Binomial Theorem

24th September 2024

Algebraic Expressions – Revisited (1/3)

variables	constants	Arithmatic operators
x, y, z	$(2+3i), \sqrt{2}, 0\cdot 3, a$	$+, -, \times, \div$

combined by arithmatic operators

Algebraic expressions

Eg: $\underline{x^2 + 2y - 320\sqrt{2}(z)}$, x, $x^2 + a/y^2$, etc

Algebraic Expressions – Revisited (2/3)

Terms: Separated by addition/subtraction operators.

$$\underbrace{x^3}_{\text{Term-1}} + \underbrace{3abx^2y}_{\text{Term-2}} + \underbrace{24x^2}_{\text{Term-3}} + \underbrace{a^3y^3}_{\text{Term-4}}$$

Algebraic Expressions – Revisited (3/3)

Monomial $\longrightarrow$ has single term (form of p)

Eg: 3, $x^{3/2}$, xy

Binomial $\longrightarrow$ has two terms (form $p \pm q$)

Eg: $(a+b)$, $(1+0.1)$, $(x/3 + 3/x)$, $(x^3 - 3xy)$

Trinomial $\longrightarrow$ has three terms (form: $p \pm q \pm r$)

Eg: $(x_1^2 + x_2^2 + x_3^2)$, $(ax^2 + bx + c)$, $(ax + by + c)$,

$(a+b+c)$, $(25x + 32xy^2 + z^{10})$.

But, the word polynomial is more nuanced. It is not any algebraic eqn. of any number of terms. A polynomial of n^{th} order and one variable must be of the form

$$a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n x^0 \text{ where } a_0 \neq 0.$$

and n is a natural number.

If $a_0 \neq 0$, it is not of the n-th order,

and if $n \notin \mathbb{N}$, it is not a polynomial at all!

Binomials

Any algebraic expression of the form $(p \pm q)$ and having two terms.

Eg: $(\sqrt{x} + y)$, $\left(\frac{a^2}{3x^2} + \frac{\sqrt{a}}{3y}\right)$

Rules for monomial exponentiation:

1. $p^{m+n} = p^m \cdot p^n$

2. $p^{m-n} = p^m / p^n$.

3. $p^{m \cdot n} = (p^m)^n$

4. $(p/q)^m = p^m / q^m$

5. $(p \cdot q)^m = p^m \cdot q^m$

6. $p^{1/m} = \sqrt[m]{p}$.

7. $p^0 = 1$ if $p \neq 0$.

$\therefore$ 0^0 is an indeterminate form.

Eg:

1. $(3x)^2 = 3^2 \cdot x^2 = 9x^2$.

2. $(x^2)^3 = x^6$.

3. $(xy)^{20} = x^{20} \cdot y^{20}$

4. $(\sqrt{x}/y)^2 = (\sqrt{x})^2/(y)^2 = \dfrac{x^{2/2}}{y^2} = \dfrac{x}{y^2}$

5. $\sqrt[3]{x^4 \cdot y^3} = (x^4 \cdot y^3)^{1/3} = (x^4)^{1/3}(y^3)^{1/3}$

$= x^{4/3} \cdot y^{3/3} = x\sqrt{x} \cdot y$

Binomial exponentiation

$$(p+q)^0 = 1 \quad \text{if } p+q \neq 0.$$

$$(p+q)^1 = p+q.$$

$$(p+q)^2 = p^2 + 2pq + q^2.$$

$$\vdots$$

$$(p+q)^n \longrightarrow \text{This is what this chapter really deals with!}$$

How do we evaluate the expansion of a binomial?

How do you evaluate/expand something like

$$(x+y)^{100} ?$$

Is it even possible?

What is the 54th term in the expansion?

In this chapter we go looking for some <u>patterns</u> that help us answering these questions.

Basic Observations

$(a+b)^0 = 1.$ one term when $n=0$

$(a+b)^1 = a+b$ two terms when $n=1.$

$(a+b)^2 = a^2+2ab+b^2$

three terms when $n=2.$

$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$

four terms when $n=3.$

⋮

OBS-1: Thus, $(n+1)$ terms in the expansion of $(a+b)^n$.

$(a+b)^0 = 1.$ $\longrightarrow$ highest power of a or b is zero. when $n = 0$

$(a+b)^1 = a+b$ $\longrightarrow$ highest power $= 1.$
$n=1$

$(a+b)^2 = a^2+2ab+b^2.$ $\longrightarrow$ highest power $= 2$
$n = 2$

$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$ $\longrightarrow$ highest power $= 3$
$n = 3.$

OBS -2: Thus, highest power of either terms of the binomial is n for $(a+b)^n$.

$(a+b)^0 = 1.$

power of a + power of b in each term = 0

$(a+b)^1 = a+b$

$1+0 = 1$ $\quad$ $0+1 = 1$

$(a+b)^2 = a^2 + 2a^1b^1 + b^2$

$2+0 = 2$ $\quad$ $1+1 = 2$ $\quad$ $0+2 = 2$

$(a+b)^3 = a^3 + 3a^2b^1 + 3a^1b^2 + b^3$

$3+0 = 3$ $\quad$ $2+1 = 3$ $\quad$ $1+2 = 3$ $\quad$ $0+3 = 3$

OBS-3:

The order of each term in the expansion of $(a+b)^n$ is $\underline{n}$.

OBS-1: There are $(n+1)$ terms in the expansion

OBS-2: The highest power of either a or b in any term is (n).

OBS-3: The order of every term in the expansion is (n).

The general form of the expansion of (a+b)^n

$$\boxed{\begin{aligned}(a+b)^n &= K_0\,(a)^n(b)^0 + K_1(a)^{n-1}(b)^1 + K_2(a)^{n-2}(b)^2 \\ &\quad + \cdots\cdots + \\ &\quad + K_{n-2}(a)^2(b)^{n-2} + K_{n-1}(a)^1(b)^{n-1} + K_n(a)^0(b)^n\end{aligned}}$$

where $k_0, k_1, \ldots, k_n$ are some constant co-efficients that we will now find.

Let us write down the coefficients for $n = 1, 2, 3, 4, 5, 6$

$\underline{n = 0}$; 1

$\underline{n = 1}$; 1 , 1

$\underline{n = 2}$; 1 , 2 , 1 .

$\underline{n = 3}$; 1 , 3 , 3 , 1 .

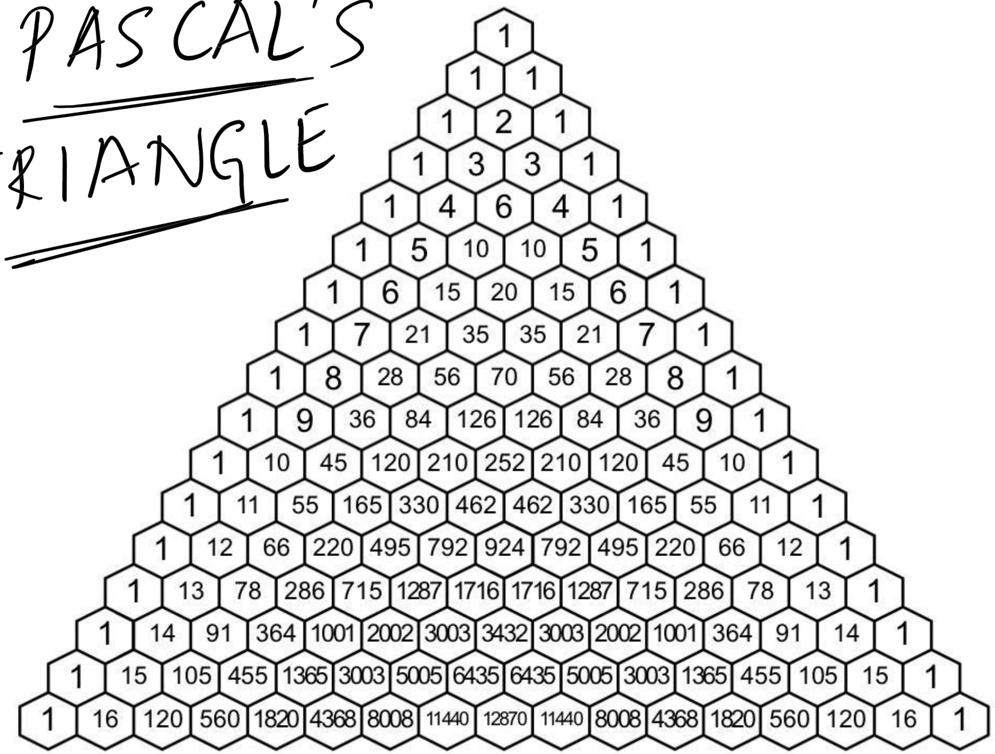
$\underline{n = 4}$; 1 , 4 , 6 , 4 , 1 .

$\underline{n = 5}$; 1 , 5 , 10 , 10 , 5 , 1 .

$\underline{n = 6}$; 1 , 6 , 15 , 20 , 15 , 6 , 1

It turns out that the co-efficients for a binomial expansion are all elements of the PASCAL'S Δ^{le}.

PASCAL'S TRIANGLE



Pascal obtained this triangle by adding the two elements right above the element we want to find.

Has remarkable properties

PASCAL's TRIANGLE.

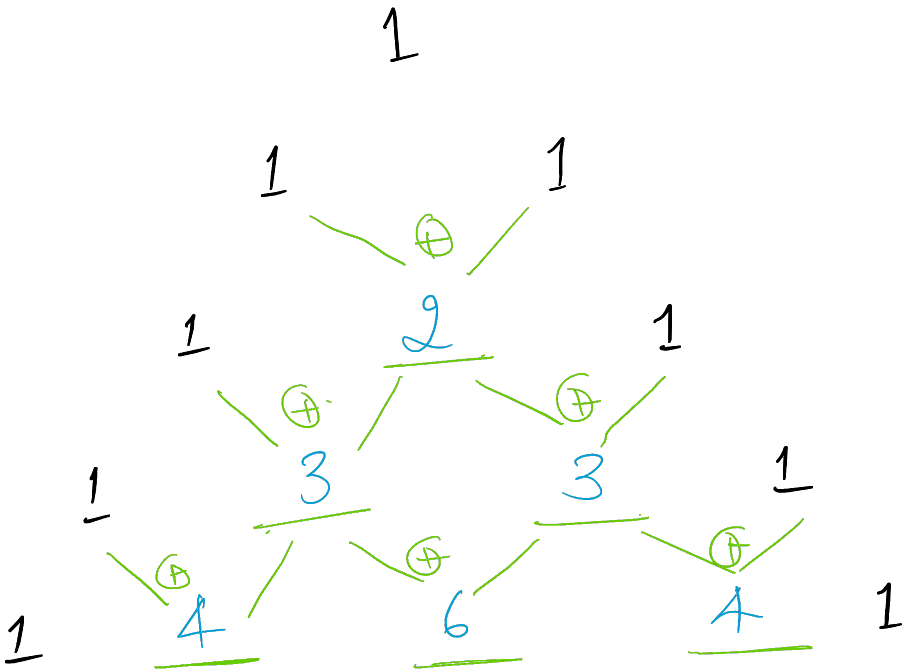
1 → row-1

1 1 → row-2.

1 1 → row-3.

1

All end and beginnin
elements of each row
= 1.



$$
=
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1.
\end{array}
$$

Every element of the PASCAL's Δle is obtained by ADDING the two elements right above it in the previous row.

Value of 0^{th} pos	of step - 0		$= 1$	$^{0}C_{0}$
0 pos	step - 1		$= 1$	$^{1}C_{0}$
1 pos	step. 1		$= 1$	$^{1}C_{1}$
0 pos	step - 2		$= 1$	$^{2}C_{0}$
1 pos	step - 2		$= 2$	$^{2}C_{1}$
2 pos	step. 2		$= 1$	$^{2}C_{2}$

From the chapter on Permutations and combinations

Theorem 6 $^{n}C_{r} + ^{n}C_{r-1} = ^{n+1}C_{r}$

Thus fourth row elements are.

$\underline{1}$ $\quad \underline{^{2}C_{0} + ^{2}C_{1}}$ $\quad \underline{^{2}C_{1} + ^{2}C_{2}}$ $\quad \underline{1}$.

But from the theorem above, if $n=2, r=1$:

$\underline{1}$ $\quad \underline{^{3}C_{1}}$. $\quad \underline{^{2}C_{1} + ^{2}C_{2}}$ $\quad \underline{1}$.

Again from the theorem above, if $n=2, r=2$:

$\underline{1}$ $\quad \underline{^{3}C_{1}}$ $\quad \underline{^{3}C_{2}}$ $\quad \underline{1}$.

$\underline{1} \quad \underline{{}^3C_1} \quad \underline{{}^3C_2} \quad \underline{1}$.

But 1 can be written as ${}^3C_0$ or ${}^3C_3$.

Thus, the fourth row elements are ${}^3C_0, {}^3C_1, {}^3C_2, {}^3C_3$.

Similarly, the fifth row elements are;

$\underline{{}^4C_0, {}^4C_1, {}^4C_2, {}^4C_3, {}^4C_4}$.

Thus, the entire pascal's Δle can be re-written in an alternate form (next slide).

$$1 = {}^{0}C_{0}$$

$$1 = {}^{1}C_{0} \qquad 1 = {}^{1}C_{0}$$

$$1 = {}^{2}C_{0} \qquad 2 = {}^{2}C_{1} \qquad 1 = {}^{2}C_{2}$$

$$1 = {}^{3}C_{0} \qquad 3 = {}^{3}C_{1} \qquad 3 = {}^{3}C_{2} \qquad 1 = {}^{3}C_{3}$$

$$1 = {}^{4}C_{0} \qquad 4 = {}^{4}C_{1} \qquad 6 = {}^{4}C_{2} \qquad 4 = {}^{4}C_{3} \qquad 1 = {}^{4}C_{4}$$

$$1 = {}^{5}C_{0} \qquad 5 = {}^{5}C_{1} \qquad 10 = {}^{5}C_{2} \qquad 10 = {}^{5}C_{3} \qquad 5 = {}^{5}C_{4} \qquad 1 = {}^{5}C_{5}$$

$$1 = {}^{6}C_{0} \qquad 6 = {}^{6}C_{1} \qquad 15 = {}^{6}C_{2} \qquad 20 = {}^{6}C_{3} \qquad 15 = {}^{6}C_{4} \qquad 6 = {}^{6}C_{5} \qquad 1 = {}^{6}C_{6}$$

$$\vdots$$

Remember when we said all co-efficients of the binomial expansion are identical to the pascal's Δ^{le}?

That means, all the co-efficients of the binomial expansion can be expressed in ${}^{n}C_{r}$ notation.

Binomial Theorem

$$(a+b)^n = {}^nC_0 (a)^n (b)^0 + {}^nC_1 (a)^{n-1} (b) + {}^nC_2 (a)^{n-2} b^2$$

$$+ \cdots\cdots + {}^nC_n (a)^{n-n} (b)^n.$$

$$\boxed{(a+b)^n = \sum_{r=0}^{n} {}^nC_r \cdot a^{n-r} \cdot b^r.}$$

Evaluate ${}^nC_r$ for small values of n w/ PASCAL's Δ^{le}.

1) Find $(x-y)^6$ through the binomial thm.

SOLUTION:

Consider $(a+b)^n = {}^nC_0 a^n b^0 + {}^nC_1 a^{n-1} b^1 + \cdots + {}^nC_n a^{n-n} b^n$.

If $a = x$, $b = -y$, $n = 6$:

$$(x-y)^6 = {}^6C_0 (x)^6 (-y)^0 + {}^6C_1 (x)^5 (-y)^1 + {}^6C_2 (x)^4 (-y)^2 + {}^6C_3 (x)^3 (-y)^3 + {}^6C_4 (x)^2 (-y)^4 + {}^6C_5 (x)^1 (-y)^5 + {}^6C_6 (x)^0 (-y)^6.$$

${}^6C_0, {}^6C_1, {}^6C_2, {}^6C_3, {}^6C_4, {}^6C_5, {}^6C_6$ can be evaluated from PASCAL'S Δle

$= 1, 6, 15, 20, 15, 6, 1.$

$$\Rightarrow \boxed{(x-y)^6 = x^6 - 6x^5y + 15x^4y^2 - 20x^3y^3 + 15x^2y^4 - 6xy^5 + y^6.}$$