal to 13 times the 13th term, then the 22nd term of the AP is

(a) 0 (b) 22 (c) 198 (d) 220

Sol. *(a)* Let the first term be a and common difference be d.

According to the question, $\quad 9 \cdot T_9 = 13 \cdot T_{13}$

$\Rightarrow \quad 9\,(a + 8d) = 13\,(a + 12d)$

$\Rightarrow \quad 9a + 72d = 13a + 156d$

$\Rightarrow \quad (9a - 13a) = 156d - 72d$

$\Rightarrow \quad -4a = 84d$

$\Rightarrow \quad a = -21d$

$\Rightarrow \quad a + 21d = 0 \quad$ ...(i)

$\therefore$ 22nd term *i.e.*, $T_{22} = [a + 21d]$

$T_{22} = 0 \quad$ [using Eq. (i)]

Q. 20 If x, $2y$ and $3z$ are in AP where the distinct numbers x, y and z are in GP, then the common ratio of the GP is

(a) 3 (b) $\frac{1}{3}$ (c) 2 (d) $\frac{1}{2}$

Sol. *(b)* Given, $x, 2y$ and $3z$ are in AP.

Then, $\quad 2y = \dfrac{x + 3z}{2}$

$$\Rightarrow \quad y = \frac{x + 3z}{4}$$

$$\Rightarrow \quad 4y = x + 3z \quad \text{...(i)}$$

and x, y, z are in GP.

Then, $$\frac{y}{x} = \frac{z}{y} = \lambda$$

$$\Rightarrow \quad y = x\lambda \text{ and } z = \lambda y = \lambda^2 x$$

On substituting these values in Eq. (i), we get

$$4(x\lambda) = x + 3(\lambda^2 x)$$

$$\Rightarrow \quad 4\lambda x = x + 3\lambda^2 x$$

$$\Rightarrow \quad 4\lambda = 1 + 3\lambda^2$$

$$\Rightarrow \quad 3\lambda^2 - 4\lambda + 1 = 0$$

$$\Rightarrow \quad (3\lambda - 1)(\lambda - 1) = 0$$

$$\therefore \quad \lambda = \frac{1}{3}, \lambda = 1$$

Q. 21 If in an AP, $S_n = q n^2$ and $S_m = qm^2$, where S_r denotes the sum of r terms of the AP, then S_q equals to

(a) $\frac{q^3}{2}$ (b) mnq (c) q^3 (d) $(m + n) q^2$

Sol. ***(c)*** Given, $S_n = qn^2$ and $S_m = qm^2$

$\therefore \quad S_1 = q, S_2 = 4q, S_3 = 9q$ and $S_4 = 16q$

Now, $$T_1 = q$$

$$\therefore \quad T_2 = S_2 - S_1 = 4q - q = 3q$$

$$T_3 = S_3 - S_2 = 9q - 4q = 5q$$

$$T_4 = S_4 - S_3 = 16q - 9q = 7q$$

So, the series is $q, 3q, 5q, 7q, \ldots$

Here, $$a = q \text{ and } d = 3q - q = 2q$$

$$\therefore \quad S_q = \frac{q}{2}[2 \times q + (q - 1)2q]$$

$$= \frac{q}{2} \times [2q + 2q^2 - 2q] = \frac{q}{2} \times 2q^2 = q^3$$

Q. 22 Let S_n denote the sum of the first n terms of an AP, if $S_{2n} = 3S_n$, then $S_{3n} : S_n$ is equal to

(a) 4 (b) 6 (c) 8 (d) 10

Sol. ***(b)*** Let first term be a and common difference be d.

Then, $$S_n = \frac{n}{2}[2a + (n - 1)d] \quad \text{...(i)}$$

$$\therefore \quad S_{2n} = \frac{2n}{2}[2a + (2n - 1)d]$$

$$S_{2n} = n[2a + (2n - 1)d] \quad \text{...(ii)}$$

$$S_{3n} = \frac{3n}{2}[2a + (3n - 1)d] \quad \text{...(iii)}$$

According to the question, $S_{2n} = 3S_n$

$\Rightarrow \quad n[2a + (2n-1)d] = 3\dfrac{n}{2}[2a + (n-1)d]$

$\Rightarrow \quad 4a + (4n-2)d = 6a + (3n-3)d$

$\Rightarrow \quad -2a + (4n - 2 - 3n + 3)d = 0$

$\Rightarrow \quad -2a + (n+1)d = 0$

$\Rightarrow \quad d = \dfrac{2a}{n+1}$ ...(iv)

Now,
$$\frac{S_{3n}}{S_n} = \frac{\frac{3n}{2}[2a + (3n-1)d]}{\frac{n}{2}[2a + (n-1)d]} = \frac{6a + (9n-3)\frac{2a}{n+1}}{2a + (n-1)\frac{2a}{n+1}}$$

$$= \frac{6an + 6a + 18an - 6a}{2an + 2a + 2an - 2a}$$

$$= \frac{24an}{4an} = \frac{S_{3n}}{S_n} = 6$$

Q. 23 The minimum value of $4^x + 4^{1-x}$, $x \in R$ is

(a) 2 (b) 4 (c) 1 (d) 0

Sol. ***(b)*** We know that, AM ≥ GM

$\Rightarrow \quad \dfrac{4^x + 4^{1-x}}{2} \ge \sqrt{4^x \cdot 4^{1-x}}$

$\Rightarrow \quad 4^x + 4^{1-x} \ge 2\sqrt{4}$

$\Rightarrow \quad 4^x + 4^{1-x} \ge 2 \cdot 2$

$\Rightarrow \quad 4^x + 4^{1-x} \ge 4$

Q. 24 Let S_n denote the sum of the cubes of the first n natural numbers and s_n denote the sum of the first n natural numbers, then $\sum\limits_{r=1}^{n} \dfrac{S_r}{S_4}$ equals to

(a) $\dfrac{n(n+1)(n+2)}{6}$ (b) $\dfrac{n(n+1)}{2}$

(c) $\dfrac{n^2 + 3n + 2}{2}$ (d) None of these

Sol. ***(a)***
$$\sum_{r=1}^{n} \frac{S_r}{S_r} = \frac{S_1}{s_1} + \frac{S_2}{s_2} + \frac{S_3}{s_3} + \dots + \frac{S_n}{s_n}$$

Let T_n be the nth term of the above series.

$$\therefore \quad T_n = \frac{S_n}{s_n} = \frac{\left[\frac{n(n+1)}{2}\right]^2}{\frac{n(n+1)}{2}}$$

$$= \frac{n(n+1)}{2} = \frac{1}{2}[n^2 + n]$$

$\therefore$ Sum of the above series $= \Sigma T_n = \dfrac{1}{2}[\Sigma n^2 + \Sigma n]$

$$= \frac{1}{2}\left[\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right] = \frac{1}{2}\cdot\frac{n(n+1)}{2}\left[\frac{(2n+1)}{3} + 1\right]$$

$$= \frac{1}{4}n(n+1)\left[\frac{2n+1+3}{3}\right] = \frac{1}{4\times 3}n(n+1)(2n+4)$$

$$= \frac{1}{12}n(n+1)(2n+4) = \frac{1}{6}n(n+1)(n+2)$$

Q. 25 If t_n denotes the nth term of the series $2+3+6+11+18+\dots$, then t_{50} is

(a) $49^2 - 1$ (b) 49^2 (c) $50^2 + 1$ (d) $49^2 + 2$

Sol. ***(d)*** Let S_n be sum of the series $2+3+6+11+18+\dots+t_{50}$.

$\therefore$ $S_n = 2+3+6+11+18+\dots+t_{50}$ ...(i)

and $S_n = 0+2+3+6+11+18+\dots+t_{49}+t_{50}$ ...(ii)

On subtracting Eq. (ii) from Eq. (i), we get

$$0 = 2+1+3+5+7+\cdots+t_{50}$$

$\Rightarrow$ $t_{50} = 2+1+3+5+7+\cdots$ upto 49 terms

$\therefore$ $t_{50} = 2 + [1+3+5+7+\cdots$ upto 49 terms$]$

$$= 2 + \frac{49}{2}[2\times 1 + 48\times 2]$$

$$= 2 + \frac{49}{2}\times[2+96]$$

$$= 2 + [49 + 49\times 48]$$

$$= 2 + 49\times 49 = 2 + (49)^2$$

Q. 26 The lengths of three unequal edges of a rectangular solid block are in GP. If the volume of the block is 216 cm^3 and the total surface area is 252 cm^2, then the length of the longest edge is

(a) 12 cm (b) 6 cm (c) 18 cm (d) 3 cm

Sol. ***(a)*** Let the length, breadth and height of rectangular solid block is $\dfrac{a}{r}$, a and ar, respectively.

$\therefore$ Volume $= \dfrac{a}{r}\times a\times ar = 216\ cm^3$

$\Rightarrow$ $a^3 = 216 \Rightarrow a^3 = 6^3$

$\therefore$ $a = 6$

Surface area $= 2\left(\dfrac{a^2}{r} + a^2 r + a^2\right) = 252$

$\Rightarrow$ $2a^2\left(\dfrac{1}{r} + r + 1\right) = 252$

$\Rightarrow$ $2\times 36\left(\dfrac{1+r^2+r}{r}\right) = 252$

$\Rightarrow$ $\dfrac{1+r^2+r}{r} = \dfrac{252}{2\times 36}$

$\Rightarrow \quad 1 + r^2 + r = \frac{126}{36} r \Rightarrow 1 + r^2 + r = \frac{21}{6} r$

$\Rightarrow \quad 6 + 6r^2 + 6r = 21r \Rightarrow 6r^2 - 15r + 6 = 0$

$\Rightarrow \quad 2r^2 - 5r + 2 = 0 \Rightarrow (2r - 1)(r - 2) = 0$

$\therefore \quad r = \frac{1}{2}, 2$

For $r = \frac{1}{2}$: Length $= \frac{a}{r} = \frac{6 \times 2}{1} = 12$

Breadth $= a = 6$

Height $= ar = 6 \times \frac{1}{2} = 3$

For $r = 2$: Length $= \frac{a}{r} = \frac{6}{2} = 3$

Breadth $= a = 6$

Height $= ar = 6 \times 2 = 12$

Fillers

Q. 27 If a, b and c are in GP, then the value of $\frac{a-b}{b-c}$ is equal to ....

Sol. Given that, a, b and c are in GP.

Then, $\frac{b}{a} = \frac{c}{b} = r$ [constant]

$\Rightarrow \quad b = ar \Rightarrow c = br$

$\therefore \quad \frac{a-b}{b-c} = \frac{a-ar}{ar-br} = \frac{a(1-r)}{r(a-b)} = \frac{a(1-r)}{r(a-ar)}$

$= \frac{a(1-r)}{ar(1-r)} = \frac{1}{r}$

$\therefore \quad \frac{a-b}{b-c} = \frac{1}{r} = \frac{a}{b}$ or $\frac{b}{c}$

Q. 28 The sum of terms equidistant from the beginning and end in an AP is equal to ....

Sol. Let AP be $a, a + d, a + 2d \cdots a + (n-1)d$

$\therefore \quad a_1 + a_n = a + a + (n-1)d$

$= 2a + (n-1)d$ ...(i)

Now, $a_2 + a_{n-1} = (a + d) + [a + (n-2)d]$

$= 2a + (n-1)d$

$a_2 + a_{n-1} = a_1 + a_n$ [using Eq. (i)]

$a_3 + a_{n-2} = (a + 2d) + [a + (n-3)d]$

$= 2a + (n-1)d$

$= a_1 + a_n$ [using Eq. (i)]

Follow this pattern, we see that the sum of terms equidistant from the beginning and end in an AP is equal to [first term + last term].

Q. 29 The third term of a GP is 4, the product of the first five terms is .....

Sol. It is given that, $T_3 = 4$

Let a and r the first term and common ration, respectively.

Then, $ar^2 = 4$ ...(i)

Product of first 5 terms $= ar \cdot ar \cdot ar^2 \cdot ar^3 \cdot ar^4$

$= a^5 r^{10} = (ar^2)^5 = (4)^5$ [using Eq. (i)]

True/False

Q. 30 Two sequences cannot be in both AP and GP together.

Sol. ***False***

Consider an AP $a, a+d, a+2d, \ldots$

Now, $$\frac{a_2}{a_1} = \frac{a+d}{a} \neq \frac{a+2d}{a+d}$$

Thus, AP is not a GP.

Q. 31 Every progression is a sequence but the converse, *i.e.*, every sequence is also a progression need not necessarily be true.

Sol. ***True***

Consider the progression $a, a+d, a+2d, \ldots$

and sequence of prime number 2, 3, 5, 7, 11,...

Clearly, progression is a sequence but sequence is not progression because it does not follow a specific pattern.

Q. 32 Any term of an AP (except first) is equal to half the sum of terms which are equidistant from it.

Sol. ***True***

Consider an AP $a, a+d, a+2d, \ldots$

Now, $$a_2 + a_4 = a + d + a + 3d$$
$$= 2a + 4d = 2a_3$$

$\Rightarrow$ $$a_3 = \frac{a_2 + a_4}{2}$$

Again, $$\frac{a_3 + a_5}{2} = \frac{a + 2d + a + 4d}{2} = \frac{2a + 6d}{2}$$
$$= a + 3d = a_4$$

Hence, the statement is true.

Q. 33 The sum or difference of two GP, is again a GP.

Sol. ***False***

Let two GP are $a, ar_1, ar_1^2, ar_2^3, \ldots$ and $b, br_2, br_2^2, br_2^3, \ldots$

Now, sum of two GP $a+b, (ar_1 + br_2), (ar_1^2 + br_2^2), \ldots$

Now, $$\frac{T_2}{T_1} = \frac{ar_1 + br_2}{a+b} \text{ and } \frac{T_3}{T_2} = \frac{ar_1^2 + br_2^2}{ar_1 + br_2}$$

$\therefore$ $$\frac{T_2}{T_1} \neq \frac{T_3}{T_2}$$

Again, difference of two GP is $a - b, ar_1 - br_2, ar_1^2 - br_2^2, \ldots$

Now, $\frac{T_2}{T_1} = \frac{ar_1 - br_2}{a - b}$ and $\frac{T_3}{T_2} = \frac{ar_1^2 - br_2^2}{ar_1 - br_2}$

$\therefore \quad \frac{T_2}{T_1} \neq \frac{T_3}{T_2}$

So, the sum or difference of two GP is not a GP.

Hence, the statement is false.

Q. 34 If the sum of n terms of a sequence is quadratic expression, then it always represents an AP.

Sol. ***False***

Let $S_n = an^2 + bn + c$

$S_1 = a + b + c$

$a_1 = a + b + c$

$S_2 = 4a + 2b + c$

$\therefore \quad a_2 = S_2 - S_1$

$= 4a + 4b + c - (a + b + c) = 3a + b$

$S_3 = 9a + 3b + c$

$\therefore \quad a_3 = S_3 - S_2 = 5a + b$

Now, $a_2 - a_1 = (3a + b) - (a + b + c) = 2a - c$

$a_3 - a_2 = (5a + b) - (3a + b) = 2a$

Now, $a_2 - a_1 \neq a_3 - a_2$

Hence, the statement is false.

Matching The Columns

Q. 35 Match the following.

	Column I		Column II
(i)	$4, 1, \frac{1}{4}, \frac{1}{16}$	(a)	AP
(ii)	2, 3, 5, 7	(b)	Sequence
(iii)	13, 8, 3, –2, –7	(c)	GP

Sol. (i) $4, 1, \frac{1}{4}, \frac{1}{16}$

$\Rightarrow \quad \frac{T_2}{T_1} = \frac{1}{4} \Rightarrow \frac{T_3}{T_2} = \frac{1}{4} \Rightarrow \frac{T_4}{T_3} = \frac{1/16}{1/4} = \frac{1}{4}$

Hence, it is a GP.

(ii) 2, 3, 5, 7

$\because \quad T_2 - T_1 = 3 - 2 = 1$

$T_3 - T_2 = 5 - 3 = 2$

$\because \quad T_2 - T_1 \neq T_3 - T_2$

Hence, it is not an AP.

Again, $\frac{T_2}{T_1} = 3/2 \Rightarrow \frac{T_3}{T_2} = 5/3$

$\because \quad \frac{T_2}{T_1} \neq \frac{T_3}{T_2}$

It is not a GP.

Hence, it is a sequence.

(iii) 13, 8, 3, – 2, – 7

$$T_2 - T_1 = 8 - 13 = -5$$
$$T_3 - T_2 = 3 - 8 = -5$$

$\because \quad T_2 - T_1 = T_3 - T_2$

Hence, it is an AP.

Q. 36 Match the following.

Column I	Column II
(i) $1^2 + 2^2 + 3^2 + \cdots + n^2$	(a) $\left[\frac{n(n+1)}{2}\right]^2$
(ii) $1^3 + 2^3 + 3^3 + \cdots + n^3$	(b) $n(n+1)$
(iii) $2 + 4 + 6 + \cdots + 2n$	(c) $\frac{n(n+1)(2n+1)}{6}$
(iv) $1 + 2 + 3 + \cdots + n$	(d) $\frac{n(n+1)}{2}$

Sol. (i) $1^2 + 2^2 + 3^2 + \cdots + n^2$

Consider the identity, $(k+1)^3 - k^3 = 3k^2 + 3k + 1$

On putting $k = 1, 2, 3, \ldots, (n-1), n$ successively, we get

$$2^3 - 1^3 = 3\cdot 1^2 + 3\cdot 1 + 1$$
$$3^3 - 2^3 = 3\cdot 2^2 + 3\cdot 2 + 1$$
$$4^3 - 3^3 = 3\cdot 3^2 + 3\cdot 3 + 1$$
$$\ldots\ldots \quad \ldots\ldots \quad \ldots\ldots$$
$$\ldots\ldots \quad \ldots\ldots \quad \ldots\ldots$$
$$n^3 - (n-1)^3 = 3\cdot (n-1)^2 + 3\cdot (n-1) + 1$$
$$(n+1)^3 - n^3 = 3\cdot n^2 + 3\cdot n + 1$$

Adding columnwise, we get

$$n^3 + 3n^2 + 3n = 3\left(\sum_{r=1}^{n} r^2\right) + 3\,\frac{n(n+1)}{2} + n \qquad \left[\because \sum_{r=1}^{n} r = \frac{n(n+1)}{2}\right]$$

$$\Rightarrow \quad 3\left(\sum_{r=1}^{n} r^2\right) = n^3 + 3n^2 + 3n - \frac{3n(n+1)}{2} + n$$

$$\Rightarrow \quad \left(\sum_{r=1}^{n} r^2\right) = \frac{2n^3 + 3n^2 + n}{2} = \frac{n(n+1)(2n+1)}{2}$$

$$\Rightarrow \quad \sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$$

Hence, $\sum_{r=1}^{n} r^2 = 1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$

(ii) $1^3 + 2^3 + 3^3 + \cdots + n^3$

Consider the identity $(k+1)^4 - k^4 = 4k^3 + 6k^2 + 4k + 1$

On putting $k = 1, 2, 3, \cdots (n-1), n$ successively, we get

$$2^4 - 1^4 = 4\cdot 1^3 + 6\cdot 1^2 + 4\cdot 1 + 1$$
$$3^4 - 2^4 = 4\cdot 2^3 + 6\cdot 2^2 + 4\cdot 2 + 1$$
$$4^4 - 3^4 = 4\cdot 3^3 + 6\cdot 3^2 + 4\cdot 3 + 1$$
$$\cdots\cdots \quad \cdots\cdots \quad \cdots\cdots$$
$$\cdots\cdots \quad \cdots\cdots \quad \cdots\cdots$$
$$n^4 - (n-1)^4 = 4(n-1)^3 + 6(n-1)^2 + 4(n-1) + 1$$
$$(n+1)^4 - n^4 = 4\cdot n^3 + 6\cdot n^2 + 4\cdot n + 1$$

Adding columnwise, we get

$$(n+1)^4 - 1^4 = 4\cdot(1^3 + 2^3 + \cdots + n^3) + 6(1^2 + 2^2 + 3^3 + \cdots + n^2) + 4(1 + 2 + 3 + \cdots + n) + (1 + 1 + \cdots + 1)\, n \text{ terms}$$

$$\Rightarrow \quad n^4 + 4n^3 + 6n^2 + 4n = 4\left(\sum_{r=1}^{n} r^3\right) + 6\left(\sum_{r=1}^{n} r^2\right) + 4\left(\sum_{r=1}^{n} r\right) + n$$

$$\Rightarrow \quad n^4 + 4n^3 + 6n^2 + 4n = 4\left(\sum_{r=1}^{n} r^3\right) + 6\left[\frac{n(n+1)(2n+1)}{6}\right] + 4\left[\frac{n(n+1)}{2}\right] + n$$

$$\Rightarrow \quad \sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}$$

$$\Rightarrow \quad \sum_{r=1}^{n} r^3 = \left[\frac{n(n+1)}{2}\right]^2 = \left(\sum_{r=1}^{n} r\right)^2$$

Hence, $$\sum_{r=1}^{n} r^3 = 1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2 = \left(\sum_{r=1}^{n} r\right)^2$$

(iii) $$2 + 4 + 6 + \cdots + 2n = 2[1 + 2 + 3 + \cdots + n]$$
$$= 2 \times \frac{n(n+1)}{2} = n(n+1)$$

(iv) Let $S_n = 1 + 2 + 3 + \cdots + n$

Clearly, it is an arithmetic series with first term, $a = 1$,

common difference, $d = 1$

and last term $= n$

$$S_n = \frac{n}{2}(1+n) = \frac{n(n+1)}{2}$$

Hence, $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$.

10

Straight Lines

Short Answer Type Questions

Q. 1 Find the equation of the straight line which passes through the point (1 – 2) and cuts off equal intercepts from axes.

Sol. Let the intercepts along the X and Y-axes are a and a respectively.

$\therefore$ Equation of the line is $\quad \dfrac{x}{a}+\dfrac{y}{a}=1 \quad \ldots$(i)

Since, the point (1, –2) lies on the line,

$$\therefore \quad \frac{1}{a}-\frac{2}{a}=1$$

$$\Rightarrow \quad \frac{1-2}{a}=1$$

$$\Rightarrow \quad a=-1$$

On putting $a=-1$ in Eq. (i), we get

$$\frac{x}{-1}+\frac{y}{-1}=1$$

$$\Rightarrow \quad x+y=-1 \Rightarrow x+y+1=0$$

Q. 2 Find the equation of the line passing through the point (5, 2) and perpendicular to the line joining the points (2, 3) and (3, –1).

Thinking Process

First of all find the slope, using the formula $=\dfrac{y_2-y_1}{x_2-x_1}$. *Then, slope of perpendicular line is* $-\dfrac{1}{m}$.

Sol. Consider the given points $A(5, 2)$, $B(2, 3)$ and $C(3, -1)$.

Slope of the line passing through the points B and C, $m_{BC}=\dfrac{-1-3}{3-2}=-4$

So, the slope of required line is $\dfrac{1}{4}$.

Since, the equation of a line passing the point $A(5, 2)$ and having slope $\dfrac{1}{4}$ is $y-2=\dfrac{1}{4}(x-5)$.

$$\Rightarrow \quad 4y-8=x-5$$

$$\Rightarrow \quad x-4y+3=0$$

Q. 3 Find the angle between the lines $y = (2 - \sqrt{3})(x + 5)$ and $y = (2 + \sqrt{3})(x - 7)$.

Thinking Process

If the angle between the lines having the slope m_1 and m_2 is θ, then $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$.

Use this formula to solve the above problem.

Sol. Given lines, $y = (2 - \sqrt{3})(x + 5)$ ...(i)

Slope of this line, $m_1 = (2 - \sqrt{3})$

and $y = (2 + \sqrt{3})(x - 7)$ ...(ii)

Slope of this line, $m_2 = (2 + \sqrt{3})$

Let θ be the angle between lines (i) and (ii), then

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$\Rightarrow \tan\theta = \left|\frac{(2 - \sqrt{3}) - (2 + \sqrt{3})}{1 + (2 - \sqrt{3})(2 + \sqrt{3})}\right| \Rightarrow \tan\theta = \left|\frac{-2\sqrt{3}}{1 + 4 - 3}\right|$$

$$\Rightarrow \tan\theta = \sqrt{3}$$

$$\Rightarrow \tan\theta = \tan\pi/3$$

$$\therefore \theta = \pi/3 = 60^\circ$$

For obtuse angle $= \pi - \pi/3 = 2\pi/3 = 120^\circ$

Hence, the angle between the lines are 60° or 120°.

Q. 4 Find the equation of the lines which passes through the point (3, 4) and cuts off intercepts from the coordinate axes such that their sum is 14.

Sol. Let the intercept along the axes be a and b.

Given, $a + b = 14 \Rightarrow b = 14 - a$

Now, the equation of line is $\frac{x}{a} + \frac{y}{b} = 1$. ...(i)

$$\Rightarrow \frac{x}{a} + \frac{y}{14 - a} = 1$$

Since, the point (3, 4) lies on the line.

$$\therefore \frac{3}{a} + \frac{4}{14 - a} = 1$$

$$\Rightarrow \frac{42 - 3a + 4a}{a(14 - a)} = 1 \Rightarrow 42 + a = 14a - a^2$$

$$\Rightarrow a^2 - 13a + 42 = 0 \Rightarrow a^2 - 7a - 6a + 42 = 0$$

$$\Rightarrow a(a - 7) - 6(a - 7) = 0 \Rightarrow (a - 7)(a - 6) = 0$$

$$\Rightarrow a - 7 = 0 \text{ or } a - 6 = 0$$

$$\therefore a = 7 \text{ or } a = 6$$

When $a = 7$, then $b = 7$

When $a = 6$, then $b = 8$

$\therefore$ The equation of line, when $a = 7$ and $b = 7$ is

$$\frac{x}{7} + \frac{y}{7} = 1 \Rightarrow x + y = 7$$

So, the equation of line, when $a = 6$ and $b = 8$ is $\frac{x}{6} + \frac{y}{8} = 1$

Q. 5 Find the points on the line $x + y = 4$ which lie at a unit distance from the line $4x + 3y = 10$.

Thinking Process

The perpendicular distance of a point (x_1, y_1) from the line $Ax + By + C = 0$, is d', where $d = \left|\frac{Ax_1 + By_1 + C}{\sqrt{A^2 + B^2}}\right|$.

Sol. Let the required point be (h, k) and point (h, k) lies on the line $x + y = 4$

i.e., $h + k = 4$ ...(i)

The distance of the point (h, k) from the line $4x + 3y = 10$ is

$$\left|\frac{4h + 3k - 10}{\sqrt{16 + 9}}\right| = 1$$

$$4h + 3k - 10 = \pm 5$$

Taking positive sign, $4h + 3k = 15$ ...(ii)

From Eq. (i) $h = 4 - k$ put in Eq. (ii), we get

$$4(4 - k) + 3k = 15$$

$\Rightarrow$ $16 - 4k + 3k = 15$

$\Rightarrow$ $k = 1$

On putting $k = 1$ in Eq. (i), we get

$$h + 1 = 4 \Rightarrow h = 3$$

So, the point is (3,1).

Taking negative sign,

$$4h + 3k - 10 = -5$$

$\Rightarrow$ $4(4 - k) + 3k = 5$

$\Rightarrow$ $16 - 4k + 3k = 5$

$\Rightarrow$ $-k = 5 - 16 = -11$

$\therefore$ $k = 11$

On putting $k = 11$ in Eq. (i), we get

$$h + 11 = 4 \Rightarrow h = -7$$

Hence, the required points are (3, 1) and (–7, 11).

Q. 6 Show that the tangent of an angle between the lines $\frac{x}{a} + \frac{y}{b} = 1$ and $\frac{x}{a} - \frac{y}{b} = 1$ is $\frac{2ab}{a^2 - b^2}$.

Sol. Given equation of lines are

$$\frac{x}{a} + \frac{y}{b} = 1 \quad \text{...(i)}$$

$\therefore$ Slope, $m_1 = -\frac{b}{a}$

and $\frac{x}{a} - \frac{y}{b} = 1$ ...(ii)

$\therefore$ Slope, $m_2 = \frac{b}{a}$

Let θ be the angle between the given lines, then

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| \Rightarrow \tan\theta = \left|\frac{-\frac{b}{a} - \frac{b}{a}}{1 + \left(\frac{-b}{a}\right)\left(-\frac{b}{a}\right)}\right|$$

$$\Rightarrow \tan\theta = \left|\frac{\frac{-2b}{a}}{\frac{a^2 - b^2}{a^2}}\right| \Rightarrow \tan\theta = \frac{2ab}{a^2 - b^2}$$

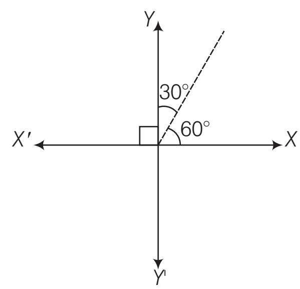
Hence proved.

Q. 7 Find the equation of lines passing through (1, 2) and making angle 30° with *Y*-axis.

Thinking Process

Equation of a line passing through the point (x_1, y_1) and having slope m is $y - y_1 = m(x - x_1)$.

Sol. Given that, angle with *Y*-axis = 30°
and angle with *X*-axis = 60°



$\therefore$ Slope of the line, $m = \tan 60° = \sqrt{3}$

So, the equation of a line passing through (1, 2) and having slope $\sqrt{3}$, is

$$y - 2 = \sqrt{3}(x - 1)$$
$$\Rightarrow \quad y - 2 = \sqrt{3}x - \sqrt{3}$$
$$\Rightarrow \quad y - \sqrt{3}x - 2 + \sqrt{3} = 0$$

Q. 8 Find the equation of the line passing through the point of intersection of $2x + y = 5$ and $x + 3y + 8 = 0$ and parallel to the line $3x + 4y = 7$.

Thinking Process

First of all solve the given equation of lines to get the point of intersection. Then, if a line having slope m_1 is parallel to another line having slope, m_2, then $m_1 = m_2$. Now, use the formula i.e., equation of a line passing through the point (x_1, y_1) with slope m is $y - y_1 = m(x - x_1)$.

Sol. Given equation of lines $\quad 2x + y = 5 \quad$ ...(i)

and $\quad x + 3y = -8 \quad$ ...(ii)

From Eq. (i), $\quad y = 5 - 2x$

Now, put the value of y in Eq. (ii), we get

$$x + 3(5 - 2x) = -8$$
$$\Rightarrow \quad x + 15 - 6x = -8$$
$$\Rightarrow \quad -5x = -23 \Rightarrow x = \frac{23}{5}$$

Now, $x = \frac{23}{5}$ put in Eq. (i), we get

$$y = 5 - \frac{46}{5} = \frac{25 - 46}{5} = \frac{-21}{5}$$

Since, the required line is parallel to the line $3x + 4y = 7$. So, slope of the line is $m = \frac{-3}{4}$.

So, the equation of the line passing through the point $\left(\frac{23}{5}, \frac{-21}{5}\right)$ having slope $\frac{-3}{4}$ is

$$y + \frac{21}{5} = \frac{-3}{4}\left(x - \frac{23}{5}\right)$$

$$\Rightarrow \quad 4y + \frac{84}{5} = -3x + \frac{69}{5}$$

$$\Rightarrow \quad 3x + 4y = \frac{84 - 69}{5} \Rightarrow 3x + 4y + \frac{15}{5} = 0$$

$$\Rightarrow \quad 3x + 4y + 3 = 0$$

Q. 9 **For what values of a and b the intercepts cut off on the coordinate axes by the line $ax + by + 8 = 0$ are equal in length but opposite in signs to those cut off by the line $2x - 3y + 6 = 0$ on the axes?**

Sol. Given equation of line $\quad ax + by + 8 = 0$

$$\Rightarrow \quad \frac{x}{\frac{-8}{a}} + \frac{y}{\frac{-8}{b}} = 1$$

So, the intercepts are $\frac{-8}{a}$ and $\frac{-8}{b}$.

and another given equation of line is $2x - 3y + 6 = 0$.

$$\Rightarrow \quad \frac{x}{-3} + \frac{y}{2} = 1$$

So, the intercepts are –3 and 2.

According to the question,

$$\frac{-8}{a} = 3 \text{ and } \frac{-8}{b} = -2$$

$$\therefore \quad a = -\frac{8}{3}, \ b = 4$$

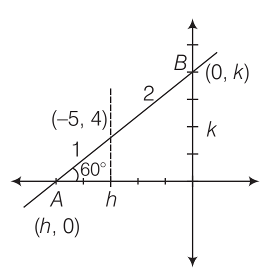
Q. 10 **If the intercept of a line between the coordinate axes is divided by the point $(-5, 4)$ in the ratio $1 : 2$, then find the equation of the line.**

Thinking Process

The coordinates of a point which divides the join of (x_1, y_1) and (x_2, y_2) in the ratio $m_1 : m_2$ internally is $\left(\frac{m_1 x_2 + m_2 x_1}{m_1 + m_2}, \frac{m_1 y_2 + m_2 y_1}{m_1 + m_2}\right)$.

Sol. Let intercept of a line are (h, k).

The coordinates of A and B are $(h, 0)$ and $(0, k)$ respectively.



$$-5 = \frac{1 \times 0 + 2 \times h}{1 + 2}$$

$$\therefore \quad -5 = \frac{2h}{3} \Rightarrow h = -\frac{15}{2}$$

$$\text{and} \quad 4 = \frac{1 \cdot k + 0 \cdot 2}{1 + 2}$$

$$\Rightarrow \quad k = 12$$

$$\therefore \quad A = \left(-\frac{15}{2}, 0\right) \text{ and } B = (0, 12)$$

Hence, the equation of a line AB is

$$y - 0 = \frac{12 - 0}{0 + 15/2}\left(x + \frac{15}{2}\right)$$

$$\Rightarrow \quad y = \frac{12 \cdot 2}{15}\left(x + \frac{15}{2}\right)$$

$$\Rightarrow \quad 5y = 8x + 60 \Rightarrow 8x - 5y + 60 = 0$$

Q. 11 **Find the equation of a straight line on which length of perpendicular from the origin is four units and the line makes an angle of 120° with the positive direction of X-axis.**

Thinking Process

The equation of the line having normal distance P from the origin and angle α which the normal makes with the positive direction of X-axis is $x\cos\alpha + y\sin\alpha = p$. Use this formula to solve the above problem.

Sol. Given that, $OC = P = 4$ units

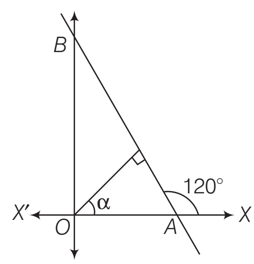
$$\angle BAX = 120°$$

Let $\angle COA = \alpha$, $\angle OCA = 90°$

$\because$ $\angle BAX = \angle COA + \angle OCA$ [exterior angle property]

$\Rightarrow$ $120° = \alpha + 90°$

$\therefore$ $\alpha = 30°$



Now, the equation of required line is

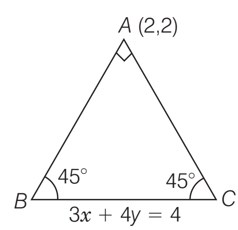
$$x\cos 30° + y\sin 30° = 4$$

$$\Rightarrow \quad x \cdot \frac{\sqrt{3}}{2} + y \cdot \frac{1}{2} = 4$$

$$\Rightarrow \quad \sqrt{3}x + y = 8$$

Q. 12 **Find the equation of one of the sides of an isosceles right angled triangle whose hypotenuse is given by $3x + 4y = 4$ and the opposite vertex of the hypotenuse is (2, 2).**

Sol. Let slope of line AC be m and slope of line BC is $\frac{-3}{4}$ and let angle between line AC and BC be θ.



$$\therefore \quad \tan\theta = \left|\frac{m + \frac{3}{4}}{1 - \frac{3m}{4}}\right| \Rightarrow \tan 45° = \pm\left[\frac{m + \frac{3}{4}}{1 - \frac{3m}{4}}\right]$$

Taking positive sign,

$$1 = \frac{m + \frac{3}{4}}{1 - \frac{3m}{4}}$$

$$\Rightarrow \quad m + \frac{3}{4} = 1 - \frac{3m}{4}$$

$$\Rightarrow \quad m + \frac{3m}{4} = 1 - \frac{3}{4}$$

$$\Rightarrow \quad \frac{7m}{4} = \frac{1}{4} \Rightarrow m = \frac{1}{7}$$

Taking negative sign,

$$1 = -\left(\frac{m + \frac{3}{4}}{1 - \frac{3m}{4}}\right) \Rightarrow 1 - \frac{3m}{4} = -m - \frac{3}{4}$$

$$\Rightarrow \quad m - \frac{3m}{4} = -1 - \frac{3}{4}$$

$$\Rightarrow \quad \frac{m}{4} = \frac{-7}{4} \Rightarrow m = -7$$

$\therefore$ Equation of side AC having slope $\left(\frac{1}{7}\right)$ is

$$y - 2 = \frac{1}{7}(x - 2)$$

$$\Rightarrow \quad 7y - 14 = x - 2$$

$$\Rightarrow \quad x - 7y + 12 = 0$$

and equation of side AB having slope (-7) is

$$y - 2 = -7(x - 2)$$

$$\Rightarrow \quad y - 2 = -7x + 14$$

$$\Rightarrow \quad 7x + y - 16 = 0$$

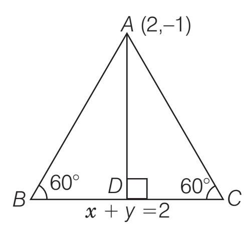
Long Answer Type Questions

Q. 13 If the equation of the base of an equilateral triangle is $x + y = 2$ and the vertex is $(2, -1)$, then find the length of the side of the triangle.

Thinking Process

Find the length of perpendicular (p) from (2, −1) to the line and use $p = l \sin 60°$, where l is the length of the side of the triangle.

Sol. Given that, equilateral ΔABC having equation of base is $x + y = 2$.



In ΔABD, $$\sin 60° = \frac{AD}{AB}$$

$$\Rightarrow \quad AD = AB \sin 60° = AB\frac{\sqrt{3}}{2}$$

$$\because \quad AD = AB\frac{\sqrt{3}}{2} \quad \dots\text{(i)}$$

Now, the length of perpendicular from (2, −1) to the line $x + y = 2$ is given by

$$AD = \left|\frac{2 + (-1) - 2}{\sqrt{1^2 + 1^2}}\right| = \frac{1}{\sqrt{2}}$$

From Eq. (i), $$\frac{1}{\sqrt{2}} = AB\frac{\sqrt{3}}{2}$$

$$AB = \sqrt{\frac{2}{3}}$$

Q. 14 A variable line passes through a fixed point P. The algebraic sum of the perpendiculars drawn from the points (2, 0), (0, 2) and (1, 1) on the line is zero. Find the coordinates of the point P.

Thinking Process

Let the slope of the line be m. Then, the equation of the line passing through the fixed point $P(x_1, y_1)$ is $y - y_1 = m(x - x_1)$. Taking the algebraic sum of perpendicular distances equal to zero, we get $y_1 - 1 = m(x_1 - 1)$. Thus, (x_1, y_1) is (1, 1).

Sol. Let slope of the line be m and the coordinates of fixed point P are (x_1, y_1).

$\therefore$ Equation of line is $y - y_1 = m(x - x_1)$ ...(i)

Since, the given points are $A(2, 0)$, $B(0, 2)$ and $C(1, 1)$.

Now, perpendicular distance from A, is

$$\frac{0 - y_1 - m(2 - x_1)}{\sqrt{1 + m^2}}$$

Perpendicular distance from B, is

$$\frac{2 - y_1 - m(0 - x_1)}{\sqrt{1 + m^2}}$$

Perpendicular distance from C, is

$$\frac{1 - y_1 - m(1 - x_1)}{\sqrt{1 + m^2}}$$

Now, $$\frac{-y_1 - 2m + mx_1 + 2 - y_1 + mx_1 + 1 - y_1 - m + mx_1}{\sqrt{1 + m^2}} = 0$$

$$\Rightarrow \quad -3y_1 - 3m + 3mx_1 + 3 = 0$$

$$\Rightarrow \quad -y_1 - m + mx_1 + 1 = 0$$

Since, (1, 1) lies on this line. So, the point P is (1, 1).

Q. 15 In what direction should a line be drawn through the point (1, 2), so that its point of intersection with the line $x + y = 4$ is at a distance $\frac{\sqrt{6}}{3}$ from the given point?

Sol. Let slope of the line be m. As, the line passes through the point $A(1, 2)$.

$\therefore$ Equation of line is $y - 2 = m(x - 1)$

$$mx - y + 2 - m = 0 \quad \text{...(i)}$$

and $$x + y - 4 = 0$$

$$\frac{x}{(4 - 2 + m)} = \frac{y}{2 - m + 4m} = \frac{1}{1 + m}$$

$$\Rightarrow \quad \frac{x}{2 + m} = \frac{y}{3m + 2} = \frac{1}{1 + m}$$

$$\Rightarrow \quad x = \frac{2 + m}{1 + m}$$

$$y = \frac{3m + 2}{1 + m}$$

So, the point of intersection is $B\left(\frac{m + 2}{m + 1}, \frac{3m + 2}{m + 1}\right)$.

Now, $$AB^2 = \left(\frac{m+2}{m+1} - 1\right)^2 + \left(\frac{3m+2}{m+1} - 2\right)^2$$

$\because$ $$AB = \frac{\sqrt{6}}{3}$$ [given]

$\therefore$ $$\left(\frac{m+2-m-1}{m+1}\right)^2 + \left(\frac{3m+2-2m-2}{m+1}\right)^2 = \frac{6}{9}$$

$\Rightarrow$ $$\left(\frac{1}{m+1}\right)^2 + \left(\frac{m}{m+1}\right)^2 = \frac{6}{9}$$

$\Rightarrow$ $$\frac{1+m^2}{(1+m)^2} = \frac{6}{9}$$

$\Rightarrow$ $$\frac{1+m^2}{1+m^2+2m} = \frac{6}{9}$$

$\Rightarrow$ $$9 + 9m^2 = 6 + 6m^2 + 12m$$

$\Rightarrow$ $$3m^2 - 12m + 3 = 0$$

$\Rightarrow$ $$m^2 - 4m + 1 = 0$$

$\therefore$ $$m = \frac{4 \pm \sqrt{16-4}}{2}$$

$$= 2 \pm \sqrt{3}$$

$$= 2 + \sqrt{3} \text{ or } 2 - \sqrt{3}$$

$\therefore$ $$\theta = 75° \text{ or } 15°$$

Q. 16 **A straight line moves so that the sum of the reciprocals of its intercepts made on axes is constant. Show that the line passes through a fixed point.**

Thinking Process

If a line is $\frac{x}{a} + \frac{y}{b} = 1$, *where* $\frac{1}{a} + \frac{1}{b} =$ *constant* $= \frac{1}{k}$ *(say). This implies that* $\frac{k}{a} + \frac{k}{b} = 1 \Rightarrow$ *line passes through the fixed point* (k, k).

Sol. Since, the intercept form of a line is $\frac{x}{a} + \frac{y}{b} = 1$.

Given that, $$\frac{1}{a} + \frac{1}{b} = \text{constant} = \frac{1}{k}$$

$\because$ $$\frac{1}{a} + \frac{1}{b} = \frac{1}{k}$$

$\Rightarrow$ $$\frac{k}{a} + \frac{k}{b} = 1$$

So, (k, k) lies on $\frac{x}{a} + \frac{y}{b} = 1$.

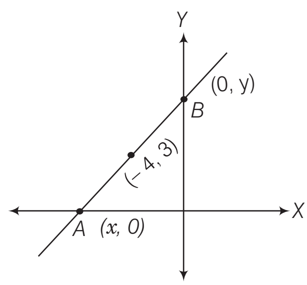
Hence, the line passes through the fixed point.

Q. 17 **Find the equation of the line which passes through the point (−4, 3) and the portion of the line intercepted between the axes is divided internally in the ratio 5 : 3 by this point.**

Thinking Process

If the point (h, k) divides the join of A (x_1, y_1) and B(x_2, y_2) internally, in the ratio $m_1 : m_2$. Then, first of all find the coordinates of A and B using section formula for internal division i.e., $h = \frac{m_1 x_2 + m_2 x_1}{m_1 + m_2}$, $k = \frac{m_1 y_2 + m_2 y_1}{m_1 + m_2}$. Then, find the equation of required line.

Sol. Since, the line intersects X and Y-axes respectively at $A(x, 0)$ and $B(0, y)$.



$$-4 = \frac{5 \times 0 + 3x}{5 + 3}$$

$\Rightarrow$ $$-4 = \frac{3x}{8} \Rightarrow x = \frac{-32}{3}$$

and $$3 = \frac{5 \cdot y + 3 \cdot 0}{5 + 3}$$

$\Rightarrow$ $$3 = \frac{5y}{8} \Rightarrow y = \frac{24}{5}$$

Since, the intercept on the X and Y-axes respectively are $a = \frac{-32}{3}$ and $b = \frac{24}{5}$.

$\therefore$ Equation of required line is

$$\frac{x}{-32/3} + \frac{y}{24/5} = 1$$

$\Rightarrow$ $$\frac{-3x}{32} + \frac{5y}{24} = 1$$

$\Rightarrow$ $$-9x + 20y = 96$$

$\Rightarrow$ $$9x - 20y + 96 = 0$$

Q. 18 **Find the equations of the lines through the point of intersection of the lines $x - y + 1 = 0$ and $2x - 3y + 5 = 0$ and whose distance from the point (3, 2) is $\frac{7}{5}$.**

Sol. Given equation of lines $x - y + 1 = 0$ ...(i)

and $2x - 3y + 5 = 0$ ...(ii)

From Eq. (i), $x = y - 1$

Now, put the value of x in Eq. (ii), we get

$$2(y-1)-3y+5=0$$
$$\Rightarrow \quad 2y-2-3y+5=0$$
$$\Rightarrow \quad 3-y=0 \Rightarrow y=3$$

$y=3$ put in Eq. (i), we get

$$x=2$$

Since, the point of intersection is (2, 3).

Let slope of the required line be m.

$\therefore$ Equation of line is $\quad y-3=m(x-2)$

$$\Rightarrow \quad mx-y+3-2m=0 \quad \ldots\text{(iii)}$$

Since, the distance from (3, 2) to line (iii) is $\frac{7}{5}$.

$$\therefore \quad \frac{7}{5}=\left|\frac{3m-2+3-2m}{\sqrt{1+m^2}}\right|$$
$$\Rightarrow \quad \frac{49}{25}=\frac{(m+1)^2}{1+m^2}$$
$$\Rightarrow \quad 49+49m^2=25(m^2+2m+1)$$
$$\Rightarrow \quad 49+49m^2=25m^2+50m+25$$
$$\Rightarrow \quad 24m^2-50m+24=0$$
$$\Rightarrow \quad 12m^2-25m+12=0$$
$$\therefore \quad m=\frac{25\pm\sqrt{625-4\cdot12\cdot12}}{24}$$
$$=\frac{25\pm\sqrt{49}}{24}=\frac{25\pm7}{24}=\frac{32}{24}\text{ or }\frac{18}{24}=\frac{4}{3}\text{ or }\frac{3}{4}$$

$\therefore$ First equation of a line is $\quad y-3=\frac{4}{3}(x-2)$

$$\Rightarrow \quad 3y-9=4x-8$$
$$\Rightarrow \quad 4x-3y+1=0$$

and second equation of line is $\quad y-3=\frac{3}{4}(x-2)$

$$\Rightarrow \quad 4y-12=3x-6$$
$$\Rightarrow \quad 3x-4y+6=0$$

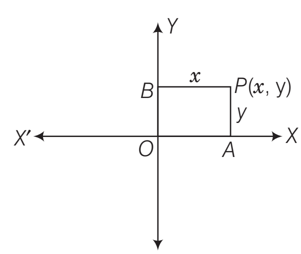
Q. 19 If the sum of the distance of a moving point in a plane from the axes is 1, then find the locus of the point.

Thinking Process

Given that $|x|+|y|=1$, which gives four sides of a square.

Sol. Let the coordinates of moving point P be (x, y).

Given that, the sum of distances of this point in a plane from the axes is 1.



$$\therefore \quad |x|+|y|=1$$
$$\Rightarrow \quad \pm x \pm y=1$$
$$\Rightarrow \quad x+y=1$$
$$\Rightarrow \quad -x-y=1$$
$$\Rightarrow \quad -x+y=1$$
$$\Rightarrow \quad x-y=1$$

So, these equations give us locus of the point which is a square.

Q. 20 P_1 and P_2 are points on either of the two lines $y-\sqrt{3}|x|=2$ at a distance of 5 units from their point of intersection. Find the coordinates of the foot of perpendiculars drawn from P_1, P_2 on the bisector of the angle between the given lines.

Thinking Process

Lines are $y=\sqrt{3}x+2$ and $y=-\sqrt{3}x+2$ according as $x\ge 0$ or $x<0$. Y-axis is the bisector of the angles between the lines. P_1, P_2 are the points on these lines at a distance of 5 units from the point of intersection of these lines which have a point on Y-axis as common foot of perpendiculars drawn from these points. The y-coordinate of the foot of the perpendiculars is given by $2+5\cos 30°$.

Sol. Given equation of lines are $y-\sqrt{3}x=2$ $[\because x \ge 0]$

and $y+\sqrt{3}x=2$ $[\because x \le 0]$

$\because$ $y=\sqrt{3}x+2$ ...(i)

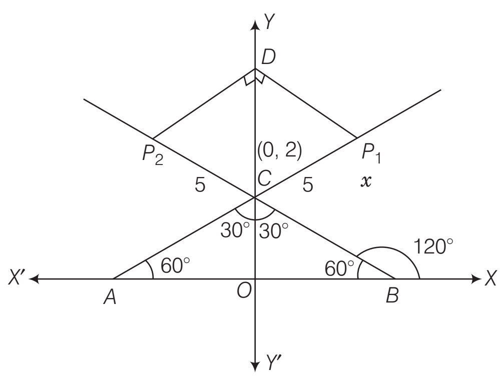
and $y=-\sqrt{3}x+2$ ...(ii)

$\Rightarrow$ $\sqrt{3}x+2=-\sqrt{3}x+2$

$\Rightarrow$ $2\sqrt{3}x=0 \Rightarrow x=0$

On putting $x=0$ in Eq. (i), we get

$$y=2$$



So, the point of intersection of line (i) and (ii) is (0, 2).

Here, $OC=2$

In ΔDEC, $\dfrac{CD}{CE}=\cos 30°$

$\therefore$ $CD=5\cos 30°$

$$=5\cdot\frac{\sqrt{3}}{2}$$

$\Rightarrow$ $OD=OC+CD=2+5\dfrac{\sqrt{3}}{2}$

So, the coordinates of the foot of perpendiculars are $\left(0, 2+\dfrac{5\sqrt{3}}{2}\right)$.

Q. 21 If p is the length of perpendicular from the origin on the line $\frac{x}{a}+\frac{y}{b}=1$ and a^2, p^2 and b^2 are in AP, the show that $a^4+b^4=0$.

Sol. Given equation of line is,

$$\frac{x}{a}+\frac{y}{b}=1 \quad \ldots\text{(i)}$$

Perpendicular length from the origin on the line (i) is given by p

i.e., $$p=\frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}}=\frac{ab}{\sqrt{a^2+b^2}}$$

$\therefore$ $$p^2=\frac{a^2b^2}{a^2+b^2}$$

Given that, a^2, p^2 and b^2 are in AP.

$\therefore$ $$2p^2=a^2+b^2$$

$\Rightarrow$ $$\frac{2a^2b^2}{a^2+b^2}=a^2+b^2$$

$\Rightarrow$ $$2a^2b^2=(a^2+b^2)^2$$

$\Rightarrow$ $$2a^2+b^2=a^4+b^4+2a^2b^2$$

$\Rightarrow$ $$a^4+b^4=0$$

Objective Type Questions

Q. 22 A line cutting off intercept -3 from the Y-axis and the tangent at angle to the X-axis is $\frac{3}{5}$, its equation is

(a) $5y-3x+15=0$ (b) $3y-5x+15=0$
(c) $5y-3x-15=0$ (d) None of the above

Sol. *(a)* Given that, $c=-3$ and $m=\frac{3}{5}$

$\therefore$ Equation of the line is $y=mx+c$

$$y=\frac{3}{5}x-3$$

$\Rightarrow$ $$5y=3x-15$$

$\Rightarrow$ $$5y-3x+15=0$$

Q. 23 Slope of a line which cuts off intercepts of equal lengths on the axes is

(a) -1 (b) 0 (c) 2 (d) $\sqrt{3}$

Sol. *(a)* Let equation of line be $\frac{x}{a}+\frac{y}{a}=1$

$\Rightarrow$ $$x+y=a$$

$\Rightarrow$ $$y=-x+a$$

$\therefore$ Required slope $=-1$

Q. 24 The equation of the straight line passing through the point (3, 2) and perpendicular to the line $y = x$ is

(a) $x - y = 5$ (b) $x + y = 5$ (c) $x + y = 1$ (d) $x - y = 1$

Sol. *(b)* Since, line passes through the point (3, 2) and perpendicular to the line $y = x$.

$\because$ Slope $(m) = -1$ [since, line is perpendicular to the line $y = x$]

$\therefore$ Equation of line which passes through (3, 2) is

$$y - 2 = -1(x - 3)$$
$$\Rightarrow \quad y - 2 = -x + 3$$
$$\Rightarrow \quad x + y = 5$$

Q. 25 The equation of the line passing through the point (1, 2) and perpendicular to the line $x + y + 1 = 0$ is

(a) $y - x + 1 = 0$ (b) $y - x - 1 = 0$

(c) $y - x + 2 = 0$ (d) $y - x - 2 = 0$

Sol. *(b)* Given point is (1, 2) and slope of the required line is 1.

$\because$ $x + y + 1 = 0 \Rightarrow y = -x - 1 \Rightarrow m_1 = -11$

$\therefore$ slope of the line $= \dfrac{-1}{-1} = 1$

$\therefore$ Equation of required line is

$$y - 2 = 1(x - 1)$$
$$\Rightarrow \quad y - 2 = x - 1$$
$$\Rightarrow \quad y - x - 1 = 0$$

Q. 26 The tangent of angle between the lines whose intercepts on the axes are $a, -b$ and $b, -a$ respectively, is

(a) $\dfrac{a^2 - b^2}{ab}$ (b) $\dfrac{b^2 - a^2}{2}$ (c) $\dfrac{b^2 - a^2}{2ab}$ (d) None of these

Sol. *(c)* Since, intercepts on the axes are $a, -b$ then equation of the line is $\dfrac{x}{a} - \dfrac{y}{b} = 1$.

$$\Rightarrow \quad \frac{y}{b} = \frac{x}{a} - 1$$
$$\Rightarrow \quad y = \frac{bx}{a} - b$$

So, the lope of this line *i.e.*, $m_1 = \dfrac{b}{a}$.

Also, for intercepts on the axes as b and $-a$, then equation of the line is

$$\frac{x}{b} - \frac{y}{a} = 1$$
$$\Rightarrow \quad \frac{y}{a} = \frac{x}{b} - 1 \Rightarrow y = \frac{a}{b}x - a$$

and slope of this line *i.e.*, $m_2 = \dfrac{a}{b}$

$$\therefore \quad \tan\theta = \frac{\dfrac{b}{a} - \dfrac{a}{b}}{1 + \dfrac{a}{b} \cdot \dfrac{b}{a}} = \frac{\dfrac{b^2 - a^2}{ab}}{2} = \frac{b^2 - a^2}{2ab}$$

Q. 27 If the line $\frac{x}{a}+\frac{y}{b}=1$ passes through the points (2, –3) and (4, –5), then (a, b) is

(a) (1, 1) (b) (–1, 1) (c) (1, –1) (d) (–1, –1)

Sol. ***(d)*** Given, line is $\frac{x}{a}+\frac{y}{b}=1$ ...(i)

Since, the points (2, –3) and (4, –5) lies on this line.

$$\therefore \quad \frac{2}{a}-\frac{3}{b}=1 \quad \text{...(ii)}$$

$$\text{and} \quad \frac{4}{a}-\frac{5}{b}=1 \quad \text{...(iii)}$$

On multiplying by 2 in Eq. (ii) and then subtracting Eq. (iii) from Eq. (ii), we get

$$-\frac{6}{b}+\frac{5}{b}=1$$

$$\Rightarrow \quad \frac{-1}{b}=1$$

$$\therefore \quad b=-1$$

On putting $b=-1$ in Eq. (ii), we get

$$\frac{2}{a}+3=1$$

$$\Rightarrow \quad \frac{2}{a}=-2 \Rightarrow a=-1$$

$$\therefore \quad (a, b)=(-1, -1)$$

Q. 28 The distance of the point of intersection of the lines $2x-3y+5=0$ and $3x+4y=0$ from the line $5x-2y=0$ is

(a) $\frac{130}{17\sqrt{29}}$ (b) $\frac{13}{7\sqrt{29}}$

(c) $\frac{130}{7}$ (d) None of these

Thinking Process

First of all find the point of intersection of the given first two lines, then get the perpendicular distance from this point to the third line. Using formula i.e., distance of a point (x_1, y_1) from the line $ax+by+c=0$ is $d=\frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$.

Sol. ***(a)*** Given equation of lines

$$2x-3y+5=0 \quad \text{...(i)}$$

$$\text{and} \quad 3x+4y=0 \quad \text{...(ii)}$$

From Eq. (ii), put the value of $x=\frac{-4y}{3}$ in Eq. (i), we get

$$2\left(\frac{-4y}{3}\right)-3y+5=0$$

$$\Rightarrow \quad -8y-9y+15=0$$

$$\Rightarrow \quad y=\frac{15}{17}$$

From Eq. (ii), $3x + 4 \cdot \frac{15}{17} = 0$

$\Rightarrow \quad x = \frac{-60}{17 \cdot 3} = \frac{-20}{17}$

So, the point of intersection is $\left(\frac{-20}{17}, \frac{15}{17}\right)$.

$\therefore$ Required distance from the line $5x - 2y = 0$ is,

$$d = \frac{\left|-5 \times \frac{20}{17} - 2\left(\frac{15}{17}\right)\right|}{\sqrt{25+4}} = \frac{\left|\frac{-100}{17} - \frac{30}{17}\right|}{\sqrt{29}} = \frac{130}{17\sqrt{29}}$$

$\left[\because \text{distance of a point } p(x_1, y_1) \text{ from the line } ax + by + c = 0 \text{ is } d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}\right]$

Q. 29 The equation of the lines which pass through the point (3, –2) and are inclined at 60° to the line $\sqrt{3}x + y = 1$ is

(a) $y + 2 = 0, \sqrt{3}x - y - 2 - 3\sqrt{3} = 0$

(b) $x - 2 = 0, \sqrt{3}x - y + 2 + 3\sqrt{3} = 0$

(c) $\sqrt{3}x - y - 2 - 3\sqrt{3} = 0$

(d) None of the above

Sol. ***(a)*** So, the given point A is (3, –2).

So, the equation of line $\sqrt{3}x + y = 1$.

$\Rightarrow \quad y = -\sqrt{3}x + 1$

$\therefore \quad$ Slope, $m_1 = -\sqrt{3}$

Let slope of the required line be m_2.

$\therefore \quad \tan\theta = \left|\frac{-\sqrt{3} - m_2}{1 - \sqrt{3}m_2}\right| \qquad \left[\because \tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|\right]$

$\Rightarrow \quad \tan 60° = \pm\left(\frac{-\sqrt{3} - m_2}{1 - \sqrt{3}m_2}\right) \qquad \ldots(i)$

$\Rightarrow \quad \sqrt{3} = \left(\frac{-\sqrt{3} - m_2}{1 - \sqrt{3}m_2}\right) \qquad$ [taking positive sign]

$\Rightarrow \quad \sqrt{3} - 3m_2 = -\sqrt{3} - m_2$

$\Rightarrow \quad 2\sqrt{3} = 2m_2$

$\Rightarrow \quad m_2 = \sqrt{3}$

$\therefore$ Equation of line passing through (3, –2) is

$y + 2 = \sqrt{3}(x - 3)$

$y - \sqrt{3}x + 2 + 3\sqrt{3} = 0$

$\Rightarrow \quad \sqrt{3}x - y - 2 - 3\sqrt{3} = 0 \qquad \ldots(i)$

[taking negative sign from Eq. (i)]

$\Rightarrow \quad \sqrt{3} - 3m_2 = \sqrt{3} + m_2$

$\Rightarrow \quad m_2 = 0$

$\therefore$ The equation of line is $y + 2 = 0\,(x - 3)$

$\Rightarrow \quad y + 2 = 0 \qquad \ldots(ii)$

So, the required equation of lines are $\sqrt{3}\,x - y - 2 - 3\sqrt{3} = 0$ and $y + 2 = 0$.

Q. 30 The equations of the lines passing through the point (1, 0) and at a distance $\frac{\sqrt{3}}{2}$ from the origin, are

(a) $\sqrt{3}x + y - \sqrt{3} = 0, \sqrt{3}x - y - \sqrt{3} = 0$
(b) $\sqrt{3}x + y + \sqrt{3} = 0, \sqrt{3}x - y + \sqrt{3} = 0$
(c) $x + \sqrt{3}y - \sqrt{3} = 0, x - \sqrt{3}y - \sqrt{3} = 0$
(d) None of the above

Sol. *(a)* Let slope of the line be m.

$\because$ Equation of line passing through (1, 0) is

$$y - 0 = m(x - 1)$$

$$\Rightarrow \quad y - mx + m = 0 \quad \dots(i)$$

Since, the distance from origin is $\frac{\sqrt{3}}{2}$.

Then,
$$\frac{\sqrt{3}}{2} = \frac{0 - 0 + m}{\sqrt{1 + m^2}}$$

$$\Rightarrow \quad \frac{\sqrt{3}}{2} = \frac{m}{\sqrt{1 + m^2}}$$

$$\Rightarrow \quad \frac{3}{4} = \frac{m^2}{1 + m^2}$$

$$\Rightarrow \quad 3 + 3m^2 = 4m^2$$

$$\Rightarrow \quad m^2 = 3$$

$$\Rightarrow \quad m = \pm\sqrt{3}$$

So, the first equation of line is

$$y = \sqrt{3}(x - 1)$$

$$\Rightarrow \quad \sqrt{3}x - y - \sqrt{3} = 0$$

and the second equation of line is

$$y = -\sqrt{3}(x - 1)$$

$$\Rightarrow \quad \sqrt{3}x + y - \sqrt{3} = 0$$

Q. 31 The distance between the lines $y = mx + c_1$ and $y = mx + c_2$ is

(a) $\frac{c_1 - c_2}{\sqrt{m^2 + 1}}$
(b) $\frac{|c_1 - c_2|}{\sqrt{1 + m^2}}$
(c) $\frac{c_2 - c_1}{\sqrt{1 + m^2}}$
(d) 0

Sol. *(b)* Given, equation of the lines are

$$y = mx + c_1 \quad \dots(i)$$

and
$$y = mx + c_2 \quad \dots(ii)$$

$\therefore$ Distance between them is given by

$$d = \frac{|c_1 - c_2|}{\sqrt{1 + m^2}}$$

Q. 32 The coordinates of the foot of perpendiculars from the point (2, 3) on the line $y = 3x + 4$ is given by

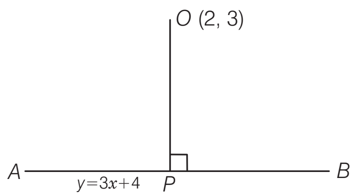
(a) $\left(\frac{37}{10}, \frac{-1}{10}\right)$ (b) $\left(-\frac{1}{10}, \frac{37}{10}\right)$ (c) $\left(\frac{10}{37}, -10\right)$ (d) $\left(\frac{2}{3}, -\frac{1}{3}\right)$

Sol. *(b)* Given, equation of the line is

$$y = 3x + 4 \quad \ldots(i)$$

$\therefore$ Slope of this line, $m_1 = 3$

So, the slope of line OP is $-\frac{1}{3}$. $[\because OP \perp AB]$



$\therefore$ Equation of line OP is

$$y - 3 = -\frac{1}{3}(x - 2)$$

$\Rightarrow$ $$3y - 9 = -x + 2$$

$\Rightarrow$ $$x + 3y - 11 = 0 \quad \ldots(ii)$$

Using the value of y from Eq. (i) in Eq. (ii), we get

$$x + 3(3x + 4) - 11 = 0$$

$\Rightarrow$ $$x + 9x + 12 - 11 = 0$$

$\Rightarrow$ $$10x + 1 = 0 \Rightarrow x = -\frac{1}{10}$$

Put $x = \frac{-1}{10}$ in Eq. (i), we get

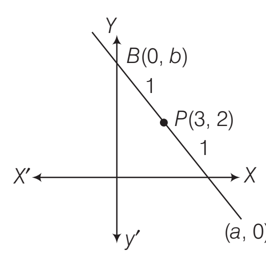
$$y = \frac{-3}{10} + 4 = \frac{-3 + 40}{10} = \frac{37}{10}$$

So, the foot of perpendicular is $\left(-\frac{1}{10}, \frac{37}{10}\right)$.

Q. 33 If the coordinates of the middle point of the portion of a line intercepted between the coordinate axes is (3, 2), then the equation of the line will be

(a) $2x + 3y = 12$ (b) $3x + 2y = 12$ (c) $4x - 3y = 6$ (d) $5x - 2y = 10$

Sol. *(a)* Since, the coordinates of the middle point are P(3, 2).



$\therefore$ $$3 = \frac{1 \cdot 0 + 1 \cdot a}{1 + 1}$$

$\Rightarrow$ $$3 = \frac{a}{2} \Rightarrow a = 6$$

Similarly, $b = 4$

$\therefore$ Equation of the line is $\frac{x}{6} + \frac{y}{4} = 1$

$\Rightarrow$ $$2x + 3y = 12$$

Q. 34 Equation of the line passing through (1, 2) and parallel to the line $y = 3x - 1$ is

(a) $y + 2 = x + 1$ (b) $y + 2 = 3(x + 1)$

(c) $y - 2 = 3(x - 1)$ (d) $y - 2 = x - 1$

Sol. *(c)* Since, the line passes through (1, 2) and parallel to the line $y = 3x - 1$.

So, slope of the required line $m = 3$. [$\because$ slope of $y = 3x - 1$ is 3]

Hence, the equation of line is

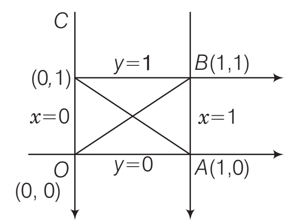
$$y - 2 = 3(x - 1)$$

Q. 35 Equations of diagonals of the square formed by the lines $x = 0$, $y = 0$, $x = 1$ and $y = 1$ are

(a) $y = x, y + x = 1$ (b) $y = x, x + y = 2$

(c) $2y = x, y + x = \frac{1}{3}$ (d) $y = 2x, y + 2x = 1$

Sol. *(a)* Equation of OB is



$$y - 0 = \frac{1-0}{1-0}(x - 0)$$

$\Rightarrow$ $y = x$

and equation of AC is

$$y - 0 = \frac{1-0}{0-1}(x - 1)$$

$\Rightarrow$ $y = -x + 1$

$\Rightarrow$ $x + y - 1 = 0$

Q. 36 For specifying a straight line, how many geometrical parameters should be known?

(a) 1 (b) 2 (c) 4 (d) 3

Sol. *(b)* Equation of straight lines are

$$y = mx + c, \text{ parameter} = 2 \quad \ldots\text{(i)}$$

$$\frac{x}{a} + \frac{y}{b} = 1, \text{ parameter} = 2 \quad \ldots\text{(ii)}$$

$$y - y_1 = m(x - x_1), \text{ parameter} = 2 \quad \ldots\text{(iii)}$$

and $$x\cos w + y\sin w = p, \text{ parameter} = 2 \quad \ldots\text{(iv)}$$

It is clear that from Eqs. (i), (ii), (iii) and (iv), for specifying a straight line clearly two parameters should be known.

Q. 37 The point (4, 1) undergoes the following two successive transformations

(i) Reflection about the line $y = x$

(ii) Translation through a distance 2 units along the positive X-axis.

Then, the final coordinates of the point are

(a) (4, 3) (b) (3, 4)

(c) (1, 4) (d) $\left(\frac{7}{2}, \frac{7}{2}\right)$

Sol. ***(b)*** Let the reflection of $A(4, 1)$ in $y = x$ is $B(h, k)$.

Now, mid-point of AB is $\left(\frac{4+h}{2}, \frac{1+k}{2}\right)$ which lies on $y = x$.

i.e., $$\frac{4+h}{2} = \frac{1+k}{2} \Rightarrow h - k = -3 \quad \text{...(i)}$$

So, the slope of line $y = x$ is 1.

$$\therefore \quad \text{Slope of } AB = \frac{h-4}{k-1}$$

$$\Rightarrow \quad 1 \cdot \left(\frac{h-4}{k-1}\right) = -1$$

$$\Rightarrow \quad h - 4 = 1 - k$$

$$\Rightarrow \quad h + k = 5 \quad \text{...(ii)}$$

$$\text{and} \quad h - k = -3$$

$$2h = 2 \Rightarrow h = 1$$

On putting $h = 1$ in Eq. (ii), we get

$$k = 4$$

So, the point is (1, 4).

Hence, after translation the point is (1 + 2, 4) or (3, 4).

Q. 38 A point equidistant from the lines $4x + 3y + 10 = 0$, $5x - 12y + 26 = 0$ and $7x + 24y - 50 = 0$ is

(a) (1, −1) (b) (1, 1) (c) (0, 0) (d) (0, 1)

Sol. ***(c)*** The given equation of lines are

$$4x + 3y + 10 = 0 \quad \text{...(i)}$$

$$\Rightarrow \quad 5x - 12y + 26 = 0 \quad \text{...(ii)}$$

$$\Rightarrow \quad 7x + 24y - 50 = 0 \quad \text{...(iii)}$$

Let the point (h, k) which is equidistant from these lines.

Distance from line (i) $= \frac{|4h + 3k + 10|}{\sqrt{16 + 9}}$

Distance from line (ii) $= \frac{|5h - 12k + 26|}{\sqrt{25 + 144}}$

Distance from the line (iii) $= \frac{|7h + 24k - 50|}{\sqrt{7^2 + 24^2}}$

So, the point (h, k) is equidistant from lines (i), (ii) and (iii).

$$\therefore \quad \frac{4h + 3k + 10}{\sqrt{16 + 9}} = \frac{5h - 12k + 26}{\sqrt{25 + 144}} = \frac{7h + 24k - 50}{\sqrt{49 + 576}}$$

$$\Rightarrow \quad \frac{|4h + 3k + 10|}{5} = \frac{|5h - 12k + 26|}{13} = \frac{|7h + 24k - 50|}{25}$$

Clearly, if $h = 0$, $k = 0$, then $\frac{10}{5} = \frac{26}{13} = \frac{50}{25} = 2$

Hence, the required point is (0, 0).

Q. 39 A line passes through (2, 2) and is perpendicular to the line $3x + y = 3$. Its y-intercept is

(a) $\frac{1}{3}$ (b) $\frac{2}{3}$ (c) 1 (d) $\frac{4}{3}$

Thinking Process

First of all find the equation of required line using the formulae. i.e., $y - y_1 = m(x - x_1)$ then put $x = 0$ to get y-intercept.

Sol. *(d)* Given line is $y = 3 - 3x$.

Then, slope of the required line $= \frac{1}{3}$

$\because$ Equation of the required line is

$$y - 2 = \frac{1}{3}(x - 2)$$

$$\Rightarrow \quad 3y - 6 = x - 2$$

$$\Rightarrow \quad x - 3y + 4 = 0$$

For y-intercept, put $x = 0$,

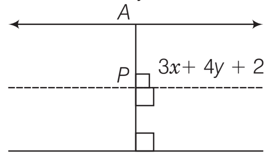
$$0 - 3y + 4 = 0$$

$$\Rightarrow \quad y = \frac{4}{3}$$

Q. 40 The ratio in which the line $3x + 4y + 2 = 0$ divides the distance between the lines $3x + 4y + 5 = 0$ and $3x + 4y - 5 = 0$ is

(a) 1 : 2 (b) 3 : 7 (c) 2 : 3 (d) 2 : 5

Sol. *(b)* Let point $A(x_1, y_1)$ lies on the line $3x + 4y + 5 = 0$, then $3x_1 + 4y_1 + 5 = 0$



Now, perpendicular distance from A to the line

$$3x + 4y + 2 = 0$$

$$\Rightarrow \quad \frac{|3x_1 + 4y_1 + 2|}{\sqrt{9 + 16}} = \frac{|-5 - 2|}{\sqrt{9 + 16}} = \frac{-7}{5}$$

Let point $B(x_2, y_2)$ lies on the line $3x + 4y - 5 = 0$ *i.e.,* $3x_2 + 4y_2 - 5 = 0$.

Now, perpendicular distance from B to the line $3x + 4y + 2 = 0$,

$$\frac{|3x_2 + 4y_2 + 2|}{\sqrt{9 + 16}} = \frac{|+ 5 - 2|}{\sqrt{9 + 16}} = \frac{3}{5}$$

Hence, the required ratio is $\frac{3}{5} : \frac{7}{5}$ *i.e.,* 3 : 7.

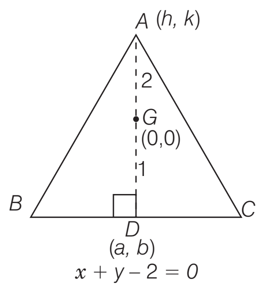
Q. 41 One vertex of the equilateral triangle with centroid at the origin and one side as $x + y - 2 = 0$ is

(a) (–1, –1)
(b) (2, 2)
(c) (–2, –2)
(d) (2, –2)

Thinking Process

Let ABC be the equilateral triangle with vertex A(h, k) and D (α, β) be the point on BC.
Then, $\frac{2\alpha + h}{3} = 0 = \frac{2\beta + k}{3}$. *Also,* $\alpha + \beta - 2 = 0$ *and* $\left(\frac{k-0}{h-0}\right)\cdot(-1) = -1$.

Sol. ***(c)*** Let *ABC* be the equilateral triangle with vertex *A*(*h*, *k*).
Let the coordinates of *D* are (α, β).



We know that, 2 : 1 from the vertex *A*.

$\therefore \quad 0 = \frac{2\alpha + h}{3}$ and $0 = \frac{2\beta + k}{3}$

$\Rightarrow \quad 2\alpha = -h$

and $\quad 2\beta = -k \quad \ldots(i)$

Also, $D(\alpha, \beta)$ lies on the line $x + y - 2 = 0$.

$\therefore \quad \alpha + \beta - 2 = 0 \quad \ldots(ii)$

$AD \perp BC$

Since, the slope of line *BC* *i.e.*, $m_{BC} = -1$

and slope of the line *AG* *i.e.*, $m_{AG} = \frac{k-0}{h-0} = \frac{k}{h}$

$\Rightarrow \quad (-1)\cdot\left(\frac{k}{h}\right) = -1$

$\Rightarrow \quad h = k \quad \ldots(iii)$

From Eqs. (i) and (iii),

$2\alpha = -h$ and $2\beta = -h$

$\therefore \quad \alpha = \beta$

From Eq. (ii), $\quad 2\alpha - 2 = 0 \Rightarrow \alpha = 1$

If $\alpha = 1$, then $\beta = 1$

From Eq. (i), $h = -2$, $k = -2$

So, the vertex *A* is (–2, –2).

Fillers

Q. 42 If a, b and c are in AP, then the straight lines $ax + by + c = 0$ will always pass through ....

Thinking Process

If a, b and c are in AP, then 2b = a + c. Use this property to solve the above problem.

Sol. Given line is $ax + by + c = 0$ ...(i)

Since, a, b and c are in AP, then

$$b = \frac{a+c}{2}$$

$$\Rightarrow \quad a - 2b + c = 0 \quad \text{...(ii)}$$

On comparing Eqs.(i) and (ii), we get

$$x = 1, y = 2 \quad \text{[using value of } b \text{ in Eq. (i)]}$$

So, $(1, -2)$ lies on the line.

Q. 43 The line which cuts off equal intercept from the axes and pass through the point (1, –2) is ....

Sol. Let equation of line is

$$\frac{x}{a} + \frac{y}{a} = 1 \quad \text{...(i)}$$

Since, this line passes through (1, –2).

$$\frac{1}{a} - \frac{2}{a} = 1$$

$$\Rightarrow \quad 1 - 2 = a \Rightarrow a = -1$$

∴ Required equation of the line is

$$-x - y = 1$$

$$\Rightarrow \quad x + y + 1 = 0$$

Q. 44 Equation of the line through thes point (3, 2) and making an angle of 45° with the line $x - 2y = 3$ are ....

Sol. Since, the given point $P(3, 2)$ and line is $x - 2y = 3$.

Slope of this line is $m_1 = \frac{1}{2}$

Let the slope of the required line is m.

Then, $$\tan\theta = \left|\frac{m - \frac{1}{2}}{1 + \frac{1}{2}m}\right|$$

$$\Rightarrow \quad 1 = \pm\left(\frac{m - \frac{1}{2}}{1 + \frac{m}{2}}\right) \quad [\because \tan 45^\circ = 1] \text{ ... (i)}$$

Taking positive sign,

$$1 + \frac{m}{2} = m - \frac{1}{2}$$

$$\Rightarrow \quad m - \frac{m}{2} = 1 + \frac{1}{2}$$

$$\Rightarrow \quad \frac{m}{2} = \frac{3}{2} \Rightarrow m = 3$$

Taking negative sign,

$$1 = -\left(\frac{m - \frac{1}{2}}{1 + \frac{m}{2}}\right)$$

$$\Rightarrow \quad 1 + \frac{m}{2} = -m + \frac{1}{2}$$

$$\Rightarrow \quad m + \frac{m}{2} = \frac{1}{2} - 1$$

$$\Rightarrow \quad \frac{3m}{2} = \frac{-1}{2} \Rightarrow m = \frac{-1}{3}$$

$\therefore$ First equation of the line is

$$y - 2 = 3(x - 3)$$

$$\Rightarrow \quad 3x - y - 7 = 0$$

and second equation of the line is

$$y - 2 = -\frac{1}{3}(x - 3)$$

$$\Rightarrow \quad 3y - 6 = -x + 3$$

$$\Rightarrow \quad x + 3y - 9 = 0$$

Q. 45 The points (3, 4) and (2, –6) are situated on the .... of the line $3x - 4y - 8 = 0$.

Sol. Given line is $\quad 3x - 4y - 8 = 0 \quad$ ...(i)

For point (3, 4), $\quad 9 - 4 \cdot 4 - 8$

$\Rightarrow \quad 9 - 16 - 8$

$\Rightarrow \quad 9 - 24$

$\Rightarrow \quad -15 < 0 \quad$ ...(i)

For point (2, –6),

$$6 + 24 - 8$$

$$22 > 0 \quad \text{...(ii)}$$

Since, the value are of opposite sign.

Hence, the points (3, 4) and (2, –6) lies on opposite side to the line.

Q. 46 A point moves so that square of its distance from the point (3, –2) is numerically equal to its distance from the line $5x - 12y = 3$. The equation of its locus is ....

Sol. Let the coordinaters of the point are (h, k),

$\therefore$ Distance between (3, – 2) and (h, k),

$$d_1^2 = (3 - h)^2 + (-2 - k)^2 \quad \text{...(i)}$$

Now, distance of the point (h, k) from the line $5x - 12y = 3$ is,

$$d_2 = \left|\frac{5h - 12k - 3}{\sqrt{25 + 144}}\right| = \left|\frac{5h - 12k - 3}{13}\right| \quad \text{...(ii)}$$

Given that, $d_1^2 = d_2$

$\Rightarrow$ $(3-h)^2 + (2+k)^2 = \dfrac{5h - 12k - 3}{13}$

$\Rightarrow$ $9 - 6h + h^2 + 4 + 4k + k^2 = \dfrac{5h - 12k - 3}{13}$

$\Rightarrow$ $h^2 + k^2 - 6h + 4k + 13 = \dfrac{5h - 12k - 3}{13}$

$\Rightarrow$ $13h^2 + 13k^2 - 78h + 52k + 169 = 5h - 12k - 3$

$\Rightarrow$ $13h^2 + 13k^2 - 83h + 64k + 172 = 0$

$\therefore$ Locus of this point is

$$13x^2 + 13y^2 - 83x + 64y + 172 = 0$$

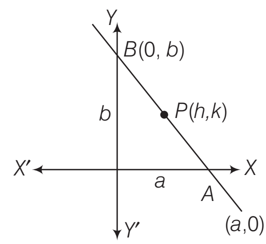
Q. 47 Locus of the mid-points of the portion of the line $x\sin\theta + y\cos\theta = p$ intercepted between the axes is ....

Sol. Given equation of the line is

$$x\sin\theta + y\cos\theta = p \quad \ldots(i)$$

Let the mid-point of AB is $p(h, k)$.

So, the mid-point of AB are $\left(\dfrac{a}{2}, \dfrac{b}{2}\right)$.



Since, the point $(a, 0)$ lies on the line (i), then

$$a\sin\theta + 0 = p$$

$\Rightarrow$ $a\sin\theta = p \Rightarrow a = \dfrac{p}{\sin\theta}$

and the point $(0, b)$ also lies on the line, then

$$0 + b\cos\theta = p$$

$\Rightarrow$ $b\cos\theta = p \Rightarrow b = \dfrac{p}{\cos\theta}$

Now, mid-point of $AB = \left(\dfrac{a}{2}, \dfrac{b}{2}\right)$ or $\left(\dfrac{p}{2\sin\theta}, \dfrac{p}{2\cos\theta}\right)$

$\because$ $\dfrac{p}{2\sin\theta} = h \Rightarrow \sin\theta = \dfrac{p}{2h}$

and $\dfrac{p}{2\cos\theta} = k \Rightarrow \cos\theta = \dfrac{p}{2k}$

$\therefore$ $\sin^2\theta + \cos^2\theta = \dfrac{p^2}{4h^2} + \dfrac{p^2}{4k^2}$

$\Rightarrow$ $1 = \dfrac{p^2}{4}\left(\dfrac{1}{h^2} + \dfrac{1}{k^2}\right)$

Locus of the mid-point is

$$4 = p^2\left(\frac{1}{x^2} + \frac{1}{y^2}\right)$$

$\Rightarrow$ $4x^2y^2 = p^2(x^2 + y^2)$

True/False

Q. 48 If the vertices of a triangle have integral coordinates, then the triangle cannot be equilateral.

Sol. ***True***

We know that, if the vertices of a triangle have integral coordinates, then the triangle cannot be equilateral. Hence, the given statement is true.

Since, in equilatteral triangle, we get $\tan 60° = \sqrt{3}$ = Slope of the line, so with integral coordinates as vertices, the triangle cannot be equilateral.

Q. 49 The points $A(-2, 1)$, $B(0, 5)$ and $C(-1, 2)$ are collinear.

Sol. ***False***

Given points are $A(-2, 1)$, $B(0, 5)$ and $C(1, 2)$.

Now, slope of $AB = \dfrac{5-1}{0+2} = 2$

Slope of $BC = \dfrac{2-5}{-1-0} = 3$

Slope of $AC = \dfrac{2-1}{-1+2} = 1$

Since, the slopes are different.

Hence, A, B and C are not collinear. So, statement is false.

Q. 50 Equation of the line passing through the point $(a\cos^3\theta, a\sin^3\theta)$ and perpendicular to the line $x\sec\theta + y\ \text{cosec}\ \theta = a$ is $x\cos\theta - y\sin\theta = a\sin 2\theta$.

Sol. ***False***

Given point $p\,(a\cos^3\theta, a\sin^3\theta)$ and the line is $x\sec\theta + y\,\text{cosec}\,\theta = a$

$\because$ Slope of this line $= \dfrac{-\sec\theta}{\text{cosec}\,\theta} = -\tan\theta$

and slope of required line $= \dfrac{1}{\tan\theta} = \cot\theta$

$\therefore$ Equation of the required line is

$$y - a\sin^3\theta = \cot\theta\,(x - a\cos^3\theta)$$

$\Rightarrow$ $y\sin\theta - a\sin^4\theta = x\cos\theta - a\cos^4\theta$

$\Rightarrow$ $x\cos\theta - y\sin\theta = a\cos^4\theta - a\sin^4\theta$

$\Rightarrow$ $x\cos\theta - y\sin\theta = a\,[(\cos^2\theta + \sin^2\theta)(\cos^2\theta - \sin^2\theta)]$

$\Rightarrow$ $x\cos\theta - y\sin\theta = a\cos^2\theta$

Hence, the given statement is false.

Q. 51 The straight line $5x + 4y = 0$ passes through the point of intersection of the straight lines $x + 2y - 10 = 0$ and $2x + y + 5 = 0$.

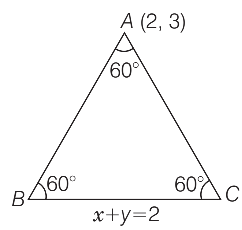
Sol. ***True***

Given that, $$x + 2y - 10 = 0 \quad \ldots(i)$$
and $$2x + y + 5 = 0 \quad \ldots(ii)$$
From Eq. (i), put the value of $x = 10 - 2y$ in Eq. (ii), we get
$$20 - 4y + y + 5 = 0$$
$$\Rightarrow \quad 20 - 3y + 5 = 0$$
$$\Rightarrow \quad y = \frac{25}{3}$$
$$\therefore \quad x + \frac{50}{3} - 10 = 0 \quad \text{[using Eq. (i)]}$$
$$\Rightarrow \quad x + \frac{20}{3} = 0 \Rightarrow x = \frac{-20}{3}$$
So, the point of intersection is $\left(-\frac{20}{3}, \frac{25}{3}\right)$.

If the line $5x + 4y = 0$ passes through the point $\left(-\frac{20}{3}, \frac{25}{3}\right)$, then this point should lie on this line.
$$\because \quad 5\left(\frac{-20}{3}\right) + \frac{4(25)}{3} = \frac{-100}{3} + \frac{100}{3} = 0$$
So, this point lies on the given line.
Hence, the statement is true.

Q. 52 The vertex of an equilateral triangle is (2, 3) and the equation of the opposite side is $x + y = 2$. Then, the other two sides are $y - 3 = (2 \pm \sqrt{3})(x - 2)$.

Sol. ***True***



Let ABC be an equilateral triangle with vertex $A(2, 3)$, and equation of BC is $x + y = 2$. *i.e.,* slope $= -1$.
Let slope of line AB is m.
Since, the angle between line AB and BC is 60°.
$$\therefore \quad \tan 60° = \left|\frac{m + 1}{1 - m}\right|$$
$$\Rightarrow \quad \sqrt{3} = \pm\left(\frac{m + 1}{1 - m}\right) \quad \text{[taking positive sign]}$$
$$\Rightarrow \quad \sqrt{3} - \sqrt{3}\, m = m + 1$$
$$\Rightarrow \quad \sqrt{3} - 1 = m + \sqrt{3} m$$
$$\Rightarrow \quad \sqrt{3} - 1 = m(1 + \sqrt{3})$$
$$\therefore \quad m = \frac{(\sqrt{3} - 1)(\sqrt{3} - 1)}{(\sqrt{3} + 1)(\sqrt{3} - 1)}$$
$$= \frac{3 + 1 - 2\sqrt{3}}{3 - 1} = \frac{4 - 2\sqrt{3}}{2} = 2 - \sqrt{3}$$
Similarly, slope of $AB = 2 + \sqrt{3}$ [taking negative sign]
$\therefore$ Equation of other two side is
$$y - 3 = (2 \pm \sqrt{3})(x - 2)$$
Hence, the statement is true.

Q. 53 The equation of the line joining the point (3, 5) to the point of intersection of the lines $4x + y - 1 = 0$ and $7x - 3y - 35 = 0$ is equidistant from the points (0, 0) and (8, 34).

Thinking Process

Equation of a line passing through the points (x_1, y_1) and (x_2, y_2) is $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$.

Sol. ***True***

Given equation of lines are

$$4x + y - 1 = 0 \quad \dots(i)$$

and

$$7x - 3y - 35 = 0 \quad \dots(ii)$$

From Eq. (i), on putting $y = 1 - 4x$ in Eq. (ii), we get

$$7x - 3 + 12x - 35 = 0$$

$$\Rightarrow \quad 19x - 38 = 0 \Rightarrow x = 2$$

On putting $x = 2$ in Eq. (i), we get

$$8 + y - 1 = 0 \Rightarrow y = -7$$

Now, the equation of a line passing through (3, 5) and (2, –7) is

$$y - 5 = \frac{-7 - 5}{2 - 3}(x - 3)$$

$$\Rightarrow \quad y - 5 = 12(x - 3)$$

$$\Rightarrow \quad 12x - y - 31 = 0 \quad \dots(iii)$$

Distance from (0, 0) to the line (iii),

$$d_1 = \frac{|-31|}{\sqrt{144 + 1}} = \frac{31}{\sqrt{145}}$$

$\therefore$ Distance from (8, 34) to the line (iii),

$$d_2 = \frac{|96 - 34 - 31|}{\sqrt{145}} = \frac{31}{\sqrt{145}}$$

$$\because \quad d_1 = d_2$$

Hence, the statement is true.

Q. 54 The line $\frac{x}{a} + \frac{y}{b} = 1$ moves in such a way that $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}$, where c is a constant. The locus of the foot of the perpendicular from the origin on the given line is $x^2 + y^2 = c^2$.

Sol. ***True***

Given that, equation of line is

$$\frac{x}{a} + \frac{y}{b} = 1 \quad \dots(i)$$

Equation of line passing through origin and perpendicular to line (i) is

$$\frac{x}{b} - \frac{y}{a} = 0 \quad \dots(ii)$$

Now, foot of perpendicular is the point of intersection of lines (i) and (ii). To find its locus we have to eliminate the variable a and b.

On squaring and adding Eqs. (i) and (ii), we get

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{2xy}{ab}+\frac{x^2}{b^2}+\frac{y^2}{a^2}-\frac{2xy}{ab}=1$$

$$\Rightarrow \quad x^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)+y^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)=1$$

$$\Rightarrow \quad \frac{x^2}{c^2}+\frac{y^2}{c^2}=1$$

$$\Rightarrow \quad x^2+y^2=c^2 \qquad \left[\because \frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{c^2}\right]$$

Hence, the statement is true.

Q. 55 The lines $ax+2y+1=0$, $bx+3y+1=0$ and $cx+4y+1=0$ are concurrent, if a, b and c are in GP.

Thinking Process

First of all find the intersection point of first two line. Then, if the lines are concurrent then this point should lies on the third line.

Sol. ***False***

Given lines are

$$ax+2y+1=0 \qquad \ldots(\text{i})$$

and

$$bx+3y+1=0 \qquad \ldots(\text{ii})$$

From Eq. (i), on putting $y=\frac{-ax-1}{2}$ in Eq. (ii), we get

$$bx-\frac{3}{2}(ax+1)+1=0$$

$$\Rightarrow \quad 2bx-3ax-3+2=0$$

$$\Rightarrow \quad x(2b-3a)=1 \Rightarrow x=\frac{1}{2b-3a}$$

Now, using $x=\frac{1}{2b-3a}$ in Eq. (i), we get

$$\frac{a}{2b-3a}+2y+1=0$$

$$\Rightarrow \quad 2y=-\left[\frac{a+2b-3a}{2b-3a}\right]$$

$$\Rightarrow \quad 2y=\frac{-(2b-2a)}{2b-3a}$$

$$\Rightarrow \quad y=\frac{(a-b)}{2b-3a}$$

So, the point of intersection is $\left(\frac{1}{2b-3a}, \frac{a-b}{2b-3a}\right)$.

Since, this point lies on $cx+4y+1=0$, then

$$\frac{c}{2b-3a}+\frac{4(a-b)}{2b-3a}+1=0$$

$$\Rightarrow \quad c+4a-4b+2b-3a=0$$

$$\Rightarrow \quad -2b+a+c=0 \Rightarrow 2b=a+c$$

Hence, the given statement is false.

Q. 56 Line joining the points (3, −4) and (−2, 6) is perpendicular to the line joining the points (−3, 6) and (9, −18).

Sol. ***False***

Given points are $A(3, -4)$, $B(-2, 6)$, $P(-3, 6)$ and $Q(9, -18)$.

Now, slope of $AB = \frac{6+4}{-2-3} = -2$

and slope of $PQ = \frac{-18-6}{9+3} = -2$

So, line AB is parallel to line PQ.

Matching The Columns

Q. 57 Match the following.

	Column I		Column II
(i)	The coordinates of the points P and Q on the line $x + 5y = 13$ which are at a distance of 2 units from the line $12x - 5y + 26 = 0$ are	(a)	$(3, 1), (-7, 11)$
(ii)	The coordinates of the point on the line $x + y = 4$, which are at a unit distance from the line $4x + 3y - 10 = 0$ are	(b)	$\left(-\frac{1}{3}, \frac{11}{3}\right), \left(\frac{4}{3}, \frac{7}{3}\right)$
(iii)	The coordinates of the point on the line joining $A(-2, 5)$ and $B(3, 1)$ such that $AP = PQ = QB$ are	(c)	$\left(1, \frac{12}{5}\right), \left(-3, \frac{16}{5}\right)$

Sol. **(i)** Let the coordinate of point $P(x_1, y_1)$ on the line $x + 5y = 13$ *i.e.*,

$$P(13 - 5y_1, y_1).$$

$\therefore$ Distance of P from the line $12x - 5y + 26 = 0$,

$$2 = \left|\frac{12(13 - 5y_1) - 5y_1 + 26}{\sqrt{144 + 25}}\right|$$

$$\Rightarrow \quad 2 = \pm \frac{156 - 60y_1 - 5y_1 + 26}{13}$$

$$\Rightarrow \quad -65y_1 = -156 \quad \text{[taking positive sign]}$$

$$\Rightarrow \quad y_1 = \frac{156}{65} = \frac{12}{5}$$

$$\because \quad x_1 = 13 - 5y_1$$

$$= 13 - 12 = 1$$

So, the coordinate of is $P\left(1, \frac{12}{5}\right)$.

Similarly, the coordinates of Q are $\left(-3, \frac{16}{5}\right)$. [taking negative sign]

(ii) Let coordinates of the point on the line $x + y = 4$ be $(4 - y_1, y_1)$.

Distance from the line $4x + 3y - 10 = 0$.

$$1 = \left| \frac{4(4 - y_1) + 3y_1 - 10}{\sqrt{16 + 9}} \right|$$

$$\Rightarrow \quad 1 = \pm \frac{16 - 4y_1 + 3y_1 - 10}{5} \quad \text{[taking negative sign]}$$

$$\Rightarrow \quad 5 = 6 - y_1$$

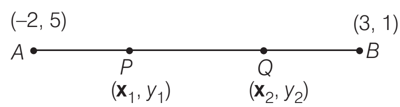
$$\Rightarrow \quad y_1 = 1$$

If $y_1 = 1$, then $x_1 = 3$

So, the point is (3, 1).

Similarly, taking negative sign the point is (–7, 11).

(iii) Given point $A(-2, 5)$ and $B(3, 1)$.



Now, the point P divides line joining the point A and B in 1 : 2.

$$\because \quad x_1 = \frac{1 \cdot 3 + 2(-2)}{1 + 2} = \frac{3 - 4}{3} = \frac{-1}{3}$$

and
$$y_1 = \frac{1 \cdot 1 + 2 \cdot 5}{1 + 2} = \frac{11}{3}$$

So, the coordinates of P are $\left(\frac{-1}{3}, \frac{11}{3}\right)$.

Thus, the point Q divided the line joining A to B in 2 : 1.

$$\because \quad x_2 = \frac{2 \cdot 3 + 1(-2)}{2 + 1} = \frac{4}{3}$$

and
$$y_2 = \frac{2 \cdot 1 + 1 \cdot 5}{2 + 1} = \frac{7}{3}$$

Hence, the coordinates of Q are $\left(\frac{4}{3}, \frac{7}{3}\right)$.

Hence, the correct matches are **(i)** → (c), **(ii)** → (a), **(iii)** → (b).

Q. 58 The value of the λ, if the lines $(2 + 3y + 4) + \lambda(6x - y + 12) = 0$ are

Column I		Column II	
(i)	parallel to Y-axis is	(a)	$\lambda = -\frac{3}{4}$
(ii)	perpendicular to $7x + y - 4 = 0$ is	(b)	$\lambda = -\frac{1}{3}$
(iii)	passes through (1, 2) is	(c)	$\lambda = -\frac{17}{41}$
(iv)	parallel to X-axis is	(d)	$\lambda = 3$

Sol. (i) Given equation of the line is

$$(2x + 3y + 4) + \lambda(6x - y + 12) = 0 \quad \ldots(i)$$

If line is parallel to Y-axis *i.e.,* it is perpendicular to X-axis

$$\therefore \quad \text{Slope} = m = \tan 90° = \infty$$

From line (i), $x(2+6\lambda)+y(3-\lambda)+4+12\lambda=0$

and slope $= \dfrac{-(2+6\lambda)}{3-\lambda}$

$$\Rightarrow \quad \frac{-2-6\lambda}{3-\lambda}=\infty$$

$$\Rightarrow \quad \frac{-2-6\lambda}{3-\lambda}=\frac{1}{0} \Rightarrow \lambda = 3$$

(ii) If the line (i) is perpendicular to the line $7x+y-4=0$ or $y=-7x+4$

$$\because \quad \frac{-(2+6\lambda)}{(3-\lambda)}(-7)=-1$$

$$\Rightarrow \quad 14+42\lambda=-3+\lambda$$

$$\Rightarrow \quad 41\lambda=-17$$

$$\Rightarrow \quad \lambda=-\frac{17}{41}$$

(iii) If the line (i) passes through the point (1, 2).

Then, $(2+6+4)+\lambda(6-2+12)=0$

$$\Rightarrow \quad 12+16\lambda=0 \Rightarrow \lambda=-\frac{3}{4}$$

(iv) If the line is parallel to X-axis the slope = 0.

Then, $\dfrac{-(2+6\lambda)}{3-\lambda}=0$

$$\Rightarrow \quad -(2+6\lambda)=0 \quad \Rightarrow \quad \lambda=-\frac{1}{3}$$

So, the correct matches are **(i)** → (d), **(ii)** → (c), **(iii)** → (a), **(iv)** → (b).

Q. 59 The equation of the line through the intersection of the lines $2x-3y=0$ and $4x-5y=2$ and

	Column I		Column II
(i)	through the point (2, 1) is	(a)	$2x-y=4$
(ii)	perpendicular to the line $x+2y+1=0$	(b)	$x+y-5=0$
(iii)	parallel to the line $3x-4y+5=0$ is	(c)	$x-y-1=0$
(iv)	equally inclined to the axes is	(d)	$3x-4y-1=0$

Sol. Given equation of the lines are

$$2x-3y=0 \quad \ldots(i)$$

and

$$4x-5y=2 \quad \ldots(ii)$$

From Eq. (i), put $x=\dfrac{3y}{2}$ in Eq. (ii), we get

$$4\cdot\frac{(3y)}{2}-5y=2$$

$$\Rightarrow \quad 6y-5y=2$$

$$\Rightarrow \quad y=2$$

Now, put $y=2$ in Eq. (i), we get

$$x=3$$

So, the intersection points are (3, 2).

(i) The equation of the line passes through the point (3, 2) and (2, 1), is

$$y - 2 = \frac{1-2}{2-3}(x - 3)$$

$\Rightarrow$ $$y - 2 = (x - 3)$$

$\Rightarrow$ $$x - y - 1 = 0$$

(ii) If the required line is perpendicular to the line $x + 2y + 1 = 0$

$\because$ Slope of the required line = 2

$\therefore$ Equation of the line is

$$y - 2 = 2(x - 3)$$

$\Rightarrow$ $$2x - y - 4 = 0$$

(iii) If the required line is parallel to the line $3x - 4y + 5 = 0$, then the slope of the required line $= \frac{3}{4}$

$\therefore$ Equation of the required line is

$$y - 2 = \frac{3}{4}(x - 3)$$

$\Rightarrow$ $$4y - 8 = 3x - 9$$

$\Rightarrow$ $$3x - 4y - 1 = 0$$

(iv) If the line is equally inclined to the *X*-axis, then

$$m = \pm \tan 45° = \pm 1$$

$\therefore$ Equation of the line is

$$y - 2 = -1(x - 3) \quad \text{[taking negative value]}$$

$\Rightarrow$ $$y - 2 = -x + 3$$

$\Rightarrow$ $$x + y - 5 = 0$$

So, the correct matches are **(a)** $\rightarrow$ (iii), **(b)** $\rightarrow$ (i), **(c)** $\rightarrow$ (iv), **(d)** $\rightarrow$ (ii).

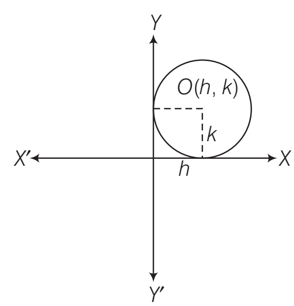
11

Conic Sections

Short Answer Type Questions

Q. 1 Find the equation of the circle which touches the both axes in first quadrant and whose radius is a.

Sol. Given that radius of the circle is a *i.e.*, $(h, k) = (a, a)$



So, the equation of required circle is

$$(x - a)^2 + (y - a)^2 = a^2$$

$$\Rightarrow \quad x^2 - 2ax + a^2 + y^2 - 2ay + a^2 = a^2$$

$$\Rightarrow \quad x^2 + y^2 - 2ax - 2ay + a^2 = 0$$

Q. 2 Show that the point (x, y) given by $x = \dfrac{2at}{1 + t^2}$ and $y = \dfrac{a(1 - t^2)}{1 + t^2}$ lies on a circle.

Sol. Given points are $x = \dfrac{2at}{1+t^2}$ and $y = \dfrac{a(1-t^2)}{1+t^2}$.

$$\because \quad x^2 + y^2 = \frac{4a^2t^2}{(1+t^2)^2} + \frac{a^2(1-t^2)^2}{(1+t^2)^2}$$

$$\Rightarrow \quad \frac{1}{a^2}(x^2 + y^2) = \frac{4t^2 + 1 + t^4 - 2t^2}{(1+t^2)^2}$$

$$\Rightarrow \quad \frac{1}{a^2}(x^2 + y^2) = \frac{t^4 + 2t^2 + 1}{(1+t^2)^2}$$

$$\Rightarrow \quad \frac{1}{a^2}(x^2 + y^2) = \frac{(1+t^2)^2}{(1+t^2)^2}$$

$\Rightarrow \quad x^2 + y^2 = a^2$, which is a required circle.

Q. 3 If a circle passes through the points (0, 0), (a, 0) and (0, b), then find the coordinates of its centre.

Thinking Process

General equation of the circle passing through the origin is $x^2 + y^2 + 2yx + 2fy = 0$.
Now, satisfied the given points to get the values of g and f. The centre of the circle is $(-g, -f)$.

Sol. Let the equation of circle is

$$x^2 + y^2 + 2gx + 2fy = 0 \quad \text{... (i)}$$

Since, this circle passes through the points A (0, 0), B (a, 0) and C (0, b).

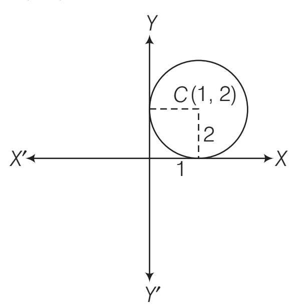
$\therefore \quad a^2 + 2ag = 0$ ... (ii)

and $\quad b^2 + 2bf = 0$ ... (iii)

From Eq. (ii), $a + 2g = 0 \Rightarrow g = -a/2$
From Eq. (iii), $b + 2f = 0 \Rightarrow f = -b/2$
Hence, the coordinates of the circle are $\left(\frac{a}{2}, \frac{b}{2}\right)$.

Q. 4 Find the equation of the circle which touches X-axis and whose centre is (1, 2).

Sol. Given that, centre of the circle is (1, 2).



$\because \quad$ Radius = 2

So, the equation of circle is

$$(x-1)^2 + (y-2)^2 = 2^2$$

$\Rightarrow \quad x^2 - 2x + 1 + y^2 - 4y + 4 = 4$

$\Rightarrow \quad x^2 - 2x + y^2 - 4y + 1 = 0$

$\Rightarrow \quad x^2 + y^2 - 2x - 4y + 1 = 0$

Q. 5 If the lines $3x + 4y + 4 = 0$ and $6x - 8y - 7 = 0$ are tangents to a circle, then find the radius of the circle.

Thinking Process

The distance between two parallel lines $ax + by + c_1 = 0$ and $ax + by + c_2 = 0$ is given by,

i.e., $d = \left|\dfrac{c_1 - c_2}{\sqrt{a^2 + b^2}}\right|$. *Use this formula to solve the above problem.*

Sol. Given lines,

$$3x - 4y + 4 = 0 \quad \text{.. (i)}$$
$$6x - 8y - 7 = 0$$

or $$3x - 4y - 7/2 = 0 \quad \text{...(ii)}$$

It is clear that lines (i) and (ii) parallel.

Now, distance between them *i.e.*,

$$d = \left|\frac{4 + 7/2}{\sqrt{9 + 16}}\right| = \left|\frac{\frac{8+7}{2}}{5}\right| = 3/2$$

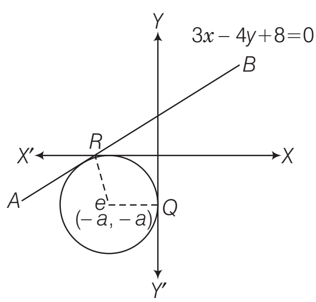
∴ Distance between these line = Diameter of these circle

∴ Diameter of the circle = 3 / 2

and radius of the circle = 3 / 4

Q. 6 Find the equation of a circle which touches both the axes and the line $3x - 4y + 8 = 0$ and lies in the third quadrant.

Sol.



Let a be the radius of the circle. Then, the coordinates of the circle are $(-a, -a)$. Now, perpendicular distance from C to the line AB = Radius of the circle

$$d = \left|\frac{-3a + 4a + 8}{\sqrt{9 + 16}}\right| = \left|\frac{a + 8}{5}\right|$$

∵ $$a = \pm\left(\frac{a + 8}{5}\right)$$

Taking positive sign, $$a = \frac{a + 8}{5}$$

⇒ $$5a = a + 8$$

⇒ $$4a = 8 \Rightarrow a = 2$$

Taking negative sign, $$a = \frac{-a - 8}{5}$$

⇒ $$5a = -a - 8$$

⇒ $$6a = -8 \Rightarrow a = -4/3$$

But $\quad a \neq -4/3$

$\because \quad a = 2$

So, the equation of circle is

$$(x+2)^2 + (y+2)^2 = 2^2 \qquad [\because a = 2]$$

$$\Rightarrow \quad x^2 + 4x + 4 + y^2 + 4y + 4 = 4$$

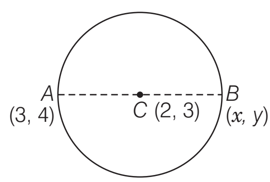
$$\Rightarrow \quad x^2 + y^2 + 4x + 4y + 4 = 0$$

Q. 7 If one end of a diameter of the circle $x^2 + y^2 - 4x - 6y + 11 = 0$ is (3, 4), then find the coordinates of the other end of the diameter.

Thinking Process

First of all get the centre of the circle from the given equation, then find the mid-point of the diameter of the circle.

Sol. Given equation of the circle is



$$x^2 + y^2 - 4x - 6y + 11 = 0.$$

$\therefore \quad 2g = -4$ and $2f = -6$

So, the centre of the circle is $(-g, -f)$ *i.e.*, (2, 3).

Since, the mid-point of *AB* is (2, 3).

Then, $\quad 2 = \dfrac{3 + x_1}{2}$

$\Rightarrow \quad 4 = 3 + x_1$

$\therefore \quad x_1 = 1$

and $\quad 3 = \dfrac{4 + y_1}{2}$

$\Rightarrow \quad 6 = 4 + y_1 \Rightarrow y_1 = 2$

So, the coordinates of other end of the diameter will be (1, 2).

Q. 8 Find the equation of the circle having (1, – 2) as its centre and passing through $3x + y = 14$, $2x + 5y = 18$.

Sol. Given that, centre of the circle is (1, – 2) and the circle passing through the lines

$$3x + y = 14 \qquad \text{... (i)}$$

and $\quad 2x + 5y = 18 \qquad \text{...(ii)}$

From Eq. (i) $y = 14 - 3x$ put in Eq. (ii), we get

$$2x + 70 - 15x = 18$$

$\Rightarrow \quad -13x = -52 \Rightarrow x = 4$

Now, $x = 4$ put in Eq. (i), we get

$$12 + y = 14 \Rightarrow y = 2$$

Since, point (4, 2) lie on these lines also lies on the circle.

$\therefore \quad$ Radius of the circle $= \sqrt{(4-1)^2 + (2+2)^2}$

$$= \sqrt{9 + 16} = 5$$

Now, equation of the circle is

$$(x-1)^2+(y+2)^2=5^2$$

$$\Rightarrow \quad x^2-2x+1+y^2+4y+4=25$$

$$\Rightarrow \quad x^2+y^2-2x+4y-20=0$$

Q. 9 If the line $y=\sqrt{3}x+k$ touches the circle $x^2+y^2=16$, then find the value of k.

Sol. Given equation of circle,

$$x^2+y^2=16$$

$\therefore$ Radius = 4 and centre = (0, 0)

Now, perpendicular from (0, 0) to line $y=\sqrt{3}\,x+k=$ Radius of the circle

$$\left|\frac{0-0+k}{\sqrt{3+1}}\right|=4$$

Since the distance from the point (m, n) to the line $Ax+By+k=0$ is $d=\left|\frac{Am+Bn+C}{A^2+B^2}\right|$

$$\Rightarrow \quad \pm\frac{k}{2}=4$$

$$\therefore \quad k=\pm 8$$

Q. 10 Find the equation of a circle concentric with the circle $x^2+y^2-6x+12y+15=0$ and has double of its area.

Sol. Given equation of the circle is

$$x^2+y^2-6x+12y+15=0 \quad \text{...(i)}$$

$$\therefore \quad 2g=-6 \Rightarrow g=-3$$

$$2f=12 \Rightarrow f=6$$

and $\quad c=15$

$$\therefore \quad \text{Centre}=(-g,-f)=(3,-6)$$

So, the centre of the required circle will be (3, – 6). [since, the circles are concentric]

Radius of the given circle

$$=\sqrt{g^2+f^2-c}$$

$$=\sqrt{9+36-15}=\sqrt{30}$$

Let radius of the required circle = r_1

$\therefore$ 2 × Area of the given circle = Area of the required circle

$$\Rightarrow \quad 2[\pi(\sqrt{30})^2]=\pi r_1^2$$

$$\Rightarrow \quad 60=r_1^2$$

$$\Rightarrow \quad r_1=\sqrt{60}$$

$$\therefore \quad \sqrt{g^2+f^2-c}=\sqrt{60}$$

$$\Rightarrow \quad 9+36-c=60$$

$$\Rightarrow \quad c=-15$$

So, the required equation of circle is $x^2+y^2-6x+12y-15=0$.

Q. 11 If the latusrectum of an ellipse is equal to half of minor axis, then find its eccentricity.

Sol. Consider the equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

$\therefore$ Length of major axis $=2a$

Length of minor axis $=2b$

and length of latusrectum $=\frac{2b^2}{a}$

Given that, $\frac{2b^2}{a}=\frac{2b}{2}$

$\Rightarrow$ $a=2b \Rightarrow b=a/2$

We know that, $b^2=a^2(1-e^2)$

$\Rightarrow$ $\left(\frac{a}{2}\right)^2=a^2(1-e^2)$

$\Rightarrow$ $\frac{a^2}{4}=a^2(1-e^2)$

$\Rightarrow$ $1-e^2=\frac{1}{4}$

$\Rightarrow$ $e^2=1-\frac{1}{4}$

$\therefore$ $e=\sqrt{\frac{3}{4}}=\sqrt{\frac{3}{2}}$

Q. 12 If the ellipse with equation $9x^2+25y^2=225$, then find the eccentricity and foci.

Thinking Process

Find the values of a and b by the given equation of ellipse, then use the formula $b^2=a^2(1-e^2)$ to get the value of e.

Sol. Given equation of ellipse, $9x^2+25y^2=225$

$\Rightarrow$ $\frac{x^2}{25}+\frac{y^2}{9}=1$

$\Rightarrow$ $a=5, b=3$

We know that, $b^2=a^2(1-e^2)$

$\Rightarrow$ $9=25(1-e^2)$

$\Rightarrow$ $\frac{9}{25}=1-e^2$

$\Rightarrow$ $e^2=1-9/25$

$\therefore$ $e=\sqrt{1-9/25}=\sqrt{\frac{25-9}{25}}$

$=\sqrt{\frac{16}{25}}=4/5$

Foci $=(\pm\, ae, 0)=(\pm\, 5\times 4/5, 0)=(\pm\, 4, 0)$

Q. 13 If the eccentricity of an ellipse is $\frac{5}{8}$ and the distance between its foci is 10, then find latusrectum of the ellipse.

Sol. Given that, eccentricity $= \frac{5}{8}$, *i.e.*, $e = \frac{5}{8}$

Let equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$,

Since the foci of this ellipse is $(\pm ae, 0)$.

$\therefore$ Distance between foci $= \sqrt{(ae + ae)^2}$

$\Rightarrow 2\sqrt{a^2e^2} = 10$ [$\because$ distance between its foci = 10]

$\Rightarrow \sqrt{a^2e^2} = 5$

$\Rightarrow a^2e^2 = 25$

$\Rightarrow a^2 = \frac{25 \times 64}{25}$

$\therefore a = 8$

We know that,

$\Rightarrow b^2 = a^2 (1 - e^2)$

$\Rightarrow b^2 = 64\left(1 - \frac{25}{64}\right)$

$\Rightarrow b^2 = 64\left(\frac{64 - 25}{64}\right)$

$b^2 = 39$

$\therefore$ Length of latusrectum of ellipse $= \frac{2b^2}{a} = 2\left(\frac{39}{8}\right) = \frac{39}{4}$

Q. 14 Find the equation of ellipse whose eccentricity is $\frac{2}{3}$, latusrectum is 5 and the centre is (0, 0).

Thinking Process

First of all find the values of a and b using the formula $b^2 = a^2 (1 - e^2)$, then get the equation of the ellipse.

Sol. Given that, $e = 2/3$ and latusrectum = 5

i.e., $\frac{2b^2}{a} = 5 \Rightarrow b^2 = \frac{5a}{2}$

We know that, $b^2 = a^2 (1 - e^2)$

$\Rightarrow \frac{5a}{2} = a^2\left(1 - \frac{4}{9}\right)$

$\Rightarrow \frac{5}{2} = \frac{5a}{9} \Rightarrow a = 9/2 \Rightarrow a^2 = \frac{81}{4}$

$\Rightarrow b^2 = \frac{5 \times 9}{2 \times 2} = \frac{45}{4}$

So, the required equation of the ellipse is $\frac{4x^2}{81} + \frac{4y^2}{45} = 1$.

Q. 15 Find the distance between the directrices of ellipse $\frac{x^2}{36}+\frac{y^2}{20}=1.$

Sol. The equation of ellipse is $\frac{x^2}{36}+\frac{y^2}{20}=1.$

On comparing this equation with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we get

$$a=6,\ b=2\sqrt{5}$$

We know that, $b^2=a^2(1-e^2)$

$\Rightarrow$ $20=36(1-e^2)$

$\Rightarrow$ $\frac{20}{36}=1-e^2$

$\therefore$ $e=\sqrt{1-\frac{20}{36}}=\sqrt{\frac{16}{36}}$

$$E=\frac{4}{6}=\frac{2}{3}$$

Now, directrices $=\left(+\frac{a}{e},-a/e\right)$

$\therefore$ $\frac{a}{e}=\frac{6}{\frac{2}{3}}=\frac{6\times 3}{2}=9$

and $-\frac{a}{e}=-9$

$\therefore$ Distance between the directrices $=|9-(-9)|=18$

Q. 16 Find the coordinates of a point on the parabola $y^2=8x$, whose focal distance is 4.

Thinking Process

The distance of a point (h, k) from the focus S is called the focal distance of the point P. The focal distance of any point P (h, k) on the parabola $y^2=4ax$ is $|h+a|$.

Sol. Given parabola is $y^2=8x$ ... (i)

On comparing this parabola to the $y^2=4ax$, we get

$$8x=4ax \Rightarrow a=2$$

$\therefore$ Focal distance $=|x+a|=4$

$\Rightarrow$ $|x+2|=4$

$\Rightarrow$ $x+2=\pm 4$

$\Rightarrow$ $x=2,-6$

But $x\neq -6$

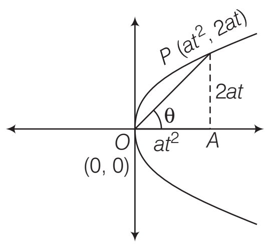
For $x=2$, $y^2=8\times 2$

$\therefore$ $y^2=16\Rightarrow y=\pm 4$

So, the points are (2, 4) and (2, – 4).

Q. 17 Find the length of the line segment joining the vertex of the parabola $y^2 = 4ax$ and a point on the parabola, where the line segment makes an angle θ to the X-axis.

Sol. Given equation of the parabola is $y^2 = 4ax$



Let the coordinates of any point P on the parabola be $(at^2, 2at)$.

In ΔPOA, $\tan\theta = \frac{2at}{at^2} = \frac{2}{t}$

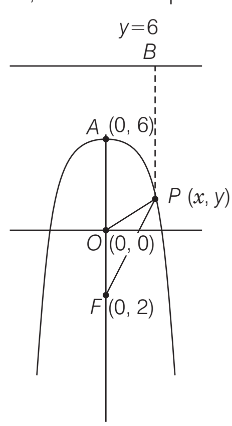
$\Rightarrow$ $\tan\theta = \frac{2}{t} \Rightarrow t = 2\cot\theta$

$\therefore$ length of $OP = \sqrt{(0 - at^2)^2 + (0 - 2at)^2}$

$$= \sqrt{a^2t^4 + 4a^2t^2}$$
$$= at\sqrt{t^2 + 4}$$
$$= 2a\cot\theta\sqrt{4\cot^2\theta + 4}$$
$$= 4a\cot\theta\sqrt{1 + \cot^2\theta}$$
$$= 4a\cot\theta \cdot \operatorname{cosec}\theta$$
$$= \frac{4a\cos\theta}{\sin\theta} \cdot \frac{1}{\sin\theta} = \frac{4a\cos\theta}{\sin^2\theta}$$

Q. 18 If the points (0, 4) and (0, 2) are respectively the vertex and focus of a parabola, then find the equation of the parabola.

Sol. Given that the coordinates, vertex of the parabola (0, 4) and focus of the parabola (0, 2).



By definition of the parabola, $PB = PF$

$$\Rightarrow \quad \left|\frac{0 + y - 6}{\sqrt{0 + 1}}\right| = \sqrt{(x - 0)^2 + (y - 2)^2}$$

$$\Rightarrow \quad |y - 6| = \sqrt{x^2 + y^2 - 4y + 4}$$

$$\Rightarrow \quad x^2 + y^2 - 4y + 4 = y^2 + 36 - 12y$$

$$\Rightarrow \quad x^2 + 8y = 32$$

Q. 19 **If the line $y = mx + 1$ is tangent to the parabola $y^2 = 4x$, then find the value of m.**

Sol. Given that, line $y = mx + 1$ is tangent to the parabola $y^2 = 4x$.

$$\therefore \quad y = mx + 1 \quad \text{...(i)}$$

and
$$y^2 = 4x \quad \text{...(ii)}$$

From Eqs. (i) and (ii),

$$m^2x^2 + 2mx + 1 = 4x$$

$$\Rightarrow \quad m^2x^2 + 2mx - 4x + 1 = 0$$

$$\Rightarrow \quad m^2x^2 + x(2m - 4) + 1 = 0$$

$$\Rightarrow \quad (2m - 4)^2 - 4m^2 \times 1 = 0$$

$$\Rightarrow \quad 4m^2 + 16 - 16m - 4m^2 = 0$$

$$\Rightarrow \quad 16m = 16$$

$$\therefore \quad m = 1$$

Q. 20 **If the distance between the foci of a hyperbola is 16 and its eccentricity is $\sqrt{2}$, then obtain the equation of the hyperbola.**

Thinking Process

First of all find the value of a and b using the given condition, then put them in $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ *to get the required equation of the hyperbola.*

Sol. Distance between the foci *i.e.*, $2ae = 16 \Rightarrow ae = 8$

and
$$e = \sqrt{2}$$

$$\therefore \quad a\sqrt{2} = 8$$

$$a = 4\sqrt{2}$$

We know that,
$$b^2 = a^2(e^2 - 1)$$

$$\Rightarrow \quad b^2 = (4\sqrt{2})^2[(\sqrt{2})^2 - 1]$$

$$= 16 \times 2\,(2 - 1)$$

$$= 32\,(2 - 1)$$

So, the equation of hyperbola is

$$\frac{x^2}{32} - \frac{y^2}{32} = 1$$

$$\Rightarrow \quad x^2 - y^2 = 32$$

Q. 21 Find the eccentricity of the hyperbola $9y^2 - 4x^2 = 36$.

Sol. Given equation of the hyperbola is

$$9y^2 - 4x^2 = 36$$

$$\Rightarrow \quad \frac{9y^2}{36} - \frac{4x^2}{36} = \frac{36}{36}$$

$$\Rightarrow \quad \frac{y^2}{4} - \frac{x^2}{9} = 1$$

$$\Rightarrow \quad -\frac{x^2}{9} + \frac{y^2}{4} = 1$$

Since, this equation in form of $-\frac{x^2}{a} + \frac{y^2}{b^2} = 1$, where $a = 3$ and $b = 2$.

$$\therefore \quad e = \sqrt{1 + \left(\frac{a}{b}\right)^2}$$

$$= \sqrt{1 + \frac{9}{4}} = \frac{\sqrt{13}}{2}$$

Q. 22 Find the equation of the hyperbola with eccentricity $\frac{3}{2}$ and foci at $(\pm 2, 0)$.

Sol. Given that eccentricity *i.e.*, $e = 3/2$ and $(\pm ae, 0) = (\pm 2, 0)$

$$\therefore \quad ae = 2$$

$$\Rightarrow \quad a \cdot \frac{3}{2} = 2 \Rightarrow a = 4/3$$

$$\because \quad b^2 = a^2 (e^2 - 1)$$

$$\Rightarrow \quad b^2 = \frac{16}{9}\left(\frac{9}{4} - 1\right)$$

$$\Rightarrow \quad b^2 = \frac{16}{4}\left(\frac{5}{4}\right) = +\frac{20}{9}$$

So, the equation of hyperbola is

$$\frac{x^2}{\frac{16}{9}} - \frac{y^2}{\frac{20}{9}} = 1$$

$$\Rightarrow \quad \frac{x^2}{4} - \frac{y^2}{5} = \frac{4}{9}$$

Long Answer Type Questions

Q. 23 If the lines $2x - 3y = 5$ and $3x - 4y = 7$ are the diameters of a circle of area 154 square units, then obtain the equation of the circle.

Thinking Process

First of all find the intersection point of the given lines, then get radius of circle from given area. Now, use formula equation of circle with centre (h, k) and radius a is $(x-h)^2 + (y-k)^2 = a^2$.

Sol. Given lines are $2x - 3y - 5 = 0$ ... (i)

and $3x - 4y - 7 = 0$

From Eqs. (i) and (ii), $$\frac{x}{21-20} = \frac{y}{-15+14} = \frac{1}{-8+9}$$

$$\Rightarrow \quad \frac{x}{1} = \frac{y}{-1} = \frac{1}{+1}$$

$$\Rightarrow \quad x = \pm 1, y = -1$$

Since the intersection point of these lines will be coordinates of the circle *i.e.,* coordinates of the circle as $(1, -1)$.

Let the radius of the circle is r.

Then $$\pi r^2 = 154$$

$$\Rightarrow \quad \frac{22}{7} \times r^2 = 154$$

$$\Rightarrow \quad r^2 = \frac{154 \times 7}{22}$$

$$\Rightarrow \quad r^2 = \frac{14 \times 7}{2} \Rightarrow r^2 = 49$$

So, the equation of circle is

$$(x-1)^2 + (y+1)^2 = 49$$

$$\Rightarrow \quad x^2 - 2x + 1 + y^2 + 2y + 1 = 49$$

$$\Rightarrow \quad x^2 + y^2 - 2x + 2y = 47$$

Q. 24 Find the equation of the circle which passes through the points (2, 3) and (4, 5) and the centre lies on the straight line $y - 4x + 3 = 0$.

Sol. Let the general equation of the circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{...(i)}$$

Since, this circle passes through the points (2, 3) and (4, 5).

$$\therefore \quad 4 + 9 + 4g + 6f + c = 0$$

$$\Rightarrow \quad 4g + 6f + c = -13 \quad \text{...(ii)}$$

and $$16 + 25 + 8g + 10f + c = 0$$

$$\Rightarrow \quad 8g + 10f + c = -41 \quad \text{...(iii)}$$

Since, the centre of the circle $(-g, -f)$ lies on the straight line $y - 4x + 3 = 0$

i.e., $$+4g - f + 3 = 0 \quad \text{...(iv)}$$

From Eq. (iv), $4g = f - 3$

On putting $4g = f - 3$ in Eq. (ii), we get

$$f - 3 + 6f + c = -13$$

$$\Rightarrow \quad 7f + c = 10 \quad \text{... (v)}$$

From Eqs. (ii) and (iii),

$$\begin{array}{r} 8g + 12f + 2c = -26 \\ 8g + 10f + c = -41 \\ - \quad - \quad - \qquad + \\ \hline 2f + c = 15 \end{array} \quad \text{...(vi)}$$

From Eqs. (ii) and (vi),

$$\begin{array}{r} 7f + c = -10 \\ 2f + c = 15 \\ - \quad - \quad - \\ \hline 5f = -25 \end{array}$$

$\therefore$ $f = -5$

Now, $c = 10 + 15 = 25$

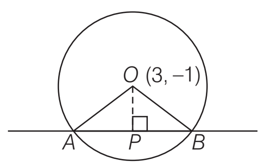
From Eq. (iv), $4g + 5 + 3 = 0$

$\Rightarrow$ $g = -2$

From Eq. (i), equation of the circle is $x^2 + y^2 - 4x - 10y + 25 = 0$.

Q. 25 **Find the equation of a circle whose centre is (3, – 1) and which cuts off a chord 6 length 6 units on the line $2x - 5y + 18 = 0$.**

Sol. Given centre of the circle is (3, – 1).



Now, $OP = \left|\dfrac{6 + 5 + 18}{\sqrt{4 + 25}}\right| = \dfrac{29}{\sqrt{29}} = \sqrt{29}$

In ΔOPB, $OB^2 = OP^2 + PB^2$ $[\because AB = 6 \Rightarrow PB = 3]$

$\Rightarrow$ $OB^2 = 29 + 9 \Rightarrow OB^2 = 38$

So, the radius of circle is $\sqrt{38}$,

$\therefore$ Equation of the circle with radius $r = \sqrt{38}$ and centre (3,–1) is

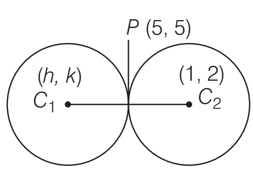
$\Rightarrow$ $(x - 3)^2 + (y + 1)^2 = 38$

$\Rightarrow$ $x^2 - 6x + 9 + y^2 + 2y + 1 = 38$

$x^2 + y^2 - 6x + 2y = 28$

Q. 26 **Find the equation of a circle of radius 5 which is touching another circle $x^2 + y^2 - 2x - 4y - 20 = 0$ at (5, 5).**

Sol. Let the coordinates of centre of the required circle are (h, k), then the centre of another circle is (1, 2).



$$\text{Radius} = \sqrt{1 + 4 + 20} = 5$$

So, it is clear that P is the mid-point of C_1C_2.

$\therefore \quad 5 = \frac{1+h}{2} \Rightarrow h = 9$

and $\quad 5 = \frac{2+k}{2} \Rightarrow k = 8$

So, the equation of and required circle is

$$(x-9)^2 + (y-8)^2 = 25$$

$\Rightarrow \quad x^2 - 18x + 81 + y^2 - 16y + 64 = 25$

$\Rightarrow \quad x^2 + y^2 - 18x - 16y + 120 = 0$

Q. 27 Find the equation of a circle passing through the point (7, 3) having radius 3 units and whose centre lies on the line $y = x - 1$.

Thinking Process

First of all let the equation of a circle with centre (h, k) and radius r is $(x-h)^2 + (y-h)^2 = r^2$, then we get the value of (h, k) using given condition.

Sol. Let equation of circle be

$$(x-h)^2 + (y-k)^2 = r^2$$

$\Rightarrow \quad (x-h)^2 + (y-k)^2 = 9 \quad \text{...(i)}$

Given that, centre (h, k) lies on the line

$$y = x - 1 \; i.e., \; k = h - 1 \quad \text{...(ii)}$$

Now, the circle passes through the point (7, 3).

$\therefore \quad (7-h)^2 + (3-k)^3 = 9$

$\Rightarrow \quad 49 - 14h + h^2 + 9 - 6k + k^2 = 9$

$\Rightarrow \quad h^2 + k^2 - 14h - 6k + 49 = 0 \quad \text{...(iii)}$

On putting $k = h - 1$ in Eq. (iii), we get

$$h^2 + (h-1)^2 - 14h - 6(h-1) + 49 = 0$$

$\Rightarrow \quad h^2 + h^2 - 2h + 1 - 14h - 6h + 6 + 49 = 0$

$\Rightarrow \quad 2h^2 - 22h + 56 = 0$

$\Rightarrow \quad h^2 - 11h + 28 = 0$

$\Rightarrow \quad h^2 - 7h - 4h + 28 = 0$

$\Rightarrow \quad h(h-7) - 4(h-7) = 0$

$\Rightarrow \quad (h-7)(h-4) = 0$

$\therefore \quad h = 4, 7$

When $\quad h = 7$, then $k = 7 - 1 = 6$

$\therefore$ Centre (7, 6)

When $\quad h = 4$, then $k = 3$

$\therefore$ Centre – (4, 3)

So, the equation of circle when centre (7, 6), is

$$(x-7)^2 + (y-6)^2 = 9$$

$\Rightarrow \quad x^2 - 14x + 49 + y^2 - 12y + 36 = 9$

$\Rightarrow \quad x^2 + y^2 - 14x - 12y + 76 = 0$

When centre (4, 3), then the equation of the circle is

$$(x-4)^2 + (y-3)^2 = 9$$

$\Rightarrow \quad x^2 - 8x + 16 + y^2 - 6y + 9 = 9$

$\Rightarrow \quad x^2 + y^2 - 8x - 6y + 16 = 0$

Q. 28 Find the equation of each of the following parabolas

(i) directrix = 0, focus at (6, 0)

(ii) vertex at (0, 4), focus at (0, 2)

(iii) focus at (– 1, – 2), directrix $x - 2y + 3 = 0$

Sol. **(i)** Given that, directrix = 0 and focus = (6, 0)

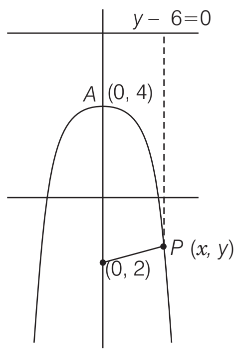
So, the equation of the parabola

$$(x - 6)^2 + y^2 = x^2$$

$$\Rightarrow \quad x^2 + 36 - 12x + y^2 = x^2$$

$$\Rightarrow \quad y^2 - 12x + 36 = 0$$

(ii) Given that, vertex = (0, 4) and focus = (0, 2)



So, the equation of parabola is

$$\sqrt{(x - 0)^2 + (y - 2)^2} = |y - 6|$$

$$\Rightarrow \quad x^2 + y^2 - 4y + 4 = y^2 - 12y + 36$$

$$\Rightarrow \quad x^2 - 4y + 12y - 32 = 0$$

$$\Rightarrow \quad x^2 + 8y - 32 = 0$$

$$\Rightarrow \quad x^2 = 32 - 8y$$

(iii) Given that, focus at (– 1, – 2) and directrix $x - 2y + 3 = 0$

So, the equation of parabola is $\sqrt{(x + 1)^2 + (y + 2)^2} = \left|\dfrac{x - 2y + 3}{\sqrt{1 + 4}}\right|$

$$\Rightarrow \quad x^2 + 2x + 1 + y^2 + 4y + 4 = \frac{1}{5}[x^2 + 4y^2 + 9 - 4xy - 12y + 6x]$$

$$\Rightarrow \quad 4x^2 + 4xy + y^2 + 4x + 32y + 16 = 0$$

Q. 29 Find the equation of the set of all points the sum of whose distances from the points (3, 0), (9, 0) is 12.

Sol. Let the coordinates of the point be (x, y), then according to the question,

$$\sqrt{(x - 3)^2 + y^2} + \sqrt{(x - 9)^2 + y^2} = 12$$

$$\Rightarrow \quad \sqrt{(x - 3)^2 + y^2} = 12 - \sqrt{(x - 9)^2 + y^2}$$

On squaring both sides, we get

$$x^2 - 6x + 9 + y^2 = 144 + (x^2 - 18x + 81 + y^2) - 24\sqrt{(x-9)^2 + y^2}$$

$\Rightarrow \quad 12x - 216 = -24\sqrt{(x-9)^2 + y^2}$

$\Rightarrow \quad x - 18 = -2\sqrt{(x-9)^2 + y^2}$

$\Rightarrow \quad x^2 - 36x + 324 = 4(x^2 - 18x + 81 + y^2)$

$\Rightarrow \quad 3x^2 + 4y^2 - 36x = 0$

Q. 30 **Find the equation of the set of all points whose distance from (0, 4) are $\frac{2}{3}$ of their distance from the line $y = 9$.**

Thinking Process

Consider the points (x, y), and apply the condition given in the problem, then get the set of all points.

Sol. Let the point be $P(x, y)$.

$\therefore$ Distance from (0, 4) $= \sqrt{x^2 + (y-4)^2}$

So, the distance from the line $y = 9$ is $\left|\frac{y-9}{\sqrt{1}}\right|$

$\therefore \quad \sqrt{x^2 + (y-4)^2} = \frac{2}{3}\left|\frac{y-9}{1}\right|$

$\Rightarrow \quad x^2 + y^2 - 8y + 10 = \frac{4}{9}(y^2 - 18y + 81)$

$\Rightarrow \quad 9x^2 + 9y^2 - 72y + 144 = 4y^2 - 72y + 324$

$\Rightarrow \quad 9x^2 + 5y^2 = 180$

Q. 31 **Show that the set of all points such that the difference of their distances from (4, 0) and (– 4, 0) is always equal to 2 represent a hyperbola.**

Sol. Let the points be $P(x, y)$.

$\therefore$ Distance of P from (4, 0) $\sqrt{(x-4)^2 + y^2}$ ... (i)

and the distance of P from (– 4, 0) $\sqrt{(x+4)^2 + y^2}$ ... (ii)

Now, $\quad \sqrt{(x+4)^2 + y^2} - \sqrt{(x-4)^2 + y^2} = 2$

$\Rightarrow \quad \sqrt{(x+4)^2 + y^2} = 2 + \sqrt{(x-4)^2 + y^2}$

On squaring both sides, we get

$$x^2 + 8x + 16 + y^2 = 4 + x^2 - 8x + 16 + y^2 + 4\sqrt{(x-4)^2 + y^2}$$

$\Rightarrow \quad 16x - 4 = 4\sqrt{(x-4)^2 + y^2}$

$\Rightarrow \quad 4(4x - 1) = 4\sqrt{(x-4)^2 + y^2}$

$\Rightarrow \quad 16x^2 - 8x + 1 = x^2 + 16 - 8x + y^2$

$\Rightarrow \quad 15x^2 - y^2 = 15$ which is a parabola.

Q. 32 Find the equation of the hyperbola with

(i) Vertices $(\pm 5, 0)$, foci $(\pm 7, 0)$

(ii) Vertices $(0, \pm 7)$, $e = \frac{7}{3}$.

(iii) Foci $(0, \pm \sqrt{10})$, passing through (2, 3).

Sol. **(i)** Given that, vertices $= (\pm 5, 0)$, foci $= (\pm 7, 0)$ and $a = \pm 5$

$\therefore \quad (\pm ae, 0) = (\pm 7, 0)$

Now $\quad ae = 7 \Rightarrow 5e = 7$

$\Rightarrow \quad e = 7/5$

$\because \quad b^2 = a^2(e^2 - 1)$

$\Rightarrow \quad b^2 = 25\left(\frac{49}{25} - 1\right)$

$\Rightarrow \quad b^2 = 25\left(\frac{49 - 25}{25}\right)$

$\Rightarrow \quad b^2 = 24$

So,the equation of parabola is

$$\frac{x^2}{25} - \frac{y^2}{24} = 1 \qquad [\because a^2 = 25 \text{ and } b^2 = 24]$$

(ii) Vertices $= (0, \pm 7)$, $e = 4/3$

$\therefore \quad b = 7, e = 4/3$

$\because \quad e^2 = 1 + \frac{a^2}{b^2}$

$\Rightarrow \quad \frac{16}{9} - 1 = \frac{a^2}{49}$

$\Rightarrow \quad \frac{7}{9} = \frac{a^2}{49} \Rightarrow a^2 = \frac{343}{9}$

So, the equation of hyperbola is

$$-\frac{x^2 \times 9}{343} + \frac{y^2}{49} = 1$$

$\Rightarrow \quad -\frac{9x^2}{7} + y^2 = 49$

$\Rightarrow \quad 9x^2 - 7y^2 + 343 = 0$

(iii) Given that, foci $= (0, \pm \sqrt{10})$

$\because \quad be = \sqrt{10}$

$\Rightarrow \quad a^2 + b^2 = 10$

$\Rightarrow \quad a^2 = 10 - b^2$

$\therefore$ Equation of the hyperbola be

$$-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{...(i)}$$

Since, this hyperbola passes through the point (2, 3).

$\therefore \quad -\frac{4}{a^2} + \frac{9}{b^2} = 1$

$\Rightarrow \quad \frac{-4}{10 - b^2} + \frac{9}{b^2} = 1$

$$\Rightarrow \quad \frac{-4b^2 + 90 - 9b^2}{b^2(10 - b^2)} = 1$$

$$\Rightarrow \quad -13b^2 + 90 = 10b^2 + b^4$$

$$\Rightarrow \quad b^4 - 23b^2 + 90 = 0$$

$$\Rightarrow \quad b^4 - 18b^2 - 5b^2 + 90 = 0$$

$$\Rightarrow \quad b^2(b^2 - 18) - 5(b^2 - 18) = 0$$

$$\Rightarrow \quad (b^2 - 18)(b^2 - 5) = 0$$

$$\Rightarrow \quad b^2 = 18 \Rightarrow b = \pm 3\sqrt{2}$$

or $\quad b^2 = 5 \Rightarrow b = \sqrt{5}$

$\therefore \quad b^2 = 18$ then $a^2 = -8$ [not possible]

When $\quad a^2 = 5$, then $b^2 = 5$

So, the equation of hyperbola is

$$-\frac{x^2}{5} + \frac{y^2}{5} = 1$$

$$\Rightarrow \quad y^2 - x^2 = 5$$

True/False

Q. 33 The line $x + 3y = 0$ is a diameter of the circle $x^2 + y^2 + 6x + 2y = 0$.

Thinking Process

If a line is the diameter of circle, then the centre of the circle should lies on line. Use this property to solve the given problem.

Sol. ***False***

Given equation of the circle is

$$x^2 + y^2 + 6x + 2y = 0$$

$\therefore \quad$ Centre $= (-3, -1)$

Since given line is $x + 3y = 0$.

$$\Rightarrow \quad -3 - 3 \neq 0$$

So, this line is not diameter of the circle.

Q. 34 The shortest distance from the point $(2, -7)$ to the circle $x^2 + y^2 - 14x - 10y - 151 = 0$ is equal to 5.

Sol. ***False***

Given circle is $x^2 + y^2 - 14x - 10y - 151 = 0$.

$\therefore \quad$ Centre $= (7, 5)$

and $\quad$ Radius $= \sqrt{49 + 25 + 151} = \sqrt{225} = 15$

So, the distance between the point $(2, -7)$ and centre of the circle is given by

$$d_1 = \sqrt{(2-7)^2 + (-7-5)^2}$$

$$= \sqrt{25 + 144} = \sqrt{169} = 13$$

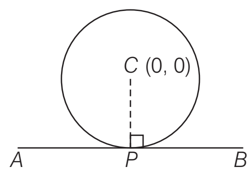
$\therefore$ Shortest distance, $d = |13 - 15| = 2$

Q. 35 If the line $lx + my = 1$ is a tangent to the circle $x^2 + y^2 = a^2$, then the point (l, m) lies on a circle.

Sol. ***True***

Given circle is $x^2 + y^2 = a^2$ ...(i)

$\therefore$ Radius of circle = a and centre = (0, 0)



$\therefore$ Distance from point (l, m) and centre is $\sqrt{(0-e)^2 + (0-m)^2} = a$

$\Rightarrow \quad l^2 + m^2 = a^2$

So, l, m lie on the circle.

Q. 36 The point (1, 2) lies inside the circle $x^2 + y^2 - 2x + 6y + 1 = 0$.

Thinking Process

If the x_1, y_1 lies inside the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$, then $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c < 0$ and it $S > 0$, then the point lies outside the circle.

Sol. ***False***

Given circle is $S \equiv x^2 + y^2 - 2x + 6y + 1 = 0$.

Since, the point is (1, 2).

Now, $\quad S_1 \equiv 1 + 4 - 2 + 12 + 1$

$\Rightarrow \quad S_1 > 0$

So, the (1, 2) lies outside the circle.

Q. 37 The line $lx + my + n = 0$ will touch the parabola $y^2 = 4ax$, if In $= am^2$.

Sol. ***True***

Given equation of a line is

$$lx + my + n = 0 \quad \text{...(i)}$$

and $\quad$ parabola $y^2 = 4ax$ ...(ii)

From Eq. (i), $x = -\left(\frac{my+n}{l}\right)$ put in Eq. (ii), we get

$$y^2 = -\frac{4a\,(my+n)}{l}$$

$\Rightarrow \quad ly^2 = -4a\,my - 4\,ax$

$\Rightarrow \quad ly^2 + 4\,amy + 4an = 0$

For tangent, $\quad D = 0$

$\Rightarrow \quad 16\,a^2m^2 = 4l \times 4\,an$

$\Rightarrow \quad 16\,a^2m^2 = 16\,anl$

$\Rightarrow \quad am^2 = nl$

Q. 38 If P is a point on the ellipse $\dfrac{x^2}{16}+\dfrac{y^2}{25}=1$ whose foci are S and S', then $PS+PS'=8$.

Sol. ***False***

Given equation of the ellipse is $\dfrac{x}{16}+\dfrac{y^2}{25}=1$.

which is in form of $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, where $b>a$

$\therefore$ Foci, $S=(0, be), S'(0, -be)$

$$\therefore \quad e=\sqrt{1-\frac{a^2}{b^2}}$$

$$=\sqrt{\frac{25-16}{25}}=3/5$$

$$\text{Foci, } S=\left(0, \frac{3\times 5}{5}\right), S'=\left(0, -\frac{3\times 5}{5}\right) \text{ i.e., } S=(0,3), S'=(0,-3)$$

Let the coordinate of point P be (x, y) then $PS+PS'=2b=2\times 5=10$

Q. 39 The line $2x+3y=12$ touches the ellipse $\dfrac{x^2}{9}+\dfrac{y^2}{4}=2$ at the point (3,2).

Sol. ***True***

Given equation of line is

$$2x+3y=12 \quad \text{...(i)}$$

and ellipse $$\frac{x^2}{9}+\frac{y^2}{4}=2 \quad \text{...(ii)}$$

Since, the equation of tangent at (x_1, y_1) is $\dfrac{xx_1}{9}+\dfrac{yy_1}{4}=2$.

$\therefore$ Tangent at (3, 2),

$$\frac{3x}{9}+\frac{2y}{4}=2$$

$$\Rightarrow \quad \frac{x}{3}+\frac{y}{2}=2$$

$\Rightarrow$ $2x+3y=12$, which is a given line.

Hence, the statement is true.

Q. 40 The locus of the point of intersection of lines $\sqrt{3}x-y-4\sqrt{3k}=0$ and $\sqrt{3}kx+ky-4\sqrt{3}=0$ for different value of k is a hyperbola whose eccentricity is 2.

Thinking Process

First of all eliminate k from the given equations of line, then get the equation of hyperbola.

Sol. ***True***

Given equations of line are

$$\sqrt{3}x-y-4\sqrt{3}k=0 \quad \text{...(i)}$$

and $$\sqrt{3}kx+ky-4\sqrt{3}=0 \quad \text{...(ii)}$$

From Eq. (i), $$4\sqrt{3}\,k=\sqrt{3}\,x-y$$

$$\Rightarrow \quad k = \frac{\sqrt{3}x - y}{4\sqrt{3}} \text{ put in Eq. (ii), we get}$$

$$\sqrt{3}x\left(\frac{\sqrt{3}x - y}{4\sqrt{3}}\right) + \left(\frac{\sqrt{3}x - y}{4\sqrt{3}}\right)y - 4\sqrt{3} = 0$$

$$\Rightarrow \quad \frac{1}{4}(\sqrt{3}x^2 - xy) + \frac{1}{4}\left(xy - \frac{y^2}{\sqrt{3}}\right) - 4\sqrt{3} = 0$$

$$\Rightarrow \quad \frac{\sqrt{3}}{4}x^2 - \frac{y^2}{4\sqrt{3}} - 4\sqrt{3} = 0$$

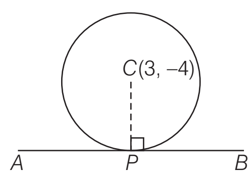
$$\Rightarrow \quad 3x^2 - y^2 - 48 = 0$$

$$\Rightarrow \quad 3x^2 - y^2 = 48, \text{ which is a hyperbola.}$$

Fillers

Q. 41 The equation of the circle having centre at (3, – 4) and touching the line $5x + 12y - 12 = 0$ is ....

Sol. The perpendicular distance from centre (3, – 4) to the line is, $d = \left|\frac{15 - 48 - 12}{\sqrt{25 + 144}}\right| = \frac{45}{13}$



So, the required equations of the circle is $(x - 3)^2 + (y + 4)^2 = \left(\frac{45}{13}\right)^2$.

Q. 42 The equation of the circle circumscribing the triangle whose sides are the lines $y = x + 2$, $3y = 4x$, $2y = 3x$ is ....

Sol. Given equations of line are

$$y = x + 2 \quad \text{...(i)}$$

$$3y = 4x \quad \text{...(ii)}$$

$$2y = 3x \quad \text{...(iii)}$$

From Eqs. (i) and (ii),

$$\frac{4x}{3} = x + 2$$

$$\Rightarrow \quad 4x = 3x + 6$$

$\Rightarrow x = 6$

On putting $x = 6$ in Eq. (i), we get

$$y = 8$$

$$\therefore \quad \text{Point, } A = (6, 8)$$

From Eqs. (i) and (iii),

$$\frac{3x}{2} = x + 2$$

$$\Rightarrow \quad 3x = 2x + 4 \Rightarrow x = 4$$

When $x = 4$, then $y = 6$

$\therefore$ Point, $B = (4, 6)$

From Eqs. (ii) and (iii) $x_1 = 0_1, \ y = 0$

Now, $C = (0, 0)$

Let the equation of circle is

$$x^2 + y^2 + 2gxn + 2fy + c = 0$$

Since, the points A (6, 8), B (4, 6) and C (0, 0) lie on this circle.

$$36 + 64 + 12g + 16f + c = 0$$

$$\Rightarrow \quad 12g + 16f + c = -100 \quad \text{...(iv)}$$

$$\text{and} \quad 16 + 36 + 8g + 12f + c = 0$$

$$\Rightarrow \quad 8g + 12f + c = -52 \quad \text{...(v)}$$

$$\Rightarrow \quad c = 0 \quad \text{...(vi)}$$

From Eqs. (iv), (v) and (vi),

$$12g + 16f = -100$$

$$\Rightarrow \quad 3g + 4f + 25 = 0$$

$$\Rightarrow \quad 2g + 3f + 13 = 0$$

$$\Rightarrow \quad \frac{g}{+52 - 75} = \frac{f}{50 - 39} = \frac{1}{9 - 8}$$

$$\Rightarrow \quad \frac{g}{-23} = \frac{f}{11} = \frac{1}{1}$$

$$\Rightarrow \quad g = -23, f = 11$$

So, the equation of circle is

$$x^2 + y^2 - 46x + 22y + 0 = 0$$

$$\Rightarrow \quad x^2 + y^2 - 46x + 22y = 0$$

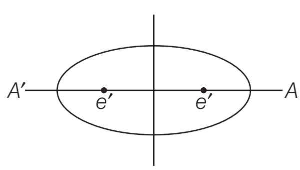
Q. 43 An ellipse is described by using an endless string which is passed over two pins. If the axes are 6 cm and 4 cm, the length of the string and distance between the pins are ....

Sol. Let equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

$\therefore$ $29 = 6$ and $2b = 4$

$\Rightarrow$ $a = 3$ and $b = 2$

We know that,



$$c^2 = a^2 - b^2 = (3)^2 - (2)^2$$

$$= 9 - 4 = 5 \Rightarrow c = \sqrt{5}$$

$\therefore$ Length of string $= AC' + C'C + AC$

$$= a + c + 2c + ac$$

$$= 2a + 2c = 6 + 2\sqrt{5}$$

$\therefore$ Distances between the pins $= 2\sqrt{5} = cc'$

Q. 44 The equation of the ellipse having foci (0, 1), (0, – 1) and minor axis of length 1 is ....

Thinking Process

First of all get the value of a and b with the help of given condition in the problem, then we get the required equation of the ellipse.

Sol. Given that, foci of the ellipse are $(0, \pm be)$.

$\because \quad be \equiv 1$

$\therefore \quad$ Length of minor axis, $2a = 1 \Rightarrow a = 1/2$

$$e^2 = 1 - \frac{a^2}{b^2}$$

$\Rightarrow \quad (be)^2 = b^2 - a^2 \Rightarrow 1 = b^2 - \frac{1}{4}$

$\Rightarrow \quad 1 + \frac{1}{4} = b^2 \Rightarrow \frac{5}{4} = b^2$

So, the equation of ellipse is

$$\frac{x^2}{\frac{1}{4}} + \frac{y^2}{5/4} = 1 \Rightarrow \frac{4x^2}{1} + \frac{4y^2}{5} = 1$$

Q. 45 The equation of the parabola having focus at (– 1, – 2) and directrix is $x - 2y + 3 = 0$, is ....

Sol. Given that, focus at $F(-1, -2)$ and directrix is $x - 2y + 3 = 0$

Let any point on the parabola be (x, y).

$\therefore \quad PF = \left|\frac{x - 2y + 3}{\sqrt{1 + 4}}\right|$

$\Rightarrow \quad (x + 1)^2 + (y + 2)^2 = \frac{(x - 2y + 3)^2}{5}$

$\Rightarrow \quad 5[x^2 + 2x + 1 + y^2 + 4y + 4] = x^2 + 4y^2 + 9 - 4xy - 12y + 6x$

$\Rightarrow \quad 4x^2 + y^2 + 4x + 32y + 16 = 0$

Q. 46 The equation of the hyperbola with vertices at $(0, \pm 6)$ and eccentricity $\frac{5}{3}$ is .... and its foci are ....

Sol. Let the equation of the hyperbola be $-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Then vertices $= (0, \pm b) = (0, \pm 6)$

$\therefore \quad b = 6$ and $e = 5/3$

$\because \quad e = \sqrt{1 + \frac{a^2}{b^2}} \Rightarrow \frac{25}{9} = 1 + \frac{a^2}{36}$

$\Rightarrow \quad \frac{25 - 9}{9} = \frac{a^2}{36} \Rightarrow 16 = \frac{a^2}{4} \Rightarrow a^2 = 48$

So, the equation of hyperbola is,

$$\frac{-x^2}{48} + \frac{y^2}{36} = 1 \Rightarrow \frac{y^2}{36} - \frac{x^2}{48} = 1$$

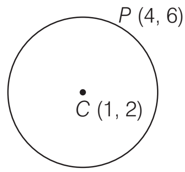
$\because \quad$ Foci $= (0, \pm be) = \left(= 0, \pm \frac{5}{3} \times 6\right) = (0, \pm 10)$

Objective Type Questions

Q. 47 The area of the circle centred at (1, 2) and passing through the point(4, 6) is

(a) 5π (b) 10π
(c) 25π (d) None of these

Sol. *(c)* Given that, centre of the circle is (1, 2).



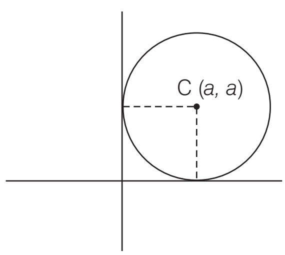
$\because \quad CP = \sqrt{9+16} = 5 = \text{Radius of the circle}$

$\therefore \quad \text{Required area} = \pi r^2 = 25\pi$

Q. 48 Equation of a circle which passes through (3, 6) and touches the axes is

(a) $x^2 + y^2 + 6x + 6y + 3 = 0$ (b) $x^2 + y^2 - 6x - 6y - 9 = 0$
(c) $x^2 + y^2 - 6x - 6y + 9 = 0$ (d) None of these

Sol. *(c)* Let centre of the circle be (a, a), then equation of the circle is $(x-a)^2 + (y-a)^2 = a^2$.



Since, the point (3, 6) lies on this circle, then

$$(3-a)^2 + (6-a)^2 = a^2$$

$$\Rightarrow \quad a^2 + 9 - 6a + 36 - 12a + a^2 = a^2$$

$$\Rightarrow \quad a^2 - 18a + 45 = 0$$

$$\Rightarrow \quad a^2 - 15a - 3a + 45 = 0$$

$$\Rightarrow \quad a(a-15) - 3(a-15) = 0$$

$$\Rightarrow \quad (a-3)(a-15) = 0$$

$$\Rightarrow \quad a = 3, a = 15$$

So, the equation of circle is

$$(x-3)^2 + (y-3)^2 = 9$$

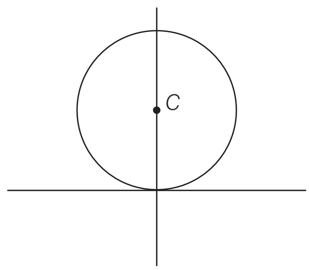
$$\Rightarrow \quad x^2 - 6x + 9 + y^2 - 6y + 9 = 9$$

$$\Rightarrow \quad x^2 + y^2 - 6x - 6y + 9 = 0$$

Q. 49 Equation of the circle with centre on the Y-axis and passing through the origin and the point (2, 3) is

(a) $x^2 + y^2 + 13y = 0$ (b) $3x^2 + 3y^2 + 13x + 3 = 0$
(c) $6x^2 + 6y^2 - 13y = 0$ (d) $x^2 + y^2 + 13x + 3 = 0$

Sol. *(c)* Let general equation of the circle is $x^2 + y^2 + 2gh + 2fy + c = 0$.



Since the point (0, 0) and (2, 3) lie on it $c = 0$.

$$\therefore \quad 4 + 9 + 4g + 6f = 0$$
$$\Rightarrow \quad 2g + 3f = -13/2$$

Since the centre lie on Y-axis, then $g = 0$.

$$\therefore \quad 3f = -13/2$$
$$\Rightarrow \quad f = -13/6$$

So, the equation of circle is

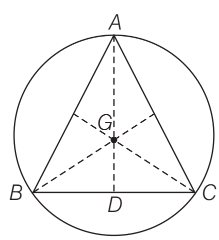
$$x^2 + y^2 - \frac{13y}{6} = 0$$
$$\Rightarrow \quad 6x^2 + 6y^2 - 13y = 0$$

Q. 50 The equation of a circle with origin as centre and passing through the vertices of an equilateral triangle whose median is of length $3a$ is

(a) $x^2 + y^2 = 9a^2$ (b) $x^2 + y^2 = 16a^2$
(c) $x^2 + y^2 = 4a^2$ (d) $x^2 + y^2 = a^2$

Sol. *(c)* Given that, length of the median $AD = 3a$

$$\because \quad \text{Radius of the circle} = \frac{3}{2} \times \text{Length of median}$$
$$= \frac{2}{3} \times 3a = 2a$$



So, the equation of the circle is $x^2 + y^2 = 4a^2$.

Q. 51 If the focus of a parabola is (0, – 3) and its directrix is $y = 3$, then its equation is

(a) $x^2 = -12y$
(b) $x^2 = 12y$
(c) $y^2 = -12x$
(d) $y^2 = 12x$

Sol. *(a)* Given that, focus of parabola at F (0, – 3) and equation of directrix is $y = 3$.

Let any point on the parabola is $P(x, y)$.

Then, $PF = |y - 3|$

$\Rightarrow \sqrt{(x-0)^2 + (y+3)^2} = |y - 3|$

$\Rightarrow x^2 + y^2 + 6y + 9 = y^2 - 6y + 9$

$\Rightarrow x^2 + 12y = 0$

$\Rightarrow x^2 = -12y$

Q. 52 If the parabola $y^2 = 4ax$ passes through the point (3, 2), then the length of its latusrectum is

(a) $\frac{2}{3}$
(b) $\frac{4}{3}$
(c) $\frac{1}{3}$
(d) 4

Sol. *(b)* Given that, parabola is

$$y^2 = 4ax \quad \text{... (i)}$$

$\therefore$ Length of latusrectum = $4a$

Since, the parabola passes through the point (3, 2).

Then, $4 = 4a(3)$

$\Rightarrow a = 1/3$

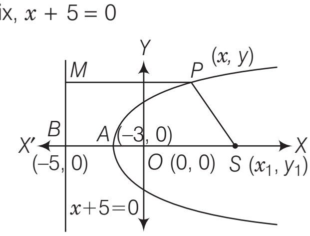
$\therefore 4a = 4/3$

Q. 53 If the vertex of the parabola is the point (– 3, 0) and the directrix is the line $x + 5 = 0$, then its equation is

(a) $y^2 = 8(x + 3)$
(b) $x^2 = 8(y + 3)$
(c) $y^2 = -8(x + 3)$
(d) $y^2 = 8(x + 5)$

Sol. *(a)* Here, vertex = (– 3, 0)

$\therefore a = -3$ and directrix, $x + 5 = 0$



Since, axis of the parabola is a line perpendicular to directrix and A is the mid-point of AS.

Then, $$-3 = \frac{x_1 - 5}{2}$$

$$\Rightarrow \quad -6 = x_1 - 5 \Rightarrow x_1 = -1,$$

$$0 = \frac{0 + y_1}{2} \Rightarrow y_1 = 0$$

$$\therefore \quad S = (-1, 0)$$

$$\because \quad PM = PS$$

$$\Rightarrow \quad |x + 5| = \sqrt{(x+1)^2 + y^2}$$

$$\Rightarrow \quad x^2 + 2x + 1 + y^2 = x^2 + 10x + 25$$

$$\Rightarrow \quad y^2 = +8x + 24$$

$$\Rightarrow \quad y^2 = +8(x + 3)$$

Q. 54 If equation of the ellipse whose focus is (1, – 1), then directrix the line $x - y - 3 = 0$ and eccentricity $\frac{1}{2}$ is

(a) $7x^2 + 2xy + 7y^2 - 10x + 10y + 7 = 0$

(b) $7x^2 + 2xy + 7y^2 + 7 = 0$

(c) $7x^2 + 2xy + 7y^2 + 10x - 10y - 7 = 0$

(d) None of the above

Sol. *(a)* Given that, focus of the ellipse is (1, – 1) and the equation of directrix is $x - y - 3 = 0$ and $e = \frac{1}{2}$

Let $P(x, y)$ and $F(1, -1)$.

$$\therefore \quad \frac{PF}{\text{Distance of } P \text{ from } (x - y - 3 = 0)} = \frac{1}{2}$$

$$\Rightarrow \quad \frac{\sqrt{(x-1)^2 + (y+1)^2}}{\frac{|x - y - 3|}{\sqrt{2}}} = \frac{1}{2}$$

$$\Rightarrow \quad \frac{2[x^2 - 2x + 1 + y^2 + 2y + 1]}{(x - y - 3)^2} = \frac{1}{4}$$

$$\Rightarrow \quad 8x^2 - 16x + 16 + 8y^2 + 16y = x^2 + y^2 + 9 - 2xy + 6y - 6x$$

$$\Rightarrow \quad 7x^2 + 7y^2 + 2xy - 10x + 10y + 7 = 0$$

Q. 55 The length of the latusrectum of the ellipse $3x^2 + y^2 = 12$ is

(a) 4 (b) 3 (c) 8 (d) $\frac{4}{\sqrt{3}}$

Thinking Process

First of all find the value of a and b from the given equation, after that get length of latusrectum by using formula $\frac{2a^2}{b}$.

Sol. *(d)* Given equation of ellipse is

$$3x^2 + y^2 = 12$$

$\Rightarrow \quad \dfrac{x^2}{4}+\dfrac{y^2}{12}=1$

$\therefore \quad a^2=4 \Rightarrow a=2$

and $\quad b^2=12 \Rightarrow b=2\sqrt{3}$

$\because \quad b>a$

$\therefore$ Length of latusrectum $=\dfrac{2\times 4}{2\sqrt{3}}=\dfrac{4}{\sqrt{3}}=\dfrac{2a^2}{b}$

Q. 56 If e is eccentricity of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ (where, $a<b$), then

(a) $b^2=a^2(1-e^2)$ (b) $a^2=b^2(1-e^2)$

(c) $a^2=b^2(e^2-1)$ (d) $b^2=a^2(e^2-1)$

Sol. ***(b)*** Given that, $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1,\ a<b$

We know that, $e=\sqrt{1-\dfrac{a^2}{b^2}} \Rightarrow e^2=\dfrac{(b^2-a^2)}{b^2}$

$\Rightarrow \quad b^2e^2=b^2-a^2$

$\Rightarrow \quad a^2=b^2(1-e^2)$

Q. 57 The eccentricity of the hyperbola whose latusrectum is 8 and conjugate axis is equal to half of the distance between the foci is

(a) $\dfrac{4}{3}$ (b) $\dfrac{4}{\sqrt{3}}$

(c) $\dfrac{2}{\sqrt{3}}$ (d) None of these

Sol. ***(c)*** Length of latusrectum of the hyperbola *i.e.,*

$$8=\frac{2b^2}{a} \Rightarrow b^2=4a \qquad \text{...(i)}$$

$\therefore$ Distance between the foci $=2ae$

Since, transverse axis be a and conjugate axis be b.

$\therefore \quad \dfrac{1}{2}(2ae)=2b$

$\Rightarrow \quad ae=2b \qquad$ ...(ii)

$\Rightarrow \quad b^2=a^2(e^2-1) \qquad$ ...(iii)

From Eqs. (i) and (ii),

$$4a=\frac{a^2e^2}{4}$$

$\Rightarrow \quad 16a=a^2e^2$

$\Rightarrow \quad 16=ae^2 \Rightarrow a=\dfrac{16}{e^2}$

$\because \quad 4a=a^2(e^2-1)$

$\Rightarrow \quad \dfrac{4}{a}=e^2-1$

$\Rightarrow \quad \dfrac{4e^2}{16}=e^2-1$

$$\Rightarrow \quad e^2\left(1-\frac{4}{16}\right)=1$$

$$\Rightarrow \quad e^2\left(\frac{12}{16}\right)=1 \Rightarrow e^2=\left(\frac{16}{12}\right)$$

$$\Rightarrow \quad e^2=\frac{4}{3} \Rightarrow e=\frac{2}{\sqrt{3}}$$

Q. 58 The distance between the foci of a hyperbola is 16 and its eccentricity is $\sqrt{2}$. Its equation is

(a) $x^2-y^2=32$ (b) $\frac{x^2}{4}-\frac{y^2}{9}=1$

(c) $2x-3y^2=7$ (d) None of these

Thinking Process

The distance between the foci of hyperbola is $2ae$ and $b^2=a^2(e^2-1)$. Use this relation to set the value of a and b.

Sol. ***(a)*** Given that, distance between the foci of hyperbola

i.e., $2ae=16 \Rightarrow ae=8$ ...(i)

and $e=\sqrt{2}$ ...(ii)

Now, $\sqrt{2}\,a=8$

$\Rightarrow \quad a=4\sqrt{2}$

$\because \quad b^2=a^2(e^2-1)$

$\Rightarrow \quad b^2=32(2-1)$

$\Rightarrow \quad b^2=32$

$\therefore \quad \frac{x^2}{32}-\frac{y^2}{32}=1$

$\Rightarrow \quad x^2-y^2=32$

Q. 59 Equation of the hyperbola with eccentricity $\frac{3}{2}$ and foci at $(\pm 2, 0)$ is

(a) $\frac{x^2}{4}-\frac{y^2}{5}=\frac{4}{9}$ (b) $\frac{x^2}{9}-\frac{y^2}{9}=\frac{4}{9}$

(c) $\frac{x^2}{4}-\frac{y^2}{9}=1$ (d) None of these

Sol. ***(a)*** Given that, eccentricity of the hyperbola, $e=3/2$

and foci $=(\pm 2, 0), (\pm ae, 0)$

$\because \quad ae=2$

$\Rightarrow \quad a\times 3/2=2 \Rightarrow a=4/3$

$\because \quad b^2=a^2(e^2-1)$

$\Rightarrow \quad b^2=\frac{16}{9}\left(\frac{9}{4}-1\right) \Rightarrow b^2=\frac{16}{9}\left(\frac{5}{4}\right)$

$\Rightarrow \quad b^2=\frac{20}{9}$

So, the equation of the hyperbola is

$$\frac{x^2}{\frac{16}{9}}-\frac{y^2}{20/9}=1 \Rightarrow \frac{x^2}{4}-\frac{y^2}{5}=\frac{4}{9}$$

12

Introduction to Three Dimensional Geometry

Short Answer Type Questions

Q. 1 Locate the following points

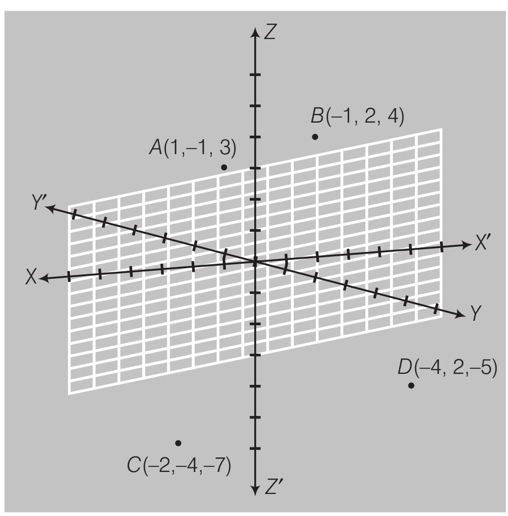
(i) (1, − 1,3) (ii) (− 1, 2, 4)

(iii) (− 2,− 4, − 7) (iv) (− 4, 2,− 5)

Sol. Given, coordinates are

(i) (1,–1,3) **(ii)** *B* (–1, 2, 4)

(iii) *C* (– 2, – 4,– 7) **(iv)** *D* (– 4, 2,– 5)



X-increment = *Y*-increment = *Z*-increment = 1

Q. 2 Name the octant in which each of the following points lies.

(i) (1, 2, 3,) (ii) (4,– 2, 3)
(iii) (4,– 2, – 5) (iv) (4, 2,– 5)
(v) (– 4, 2, 5) (iv) (– 3, – 1, 6)
(vii) (2,– 4, – 7) (viii) (– 4, 2, – 5).

Sol. **(i)** Point (1, 2 ,3) lies in first quadrant. **(ii)** (4, – 2, 3) in fourth octant.
(iii) (4, – 2, – 5) in eight octant. **(iv)** (4, 2, – 5) in fifth octant.
(v) (– 4, 2, 5) in second octant. **(vi)** (– 3, – 1, 6) in third octant.
(vii) (2,– 4, – 7) in eight octant. **(viii)** (– 4, 2, – 5) in sixth octant.

Q. 3 If *A*, *B*, *C* be the feet of perpendiculars from a point *P* on the *X*, *Y* and *Z*-axes respectively, then find the coordinates of *A*, *B* and *C* in each of the following where the point *P* is

(i) A (3, 4, 2) (ii) *B* (– 5, 3, 7)
(iii) *C* (4, – 3, – 5)

Sol. The coordinates of *A*, *B* and *C* are the following
(i) *A* (3, 0, 0), *B* (0,4, 0), *C* (0,0, 2)
(ii) *A* (– 5, 0, 0), *B* (0, 3, 0), *C* (0,0, 7)
(iii) *A* (4, 0, 0), *B* (0, – 3, 0), *C* (0,0,– 5)

Q. 4 If *A*, *B*, and *C* be the feet of perpendiculars from a point *P* on the *XY*, *YZ* and *ZX*-planes respectively, then find the coordinates of *A*, *B* and *C* in each of the following where the point *P* is

(i) (3, 4, 5) (ii) (– 5, 3, 7)
(iii) (4, – 3, – 5)

Sol. We know that, on *XY*-plane $z = 0$, on *YZ*-plane, $x = 0$ and on *ZX*-plane, $y = 0$. Thus, the coordinate of *A*, *B* and *C* are following
(i) *A* (3, 4, 0), *B* (0, 4, 5), *C* (3, 0, 5)
(ii) *A* (– 5, 3, 0), *B* (0, 3, 7), *C* (– 5, 0, 7)
(iii) *A* (4,– 3, 0), *B* (0, – 3, – 5), *C* (4, 0,– 5)

Q. 5 How far apart are the points (2, 0, 0) and (–3, 0, 0)?

Thinking Process

Distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2)

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$

Sol. Given points, *A* (2, 0, 0) and *B* (– 3, 0, 0)

$$AB = \sqrt{(2 + 3)^2 + 0^2 + 0^2} = 5$$

Q. 6 Find the distance from the origin to(6, 6, 7).

Sol. Distance from origin to the point (6, 6,7)

$$= \sqrt{(0-6)^2 + (0-6)^2 + (0-7)^2} \qquad [\because d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}]$$

$$= \sqrt{36 + 36 + 49}$$

$$= \sqrt{121} = 11$$

Q. 7 Show that, if $x^2 + y^2 = 1$, then the point $(x, y, \sqrt{1 - x^2 - y^2})$ is at a distance 1 unit form the origin.

Sol. Given that, $x^2 + y^2 = 1$

$\therefore$ Distance of the point $(x, y, \sqrt{1 - x^2 - y^2})$ from origin is given as

$$d = \left| \sqrt{x^2 + y^2 + (\sqrt{1 - x^2 - y^2})^2} \right|$$

$$= \left| \sqrt{x^2 + y^2 + 1 - x^2 - y^2} \right| = 1$$

Hence proved.

Q. 8 Show that the point A (1, – 1, 3), B (2, – 4, 5) and C (5, – 13, 11) are collinear.

Thinking Process

If the three points A, B, and C are collinear, then AB + BC = AC.

Sol. Given points, A (1, – 1, 3), B (2, – 4, 5) and C (5, – 13, 11).

$$AB = \sqrt{(1-2)^2 + (-1+4)^2 + (3-5)^2}$$

$$= \sqrt{1 + 9 + 4} = \sqrt{14}$$

$$BC = \sqrt{(2-5)^2 + (-4+13)^2 + (5-11)^2}$$

$$= \sqrt{9 + 81 + 36} = \sqrt{126}$$

$$AC = \sqrt{(1-5)^2 + (-1+13)^2 + (3-11)^2}$$

$$= \sqrt{16 + 144 + 64} = \sqrt{224}$$

$\because \quad AB + BC = AC$

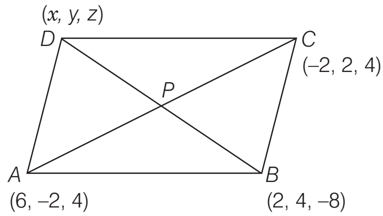
$\Rightarrow \quad \sqrt{14} + \sqrt{126} = \sqrt{224}$

$\Rightarrow \quad \sqrt{14} + 3\sqrt{14} = 4\sqrt{14}$

So, the points A, B and C are collinear.

Q. 9 Three consecutive vertices of a parallelogram *ABCD* are *A* (6, − 2, 4), *B* (2, 4, −8) and *C* (− 2, 2, 4). Find the coordinates of the fourth vertex.

Sol. Let the coordinates of the fourth vertices *D* (x, y, z).



Mid-points of diagonal *AC*,

$$x = \frac{x_1 + x_2}{2},\ y = \frac{y_1 + y_2}{2},\ z = \frac{z_1 + z_2}{2}$$

and
$$x = \frac{6 - 2}{2} = 2,\ y = \frac{-2 + 2}{2} = 0,\ z = \frac{4 + 4}{2} = 4$$

Since, the mid-point of *AC* are (2, 0, 4).

Now, mid-point of *BD*, $2 = \frac{x + 2}{2} \Rightarrow x = 2$

$\Rightarrow$ $$0 = \frac{y + 4}{2} \Rightarrow y = -4$$

$\Rightarrow$ $$4 = \frac{z - 8}{2} \Rightarrow z = 16$$

So, the coordinates of fourth vertex *D* is (2, – 4, 16).

Q. 10 Show that the ΔABC with vertices *A* (0, 4, 1), *B* (2, 3, – 1) and *C* (4, 5, 0) is right angled.

Thinking Process

In a right angled triangle sum of the square of two sides is equal to square of third side.

Sol. Given that, the vertices of the ΔABC are *A* (0, 4, 1), *B* (2, 3, – 1) and *C* (4, 5, 0).

Now,
$$AB = \sqrt{(0-2)^2 + (4-3)^3 + (1+1)^2}$$
$$= \sqrt{4 + 1 + 4} = 3$$
$$BC = \sqrt{(2-4)^2 + (3-5)^2 + (-1-0)^2}$$
$$= \sqrt{4 + 4 + 1} = 3$$
$$AC = \sqrt{(0-4)^2 + (4-5)^2 + (1-0)^2}$$
$$= \sqrt{16 + 1 + 1} = \sqrt{18}$$

$\because$ $$AC^2 = AB^2 + BC^2$$

$\Rightarrow$ $$18 = 9 + 9$$

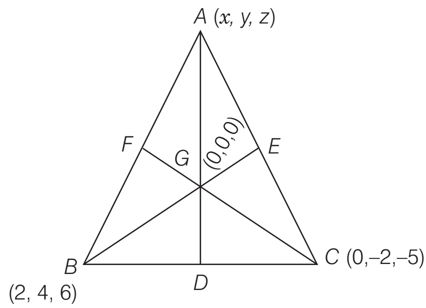
Hence, vertices ΔABC is a right angled triangle.

Q. 11 Find the third vertex of triangle whose centroid is origin and two vertices are (2, 4, 6) and (0, – 2, 5).

Thinking Process

The vertices of the ΔABC are $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$, then the coordinates of the centroid G are $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$.

Sol. Let third vertex of ΔABC *i.e.,* is $A(x, y, z)$.



Given that, the coordinate of centroid G are (0, 0, 0).

$$\because \quad 0 = \frac{x + 2 + 0}{3} \Rightarrow x = -2$$

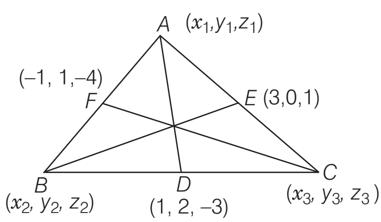
$$0 = \frac{y + 4 - 2}{3} \Rightarrow y = -2$$

$$0 = \frac{z + 6 - 5}{2} \Rightarrow z = -1$$

Hence, the third vertex of triangle is (– 2, – 2, –1).

Q. 12 Find the centroid of a triangle, the mid-point of whose sides are D (1, 2, – 3), E (3, 0, 1) and F (– 1, 1, – 4).

Sol. Given that, mid-points of sides are D (1, 2, – 3), E (3, 0, 1) and F (– 1, 1, – 4).



Let the vertices of the ΔABC are $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_3)$ and $C(x_3, y_3, z_3)$.
Then, mid-point of BC are (1, 2, – 3).

$$\therefore \quad 1 = \frac{x_2 + x_3}{2} \Rightarrow x_2 + x_3 = 2 \quad \text{...(i)}$$

$$2 = \frac{y_2 + y_3}{2} \Rightarrow y_2 + y_3 = 4 \quad \text{...(ii)}$$

$$-3 = \frac{z_2 + z_3}{2} \Rightarrow z_2 + z_3 = -6 \quad \text{...(iii)}$$

Similarly for the sides *AB* and *AC*,

$\Rightarrow \quad -1 = \dfrac{x_1 + x_2}{2} \Rightarrow x_1 + x_2 = -2$ ...(iv)

$\Rightarrow \quad 1 = \dfrac{y_1 + y_2}{2} \Rightarrow y_1 + y_2 = 2$ ...(v)

$\Rightarrow \quad -4 = \dfrac{z_1 + z_2}{2} \Rightarrow z_1 + z_2 = -8$ ...(vi)

$\Rightarrow \quad 3 = \dfrac{x_1 + x_3}{2} \Rightarrow x_1 + x_3 = 6$ ...(vii)

$\Rightarrow \quad 0 = \dfrac{y_1 + y_3}{2} \Rightarrow y_1 + y_3 = 0$ ...(viii)

$\Rightarrow \quad 1 = \dfrac{z_1 + z_3}{2} \Rightarrow z_1 + z_3 = 2.$ ...(ix)

On adding Eqs. (i) and (iv), we get

$$x_1 + 2x_2 + x_3 = 0 \quad \text{...(x)}$$

On adding Eqs. (ii) and (v), we get

$$y_1 + 2y_2 + y_3 = 6 \quad \text{...(xi)}$$

On adding Eqs. (iii) and (vi), we get

$$z_1 + 2z_2 + z_3 = -14 \quad \text{...(xii)}$$

From Eqs. (vii) and (x),

$$2x_2 = -6 \Rightarrow x_2 = -3$$

If $x_2 = -3$, then $x_3 = 5$

If $x_3 = 5$, then $x_1 = 1$, $\quad x_2 = -3, x_3 = 5$

From Eqs. (xi) and (viii),

$$2y_2 = 6 \Rightarrow y_2 = 3$$

If $y_2 = 3$, then $y_1 = -1$ $\quad$ If $y_1 = -1$, then $y_3 = 1$, $\quad y_2 = 3, y_3 = 1$

From Eqs. (xii) and (ix),

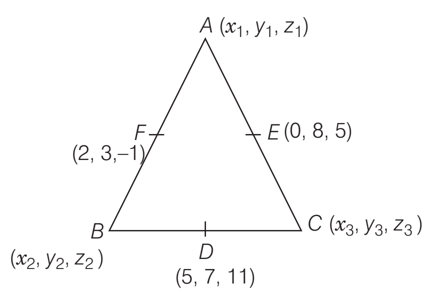
$$2z_2 = -16 \Rightarrow z_2 = -8$$
$$z_2 = -8, \text{ then } z_1 = 0$$
$$z_1 = 0, \text{ then } z_3 = 2$$
$$z_1 = 0, z_2 = -8, z_3 = 2$$

So, the points are *A* (1 – 1,0), *B* (– 3, 3,– 8) and *C* (5, 1, 2).

$\therefore$ Centroid of the triangle $= G\left(\dfrac{1-3+5}{3}, \dfrac{-1+3+1}{3}, \dfrac{0-8+2}{3}\right)$ *i.e.*, *G* (1, 1,– 2)

Q. 13 The mid-points of the sides of a triangle are (5, 7, 11), (0, 8, 5) and (2, 3, – 1). Find its vertices.

Sol. Let vertices of the ΔABC are $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$, then the mid-point of *BC* (5, 7, 11).



$$5 = \frac{x_2 + x_3}{2} \Rightarrow x_2 + x_3 = 10 \quad \text{...(i)}$$

$$7 = \frac{y_2 + y_3}{2} \Rightarrow y_2 + y_3 = 14 \quad \text{...(ii)}$$

$$11 = \frac{z_2 + z_3}{2} \Rightarrow z_2 + z_3 = 22 \quad \text{...(iii)}$$

Similarly for the sides *AB* and *AC*,

$$2 = \frac{x_1 + x_2}{2} \Rightarrow x_1 + x_2 = 4 \quad \text{...(iv)}$$

$$3 = \frac{y_1 + y_2}{2} \Rightarrow y_1 + y_2 = 6 \quad \text{...(v)}$$

$$-1 = \frac{z_1 + z_2}{2} \Rightarrow z_1 + z_2 = -2 \quad \text{...(vi)}$$

$$0 = \frac{x_1 + x_3}{2} \Rightarrow x_1 + x_3 = 0 \quad \text{...(vii)}$$

$$8 = \frac{y_1 + y_3}{2} \Rightarrow y_1 + y_3 = 16 \quad \text{...(viii)}$$

$$5 = \frac{z_1 + z_3}{2} \Rightarrow z_1 + z_3 = 10 \quad \text{...(ix)}$$

From Eqs. (i) and (iv),

$$x_1 + 2x_2 + x_3 = 14 \quad \text{...(x)}$$

From Eqs. (ii) and (v),

$$y_1 + 2y_2 + y_3 = 20 \quad \text{...(xi)}$$

From Eqs. (iii) and (vi),

$$z_1 + 2z_2 + z_3 = 20 \quad \text{...(xii)}$$

From Eqs. (vii) and (x),

$$2x_2 = 14 \Rightarrow x_2 = 7$$
$$x_2 = 7, \text{ then } x_3 = 10 - 7 = 3$$
$$x_3 = 3, \text{ then } x_1 = -3$$
$$x_1 = -3, x_2 = 7, x_3 = 3$$

From Eqs. (viii) and (xi),

$$2y_2 = 4 \Rightarrow y_2 = 2$$
$$y_2 = 2, \text{ then } y_1 = 4$$
$$y_1 = 4, \text{ then } y_3 = 12$$
$$y_1 = 4, y_2 = 2, y_3 = 12$$

From Eqs. (ix) and (xii),

$$2z_2 = 10 \Rightarrow z_2 = 5$$
$$z_2 = 5, \text{ then } z_1 = -7$$
$$z_1 = -7, \text{ then } z_3 = 17$$
$$z_1 = -7, z_2 = 5, z_3 = 17$$

So, the vertices are *A* (– 3, 4, – 7), *B* (7, 2, 5) and *C* (3, 12, 17).

Q. 14 If the vertices of a parallelogram *ABCD* are *A* (1, 2, 3), *B* (– 1, – 2, – 1) and *C* (2, 3, 2), then find the fourth vertex *D*.

Thinking Process

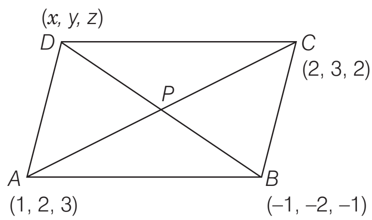
The diagonal of a parallelogram have the same vertices. Use this property to solve the problem.

Sol. Let the fourth vertex of the parallelogram *ABCD* is *D* (*x*, *y*, *z*). Then, the mid-point of *AC* are

$$P\left(\frac{1+2}{2}, \frac{2+3}{2}, \frac{3+2}{2}\right) \text{ i.e., } P\left(\frac{3}{2}, \frac{5}{2}, \frac{5}{2}\right).$$

Now, mid-point of BD,

$$\frac{3}{2} = \frac{-1 + x}{2} \Rightarrow x = 4$$
$$\frac{5}{2} = \frac{-2 + y}{2} \Rightarrow y = 7$$
$$\frac{5}{2} = \frac{-1 + z}{2} \Rightarrow z = 6$$



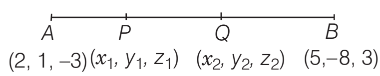
So, the coordinates of fourth vertex is (4, 7, 6).

Q. 15 Find the coordinate of the points which trisect the line segment joining the points A (2, 1, –3) and B (5, – 8, 3).

Thinking Process

If point P divided line segment joint the point $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ in $m_1 : m_2$ internally then the coordinate of P are $\left(\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_1y_2 + m_2y_1}{m_1 + m_2}, \frac{m_1z_2 + m_2z_1}{m_1 + m_2}\right)$

Sol. Let the $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ trisect line segment AB.



Since, the point P divided line AB in 1 : 2 internally, then

$$x_1 = \frac{2 \times 2 + 1 \times 5}{1 + 2} = \frac{9}{3} = 3$$
$$y_1 = \frac{2 \times 1 + 1 \times (-8)}{3} = \frac{-6}{3} = -2$$
$$z_1 = \frac{2 \times (-3) + 1 \times 3}{3} = \frac{-6 + 3}{3} = \frac{-3}{3} = -1$$

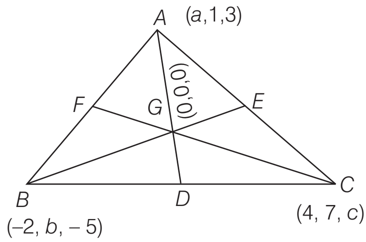
Since, the point Q divide the line segment AB in 2 : 1, then

$$x_2 = \frac{1 \times 2 + 2 \times 5}{3} = 4,$$
$$y_2 = \frac{1 \times 1 + (-8 \times 2)}{3} = -5$$
$$z_2 = \frac{1 \times (-3) + 2 \times 3}{3} = -1$$

So, the coordinates of P are (3, – 2, – 1) and the coordinates of Q are (4, – 5, 1).

Q. 16 If the origin is the centroid of a ΔABC having vertices $A(a, 1, 3)$, $B(-2, b, -5)$ and $C(4, 7, c)$, then find the values of a, b, c.

Sol. Given that origin is the centroid of the ΔABC *i.e.*, $G(0, 0, 0)$.



$\because$ $$0 = \frac{a - 2 + 4}{3} \Rightarrow a = -2$$

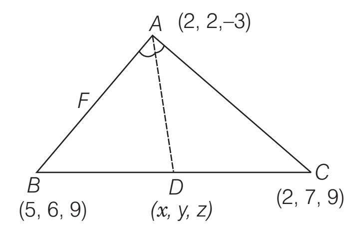
$$0 = \frac{1 + b + 7}{3} \Rightarrow b = -8$$

$$0 = \frac{3 - 5 + c}{3} \Rightarrow c = +2$$

$\therefore$ $$a = -2, b = -8 \text{ and } c = 2$$

Q. 17 If $A(2, 2, -3)$, $B(5, 6, 9)$, $C(2, 7, 9)$ be the vertices of a triangle. The internal bisector of the angle A meets BC at the point D, then find the coordinates of D.

Sol. Let the coordinates of D are (x, y, z).



$$AB = \sqrt{9 + 16 + 144} = \sqrt{169} = 13$$

$$AC = \sqrt{0 + 25 + 144} = \sqrt{169} = \sqrt{13}$$

$\Rightarrow$ $$\frac{AB}{AC} = \frac{13}{13} \Rightarrow AB = AC$$

$$\frac{BD}{DC} = \frac{1}{1} \Rightarrow BD = DC$$

Since, D is divide the line BC in two equal parts. So, D is the mid-point of BC.

$\therefore$ $$x = \frac{5 + 2}{2} = 7/2$$

$\Rightarrow$ $$y = \frac{6 + 7}{2} = 13/2$$

$\Rightarrow$ $$z = \frac{9 + 9}{2} = 9$$

So, the coordinates of D are $\left(\frac{7}{2}, \frac{13}{2}, 9\right)$.

Long Answer Type Questions

Q. 18 Show that the three points A (2, 3, 4), B (1, 2, – 3) and C (– 4, 1,– 10) are collinear and find the ratio in which C divides AB.

Sol. Given points are A (2, 3, 4), B (– 1, 2, – 3) and C (– 4, 1,– 10).

$$\therefore \quad AB = \sqrt{(2+1)^2 + (3-2)^2 + (4+3)^2}$$
$$= \sqrt{9 + 1 + 49} = \sqrt{59}$$
$$BC = \sqrt{(-1+4)^2 + (2-1)^2 + (-3+10)^2}$$
$$= \sqrt{9 + 1 + 49} = \sqrt{59}$$
$$AC = \sqrt{(2+4)^2 + (3-1)^2 + (4+10)^2}$$
$$= \sqrt{36 + 4 + 196}$$
$$= \sqrt{236} = 2\sqrt{59}$$

Now, $\quad AB + BC = \sqrt{59} + \sqrt{59} = 2\sqrt{59}$

$\because \quad AB + BC = AC$

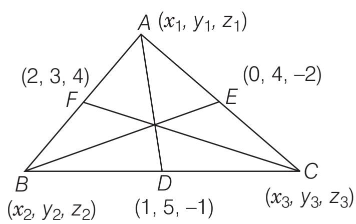
Hence, the points A, B and C are collinear.

Now, $\quad AC : BC = 2\sqrt{59} : \sqrt{59} = 2 : 1$

So, C divide AB in 2 :1 externally.

Q. 19 The mid-point of the sides of a triangle are (1, 5, – 1), (0, 4, – 2) and (2, 3, 4). Find its vertices and also find the centroid of the triangle.

Sol. Let the vertices of ΔABC are $A\,(x_1, y_1, z_1)$, $B\,(x_2, y_2, z_2)$ and $C\,(x_3, y_3, z_3)$.



Since, the mid-point of side BC is D (1, 5, –1).

Then, $\quad \dfrac{x_2 + x_3}{2} = 1 \Rightarrow x_2 + x_3 = 2 \qquad$ ...(i)

$$\frac{y_2 + y_3}{2} = 5 \Rightarrow y_2 + y_3 = 10 \qquad \text{...(ii)}$$
$$\frac{z_2 + z_3}{2} = -1 \Rightarrow z_2 + z_3 = -2 \qquad \text{...(iii)}$$

Similarly, the mid-points of AB and AC are F (2, 3, 4) and E (0, 4, – 2),

$$\frac{x_1 + x_2}{2} = 2 \Rightarrow x_1 + x_2 = 4 \qquad \text{...(iv)}$$
$$\frac{y_1 + y_2}{2} = 3 \Rightarrow y_1 + y_2 = 6 \qquad \text{...(v)}$$

and $\quad \dfrac{z_1 + z_2}{2} = 4 \Rightarrow z_1 + z_2 = 8 \qquad$ ...(vi)

Now, $\frac{x_1 + x_3}{2} = 0 \Rightarrow x_1 + x_3 = 0$ ...(vii)

$\frac{y_1 + y_3}{2} = 4 \Rightarrow y_1 + y_3 = 8$ ...(viii)

$\frac{z_1 + z_3}{2} = -2 \Rightarrow z_1 + z_3 = -4$ ...(ix)

From Eqs. (i) and (iv),

$x_1 + 2x_2 + x_3 = 6$ ...(x)

From Eqs. (ii) and (v),

$y_1 + 2y_2 + y_3 = 16$ ...(xi)

From Eqs. (iii) and (vi),

$z_1 + 2z_2 + z_3 = 6$ ...(xii)

From Eqs. (vii) and (x),

$2x_2 = 6 \Rightarrow x_2 = 3$

$x_2 = 3$, then $x_3 = -1$

$x_3 = -1$,

Then, $x_1 = 1 \Rightarrow x_1 = 1, x_2 = 3, x_2 = -1$

From Eqs. (viii) and (xi),

$2y_2 = 8 \Rightarrow y_2 = 4$

$y_2 = 4$,

Then, $y_1 = 2$

$y_1 = 2$,

Then, $y_3 = 6$,

$\Rightarrow$ $y_1 = 2, y_2 = 4, y_3 = 6$

From Eqs. (ix) and (xii),

$2z_2 = 10 \Rightarrow z_2 = 5$

$z_2 = 5$,

Then, $z_1 = 3$

$z_1 = 3$,

Then, $z_3 = -7$

$\Rightarrow$ $z_1 = 3, z_2 = 5, z_3 = -7$

So, the vertices of the triangle A (1, 2, 3), B (3, 4, 5) and C (– 1, 6, –7).

Hence, centroid of the triangle $G\left(\frac{1+3-1}{3}, \frac{2+4+6}{3}, \frac{3+5-7}{3}\right)$ *i.e.*, G (1, 4, 1/3).

Q. 20 Prove that the points (0, –1, – 7), (2, 1, – 9) and (6, 5, – 13) are collinear. Find the ratio in which the first point divides the join of the other two.

Thinking Process

First of all find the value of AB, AC and BC using distance formula i.e., $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2) + (z_1 - z_2)^2}$, *then show that* $AB + BC = AC$ *for collinearity of the points A, B and C.*

Sol. Given points are $A(0, -1, -7)$, B (2, 1, – 9) and C (6, 5, – 13)

$$AB = \sqrt{(0-2)^2 + (-1-1)^2 + (-7+9)^2} = \sqrt{4+4+4} = 2\sqrt{3}$$

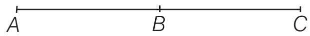
$$BC = \sqrt{(2-6)^2 + (1-5)^2 + (-9+13)^2} = \sqrt{16+16+16} = 4\sqrt{3}$$

$$AC = \sqrt{(0-6)^2 + (-1-5)^2 + (-7+13)^2} = \sqrt{36+36+36} = 6\sqrt{3}$$

$\because \quad AB + BC = 2\sqrt{3} + 4\sqrt{3} = 6\sqrt{3}$

So, $\quad AB + BC = AC$

Hence, the points A, B and C are collinear.



$$AB : AC = 2\sqrt{3} : 6\sqrt{3} = 1 : 3$$

So, point A divide B and C in $1 : 3$ externally.

Q. 21 What are the coordinates of the vertices of a cube whose edge is 2 units, one of whose vertices coincides with the origin and the three edges passing through the origin, coincides with the positive direction of the axes through the origin?

Sol. The coordinates of the cube which edge is 2 units, are (2, 0, 0), (2, 2, 0), (0, 2, 0), (0, 2, 2),(0, 0, 2), (2, 0, 2), (0, 0, 0) and (2, 2, 2).

Objective Type Questions

Q. 22 The distance of point $P(3, 4, 5)$ from the YZ-plane is

(a) 3 units (b) 4 units
(c) 5 units (d) 550

Sol. ***(a)*** Given, point is P (3, 4, 5).

Distance of P from YZ-plane, $\quad [\because YZ\text{-plane}, x = 0]$

$$d = \sqrt{(0-3)^2 + (4-4)^2 + (5-5)^2} = 3$$

Q. 23 What is the length of foot of perpendicular drawn from the point P (3, 4, 5) on Y-axis?

(a) $\sqrt{41}$ (b) $\sqrt{34}$
(c) 5 (d) None of these

Sol. ***(b)*** We know that, on the Y-axis, $x = 0$ and $z = 0$.

$\therefore$ Point A (0, 4, 0),

$$PA = \sqrt{(0-3)^2 + (4-4)^2 + (0-5)^2}$$
$$= \sqrt{9 + 0 + 25} = \sqrt{34}$$

Q. 24 Distance of the point (3, 4, 5) from the origin (0, 0, 0) is

(a) $\sqrt{50}$ (b) 3
(c) 4 (d) 5

Sol. ***(a)*** Given, points P (3, 4, 5) and O (0, 0, 0),

$$PO = \sqrt{(0-3)^2 + (0-4)^2 + (0-5)^2}$$
$$= \sqrt{9 + 16 + 25} = \sqrt{50}$$

Q. 25 If the distance between the points $(a, 0, 1)$ and $(0, 1, 2)$ is $\sqrt{27}$, then the value of a is

(a) 5
(b) ± 5
(c) -5
(d) None of these

Sol. *(b)* Given, the points are $A(a, 0, 1)$ and $B(0, 1, 2)$.

$\therefore \quad AB = \sqrt{(a-0)^2 + (0-1)^2 + (1-2)^2}$

$\Rightarrow \quad \sqrt{27} = \sqrt{a^2 + 1 + 1}$

$\Rightarrow \quad 27 = a^2 + 2$

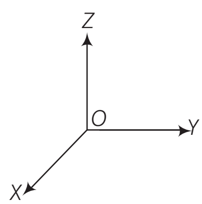
$\Rightarrow \quad a^2 = 25$

$\Rightarrow \quad a = \pm 5$

Q. 26 X-axis is the intersection of two planes

(a) XY and XZ
(b) YZ and ZX
(c) XY and YZ
(d) None of these

Sol. *(a)* We know that, on the XY and XZ-planes, the line of intersection is X-axis.



Q. 27 Equation of Y-axis is considered as

(a) $x = 0, y = 0$
(b) $y = 0, z = 0$
(c) $z = 0, x = 0$
(d) None of these

Sol. *(c)* On the Y-axis, $x = 0$ and $z = 0$.

Q. 28 The point $(-2, -3, -4)$ lies in the

(a) first octant
(b) seventh octant
(c) second octant
(d) eight octant

Sol. *(b)* The point $(-2, -3, -4)$ lies in seventh octant.

Q. 29 A plane is parallel to YZ-plane, so it is perpendicular to

(a) X-axis
(b) Y-axis
(c) Z-axis
(d) None of these

Sol. *(a)* A plane is parallel to YZ-plane, so it is perpendicular to X-axis.

Q. 30 The locus of a point for which $y = 0$ and $z = 0$, is

(a) equation of X-axis
(b) equation of Y-axis
(c) equation at Z-axis
(d) None of these

Sol. *(a)* We know that, equation on the X-axis, $y = 0$, $z = 0$.
So, the locus of the point is equation of X-axis.

Q. 31 The locus of a point for which $x = 0$ is

(a) XY-plane
(b) YZ-plane
(c) ZX-plane
(d) None of these

Sol. *(b)* On the YZ-plane, $x = 0$, hence the locus of the point is YZ-plane.

Q. 32 If a parallelopiped is formed by planes drawn through the points (5, 8, 10) and (3, 6, 8) parallel to the coordinate planes, then the length of diagonal of the parallelopiped is

(a) $2\sqrt{3}$
(b) $3\sqrt{2}$
(c) $\sqrt{2}$
(d) $\sqrt{3}$

Sol. *(a)* Given points of the parallelopiped are A (5, 8, 10) and B (3, 6, 8).

$$\therefore \quad AB = \sqrt{(5-3)^2 + (6-8)^2 + (10-8)^2}$$
$$= \sqrt{4 + 4 + 4} = 2\sqrt{3}$$

Q. 33 L is the foot of the perpendicular drawn from a point P (3, 4, 5) on the XY-plane. The coordinates of point L are

(a) (3, 0, 0)
(b) (0, 4, 5)
(c) (3, 0, 5)
(d) None of these

Sol. *(d)* We know that, on the XY-plane $z = 0$.
Hence, the coordinates of the points L are (3, 4, 0).

Q. 34 L is the foot of the perpendicular drawn from a point (3, 4, 5) on X-axis. The coordinates of L are

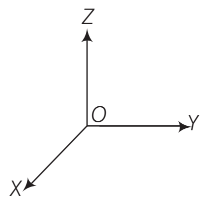
(a) (3, 0, 0)
(b) (0, 4, 0)
(c) (0, 0, 5)
(d) None of these

Sol. *(a)* On the X-axis, $y = 0$ and $z = 0$
Hence, the required coordinates are (3, 0, 0).

Fillers

Q. 35 The three axes OX, OY and OZ determine ....

Sol. The three axes OX, OY and OZ determine three coordinates planes.



Q. 36 The three planes determine a rectangular parallelopiped which has .... of rectangular faces.

Sol. Three points

Q. 37 The coordinates of a point are the perpendicular distance from the .... on the respectives axes.

Sol. Given points

Q. 38 The three coordinate planes divide the space into .... parts.

Sol. Eight parts

Q. 39 If a point P lies in YZ-plane, then the coordinates of a point on YZ-plane is of the form ....

Sol. We know that, on YZ-plane, $x = 0$. So, the coordinates of the required point is $(0, y, z)$.

Q. 40 The equation of YZ-plane is ....

Sol. The equation of YZ-plane is $x = 0$.

Q. 41 If the point P lies on Z-axis, then coordinates of P are of the form ....

Sol. On the Z-axis, $x = 0$ and $y = 0$. So, the required coordinates are $(0, 0, z)$.

Q. 42 The equation of Z-axis, are ....

Sol. The equation of Z-axis, $x = 0$ and $y = 0$.

Q. 43 A line is parallel to XY-plane if all the points on the line have equal ....

Sol. z-coordinates.

Q. 44 A line is parallel to X-axis, if all the points on the line have equal ....

Sol. y and z-coordinates.

Q. 45 $x = a$ represent a plane parallel to ....

Sol. $x = a$ represent a plane parallel to YZ-plane.

Q. 46 The plane parallel to YZ-plane is perpendicular to ....

Sol. The plane parallel to YZ-plane is perpendicular to X-axis.

Q. 47 The length of the longest piece of a string that can be stretched straight in a rectangular room whose dimensions are 10, 13 and 8 units are ....

Sol. Given dimensions are $a = 10$, $b = 13$ and $c = 8$.

$$\therefore \quad \text{Required length} = \sqrt{a^2 + b^2 + c^2}$$
$$= \sqrt{100 + 169 + 64} = \sqrt{333}$$

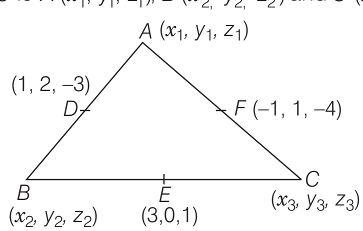
Q. 48 If the distance between the points $(a, 2, 1)$ and $(1, -1, 1)$ is 5, then a ....

Sol. Given points are $(a, 2, 1)$ and $(1, -1, 1)$.

$$\therefore \quad \sqrt{(a-1)^2 + (2+1)^2 + (1-1)^2} = 5$$
$$\Rightarrow \quad (a-1)^2 + 9 + 0 = 25$$
$$\Rightarrow \quad a^2 - 2a + 1 + 9 = 25$$
$$\Rightarrow \quad a^2 - 2a - 15 = 0$$
$$\Rightarrow \quad a^2 - 5a + 3a - 15 = 0$$
$$\Rightarrow \quad a(a-5) + 3(a-5) = 0$$
$$\Rightarrow \quad (a-5)(a+3) = 0$$
$$\Rightarrow \quad a - 5 = 0 \text{ or } a + 3 = 0$$
$$\therefore \quad a = +5 \text{ or } -3$$

Q. 49 If the mid-points of the sides of a triangle AB, BC and CA are $D\,(1, 2, -3)$, $E\,(3, 0, 1)$ and $F\,(-1, 1, -4)$, then the centroid of the ΔABC is ....

Sol. Let the vertices of ΔABC is $A\,(x_1, y_1, z_1)$, $B\,(x_2, y_2, z_2)$ and $C\,(x_3, y_3, z_3)$.



Since, D is the mid-point of AB, then

$$\frac{x_1 + x_2}{2} = 1 \Rightarrow x_1 + x_2 = 2 \quad ...(i)$$

$$\frac{y_1 + y_2}{2} = 2 \Rightarrow y_1 + y_2 = 4 \quad ...(ii)$$

$$\frac{z_1 + z_2}{2} = -3 \Rightarrow z_1 + z_2 = -6 \quad ...(iii)$$

Similarly, E and F are the mid-points of sides BC and AC, respectively.

$$\frac{x_2 + x_3}{2} = 3 \Rightarrow x_2 + x_3 = 6 \quad ...(iv)$$

$$\frac{y_2 + y_3}{2} = 0 \Rightarrow y_2 + y_3 = 0 \quad ...(v)$$

$$\frac{z_2 + z_3}{2} = 1 \Rightarrow z_2 + z_3 = 2 \quad ...(vi)$$

$$\frac{x_1 + x_3}{2} = -1 \Rightarrow x_1 + x_3 = -2 \quad ...(vii)$$

$$\frac{y_1 + y_3}{2} = 1 \Rightarrow y_1 + y_3 = 2 \quad ...(viii)$$

$$\frac{z_1 + z_3}{2} = -4 \Rightarrow z_1 + z_3 = -8 \quad ...(ix)$$

From Eqs. (i) and (iv),

$$x_1 + 2x_2 + x_3 = 8 \quad ...(x)$$

From Eqs. (ii) and (v),

$$y_1 + 2y_2 + y_3 = 4 \quad ...(xi)$$

From Eqs. (iii) and (vi),

$$z_1 + 2z_2 + z_3 = -4 \quad ...(xii)$$

From Eqs. (vii) and (x),

$$2x_2 = 10 \Rightarrow x_2 = 5$$

$\Rightarrow$ $x_2 = 5$, then $x_3 = 1$

If $x_3 = 1$, then $x_1 = -3$

$\because$ $x_1 = -3,\ x_2 = 5,\ x_3 = 1$

From Eqs. (viii) and (xi),

$$2y_2 = 2 \Rightarrow y_2 = 1$$

If $y_2 = 1$, then $y_3 = -1$

If $y_3 = -1$, then $y_1 = 3$

$\because$ $y_1 = 3,\ y_2 = 1,\ y_3 = -1$

From Eqs. (ix) and (xii),

$$2z_2 = 4 \Rightarrow z_2 = 2$$

If $z_2 = 2$, then $z_3 = 0$

If $z_3 = 0$, then $z_1 = -8$

$\because$ $z_1 = -8, z_2 = 2,\ z_3 = 0$

So, the vertices of ΔABC are A (– 3, 3, – 8), B (5, 1, 2) and C (1, – 1, 0).

Hence, coordinates of centroid of ΔABC, $G\left(\frac{-3+5+1}{3}, \frac{3+1-1}{3}, \frac{-8+2+0}{3}\right)$

i.e., G (1, 1, – 2).

Q. 50 Match each item given under the Column I to its correct answer given under Column II.

	Column I		Column II
(i)	In -XY-plane	(a)	Ist octant
(ii)	Point (2, 3, 4) lies in the	(b)	YZ-plane
(iii)	Locus of the points having X coordinate 0 is	(c)	z-coordinate is zero
(iv)	A line is parallel to X-axis if and only	(d)	Z-axis
(v)	If $X = 0$, $y = 0$ taken together will represent the	(e)	plane parallel to XY-plane
(vi)	$z = c$ represent the plane	(f)	if all the points on the line have equal y and z-coordinates
(vii)	Planes $X = a$, $Y = b$ represent the line	(f)	from the point on the respective
(viii)	Coordinates of a point are the distances from the origin to the feet of perpendiculars	(h)	parallel to Z-axis
(ix)	A ball is the solid region in the space enclosed by a	(i)	disc
(x)	Region in the plane enclosed by a circle is known as a	(j)	sphere

Sol. **(i)** In XY-plane, z-coordinates is zero.

(ii) The point (2, 3, 4) lies in 1st octant .

(iii) Locus of the points having x-coordinate is zero is YZ-plane.

(iv) A line is parallel to X-axis if and only if all the points on the line have equal y and z -coordinates.

(v) $x = 0$, $y = 0$ represent Z-axis.

(vi) $z = c$ represent the plane parallel to XY-plane.

(vii) The planes $x = a$, $y = b$ represent the line parallel to Z-axis.

(viii) Coordinates of a point are the distances from the origin to the feet of perpendicular from the point on the respective.

(ix) A ball is the solid region in the space enclosed by a sphere.

(x) The region in the plane enclosed by a circle is known as a disc.

13

Limits and Derivatives

Short Answer Type Questions

Q. 1 Evaluate $\lim_{x \to 3} \frac{x^2 - 9}{x - 3}$.

Sol. Given,

$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = \lim_{x \to 3} \frac{x^2 - (3)^2}{x - 3}$$

$$= \lim_{x \to 3} \frac{(x + 3)(x - 3)}{(x - 3)} = \lim_{x \to 3} (x + 3)$$

$$= 3 + 3 = 6$$

Q. 2 Evaluate $\lim_{x \to 1/2} \frac{4x^2 - 1}{2x - 1}$.

Sol. Given,

$$\lim_{x \to 1/2} \frac{4x^2 - 1}{2x - 1} = \lim_{x \to 1/2} \frac{(2x)^2 - (1)^2}{2x - 1}$$

$$= \lim_{x \to 1/2} \frac{(2x + 1)(2x - 1)}{(2x - 1)} = \lim_{x \to 1/2} (2x + 1)$$

$$= 2 \times \frac{1}{2} + 1 = 1 + 1 = 2$$

Q. 3 Evaluate $\lim_{h \to 0} \frac{\sqrt{x + h} - \sqrt{x}}{h}$.

Sol. Given,

$$\lim_{h \to 0} \frac{\sqrt{x + h} - \sqrt{x}}{h} = \lim_{h \to 0} \frac{(x + h)^{1/2} - (x)^{1/2}}{x + h - x}$$

$$= \lim_{h \to 0} \frac{(x + h)^{1/2} - (x)^{1/2}}{(x + h) - x} \qquad \left[\because \lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}\right]$$

$$= \frac{1}{2} x^{\frac{1}{2} - 1} = \frac{1}{2} x^{-1/2} \qquad [\because h \to 0 \Rightarrow x + h \to x]$$

$$= \frac{1}{2\sqrt{x}}$$

Q. 4 Evaluate $\lim\limits_{x \to 0} \dfrac{(x+2)^{1/3} - 2^{1/3}}{x}$.

Sol. Given, $\lim\limits_{x \to 0} \dfrac{(x+2)^{1/3} - 2^{1/3}}{x} = \lim\limits_{x \to 0} \dfrac{(x+2)^{1/3} - 2^{1/3}}{(x+2) - 2}$

$$= \frac{1}{3} \times 2^{\frac{1}{3} - 1}$$

$$= \frac{1}{3} \times (2)^{-2/3} \qquad \left[\because \lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}\right]$$

$$= \frac{1}{3\,(2)^{2/3}} \qquad [\because x \to 0 \Rightarrow x + 2 \to 2]$$

Q. 5 Evaluate $\lim\limits_{x \to 0} \dfrac{(1+x)^6 - 1}{(1+x)^2 - 1}$.

Sol. Given, $\lim\limits_{x \to 0} \dfrac{(1+x)^6 - 1}{(1+x)^2 - 1} = \lim\limits_{x \to 0} \dfrac{\dfrac{(1+x)^6 - 1}{x}}{\dfrac{(1+x)^2 - 1}{x}}$ [dividing numerator and denominator by x]

$$= \lim_{x \to 0} \frac{\dfrac{(1+x)^6 - 1}{(1+x) - 1}}{\dfrac{(1+x)^2 - 1}{(1+x) - 1}} \qquad [\because x \to 0 \Rightarrow 1 + x \to 1]$$

$$= \frac{\lim\limits_{x \to 0} \dfrac{(1+x)^6 - (1)^6}{(1+x) - 1}}{\lim\limits_{x \to 0} \dfrac{(1+x)^2 - (1)^2}{(1+x) - 1}} \qquad \left[\because \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)}\right]$$

$$= \frac{6\,(1)^{6-1}}{2\,(1)^{2-1}} \qquad \left[\therefore \lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}\right]$$

$$= \frac{6 \times 1}{2 \times 1} = \frac{6}{2} = 3$$

Q. 6 Evaluate $\lim\limits_{x \to a} \dfrac{(2+x)^{5/2} - (a+2)^{5/2}}{x - a}$.

Sol. Given, $\lim\limits_{x \to a} \dfrac{(2+x)^{5/2} - (a+2)^{5/2}}{x - a} = \lim\limits_{x \to a} \dfrac{(2+x)^{5/2} - (a+2)^{5/2}}{(2+x) - (a+2)}$

$$= \frac{5}{2}(a+2)^{\frac{5}{2} - 1} \qquad \left[\because \lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}\right]$$

$$= \frac{5}{2}(a+2)^{3/2} \qquad [\because x \to a \Rightarrow x + 2 \to a + 2]$$

Q. 7 Evaluate $\lim_{x \to 1} \dfrac{x^4 - \sqrt{x}}{\sqrt{x} - 1}$.

Sol. Given, $\lim_{x \to 1} \dfrac{x^4 - \sqrt{x}}{\sqrt{x} - 1} = \lim_{x \to 1} \dfrac{\sqrt{x}[(x)^{7/2} - 1]}{\sqrt{x} - 1}$

$$= \lim_{x \to 1} \frac{(x)^{7/2} - 1}{\sqrt{x} - 1} \cdot \lim_{x \to 1} \sqrt{x} \qquad \left[\because \lim_{x \to a} f(x) \cdot g(x) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)\right]$$

$$= \lim_{x \to 1} \frac{\dfrac{x^{7/2} - 1}{x - 1}}{\dfrac{(x)^{1/2} - 1}{x - 1}} \cdot 1$$

$$= \frac{\lim\limits_{x \to 1} \dfrac{x^{7/2} - 1}{x - 1}}{\lim\limits_{x \to 1} \dfrac{(x)^{1/2} - 1}{x - 1}} \qquad \left[\because \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)}\right]$$

$$= \frac{\dfrac{7}{2}(1)^{\frac{7}{2} - 1}}{\dfrac{1}{2}(1)^{\frac{1}{2} - 1}} = \frac{\dfrac{7}{2}}{\dfrac{1}{2}} = 7$$

Q. 8 Evaluate $\lim_{x \to 2} \dfrac{x^2 - 4}{\sqrt{3x - 2} - \sqrt{x + 2}}$.

Sol. Given, $\lim_{x \to 2} \dfrac{x^2 - 4}{\sqrt{3x - 2} - \sqrt{x + 2}} = \lim_{x \to 2} \dfrac{(x^2 - 4)\sqrt{3x - 2} + \sqrt{x + 2})}{(\sqrt{3x - 2} - \sqrt{x + 2})(\sqrt{3x - 2} - \sqrt{x + 2})}$

$$= \lim_{x \to 2} \frac{(x^2 - 4)(\sqrt{3x - 2} + \sqrt{x + 2})}{(\sqrt{3x - 2})^2 - (\sqrt{x + 2})^2}$$

$$[\because (a + b)(a - b) = a^2 - b^2]$$

$$= \lim_{x \to 2} \frac{(x^2 - 4)(\sqrt{3x - 2} + \sqrt{x + 2})}{(3x - 2) - (x + 2)}$$

$$= \lim_{x \to 2} \frac{(x^2 - 4)(\sqrt{3x - 2} + \sqrt{x + 2})}{3x - 2 - x - 2}$$

$$= \lim_{x \to 2} \frac{(x^2 - 4)(\sqrt{3x - 2} + \sqrt{x + 2})}{2x - 4}$$

$$= \lim_{x \to 2} \frac{(x + 2)(x - 2)(\sqrt{3x - 2} + \sqrt{x + 2})}{2\,(x - 2)}$$

$$= \lim_{x \to 2} \frac{(x + 2)\,(\sqrt{3x - 2} + \sqrt{x + 2})}{2}$$

$$= \frac{(2 + 2)(\sqrt{6 - 2} + \sqrt{2 + 2})}{2}$$

$$= \frac{4\,(2 + 2)}{2} = 8$$

Q. 9 Evaluate $\lim_{x \to \sqrt{2}} \dfrac{x^4 - 4}{x^2 + 3\sqrt{2}x - 8}$.

Sol. Given, $\lim_{x \to \sqrt{2}} \dfrac{x^4 - 4}{x^2 + 3\sqrt{2}x - 8} = \lim_{x \to \sqrt{2}} \dfrac{(x^2)^2 - (2)^2}{x^2 + 3\sqrt{2}x - 8}$

$$= \lim_{x \to \sqrt{2}} \frac{(x^2 - 2)(x^2 + 2)}{x^2 + 4\sqrt{2}x - \sqrt{2}x - 8}$$

$$= \lim_{x \to \sqrt{2}} \frac{(x - \sqrt{2})(x + \sqrt{2})(x^2 + 2)}{x(x + 4\sqrt{2}) - \sqrt{2}(x + 4\sqrt{2})}$$

$$= \lim_{x \to \sqrt{2}} \frac{(x - \sqrt{2})(x + \sqrt{2})(x^2 + 2)}{(x - \sqrt{2})(x + 4\sqrt{2})}$$

$$= \lim_{x \to \sqrt{2}} \frac{(x + \sqrt{2})(x^2 + 2)}{(x + 4\sqrt{2})}$$

$$= \frac{(\sqrt{2} + \sqrt{2})[(\sqrt{2})^2 + 2]}{(\sqrt{2} + 4\sqrt{2})}$$

$$= \frac{2\sqrt{2}(2 + 2)}{5\sqrt{2}} = \frac{8}{5}$$

Q. 10 Evaluate $\lim_{x \to 1} \dfrac{x^7 - 2x^5 + 1}{x^3 - 3x^2 + 2}$.

Sol. Given, $\lim_{x \to 1} \dfrac{x^7 - 2x^5 + 1}{x^3 - 3x^2 + 2}$ $\left[\dfrac{0}{0} \text{ form}\right]$

$$= \lim_{x \to 1} \frac{x^7 - x^5 - x^5 + 1}{x^3 - x^2 - 2x^2 + 2}$$

$$= \lim_{x \to 1} \frac{x^5(x^2 - 1) - 1(x^5 - 1)}{x^2(x - 1) - 2(x^2 - 1)}$$

On dividing numerator and denominator by $(x - 1)$, then

$$= \lim_{x \to 1} \frac{\dfrac{x^5(x^2 - 1)}{(x - 1)} - \dfrac{1(x^2 - 1)}{(x - 1)}}{\dfrac{x^2(x - 1)}{(x - 1)} - \dfrac{2(x^2 - 1)}{(x - 1)}}$$

$$= \frac{\lim_{x \to 1} x^5(x + 1) - \lim_{x \to 1} \left(\dfrac{x^5 - 1}{x - 1}\right)}{\lim_{x \to 1} x^2 - \lim_{x \to 1} (x + 1)}$$

$$= \frac{1 \times 2 - 5 \times (1)^4}{1 - 2 \times 2} = \frac{2 - 5}{1 - 4}$$

$$= \frac{-3}{-3} = 1$$

Q. 11 Evaluate $\lim_{x \to 0} \frac{\sqrt{1+x^3} - \sqrt{1-x^3}}{x^2}$.

Sol. Given, $\lim_{x \to 0} \frac{\sqrt{1+x^3} - \sqrt{1-x^3}}{x^2} = \lim_{x \to 0} \frac{\sqrt{1+x^3} - \sqrt{1-x^3}}{x^2} \cdot \frac{\sqrt{1+x^3} + \sqrt{1-x^3}}{\sqrt{1+x^3} + \sqrt{1-x^3}}$

$$= \lim_{x \to 0} \frac{(1+x^3) - (1-x^3)}{x^2(\sqrt{1+x^3} + \sqrt{1-x^3})}$$

$$= \lim_{x \to 0} \frac{1+x^3 - 1 + x^3}{x^2(\sqrt{1+x^3} + \sqrt{1-x^3})}$$

$$= \lim_{x \to 0} \frac{2x^3}{x^2(\sqrt{1+x^3} + \sqrt{1-x^3})}$$

$$= \lim_{x \to 0} \frac{2x}{(\sqrt{1+x^3} + \sqrt{1-x^3})}$$

$$= 0$$

Q. 12 Evaluate $\lim_{x \to -3} \frac{x^3 + 27}{x^5 + 243}$.

Sol. Given, $\lim_{x \to -3} \frac{x^3 + 27}{x^5 + 243} = \lim_{x \to -3} \frac{\frac{x^3 + 27}{x + 3}}{\frac{x^5 + 243}{x + 3}}$

$$= \lim_{x \to -3} \frac{\frac{x^3 - (-3)^3}{x - (-3)}}{\frac{x^5 - (-3)^5}{x - (-3)}} = \frac{\lim_{x \to -3} \frac{x^3 - (-3)^3}{x - (-3)}}{\lim_{x \to -3} \frac{x^5 - (-3)^5}{x - (-3)}} \quad \left[\because \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}\right]$$

$$= \frac{3(-3)^{3-1}}{5(-3)^{5-1}} = \frac{3(-3)^2}{5(-3)^4} \quad \left[\because \lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}\right]$$

$$= \frac{3}{5(-3)^2} = \frac{3}{45} = \frac{1}{15}$$

Q. 13 Evaluate $\lim_{x \to 1/2} \left(\frac{8x - 3}{2x - 1} - \frac{4x^2 + 1}{4x^2 - 1}\right)$.

Sol. Given, $\lim_{x \to 1/2} \left(\frac{8x - 3}{2x - 1} - \frac{4x^2 + 1}{4x^2 - 1}\right) = \lim_{x \to 1/2} \left[\frac{(8x - 3)(2x + 1) - (4x^2 + 1)}{(4x^2 - 1)}\right]$

$$= \lim_{x \to 1/2} \left[\frac{16x^2 + 8x - 6x - 3 - 4x^2 - 1}{4x^2 - 1}\right]$$

$$= \lim_{x \to 1/2} \left[\frac{12x^2 + 2x - 4}{4x^2 - 1}\right]$$

$$= \lim_{x \to 1/2} \frac{2(6x^2 + x - 2)}{4x^2 - 1}$$

$$= \lim_{x \to 1/2} \frac{2\,(6x^2 + 4x - 3x - 2)}{4x^2 - 1}$$

$$= \lim_{x \to 1/2} \frac{2\,[2x(3x + 2) - 1\,(3x + 2)]}{4x^2 - 1}$$

$$= \lim_{x \to 1/2} \frac{2\,[(3x + 2)(2x - 1)]}{(2x)^2 - (1)^2}$$

$$= \lim_{x \to 1/2} \frac{2\,(3x + 2)\,(2x - 1)}{(2x - 1)\,(2x + 1)}$$

$$= \lim_{x \to 1/2} \frac{2\,(3x + 2)}{2x - 1} = \frac{2\left(3 \times \frac{1}{2} + 2\right)}{2 \times \frac{1}{2} + 1}$$

$$= \frac{3}{2} + 2 = \frac{7}{2}$$

Q. 14 Find the value of n, if $\lim_{x \to 2} \frac{x^n - 2^n}{x - 2} = 80$, $n \in N$.

Sol. Given, $\lim_{x \to 2} \frac{x^n - 2^n}{x - 2} = 80$

$\Rightarrow$ $n(2)^{n-1} = 80$ $\left[\because \lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}\right]$

$\Rightarrow$ $n(2)^{n-1} = 5 \times 16$

$\Rightarrow$ $n \times 2^{n-1} = 5 \times (2)^4$

$\Rightarrow$ $n \times 2^{n-1} = 5 \times (2)^{5-1}$

$\therefore$ $n = 5$

Q. 15 Evaluate $\lim_{x \to 0} \frac{\sin 3x}{\sin 7x}$.

Sol. Given,

$$\lim_{x \to 0} \frac{\frac{\sin 3x}{3x} \cdot 3x}{\frac{\sin 7x}{7x} \cdot 7x} = \frac{\lim_{x \to 0} \frac{\sin 3x}{3x}}{\lim_{x \to 0} \frac{\sin 7x}{7x}} \cdot \frac{3x}{7x}$$

$$= \frac{3}{7} \cdot \frac{\lim_{x \to 0} \frac{\sin 3x}{3x}}{\lim_{x \to 0} \frac{\sin 7x}{7x}} \qquad \left[\because \lim_{x \to 0} \frac{\sin x}{x} = 1\right]$$

$$= \frac{3}{7} \qquad [\because x \to 0 \Rightarrow (kx \to 0), \text{ here } k \text{ is real number}]$$

Q. 16 Eavaluate $\lim_{x \to 0} \dfrac{\sin^2 2x}{\sin^2 4x}$.

Sol. Given,
$$\lim_{x \to 0} \frac{\sin^2 2x}{\sin^2 4x} = \lim_{x \to 0} \frac{\sin^2 2x}{[\sin 2\,(2x)]^2}$$
$$= \lim_{x \to 0} \frac{\sin^2 2x}{(2\sin 2x \cos 2x)^2}$$
$$= \lim_{x \to 0} \frac{\sin^2 2x}{4\sin^2 2x \cos^2 2x} \quad [\because \sin 2\theta = 2\sin\theta\cos\theta]$$
$$= \lim_{x \to 0} \frac{1}{4\cos^2 2x} = \frac{1}{4} \quad [\because \cos 0 = 1]$$

Q. 17 Evaluate $\lim_{x \to 0} \dfrac{1 - \cos 2x}{x^2}$.

Sol. Given,
$$\lim_{x \to 0} \frac{1 - \cos 2x}{x^2} = \lim_{x \to 0} \frac{1 - 1 + 2\sin^2 x}{x^2} \quad [\because \cos 2x = 1 - 2\sin^2 x]$$
$$= \lim_{x \to 0} \frac{2\sin^2 x}{x^2} = 2\lim_{x \to 0} \frac{\sin^2 x}{x^2}$$
$$= 2\lim_{x \to 0} \left(\frac{\sin x}{x}\right)^2 \quad \left[\because \lim_{x \to 0} \frac{\sin x}{x} = 1\right]$$
$$= 2 \times 1 = 2$$

Q. 18 Evaluate $\lim_{x \to 0} \dfrac{2\sin x - \sin 2x}{x^3}$.

Sol. Given,
$$\lim_{x \to 0} \frac{2\sin x - \sin 2x}{x^3} = \lim_{x \to 0} \frac{2\sin x - 2\sin x \cos x}{x^3} \quad [\because \sin 2x = 2\sin x \cos x]$$
$$= \lim_{x \to 0} \frac{2\sin x(1 - \cos x)}{x^3}$$
$$= 2\lim_{x \to 0} \frac{\sin x}{x} \cdot \lim_{x \to 0} \left(\frac{1 - \cos x}{x^2}\right)$$
$$= 2 \cdot 1 \lim_{x \to 0} \frac{1 - \cos x}{x^2} \quad \left[\because \lim_{x \to 0} \frac{\sin x}{x} = 1\right]$$
$$= 2\lim_{x \to 0} \frac{1 - 1 + 2\sin^2 \frac{x}{2}}{x^2} = 2\lim_{x \to 0} \frac{2\sin^2 \frac{x}{2}}{4 \times \frac{x^2}{4}}$$
$$= \frac{2 \cdot 2}{4} \lim_{x \to 0} \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}}\right)^2 = \lim_{x \to 0} \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}}\right)^2 = 1$$

Q. 19 Evaluate $\lim\limits_{x \to 0} \dfrac{1 - \cos mx}{1 - \cos nx}$.

Sol. Given, $\lim\limits_{x \to 0} \dfrac{1-\cos mx}{1-\cos nx} = \lim\limits_{x \to 0} \dfrac{1-1+2\sin^2 \frac{mx}{2}}{1-1+2\sin^2 \frac{nx}{2}}$ $\left[\because \cos mx = 1 - 2\sin^2 \frac{mx}{2} \text{ and } \sin nx = 1 - 2\sin^2 \frac{nx}{2}\right]$

$$= \lim_{x \to 0} \frac{\sin^2 \frac{mx}{2}}{\sin^2 \frac{nx}{2}} = \lim_{x \to 0} \frac{\dfrac{\sin^2 \frac{mx}{2}}{\left(\frac{mx}{2}\right)^2} \cdot \left(\frac{mx}{2}\right)^2}{\dfrac{\sin^2 \frac{nx}{2}}{\left(\frac{nx}{2}\right)^2} \cdot \left(\frac{nx}{2}\right)^2} = \frac{\lim\limits_{x \to 0} \left(\dfrac{\sin \frac{mx}{2}}{\frac{mx}{2}}\right)^2}{\lim\limits_{x \to 0} \left(\dfrac{\sin \frac{nx}{2}}{\frac{nx}{2}}\right)^2} \cdot \frac{\frac{m^2 x^2}{4}}{\frac{n^2 x^2}{4}}$$

$$= \frac{m^2}{n^2} \cdot \frac{\lim\limits_{x \to 0} \left(\dfrac{\sin \frac{mx}{2}}{\frac{mx}{2}}\right)^2}{\lim\limits_{x \to 0} \left(\dfrac{\sin \frac{nx}{2}}{\frac{nx}{2}}\right)^2} = \frac{m^2}{n^2} \qquad \left[\because \lim_{x \to 0} \frac{\sin x}{x} = 1\right]$$

$[\because x \to 0 \Rightarrow k\,x \to 0]$

Q. 20 Evaluate $\lim\limits_{x \to \pi/3} \dfrac{\sqrt{1 - \cos 6x}}{\sqrt{2}\left(\frac{\pi}{3} - x\right)}$.

Sol. Given, $\lim\limits_{x \to \pi/3} \dfrac{\sqrt{1-\cos 6x}}{\sqrt{2}\left(\frac{\pi}{3} - x\right)} = \lim\limits_{x \to \pi/3} \dfrac{\sqrt{1-1+2\sin^2 3x}}{\sqrt{2}\left(\frac{\pi}{3} - x\right)}$ $[\because \cos 2x = 1 - 2\sin^2 x]$

$$= \lim_{x \to \pi/3} \frac{\sqrt{2}\sin 3x}{\sqrt{2}\left(\frac{\pi}{3} - x\right)} = \lim_{x \to \pi/3} \frac{\sin 3x}{\frac{\pi}{3} - x}$$

$$= \lim_{x \to \pi/3} \frac{\sin(\pi - 3x)}{\frac{\pi - 3x}{3}} \qquad [\because \sin(\pi - \theta) = \sin\theta]$$

$$= 3 \lim_{x \to \pi/3} \frac{\sin(\pi - 3x)}{(\pi - 3x)} = 3 \times 1 \qquad \left[\because \lim_{x \to 0} \frac{\sin x}{x} = 1\right]$$

$$= 3 \qquad \left[\because x \to \frac{\pi}{3} \Rightarrow \left(x - \frac{\pi}{3}\right) \to 0\right]$$

Q. 21 Evaluate $\lim_{x \to \frac{\pi}{4}} \dfrac{\sin x - \cos x}{x - \dfrac{\pi}{4}}$.

Sol. Given, $\lim_{x \to \pi/4} \dfrac{\sqrt{2}\left(\sin x \cdot \dfrac{1}{\sqrt{2}} - \cos x \cdot \dfrac{1}{\sqrt{2}}\right)}{\left(x - \dfrac{\pi}{4}\right)} = \lim_{x \to \pi/4} \dfrac{\sqrt{2}\left(\sin x \cos \dfrac{\pi}{4} - \cos x \cdot \sin \dfrac{\pi}{4}\right)}{\left(x - \dfrac{\pi}{4}\right)}$

$$= \lim_{x \to \pi/4} \frac{\sqrt{2}\left\{\sin\left(x - \frac{\pi}{4}\right)\right\}}{\left(x - \frac{\pi}{4}\right)}$$

$$= \sqrt{2} \lim_{x \to \pi/4} \frac{\sin\left(x - \frac{\pi}{4}\right)}{\left(x - \frac{\pi}{4}\right)} = \sqrt{2} \qquad \left[\because \lim_{x \to 0} \frac{\sin x}{x} = 1\right]$$

$$\left[\because x \to \frac{\pi}{4} \Rightarrow \left(x - \frac{\pi}{4}\right) \to 0\right]$$

Q. 22 Evaluate $\lim_{x \to \pi/6} \dfrac{\sqrt{3}\sin x - \cos x}{x - \dfrac{\pi}{6}}$.

Sol. Given, $\lim_{x \to \pi/6} \dfrac{\sqrt{3}\sin x - \cos x}{x - \dfrac{\pi}{6}} = \lim_{x \to \pi/6} \dfrac{2\left(\dfrac{\sqrt{3}}{2}\sin x - \dfrac{1}{2}\cos x\right)}{\left(x - \dfrac{\pi}{6}\right)}$

$$= \lim_{x \to \pi/6} \frac{2\left(\sin x \cos \frac{\pi}{6} - \cos x \sin \frac{\pi}{6}\right)}{\left(x - \frac{\pi}{6}\right)} = 2 \lim_{x \to \pi/6} \frac{\sin\left(x - \frac{\pi}{6}\right)}{\left(x - \frac{\pi}{6}\right)}$$

$[\because \sin A \cos B - \cos A \sin B = \sin (A - B)]$

$$= 2 \qquad \left[\because \lim_{x \to 0} \frac{\sin x}{x} = 1\right]$$

$$\left[\because x \to \frac{\pi}{6} \Rightarrow \left(x - \frac{\pi}{6}\right) \to 0\right]$$

Q. 23 Evaluate $\lim_{x \to 0} \dfrac{\sin 2x + 3x}{2x + \tan 3x}$.

Sol. Given, $\lim_{x \to 0} \dfrac{\sin 2x + 3x}{2x + \tan 3x} = \lim_{x \to 0} \dfrac{\dfrac{\sin 2x + 3x}{2x} \cdot 2x}{\dfrac{2x + \tan 3x}{3x} \cdot 3x}$

$$= \lim_{x \to 0} \frac{\left(\frac{\sin 2x}{2x} + \frac{3x}{2x}\right)2x}{\left(\frac{2x}{3x} + \frac{\tan 3x}{3x}\right)3x} = \lim_{x \to 0} \frac{\frac{\sin 2x}{2x} + \frac{3}{2}}{\frac{2}{3} + \frac{\tan 3x}{3x}} \cdot \frac{2}{3}$$

$$= \frac{2}{3} \lim_{x \to 0} \frac{\frac{\sin 2x}{2x} + \frac{3}{2}}{\frac{2}{3} + \lim_{x \to 0} \frac{\tan 3x}{3x}}$$

$$= \frac{2}{3}\left(\frac{1 + \frac{3}{2}}{\frac{2}{3} + 1}\right) \quad \left[\because \lim_{x \to 0} \frac{\sin x}{x} = 1 \text{ and } \lim_{x \to 0} \frac{\tan x}{x} = 1\right]$$

$$= \frac{2}{3} \times \frac{\frac{5}{2}}{\frac{5}{3}} = \frac{2}{3} \times \frac{5}{2} \times \frac{3}{5} = 1$$

Q. 24 Evaluate $\lim_{x \to a} \frac{\sin x - \sin a}{\sqrt{x} - \sqrt{a}}$.

Sol. Given, $\lim_{x \to a} \frac{\sin x - \sin a}{\sqrt{x} - \sqrt{a}} = \lim_{x \to 0} \frac{2\cos\left(\frac{x + a}{2}\right)\sin\left(\frac{x - a}{2}\right)}{\sqrt{x} - \sqrt{a}}$

$$\left[\because \sin C - \sin D = 2\cos\frac{C + D}{2} . \sin\frac{C - D}{2}\right]$$

$$= \lim_{x \to a} \frac{2\cos\left(\frac{x + a}{2}\right)\sin\left(\frac{x - a}{2}\right)(\sqrt{x} + \sqrt{a})}{(\sqrt{x} - \sqrt{a})\ (\sqrt{x} + \sqrt{a})}$$

$$= \lim_{x \to 0} \frac{2\cos\left(\frac{x + a}{2}\right)\sin\left(\frac{x - a}{2}\right)\ (\sqrt{x} + \sqrt{a})}{x - a}$$

$$= 2 \lim_{x \to a} \cos\left(\frac{x + a}{2}\right)\ (\sqrt{x} + \sqrt{a}) \lim_{x \to 0} \frac{\sin\left(\frac{x - a}{2}\right)}{2\left(\frac{x - a}{2}\right)}$$

$$= 2 \lim_{x \to 0} \cos\left(\frac{x + a}{2}\right)(\sqrt{x} + \sqrt{a}) \cdot \frac{1}{2} \lim_{x \to 0} \frac{\sin\left(\frac{x - a}{2}\right)}{\left(\frac{x - a}{2}\right)}$$

$$= 2 \cdot \cos\frac{a}{2} \cdot \sqrt{a} \cdot \frac{1}{2} \quad \left[\because \lim_{x \to 0} \frac{\sin x}{x} = 1\right]$$

$$= \sqrt{a}\cos\frac{a}{2}$$

Q. 25 Evaluate $\lim_{x \to \pi/6} \frac{\cot^2 x - 3}{\operatorname{cosec} x - 2}$.

Sol. Given $\lim_{x \to \pi/6} \frac{\cot^2 x - 3}{\operatorname{cosec} x - 2} = \lim_{x \to \pi/6} \frac{\operatorname{cosec}^2 x - 1 - 3}{\operatorname{cosec} x - 2}$ $\quad [\because \operatorname{cosec}^2 x = 1 + \cot^2 x]$

$$= \lim_{x \to \pi/6} \frac{\operatorname{cosec}^2 x - 4}{\operatorname{cosec} x - 2} = \lim_{x \to \pi/6} \frac{(\operatorname{cosec} x)^2 - (2)^2}{\operatorname{cosec} x - 2}$$

$$= \lim_{x \to \pi/6} \frac{(\operatorname{cosec} x + 2)(\operatorname{cosec} x - 2)}{(\operatorname{cosec} x - 2)} = \lim_{x \to \pi/6} (\operatorname{cosec} x + 2)$$

$$= \operatorname{cosec}\frac{\pi}{6} + 2 = 2 + 2 = 4$$

Q. 26 Evaluate $\lim\limits_{x \to 0} \dfrac{\sqrt{2} - \sqrt{1+\cos x}}{\sin^2 x}$.

Sol. Given, $\lim\limits_{x \to 0} \dfrac{\sqrt{2} - \sqrt{1+\cos x}}{\sin^2 x} = \lim\limits_{x \to 0} \dfrac{\sqrt{2} - \sqrt{1+2\cos^2 \frac{x}{2} - 1}}{\sin^2 x}$ $\left[\because \cos x = 2\cos^2 \frac{x}{2} - 1\right]$

$= \lim\limits_{x \to 0} \dfrac{\sqrt{2} - \sqrt{2\cos^2 \frac{x}{2}}}{\sin^2 x}$ $\left[\because \sin x = 2\sin \frac{x}{2} \cos \frac{x}{2}\right]$

$= \lim\limits_{x \to 0} \dfrac{\sqrt{2}\left(1 - \cos \frac{x}{2}\right)}{\sin^2 x} = \lim\limits_{x \to 0} \dfrac{\sqrt{2}\left(1 - 1 + 2\sin^2 \frac{x}{4}\right)}{\sin^2 x}$

$= \lim\limits_{x \to 0} \dfrac{\sqrt{2}\left(2\sin^2 \frac{x}{4}\right)}{\sin^2 x} = \lim\limits_{x \to 0} 2\sqrt{2} \dfrac{\sin^2 \frac{x}{4}}{\left(\frac{x}{4}\right)^2} \cdot \dfrac{\left(\frac{x}{4}\right)^2}{\sin^2 x}$

$= 2\sqrt{2} \lim\limits_{x \to 0} \left(\dfrac{\sin \frac{x}{4}}{\frac{x}{4}}\right)^2 \cdot \lim\limits_{x \to 0} \left(\dfrac{x}{\sin x}\right)^2 \cdot \dfrac{1}{16}$

$= 2\sqrt{2} \cdot 1 \cdot 1 \cdot \dfrac{1}{16} = \dfrac{1}{4\sqrt{2}}$

Q. 27 Evaluate $\lim\limits_{x \to 0} \dfrac{\sin x - 2\sin 3x + \sin 5x}{x}$.

Sol. Given,

$\lim\limits_{x \to 0} \dfrac{\sin x - 2\sin 3x + \sin 5x}{x} = \lim\limits_{x \to 0} \dfrac{\sin 5x + \sin x - 2\sin 3x}{x}$

$= \lim\limits_{x \to 0} \dfrac{2\sin 3x \cos 2x - 2\sin 3x}{x} = \lim\limits_{x \to 0} \dfrac{2\sin 3x(\cos 2x - 1)}{x}$

$= \lim\limits_{x \to 0} \dfrac{2\sin 3x}{\frac{1}{3} \times 3x}(\cos 2x - 1) = 6 \lim\limits_{x \to 0} \dfrac{\sin 3x}{3x}(\cos 2x - 1)$

$= 6 \lim\limits_{x \to 0} \dfrac{\sin 3x}{3x} \cdot \lim\limits_{x \to 0} (\cos 2x - 1) = 6 \times 1 \times 0 = 0$

Q. 28 If $\lim\limits_{x \to 1} \dfrac{x^4 - 1}{x - 1} = \lim\limits_{x \to k} \dfrac{x^3 - k^3}{x^2 - k^2}$, then find the value of k.

Sol. Given, $\lim\limits_{x \to 1} \dfrac{x^4 - 1}{x - 1} = \lim\limits_{x \to k} \dfrac{x^3 - k^3}{x^2 - k^2}$

$\Rightarrow \quad 4(1)^{4-1} = \lim\limits_{x \to k} \dfrac{\dfrac{x^3 - k^3}{x - k}}{\dfrac{x^2 - k^2}{x - k}}$ $\left[\because \lim\limits_{x \to a} \dfrac{x^n - a^n}{x - a} = na^{n-1}\right]$

$$\Rightarrow \quad 4 = \frac{\lim_{x \to k} \frac{x^3 - k^3}{x - k}}{\lim_{x \to k} \frac{x^2 - k^2}{x - k}} 1 \qquad \left[\because \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}\right]$$

$$\Rightarrow \quad 4 = \frac{3k^2}{2k} \Rightarrow 4 = \frac{3}{2}k$$

$$\therefore \quad k = \frac{4 \times 2}{3} = \frac{8}{3}$$

Differentiate each of the functions w.r.t. x in following questions

Q. 29 $\dfrac{x^4 + x^3 + x^2 + 1}{x}$

Sol. $$\frac{d}{dx}\left(\frac{x^4 + x^3 + x^2 + 1}{x}\right) = \frac{d}{dx}\left(x^3 + x^2 + x + \frac{1}{x}\right)$$

$$= \frac{d}{dx}x^3 + \frac{d}{dx}x^2 + \frac{d}{dx}x + \frac{d}{dx}\left(\frac{1}{x}\right)$$

$$= 3x^2 + 2x + 1 + \left(-\frac{1}{x^2}\right)$$

$$= 3x^2 + 2x + 1 - \frac{1}{x^2}$$

$$= \frac{3x^4 + 2x^3 + x^2 - 1}{x^2}$$

Q. 30 $\left(x + \dfrac{1}{x}\right)^3$

Sol. Let $$y = \left(x + \frac{1}{x}\right)^3$$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}\left(x + \frac{1}{x}\right)^3 = 3\left(x + \frac{1}{x}\right)^{3-1} \frac{d}{dx}\left(x + \frac{1}{x}\right) \qquad \text{[by chain rule]}$$

$$= 3\left(x + \frac{1}{x}\right)^2 \left(1 - \frac{1}{x^2}\right)$$

$$= 3x^2 - \frac{3}{x^2} - \frac{3}{x^4} + 3$$

Q. 31 $(3x + 5)(1 + \tan x)$

Sol. Let $$y = (3x + 5)(1 + \tan x)$$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}[(3x + 5)(1 + \tan x)]$$

$$= (3x + 5)\frac{d}{dx}(1 + \tan x) + (1 + \tan x)\frac{d}{dx}(3x + 5) \qquad \text{[by product rule]}$$

$$= (3x + 5)(\sec^2 x) + (1 + \tan x) \cdot 3$$

$$= (3x + 5)\sec^2 x + 3(1 + \tan x)$$

$$= 3x\sec^2 x + 5\sec^2 x + 3\tan x + 3$$

Q. 32 $(\sec x - 1)(\sec x + 1)$

Sol. Let

$$y = (\sec x - 1)(\sec x + 1)$$

$$y = (\sec^2 - 1) \qquad [\because (a+b)(a-b) = a^2 - b^2]$$

$$= \tan^2 x$$

$$\therefore \quad \frac{dy}{dx} = 2\tan x \cdot \frac{d}{dx}\tan x$$

$$= 2\tan x \cdot \sec^2 x \qquad \text{[by chain rule]}$$

Q. 33 $\dfrac{3x+4}{5x^2 - 7x + 9}$

Sol. Let

$$y = \frac{3x+4}{5x^2 - 7x + 9}$$

$$\therefore \quad \frac{dy}{dx} = \frac{(5x^2 - 7x + 9)\frac{d}{dx}(3x+4) - (3x+4)\frac{d}{dx}(5x^2 - 7x + 9)}{(5x^2 - 7x + 9)^2} \qquad \text{[by quotient rule]}$$

$$= \frac{(5x^2 - 7x + 9) \cdot 3 - (3x+4)(10x - 7)}{(5x^2 - 7x + 9)^2}$$

$$= \frac{15x^2 - 21x + 27 - 30x^2 + 21x - 40x + 28}{(5x^2 - 7x + 9)^2}$$

$$= \frac{-15x^2 - 40x + 55}{(5x^2 - 7x + 9)^2}$$

$$= \frac{55 - 15x^2 - 40x}{(5x^2 - 7x + 9)^2}$$

Q. 34 $\dfrac{x^5 - \cos x}{\sin x}$

Sol. Let

$$y = \frac{x^5 - \cos x}{\sin x}$$

$$\therefore \quad \frac{dy}{dx} = \frac{\sin x \frac{d}{dx}(x^5 - \cos x) - (x^5 - \cos x)\frac{d}{dx}\sin x}{(\sin x)^2} \qquad \text{[by quotient rule]}$$

$$= \frac{\sin x(5x^4 + \sin x) - (x^5 - \cos x)\cos x}{\sin^2 x}$$

$$= \frac{5x^4 \sin x + \sin^2 x - x^5 \cos x + \cos^2 x}{\sin^2 x}$$

$$= \frac{5x^4 \sin x - x^5 \cos x + \sin^2 x + \cos^2 x}{\sin^2 x}$$

$$= \frac{5x^4 \sin x - x^5 \cos x + 1}{\sin^2 x}$$

Q. 35 $\dfrac{x^2 \cos \frac{\pi}{4}}{\sin x}$

Sol. Let
$$y = \frac{x^2 \cos \frac{\pi}{4}}{\sin x} = \frac{\frac{x^2}{\sqrt{2}}}{\sin x}$$
$$y = \frac{1}{\sqrt{2}} \cdot \frac{x^2}{\sin x}$$
$$\therefore \quad \frac{dy}{dx} = \frac{1}{\sqrt{2}} \left[\frac{\sin x \cdot \frac{d}{dx} x^2 - x^2 \frac{d}{dx} \sin x}{\sin^2 x} \right] \quad \text{[by quotient rule]}$$
$$= \frac{1}{\sqrt{2}} \left[\frac{\sin x \cdot 2x - x^2 \cdot \cos x}{\sin^2 x} \right]$$
$$= \frac{1}{\sqrt{2}} \cdot \frac{2x \sin x - x^2 \cos x}{\sin^2 x}$$
$$= \frac{x}{\sqrt{2}} [2 \operatorname{cosec} x - x \cot x \operatorname{cosec} x]$$
$$= \frac{x}{\sqrt{2}} \operatorname{cosec} [2 - x \cot x]$$

Q. 36 $(ax^2 + \cot x)(p + q \cos x)$

Sol. Let $y = (ax^2 + \cot x)(p + q \cos x)$
$$\therefore \quad \frac{dy}{dx} = (ax^2 + \cot x) \frac{d}{dx}(p + q \cos x) + (p + q \cos x) \frac{d}{dx}(ax^2 + \cot x) \quad \text{[by product rule]}$$
$$= (ax^2 + \cot x)(-q \sin x) + (p + q \cos x)(2ax - \operatorname{cosec}^2 x)$$
$$= -q \sin x(ax^2 + \cot x) + (p + q \cos x)(2ax - \operatorname{cosec}^2 x)$$

Q. 37 $\dfrac{a + b \sin x}{c + d \cos x}$

Sol. Let
$$y = \frac{a + b \sin x}{c + d \cos x}$$
$$\therefore \quad \frac{dy}{dx} = \frac{(c + d \cos x) \frac{d}{dx}(a + b \sin x) - (a + b \sin x) \frac{d}{dx}(c + d \cos x)}{(c + d \cos x)^2} \quad \text{[by quotinet rule]}$$
$$= \frac{(c + d \cos x)(b \cos x) - (a + b \sin x)(-d \sin x)}{(c + d \cos x)^2}$$
$$= \frac{bc \cos x + bd \cos^2 x + ad \sin x + bd \sin^2 x}{(c + d \cos x)^2}$$
$$= \frac{bc \cos x + ad \sin x + bd(\cos^2 x + \sin^2 x)}{(c + d \cos x)^2}$$
$$= \frac{bc \cos x + ad \sin x + bd}{(c + d \cos x)^2}$$

Q. 38 $(\sin x+\cos x)^2$

Sol. Let $y=(\sin x+\cos x)^2$

$\therefore$ $\frac{dy}{dx}=2\,(\sin x+\cos x)(\cos x-\sin x)$

$=2\,(\cos^2 x-\sin^2 x)$ [by chain rule]

$=2\cos 2x$ [$\because \cos 2x=\cos^2 x-\sin^2 x$]

Q. 39 $(2x-7)^2(3x+5)^3$

Sol. Let $y=(2x-7)^2(3x+5)^3$

$\frac{dy}{dx}=(2x-7)^2\frac{d}{dx}(3x+5)^3+(3x+5)^3\frac{d}{dx}(2x-7)^2$ [by product rule]

$=(2x-7)^2\,(3)\,(3x+5)^2\,(3)+(3x+5)^3 2\,(2x-7)(2)$ [by chain rule]

$=9(2x-7)^2(3x+5)^2+4(3x+5)^3(2x-7)$

$=(2x-7)(3x+5)^2\,[9\,(2x-7)+4\,(3x+5)]$

$=(2x-7)(3x+5)^2(18x-63+12x+20)$

$=(2x-7)(3x+5)^2(30x-43)$

Q. 40 $x^2\sin x+\cos 2x$

Sol. Let $y=x^2\sin x+\cos 2x$

$\frac{dy}{dx}=\frac{d}{dx}(x^2\sin x)+\frac{d}{dx}\cos 2x$

$=x^2\cdot\cos x+\sin x\,2x+(-\sin 2x)\cdot 2$ [by product rule]

$=x^2\cos x+2x\sin x-2\sin 2x$ [by chain urle]

Q. 41 $\sin^3 x\cos^3 x$

Sol. Let $y=\sin^3 x\cos^3 x$

$\therefore$ $\frac{dy}{dx}=\sin^3 x\cdot\frac{d}{dx}\cos^3 x+\cos^3 x\frac{d}{dx}\sin^3 x$ [by product rule]

$=\sin^3 x\cdot 3\cos^2 x(-\sin x)+\cos^3 x\cdot 3\sin^2 x\cos x$ [by chain rule]

$=-\,3\cos^2 x\sin^4 x+3\sin^2 x\cos^4 x$

$=3\sin^2 x\cos^2 x(\cos^2 x-\sin^2 x)$

$=3\sin^2 x\cos^2 x\cos 2x$

$=\frac{3}{4}(2\sin x\cos x)^2\cos 2x$

$=\frac{3}{4}\sin^2 2x\cos 2x$

Q. 42 $\dfrac{1}{ax^2+bx+c}$

Sol. Let $$y = \frac{1}{ax^2+bx+c} = (ax^2+bx+c)^{-1}$$

$$\therefore \quad \frac{dy}{dx} = -(ax^2+bx+c)^{-2}(2ax+b) \quad \text{[by chain rule]}$$

$$= \frac{-(2ax+b)}{(ax^2+bx+c)^2}$$

Long Answer Type Questions

Differentiate each of the functions with respect to x in following questions using first principle.

Q. 43 $\cos(x^2+1)$

Sol. Let $f(x) = \cos(x^2+1)$ and $f(x+h) = \cos\{(x+h)^2+1\}$

$$\therefore \quad \frac{d}{dx}f(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{\cos\{(x+h)^2+1\}-\cos(x^2+1)}{h}$$

$$= \lim_{h\to 0}\frac{-2\sin\left\{\frac{(x+h)^2+1+x^2+1}{2}\right\}\sin\left\{\frac{(x+h)^2+1-x^2-1}{2}\right\}}{h}$$

$$\left[\because \cos C - \cos D = -2\sin\frac{C+D}{2}\cdot\sin\frac{C-D}{2}\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[-2\sin\left\{\frac{(x+h)^2+x^2+2}{2}\right\}\sin\left\{\frac{(x+h)^2-x^2}{2}\right\}\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[-2\sin\left\{\frac{(x+h)^2+x^2+2}{2}\right\}\sin\left\{\frac{x^2+h^2+2xh-x^2}{2}\right\}\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[-2\sin\left\{\frac{(x+h)^2+x^2+2}{2}\right\}\sin\left\{\frac{h^2+2hx}{2}\right\}\right]$$

$$= -2\lim_{h\to 0}\sin\left\{\frac{(x+h)^2+x^2+2}{2}\right\}\lim_{h\to 0}\left\{\frac{\sin h\left(\frac{h+2x}{2}\right)}{h\left(\frac{h+2x}{2}\right)}\right\}\times\left(\frac{h+2x}{2}\right)$$

$$= -2\lim_{h\to 0}\sin\left\{\frac{(x+h)^2+x^2+2}{2}\right\}\lim_{h\to 0}\left(\frac{h+2x}{2}\right) \quad \left[\because \lim_{x\to 0}\frac{\sin x}{x}=1\right]$$

$$= -2x\sin(x^2+1) \quad [\because x\to 0 \Rightarrow kx\to 0]$$

Q. 44 $\dfrac{ax+b}{cx+d}$

Sol. Let
$$f(x) = \frac{ax + b}{cx + d}$$
$$f(x + h) = \frac{a(x + h) + b}{c(x + h) + d}$$
$$\therefore \quad \frac{d}{dx} f(x) = \lim_{h \to 0} \frac{1}{h}[f(x + h) - f(x)]$$
$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{a(x + h) + b}{c(x + h) + d} - \frac{ax + b}{cx + d}\right]$$
$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{ax + b + ah}{c(x + h) + d} - \frac{ax + b}{cx + d}\right]$$
$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{(ax + ah + b)(cx + d) - (ax + b\{c(x + h) + d\}}{\{c(x + h) + d\}(cx + d)}\right]$$
$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{(ax + ah + b)(cx + d) - (ax + b)(cx + ch + d)}{\{c(x + h) + d)\}(cx + d)}\right]$$
$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{acx^2 + achx + bcx + adx + adh + bd - \{acx^2 + achx + adx + bcx + bch + bd\}}{\{c(x + h) + d\}(cx + d)}\right]$$
$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{acx^2 + achx + bcx + adx + adh + bd - acx^2 - achx - adx - bcx - bch - bd\}}{\{c(x + h) + d\}(cx + d)}\right]$$
$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{adh - bch}{\{c(x + h) + d\}(cx + d)}\right]$$
$$= \lim_{h \to 0} \frac{ac - bd}{\{c(x + h) + d\}(cx + d)}$$
$$= \frac{ac - bd}{(cx + d)^2}$$

Q. 45 $x^{2/3}$

Sol. Let
$$f(x) = x^{2/3}$$
$$f(x + h) = (x + h)^{2/3}$$
Now,
$$\frac{d}{dx} f(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$
$$= \lim_{h \to 0} \frac{1}{h}[(x + h)^{2/3} - x^{2/3}]$$
$$= \lim_{h \to 0} \frac{1}{h}\left[x^{2/3}\left(1 + \frac{h}{x}\right)^{2/3} - x^{2/3}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[x^{2/3}\left(1 + \frac{h}{x} \cdot \frac{2}{3} + \frac{2}{3}\left(\frac{2}{3} - 1\right)\frac{h^2}{x^2} + \cdots\right) - 1\right]$$

$$\left[\because (1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[x^{2/3}\left(\frac{2}{3} \cdot \frac{h}{x} - \frac{2}{9} \cdot \frac{h^2}{x^2} + \cdots\right)\right]$$

$$= \lim_{h \to 0} \frac{x^{2/3}}{h} \cdot \frac{2}{3}\frac{h}{x}\left(1 - \frac{1}{3} \cdot \frac{h}{x} + \cdots\right)$$

$$= \frac{2}{3}x^{2/3 - 1} = \frac{2}{3}x^{-1/3}$$

Alternate Method

Let $f(x) = x^{2/3}$

$f(x + h) = (x + h)^{2/3}$

$\therefore$ $\frac{d}{dx} f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{(h \to 0} \left[\frac{(x+h)2/3 - x^{2/3}}{h}\right] = \lim_{(x+h) \to x}\left[\frac{(x+h)2/3 - x^{2/3}}{(x+h) - x}\right]$$

$$= \frac{2}{3}(x)^{2/3-1} \qquad \left[\because \lim_{x \to a} \frac{x^x - a^n}{x - a} = na^{n-1}\right]$$

$$= \frac{2}{3}x^{-1/3}$$

Q. 46 $x \cos x$

Sol. Let $f(x) = x \cos x$

$\therefore$ $f(x + h) = (x + h) \cos (x + h)$

$\therefore$ $\frac{d}{dx} f(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{1}{h}[(x + h)\cos(x + h) - x\cos x]$$

$$= \lim_{h \to 0} \frac{1}{h}[x\cos(x + h) + h\cos(x + h) - x\cos x]$$

$$= \lim_{h \to 0} \frac{1}{h} [x\{\cos(x + h) - \cos x\} + h\cos(x + h)]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[x\left\{-2\sin\left(\frac{2x + h}{2}\right)\sin\frac{h}{2}\right\} + h\cos(x + h)\right]$$

$$= \lim_{h \to 0}\left[-2x\sin\left(x + \frac{h}{2}\right)\frac{\sin\frac{h}{2}}{h} + \cos(x + h)\right]$$

$$\left[\because \cos C - \cos D = -2\sin\frac{C + D}{2} \cdot \sin\frac{C - D}{2}\right]$$

$$= -2\lim_{h \to 0} x\sin\left(x + \frac{h}{2}\right)\lim_{h \to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}} \cdot \frac{1}{2} + \lim_{h \to 0}\cos(x + h)$$

$$= -2 \cdot \frac{1}{2}x\sin x + \cos x$$

$$= \cos x - x\sin x$$

Evaluate each of the following limits in following questions

Q. 47 $\lim_{y \to 0} \dfrac{(x+y)\sec(x+y) - x\sec x}{y}$

Sol. Given, $\lim_{y \to 0} \dfrac{(x+y)\sec(x+y) - x\sec x}{y}$

$$= \lim_{y \to 0} \frac{\dfrac{x+y}{\cos(x+y)} - \dfrac{x}{\cos x}}{y}$$

$$= \lim_{y \to 0} \frac{(x+y)\cos x - x\cos(x+y)}{y\cos x\cos(x+y)}$$

$$= \lim_{y \to 0} \left[\frac{x\cos x + y\cos x - x\cos(x+y)}{y\cos x\cos(x+y)}\right]$$

$$= \lim_{y \to 0} \left[\frac{x\cos x - x\cos(x+y) + y\cos x}{y\cos x\cos(x+y)}\right]$$

$$= \lim_{y \to 0} \frac{x\{\cos x - \cos(x+y)\} + y\cos x}{y\cos x\cos(x+y)}$$

$$= \lim_{y \to 0} \frac{x\left[-2\sin\left(x + \dfrac{y}{2}\right)\sin\left(\dfrac{-y}{2}\right)\right] + y\cos x}{y\cos x\cos(x+y)}$$

$$\left[\because \cos C - \cos D = -2\sin\frac{C+D}{2} \cdot \sin\frac{C-D}{2}\right]$$

$$= \lim_{y \to 0} \left[\frac{x\left\{2\sin\left(x + \dfrac{y}{2}\right)\sin\dfrac{y}{2}\right\} + y\cos x}{y\cos x\cos(x+y)}\right]$$

$$= \lim_{y \to 0} \frac{2x\sin\left(x + \dfrac{y}{2}\right)}{\cos x\cos(x+y)} \cdot \lim_{y \to 0} \frac{\sin\dfrac{y}{2}}{\dfrac{y}{2}} \cdot \frac{1}{2} + \lim_{y \to 0} \sec(x+y)$$

$$\left[\because \lim_{x \to 0} \frac{\sin x}{x} = 1 \text{ and } x \to 0 \Rightarrow kx \to 0\right]$$

$$= \lim_{y \to 0} \frac{2x\sin\left(x + \dfrac{y}{2}\right)}{\cos x\cos(x+y)} \cdot \frac{1}{2} + \lim_{y \to 0} \sec(x+y)$$

$$= \frac{2x\sin x}{\cos x\cos x} \cdot \frac{1}{2} + \sec x$$

$$= x\tan x\sec x + \sec x$$

$$= \sec x\,(x\tan x + 1)$$

Q. 48 $\lim\limits_{x \to 0} \dfrac{\sin(\alpha + \beta)x + \sin(\alpha - \beta)x + \sin 2\alpha x}{\cos 2\beta x - \cos 2\alpha\, x} \cdot x$

Sol. Given, $\lim\limits_{x \to 0} \dfrac{[\sin(\alpha + \beta)x + \sin(\alpha - \beta)x + \sin 2\alpha x]}{\cos 2\beta x - \cos 2\alpha x} \cdot x$

$$= \lim_{x \to 0} \frac{[2\sin\alpha x \cdot \cos\beta x + \sin 2\alpha x]}{\cos 2\beta x - \cos 2\alpha x} \cdot x \quad \left[\because \sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}\right]$$

$$= \lim_{x \to 0} \frac{[2\sin\alpha x\cos\beta x + \sin 2\alpha x]x}{2\sin(\alpha + \beta)x\,\sin(\alpha - \beta)x} \quad \left[\because \cos C - \cos D = 2\sin\frac{C+D}{2}\cdot\sin\frac{D-C}{2}\right]$$

$$= \lim_{x \to 0} \frac{[2\sin\alpha x\cos\beta x + 2\sin\alpha x\cos\alpha x]x}{2\sin(\alpha + \beta)x\,\sin(\alpha - \beta)x}$$

$$= \lim_{x \to 0} \frac{2\sin\alpha x[\cos\beta x + \cos\alpha x]x}{2\sin(\alpha + \beta)x\,\sin(\alpha - \beta)x}$$

$$= \lim_{x \to 0} \frac{\sin\alpha x\left[2\cos\left(\frac{\alpha+\beta}{2}\right)x\cos\left(\frac{\alpha-\beta}{2}\right)x\right]x}{2\sin\left(\frac{\alpha+\beta}{2}\right)x\cos\left(\frac{\alpha+\beta}{2}\right)x \cdot 2\sin\left(\frac{\alpha-\beta}{2}\right)x\cos\left(\frac{\alpha-\beta}{2}\right)x}$$

$$\left[\because \cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2} \text{ and } \sin 2\theta = 2\sin\theta\cos\theta\right]$$

$$= \lim_{x \to 0} \frac{\sin\alpha x \cdot x}{2\sin\left(\frac{\alpha+\beta}{2}\right)x\sin\left(\frac{\alpha-\beta}{2}\right)x}$$

$$= \frac{1}{2}\lim_{x \to 0} \frac{\frac{\sin\alpha x}{\alpha x} \cdot x \cdot (\alpha x)}{2\frac{\sin\left(\frac{\alpha+\beta}{2}\right)x}{\left(\frac{\alpha+\beta}{2}\right)x} \cdot \frac{\sin\left(\frac{\alpha-\beta}{2}\right)x}{\left(\frac{\alpha-\beta}{2}\right)x} \cdot \left(\frac{\alpha+\beta}{2}\right)x \cdot \left(\frac{\alpha-\beta}{2}\right)x}$$

$$= \frac{\frac{1}{2}\lim\limits_{x \to 0}\frac{\sin\alpha x}{\alpha x}\cdot\alpha x^2}{\lim\limits_{x \to 0}\frac{\sin\left(\frac{\alpha+\beta}{2}\right)x}{\left(\frac{\alpha+\beta}{2}\right)x}\lim\limits_{x \to 0}\frac{\sin\left(\frac{\alpha-\beta}{2}\right)x}{\left(\frac{\alpha-\beta}{2}\right)x}\cdot\left(\frac{\alpha^2-\beta^2}{4}\right)x^2}$$

$$\left[\because \lim_{x\to 0}\frac{\sin x}{x} = 1 \text{ and } x \to 0 \Rightarrow kx \to 0\right]$$

$$= \frac{1}{2}\cdot\frac{\alpha \cdot 4}{\alpha^2 - \beta^2}\left[\frac{\lim\limits_{x \to 0}\frac{\sin\alpha x}{\alpha x}}{\lim\limits_{x \to 0}\frac{\sin\left(\frac{\alpha+\beta}{2}\right)x}{\left(\frac{\alpha+\beta}{2}\right)x}\;\lim\limits_{x \to 0}\frac{\sin\left(\frac{\alpha-\beta}{2}\right)x}{\left(\frac{\alpha-\beta}{2}\right)x}}\right]$$

$$= \frac{1}{2}\cdot\frac{4\alpha}{\alpha^2 - \beta^2} = \frac{2\alpha}{\alpha^2 - \beta^2}$$

Q. 49 $\lim_{x \to \pi/4} \dfrac{\tan^3 x - \tan x}{\cos\left(x + \dfrac{\pi}{4}\right)}$

Sol. Given, $\lim_{x \to \pi/4} \dfrac{\tan^3 x - \tan x}{\cos\left(x + \dfrac{\pi}{4}\right)}$ $\left[\dfrac{0}{0} \text{ form}\right]$

$$= \lim_{x \to \pi/4} \frac{\tan x (\tan^2 x - 1)}{\cos\left(x + \frac{\pi}{4}\right)} = \lim_{x \to \pi/4} \tan x \cdot \lim_{x \to \pi/4} \left(\frac{1 - \tan^2 x}{\cos\left(x + \frac{g\pi}{4}\right)}\right)$$

$$= -1 \times \lim_{x \to \pi/4} \frac{(1 + \tan x)(1 - \tan x)}{\cos\left(x + \frac{\pi}{4}\right)} \qquad [\because a^2 - b^2 = (a+b)(a-b)]$$

$$= -\lim_{x \to \pi/4} (1 + \tan x) \lim_{x \to \pi/4} \left[\frac{\cos x - \sin x}{\cos x \cdot \cos\left(x + \frac{\pi}{4}\right)}\right]$$

$$= -(1+1) \times \lim_{x \to \pi/4} \frac{\sqrt{2}\left[\frac{1}{\sqrt{2}} \cdot \cos x - \frac{1}{\sqrt{2}} \cdot \sin x\right]}{\cos x \cdot \cos\left(x + \frac{\pi}{4}\right)} = -2\sqrt{2} \lim_{x \to \pi/4} \left[\frac{\cos\frac{\pi}{4} \cdot \cos x - \sin\frac{\pi}{4} \cdot \sin x}{\cos x \cdot \cos\left(x + \frac{\pi}{4}\right)}\right]$$

$$[\because \cos A \cdot \cos B - \sin A \sin B = \cos(A + B)]$$

$$= -2\sqrt{2} \lim_{x \to \pi/4} \frac{\cos\left(x + \frac{\pi}{4}\right)}{\cos x \cdot \cos\left(x + \frac{\pi}{4}\right)} = -2\sqrt{2} \times \frac{1}{\frac{1}{\sqrt{2}}} = -2\sqrt{2} \times \sqrt{2} = -4$$

Q. 50 $\lim_{x \to \pi} \dfrac{1 - \sin\dfrac{x}{2}}{\cos\dfrac{x}{2}\left(\cos\dfrac{x}{4} - \sin\dfrac{x}{4}\right)}$

Sol. Given, $\lim_{x \to \pi} \dfrac{1 - \sin\dfrac{x}{2}}{\cos\dfrac{x}{2}\left(\cos\dfrac{x}{4} - \sin\dfrac{x}{4}\right)}$

$$= \lim_{x \to \pi} \frac{\cos^2\frac{x}{4} + \sin^2\frac{x}{4} - 2 \cdot \sin\frac{x}{4} \cdot \cos\frac{x}{4}}{\cos\frac{x}{2} \cdot (\cos\frac{x}{4} - \sin\frac{x}{4})} \qquad [\because \sin^2\theta + \cos^2\theta = 1 \sin 2\theta = 2\sin\theta\cos\theta]$$

$$= \lim_{x \to \pi} \frac{\left(\cos\frac{x}{4} - \sin\frac{x}{4}\right)^2}{\left(\cos^2\frac{x}{4} - \sin^2\frac{x}{4}\right)\left(\cos\frac{x}{4} - \sin\frac{x}{4}\right)} \qquad [\because \cos^2 2\theta = \cos^2\theta - \sin^2\theta]$$

$$= \lim_{x\to\pi} \frac{\left(\cos\frac{x}{4} - \sin\frac{x}{4}\right)}{\left(\cos\frac{x}{4} + \sin\frac{x}{4}\right)\left(\cos\frac{x}{4} - \sin\frac{x}{4}\right)} \qquad [\because a^2 - b^2 = (a+b)(a-b)]$$

$$\lim_{x\to\pi} \frac{1}{\cos\frac{x}{4} + \sin\frac{x}{4}} = \frac{1}{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

Q. 51 Show that $\lim_{x \to \pi/4} \frac{|x-4|}{x-4}$ does not exist,

Sol. Given, $\lim_{x \to \pi/4} \frac{|x-4|}{x-4}$

$$\text{LHL} = \lim_{x \to \frac{\pi^-}{4}} \frac{-(x-4)}{x-4} \qquad [\because |x-4| = -(x-4),\ x < 4]$$

$$= -1$$

$$\text{RHL} = \lim_{x \to \frac{\pi^+}{4}} \frac{(x-4)}{x-4} = 1 \qquad [\because |x-4| = (x-4),\ x > 4]$$

$\therefore$ LHL $\neq$ RHL

So, limit does not exist.

Q. 52 If $f(x) = \begin{cases} \frac{k\cos x}{\pi - 2x}, & \text{when } x \neq \frac{\pi}{2} \\ 3, & \text{when } x = \frac{\pi}{2} \end{cases}$ and $\lim_{x \to \pi/2} f(x) = f\left(\frac{\pi}{2}\right)$, then find the value of k.

Sol. Given, $f(x) = \begin{cases} \frac{k\cos x}{\pi - 2x}, & x \neq \frac{\pi}{2} \\ 3, & x = \frac{\pi}{2} \end{cases}$

$$\therefore \quad \text{LHL} = \lim_{x \to \frac{\pi^-}{2}} \frac{k\cos x}{\pi - 2x} = \lim_{h \to 0} \frac{k\cos\left(\frac{\pi}{2} - h\right)}{\pi - 2\left(\frac{\pi}{2} - h\right)}$$

$$= \lim_{h \to 0} \frac{k\sin h}{\pi - \pi + 2h} = \lim_{h \to 0} \frac{k\sin h}{2h}$$

$$= \frac{k}{2} \lim_{h \to 0} \frac{\sin h}{h} = \frac{k}{2} \cdot 1 = \frac{k}{2} \qquad \left[\because \lim_{h \to 0} \frac{\sin x}{x} = 1\right]$$

$$\text{RHL} = \lim_{x \to \frac{\pi^+}{2}} \frac{k\cos x}{\pi - 2x} = \lim_{x \to \frac{\pi}{2}^+} \frac{k\cos\left(\frac{\pi}{2} + h\right)}{\pi - 2\left(\frac{\pi}{2} + h\right)}$$

$$= \lim_{h \to 0} \frac{-k\sin h}{\pi - \pi - 2h} = \lim_{h \to 0} \frac{k\sin h}{2h} = \frac{k}{2} \lim_{h \to 0} \frac{\sin h}{2h} = \frac{k}{2} \text{ and } f\left(\frac{\pi}{2}\right) = 3$$

It is given that, $\lim_{x \to \pi/2} f(x) = f\left(\frac{\pi}{2}\right) \Rightarrow \frac{k}{2} = 3$

$\therefore$ $k = 6$

Q. 53 If $f(x) = \begin{cases} x + 2, & x \le -1 \\ cx^2, & x > -1 \end{cases}$, then find c when $\lim_{x \to -1} f(x)$ exists.

Sol. Given,
$$f(x) = \begin{cases} x + 2, & x \le -1 \\ cx^2, & x > -1 \end{cases}$$

$$\text{LHL} = \lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} (x + 2)$$

$$= \lim_{h \to 0} (-1 - h + 2) = \lim_{h \to 0} (1 - h) = 1$$

$$\text{RHL} = \lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} cx^2 = \lim_{h \to 0} c(-1 + h)^2$$

$\therefore$ $= c$

If $\lim_{x \to -1} f(x)$ exist, then LHL = RHL

$\therefore$ $c = 1$

Objective Type Questions

Q. 54 $\lim_{x \to \pi} \dfrac{\sin x}{x - \pi}$ is equal to

(a) 1
(b) 2
(c) −1
(d) −2

Sol. ***(c)*** Given, $\lim_{x \to \pi} \dfrac{\sin x}{x - \pi} = \lim_{x \to \pi} \dfrac{\sin(\pi - x)}{-(\pi - x)}$

[$\because \sin\theta = \sin(\pi - \theta)$]

$$= -\lim_{x \to \pi} \frac{\sin(\pi - x)}{(\pi - x)} = -1 \quad \left[\because \lim_{x \to 0} \frac{\sin x}{x} = 1 \text{ and } \pi - x \to 0 \Rightarrow x \to \pi\right]$$

Q. 55 $\lim_{x \to 0} \dfrac{x^2 \cos x}{1 - \cos x}$ is equal to

(a) 2
(b) $\dfrac{3}{2}$
(c) $\dfrac{-3}{2}$
(d) 1

Sol. ***(a)*** Given, $\lim_{x \to 0} \dfrac{x^2 \cos x}{1 - \cos x} = \lim_{x \to 0} \dfrac{x^2 \cos x}{2 \sin^2 \dfrac{x}{2}}$ $\left[\because 1 - \cos x = 2 \sin^2 \dfrac{x}{2}\right]$

$$= 2 \lim_{x \to 0} \frac{\left(\frac{x}{2}\right)^2}{\sin^2 \frac{x}{2}} \cdot \lim_{x \to 0} \cos x = 2 \cdot 1 = 2$$

Q. 56 $\lim_{x \to 0} \dfrac{(1+x)^n - 1}{x}$ is equal to

(a) n (b) 1

(c) $-n$ (d) 0

Sol. *(a)* Given, $\lim_{x \to 0} \dfrac{(1+x)^n - 1}{x} = \lim_{x \to 0} \dfrac{(1+x)^n - 1}{(1+x)-1} = \lim_{x \to 0} \dfrac{(1+x)^n - 1}{(1+x)-1}$

$$= \lim_{x \to 0} \frac{(1+x)^n - 1^n}{(1+x)-1} = \lim_{(1+x) \to 1} \frac{(1+x)^n - 1^n}{(1+x)-1}$$

$$= n \cdot (1)^{n-1} = n \qquad \left[\because \lim_{x \to a} \frac{x^n - a^n}{x-a} = na^{n-1}\right]$$

Q. 57 $\lim_{x \to 1} \dfrac{x^m - 1}{x^n - 1}$ is equal to

(a) 1 (b) $\dfrac{m}{n}$

(c) $-\dfrac{m}{n}$ (d) $\dfrac{m^2}{n^2}$

Sol. *(b)* Given,
$$\lim_{x \to 1} \frac{x^m - 1}{x^n - 1} = \lim_{x \to 1} \frac{\dfrac{x^m - 1}{x-1}}{\dfrac{x^n - 1}{x-1}} = \frac{\lim_{x \to 1} \dfrac{x^m - 1^m}{x-1}}{\lim_{x \to 1} \dfrac{x^n - 1^n}{x-1}}$$

$$= \frac{m\,(1)^{m-1}}{n\,(1)^{n-1}} = \frac{m}{n} \qquad \left[\because \lim_{x \to a} \frac{x^n - a^n}{x-a} = na^{n-1}\right]$$

Q. 58 $\lim_{\theta \to 0} \dfrac{1 - \cos 4\theta}{1 - \cos 6\theta}$ is equal to

(a) $\dfrac{4}{9}$ (b) $\dfrac{1}{2}$

(c) $\dfrac{-1}{2}$ (d) -1

Sol. *(a)* Given, $\lim_{\theta \to 0} \dfrac{1 - \cos 4\theta}{1 - \cos 6\theta} = \lim_{\theta \to 0} \dfrac{2\sin^2 2\theta}{2\sin^2 3\theta}$ $\quad [\because 1 - \cos 2\theta = 2\sin^2 \theta]$

$$= \frac{\lim_{\theta \to 0} \dfrac{\sin^2 2\theta}{(2\theta)^2} \cdot (2\theta)^2}{\lim_{\theta \to 0} \dfrac{\sin^2 3\theta}{(3\theta)^2} \cdot (3\theta)^2} = \frac{4}{9} \cdot \frac{\lim_{\theta \to 0} \left(\dfrac{\sin 2\theta}{2\theta}\right)^2}{\lim_{\theta \to 0} \left(\dfrac{\sin 3\theta}{3\theta}\right)^2} \qquad \left[\because \lim_{x \to 0} \frac{\sin x}{x} = 1 \text{ and } x \to 0 \Rightarrow kx \to 0\right]$$

$$= \frac{4}{9}$$

Q. 59 $\lim\limits_{x \to 0} \dfrac{\text{cosec}\, x - \cot x}{x}$ is equal to

(a) $\dfrac{-1}{2}$ (b) 1 (c) $\dfrac{1}{2}$ (d) 1

Sol. ***(c)*** Given,

$$\lim_{x \to 0} \frac{\text{cosec}\, x - \cot x}{x}$$

$$= \lim_{x \to 0} \frac{\frac{1}{\sin x} - \frac{\cos x}{\sin x}}{x} = \lim_{x \to 0} \frac{1 - \cos x}{x \cdot \sin x}$$

$$= \lim_{x \to 0} \frac{2\sin^2 \frac{x}{2}}{x \cdot 2\sin\frac{x}{2}\cos\frac{x}{2}} = \lim_{x \to 0} \frac{\tan\frac{x}{2}}{x}$$

$$= \lim_{x \to 0} \frac{\tan\frac{x}{2}}{\frac{x}{2}} \cdot \frac{1}{2} = \frac{1}{2} \qquad \left[\because \lim_{\theta \to 0} \frac{\tan\theta}{\theta} = 1\right]$$

Q. 60 $\lim\limits_{x \to 0} \dfrac{\sin x}{\sqrt{x+1} - \sqrt{1-x}}$ is equal to

(a) 2 (b) 0 (c) 1 (d) –1

Sol. ***(c)*** Given,

$$\lim_{x \to 0} \frac{\sin x}{\sqrt{x+1} - \sqrt{1-x}}$$

$$= \lim_{x \to 0} \frac{\sin x}{\sqrt{x+1} - \sqrt{1-x}} \cdot \frac{\sqrt{x+1} + \sqrt{1-x}}{\sqrt{x+1} + \sqrt{1-x}}$$

$$= \lim_{x \to 0} \frac{\sin x(\sqrt{x+1} + \sqrt{1-x})}{(x+1) - (1-x)}$$

$$= \lim_{x \to 0} \frac{\sin x(\sqrt{x+1} + \sqrt{1-x})}{x + 1 - 1 + x}$$

$$= \frac{1}{2} \lim_{x \to 0} \frac{\sin x}{x} \lim_{x \to 0} (\sqrt{x+1} + \sqrt{1-x})$$

$$= \frac{1}{2} \cdot 1 \cdot 2 = 1 \qquad \left[\because \lim_{x \to 0} \frac{\sin x}{x} = 1\right]$$

Q. 61 $\lim\limits_{x \to \pi/4} \dfrac{\sec^2 x - 2}{\tan x - 1}$ is

(a) 3 (b) 1 (c) 0 (d) 2

Sol. ***(d)*** Given,

$$\lim_{x \to \pi/4} \frac{\sec^2 x - 2}{\tan x - 1}$$

$$= \lim_{x \to \pi/4} \frac{1 + \tan^2 x - 2}{\tan x - 1} = \lim_{x \to \pi/4} \frac{\tan^2 x - 1}{\tan x - 1}$$

$$= \lim_{x \to \pi/4} \frac{(\tan x + 1)(\tan x - 1)}{(\tan x - 1)} = \lim_{x \to \pi/4} (\tan x + 1)$$

$$= 2$$

Q. 62 $\lim_{x \to 1} \dfrac{(\sqrt{x} - 1)(2x - 3)}{2x^2 + x - 3}$ is equal to

(a) $\dfrac{1}{10}$ (b) $\dfrac{-1}{10}$ (c) 1 (d) None of these

Sol. ***(b)*** Given, $\lim_{x \to 1} \dfrac{(\sqrt{x} - 1)(2x - 3)}{2x^2 + x - 3} = \lim_{x \to 1} \dfrac{(\sqrt{x} - 1)(2x - 3)}{(2x + 3)(x - 1)}$

$$= \lim_{x \to 1} \frac{(\sqrt{x} - 1)(2x - 3)}{(2x + 3)(\sqrt{x} - 1)(\sqrt{x} + 1)}$$

$$= \lim_{x \to 1} \frac{2x - 3}{(2x + 3)(\sqrt{x} + 1)} = \frac{-1}{5 \times 2} = \frac{-1}{10}$$

Q. 63 If $f(x) = \begin{cases} \dfrac{\sin[x]}{[x]}, & [x] \neq 0 \\ 0, & [x] = 0 \end{cases}$, where $[\cdot]$ denotes the greatest integer function, then $\lim_{x \to 0} f(x)$ is equal to

(a) 1 (b) 0 (c) -1 (d) Does not exist

Sol. ***(d)*** Given, $f(x) = \begin{cases} \dfrac{\sin[x]}{[x]}, & [x] \neq 0 \\ 0, & [x] = 0 \end{cases}$

$\therefore$ $\text{LHL} = \lim_{x \to 0^-} f(x)$

$$= \lim_{x \to 0^-} \frac{\sin[x]}{[x]} = \lim_{h \to 0} \frac{\sin[0 - h]}{[0 - h]}$$

$$= \lim_{h \to 0} \frac{-\sin[-h]}{[-h]} = -1$$

$$\text{RHL} = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{\sin[x]}{[x]}$$

$$= \lim_{x \to 0^+} \frac{\sin[0 + h]}{[0 + h]} = \lim_{h \to 0} \frac{\sin[h]}{[h]} = 1$$

$\because$ $\text{LHL} \neq \text{RHL}$

So, limit does not exist.

Q. 64 $\lim_{x \to 0} \dfrac{|\sin x|}{x}$ is equal to

(a) 1 (b) $= -1$

(c) Does not exist (d) None of these

Sol. ***(c)*** Given, $\text{limit} = \lim_{x \to 0} \dfrac{|\sin x|}{x}$

$\therefore$ $\text{LHL} = \lim_{x \to 0^-} \left(\dfrac{-\sin x}{x} \right) = -\lim_{x \to 0^-} \dfrac{\sin x}{x} = -1$

$$\text{RHL} = \lim_{x \to 0^+} \frac{\sin x}{x} = 1$$

$\because$ $\text{LHL} \neq \text{RHL}$

So , limit does not exist.

Q. 65 If $f(x)=\begin{cases}x^2-1, & 0<x<2\\ 2x+3, & 2\le x<3\end{cases}$, then the quadratic equation whose roots are $\lim\limits_{x\to 2^-} f(x)$ and $\lim\limits_{x\to 2^+} f(x)$ is

(a) $x^2-6x+9=0$
(b) $x^2-7x+8=0$
(c) $x^2-14x+49=0$
(d) $x^2-10x+21=0$

Sol. ***(d)*** Given, $f(x)=\begin{cases}x^2-1, & 0<x<2\\ 2x+3, & 2\le x<3\end{cases}$

$$\therefore \quad \lim_{x\to 2^-} f(x)=\lim_{x\to 2^-}(x^2-1)$$
$$=\lim_{h\to 0}[(2-h)^2-1]=\lim_{h\to 0}(4+h^2-4h-1)$$
$$=\lim_{h\to 0}(h^2-4h+3)=3$$
$$\text{and} \quad \lim_{x\to 2^+} f(x)=\lim_{x\to 2^+}(2x+3)$$
$$=\lim_{h\to 0}[2(2+h)+3]=\lim_{h\to 0}(4+2h+3)=7$$

So, the quadratic equation whose roots are 3 and 7 is $x^2-(3+7)x+3\times 7=0$ *i.e.*, $x^2-10x+21=0$.

Q. 66 $\lim\limits_{x\to 0}\dfrac{\tan 2x-x}{3x-\sin x}$ is equal to

(a) 2
(b) $\frac{1}{2}$
(c) $\frac{-1}{2}$
(d) $\frac{1}{4}$

Sol. ***(b)*** Given,
$$\lim_{x\to 0}\frac{\tan 2x-x}{3x-\sin x}=\lim_{x\to 0}\frac{x\left[\frac{\tan 2x}{x}-1\right]}{x\left[3-\frac{\sin x}{x}\right]}$$
$$=\frac{\lim\limits_{x\to 0}2\times\frac{\tan 2x}{2x}-1}{3-\lim\limits_{x\to 0}\frac{\sin x}{x}}=\frac{2-1}{3-1}=\frac{1}{2}$$

Q. 67 If $f(x)=x-[x]$, $\in R$, then $f'\left(\frac{1}{2}\right)$ is equal to

(a) $\frac{3}{2}$
(b) 1
(c) 0
(d) -1

Sol. ***(b)*** Given, $f(x)=x-[x]$

Now, first we have to check the differentiability of $f(x)$ at $x=\frac{1}{2}$.

$$\therefore \quad Lf'\left(\frac{1}{2}\right)=\text{LHD}=\lim_{h\to 0}\frac{f\left(\frac{1}{2}-h\right)-f\left(\frac{1}{2}\right)}{-h}$$

$$= \lim_{n \to 0} \frac{\left(\frac{1}{2}-h\right)-\left(\frac{1}{2}-h\right)-\frac{1}{2}+\left(\frac{1}{2}\right)}{h} = \lim_{h \to 0} \frac{\frac{1}{2}1 - h - 0 - \frac{1}{2} + 0 = 1}{h}$$

and $\quad Rf'\left(\frac{1}{2}\right) = \text{RHD} \lim_{h \to 0} \frac{f\left(\frac{1}{2}+h\right)-f\left(\frac{1}{2}\right)}{h}$

$$= \lim_{h \to 0} \frac{\left(\frac{1}{2}+h\right)-\left(\frac{1}{2}+h\right)-\frac{1}{2}+\left(\frac{1}{2}\right)}{h} = \lim_{h \to 0} \frac{\frac{1}{2}+h-0-\frac{1}{2}+0}{h} = 1$$

$\because \quad$ LHL = RHD

$\therefore \quad f'\left(\frac{1}{2}\right) = 1$

Q. 68 If $y = \sqrt{x} + \dfrac{1}{\sqrt{x}}$, then $\dfrac{dy}{dx}$ at $x = 1$ is equal to

(a) 1 (b) $\frac{1}{2}$ (c) $\frac{1}{\sqrt{2}}$ (d) 0

Sol. ***(d)*** Given, $\quad y = \sqrt{x} + \dfrac{1}{\sqrt{x}}$

Now, $\quad \dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x}} - \dfrac{1}{2x^{3/2}}$

$\therefore \quad \left(\dfrac{dy}{dx}\right)_{x=1} = \dfrac{1}{2} - \dfrac{1}{2} = 0$

Q. 69 If $f(x) = \dfrac{x-4}{2\sqrt{x}}$, then $f'(1)$ is equal to

(a) $\frac{5}{4}$ (b) $\frac{4}{5}$ (c) 1 (d) 0

Sol. ***(a)*** Given, $\quad f(x) = \dfrac{x-4}{2\sqrt{x}}$

Now, $\quad f'(x) = \dfrac{2\sqrt{x} - (x-4)\cdot 2 \cdot \frac{1}{2\sqrt{x}}}{4x}$

$$= \frac{2x-(x-4)}{4x^{3/2}} = \frac{2x - x + 4}{4x^{3/2}}$$

$$= \frac{x+4}{4x^{3/2}}$$

$\therefore \quad f'(1) = \dfrac{1+4}{4 \times (1)^{3/2}} = \dfrac{5}{4}$

Q. 70 If $y = \dfrac{1+\frac{1}{x^2}}{1-\frac{1}{x^2}}$, then $\dfrac{dy}{dx}$ is equal to

(a) $\dfrac{-4x}{(x^2-1)^2}$ (b) $\dfrac{-4x}{x^2-1}$

(c) $\dfrac{1-x^2}{4x}$ (d) $\dfrac{4x}{x^2-1}$

Sol. ***(a)*** Given,
$$y = \frac{1+\frac{1}{x^2}}{1-\frac{1}{x^2}} \Rightarrow y = \frac{x^2+1}{x^2-1}$$

$$\therefore \quad \frac{dy}{dx} = \frac{(x^2-1)2x-(x^2+1)(2x)}{(x^2-1)^2} \quad \text{[by quotient rule]}$$

$$\frac{dy}{dx} = \frac{2x(x^2-1-x^2-1)}{(x^2-1)^2}$$

$$= \frac{2x(-2)}{(x^2-1)^2} = \frac{-4x}{(x^2-1)^2}$$

Q. 71 If $y = \dfrac{\sin x + \cos x}{\sin x - \cos x}$, then $\dfrac{dy}{dx}$ at $x = 0$ is equal to

(a) -2 (b) 0 (c) $\frac{1}{2}$ (d) Does not exist

Sol. ***(a)*** Given, $y = \dfrac{\sin x + \cos x}{\sin x - \cos x}$

$$\therefore \quad \frac{dy}{dx} = \frac{(\sin x - \cos x)(\cos x - \sin x) - (\sin x + \cos x)(\cos x + \sin x)}{(\sin x - \cos x)^2} \quad \text{[by quotient rule]}$$

$$= \frac{-(\sin x - \cos x)^2 - (\sin x + \cos x)^2}{(\sin x - \cos x)^2}$$

$$= \frac{-[(\sin x - \cos x)^2 + (\sin x + \cos x)^2]}{(\sin x - \cos x)^2}$$

$$= \frac{-[\sin^2 x + \cos^2 x - 2\sin x\cos x + \sin^2 x + \cos^2 x + 2\sin x\cos x]}{(\sin x - \cos x)^2}$$

$$= \frac{-2}{(\sin x - \cos x)^2}$$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{x=0} = -2$$

Q. 72 If $y = \dfrac{\sin(x+9)}{\cos x}$, then $\dfrac{dy}{dx}$ at $x = 0$ is equal to

(a) $\cos 9$ (b) $\sin 9$ (c) 0 (d) 1

Sol. ***(a)*** Given,
$$y = \frac{\sin(x+9)}{\cos x}$$

$$\therefore \quad \frac{dy}{dx} = \frac{\cos x\cos(x+9) - \sin(x+9)(-\sin x)}{(\cos x)^2} \quad \text{[by quotient rule]}$$

$$= \frac{\cos x\cos(x+9) + \sin x\sin(x+9)}{\cos^2 x}$$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{x=0} = \frac{\cos 9}{1}$$

$$= \cos 9$$

Q. 73 If $f(x) = 1 + x + \frac{x^2}{2} + \ldots + \frac{x^{100}}{100}$, then $f'(1)$ is equal to

(a) $\frac{1}{100}$ (b) 100

(c) 0 (d) Does not exist

Sol. ***(b)*** Given, $f(x) = 1 + x + \frac{x^2}{2} + \ldots + \frac{x^{100}}{100}$

$\therefore$ $f'(x) = 0 + 1 + 2 \times \frac{x}{2} + \ldots + 100\frac{x^{99}}{100}$

$f'(x) = 1 + x + x^2 + \ldots + x^{99}$

Now, $f'(1) = 1 + 1 + 1 + \ldots + 1$ (100 times)

$= 100$

Q. 74 If $f(x) = \frac{x^n - a^n}{x - a}$ for some constant a, then $f'(a)$ is equal to

(a) 1 (b) 0 (c) $\frac{1}{2}$ (d) Does not exist

Sol. ***(d)*** Given, $f(x) = \frac{x^n - a^n}{x - a}$

$\therefore$ $f'(x) = \frac{(x - a)nx^{n-1} - (x^n - a^n)(1)}{(x - a)^2}$ [by quotient rule]

$\Rightarrow$ $f'(x) = \frac{nx^{n-1}(x - a) - x^n + a^n}{(x - a)^2}$

Now, $f'(a) = \frac{na^{n-1}(0) - a^n + a^n}{(x - a)^2}$

$\Rightarrow$ $f'(a) = \frac{0}{0}$

So, $f'(a)$ does not exist,

Since, $f(x)$ is not defined at $x = a$.

Hence, $f'(x)$ at $x = a$ does not exist.

Q. 75 If $f(x) = x^{100} + x^{99} + \ldots + x + 1$, then $f'(1)$ is equal to

(a) 5050 (b) 5049

(c) 5051 (d) 50051

Sol. ***(a)*** Given, $f(x) = x^{100} + x^{99} + \ldots + x + 1$

$\therefore$ $f'(x) = 100x^{99} + 99x^{98} + \ldots + 1 + 0$

$= 100x^{99} + 99x^{98} + \ldots + 1$

Now, $f'(1) = 100 + 99 + \ldots + 1$

$= \frac{100}{2}[2 \times 100 + (100 - 1)(-1)]$

$= 50[200 - 99]$ $\left[\because S_n = \frac{n}{2}\{2a + (n - 1)d\}\right]$

$= 50 \times 101$

$= 5050$

Q. 76 If $f(x) = 1 - x + x^2 - x^3 + \ldots - x^{99} + x^{100}$, then $f'(1)$ is equal to

(a) 150 (b) -50 (c) -150 (d) 50

Sol. *(d)* Given, $f(x) = 1 - x + x^2 - x^3 + \ldots - x^{99} + x^{100}$

$$f'(x) = 0 - 1 + 2x - 3x^2 + \ldots - 99x^{98} + 100x^{99}$$
$$= -1 + 2x - 3x^2 + \ldots - 99x^{98} + 100x^{99}$$

$\therefore$ $f'(1) = -1 + 2 - 3 + \ldots - 99 + 100$

$$= (-1 - 3 - 5 - \ldots - 99) + (2 + 4 + \ldots + 100) \quad \left[\because S_n = \frac{n}{2}\{2a + (n-1)d\}\right]$$
$$= -\frac{50}{2}[2 \times 1 + (50 - 1)2] + \frac{50}{2}[2 \times 2 + (50 - 1)2]$$
$$= -25[2 + 49 \times 2] + 25[4 + 49 \times 2]$$
$$= -25(2 + 98) + 25(4 + 98)$$
$$= -25 \times 100 + 25 \times 102$$
$$= -2500 + 2550$$
$$= 50$$

Fillers

Q. 77 If $f(x) = \dfrac{\tan x}{x - \pi}$, then $\lim\limits_{x \to \pi} f(x) =$ ....

Sol. Given, $f(x) = \dfrac{\tan x}{x - \pi} = \lim\limits_{x \to \pi} \dfrac{\tan x}{x - \pi} = \lim\limits_{\pi - x \to 0} \dfrac{-\tan(\pi - x)}{-(\pi - x)}$ $\left[\because \lim\limits_{x \to 0} \dfrac{\tan x}{x} = 1\right]$

$= 1$

Q. 78 $\lim\limits_{x \to 0} \left(\sin mx \cot \dfrac{x}{\sqrt{3}}\right) = 2$, then $m =$ ......

Sol. Given, $\lim\limits_{x \to 0} \left(\sin mx \cot \dfrac{x}{\sqrt{3}}\right) = 2$

$$\Rightarrow \lim_{x \to 0} \frac{\sin mx}{mx} \cdot mx \cdot \frac{1}{\tan \frac{x}{\sqrt{3}}} = 2$$

$$\Rightarrow \lim_{x \to 0} \frac{\sin mx}{mx} \cdot mx \cdot \frac{\frac{x}{\sqrt{3}}}{\tan \frac{x}{\sqrt{3}}} \cdot \frac{1}{\frac{x}{\sqrt{3}}} = 2$$

$$\Rightarrow \lim_{x \to 0} \frac{\sin mx}{mx} \cdot \lim_{x \to 0} \frac{\frac{x}{\sqrt{3}}}{\tan \frac{x}{\sqrt{3}}} \cdot \lim_{x \to 0} \frac{mx}{\frac{x}{\sqrt{3}}} = 2$$

$$\Rightarrow \sqrt{3}x = 2$$

$$\therefore m = \frac{2\sqrt{3}}{3}$$

Q. 79 If $y = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$, then $\frac{dy}{dx} = \ldots\ldots$

Sol. Given,

$$y = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

$$\therefore \quad \frac{dy}{dx} = 0 + 1 + \frac{2x}{2} + \frac{3x^2}{6} + \frac{4x^3}{4!}$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots$$

$$= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$= y$$

Q. 80 $\lim_{x \to 3^+} \frac{x}{[x]} = \ldots$

Sol. Given,

$$\lim_{x \to 3^+} \frac{x}{[x]} = \lim_{h \to 0} \frac{(3 + h)}{[3 + h]}$$

$$= \lim_{h \to 0} \frac{(3 + h)}{3} = 1$$

14

Mathematical Reasoning

Short Answer Type Questions

Q. 1 Which of the following sentences are statements? Justify

(i) A triangle has three sides.

(ii) 0 is a complex number.

(iii) Sky is red.

(iv) Every set is an infinite set.

(v) $15 + 8 > 23$.

(vi) $y + 9 = 7$

(vii) Where is your bag?

(viii) Every square is a rectangle.

(ix) Sum of opposite angles of a cyclic quadrilateral is 180°.

(x) $\sin^2 x + \cos^2 x = 0$

Sol. As we know, a statement is a sentence which is either true or false but not both simultaneously.

(i) It is true statement.

(ii) It is true statement.

(iii) It is false statement.

(iv) It is false statement.

(v) It is false statement.

(vi) $y + 9 = 7$

It is not considered as a statement, since the value of y is not given.

(vii) It is a question, so it is not a statement.

(viii) It is a true statement.

(ix) It is true statement.

(x) It is false statement.

Q. 2 Find the component statements of the following compound statements.

(i) Number 7 is prime and odd.
(ii) Chennai is in India and is the capital of Tamil Nadu.
(iii) The number 100 is divisible by 3, 11 and 5.
(iv) Chandigarh is the capital of Haryana and UP.
(v) $\sqrt{7}$ is a rational number or an irrational number.
(vi) 0 is less than every positive integer and every negative integer.
(vii) Plants use sunlight, water and carbon dioxide for photosynthesis.
(viii) Two lines in a plane either intersect at one point or they are parallel.
(ix) A rectangle is a quadrilateral or a 5 sided polygon.

Sol. **(i)** p : Number 7 is prime.
q : Number 7 is odd.

(ii) P: Chennai is in India.
q : Chennai is capital of Tamil Nadu.

(iii) p : 100 is divisible by 3.
q : 100 is divisible by 11.
r : 100 is divisible by 5.

(iv) p : Chandigarh is capital of Haryana.
q : Chandigarh is capital of UP.

(v) p : $\sqrt{7}$ is a rational number.
q : $\sqrt{7}$ is an irrational number.

(vi) p : 0 is less than every positive integer.
q : 0 is less than every negative integer.

(vii) p : Plants use sunlight for photosysthesis.
q : Plants use water for photosynthesis.
r : Plants use carbon dioxide for photosysthesis.

(viii) p : Two lines in a plane intersect at one point.
q : Two lines in a plane are parallel.

(ix) p : A rectangli, is a quadrilateral.
q : A rectangle is a 5-sided polygon.

Q. 3 Write the component statements of the following compound statements and check whether the compound statement is true or false.

(i) 57 is divisible by 2 or 3.
(ii) 24 is a multiple of 4 and 6.
(iii) All living things have two eyes and two legs.
(iv) 2 is an even number and a prime number.

Sol. **(i)** Given compound statement is of the form 'pvq'. Since, the statement 'pvq' has the truth value T whenever either p or q or both have the truth value T.

So, it is true statement.

Its component statements are

p : 57 is divisible by 2. [false]

q : 57 is divisible by 3. [true]

(ii) Given compound statement is of the form '$p \wedge q$'. Since, the statement '$p \wedge q$' have the truth value T whenever both p and q have the truth value T.

So, it is a true statement.

Its component statements are

p : 24 is multiple of 4 [true]

q : 24 is multiple of 6. [true]

(iii) It is a false statement. Since '$p \wedge q$' has truth value F whenever either p or q or both have the truth value F.

Its component statements are

p : All living things have two eyes. [false]

q : All living things have two legs. [false]

(iv) It is a true statement.

Its component statements are

p : 2 is an even number. [true]

q : 2 is a prime number. [true]

Q. 4 Write the negative on the following simple statements.

(i) The number 17 is prime.

(ii) $2+7=6$.

(iii) Violets are blue.

(iv) $\sqrt{5}$ is a rational number.

(v) 2 is not a prime number.

(vi) Every real number is an irrational number.

(vii) Cow has four legs.

(viii) A leap year has 366 days.

(ix) All similar triangles are congruent.

(x) Area of a circle is same as the perimeter of the circle.

Sol. **(i)** The number 17 is not prime.

(ii) $2+7 \neq 6$.

(iii) Violets are not blue.

(iv) $\sqrt{5}$ is not a rational number.

(v) 2 is a prime number.

(vi) Every real number is not an irrational number.

(vii) Cow has not four legs.

(viii) A leap year has not 366 days.

(ix) There exist similar triangles which are not congruent.

(x) Area of a circle is not same as the perimeter of the circle.

Q. 5 Translate the following statements into symbolic form

(i) Rahul passed in Hindi and English.

(ii) x and y are even integers.

(iii) 2, 3 and 6 are factors of 12.

(iv) Either x or $x+1$ is an odd integer.

(v) A number is either divisible by 2 or 3.

(vi) Either $x = 2$ or $x = 3$ is a root of $3x^2 - x - 10 = 0$.

(vii) Students can take Hindi or English as an optional paper.

Sol. **(i)** p : Rahul passed in Hindi.

q : Rahul passed in English.

$p \wedge q$: Rahul passed in Hindi and English.

(ii) p : x is even integers.

q : y is even integers.

$p \cap q$: x and y are even integers.

(iii) p : 2 is factor of 12.

q : 3 is factor of 12.

r : 6 is factor of 12.

$p \wedge q \wedge r$: 2, 3 and 6 are factor of 12.

(iv) p : x is an odd integer.

q : $(x + 1)$ is an odd integer.

$p \vee q$: Either x or $(x + 1)$ is an odd integer.

(v) p : A number is divisible by 2.

q : A number is divisible by 3.

$p \vee q$: A number is either divisible by 2 or 3.

(vi) p : $x = 2$ is a root of $3x^2 - x - 10 = 0$.

q : $x = 3$ is a root of $3x^2 - x - 10 = 0$.

$p \vee q$: Either $x = 2$ or $x = 3$ is a root of $3x^2 - x - 10 = 0$.

(vii) p : Students can take Hindi as an optional paper.

q : Students can take English as an optional subject.

$p \vee q$: Students can take Hindi or English as an optional paper.

Q. 6 Write down the negation of following compound statements.

(i) All rational numbers are real and complex.

(ii) All real numbers are rationals or irrationals.

(iii) $x = 2$ and $x = 3$ are roots of the quadratic equation $x^2 - 5x + 6 = 0$.

(iv) A triangle has either 3-sides or 4-sides.

(v) 35 is a prime number or a composite number.

(vi) All prime integers are either even or odd.

(vii) $|x|$ is equal to either x or $-x$.

(viii) 6 is divisible by 2 and 3.

Thinking Process

Use (i) $\sim(p \wedge q) = \sim p \vee \sim q$

(ii) $\sim(p \vee q) = \sim p \wedge \sim q$

Sol. **(i)** Let p : All rational numbers are real.

q : All rational numbers are complex.

$\sim p$: All rational number are not real.

$\sim q$: All rational numbers are not complex.

$\sim(p \wedge q)$: All rational numbers are not real or not complex. $[\because \sim(p \wedge q) = \sim p \vee \sim q]$

(ii) Let p : All real numbers are rationals.

q : All real numbers are irrational.

Then, the negation of the above statement is given by

$\sim(p \vee q)$: All real numbers are not rational and all real numbers are not irrational.

$[\because \sim(p \vee q) = \sim p \wedge \sim q]$

(iii) Let p : $x = 2$ is root of quadratic equation $x^2 - 5x + 6 = 0$.

q : $x = 3$ is root of quadratic equation $x^2 - 5x + 6 = 0$.

Then, the negation of conjunction of above statement is given by

$\sim(p \wedge q)$: $x = 2$ is not a root of quadratic equation $x^2 - 5x + 6 = 0$ or $x = 3$ is not a root of the quadratic equation $x^2 - 5x + 6 = 0$.

(iv) Let p : A triangle has 3-sides.

q : A triangle has 4-sides.

Then, negation of disjunction of the above statement is given by

$\sim(p \vee q)$: A triangle has neither 3-sides nor 4-sides.

(v) Let p : 35 is a prime number.

q : 35 is a composite number.

Then, negation of disjunction of the above statement is given by

$\sim(p \vee q)$: 35 is not a prime number and it is not a composite number.

(vi) Let p : All prime integers are even.

q : All prime integers are odd.

Then negation of disjunction of the above statement is given by
$\sim(p \vee q)$: All prime integers are not even and all prime integers are not odd.

(vii) Let p : $|x|$ is equal to x.

q : $|x|$ is equal to $-x$.

Then negation of disjunction of the above statement is given by

$\sim(p \vee q)$: $|x|$ is not equal to x and it is not equal to $-x$.

(viii) Let p : 6 is divisible by 2.

q : 6 is divisible by 3.

Then, negation of conjunction of above statement is given by

$\sim(p \wedge q)$: 6 is not divisible by 2 or it is not divisible by 3

Q. 7 Rewrite each of the following statements in the form of conditional statements.

(i) The square of an odd number is odd.
(ii) You will get a sweet dish after the dinner.
(iii) You will fail, if you will not study.
(iv) The unit digit of an integer is 0 or 5, if it is divisible by 5.
(v) The square of a prime number is not prime.
(vi) $2b = a + c$, if a, b and c are in AP.

Sol. We know that, some of the common expressions of conditional statement $p \rightarrow q$ are

(i) if p, then q
(ii) q if p
(iii) p only if q
(iv) p is sufficient for q
(v) q is necesary for p
(vi) $\sim q$ implies $\sim p$

So, use above information to get the answer

(i) If the number is odd number, then its square is odd number.
(ii) If you take the dinner, then you will get sweet dish.
(iii) If you will not study, then you will fail.
(iv) If an integer is divisible by 5, then its unit digits are 0 or 5.
(v) If the number is prime, then its square is not prime.
(vi) If a, b and c are in AP, then $2b = a + c$.

Q. 8 Form the biconditional statement $p \leftrightarrow q$, where

(i) p : The unit digits of an integer is zero.
q : It is divisible by 5.
(ii) p : A natural number n is odd.
q : Natural number n is not divisible by 2.
(iii) p : A triangle is an equilateral triangle.
q : All three sides of a triangle are equal.

Sol. **(i)** $p \leftrightarrow q$: The unit digit of on integer is zero, if and only if it is divisible by 5.
(ii) $p \leftrightarrow q$: A natural number no odd if and only if it is not divisible by 2.
(iii) $p \leftrightarrow q$: A triangle is an equilateral triangle if and only if all three sides of triangle are equal.

Q. 9 Write down the contrapositive of the following statements.

(i) If $x = y$ and $y = 3$, then $x = 3$.
(ii) If n is a natural number, then n is an integer.
(iii) If all three sides of a triangle are equal, then the triangle is equilateral.
(iv) If x and y are negative integers, then xy is positive.
(v) If natural number n is divisible by 6, then n is divisible by 2 and 3.
(vi) If it snows, then the weather will be cold.
(vii) If x is a real number such that $0 < x < 1$, then $x^2 < 1$.

Thinking Process

We know that, the statement $(\sim q) \to (\sim p)$ is called contrapositive of the statement $p \to q$.

Sol. **(i)** If $x \neq 3$, then $x \neq y$ or $y \neq 3$.
(ii) If n is not an integer, then it is not a natural number.
(iii) If the triangle is not equilateral, then all three sides of the triangle are not equal.
(iv) If xy is not positive integer, then either x or y is not negative integer.
(v) If natural number n is not divisible by 2 or 3, then n is not divisible by 6.
(vi) The weather will not be cold, if it does not snow.
(vii) If $x^2 \nless 1$, then x is not a real number such that $0 < x < 1$.

Q. 10 Write down the converse of following statements.

(i) If a rectangle 'R' is a square, then R is a rhombus.
(ii) If today is Monday, then tomorrow is Tuesday.
(iii) If you go to Agra, then you must visit Taj Mahal.
(iv) If sum of squares of two sides of a triangle is equal to the square of third side of a triangle, then the triangle is right angled.
(v) If all three angles of a triangle are equal, then the triangle is equilateral.
(vi) If $x : y = 3 : 2$, then $2x = 3y$.
(vii) If S is a cyclic quadrilateral, then the opposite angles of S are supplementary.
(viii) If x is zero, then x is neither positive nor negative.
(ix) If two triangles are similar, then the ratio of their corresponding sides are equal.

Thinking Process

We know that, the converse of the statement "$p \to q$" is "$(q) \to (p)$".

Sol. **(i)** If thes rectangle 'R' is rhombus, then it is square.
(ii) If tomorrow is Tuesday, then today is Monday.
(iii) If you must visit Taj Mahal, you go to Agra.
(iv) If the triangle is right angle, then sum of squares of two sides of a triangle is equal to the square of third side.
(v) If the triangle is equilateral, then all three angles of triangle are equal.

(vi) If $2x = 3y$, then $x:y = 3:2$

(vii) If the opposite angles of a quadrilateral are supplementary, then S is cyclic.

(viii) If x is neither positive nor negative, then x is 0.

(ix) If the ratio of corresponding sides of two triangles are equal, then triangles are similar.

Q. 11 Identify the quantifiers in the following statements.

(i) There exists a triangle which is not equilateral.

(ii) For all real numbers x and y, $xy = yx$.

(iii) There exists a real number which is not a rational number.

(iv) For every natural number x, $x + 1$ is also a natural number.

(v) For all real numbers x with $x > 3$, x^2 is greater than 9.

(vi) There exists a triangle which is not an isosceles triangle.

(vii) For all negative integers x, x^3 is also a negative integers.

(viii) There exists a statement in above statements which is not true.

(ix) There exists an even prime number other than 2.

(x) There exists a real number x such that $x^2 + 1 = 0$.

Sol. Quantifier are the phrases like 'There exist' and 'For every', 'For all' etc.

(i) There exists

(ii) For all

(iii) There exists

(iv) For every

(v) For all

(vi) There exists

(vii) For all

(viii) There exists

(ix) There exists

(x) There exists

Q. 12 Prove by direct method that for any integer 'n', $n^3 - n$ is always even.

Thinking Process

We know that, in direct method to show a statement, if p then q is true, we assume p is true and show q is true i.e., $p \rightarrow q$.

Sol. Here, two cases arise

Case **I** When n is even,

Let $n = 2K, K \in N$

$\Rightarrow$ $n^3 - n = (2K)^3 - (2K) = 2K(4K^2 - 1)$

$= 2\lambda$, where $\lambda = K(4K^2 - 1)$

Thus, $(n^3 - n)$ is even when n is even.

Case **II** When n is odd,

Let $n = 2K + 1, K \in N$

$\Rightarrow$ $n^3 - n = (2K+1)^3 - (2K+1)$

$= (2K+1)[(2K+1)^2 - 1]$

$= (2K+1)[4K^2 + 1 + 4K - 1]$

$= (2k+1)(4K^2 + 4K)$

$= 4K(2K+1)(K+1)$

$= 2\mu$, when $\mu = 2K(K+1)(2K+1)$

Then, $n^3 - n$ is even when n is odd.

So, $n^3 - n$ is always even.

Q. 13 Check validity of the following statement.

(i) p : 125 is divisible by 5 and 7.

(ii) q : 131 is a multiple of 3 or 11.

Sol. **(i)** p : 125 is divisible by 5 and 7.

Let q : 125 is divisible by 5.

r : 125 is divisible by 7.

q is true, r is false.

$\Rightarrow$ $q \wedge r$ is false.

[since, $p \wedge q$ has the truth value F (false) whenever either p or q or both have the truth value F.]

Hence, p is not valid.

(ii) p : 131 is a multiple of 3 or 11.

Let q : 131 is multiple of 3.

r : 131 is a multiple of 11.

p is true, r is false.

$\Rightarrow$ $p \vee r$ is true.

[since, $p \vee q$ has the truth value T (true) whenever either p or q or both have the truth value T]

Hence, q is valid.

Q.14 Prove the following statement by contradiction method

p : The sum of an irrational number and a rational number is irrational.

Sol. Let p is false *i.e.*, sum of an irrational and a rational number is rational.

Let $\sqrt{m}$ is irrational and n is rational number.

$\Rightarrow$ $\sqrt{m} + n = r$ [rational]

$\Rightarrow$ $\sqrt{m} = r - n$

$\sqrt{m}$ is irrational, where as $(r - n)$ is rational. This is contradiction.

Then, our supposition is wrong.

Hence, p is true.

Q. 15 Prove by direct method that for any real number x, y if $x = y$, then $x^2 = y^2$.

Thinking Process

In direct method assume p is true and show q is true i.e., $p \Rightarrow q$.

Sol. Let $p : x = y$, $x, y \in R$

On squaring both sides,

$$x^2 = y^2 : q \quad \text{[say]}$$

$$p \Rightarrow q$$

Hence, we have the result.

Q. 16 Using contrapositive method prove that, if n^2 is an even integer, then n is also an even integer.

Thinking Process

In contrapositive method assume $\sim q$ is true and show $\sim p$ is true *i.e.,* $\sim q \Rightarrow \sim p$.

Sol. Let $p : n^2$ is an even integer.

$q : n$ is also an even integer.

Let $\sim p$ is true *i.e.,* n is not an even integer.

$\Rightarrow n^2$ is not an even integer. [since, square of an odd integer is odd]

$\Rightarrow \sim p$ is true.

Therefore, $\sim q$ is true $\Rightarrow \sim p$ is true. **Hence proved.**

Objective Type Questions

Q. 17 Which of the following is a statement?

(a) x is a real number

(b) Switch off the fan

(c) 6 is a natural number

(d) Let me go

Sol. *(c)* As we know a statement is a sentence which is either true or false.

So, 6 is a natural number, which is true.

Hence, it is a statement.

Q. 18 Which of the following is not a statement.

(a) Smoking is injurious to health

(b) $2 + 2 = 4$

(c) 2 is the only even prime number

(d) Come here

Sol. *(d)* 'Come here' is not a statement. Since, no sentence can be called a statement, if it is an order.

Q. 19 The connective in the statement '$2 + 7 > 9$ or $2 + 7 < 9$' is

(a) and (b) or

(c) $>$ (d) $<$

Sol. *(b)* In '$2 + 7 > 9$ or $2 + 7 < 9$', or is the connective.

Q. 20 The connective in the statement "Earth revolves round the Sun and Moon is a satellite of earth" is

(a) or (b) Earth

(c) Sun (d) and

Sol. *(d)* Connective word is 'and'.

Q. 21 The negation of the statement "A circle is an ellipse" is

(a) An ellipse is a circle
(b) An ellipse is not a circle
(c) A circle is not an ellipse
(d) A circle is an ellipse

Sol. *(c)* Let p : A circle is an ellipse.
$\sim p$: A circle is not an ellipse.

Q. 22 The negation of the statement "7 is greater than 8" is

(a) 7 is equal to 8
(b) 7 is not greater than 8
(c) 8 is less than 7
(d) None of these

Sol. *(b)* Let p : 7 is greater than 8.
$\sim p$: 7 is not greater than 8.

Q. 23 The negation of the statement "72 is divisible by 2 and 3" is

(a) 72 is not divisible by 2 or 72 is not divisible by 3
(b) 72 is not divisible by 2 and 72 is not divisible by 3
(c) 72 is divisible by 2 and 72 is not divisible by 3
(d) 72 is not divisible by 2 and 72 is divisible by 3

Sol. *(b)* Let p : 72 is divisible by 2 and 3.
Let q : 72 is divisible by 2.
r : 72 is divisible by 3.
$\sim q$: 72 is not divisible by 2.
$\sim r$: 72 is not divisible by 3.
$\sim (q \wedge r) : \sim q \vee \sim r$
$\Rightarrow$ 72 is not divisible by 2 or 72 is not divisible by 3.

Q. 24 The negation of the statement "Plants take in CO_2 and give out O_2" is

(a) Plants do not take in CO_2 and do not given out O_2
(b) Plants do not take in CO_2 or do not give out O_2
(c) Plants take is CO_2 and do not give out O_2
(d) Plants take in CO_2 or do not give out O_2

Sol. *(b)* Let p : Plants take in CO_2 and give out O_2.
Let q : Plants take in CO_2.
r : Plants give out O_2.
$\sim q$: Plants do not take in CO_2.
$\sim r$: Plants do not give out O_2.
$\sim (q \wedge r)$: Plants do not take in CO_2 or do not give out O_2.

Q. 25 The negative of the statement "Rajesh or Rajni lived in Bangaluru" is

(a) Rajesh did not live in Bengaluru or Rajni lives in Bengaluru
(b) Rajesh lives in Bengaluru and Rajni did not live in Bengaluru
(c) Rajesh did not live in Bengaluru and Rajni did not live in Bengaluru
(d) Rajesh did not live in Bengaluru or Rajni did not live in Bengaluru

Sol. ***(c)*** Let p : Rajesh or Rajni lived in Bengaluru.

and q : Rajesh lived in Bengaluru.

r : Rajni lived in Bengaluru.

$\sim q$: Rajesh did not live in Bengaluru.

$\sim r$: Rajni did not live in Bengaluru.

$\sim (q \vee r)$: Rajesh did not live in Bengaluru and Rajni did not live in Bengaluru.

Q. 26 The negation of the statement "101 is not a multiple of 3" is

(a) 101 is a multiple of 3
(b) 101 is a multiple of 2
(c) 101 is an odd number
(d) 101 is an even number

Sol. ***(a)*** Let p : 101 is not a multiple of 3.

$\sim p$: 101 is a multiple of 3.

Q. 27 The contrapositive of the statement
"If 7 is greater than 5, then 8 is greater than 6" is

(a) If 8 is greater than 6, then 7 is greater than 5
(b) If 8 is not greater than 6, then 7 is greater than 5
(c) If 8 is not greater than 6, then 7 is not greater than 5
(d) If 8 is greater than 6, then 7 is not greater than 5

Sol. ***(c)*** Let p : 7 is greater than 5.

and q : 8 is greater than 6.

$\therefore\ p \rightarrow q$

$\sim p$: 7 is not greater than 5.

$\sim q$: 8 is not greater than 6.

$(\sim q) \rightarrow (\sim p)$ *i.e.*, If 8 is not greater than 6, then 7 is not greater than 5.

Q. 28 The converse of the statement "If $x > y$, then $x + a > y + a$" is

(a) If $x < y$, then $x + a < y + a$
(b) If $x + a > y + a$, then $x > y$
(c) If $x < y$, then $x + a < y + a$
(d) If $x > y$, then $x + a < y + a$

Sol. ***(b)*** Let

$$p : x > y$$
$$q : x + a > y + a$$
$$p \rightarrow q$$

Converse of the above statement is

$$q \rightarrow p$$

i.e., If $x + a > y + a$, then $x > y$.

Q. 29 The converse of the statement "If sun is not shining, then sky is filled with clouds" is

(a) If sky is filled with clouds, then the Sun is not shining
(b) If Sun is shining, then sky is filled with clouds
(c) If sky is clear, then Sun is shining
(d) If Sun is not shining, then sky is not filled with clouds

Sol. *(a)* Let p : Sun is not shining.
and q : Sky is filled with clouds.
Converse of the above statement $p \rightarrow q$ is $q \rightarrow p$.
If sky is filled with clouds, then the Sun is not shining.

Q. 30 The contrapositive of the statement "If p, then q", is

(a) if q, then p
(b) if p, then $\sim q$
(c) if $\sim q$. then $\sim p$
(d) if $\sim p$, then $\sim q$

Sol. *(c)* $p \rightarrow q$
If p, then q
Contrapositive of the statement $p \rightarrow q$ is $(\sim q) \rightarrow (\sim p)$.
If $\sim q$, then $\sim p$.

Q. 31 The statement "If x^2 is not even, then x is not even" is converse of the statement

(a) If x^2 is odd, then x is even
(b) If x is not even, then x^2 is not even
(c) If x is even, then x^2 is even
(d) If x is odd, then x^2 is even

Sol. *(b)* Let p : x^2 is not even.
and q : x is not even.
Converse of the statement $p \rightarrow q$ is $q \rightarrow p$.
i.e., If x is not even, then x^2 is not even.

Q. 32 The contrapositive of statement 'If Chandigarh is capital of Punjab, then Chandigarh is in India' is

(a) if Chandigarh is not in India, then Chandigarh is not the capital of Punjab
(b) if Chandigarh is in India, then Chandigarh is Capital of Punjab
(c) if Chandigarh is not capital of Punjab, then Chandigarh is not capital of India
(d) if Chandigarh is capital of Punjab, then Chandigarh is not is India

Sol. *(a)* Let p : Chandigarh is capital of Punjab.
and q : Chandigarh is in India.
$\sim p$: Chandigarh is not capital of Punjab.
$\sim q$: Chandigarh is not in India.
Contrapositive of the statement $p \rightarrow q$ is
if $(\sim q)$, then $(\sim p)$.
If Chandigarh is not in India, then Chandigarh is not the capital of Punjab.

Q. 33 Which of the following is the conditional $p \rightarrow q$?

(a) q is sufficient for p
(b) p is necessary for q
(c) p only if q
(d) if q then p

Sol. ***(c)*** '$p \rightarrow q$' is same as 'p only if q'.

Q. 34 The negation of the statement "The product of 3 and 4 is 9" is

(a) it is false that the product of 3 and 4 is 9
(b) the product of 3 and 4 is 12
(c) the product of 3 and 4 is not 12
(d) it is false that the product of 3 and 4 is not 9

Sol. ***(a)*** The negation of the above statement is 'It is false that the product of 3 and 4 is 9'.

Q. 35 Which of the following is not a negation of "A nature number is greater than zero"

(a) A natural number is not greater than zero
(b) It is false that a natural number is greater than zero
(c) It is false that a natural number is not greater than zero
(d) None of the above

Sol. ***(c)*** The false negation of the given statement is "It is false that a natural number is not greater than zero".

Q. 36 Which of the following statement is a conjunction?

(a) Ram and Shyam are friends
(b) Both Ram and Shyam are tall
(c) Both Ram and Shyam are enemies
(d) None of the above

Sol. ***(d)*** If two simple statements p and q are connected by the word 'and', then the resulting compound statement p and q is called a conjuction of p and q

Here, none of the given statement is conjunction

Q. 37 State whether the following sentences are statements or not

(i) The angles opposite to equal sides of a triangle are equal.
(ii) The moon is a satellites of Earth.
(iii) May God bless you.
(iv) Asia is a continent.
(v) How are you?

Sol. **(i)** It is a statement.
(ii) It is a statement.
(iii) It is not a statement, since it is an exclamations.
(iv) It is a statement.
(v) It is not a statement, since it is a questions.

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Statistics

Short Answer Type Questions

Q. 1 Find the mean deviation about the mean of the distribution.

Size	20	21	22	23	24
Frequency	6	4	5	1	4

Sol.

Size	Frequency	$f_i\ x_i$	$d_i = \lvert x_i - x \rvert$	$f_i\ d_i$
20	6	120	1.65	9.90
21	4	84	0.65	2.60
22	5	110	0.35	1.75
23	1	23	1.35	1.35
24	4	96	2.35	9.40
Total	20	433		25

Now, $$\bar{x} = \frac{\Sigma f_i x_i}{\Sigma f_i} = \frac{433}{20} = 21.65$$

$\therefore$ $$\text{MD} = \frac{\Sigma f_i \lvert x_i - \bar{x} \rvert}{\Sigma f_i} = \frac{25}{20} = 1.25$$

Q. 2 Find the mean deviation about the median of the following distribution.

Marks obtained	10	11	12	14	15
Number of students	2	3	8	3	4

Sol.

Marks obtained	f_i	cf	$d_i = \lvert x_i - M_e \rvert$	$f_i d_i$
10	2	2	2	4
11	3	5	1	3
12	8	13	0	0
14	3	16	2	6
15	4	20	3	12
Total	$\Sigma f_i = 20$			$\Sigma f_i d_i = 25$

Now, $M_e = \left(\frac{20+1}{2}\right)$th item $= \left(\frac{21}{2}\right) = 10.5$th item

$\therefore$ $M_e = 12$

$\therefore$ $\text{MD} = \frac{\sum f_i d_i}{\sum f_i} = \frac{25}{20} = 1.25$

Q. 3 Calculate the mean deviation about the mean of the set of first n natural numbers when n is an odd number.

Sol. Consider first natural number when n is an odd *i.e.*, 1, 2, 3, 4, ...n, [odd].

$$\text{Mean } \bar{x} = \frac{1+2+3+\ldots+n}{n} = \frac{n(n+1)}{2n} = \frac{n+1}{2}$$

$$\therefore \quad \text{MD} = \frac{\left|1-\frac{n+1}{2}\right| + \left|2-\frac{n+1}{2}\right| + \left|3-\frac{n+1}{2}\right| + \ldots + \left|n-\frac{n+1}{2}\right|}{n}$$

$$= \frac{\left|-\frac{n+1}{2}\right| + \left|2-\frac{n+1}{2}\right| + \ldots + \left|\frac{n-1}{2}-\frac{n+1}{2}\right| + \left|\frac{n+1}{2}-\frac{n+1}{2}\right| + \left|\frac{n+3}{2}-\frac{n+1}{2}\right| + \ldots + \left|\frac{2n-2}{2}-\frac{n+1}{2}\right| + \left|n-\frac{n+1}{2}\right|}{n}$$

$$= \frac{2}{n}\left[1+2+\ldots+\frac{n-3}{2}+\frac{n-1}{2}\right] \quad \left(\frac{n-1}{2}\right) \text{terms}$$

$$= \frac{2}{n}\left[\frac{\left(\frac{n-1}{2}\right)\left(\frac{n-1}{2}+1\right)}{2}\right] \qquad \left[\because \text{ sum of first } n \text{ natural numbers} = \frac{n(n+1)}{2}\right]$$

$$= \frac{2}{n}\cdot\frac{1}{2}\left[\left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)\right] = \frac{1}{n}\left(\frac{n^2-1}{4}\right) = \frac{n^2-1}{4n}$$

Q. 4 Calculate the mean deviation about the mean of the set of first n natural numbers when n is an even number.

Sol. Consider first n natural number, when n is even *i.e.*, 1, 2, 3, 4, ...n. [even]

$$\therefore \quad \text{Mean } \bar{x} = \frac{1+2+3+\ldots+n}{n} = \frac{n(n+1)}{2n} = \frac{n+1}{2}$$

$$\text{MD} = \frac{1}{n}\left[\left|1-\frac{n+1}{2}\right| + \left|2-\frac{n+1}{2}\right| + \left|3-\frac{n+1}{2}\right|\right] + \left|\frac{n-2}{2}-\frac{n+1}{2}\right| + \left|\frac{n}{2}-\frac{n+1}{2}\right| + \left|\frac{n+2}{2}-\frac{n+1}{2}\right| + \ldots + \left|n-\frac{n+1}{2}\right|$$

$$= \frac{1}{n}\left[\left|\frac{1-n}{2}\right| + \left|\frac{3-n}{2}\right| + \left|\frac{5-n}{2}\right| + \ldots + \left|\frac{-3}{2}\right| + \left|\frac{1}{2}\right| + \ldots + \left|\frac{n-1}{2}\right|\right]$$

$$= \frac{2}{n}\left[\frac{1}{2}+\frac{3}{2}+\ldots+\frac{n-1}{2}\right]\left(\frac{n}{2}\right) \text{terms}$$

$$= \frac{1}{n}\cdot\left(\frac{n}{2}\right)^2 \qquad [\because \text{ sum of first } n \text{ natural numbers} = n^2]$$

$$= \frac{1}{n}\cdot\frac{n^2}{4} = \frac{n}{4}$$

Q. 5 Find the standard deviation of first n natural numbers.

Sol.

x_i	1	2	3	4	5	...	...	n
x_i^2	1	4	9	16	25	...	...	n^2

Now, $\Sigma x_i = 1 + 2 + 3 + 4 + \dots + n = \dfrac{n(n+1)}{2}$

and $\Sigma x_i^2 = 1^2 + 2^2 + 3^2 + \dots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

$$\therefore \quad \sigma = \sqrt{\frac{\Sigma x_i^2}{N} - \left(\frac{\Sigma x_i}{N}\right)^2}$$

$$= \sqrt{\frac{n(n+1)(2n+1)}{6n} - \frac{n^2(n+1)^2}{4n^2}}$$

$$= \sqrt{\frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4}}$$

$$= \sqrt{\frac{2(2n^2+3n+1) - 3(n^2+2n+1)}{12}}$$

$$= \sqrt{\frac{4n^2+6n+2-3n^2-6n-3}{12}}$$

$$= \sqrt{\frac{n^2-1}{12}}$$

Q. 6 The mean and standard deviation of some data for the time taken to complete a test are calculated with the following results

Number of observation = 25, mean = 18.2 s, standard, deviation = 3.25 s

Further, another set of 15 observations $x_1\ x_2 \dots x_{15}$, also in seconds, is now available and we have $\sum_{i=1}^{15} x_i = 279$ and $\sum_{i=1}^{15} x_i^2 = 5524$. Calculate the standard derivation based on all 40 observations.

Sol. Given, $n_i = 25,\ \bar{x}_i = 18.2,\ \sigma_1 = 3.25,$

$$n_2 = 15,\ \sum_{i=1}^{15} x_i = 279 \text{ and } \sum_{i=1}^{15} x_i^2 = 5524$$

For first set, $\Sigma x_i = 25 \times 18.2 = 455$

$$\therefore \quad \sigma_1^2 = \frac{\Sigma x_i^2}{25} - (18.2)^2$$

$$\Rightarrow \quad (3.25)^2 = \frac{\Sigma x_i^2}{25} - 331.24$$

$$\Rightarrow \quad 10.5625 + 331.24 = \frac{\Sigma x_i^2}{25}$$

$$\Rightarrow \quad \Sigma x_i^2 = 25 \times (10.5625 + 331.24)$$

$$= 25 \times 341.8025$$

$$= 8545.0625$$

For combined SD of the 40 observations $n = 40$, Now $\Sigma x_i^2 = 5524 + 8545.0625 = 14069.0625$

and $\Sigma x_i = 455 + 279 = 734$

$\therefore$ $$\text{SD} = \sqrt{\frac{14069.0625}{40} - \left(\frac{734}{40}\right)^2}$$

$$= \sqrt{351.726 - (18.35)^2}$$

$$= \sqrt{351.726 - 336.7225}$$

$$= \sqrt{15.0035} = 3.87$$

Q. 7 The mean and standard deviation of a set of n_1 observations are $\bar{x}_1$ and s_1, respectively while the mean and standard deviation of another set of n_2 observations are $\bar{x}_2$ and s_2, respectively. Show that the standard deviation of the combined set of $(n_1 + n_2)$ observations is given by

$$\text{SD} = \sqrt{\frac{n_1(s_1)^2 + n_2(s_2)^2}{n_1 + n_2} + \frac{n_1 n_2(\bar{x}_1 - \bar{x}_2)^2}{(n_1 - n_2)^2}}.$$

Sol. Let $x_i,\ i = 1, 2, 3\ \ldots, n_1$ and $y_j,\ j = 1, 2, 3, \ldots, n_2$

$\therefore$ $$\bar{x}_1 = \frac{1}{n_1}\sum_{i=1}^{n_1} x_i \text{ and } \bar{x}_2 = \frac{1}{n_2}\sum_{j=1}^{n_2} y_j$$

$\Rightarrow$ $$\sigma_1^2 = \frac{1}{n_1}\sum_{i=1}^{n_1}(x_i - \bar{x}_1)^2$$

and $$\sigma_2^2 = \frac{1}{n_2}\sum_{j=1}^{n}(y_j - \bar{x}_2)^2$$

Now, mean $\bar{x}$ of the given series is given by

$$\bar{x} = \frac{1}{n_1 + n_2}\left[\sum_{i=1}^{n_1} x_i + \sum_{j=1}^{n_2} y_j\right] = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}$$

The variance σ^2 of the combined series is given by

$$\sigma^2 = \frac{1}{n_1 + n_2}\left[\sum_{i=1}^{n_1}(x_i - \bar{x})^2 + \sum_{j=1}^{n_2}(y_j - \bar{x})^2\right]$$

Now, $$\sum_{i=1}^{n_1}(x_i - \bar{x})^2 = \sum_{i=1}^{n_1}(x_i - \bar{x}_j + \bar{x}_j - \bar{x})^2$$

$$= \sum_{i=1}^{n_1}(x_i - \bar{x}_j)^2 + n_1(\bar{x}_j - \bar{x})^2 + 2(\bar{x}_j - \bar{x})\sum_{i=1}^{n_1}(x_i - \bar{x}_j)^2$$

But $$\sum_{i=1}^{n_1}(x_i - \bar{x}_i) = 0$$

[algebraic sum of the deviation of values of first series from their mean is zero]

Also, $$\sum_{i=1}^{n_1}(x_i - \bar{x})^2 = n_1 s_1^2 + n_1(\bar{x}_1 - \bar{x})^2 = n_1 s_1^2 + n_1 d_1^2$$

Where, $$d_1 = (\bar{x}_1 - \bar{x})$$

Similarly, $\sum_{j=1}^{n_2}(y_j - \bar{x})^2 = \sum_{j=1}^{n_2}(y_j - \bar{x}_i + \bar{x}_i - \bar{x})^2 = n_2 s_2^2 + n_2 d_2^2$

where, $d_2 = \bar{x}_2 - \bar{x}$

Combined SD, $\sigma = \sqrt{\frac{[n_1(s_1^2 + d_1^2) + n_2(s_2^2 + d_2^2)]}{n_1 + n_2}}$

where, $d_1 = \bar{x}_1 - \bar{x} = \bar{x}_1 - \left(\frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}\right) = \frac{n_2(\bar{x}_1 - \bar{x}_2)}{n_1 + n_2}$

and $d_2 = \bar{x}_2 - \bar{x} = \bar{x}_2 - \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2} = \frac{n_1(\bar{x}_2 - \bar{x}_1)}{n_1 + n_2}$

$\therefore$ $\sigma^2 = \frac{1}{n_1 + n_2}\left[n_1 s_1^2 + n_2 s_2^2 + \frac{n_1 n_2 (\bar{x}_1 - \bar{x}_2)^2}{(n_1 + n_2)^2} + \frac{n_2 n_1 (\bar{x}_2 - \bar{x}_1)^2}{(n_1 + n_2)^2}\right]$

Also, $\sigma = \sqrt{\frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2} + \frac{n_1 n_2 (\bar{x}_1 - \bar{x}_2)^2}{(n_1 + n_2)^2}}$

Q. 8 Two sets each of 20 observations, have the same standard deviation 5. The first set has a mean 17 and the second mean 22.
Determine the standard deviation of the x sets obtained by combining the given two sets.

Sol. Given, $n_1 = 20$, $\sigma_1 = 5$, $\bar{x}_1 = 17$ and $n_2 = 20$, $\sigma_2 = 5$, $\bar{x}_2 = 22$

We know that, $\sigma = \sqrt{\frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2} + \frac{n_1 n_2 (\bar{x}_1 - \bar{x}_2)^2}{(n_1 + n_2)^2}}$

$$= \sqrt{\frac{20 \times (5)^2 + 20 \times (5)^2}{20 + 20} + \frac{20 \times 20\,(17 - 22)^2}{(20 + 20)^2}}$$

$$= \sqrt{\frac{1000}{40} + \frac{400 \times 25}{1600}} = \sqrt{25 + \frac{25}{4}} = \sqrt{\frac{125}{4}} = \sqrt{31.25} = 5.59$$

Q.9 The frequency distribution

x	A	$2A$	$3A$	$4A$	$5A$	$6A$
f	2	1	1	1	1	1

where, A is a positive integer, has a variance of 160. Determine the value of A.

Sol.

x	f_i	$f_i\ x_i$	$f_i\ x_i^2$
A	2	$2A$	$2A^2$
$2A$	1	$2A$	$4A^2$
$3A$	1	$3A$	$9A^2$
$4A$	1	$4A$	$16A^2$
$5A$	1	$5A$	$25A^2$
$6A$	1	$6A$	$36A^2$
Total	7	$22A$	$92A^2$
	$n = 7$	$\Sigma f_i n_i = 22A$	$\Sigma f_i n_i^2 = 92A^2$

$$\therefore \quad \sigma^2 = \frac{\Sigma f_i x_i^2}{n} - \left(\frac{\Sigma f_i x_i}{n}\right)^2$$

$$\Rightarrow \quad 160 = \frac{92A^2}{7} - \left(\frac{22A}{7}\right)^2$$

$$\Rightarrow \quad 160 = \frac{92A^2}{7} - \frac{484A^2}{49}$$

$$\Rightarrow \quad 160 = (644 - 484)\frac{A^2}{49}$$

$$\Rightarrow \quad 160 = \frac{160A^2}{49} \Rightarrow A^2 = 49$$

$$\therefore \quad A = 7$$

Q. 10 For the frequency distribution

x	2	3	4	5	6	7
f	4	9	16	14	11	6

Find the standard distribution.

Sol.

x_i	f_i	$d_i = x_i - 4$	$f_i\, d_i$	$f_i\, d_i^2$
2	4	–2	–8	16
3	9	–1	–9	9
4	16	0	0	0
5	14	1	14	14
6	11	2	22	44
7	6	3	18	54
Total	60		$\Sigma f_i d_i = 37$	$\Sigma f_i d_i^2 = 137$

$$\therefore \quad \text{SD} = \sqrt{\frac{\Sigma f_i d_i^2}{N} - \left(\frac{\Sigma f_i d_i}{N}\right)^2}$$

$$= \sqrt{\frac{137}{60} - \left(\frac{37}{60}\right)^2}$$

$$= \sqrt{2.2833 - (0.616)^2}$$

$$= \sqrt{2.2833 - 0.3794}$$

$$= \sqrt{1.9037} = 1.38$$

Q. 11 There are 60 students in a class. The following is the frquency distribution of the marks obtained by the students in a test.

Marks	0	1	2	3	4	5
Frequency	$x - 2$	x	x^2	$(x+1)^2$	$2x$	$x+1$

where, x is positive integer. Determine the mean and standard deviation of the marks

Sol. $\therefore$ Sum of frequencies,

$$x - 2 + x + x^2 + (x+1)^2 + 2x + x + 1 = 60$$

$$\Rightarrow \quad 2x - 2 + x^2 + x^2 + 1 + 2x + 2x + x + 1 = 60$$

$$\Rightarrow \quad 2x^2 + 7x = 60$$

$$\Rightarrow \quad 2x^2 + 7x - 60 = 0$$

$$\Rightarrow \quad 2x^2 + 15x - 8x - 60 = 0$$

$$\Rightarrow \quad x(2x + 15) - 4(2x + 15) = 0$$

$$\Rightarrow \quad (2x + 15)(x - 4) = 0$$

$$\Rightarrow \quad x = -\frac{15}{2}, 4$$

$$\Rightarrow \quad x = -\frac{15}{2} \quad \text{[inaddmisible]} \; [\because x \in I^+]$$

x_i	f_i	$d_i = x_i - 3$	$f_i d_i$	$f_i d_i^2$
0	2	–3	–6	18
1	4	–2	–8	16
2	16	–1	–16	16
$A = 3$	25	0	0	0
4	8	1	8	8
5	5	2	10	20
Total	$\Sigma f_i = 60$		$\Sigma f_i d_i = -12$	$\Sigma f_i d_i^2 = 78$

$$\text{Mean} = A + \frac{\Sigma f_i d_i}{\Sigma f_i} = 3 + \left(\frac{-12}{60}\right) = 2.8$$

$$\sigma = \sqrt{\frac{\Sigma f_i d_i^2}{\Sigma f_i} - \left(\frac{\Sigma f_i d_i}{\Sigma f_i}\right)^2} = \sqrt{\frac{78}{60} - \left(\frac{-12}{60}\right)^2}$$

$$= \sqrt{1.3 - 0.04} = \sqrt{1.26} = 1.12$$

Q. 12 The mean life of a sample of 60 bulbs was 650 h and the standard deviation was 8 h. If a second sample of 80 bulbs has a mean life of 660 h and standard deviation 7 h, then find the over all standard deviation.

Sol. Here, $n_1 = 60$, $\bar{x}_1 = 650$, $s_1 = 8$ and $n_2 = 80$, $\bar{x}_2 = 660$, $s_2 = 7$

$$\therefore \quad \sigma = \sqrt{\frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2} + \frac{n_1 n_2 (\bar{x}_1 - \bar{x}_2)^2}{(n_1 + n_2)^2}}$$

$$= \sqrt{\frac{60 \times (8)^2 + 80 \times (7)^2}{60 + 80} + \frac{60 \times 80(650 - 660)^2}{(60 + 80)^2}}$$

$$= \sqrt{\frac{6 \times 64 + 8 \times 49}{14} + \frac{60 \times 80 \times 100}{140 \times 140}}$$

$$= \sqrt{\frac{192 + 196}{7} + \frac{1200}{49}} = \sqrt{\frac{388}{7} + \frac{1200}{49}}$$

$$= \sqrt{\frac{2716 + 1200}{49}} = \sqrt{\frac{3916}{49}} = \frac{62.58}{7} = 8.9$$

Q. 13 If mean and standard deviation of 100 items are 50 and 4 respectively, then find the sum of all the item and the sum of the squares of item.

Sol. Here, $\bar{x} = 50, n = 100$ and $\sigma = 4$

$$\therefore \quad \frac{\Sigma x_i}{100} = 50$$

$$\Rightarrow \quad \Sigma x_i = 5000$$

and

$$\sigma^2 = \frac{\Sigma f_i x_i^2}{\Sigma f_i} - \left(\frac{\Sigma f_i x_i}{\Sigma f_i}\right)^2$$

$$\Rightarrow \quad (4)^2 = \frac{\Sigma f_i x_i^2}{100} - (50)^2$$

$$\Rightarrow \quad 16 = \frac{\Sigma f_i x_i^2}{100} - 2500$$

$$\Rightarrow \quad \frac{\Sigma f_i x_i^2}{100} = 16 + 2500 = 2516$$

$$\therefore \quad \Sigma f_i x_i^2 = 251600$$

Q. 14 If for distribution $\Sigma(x-5) = 3$, $\Sigma(x-5)^2 = 43$ and total number of item is 18. Find the mean and standard deviation.

Sol. Given, $n = 18$, $\Sigma(x-5) = 3$ and $\Sigma(x-5)^2 = 43$

$$\therefore \quad \text{Mean} = A + \frac{\Sigma(x-5)}{18}$$

$$= 5 + \frac{3}{18} = 5 + 0.1666 = 5.1666 = 5.17$$

and

$$\text{SD} = \sqrt{\frac{\Sigma(x-5)^2}{n} - \left(\frac{\Sigma(x-5)}{n}\right)^2}$$

$$= \sqrt{\frac{43}{18} - \left(\frac{3}{18}\right)^2}$$

$$= \sqrt{2.3944 - (0.166)^2} = \sqrt{2.3944 - 0.2755} = 1.59$$

Q. 15 Find the mean and variance of the frequency distribution given below.

x	$1 \le x \le 3$	$3 \le x \le 5$	$5 \le x \le 7$	$7 \le x \le 10$
f	6	4	5	1

Sol.

x	f_i	x_i	$f_i x_i$	$f_i x_i^2$
1-3	6	2	12	24
3-5	4	4	16	64
5-7	5	6	30	180
7-10	1	8.5	8.5	72.25
Total	$n = 16$		$\Sigma f_i x_i = 66.5$	$\Sigma f_i x_i^2 = 340.25$

$$\therefore \quad \text{Mean} = \frac{\Sigma f_i x_i}{\Sigma f_i} = \frac{66.5}{16} = 4.15$$

and variance $= \sigma^2 = \frac{\Sigma f_i x_i^2}{\Sigma f_i} - \left(\frac{\Sigma f_i x_i}{\Sigma f_i}\right)^2$

$= \frac{340.25}{16} - (4.15)^2$

$= 21.2656 - 17.2225 = 4.043$

Long Answer Type Questions

Q. 16 Calculate the mean deviation about the mean for the following frequency distribution.

Class interval	0-4	4-8	8-12	12-16	16-20
Frequency	4	6	8	5	2

Sol.

Class internal	f_i	x_i	$f_i x_i$	$d_i = \lvert x_i - \bar{x} \rvert$	$f_i d_i$
0-4	4	2	8	7.2	28.8
4-8	6	6	36	3.2	19.2
8-12	8	10	80	0.8	6.4
12-16	5	14	70	4.8	24.0
16-20	2	18	36	8.8	17.6
Total	$\Sigma f_i = 25$		$\Sigma f_i x_i = 230$		$\Sigma f_i d_i = 96$

$\therefore$ Mean $= \frac{\Sigma f_i x_i}{\Sigma f_i} = \frac{230}{25} = 9.2$

and mean deviation $= \frac{\Sigma f_i d_i}{\Sigma f_i} = \frac{96}{25} = 3.84$

Q. 17 Calculate the mean deviation from the median of the following data.

Class interval	0-6	6-12	12-18	18-24	24-30
Frequency	4	5	3	6	2

Sol.

Class interval	f_i	x_i	cf	$d_i = \lvert x_i - m_d \rvert$	$f_i d_i$
0-6	4	3	4	11	44
6-12	5	9	9	5	25
12-18	3	15	12	1	3
18-24	6	21	18	7	42
24-30	2	27	20	13	26
Total	$N = 20$				$\Sigma f_i d_i = 140$

$\because \frac{N}{2} = \frac{20}{2} = 10$

So, the median class is 12-18.

$\therefore$ $$\text{Median} = l + \frac{\frac{N}{2} - cf}{f} \times i$$
$$= 12 + \frac{6}{3}(10 - 9)$$
$$= 12 + 2 = 14$$
$$\text{MD} = \frac{\Sigma f_i d_i}{\Sigma f_i} = \frac{140}{20} = 7$$

Q. 18 Determine the mean and standard deviation for the following distribution.

Marks	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Frequency	1	6	6	8	8	2	2	3	0	2	1	0	0	0	1

Sol.

Marks	f_i	$f_i x_i$	$d_i = x_i - \bar{x}$	$f_i d_i$	$f_i d_i^2$
2	1	2	$2-6=-4$	-4	16
3	6	18	$3-6=-3$	-18	54
4	6	24	$4-6=-2$	-12	24
5	8	40	$5-6=-1$	-8	8
6	8	48	$6-6=0$	0	0
7	2	14	$7-6=1$	2	2
8	2	16	$8-6=2$	4	8
9	3	27	$9-6=3$	9	27
10	0	0	$10-6=4$	0	0
11	2	22	$11-6=5$	10	50
12	1	12	$12-6=6$	6	36
13	0	0	$13-6=7$	0	0
14	0	0	$14-6=8$	0	0
15	0	0	$15-6=9$	0	0
16	1	16	$16-6=10$	10	100
Total	$\Sigma f_i = 40$	$\Sigma f_i x_i = 239$		$\Sigma f_i d_i = -1$	$\Sigma f_i x_i^2 = 325$

$\therefore$ $$\text{Mean } \bar{x} = \frac{\Sigma f_i x_i}{\Sigma f_i} = \frac{239}{40} = 5.975 \approx 6$$

and $$\sigma = \sqrt{\frac{\Sigma f_i d_i^2}{\Sigma f_i} - \left(\frac{\Sigma f_i d_i}{\Sigma f_i}\right)^2} = \sqrt{\frac{325}{40} - \left(\frac{-1}{40}\right)^2}$$
$$= \sqrt{8.125 - 0.000625} = \sqrt{8.124375} = 2.85$$

Q. 19 The weights of coffee in 70 jars is shown in the following table

Weight (in g)	**Frequency**
200-201	13
201-202	27
202-203	18
203-204	10
204-205	1
205-206	1

Determine variance and standard deviation of the above distribution.

Sol.

CI	f_i	x_i	$d_i = x_i - \bar{x}$	$f_i d_i$	$f_i d_i^2$
200-201	13	200.5	−2	−26	52
201-202	27	201.5	−1	−27	27
202-203	18	202.5	0	0	0
203-204	10	203.5	1	10	10
204-205	1	204.5	2	2	4
205-206	1	205.5	3	3	9
	$\Sigma f_i = 70$			$\Sigma f_i d_i = -38$	$\Sigma f_i d_i^2 = 102$

$\therefore$ $$\sigma^2 = \frac{\Sigma f_i d_i^2}{\Sigma f_i} - \left(\frac{\Sigma f_i d_i}{\Sigma f_i}\right)^2 = \frac{102}{70} - \left(\frac{-38}{70}\right)^2$$

$$= 1.4571 - 0.2916 = 1.1655$$

Now, $$\sigma = \sqrt{1.1655} = 1.08 \text{ g}$$

Q. 20 Determine mean and standard deviation of first n terms of an AP whose first term is a and common difference is d.

Sol.

x_i	$x_i - a$	$(x_i - a)^2$
a	0	0
$a + d$	d	d^2
$a + 2d$	$2d$	$4d^2$
.....	.....	$9d^2$
.....	.....	.....
.....	.....	.....
$a + (n-1)d$	$(n-1)d$	$(n-1)^2 d^2$
$\Sigma x_i = \frac{n}{2}[2a + (n-1]$		

$\because$ $$\text{Mean} = \frac{\Sigma x_i}{n} = \frac{1}{n}\left[\frac{n}{2}(2a + (n-1)d\right]$$

$$= a + \frac{(n-1)}{2}d$$

$\therefore \quad \Sigma(x_i - a) = d\,[1 + 2 + 3 + \ldots + (n-1)d]$

$$= d\,\frac{(n-1)n}{2}$$

and $\quad \Sigma(x_i - a)^2 = d^2\,[1^2 + 2^2 + 3^2 + \ldots + (n-1)^2]$

$$= \frac{d^2(n-1)n(2n-1)}{6}$$

$$\sigma = \sqrt{\frac{(x_i - a)^2}{n} - \left(\frac{x_i - a}{n}\right)^2}$$

$$= \sqrt{\frac{d^2(n-1)(n)(2n-1)}{6n} - \left[\frac{d(n-1)\,n}{2n}\right]^2}$$

$$= \sqrt{\frac{d^2(n-1)(2n-1)}{6} - \frac{d^2(n-1)^2}{4}}$$

$$= d\sqrt{\frac{(n-1)(2n-1)}{6} - \frac{(n-1)^2}{4}}$$

$$= d\sqrt{\frac{(n-1)}{2}\left(\frac{2n-1}{3} - \frac{n-1}{2}\right)}$$

$$= d\sqrt{\frac{(n-1)}{2}\left[\frac{4n-2-3n+3}{6}\right]}$$

$$= d\sqrt{\frac{(n-1)(n+1)}{12}} = d\sqrt{\frac{(n^2-1)}{12}}$$

Q. 21 Following are the marks obtained, out of 100, by two students Ravi and Hashina in 10 tests

Ravi	25	50	45	30	70	42	36	48	35	60
Hashina	10	70	50	20	95	55	42	60	48	80

Who is more intelligent and who is more consistent?

Sol. For Ravi,

x_i	$d_i = x_i - 45$	d_i^2
25	−20	400
50	5	25
45	0	0
30	−15	225
70	25	625
42	−3	9
36	−9	81
48	3	9
35	−10	100
60	15	225
Total	$\Sigma d_i = -14$	$\Sigma d^2{}_i = 1699$

$$\sigma = \sqrt{\frac{\Sigma d^2{}_i}{n} - \left(\frac{\Sigma d_i}{n}\right)^2}$$

$$= \sqrt{\frac{1699}{10} - \left(\frac{-14}{10}\right)^2} = \sqrt{169.9 - 0.0196}$$

$$= \sqrt{169.88} = 13.03$$

Now, $\bar{x} = A + \frac{\Sigma d_i}{\Sigma f_i} = 45 - \frac{14}{10} = 43.6$

For Hashina,

x_i	$d_i = x_i - 55$	d_i^2
10	−45	2025
70	25	625
50	−5	25
20	−35	1225
95	40	1600
55	0	0
42	−13	169
60	5	25
48	−7	49
80	25	625
Total	$\Sigma d_i = 0$	$\Sigma d_i^2 = 6368$

∵ Mean $= 55$

∴ $\sigma = \sqrt{\frac{6368}{10}} = \sqrt{636.8} = 25.2$

For Ravi, $CV = \frac{\sigma}{\bar{x}} \times 100 = \frac{13.03}{43.6} \times 100 = 29.88$

For Hashina, $CV = \frac{\sigma}{\bar{x}} \times 100 = \frac{25.2}{55} \times 100 = 45.89$

Hence, Hashina is more consistent and intelligent.

Q. 22 Mean and standard deviation of 100 observations were found to be 40 and 10, respectively. If at the time of calculation two observations were wrongly taken as 30 and 70 in place of 3 and 27 respectively, then find the correct standard deviation.

Sol. Given, $n = 100, \bar{x} = 40, \sigma = 10$ and $\bar{x} = 40$

∴ $\frac{\Sigma x_i}{n} = 40$

⇒ $\frac{\Sigma x_i}{100} = 40$

⇒ $\Sigma x_i = 4000$

Now, Corrected $\Sigma x_i = 4000 - 30 - 70 + 3 + 27$

∴ $= 4030 - 100 = 3930$

Corrected mean $= \frac{2930}{100} = 39.3$

Now, $\sigma^2 = \frac{\Sigma x_i^2}{n} - (40)^2$

$\Rightarrow$ $100 = \frac{\Sigma x_i^2}{100} - 1600$

$\Rightarrow$ $\Sigma x_i^2 = 170000$

Now, Corrected $\Sigma x_i^2 = 170000 - (30)^2 - (70)^2 + 3^2 + (27)^2$

$= 164939$

$\therefore$ Corrected $\sigma = \sqrt{\frac{164939}{100} - (39.3)^2}$

$= \sqrt{1649.39 - 39.3 \times 39.3}$

$= \sqrt{1649.39 - 1544.49}$

$= \sqrt{104.9} = 10.24$

Q. 23 While calculating the mean and variance of 10 readings, a student wrongly used the reading 52 for the correct reading 25. He obtained the mean and variance as 45 and 16, respectively. Find the correct mean and the variance.

Sol. Given, $n = 10$, $\bar{x} = 45$ and $\sigma^2 = 16$

$\therefore$ $\bar{x} = 45 \Rightarrow \frac{\Sigma x_i}{n} = 45$

$\Rightarrow$ $\frac{\Sigma x_i}{10} = 45 \Rightarrow \Sigma x_i = 450$

Corrected $\Sigma x_i = 450 - 52 + 25 = 423$

$\therefore$ $\bar{x} = \frac{423}{10} = 42.3$

$\Rightarrow$ $\sigma^2 = \frac{\Sigma x_i^2}{n} - \left(\frac{\Sigma x_i}{n}\right)^2$

$\Rightarrow$ $16 = \frac{\Sigma x_i^2}{10} - (45)^2$

$\Rightarrow$ $\Sigma x_i^2 = 10\,(2025 + 16)$

$\Rightarrow$ $\Sigma x_i^2 = 20410$

$\therefore$ Corrected $\Sigma x_i^2 = 20410 - (52)^2 + (25)^2 = 18331$

and corrected $\sigma^2 = \frac{18331}{10} - (42.3)^2 = 43.81$

Objective Type Questions

Q. 24 The mean deviation of the data 3, 10, 10, 4, 7, 10, 5 from the mean is

(a) 2 (b) 2.57
(c) 3 (d) 3.75

Sol. *(b)* Given, observations are 3, 10, 10, 4, 7, 10 and 5.

$$\therefore \quad \bar{x} = \frac{3+10+10+4+7+10+5}{7} = \frac{49}{7} = 7$$

x_i	$d_i = \lvert x_i - \bar{x} \rvert$
3	4
10	3
10	3
4	3
7	0
10	3
5	2
Total	$\Sigma d_i = 18$

Now, $$\text{MD} = \frac{\Sigma d_i}{N} = \frac{18}{7} = 2.57$$

Q. 25 Mean deviation for n observations $x_1, x_2, \ldots, x_n$ from their mean $\bar{x}$ is given by

(a) $\sum_{i=1}^{n} (x_i - \bar{x})$ (b) $\frac{1}{n}\sum_{i=1}^{n} \lvert x_i - \bar{x} \rvert$

(c) $\sum_{i=1}^{n} (x_i - \bar{x})^2$ (d) $\frac{1}{n}\sum_{i=1}^{n} (x_i - \bar{x})^2$

Sol. *(b)* $\text{MD} = \frac{1}{n}\sum_{i=1}^{n} \lvert x_i - \bar{x} \rvert$

Q. 26 When tested, the lives (in hours) of 5 bulbs were noted as follows

1357, 1090, 1666, 1494, 1623

The mean deviations (in hours) from their mean is

(a) 178 (b) 179 (c) 220 (d) 356

Sol. *(a)* Since, the lives of 5 bulbs are 1357, 1090, 1666, 1494 and 1623.

$$\therefore \quad \text{Mean} = \frac{1357 + 1090 + 1666 + 1494 + 1623}{5} = \frac{7230}{5} = 1446$$

x_i	$d_i = \lvert x_i - \bar{x} \rvert$
1357	89
1090	356
1666	220
1494	48
1623	177
Total	$\Sigma d_i = 890$

$$MD = \frac{\Sigma d_i}{N} = \frac{890}{5} = 178$$

Q. 27 Following are the marks obtained by 9 students in a mathematics test

50, 69, 20, 33, 53, 39, 40, 65, 59

The mean deviation from the median is

(a) 9 (b) 10.5
(c) 12.67 (d) 14.76

Sol. *(c)* Since, marks obtained by 9 students in Mathematics are 50, 69, 20, 33, 53, 39, 40, 65 and 59.

Rewrite the given data in ascending order.

20,33, 39, 40, 50, 53, 59, 65, 69,

Here, $n = 9$ [odd]

$\therefore$ Median $= \left(\frac{9+1}{2}\right)$ term = 5th term

$Me = 50$

x_i	$d_i = \lvert x_i - Me \rvert$
20	30
33	17
39	11
40	10
50	0
53	3
59	9
65	15
69	19
$N = 2$	$\Sigma d_i = 114$

$\therefore$ $MD = \frac{114}{9} = 12.67$

Q. 28 The standard deviation of data 6, 5, 9, 13, 12, 8 and 10 is

(a) $\sqrt{\frac{52}{7}}$ (b) $\frac{52}{7}$
(c) $\sqrt{6}$ (d) 6

Sol. *(a)* Given, data are 6, 5, 9, 13, 12, 8, and 10.

x_i	x_i^2
6	36
5	25
9	81
13	169
12	144
8	64
10	100
$\Sigma x_i = 63$	$\Sigma x_i^2 = 619$

$$\therefore \quad SD = \sigma = \sqrt{\frac{\Sigma x_i^2}{N} - \left(\frac{\Sigma x_i}{N}\right)^2} = \sqrt{\frac{619}{7} - \left(\frac{63}{7}\right)^2}$$

$$= \sqrt{\frac{7 \times 619 - 3969}{49}}$$

$$= \sqrt{\frac{4333 - 3969}{49}}$$

$$= \sqrt{\frac{364}{49}} = \sqrt{\frac{52}{7}}$$

Q. 29 If $x_1, x_2, \ldots, x_n$ be n observations and $\bar{x}$ be their arithmetic mean. Then, formula for the standard deviation is given by

(a) $\Sigma(x_i - \bar{x})^2$
(b) $\frac{\Sigma(x_i - \bar{x})^2}{n}$
(c) $\sqrt{\frac{\Sigma(x_i - \bar{x})^2}{n}}$
(d) $\sqrt{\frac{\Sigma x_i^2}{n} + \bar{x}^{-2}}$

Sol. *(c)* SD is given by

$$\sigma = \sqrt{\frac{\Sigma(x_i - \bar{x})^2}{n}}$$

Q. 30 If the mean of 100 observations is 50 and their standard deviation is 5, than the sum of all squares of all the observations is

(a) 50000
(b) 250000
(c) 252500
(d) 255000

Sol. *(c)* Given, $\bar{x} = 50, n = 100$ and $\sigma = 5$

$$\Sigma x_i^2 = ?$$

$$\because \quad \bar{x} = \frac{\Sigma x_i}{n}$$

$$\Rightarrow \quad 50 = \frac{\Sigma x_i}{100}$$

$$\therefore \quad \Sigma x_i = 50 \times 100 = 5000$$

Now, $\sigma = \sqrt{\dfrac{\Sigma x_i^2}{n} - \left(\dfrac{\Sigma x_i}{n}\right)^2} \Rightarrow \sigma^2 = \dfrac{\Sigma x_i^2}{n} - (\bar{x})^2$

$\Rightarrow$ $25 = \dfrac{\Sigma x_i^2}{100} - (50)^2 \Rightarrow 25 = \dfrac{\Sigma x_i^2}{100} - 2500$

$\Rightarrow$ $2525 = \dfrac{\Sigma x_i^2}{100}$

$\therefore$ $\Sigma x_i^2 = 252500$

Q. 31 If a, b, c, d and e be the observations with mean m and standard deviation s, then find the standard deviation of the observations $a+k$, $b+k$, $c+k$, $d+k$ and $e+k$ is

(a) s (b) ks (c) $s+k$ (d) $\dfrac{s}{k}$

Sol. ***(a)*** Given observations are a, b, c, d and e.

$$\text{Mean} = m = \frac{a+b+c+d+e}{5}$$

$$\Sigma x_i = a+b+c+d+e = 5m$$

Now, $$\text{mean} = \frac{a+k+b+k+c+k+d+k+e+k}{5}$$

$$= \frac{(a+b+c+d+e)+5k}{5} = m+k$$

$\therefore$ $$\text{SD} = \sqrt{\frac{\Sigma(x_i+k)^2}{n} - (m+k)^2}$$

$$= \sqrt{\frac{\Sigma(x_i^2+k^2+2k\,x_i)}{n} - (m^2+k^2+2mk)}$$

$$= \sqrt{\frac{\Sigma x_i^2}{n} - m^2 + \frac{2k\Sigma x_i}{n} - 2mk}$$

$$= \sqrt{\frac{\Sigma x_i^2}{n} - m^2 + 2km - 2mk} \qquad \left[\because \frac{\Sigma x_i}{n} = m\right]$$

$$= \sqrt{\frac{\Sigma x_i^2}{n} - m^2}$$

$$= s$$

Q. 32 If x_1, x_2, x_3, x_4 and x_5 be the observations with mean m and standard deviation s then, the standard deviation of the observations kx_1, kx_2, kx_3, kx_4 and kx_5 is

(a) $k+s$ (b) $\dfrac{s}{k}$ (c) ks (d) s

Sol. ***(c)*** Here, $$m = \frac{\Sigma x_i}{5},\ s = \sqrt{\frac{\Sigma x_i^2}{5} - \left(\frac{\Sigma x_i}{5}\right)^2}$$

$\therefore$ $$\text{SD} = \sqrt{\frac{k^2\Sigma x_i^2}{5} - \left(\frac{k\Sigma x_i}{5}\right)^2}$$

$$= \sqrt{\frac{k^2\Sigma x_i^2}{5} - k^2\left(\frac{\Sigma x_i}{5}\right)^2} = \sqrt{\left(\frac{\Sigma x_i^2}{5}\right) - \left(\frac{\Sigma x_i}{5}\right)^2} = ks$$

Q. 33 Let $x_1, x_2, \ldots x_n$ be n observations. Let $w_i = lx_i + k$ for $i = 1, 2, \ldots, n$, where l and k are constants. If the mean of x_i's is 48 and their standard deviation is 12, the mean of w_i's is 55 and standard deviation of w_i's is 15, then the value of l and k should be

(a) $l = 1.25, k = -5$ (b) $l = -1.25, k = 5$
(c) $l = 2.5, k = -5$ (d) $l = 2.5, k = 5$

Sol. *(a)* Given, $w_i = x_i + k$, $\bar{x}_i = 48$, $sx_i = 12$, $w_i = 55$ and $sw_i = 15$

Then, $\bar{w}_i = \bar{x}_i + k$

[where, $\bar{w}_i$ is mean w_i's and $\bar{x}_i$ is mean of x_i's]

$\Rightarrow$ $55 = 48 + k$ ...(i)

Now, SD of w_i = SD of x_i

$\Rightarrow$ $15 = 12$

$\Rightarrow$ $l = \dfrac{15}{12}$

$= 1.25$ ...(ii)

From Eqs. (i) and (ii), $k = 55 - 1.25 \times 48$

$= -5$

Q. 34 The standard deviations for first natural numbers is

(a) 5.5 (b) 3.87 (c) 2.97 (d) 2.87

Sol. *(d)* We know that, SD of first n natural number $= \sqrt{\dfrac{n^2 - 1}{12}}$

$\therefore$ SD of first 10 natural numbers $= \sqrt{\dfrac{(10)^2 - 1}{12}}$

$= \sqrt{\dfrac{100 - 1}{12}} = \sqrt{\dfrac{99}{12}} = \sqrt{8.25} = 2.87$

Q. 35 Consider the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10. If 1 is added to each number the variance of the numbers, so obtained is

(a) 6.5 (b) 2.87 (c) 3.87 (d) 8.25

Sol. *(d)* Given numbers are 1, 2, 3, 4, 5, 6, 7, 8, 9 and 10.

If 1 is added to each number, then observations will be 2, 3, 4, 5, 6, 7, 8, 9, 10 and 11.

$\therefore$ $\Sigma x_i = 2 + 3 + 4 + \ldots + 11$

$= \dfrac{10}{2}[2 \times 2 + 9 \times 1] = 5[4 + 9] = 65$

and $\Sigma x_i^2 = 2^2 + 3^2 + 4^2 + 5^2 + \ldots + 11^2$

$= (1^2 + 2^2 + 3^2 + \ldots + 11^2) - (1^2)$

$= \dfrac{11 \times 12 \times 23}{6} - 1$

$= \dfrac{11 \times 12 \times 23 - 6}{6} = 505$

$$\therefore \quad s^2 = \frac{\Sigma x_i^2}{n} - \left(\frac{\Sigma x_i}{n}\right)^2 = \frac{505}{10} - \left(\frac{65}{10}\right)^2$$
$$= 50.5 - (6.5)^2$$
$$= 50.5 - 42.25$$
$$= 8.25$$

Q. 36 Consider the first 10 positive integers. If we multiply each number by –1 and, then add 1 to each number, the variance of the numbers, so obtained is

(a) 8.25 (b) 6.5 (c) 3.87 (d) 2.87

Sol. ***(a)*** Since, the first 10 positive integers are1, 2, 3, 4, 5, 6, 7, 8, 9 and 10.

On multiplying each number by –1, we get

$$-1, -2, -3, -4, -5, -6, -7, -8, -9, -10$$

On adding 1 in each number, we get

$$0, -1, -2, -3, -4, -5, -6, -7, -8, -9$$

$$\therefore \quad \Sigma x_i = 0 - 1 - 2 - 3 - 4 - 5 - 6 - 7 - 8 - 9$$
$$= -\frac{9 \times 10}{2}$$
$$= -45$$

and
$$\Sigma x_i^2 = 0^2 + (-1)^2 + (-2)^2 + \dots + (-9)^2$$
$$= \frac{9 \times 10 \times 19}{6}$$
$$= 285$$

$$\therefore \quad SD = \sqrt{\frac{285}{10} - \left(\frac{-45}{10}\right)^2} = \sqrt{\frac{285}{10} - \frac{2025}{100}}$$
$$= \sqrt{\frac{2850 - 2025}{100}} = \sqrt{8.25}$$

Now, $\quad$ variance $= (SD)^2 = (\sqrt{8.25})^2 = 8.25$

Q. 37 The following information relates to a sample of size 60, $\Sigma x^2 = 18000$, and $\Sigma x = 960$. Then, the variance is

(a) 6.63 (b) 16 (c) 22 (d) 44

Sol. ***(d)***
$$\text{Variance} = \frac{\Sigma x_i^2}{n} - \left(\frac{\Sigma x_i}{n}\right)^2$$
$$= \frac{18000}{60} - \left(\frac{960}{60}\right)^2 = 300 - 256 = 44$$

Q. 38 If the coefficient of variation of two distributions are 50, 60 and their arithmetic means are 30 and 25 respectively, then the difference of their standard deviation is

(a) 0 (b) 1

(c) 1.5 (d) 2.5

Sol. ***(a)*** Here $\quad CV_1 = 50,\ CV_2 = 60,\ \bar{x}_1 = 30$ and $\bar{x}_2 = 25$

$$\therefore \quad CV_1 = \frac{\sigma_1}{\bar{x}_1} \times 100 \Rightarrow 50 = \frac{\sigma_1}{30} \times 100$$

$$\therefore \quad \sigma_1 = \frac{30 \times 50}{100} = 15 \text{ and } CV_2 = \frac{\sigma_2}{\bar{x}_2} \times 100$$

$$\Rightarrow \quad 60 = \frac{\sigma_2}{25} \times 100$$

$$\therefore \quad \sigma_2 = \frac{60 \times 25}{100} = 15$$

Now, $\sigma_1 - \sigma_2 = 15 - 15 = 0$

Q. 39 The standard deviation of some temperature data in °C is 5. If the data were converted into °F, then the variance would be

(a) 81 (b) 57 (c) 36 (d) 25

Sol. ***(a)*** Given, $\sigma_C = 5 \Rightarrow \frac{5}{9}(F - 32) = C$

$$F = \frac{9C}{5} + 32$$

$$\sigma_F = \frac{9}{5}\sigma_C = \frac{9}{5} \times 5 = 9$$

Here, $\sigma_F^{\,2} = (9)^2 = 81$

Fillers

Q. 40 Coefficient of variation $= \frac{\dots}{\text{Mean}} \times 100$

Sol. $CV = \frac{SD}{\text{Mean}} \times 100$

Q. 41 If $\bar{x}$ is the mean of n values of x, then $\sum_{i=1}^{n}(x_1 - \bar{x})$ is always equal to .... If a has any value other than $\bar{x}$, then $\sum_{i=1}^{n}(x_i - \bar{x})^2$ is .... than $\sum(x_i - a)^2$

Sol. If $\bar{x}$ is the mean of n values of x, then $\sum_{i=1}^{n}(x_i - \bar{x}) = 0$ and if a has any value other than $\bar{x}$, then $\sum_{i=1}^{n}(x_i - \bar{x})^2$ is less than $\sum(x_i - a)^2$.

Q. 42 If the variance of a data is 121, then the standard deviation of the data is ....

Sol. If the variance of a data is 121.

Then, $$SD = \sqrt{\text{Variance}}$$
$$= \sqrt{121} = 11$$

Q. 43 The standard deviation of a data is ...... of any change in origin but is .... of change of scale.

Sol. The standard deviation of a data is independent of any change in origin but is dependent of change of scale.

Q. 44 The sum of squares of the deviations of the values of the variable is .... when taken about their arithmetic mean.

Sol. The sum of the squares of the deviations of the values of the variable is minimum when taken about their arithmetic mean.

Q. 45 The mean deviation of the data is .... when measured from the median.

Sol. The mean deviation of the data is least when measured from the median.

Q. 46 The standard deviation is .... to the mean deviation taken from the arithmetic mean.

Sol. The SD is greater than or equal to the mean deviation taken from the arithmetic mean.

16

Probability

Short Answer Type Questions

Q. 1 If the letters of the word 'ALGORITHM' are arranged at random in a row what is the probability the letters 'GOR' must remain together as a unit?

Sol. Number of letters in the word 'ALGORITHM' = 9

If 'GOR' remain together, then considered it as 1 group.

$\therefore$ Number of letters = 6 + 1 = 7

Number of word, if 'GOR' remain together = 7!

Total number of words from the letters of the word 'ALGORITHM' = 9!

$\therefore$ Required probability $= \dfrac{7!}{9!} = \dfrac{1}{72}$

Q. 2 Six new employees, two of whom are married to each other, are to be assigned six desks that are lined up in a row. If the assignment of employees to desks is made randomly, what is the probability that the married couple will have non-adjacent desks?

Sol. Let the couple occupied adjacent desks consider those two as 1.

There are (4 + 1) *i.e.*, 5 persons to be assigned.

$\therefore$ Number of ways of assigning these five person = $5! \times 2!$

Total number of ways of assigning 6 persons = 6!

$\therefore$ Probability that the couple has adjacent desk $= \dfrac{5! \times 2!}{6!} = \dfrac{2}{6} = \dfrac{1}{3}$

Probability that the married couple will have non-adjacent desks $= 1 - \dfrac{1}{3} = \dfrac{2}{3}$

Q. 3 If an integer from 1 through 1000 is chosen at random, then find the probability that the integer is a multiple of 2 or a multiple of 9.

Sol. Multiple of 2 from 1 to 1000 are 2, 4, 6, 8, ..., 1000

Let n be the number of terms of above series.

$\therefore \quad n\text{th term} = 1000$

$\Rightarrow \quad 2 + (n-1)2 = 1000$

$\Rightarrow \quad 2 + 2n - 2 = 1000$

$\Rightarrow \quad 2n = 1000$

$\therefore \quad n = 500$

Since, the number of multiple of 2 are 500.

So, the multiple of 9 are 9, 18, 27, ..., 999

Let m be the number of term in above series.

$\therefore \quad m\text{th term} = 999$

$\Rightarrow \quad 9 + (m-1)9 = 999$

$\Rightarrow \quad 9 + 9m - 9 = 999$

$\Rightarrow \quad 9m = 999$

$\therefore \quad m = 111$

Since, the number of multiple of 9 are 111. So, the multiple of 2 and 9 both are 18, 36, ..., 990

Let p be the number of terms in above series.

$\therefore \quad p\text{th term} = 990$

$\Rightarrow \quad 18 + (p-1)18 = 990$

$\Rightarrow \quad 18 + 18p - 18 = 990$

$\Rightarrow \quad 18p = 990$

$\therefore \quad p = \frac{990}{18} = 55$

Since, the number of multiple of 2 and 9 are 55.

$\therefore$ Number of multiple of 2 or 9 = 500 + 111 − 55 = 556

$\therefore$ Required probability $= \frac{n(E)}{n(S)} = \frac{556}{1000} = 0.556$

Q. 4 An experiment consists of rolling a die until a 2 appears.

(i) How many elements of the sample space correspond to the event that the 2 appears on the kth roll of the die?

(ii) How many elements of the sample space correspond to the event that the 2 appears not later than the kth roll of the die?

Sol. In a through of a die there is 6 sample points.

(i) If 2 appears on the kth roll of the die.

So, first $(k-1)$ roll have 5 outcomes each and kth roll results 2 *i.e.,* 1 outcome.

$\therefore$ Number of element of sample space correspond to the event that 2 appears on the kth roll of the die $= 5^{k-1}$

(ii) If we consider that 2 appears not later than kth roll of the die, then it is possible that 2 comes in first throw *i.e.,* 1 outcome.

If 2 does not appear in first throw, then outcomes will be 5 and 2 comes in second throw *i.e.,* 1 outcome, possible outcome $= 5 \times 1 = 5$

Similarly, if 2 does not appear in second throw and appears in third throw.

$\therefore$ Possible outcomes $= 5 \times 5 \times 1$

Given, series $= 1 + 5 + 5 \times 5 + 5 \times 5 \times 5 + ... + 5^{k-1}$

$$= 1 + 5 + 5^2 + 5^3 + ... + 5^{k-1}$$

$$= \frac{1(5^k - 1)}{5 - 1} = \frac{5^k - 1}{4}$$

Q. 5 **A die is loaded in such a way that each odd number is twice as likely to occur as each even number. Find $P(G)$, where G is the event that a number greater than 3 occurs on a single roll of the die.**

Sol. It is given that, 2 × Probability of even number = Probability of odd number

$\Rightarrow$ $P(O) = 2P(E)$

$\Rightarrow$ $P(O) : P(E) = 2 : 1$

$\therefore$ Probability of occurring odd number, $P(O) = \frac{2}{2+1} = \frac{2}{3}$

and probability of occurring 5each number,

$$P(E) = \frac{1}{2+1} = \frac{1}{3}$$

Now, G be the even that a number greater than 3 occur in a single roll of die.

So, the possible outcomes are 4, 5 and 6 out of which two are even and one odd.

$\therefore$ Required probability $= P(G) = 2 \times P(E) \times P(O)$

$$= 2 \times \frac{1}{3} \times \frac{2}{3} = \frac{4}{9}$$

Q. 6 **In a large metropolitan area, the probabilities are 0.87, 0.36, 0.30 that a family (randomly chosen for a sample survey) owns a colour television set, a black and white television set or both kinds of sets. What is the probability that a family owns either anyone or both kinds of sets?**

Sol. Let E_1 be the event that family own colour television set and E_2 be the event that family owns a black and white television set.

It is given that, $P(E_1) = 0.87$

$P(E_2) = 0.36$

and $P(E_1 \cap E_2) = 0.30$

We have to find probability that a family owns either anyone or both kind of sets *i.e.,* $P(E_1 \cup E_2)$.

Now, $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$ [by addition theorem]

$$= 0.87 + 0.36 - 0.30$$

$$= 0.93$$

Q. 7 **If A and B are mutually exclusive events, $P(A) = 0.35$ and $P(B) = 0.45$, then find**

(i) $P(A')$ (ii) $P(B')$

(iii) $P(A \cup B)$ (iv) $P(A \cap B)$

(v) $P(A \cap B')$ (vi) $P(A' \cap B')$

Sol. Since, it is given that, A and B are mutually exclusive events.

$\therefore \quad P(A \cap B) = 0 \quad [\because A \cap B = \phi]$

and $\quad P(A) = 0.35,\ P(B) = 0.45$

(i) $P(A') = 1 - P(A) = 1 - 0.35 = 0.65$

(ii) $P(B') = 1 - P(B) = 1 - 0.45 = 0.55$

(iii) $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.35 + 0.45 - 0 = 0.80$

(iv) $P(A \cap B) = 0$

(v) $P(A \cap B') = P(A) - P(A \cap B) = 0.35 - 0 = 0.35$

(vi) $P(A' \cap B') = P(A \cup B)' = 1 - P(A \cup B) = 1 - 0.8 = 0.2$

Q. 8 A team of medical students doing their internship have to assist during surgeries at a city hospital. The probabilities of surgeries rated as very complex, complex, routine, simple or very simple are respectively, 0.15, 0.20, 0.31, 0.26 and 0.08. Find the probabilities that a particular surgery will be rated

(i) complex or very complex.

(ii) neither very complex nor very simple.

(iii) routine or complex.

(iv) routine or simple.

Sol. Let E_1, E_2, E_3, E_4 and E_5 be the event that surgeries are rated as very complex, complex, routine, simple or very simple, respectively.

$\therefore \quad P(E_1) = 0.15,\ P(E_2) = 0.20,\ P(E_3) = 0.31,\ P(E_4) = 0.26,\ P(E_5) = 0.08$

(i) P (complex or very complex) $= P(E_1 \text{ or } E_2)$

$= P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$

$= 0.15 + 0.20 - 0$ [$P(E_1 \cap E_2) = 0$ because all events are independent]

$= 0.35$

(ii) P (neither very complex nor very simple), $(P(E_1' \cap E_5') = P(E_1 \cup E_5)'$

$= 1 - P(E_1 \cup E_5)$

$= 1 - [P(E_1) + P(E_5)]$

$= 1 - (0.15 + 0.08)$

$= 1 - 0.23$

$= 0.77$

(iii) P (routine or complex) $= P(E_3 \cup E_2) = P(E_3) + P(E_2)$

$= 0.31 + 0.20 = 0.51$

(iv) P (routine or simple) $= P(E_3 \cup E_4) = P(E_3) + P(E_4)$

$= 0.31 + 0.26 = 0.57$

Q. 9 Four candidates A, B, C and D have applied for the assignment to coach a school cricket team. If A is twice as likely to be selected as B and B and C are given about the same chance of being selected, while C is twice as likely to be selected as D, then what are the probabilities that

(i) C will be selected?

(ii) A will not be selected?

Sol. It is given that A is twice as likely to be selected as D.

$$P(A) = 2P(B)$$

$$\Rightarrow \quad \frac{P(A)}{2} = P(B)$$

while C is twice as likely to be selected as D.

$$P(C) = 2P(D) \Rightarrow P(B) = 2P(D)$$

$$\Rightarrow \quad \frac{P(A)}{2} = 2P(D) \Rightarrow P(D) = \frac{P(A)}{4}$$

B and C are given about the same chance of being selected.

$$P(B) = P(C)$$

Now, sum of probability = 1

$$P(A) + P(B) + P(C) + P(D) = 1$$

$$P(A) + \frac{P(A)}{2} + P\frac{(A)}{2} + \frac{P(A)}{4} = 1$$

$$\Rightarrow \quad \frac{4P(A) + 2P(A) + 2P(A) + P(A)}{4} = 1$$

$$\Rightarrow \quad 9P(A) = 4 \Rightarrow P(A) = \frac{4}{9}$$

(i) $P(C \text{ will be selected}) = P(C) = P(B) = \frac{P(A)}{2}$

$$= \frac{4}{9 \times 2} \qquad \left[\because P(A) = \frac{9}{4}\right]$$

$$= \frac{2}{9}$$

(ii) $P(A \text{ will not be selected}) = P(A') = 1 - P(A) = 1 - \frac{4}{9} = \frac{5}{9}$

Q. 10 One of the four persons John, Rita, Aslam or Gurpreet will be promoted next month. Consequently the sample space consists of four elementary outcomes S = {John promoted, Rita promoted, Aslam promoted, Gurpreet promoted}. You are told that the chances of John's promotion is same as that of Gurpreet Rita's chances of promotion are twice as likely as Johns. Aslam's chances are four times that of John.

(i) Determine

P (John promoted), P (Rita promoted),

P (Aslam promoted), P (Gurpreet promoted).

(ii) If A = {John promoted or Gurpreet promoted}, find $P(A)$

Sol. Let E_1 = John promoted

E_2 = Rita promoted

E_3 = Aslam promoted

E_4 = Gurpreet promoted

Given, sample space, S = {John promoted, Rita promoted, Aslam promoted, Gurpreet promoted}

i.e., $$S = \{E_1, E_2, E_3, E_4\}$$

It is given that, chances of John's promotion is same as that of Gurpreet.

$$P(E_1) = P(E_4)$$

Rita's chances of promotion are twice as likely as John.

$$P(E_2) = 2P(E_1)$$

And Aslam's chances of promotion are four times that of John.

$$P(E_3) = 4P(E_1)$$

Now, $P(E_1) + P(E_2) + P(E_3) + P(E_4) = 1$

$\Rightarrow$ $P(E_1) + 2P(E_1) + 4P(E_1) + P(E_1) = 1$

$\Rightarrow$ $8P(E_1) = 1$

$\therefore$ $P(E_1) = \frac{1}{8}$

(i) P (John promoted) $= P(E_1) = \frac{1}{8}$

P (Rita promoted) $= P(E_2) = 2P(E_1) = 2 \times \frac{1}{8} = \frac{2}{8} = \frac{1}{4}$

P (Aslam promoted) $= P(E_3) = 4P(E_1) = 4 \times \frac{1}{8} = \frac{1}{2}$

P (Gurpreet promoted) $= P(E_4) = P(E_1) = \frac{1}{8}$

(ii) A = John promoted or Gurpreet promoted

$$A = E_1 \cup E_4$$

$$P(A) = P(E_1 \cup E_4) = P(E_1) + P(E_4) - P(E_1 \cap E_4)$$

$$= \frac{1}{8} + \frac{1}{8} - 0 \qquad [\because P(E_1 \cap E_4) = 0]$$

$$= \frac{2}{8} = \frac{1}{4}$$

Q. 11 The accompanying Venn diagram shows three events, A, B and C and also the probabilities of the various intersections [for instance, $P(A \cap B) = 0.7$]. Determine

(i) $P(A)$ (ii) $P(B \cap \bar{C})$

(iii) $P(A \cup B)$ (iv) $P(A \cap \bar{B})$

(v) $P(B \cap C)$ (vi) Probability of exactly one of the three occurs.

Sol. From the above Venn diagram,

(i) $P(A) = 0.13 + 0.07 = 0.20$

(ii) $P(B \cap \bar{C}) = P(B) - P(B \cap C) = 0.07 + 0.10 + 0.15 - 0.15 = 0.07 + 0.10 = 0.17$

(iii) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

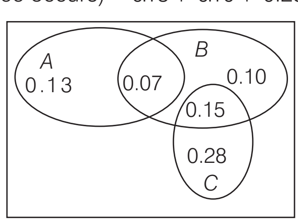
$= 0.13 + 0.07 + 0.07 + 0.10 + 0.15 - 0.07$

$= 0.13 + 0.07 + 0.10 + 0.15 = 0.45$

(iv) $P(A \cap \bar{B}) = P(A) - P(A \cap B) = 0.13 + 0.07 - 0.07 = 0.13$

(v) $P(B \cap C) = 0.15$

(vi) P (exactly one of the three occurs) $= 0.13 + 0.10 + 0.28 = 0.51$



Long Answer Type Questions

Q. 12 One urn contains two black balls (labelled B_1 and B_2) and one white ball. A second urn contains one black ball and two white balls (labelled W_1 and W_2). Suppose the following experiment is performed. One of the two urns is chosen at random. Next a ball is randomly chosen from the urn. Then, a second ball is chosen at random from the same urn without replacing the first ball.

(i) Write the sample space showing all possible outcomes.

(ii) What is the probability that two black balls are chosen?

(iii) What is the probability that two balls of opposite colour are chosen?

Sol. It is given that one of the two urn is chosen, then a ball is randomly chosen from the urn, then a second ball is chosen at random from the same urn without replacing the first ball.

(i) $\therefore$ Sample space $S = \{B_1B_2, B_1W, B_2B_1, B_2W, WB_1, WB_2, BW_1, BW_2, W_1B, W_1W_2, W_2B, W_2W_1\}$

$\therefore$ total sample point = 12

(ii) If two black ball are chosen.

So, the favourable events are $B_1\,B_2$, $B_2\,B_1$ *i.e.,* 2

$\therefore$ Required probability $= \frac{2}{12} = \frac{1}{6}$

(iii) If two balls of opposite colour are chosen.

So, the favourable events are $B_1\,W_1$, $B_2\,W_1$, WB_1, WB_2, BW_1, BW_2, W_1B_1, W_2B *i.e.,* 8.

$\therefore$ Required probability $= \frac{8}{12} = \frac{2}{3}$

Q.13 A bag contains 8 red and 5 white balls. Three balls are drawn at random. Find the probability that

(i) all the three balls are white.

(ii) all the three balls are red.

(iii) one ball is red and two balls are white.

Sol. $\because$ Number of red balls = 8
and number of white balls = 5

(i) P (all the three balls are white) $= \frac{^5C_3}{^{13}C_3} = \frac{\frac{5!}{3!2!}}{\frac{13!}{3!10!}} = \frac{5!}{3!2!} \times \frac{3!10!}{13!}$

$$= \frac{5 \times 4 \times 3 \times 2!}{2!} \times \frac{10!}{13 \times 12 \times 11 \times 10!} = \frac{5 \times 4 \times 3}{13 \times 12 \times 11}$$

$$= \frac{5}{13 \times 11} = \frac{5}{143}$$

$$= \frac{5 \times 4 \times 3}{13 \times 12 \times 11} = \frac{5}{13 \times 11} = \frac{5}{143}$$

(ii) P (all the three balls are red)

$$= \frac{{}^8C_3}{{}^{13}C_3} = \frac{\frac{8!}{3!5!}}{\frac{13!}{3!10!}} = \frac{8!}{3! \times 5!} \times \frac{3!10!}{13!}$$

$$= \frac{8 \times 7 \times 6 \times 5!}{5!} \times \frac{10!}{13 \times 12 \times 11 \times 10!}$$

$$= \frac{8 \times 7 \times 6}{13 \times 12 \times 11} = \frac{28}{143}$$

(iii) P (one ball is red and two balls are white)

$$= \frac{{}^8C_1 \times {}^5C_2}{{}^{13}C_3} = \frac{8 \times 10}{13 \times 6 \times 11} = \frac{40}{143}$$

Q. 14 If the letters of the word 'ASSASSINATION' are arranged at random. Find the probability that

(i) four S's come consecutively in the word.

(ii) two I's and two N's come together.

(iii) all A's are not coming together.

(iv) no two A's are coming together.

Sol. Total number of letters in the word 'ASSASSINATION' are 13.
Out of which 3A's, 4S's, 2 I's, 2 N's, 1 T's and 10.

(i) If four S's come consecutively in the word, then we considers these 4 S's as 1 group. Now, the number of laters is 10.

S	S	S	S	A	A	A	I	I	N	N	T	O
1				9								

Number of words when all S's are together $= \frac{10!}{3!2!2!}$

Total number of word using letters of the word 'ASSASSINATION'

$$= \frac{13!}{3!4!2!2!}$$

$\therefore$ Required probability $= \dfrac{\frac{10!}{3!2!2! \times 13!}}{3!4!2!2!}$

$$= \frac{10! \times 4!}{13!} = \frac{4!}{13 \times 12 \times 11} = \frac{24}{1716} = \frac{2}{143}$$

(ii) If 2 I's and 2 N's come together, then there as 10 alphabets.

Number of word when 2 I's and 2 N's are come together

$$= \frac{10!}{3!4!} \times \frac{4!}{2!2!}$$

$\therefore$ Required probability $= \dfrac{\frac{10!4!}{3!4!2!2!}}{\frac{13!}{3!4!2!2!}} = \dfrac{4!10!}{2!2!3!4!} \times \dfrac{3!4!2!2!}{13!}$

$$= \frac{4!10!}{13!} = \frac{4!}{13 \times 12 \times 11} = \frac{24}{13 \times 12 \times 11} = \frac{2}{143}$$

(iii) If all A's are coming together, then there are 11 alphabets.
Number of words when all A's come together

$$= \frac{11!}{4!2!2!}$$

Probability when all A's come together

$$= \frac{\frac{11!}{4!2!2!}}{\frac{13!}{4!3!2!2!}} = \frac{11!}{4!2!2!} \times \frac{4!3!2!2!}{13!} = \frac{11! \times 3!}{13!} = \frac{6}{13 \times 12} = \frac{1}{26}$$

Required probability when all A's does not come together

$$= 1 - \frac{1}{26} = \frac{25}{26}$$

(iv) If no two A's are together, then first we arrange the alphabets except A's.

	S		S		S		S		I		N		T		I		O		N	

All the alphabets except A's are arranged in $\frac{10!}{4!2!2!}$.

There are 11 vacant places between these alphabets.

So, 3 A's can be place in 11 places in ${}^{11}C_3$ ways $= \frac{11!}{3!8!}$

$\therefore$ Total number of words when no two A's together

$$= \frac{11!}{3!8!} \times \frac{10!}{4!2!2!}$$

$$\text{Required probability} = \frac{11! \times 10!}{3!8!4!2!2!} \times \frac{4!3!2!2!}{13!} = \frac{10!}{8! \times 13 \times 12}$$

$$= \frac{10 \times 9}{13 \times 12} = \frac{90}{156} = \frac{15}{26}$$

Q. 15 If a card is drawn from a deck of 52 cards, then find the probability of getting a king or a heart or a red card.

Sol. $\because$ Number of possible event = 52

and favourable events = 4 king + 13 heart + 26 red – 13 – 2 = 28

$\therefore$ $\text{Required probability} = \frac{28}{52} = \frac{7}{13}$

Q.16 A sample space consists of 9 elementary outcomes $E_1, E_2, \ldots, E_9$ whose probabilities are

$$P(E_1) = P(E_2) = 0.08,\ P(E_3) = P(E_4) = P(E_5) = 0.1$$
$$P(E_6) = P(E_7) = 0.2,\ P(E_8) = P(E_9) = 0.07$$

Suppose $A = \{E_1, E_5, E_8\}, B = \{E_2, E_5, E_8, E_9\}$

(i) Calculate $P(A)$, $P(B)$ and $P(A \cap B)$.

(ii) Using the addition law of probability, calculate $P(A \cup B)$.

(iii) List the composition of the event $A \cup B$ and calculate $P(A \cup B)$ by adding the probabilities of the elementary outcomes.

(iv) Calculate $P(\bar{B})$ from $P(B)$, also calculate $P(\bar{B})$ directly from the elementary outcomes of $\bar{B}$.

Sol. Given, $S = \{E_1, E_2, E_3, E_4, E_5, E_6, E_7, E_8, E_9\}$

$A = \{E_1, E_5, E_8\}$, $B = \{E_2, E_5, E_8, E_9\}$

$P(E_1) = P(E_2) = 0.08$

$P(E_3) = P(E_4) = P(E_5) = 0.1$

$P(E_6) = P(E_7) = 2$, $P(E_8) = P(E_9) = 0.07$

(i) $P(A) = P(E_1) + P(E_5) + P(E_8)$

$= 0.08 + 0.1 + 0.07 = 0.25$

(ii) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ ... (i)

Now, $P(B) = P(E_2) + P(E_5) + P(E_8) + P(E_9)$

$= 0.08 + 0.1 + 0.07 + 0.07 = 0.32$

$A \cap B = \{E_5, E_8\}$

$P(A \cap B) = P(E_5) + P(E_8) = 0.1 + 0.7 = 0.17$

On substituting these values in Eq.(i), we get

$P(A \cup B) = 0.25 + 0.32 - 0.17 = 0.40$

(iii) $A \cup B = \{E_1, E_2, E_5, E_8, E_9\}$

$P(A \cup B) = P(E_1) + P(E_2) + P(E_5) + P(E_8) + P(E_9)$

$= 0.08 + 0.08 + 0.1 + 0.07 + 0.07 = 0.40$

(iv) $\because P(\overline{B}) = 1 - P(B) = 1 - 0.32 = 0.68$

and $\overline{B} = \{E_1, E_3, E_4, E_6, E_7\}$

$\therefore$ $P(\overline{B}) = P(E_1) + P(E_3) + P(E_4) + P(E_6) + P(E_7)$

$= 0.08 + 0.1 + 0.1 + 0.2 + 0.2 = 0.68$

Q. 17 Determine the probability p, for each of the following events.

(i) An odd number appears in a single toss of a fair die.

(ii) Atleast one head appears in two tosses of a fair coin.

(iii) A king, 9 of hearts or 3 of spades appears in drawing a single card from a well shuffled ordinary deck of 52 cards.

(iv) The sum of 6 appears in a single toss of a pair of fair dice.

Sol. **(i)** When a die is throw the possible outcomes are

$S = \{1, 2, 3, 4, 5, 6\}$ out of which 1, 3, 5 are odd,

$\therefore$ Required probability $= \frac{3}{6} = \frac{1}{2}$

(ii) When a fair coin is tossed two times the sample space is

$S = \{HH, HT, TH, TT\}$

In at least one head favourable enonts are HH, HT, TH

$\therefore$ Required probability $= \frac{3}{4}$

(iii) Total cards = 52

Favourable = 4 king + 2 of heart + 3 of spade = 4 + 1 + 1 = 6

$\therefore$ Required probability $= \frac{6}{52} = \frac{3}{26}$

(iv) When a pair of dice is rolled total sample parts are 36. Out of which (1, 5), (5, 1), (2, 4), (4, 2) and (3, 3).

$\therefore$ Required probability $= \frac{5}{36}$

Objective Type Questions

Q.18 In a non-leap year, the probability of having 53 Tuesday or 53 Wednesday is

(a) $\frac{1}{7}$ (b) $\frac{2}{7}$ (c) $\frac{3}{7}$ (d) None of these

Sol. *(a)* In a non-leap year' there are 365 days which have 52 weeks and 1 day. If this day is a Tuesday or Wednesday, then the year will have 53 Tuesday or 53 Wednesday.

$\therefore$ Required probability $= \frac{1}{7}$

Q. 19 Three numbers are chosen from 1 to 20. Find the probability that they are not consecutive

(a) $\frac{186}{190}$ (b) $\frac{187}{190}$ (c) $\frac{188}{190}$ (d) $\frac{18}{^{20}C_3}$

Sol. *(b)* Since, the set of three consecutive numbers from 1 to 20 are 123, 234, 345, ..., 18, 19, 20 *i.e.,* 18.

$$P \text{ (numbers are consecutive)} = \frac{18}{^{20}C_3} = \frac{18}{1140} = \frac{3}{190}$$

$$P \text{ (three numbers are not consecutive)} = 1 - \frac{3}{190} = \frac{187}{190}$$

Q. 20 While shuffling a pack of 52 playing cards, 2 are accidentally dropped. Find the probability that the missing cards to be of different colours.

(a) $\frac{29}{52}$ (b) $\frac{1}{2}$ (c) $\frac{26}{51}$ (d) $\frac{27}{51}$

Sol. *(c)* Since,in a back of 52 cards 26 are red colour and 26 are black colour.

$$\therefore P \text{ (both cards of opposite colour)} = \frac{26}{52} \times \frac{26}{51} + \frac{26}{52} \times \frac{26}{51}$$

$$= 2 \times \frac{26}{52} \times \frac{26}{51} = \frac{26}{51}$$

Q. 21 If seven persons are to be seated in a row. Then, the probability that two particular persons sit next to each other is

(a) $\frac{1}{3}$ (b) $\frac{1}{6}$ (c) $\frac{2}{7}$ (d) $\frac{1}{2}$

Sol. *(c)* If two persons sit next to each other, then consider these two person as 1 group.

Now, we have to arrange 6 persons.

$\therefore$ Number of arrangement $= 2! \times 6!$

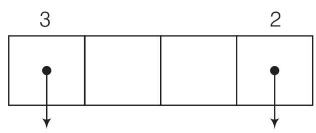
Total number of arrangement of 7 persons $= 7!$

$$\text{Required probability} = \frac{2!6!}{7!} = \frac{2}{7}$$

Q. 22 If without repetition of the numbers, four-digit numbers are formed with the numbers 0, 2, 3 and 5, then the probability of such a number divisible by 5 is

(a) $\frac{1}{5}$ (b) $\frac{4}{5}$ (c) $\frac{1}{30}$ (d) $\frac{5}{9}$

Sol. ***(d)*** We have, to form four-digit number using the digit 0, 2, 3 and 5 which are divisible by 5.



If 0 is fixed at units place $= 3 \times 2 \times 1 = 6$

If 5 is fixed at units place $= 2 \times 2 \times 1 = 4$

Total four-digit numbers divisible by 5 $= 6 + 4 = 10$

$\therefore$ Required probability $= \frac{10}{18} = \frac{5}{9}$

Q. 23 If A and B are mutually exclusive events, then

(a) $P(A) \leq P(\overline{B})$ (b) $P(A) \geq P(\overline{B})$

(c) $P(A) < P(\overline{B})$ (d) None of these

Sol. ***(a)*** For mutually exclusive events,

$$P(A \cap B) = 0$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\Rightarrow \quad P(A \cup B) = P(A) + P(B)$$

$$\Rightarrow \quad P(A) + P(B) \leq 1$$

$$\Rightarrow \quad P(A) + 1 - P(\overline{B}) \leq 1 \qquad [\because P(B) = 1 - P(\overline{B})]$$

$$\therefore \quad P(A) \leq P(\overline{B})$$

Q. 24 If $P(A \cup B) = P(A \cap B)$ for any two events A and B, then

(a) $P(A) = P(B)$ (b) $P(A) > P(B)$

(c) $P(A) < P(B)$ (d) None of these

Sol. ***(a)*** Given, $P(A \cup B) = P(A \cap B)$

$$P(A) + P(B) - P(A \cap B) = P(A \cap B)$$

$$\Rightarrow \quad [P(A) - P(A \cap B)] + [P(B) - P(A \cap B)] = 0$$

But $\quad P(A) - P(A \cap B) \geq 0$

and $\quad P(B) - P(A \cap B) \geq 0 \qquad [\because P(A \cap B) \leq P(A) \text{ or } P(B)]$

$\Rightarrow \quad P(A) - P(A \cap B) = 0$

and $\quad P(B) - P(A \cap B) = 0$

[since, sum of two non-negative numbers can be zero only when these numbers aree zero]

$\Rightarrow \quad P(A) = P(A \cap B)$

and $\quad P(B) = P(A \cap B)$

$\therefore \quad P(A) = P(B)$

Q. 25 If 6 boys and 6 girls sit in a row at random, then the probability that all the girls sit together is

(a) $\frac{1}{432}$ (b) $\frac{12}{431}$

(c) $\frac{1}{132}$ (d) None of these

Sol. ***(c)*** If all the girls sit together, then considered it as 1 group.

$\therefore$ Arrangement of $6+1=7$ person in a row is $7!$ and the girls interchanges their seets in 6! ways.

$\therefore$ Required probability $= \frac{6!7!}{12!} = \frac{1}{132}$

Q. 26 If a single letter is selected at random from the word 'PROBABILITY', then the probability that it is a vowel is

(a) $\frac{1}{3}$ (b) $\frac{4}{11}$ (c) $\frac{2}{11}$ (d) $\frac{3}{11}$

Sol. ***(b)*** Total number of alphabet in the word probability = 11

Number of vowels = 4

P(letter is vowel) $= \frac{4}{11}$

Q. 27 If the probabilities for A to fail in an examination is 0.2 and that for B is 0.3, then the probability that either A or B fails is

(a) > 0.5 (b) 0.5 (c) ≤ 0.5 (d) 0

Sol. ***(c)*** Given, $P(A \text{ fail}) = 0.2$

and $P(B \text{ fail}) = 0.3$

$\therefore$ $P(\text{either } A \text{ or } B \text{ fail}) \leq P(A \text{ fail}) + P(B \text{ fail})$

$\leq 0.2 + 0.3$

≤ 0.5

Q. 28 The probability that atleast one of the events A and B occurs is 0.6. If A and B occur simultaneously with probability 0.2, then $P(\overline{A}) + P(\overline{B})$ is equal to

(a) 0.4 (b) 0.8 (c) 1.2 (d) 1.6

Sol. ***(c)*** Given, $P(A \cup B) = 0.6$ and $P(A \cap B) = 0.2$

$\therefore$ $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\Rightarrow$ $0.6 = P(A) + P(B) - 0.2$

$\Rightarrow$ $P(A) + P(B) = 0.8$

$\therefore$ $P(\overline{A}) + P(\overline{B}) = 1 - P(A) + 1 - P(B)$

$= 2 - [P(A) + P(B)]$

$= 2 - 0.8 = 1.2$

Q. 29 If M and N are any two events, the probability that atleast one of them occurs is

(a) $P(M) + P(N) - 2P(M \cap N)$
(b) $P(M) + P(N) - P(M \cap N)$
(c) $P(M) + P(N) + P(M \cap N)$
(d) $P(M) + P(N) + 2P(M \cap N)$

Sol. ***(b)*** If M and N are any two events.

$$\therefore \quad P(M \cup N) = P(M) + P(N) - P(M \cap N)$$

True/False

Q. 30 The probability that a person visiting a zoo will see the giraffee is 0.72, the probability that he will see the bears is 0.84 and the probability that he will see both is 0.52.

Sol. ***False***

P (to see girafee) = 0.72
P (to see bear) = 0.84
P (to see giraffee and bear) = 0.52
P (to see giraffee or bear) = P (giraffee) + P (bear) – P (giraffee and bear)
= 0.72 + 0.84 – 0.52
= 1.04

which is not possible. Hence statement is false.

Q. 31 The probability that a student will pass his examination is 0.73, the probability of the student getting a compartment is 0.13 and the probability that the student will either pass or get compartment is 0.96.

Sol. ***False***

Let A = Student will pass examination
B = Student will getting compartment
$P(A) = 0.73$ and $P(A \text{ or } B) = 0.96$ and $P(B) = 0.13$

$$\therefore \quad P(A \text{ or } B) = P(A) + P(B) = 0.73 + 0.13 = 0.86$$

But $P(A \text{ or } B) = 0.96$

Hence, it is **false** statement.

Q. 32 The probabilities that a typist will make 0, 1, 2, 3, 4 and 5 or more mistakes in typing a report are respectively, 0.12, 0.25, 0.36, 0.14, 0.08 and 0.11.

Sol. ***False***

Sum of these probabilities must be equal to 1.

$$P(0) + P(1) + P(2) + P(3) + P(4) + P(5)$$
$$= 0.12 + 0.25 + 0.36 + 0.14 + 0.08 + 0.11 = 1.06$$

which is greater than 1,
So, it is **false** statement.

Q. 33 If A and B are two candidates seeking admission in an engineering college. The probability that A is selected is 0.5 and the probability that both A and B are selected is at most 0.3. Is it possible that the probability of B getting selected is 0.7?

Sol. ***False***

Here, $P(A) = 0.5,\ P(A \cap B) \le 0.3$

Now, $P(A) \times P(B) \le 0.3$

$\Rightarrow \quad 0.5 \times P(B) \le 0.3$

$\Rightarrow \quad P(B) \le 0.6$

Hence, it is **false** statement.

Q. 34 The probability of intersection of two events A and B is always less than or equal to those favourable to the event A.

Sol. ***True***

$$P(A \cap B) \le P(A)$$

Hence, it is **true** statement.

Q. 35 The probability of an occurrence of event A is 0.7 and that of the occurrence of event B is 0.3 and the probability of occurence of both is 0.4.

Sol. ***False***

Here, $P(A) = 0.7$

and $P(B) = 0.3$

$\therefore \quad P(A \cap B) = P(A) \times P(B)$

$= 0.7 \times 0.3 = 0.21$

Hence, it is **false** statement.

Q. 36 The sum of probabilities of two students getting distinction in their final examinations is 1.2.

Sol. ***True***

Since, these two events not related to the same sample space.

So, sum of probabilities of two students getting distinction in their final examination may be 1.2.

Hence, it is **true** statement.

Fillers

Q. 37 The probability that the home team will win an upcoming football game is 0.77, the probability that it will tie the game is 0.08 and the probability that it will lose the game is ... .

Sol. P (lossing) $= 1 - (0.77 + 0.08) = 0.15$

Q. 38 If e_1, e_2, e_3 and e_4 are the four elementary outcomes in a sample space and $P(e_1) = 0.1$, $P(e_2) = 0.5$ and $P(e_3) = 0.1$, then the probability of e_4 is .... .

Sol. $\because$ $P(e_1) + P(e_2) + P(e_3) + P(e_4) = 1$

$\Rightarrow$ $0.1 + 0.5 + 0.1 + P(e_4) = 1$

$\Rightarrow$ $0.7 + P(e_4) = 1$

$\therefore$ $P(e_4) = 0.3$

Q. 39 If $S = \{1, 2, 3, 4, 5, 6\}$ and $E = \{1, 3, 5\}$, then $\bar{E}$ is .... .

Sol. Here, $S = \{1, 2, 3, 4, 5, 6\}$

and $E = \{1, 3, 5\}$

$\therefore$ $\bar{E} = S - E = \{2, 4, 6\}$

Q. 40 If A and B are two events associated with a random experiment such that $P(A) = 0.3$, $P(B) = 0.2$ and $P(A \cap B) = 0.1$, then the value of $P(A \cap \bar{B})$ is ... .

Sol. $P(A \cap \bar{B}) = P(A) - P(A \cap B) = 0.3 - 0.1 = 0.2$

Q. 41 The probability of happening of an event A is 0.5 and that of B is 0.3. If A and B are mutually exclusive events, then the probability of neither A nor B is ....

Sol. $P(\bar{A} \cap \bar{B}) = P(\overline{A \cup B}) = 1 - P(A \cup B)$

$= 1 - [P(A) + P(B)]$ [since, A and B are mutually exclusive]

$= 1 - (0.5 + 0.3) = 1 - 0.8 = 0.2$

Matching The Columns

Q. 42 Match the following.

	Column I		Column II
(i)	0.95	(a)	An incorrect assignment
(ii)	0.02	(b)	No chance of happening
(iii)	– 0.3	(c)	As much chance of happening as not
(iv)	0.5	(d)	Very likely to happen
(v)	0	(e)	Very little chance of happening

Sol. **(i)** 0.95 is very likely to happen, so it is close to 1.

(ii) 0.02 very little chance of happening because probability is very low.

(iii) – 0.3 an incorrect assignment because probability of any events lie between 0 and 1.

(iv) 0.5, as much chance of happening as not because sum of chances of happening and not happening is zero.

(v) 0, no chance of happening.

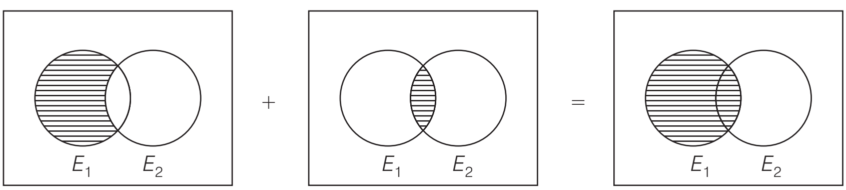
Q. 43 Match the following.

	Column I		Column II
(i)	If E_1 and E_2 are the two mutually exclusive events	(a)	$E_1 \cap E_2 = E_1$
(ii)	If E_1 and E_2 are the mutually exclusive and exhaustive events	(b)	$(E_1 - E_2) \cup (E_1 \cap E_2) = E_1$
(iii)	If E_1 and E_2 have common outcomes, then	(c)	$E_1 \cap E_2 = \phi, E_1 \cup E_2 = S$
(iv)	If E_1 and E_2 are two events such that $E_1 \subset E_2$	(d)	$E_1 \cap E_2 = \phi$

Sol. **(i)** If E_1 and E_2 are two mutually exclusive event, then $E_1 \cap E_2 = \phi$.

(ii) If E_1 and E_2 are mutually exclusive and exhaustive events, then $E_1 \cap E_2 = \phi$ and $E_1 \cup E_2 = S$.

(iii) If E_1 and E_2 have common outcomes, then $(E_1 - E_2) \cup (E_1 \cap E_2) = E_1$



$E_1 - E_2$ $\quad$ $E_1 \cap E_2$ $\quad$ E_1

(iv) If E_1 and E_2 are two events such that $E_1 \subset E_2 \Rightarrow E_1 \cap E_2 = E_1$