JEE MATHEMATICS

DPS-1

DAILY PRACTICE SHEET

Class XI

Sets, Relations and Functions

SOLUTIONS

1. **(b) :** Using commutative law $(A|B) \cup (B|A) = (B|A) \cup (A|B)$
$= (A \cup B) \mid (B \cap A)$

2. **(b) :** Given, $A = \{3, 6, 9, 12\}$
$= \{x : x = 3n, n \in N \text{ and } 1 \le n \le 4\}$
and $B = \{1, 4, 9, ..., 100\} = \{x : x = n^2, n \in N \text{ and } 1 \le n \le 10\}$

3. **(c) :** A is a set of numbers. B is a set of sets.
So, $A = B$ is impossible. Also, $B \not\subset C$ since $\{2\} \notin C$
$A \in C$ is correct since $\{1, 2\} \in C$.
$A \subset C$ is not true since A is set of numbers and C is a set of sets.

4. **(d) :** Let number of elements in S be x.
i.e., $n(S) = x$

As $\bigcup_{i=1}^{30} A_i = \bigcup_{j=1}^{n} B_i = S$

$\Rightarrow$ $5 \times 30 = 10x$

$\Rightarrow$ $x = \dfrac{150}{10} = 15$

$\Rightarrow$ $x = 15$ ...(i)

and $\bigcup_{j=1}^{n} B_j = S \Rightarrow \sum_{j=1}^{n} (B_j) = 9x$

$\Rightarrow$ $3n = 9x$ ...(ii)

Putting $x = 15$ in (ii), we get

$n = \dfrac{9 \times 15}{3} = 45 \Rightarrow n = 45$

5. **(d) :** Let A and B be the sets of persons who can speak Hindi and Bengali respectively.
$n(A \cup B) = 1000$, $n(A) = 750$, $n(B) = 400$.
Number of persons who can speak both Hindi and Bengali
$= n(A \cap B) = n(A) + n(B) - n(A \cup B)$.
$= 750 + 400 - 1000 = 150$
Number of persons who can speak Hindi only
$= n(A - B) = n(A) - n(A \cap B) = 750 - 150 = 600$
Number of persons who can speak Bengali only
$= n(B - A) = n(B) - n(B \cap A) = 400 - 150 = 250$

6. **(d) :** Any element of S has got three possibilities either it is in A or in B or none. Thus each has 3 choices. For four elements we have $3^4 = 81$ choices. These are choices for ordered pair (A, B). Only one pair is counted once, *i.e.*, (ϕ, ϕ).

The number of unordered pairs $= \dfrac{3^4 - 1}{2} + 1 = 41$

7. **(c) :** Let x be total number of families in the town. Let set P be the families who own a cell phone and C be the families who own a scooter.
$n(P) = 25\%$, $n(C) = 15\%$, $n(P' \cap C') = 65\% \Rightarrow n(P \cup C) = 35\%$
$\Rightarrow$ $n(P \cap C) = n(P) + n(C) - n(P \cup C)$
$= (25 + 15 - 35)\% = 5\%$
$\therefore$ 5% of $x = 1500$
$\Rightarrow$ $x = \dfrac{1500}{5} \times 100 = 30000$
Hence, total number of families in the town = 30000.

8. **(c) :** $n[(A \cap B)' \cap A]$
$= n[(A' \cup B') \cap A]$ [by De Morgan's law]
$= n[(A' \cap A) \cup (B' \cap A)]$ [by distributive law]
$= n(A - B) = 8 - 2 = 6$.

9. **(c) :** Let U be the set of all consumers who were questioned, A be the set of consumers who liked product P_1 and B be the set of consumers who liked the product P_2.
It is given that $n(U) = 2000$,
$n(A) = 1720$, $n(B) = 1450$.
$n(A \cup B) = n(A) + n(B) - n(A \cap B)$
$n(A \cup B) = 1720 + 1450 - n(A \cap B)$
$= 3170 - n(A \cap B)$
Since, $A \cup B \subseteq U$
$\therefore$ $n(A \cup B) \le n(U)$
$\Rightarrow$ $3170 - n(A \cap B) \le 2000 \Rightarrow n(A \cap B) \ge 1170$

10. **(c) :** $n(A) = 1000$, $n(B) = 500$, $n(A \cap B) \ge 1$,
$n(A \cup B) = p$, $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
$\Rightarrow p = 1000 + 500 - n(A \cap B)$
But $1 \le n(A \cap B) \le 500$
Hence, $p = 1000 + 500 - 1 = 1499$
and $p = 1000 + 500 - 500 = 1000$
$\therefore$ $1000 \le p \le 1499$

11. **(c) :** We know that every subset of $A \times A$ is a relation on A.
So, Statement-1 is true but Statement-2 is false.

12. **(b) :** Given $n(A) = 4$, $n(B) = 5$, $n(A \cap B) = 3$
$n[(A \times B) \cap (B \times A)] = n[(A \cap B) \times (B \cap A)]$
$= n(A \cap B) \times n(B \cap A) = 3 \times 3 = 9$
[$\therefore$ $n(A \times B) = n(A) \times n(B)$]

13. **(a) :** Given $R = \{(x, y) : x + 2y = 8 \ \forall \ x, y \in N\}$
$\therefore$ $x + 2y = 8 \Rightarrow y = \dfrac{8 - x}{2}$

When $x = 1$, $y = \dfrac{7}{2} \notin N$ $\quad\therefore \left(1, \dfrac{7}{2}\right) \notin$ Relation (R)

When $x = 2$, $y = 3 \in N$ $\quad\therefore (2, 3) \in R$

When $x = 3$, $y = \dfrac{5}{2} \notin N$ $\quad\therefore \left(3, \dfrac{5}{2}\right) \notin R$

When $x = 4$, $y = 2 \in N$ $\quad\therefore (4, 2) \in R$

When $x = 5$, $y = \dfrac{3}{2} \notin N$ $\quad\therefore \left(5, \dfrac{3}{2}\right) \notin R$

When $x = 6$, $y = 1 \in N$ $\quad\therefore (6, 1) \in R$

$\therefore$ $R = \{(2, 3), (4, 2), (6, 1)\}$
$\therefore$ Range of relation (R) = $\{1, 2, 3\}$

14. **(a) :** Total number of relations from A to $B = 2^{n(A)\times n(B)}$
$= 2^{5\times 7} = 2^{35}$

15. **(d) :** Given, $n(U) = 800, n(A) = 300, n(B) = 400, n(A\cap B) = 100$
Now, $n(A^c \cap B^c) = n(A\cup B)^c = n(U) - n(A\cup B)$
$= n(U) - [n(A) + n(B) - n(A\cap B)]$
$= 800 - (300 + 400 - 100) = 200$
We know that relation from the set X to set $Y = 2^{n(X)\times n(Y)}$
$\therefore$ Number of relations on $A^c \cap B^c = 2^{n(A^c\cap B^c)\times n(A^c\cap B^c)}$
$= 2^{200\times 200} = 2^{4\times 10^4} = 2^{(40000)} = 16^{10000}$

16. **(a) :** $R = \{(x, y) : y = 3x, x \in A\}$
$\therefore$ $R = \{(1,3), (2,6), (3,9), (4,12)\}$
$\therefore$ Domain of the relation = {1, 2, 3, 4}
and Range of the relation = {3, 6, 9, 12}.

17. **(c) :** We have $a + 4b = 15$
$\therefore$ $a = 15 - 4b$, now putting b = 1, 2, 3, we get
a = 11, 7, 3. Also for $b \geq 4$, $a \notin N$.
$\therefore$ $R = \{(11, 1), (7, 2), (3, 3)\}$
$\therefore$ Range(R) = set of second elements of distinct ordered pairs = {1, 2, 3}

18. **(a) :** Given, $R = \{(x, y) : y = 2x - 1; x, y \in A\}$
$\Rightarrow$ $R = \{(1, 1), (2, 3), (3, 5)\}$ $\therefore$ Domain (R) = {1, 2, 3}

19. **(c) :** We know that $f(x + y) = f(x) f(y) \Rightarrow f(x) = a^x, f(1) = 2 \Rightarrow a = 2$
$\therefore$ $f(x) = 2^x$
$$\sum_{m=1}^{n} f(k+m) = \sum_{m=1}^{n} 2^{k+m}$$
$= 2^{k+1}(1 + 2 + \ldots + 2^{n-1})$
$= 2^{k+1}(2^n - 1) = 16(2^n - 1)$ (given) $\therefore$ $k = 3$.

20. **(d) :** $f(x)$ is defined if $x! > 0$
$\log_{0.5} x! \geq 0, \dfrac{\log x!}{\log 0.5} > 0, \log x! \leq 0$
$x! \leq e^0, \ x! \leq 1$
$\therefore$ Domain of the function $f(x)$ is {0, 1}

21. **(c) :** $f(x) = \dfrac{1}{2} - \tan\left(\dfrac{\pi x}{2}\right), \ -1 < x < 1$
Since $f(x)$ is defined in the given interval $-1 < x < 1$
$\therefore$ Domain of $f = (-1, 1)$ ...(i)
$g(x) = \sqrt{3 + 4x - 4x^2}$
Since $g(x)$ is defined when $3 + 4x - 4x^2 \geq 0$
$\Rightarrow$ $4x^2 - 4x - 3 \leq 0 \Rightarrow 4x^2 - 6x + 2x - 3 \leq 0$
$\Rightarrow$ $(2x - 3)(2x + 1) \leq 0$
$\Rightarrow$ $\dfrac{-1}{2} \leq x \leq \dfrac{3}{2} \Rightarrow$ Domain of $g = \left[\dfrac{-1}{2}, \dfrac{3}{2}\right]$ ...(ii)
From (i) and (ii), we get
Domain of $(f + g) = \left[-\dfrac{1}{2}, 1\right)$.

22. **(d) :** $f(x)$ to be defined
(i) $x > 0$
(ii) $\log_2 x > 0 \Rightarrow x > 2^0 \Rightarrow x > 1$
(iii) $\log_2 (\log_2 x) > 0 \Rightarrow \log_2 x > 1 \therefore x > 2$
(iv) $\log_2 \log_2 (\log_2 x) > 0 \Rightarrow \log_2 (\log_2 x) > 1$
$\Rightarrow$ $\log_2 x > 2 \Rightarrow x > 2^2$
$\therefore$ $\log_2 \log_2 \log_2 \ldots \log_2 x$ (n times) > 0
$\Rightarrow$ $x = 2^{2^{2^{\cdot^{\cdot^{\cdot 2}}}}}$ ($(n-1)$ times)
$\therefore$ $D_f = \left(2^{2^{2^{\cdot^{\cdot^{\cdot 2}}}}}, \infty\right)$ ($(n-1)$ times)

23. **(a) :** Clearly, $f(x) = g(x)$ for all $x \in R - \{2\}$.
But, $f(x)$ and $g(x)$ have different domains.
Infact, domain of $f = R - \{2\}$ and domain of $g = R$. Therefore, $f \neq g$.

24. **(d) :** It is the sum of an infinite G.P. with common ratio x and sum $\dfrac{1}{1-x}$.
Domain is $(-1, 1)$. The range is $\left(\dfrac{1}{2}, \infty\right)$.

25. **(d) :** Given, $f(x) = 2x^2$
$\Rightarrow$ $f(3.8) = 2 \times 3.8 \times 3.8 = 28.88$
$\Rightarrow$ $f(4) = 2 \times 4 \times 4 = 32$
$\therefore$ $\dfrac{f(3.8) - f(4)}{3.8 - 4} = \dfrac{28.88 - 32}{3.8 - 4} = \dfrac{3.12}{0.2} = 15.6$

26. **(c) :** The point $(1, b)$ is always on the graph of the function.

27. **(a) :** The equation implies $(f(x) - 1)\left(f\left(\dfrac{1}{x}\right) - 1\right) = 1$
If $g(x) = f(x) - 1$, then $g(x)\ g\left(\dfrac{1}{x}\right) = 1$
$\Rightarrow g(x) = \pm x^n, \ f(x) = 1 \pm x^n, f(-2) = 33$
$\Rightarrow$ $1 \pm (-2)^n = 33$
$\Rightarrow n = 5$
$\therefore f(x) = 1 - x^5, f(1) = 0$

28. **(c) :** $\dfrac{4 - x^2}{2 + [x]} \geq 0 \Rightarrow 4 - x^2 \geq 0, 2 + [x] > 0$
$\Rightarrow x \in [-2, 2]$ and $x \in [-1, \infty) \Rightarrow x \in [-1, 2]$ ...(i)
On the other hand, $4 - x^2 < 0, 2 + [x] < 0$
$x \in R - [-2, 2]$ and $x \in (-\infty, -2) \Rightarrow x \in (-\infty, -2)$ ...(ii)
From (i) and (ii), we get $(-\infty, -2) \cup [-1, 2]$

29. **(b) :** $f(x) = \tan\sqrt{\dfrac{\pi^2}{9} - x^2}$
$f(x)$ is real valued function when $\dfrac{\pi^2}{9} - x^2 \geq 0$
$\Rightarrow$ $x^2 \leq \dfrac{\pi^2}{9} \Rightarrow x \in \left[-\dfrac{\pi}{3}, \dfrac{\pi}{3}\right]$
$\therefore$ $\dfrac{-\pi}{3} \leq x \leq \dfrac{\pi}{3}$
$\Rightarrow$ $0 \leq x^2 \leq \dfrac{\pi^2}{9} \Rightarrow \dfrac{-\pi^2}{9} \leq -x^2 \leq 0$
$\Rightarrow$ $0 \leq \dfrac{\pi^2}{9} - x^2 \leq \dfrac{\pi^2}{9} \Rightarrow 0 \leq \sqrt{\dfrac{\pi^2}{9} - x^2} \leq \dfrac{\pi}{3}$
$\because$ $\tan x$ is increasing in $x \in \left[0, \dfrac{\pi}{3}\right]$
$\therefore$ $\tan 0 \leq \tan\sqrt{\dfrac{\pi^2}{9} - x^2} \leq \tan\dfrac{\pi}{3}$
$\Rightarrow$ $0 \leq \tan\sqrt{\dfrac{\pi^2}{9} - x^2} \leq \sqrt{3}$
$\therefore$ Range of the function $= \left[0, \sqrt{3}\right]$

30. **(c) :** $f(x) = \dfrac{\alpha x^2}{x+1}, x \neq -1;\ f(a) = a$

$\Rightarrow \quad a = \dfrac{\alpha a^2}{a+1} \Rightarrow \alpha = \dfrac{1+a}{a} = \dfrac{1}{a} + 1$

31. **(b, c)** **:** Here, $f(x) = x^2$ and $g(x) = \sqrt{x}$, therefore,

$(fog)(x) = f(g(x)) = f(\sqrt{x}) = (\sqrt{x})^2 = x$

for all x, $D_{fog} = D_g = [0, \infty)$

$(\because R_g = [0, \infty) \subset D_f,\ \therefore D_{fog} = D_g)$

Also $(gof)(x) = g(f(x)) = g(x^2)$

$= \sqrt{x^2} = |x|$

for all x, $D_{gof} = D_f = \mathbf{R}$.

$(\because R_f = [0, \infty) = D_g,\ \therefore D_{gof} = D_f)$

32. **(a, d) :** $y = f(x) = \dfrac{x+2}{x-1}$

$\Rightarrow \quad y(x-1) = x+2 \quad \Rightarrow x(y-1) = y+2$

$\Rightarrow \quad x = \dfrac{y+2}{y-1} \Rightarrow x = f(y)$

again $f(1)$ does not exist as domain is $R - \{1\}$

$\dfrac{dy}{dx} = -\dfrac{1}{(x-1)^2} \Rightarrow f(x)$ is decreasing for all $x \in R - \{1\}$

Also f is a rational function of x.

33. **(b, c)** **:** $\because \quad f(x) = \left[\dfrac{1}{[x]}\right]$

$\therefore \quad f(x)$ is not defined when $[x] = 0$ *i.e.,* when $0 \le x < 1$

$\therefore \quad D_f = R - [0, 1) = (-\infty, 0) \cup [1, \infty)$

When $x = 1, f(x) = 1$

When $x > 1,\ 0 < \dfrac{1}{[x]} < 1 \ \Rightarrow \left[\dfrac{1}{[x]}\right] = 0$

When $-1 \le x < 0,\ [x] = -1 \ \Rightarrow \left[\dfrac{1}{[x]}\right] = -1$

When $-\infty < x < -1,\ [x] \le -2$

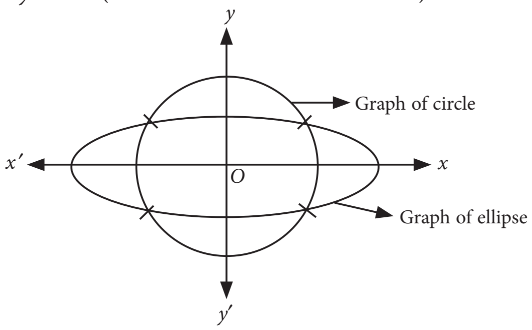
$\Rightarrow -1 < \dfrac{1}{[x]} < 0 \ \Rightarrow \left[\dfrac{1}{[x]}\right] = -1$

Hence, $R_f = \{0\} \cup \{-1\} \cup \{1\}$

34. **(b, c)** **:** Given $x^2 + 16y^2 = 144$

$\Rightarrow \quad \dfrac{x^2}{12^2} + \dfrac{y^2}{3^2} = 1$ (An ellipse whose major axis is x-axis) ...(i)

and $x^2 + y^2 = 25$ (a circle with radius $r = 5$ units) ...(ii)



From the graph of A & B, we note that $A \cap B$ have four points. We can solve the equation of circle & ellipse to find $A \cap B$.

35. **(a, b, c, d) :** (a) $\because$ Periods of $\sin x$ and $|\sin x|$ are 2π and π respectively.

$\therefore \quad$ Period of $f(x)$ = LCM of $\{2\pi, \pi\} = 2\pi$

(b) Periods of $\sin x$, $\cos x$, $\sec x$ and $\operatorname{cosec} x$ are 2π.
Hence, period of $g(x) = 2\pi$

(c) Period of $h(x)$ = LCM $\{2\pi, 2\pi\} = 2\pi$

(d) $p(x) = [x] + \left[x + \dfrac{1}{3}\right] + \left[x + \dfrac{2}{3}\right]$

$- \left(x + \left(x + \dfrac{1}{3}\right) + \left(x + \dfrac{2}{3}\right)\right) + 11$

$= 11 - \{x\} - \left\{x + \dfrac{1}{3}\right\} - \left\{x + \dfrac{2}{3}\right\}$

$\therefore \quad$ Period of $p(x)$ is 1.

36. **(b, d) :** Domain of $f = R$, Domain of $g = R - [-1, 0)$; because $-1 \le x < 0 \Rightarrow 1 + [x] = 0$

$\therefore \quad$ Domain of $f - g = R - [-1, 0)$; since $e^{-x} > 0$

Now, $(1 + [x])y > 0$

Either $y > 0 \Rightarrow 1 + [x] > 0$ or $y < 0 \Rightarrow 1 + [x] < 0$

$\Rightarrow \quad y \in R - \{0\}$

37. **(b, c)** **:** As $(0, 0)$ and $(x, g(x))$ are two vertices of an equilateral triangle, therefore length of the side of Δ is

$= \sqrt{(x-0)^2 + (g(x)-0)^2} = \sqrt{x^2 + (g(x))^2}$

$\therefore \quad$ The area of equilateral $\Delta = \dfrac{\sqrt{3}}{4}(x^2 + (g(x))^2)$

Also, area $= \dfrac{\sqrt{3}}{4}$ (Given)

$\therefore \quad$ We get, $\dfrac{\sqrt{3}}{4}(x^2 + (g(x))^2) = \dfrac{\sqrt{3}}{4}$

$\Rightarrow \quad (g(x))^2 = 1 - x^2 \ \Rightarrow \ g(x) = \pm\sqrt{1 - x^2}$

38. **(a, c, d) :** $U = \{x : x \in N$ and $2 \le x \le 12\}$,

$U = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

$A = \{x : x$ is an even prime$\}$

$\therefore \quad A = \{2\}$

$B = \{x : x$ is a factor of $24\}$

$\therefore \quad B = \{2, 3, 4, 6, 8, 12\}$

Now, $A - B$ is a set of those elements which are present in A but not in B

$\therefore \quad A - B = \{2\} - \{2, 3, 4, 6, 8, 12\} = \phi$

$\therefore \quad A - B$ is an empty set is a true statement.

Again $A - B$ means the elements of such set belongs to only A so cannot be equal to $B \cap A'$.

So, the statement $A - B = B \cap A'$ is not correct.

Again, $A' = \{3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$,

$B' = \{5, 7, 9, 10, 11\}$

$\therefore \quad A' - B' = A' - (A' \cap B') = \{3, 4, 6, 8, 12\} = B - A$, which is true and $(A \cap B)' = A' \cup B'$ is the De morgan's law, which is also true.

39. **(c) : (A) – (r), (B) – (s), (C) – (q), (D) – (p)**

(A) $f(x) = \log(x(ax^2 + bx + c) + (ax^2 + bx + c))$

$= \log(x+1)(ax^2 + bx + c)$

$\therefore$ Domain of $f(x) = (-1, \infty)$ as $ax^2 + bx + c > 0 \ \forall\ x$ (since $a > 0, \Delta < 0$)

(B) For $\ln(\tan^{-1}(x-1)(x-2)(x-3)x(e^x - 1))$ to be defined $x(e^x - 1)(x-1)(x-2)(x-3) > 0$

$\Rightarrow \quad x \in (1, 2) \cup (3, \infty)$

(C) $y = f(x) = \frac{x-1}{x+3}$ for $x \neq 2, -3$

and $x = \frac{3y+1}{1-y}$ for $y \neq 1$ and $x \to 2 \Rightarrow y \to \frac{1}{5}$

$\therefore$ Range of $f(x) = R - \left\{1, \frac{1}{5}\right\}$

(D) $f(x) = 1 - \cos^2\frac{x}{4} + \cos\frac{x}{4} = \frac{5}{4} - \left(\cos\frac{x}{4} - \frac{1}{2}\right)^2$

$\Rightarrow -1 \le f(x) \le \frac{5}{4}$

40. **(b) :** **(A) – (s), (B) – (s), (C) – (q), (D) – (q)**

(A) Given, $f(0) = 2, f(1) = 3$
and $f(x + 2) = 2f(x) - f(x + 1)$
Put $x = 0$
$f(2) = 2f(0) - f(1) = (2 \times 2) - 3 = 1$
Put $x = 1$
$f(3) = 2f(1) - f(2) = (2 \times 3) - 1 = 5$
Put $x = 2$
$f(4) = 2f(2) - f(3) = (2 \times 1) - 5 = -3$
Put $x = 3$
$f(5) = 2f(3) - f(4) = (2 \times 5) - (-3) = 13$

(B) $\sqrt{13} > 0 \Rightarrow f(\sqrt{13}) = (\sqrt{13})^2 = 13$

(C) Given, $f(x) + 2f(1 - x) = x^2 + 2, \forall x \in R$ ...(i)
Replace x by $1 - x$
$f(1 - x) + 2f(x) = (1 - x)^2 + 2$ ...(ii)
Multiply eq. (ii) by 2, we get
$2f(1 - x) + 4f(x) = 2(1 - x)^2 + 4$ ...(iii)
(iii) – (i) $\Rightarrow$ $3f(x) = 2(1 - x)^2 - x^2 + 2$
Put $x = 5$, we get $f(5) = 3$

(D) We have $f(x) = \frac{4^x}{4^x + 2}$

$$\Rightarrow f(1-x) = \frac{4^{1-x}}{4^{1-x}+2} = \frac{\frac{4}{4^x}}{\frac{4}{4^x}+2} = \frac{4}{4+2\cdot 4^x} = \frac{2}{4^x+2}$$

$\therefore$ $f(x) + f(1 - x) = 1$

$\Rightarrow f\left(\frac{1}{7}\right) + f\left(\frac{6}{7}\right) = 1$ ...(i)

$f\left(\frac{2}{7}\right) + f\left(\frac{5}{7}\right) = 1$ ...(ii)

$f\left(\frac{3}{7}\right) + f\left(\frac{4}{7}\right) = 1$ ...(iii)

Adding (i), (ii) and (iii), we get

$$\sum_{k=1}^{6} f\left(\frac{k}{7}\right) = 1+1+1 = 3$$

41. **(b) :** Let P, Q and R denote the sets of families buying newspaper A, B and C respectively. Let U be the universal set. Then,
$n(P)$ = 40% of 10,000 = 4000, $n(Q)$ = 20% of 10,000 = 2000, $n(R)$ = 10% of 10,000 = 1000,
$n(P \cap Q)$ = 5% of 10,000 = 500, $n(Q \cap R)$ = 3% of 10,000 = 300, $n(R \cap P)$ = 4% of 10,000 = 400
$n(P \cap Q \cap R)$ = 2% of 10,000 = 200 and $n(U)$ = 10,000
Required number
$= n(P \cap Q' \cap R') = n(P \cap (Q \cup R)')$
$= n(P) - n[P \cap (Q \cup R)]$ $\quad [\because n(A \cap B') = n(A) - n(A \cap B)]$
$= n(P) - n[(P \cap Q) \cup (P \cap R)]$
$= n(P) - [n(P \cap Q) + n(P \cap R) - n\{(P \cap Q) \cap (P \cap R)\}]$
$= n(P) - [n(P \cap Q) + n(P \cap R) - n(P \cap Q \cap R)]$
$= 4000 - (500 + 400 - 200) = 3300$

42. **(c) :** Required number
$= n(P' \cap Q' \cap R') = n[(P \cup Q \cup R)']$
$= n(U) - n(P \cup Q \cup R)$
$= n(U) - [n(P) + n(Q) + n(R) - n(P \cap Q) - n(Q \cap R) - n(R \cap P) + n(P \cap Q \cap R)]$
$= 10000 - [4000 + 2000 + 1000 - 500 - 300 - 400 + 200] = 4000$

43. **(b) :** Dividing throughout by $xy(x^2 - y^2)$, we get

$$\frac{f(x)f(y)}{x.y} = \frac{f(x+y)}{x+y} + \frac{f(x-y)}{x-y}$$

Let $F(x) = \frac{f(x)}{x}$

Then, $F(0) = \lim_{x\to 0}\frac{f(x)}{x} = \frac{f'(0)}{1} = 2$

$F(x)F(y) = F(x) + F(y)$
Put $y = 0$
$F(x)F(0) = F(x) + F(0)$
$\Rightarrow$ $2F(x) = F(x) + 2$
$\Rightarrow$ $F(x) = 2 \Rightarrow f(x) = 2x$
$\therefore$ $f(x)$ is an odd function

44. **(d) :** $f(\ln 2) = 2\ln 2 = \ln 4$

45. **(b) :** $\lim_{x\to 0}\left[\frac{f(x)}{\sin x}\right] = \lim_{x\to 0}\left[\frac{2x}{\sin x}\right] = 2$

46. **(4) :** $|x| \le \frac{3\pi}{4}$. Number of integers belonging to domain = 4

47. **(2) :** Putting $x = 0, y = 0$, we get $f(1) = 1$
Putting $x = 1, y = 1$, we get $f(2) = 2$
Similarly, we get $f(10) = 10$ $\therefore \frac{f(10)}{5} = 2$

48. **(3) :** Now $f(x) = \frac{x^4 + x^2 + 1}{x^2 - x + 1} \Rightarrow f(x) = x^2 + x + 1$

so, $f(\omega^n) = \omega^{2n} + \omega^n + 1 = 3$
($\because$ $\omega^n = 1$, when n is a multiple of 3).

49. **(3) :** $f(x) = [x] + \sum_{k=1}^{2008}\frac{x + k - [x+k]}{2008}$

$= [x] + \frac{1}{2008}\sum_{k=1}^{2008}(x + k - ([x] + k))$

$= [x] + \frac{1}{2008}[2008(x - [x])]$

$\Rightarrow$ $f(x) = [x] + x - [x] = x$ $\therefore f(3) = 3$

50. **(5) :** Replacing $\frac{1-x}{1+x}$ by x, we have

$f(x) \cdot f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$

$\Rightarrow f(x) = \pm x^n + 1 \Rightarrow f(x) = x^3 + 1$ $\quad (\because f(3) = 28)$

Now $\sum_{n=1}^{10}(f(n) - 1) = \sum_{n=1}^{10} n^3 = 55^2 = 3025$

$\Rightarrow \frac{1}{605}\left[\sum_{n=1}^{10}(f(n)-1)\right] = 5$

JEE MATHEMATICS

DPS-2

DAILY PRACTICE SHEET

Class XI

Trigonometric Functions and Principle of Mathematical Induction

SOLUTIONS

1. **(a) :** $\dfrac{\tan A}{1-\cot A}+\dfrac{\cot A}{1-\tan A}$

$=\dfrac{\sin^2 A}{\cos A(\sin A-\cos A)}+\dfrac{\cos^2 A}{\sin A(\cos A-\sin A)}$

$=\dfrac{\sin^3 A-\cos^3 A}{(\sin A-\cos A)\cos A\sin A}$

$=\dfrac{(\sin A-\cos A)(\sin^2 A+\sin A\cos A+\cos^2 A)}{(\sin A-\cos A)\sin A\cos A}$

$=\dfrac{1+\sin A\cos A}{\sin A\cos A}=1+\sec A\ \text{cosec}A$

2. **(a) :** Angle traced by the hour hand in 12 hours = 360°

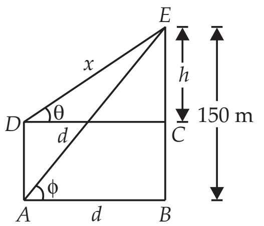
Angle traced by hour hand in 4 hrs 25 mins *i.e.*,

$4\dfrac{5}{12}\text{hrs}=\dfrac{53}{12}\text{hrs}=\left(\dfrac{360}{12}\times\dfrac{53}{12}\right)^{\circ}=\left(132\dfrac{1}{2}\right)^{\circ}$

Angled traced by minute hand in 25 minutes $=\dfrac{360^\circ}{60}\times 25=150^\circ$

$\therefore$ Required angle $=\left(150-132\dfrac{1}{2}\right)^{\circ}=17\dfrac{1}{2}^{\circ}$

3. **(d) :** Given that, $\tan\theta=\dfrac{4}{3}$ and $\tan\phi=\dfrac{5}{2}$



In ΔABE, $\tan\phi=\dfrac{150}{d}$

$\Rightarrow\ d=\dfrac{150}{\tan\phi}=150\times\dfrac{2}{5}=60$ m

In ΔDCE, $\tan\theta=\dfrac{h}{d}$

$\Rightarrow\ \dfrac{4}{3}=\dfrac{h}{60}\ \Rightarrow\ h=80$ m

Now in ΔDCE, $DE^2=DC^2+CE^2$

$\Rightarrow\ x^2=60^2+80^2=10000\Rightarrow x=100$ m

4. **(d) :** $\sin^2 5°+\sin^2 10°+\sin^2 15°+\ldots.+\sin^2 90°$

$=\sin^2 5°+\sin^2 10°+\ldots.+\sin^2 35°+\sin^2 40°+\sin^2 45°$
$+\sin^2(90°-40°)+\sin^2(90°-35°)+\ldots+\sin^2(90°-15°)+$
$\sin^2(90°-10°)+\sin^2(90°-5°)+\sin^2 90°$

$=\sin^2 5°+\sin^2 10°+\ldots.+\sin^2 35°+\sin^2 40°+\sin^2 45°$
$+\cos^2 40°+\cos^2 35°+\ldots+\cos^2 10°+\cos^2 5°+\sin^2 90°$

$=[(\sin^2 5°+\cos^2 5°)+(\sin^2 10°+\cos^2 10°)+\ldots$
$+(\sin^2 40°+\cos^2 40°)]+\sin^2 45°+\sin^2 90°$

$=[\underbrace{1+1+\ldots..+1}_{8\text{ times}}]+\left(\dfrac{1}{\sqrt{2}}\right)^2+1$

$=8+\dfrac{1}{2}+1=\dfrac{19}{2}=9.5$

5. **(a) :** $f(x)=\cos(\log_e x)$

Now, $f(x)\cdot f(y)-\dfrac{1}{2}\left[f(x/y)+f(xy)\right]$

$=\cos\log_e x\cdot\cos\log_e y-\dfrac{1}{2}\left[\cos\log_e\dfrac{x}{y}+\cos\log_e xy\right]$

$=\cos\log_e x\cdot\cos\log_e y-$

$\dfrac{1}{2}\left[2\cos\left(\dfrac{\log_e x/y+\log_e xy}{2}\right)\times\cos\left(\dfrac{\log_e xy-\log_e x/y}{2}\right)\right]$

$=\cos\log_e x\cdot\cos\log_e y-[\cos\log_e x\cdot\cos\log_e y]=0.$

6. **(b) :** Given that $\sec^2\theta=\dfrac{4xy}{(x+y)^2}$

$\therefore\ \cos^2\theta=\dfrac{(x+y)^2}{4xy}$ is possible only when $x=y$. We have $(x+y)^2=(x-y)^2+4xy$. If $x\neq y$, we deduce $(x+y)^2>4xy$. Now if xy is negative, $\dfrac{(x+y)^2}{4xy}$ is negative and it cannot be equal to $\cos^2\theta$, since $0\le\cos^2\theta\le 1$. Also if xy be positive, $\dfrac{(x+y)^2}{4xy}>1$ and it cannot also be equal to $\cos^2\theta$.

If however $x=y$ then $(x+y)^2=4xy$ and hence $\dfrac{(x+y)^2}{4xy}=1$ and $\cos^2\theta$ may have this value. It is also clear that x or y cannot be equal to zero.

7. **(a) :** $2\sin x=\sqrt{\dfrac{p}{q}}+\sqrt{\dfrac{q}{p}}$

$\therefore\ -2\le\sqrt{\dfrac{p}{q}}+\sqrt{\dfrac{q}{p}}\le 2$ $\qquad(\because\ -1\le\sin x\le 1)$

$\Rightarrow\ -2\sqrt{pq}\le p+q\le 2\sqrt{pq}$

$\Rightarrow\ 0\le\left(\sqrt{p}+\sqrt{q}\right)^2$ or $\left(\sqrt{p}-\sqrt{q}\right)^2\le 0\ \Rightarrow\ p=q$

8. **(d) :** Since, $\dfrac{1}{\cos 20°}-\dfrac{\sqrt{3}}{\sin 20°}=\dfrac{\sin 20°-\sqrt{3}\cos 20°}{\sin 20°\cos 20°}$

$=\dfrac{\frac{1}{2}\sin 20°-\frac{\sqrt{3}}{2}\cos 20°}{\frac{1}{2}\sin 20°\cos 20°}=\dfrac{\sin 20°\cos 60°-\cos 20°\sin 60°}{\frac{1}{4}(2\sin 20°\cos 20°)}$

$=\dfrac{4\sin(20°-60°)}{\sin 40°}=\dfrac{-4\sin 40°}{\sin 40°}=-4$

9. **(d) :** $u=(1+\cos\theta)(1+\cos 2\theta)-\sin\theta\sin 2\theta$

$=1+\cos 2\theta+\cos\theta+\cos 2\theta\cos\theta-\sin 2\theta\sin\theta$

$= (1 + \cos 2\theta) + (\cos 3\theta + \cos \theta)$
$= 2\cos^2 \theta + 2\cos 2\theta \cos\theta$
$= 2 \cos \theta (\cos \theta + \cos 2\theta)$

Now, $v = \sin\theta (1 + \cos 2\theta) + (1 + \cos\theta) \sin 2\theta$
$= \sin \theta + \sin 2\theta + \sin \theta \cos 2\theta + \sin 2\theta \cos \theta$
$= \sin 2\theta + (\sin 3\theta + \sin \theta)$
$= 2 \sin \theta \cos \theta + 2 \sin 2\theta \cos \theta$
$= 2 \cos \theta (\sin \theta + \sin 2\theta)$

$\therefore\ u^2 + v^2 = 4 \cos^2\theta [(\cos \theta + \cos 2\theta)^2 + (\sin \theta + \sin 2\theta)^2]$

$$= 4\cos^2\theta\left[4\cos^2\frac{3}{2}\theta\cdot\cos^2\frac{\theta}{2}+4\sin^2\frac{3}{2}\theta\cdot\cos^2\frac{\theta}{2}\right]$$

$$=16\cos^2\theta\cdot\cos^2\frac{\theta}{2}\left(\cos^2\frac{3}{2}\theta+\sin^2\frac{3}{2}\theta\right)$$

$$u^2+v^2=4(1+\cos 2\theta)(1+\cos\theta)$$

10. (d) : $\log \sin 1°\cdot\log \sin 2°... \log \sin 90°... \log \sin 179°$
$= \log \sin1°\cdot\log \sin2°... \log1 ... \log \sin179° = 0$

11. (a) : $\sqrt{2(1+\cos\theta)}=\sqrt{2\times 2\cos^2\frac{\theta}{2}}\ =2\cos\frac{\theta}{2}$

$\therefore\ \sqrt{2+\sqrt{2(1+\cos\theta)}}=\sqrt{2+2\cos\frac{\theta}{2}}$

$$=\sqrt{2\left(1+\cos\frac{\theta}{2}\right)}=\sqrt{2\times2\cos^2\frac{\theta}{4}}\ =2\cos\frac{\theta}{4}=2\cos\frac{\theta}{2^2}$$

and $\sqrt{2+\sqrt{2+\sqrt{2(1+\cos\theta)}}}$

$$=\sqrt{2+2\cos\frac{\theta}{4}}=\sqrt{2\times2\cos^2\frac{\theta}{8}}=2\cos\frac{\theta}{8}=2\cos\frac{\theta}{2^3}$$

Continuing the process, we get, if there are n number of 2's

$$\sqrt{2+\sqrt{2+\sqrt{2+...+\sqrt{2(1+\cos\theta)}}}}\ =2\cos\frac{\theta}{2^n}$$

12. (d) : $\frac{1}{\cos^2\alpha}+\frac{1}{1+\sin^2\alpha}+\frac{2}{1+\sin^4\alpha}+\frac{4}{1+\sin^8\alpha}$

$$=\frac{1}{1-\sin^2\alpha}+\frac{1}{1+\sin^2\alpha}+\frac{2}{1+\sin^4\alpha}+\frac{4}{1+\sin^8\alpha}$$

$$=\frac{4}{1-\sin^8\alpha}+\frac{4}{1+\sin^8\alpha}=\frac{8}{1-\sin^{16}\alpha}=\frac{8}{1-\frac{1}{5}}=\frac{8\times5}{4}=10$$

13. (a) : $\frac{2}{\sqrt{2+\sqrt{2+\sqrt{2(1+\cos4x)}}}}=\frac{2}{\sqrt{2+\sqrt{2+\sqrt{4\cos^2 2x}}}}$

$$=\frac{2}{\sqrt{2+\sqrt{4\cdot\cos^2 x}}}=\frac{2}{\sqrt{2+2\cos x}}=\frac{2}{\sqrt{4\cdot\cos^2\frac{x}{2}}}=\sec\frac{x}{2}$$

14. (d) : $\sin x + 2\sin2x - \sin3x = 3$
$\Rightarrow\ \sin x + 4\sin x\cos x - 3\sin x + 4 \sin^3 x = 3$
$\Rightarrow\ \sin x\{-2 + 4\cos x + 4\sin^2 x\} = 3$
$\Rightarrow\ \sin x(-2 + 4\cos x + 4 - 4\cos^2 x) = 3$
$\Rightarrow\ \sin x\{2 - (4\cos^2 x - 4\cos x + 1) + 1\} = 3$
$\Rightarrow\ \sin x\{3 - (1 - 2\cos x)^2\} = 3$
$\Rightarrow\ \sin x = 1, 2\cos x - 1 = 0$
$\Rightarrow\ x=\frac{\pi}{3}$ and $x=\frac{\pi}{2}$

So, no common solution

$\therefore$ There is no value of x which satisfies the equation.

15. (d) : Let $a = 2 \sin^4 x + 18 \cos^2 x$
and $b = 2 \cos^4 x + 18 \sin^2 x$
Now, $a - b = 2(\sin^4 x - \cos^4 x) + 18(\cos^2 x - \sin^2 x)$
$= 2(\sin^2 x - \cos^2 x) + 18(\cos^2 x - \sin^2 x) = 16 \cos2x$
$a + b = 2(\sin^4 x + \cos^4 x) + 18(\cos^2 x + \sin^2 x)$
$= 2\{(1 - 2 \sin^2 x \cos^2 x)\} + 18$
$= 20 - \sin^2 2x = 19 + \cos^2 2x$

The given equation becomes, $|\sqrt{a}-\sqrt{b}|=1$

On squaring both sides, we get

$a+b-2\sqrt{ab}=1$
$\Rightarrow\ (a + b - 1)^2 = 4ab \Rightarrow (a + b)^2 - 2(a + b) + 1 = 4ab$
$\Rightarrow\ (a - b)^2 - 2(a + b) + 1 = 0$
$\Rightarrow\ 256 \cos^2 2x - 2(19 + \cos^2 2x) + 1 = 0$
$\Rightarrow\ 254 \cos^2 2x - 37 = 0$

$\cos^2 2x = \frac{37}{254}=\lambda(\text{say})$ where $|\lambda| \le 1$

So, $\cos 2x = \pm\sqrt{\lambda}$

Since, $x \in [0, 2\pi]\ \therefore\ 2x \in [0, 4\pi]$ hence, we have 4 values of x.

16. (d) : Given, $\cos^4 x + \sin^4 x = 2 \cos(2x + \pi)\cos(2x - \pi)$
$\Rightarrow\ (\cos^2 x + \sin^2 x)^2 - 2\sin^2 x \cos^2 x = \cos 4x + \cos 2\pi$
$\Rightarrow\ 1-\frac{1}{2}\sin^2 2x=\cos4x+1$
$\Rightarrow\ -\frac{1}{2}\left(\frac{1-\cos4x}{2}\right)=\cos4x\ \Rightarrow\ -\frac{1}{4}=\frac{3}{4}\cos4x$
$\Rightarrow\ \cos4x=-\frac{1}{3}\ \Rightarrow\ \sin4x=\frac{2\sqrt{2}}{3}$

Again, $\cos4x=-\frac{1}{3}$

$\Rightarrow\ 1-2\sin^2 2x=-\frac{1}{3}\ \Rightarrow\ \sin2x=\sqrt{\frac{2}{3}}$

$\therefore\ x=\frac{n\pi}{2}+\frac{(-1)^n}{2}\sin^{-1}\left(\sqrt{\frac{2}{3}}\right)$

17. (d) : We know, $\cot\alpha-\tan\alpha\ =\ 2\cot2\alpha$
and $2 (\cot 2\alpha - \tan 2\alpha) = 2^2 \cot 2^2\alpha$

$2^2(\cot2^2\alpha-\tan2^2\alpha)\ =\ 2^3\cot2^3\alpha$
$\vdots\quad\vdots\quad\vdots\quad\vdots\quad\vdots$

By adding all identities, we get

$$\cot\alpha-\tan\alpha-2\tan2\alpha-2^2\tan2^2\alpha-...-2^{n-1}\tan2^{n-1}\alpha\ =\ 2^n\cot2^n\alpha$$

$$\Rightarrow\ \tan\alpha\ +\ 2\tan(2\alpha)\ +\ 4\tan(4\alpha)\ +....+\ 2^{n-1}\tan(2^{n-1}\alpha)\ +\ 2^n\cot2^n\alpha\ =\ \cot\alpha$$

18. (c) : On simplification, we get

$\sqrt{3}\sin x+\cos x=2\cos2x\ \Rightarrow\ \frac{\sqrt{3}}{2}\sin x+\frac{1}{2}\cos x=\cos2x$

$\therefore$ We have, $\cos\left(x-\frac{\pi}{3}\right)=\cos2x$

We have, $2x\pm\left(x-\frac{\pi}{3}\right)=2k\pi, k\in Z$

Then, $x=(6k-1)\frac{\pi}{3}$ or $(6k+1)\frac{\pi}{9}, k\in Z$

The values of x in $(-\pi, \pi)$ are $-\frac{\pi}{3},\frac{\pi}{9},\frac{7\pi}{9},-\frac{5\pi}{9}$

Their sum turns to be zero.

19. **(a) :** We have, $\dfrac{\tan 3x - \tan 2x}{1+\tan 3x \cdot \tan 2x} = 1$

$\Rightarrow \quad \tan(3x - 2x) = 1$

$\Rightarrow \quad \tan x = 1 = \tan\left(\dfrac{\pi}{4}\right)$

$\Rightarrow \quad x = n\pi + \dfrac{\pi}{4}, n \in Z$

But if $x = n\pi + \dfrac{\pi}{4}$, then

$\tan 2x = \tan 2\left(n\pi + \dfrac{\pi}{4}\right) = \tan\left(2n\pi + \dfrac{\pi}{2}\right)$

$= \tan\left(\dfrac{\pi}{2}\right) = \infty$

Thus, there is no value exist for which given equation holds. Hence, solution set = ϕ.

20. **(b) :** a, b, c are roots of equation

$x\sin\theta + y\sin 2\theta + z\sin 3\theta = \sin 4\theta$

$\Rightarrow \quad x\sin\theta + y(2\sin\theta\cos\theta) + z(3\sin\theta - 4\sin^3\theta)$
$= 4\sin\theta\cos\theta\cos 2\theta$

$\Rightarrow \cos^3\theta - \dfrac{z}{2}\cos^2\theta - \dfrac{y+2}{4}\cos\theta + \dfrac{z-x}{8} = 0$

21. **(b) :** $k = \dfrac{2\tan x}{1+\tan^2 x} = \sin 2x \quad (\because 0 < k < 1)$

$\Rightarrow \quad 2A = \pi - 2B \Rightarrow C = \dfrac{\pi}{2}$ and $\sin C = 1$

22. **(b) :** We have by cosine rule

$(x^2 - 1)^2 + (x^2 + x + 1)^2 - (2x + 1)^2$
$= 2(x^2 + x + 1)(x^2 - 1)\cos\dfrac{\pi}{6}$

$\Rightarrow \quad (x^2 - 1)^2 + \{(x^2 + 3x + 2)(x^2 - x)\}$
$= 2(x^2 + x + 1)(x^2 - 1) \times \dfrac{\sqrt{3}}{2}$

$\Rightarrow \quad (x^2 - 1)^2 + (x + 2)(x^2 - 1)x = (x^2 + x + 1)(x^2 - 1)\sqrt{3}$

$\Rightarrow \quad (x^2 - 1)\{x^2 - 1 + x^2 + 2x\} = (x^2 + x + 1)(x^2 - 1)\sqrt{3}$

As $x \neq \pm 1$, we have

$2x^2 + 2x - 1 = (x^2 + x + 1)\sqrt{3}$

$\Rightarrow \quad 2(x^2 + x) - 1 = \sqrt{3}(x^2 + x) + \sqrt{3}$

$\Rightarrow \quad (2 - \sqrt{3})(x^2 + x) = \sqrt{3} + 1 \Rightarrow x^2 + x = \dfrac{\sqrt{3}+1}{2-\sqrt{3}}$

$\Rightarrow \quad x^2 + x = (\sqrt{3}+1)(2+\sqrt{3})$

$\Rightarrow \quad x^2 + x - (\sqrt{3}+1)(2+\sqrt{3}) = 0$

$\Rightarrow \quad \{x - (\sqrt{3}+1)\}\{x + (2+\sqrt{3})\} = 0$

$\Rightarrow \quad x = \sqrt{3} + 1 \quad$ or $\quad x = -(2 + \sqrt{3})$ (Rejected)

23. **(b) :** From given, we have

$\tan(A + B) = \tan\dfrac{\pi}{4}$

$\therefore \quad \tan A + \tan B = 1 - \tan A \tan B$

$\Rightarrow \quad \tan A + \tan B + \tan A \tan B = 1$

$\Rightarrow \quad (1 + \tan A) + \tan B(1 + \tan A) = 1 + 1$
(by adding 1 on both sides)

$\Rightarrow \quad (1 + \tan A)(1 + \tan B) = 2$

24. **(a) :** $\tan\left(\dfrac{B}{4} + \dfrac{C}{4}\right) = \tan\left(\dfrac{\pi - A}{4}\right)$

$= \dfrac{\tan\frac{\pi}{4} - \tan\frac{A}{4}}{1 + \tan\frac{\pi}{4}\tan\frac{A}{4}} = \dfrac{1 - \tan\frac{A}{4}}{1 + \tan\frac{A}{4}} = \dfrac{\cos\frac{A}{4} - \sin\frac{A}{4}}{\cos\frac{A}{4} + \sin\frac{A}{4}}$

$= \dfrac{\cos^2\frac{A}{4} - \sin^2\frac{A}{4}}{1 + 2\sin\frac{A}{4}\cos\frac{A}{4}} = \dfrac{\cos\frac{A}{2}}{1 + \sin\frac{A}{2}}$

25. **(d) :** Given, $P(n) = \dfrac{n^5}{5} + \dfrac{n^3}{3} + \dfrac{7n}{15}$

Let $n = 1 \therefore P(1) = \dfrac{1}{5} + \dfrac{1}{3} + \dfrac{7}{15} = \dfrac{15}{15}$ (a natural number, true)

$n = 2 \therefore P(2) = \dfrac{32}{5} + \dfrac{8}{3} + \dfrac{14}{15} = \dfrac{150}{15} = 10$ (a natural number)

$\therefore \quad P(n)$ is true for $n = 2$

Similarly, by induction, we say that $P(n)$ is a natural number $\forall\, n \in N$.

26. **(b) :** Let $P(k)$ is true.

$P(k) = \displaystyle\sum_{k=1}^{n} k(k!)$

(Note : $k = 1, ..., r$ or, $k = 1, ..., n$ have same meaning as we have consider $r, n \in N$)

$= \displaystyle\sum_{k=1}^{n} (k+1-1)(k!)$

$= \displaystyle\sum_{k=1}^{n} (k+1)! - \sum_{k=1}^{n} (k!)$ for $k = n, n - 1, ..., 1$

$= ((n + 1)! - n! + n! - (n - 1)! + (n - 1)! - (n - 2)! + ...$
$+ 3! - 2! + 2! - 1!)$

$\therefore \quad P(k) = (n + 1)! - 1!.$

27. **(a) :** It can be checked by using PMI that statement-2 is true for all $n \in N$.
So, statement-2 is correct.
Now, $(n + 1)^7 - n^7 - 1 = \{(n + 1)^7 - (n + 1)\} - \{n^7 - n\}$
Using statement-2, $\{n + 1)^7 - (n + 1)$ and $\{n^7 - n)$ both are divisible by 7. Therefore, $(n + 1)^7 - n^7 - 1$ is also divisible by 7.
So, statement-1 is correct and statement-2 is a correct explanation for statement-1.

28. **(d) :** The first principle of mathematical induction states that if the basis step and the inductive states are prove, then $p(n)$ is true for all natural number $n \leq m$. As a first step, it is often $p(m+1)$ in terms of $p(n)$.

29. **(b) :** $P(n) : 2^{n+2} < 3^n$

Let $n = 1$, $P(1)$: $2^3 < 3^1$
i.e. $P(1) = 8 < 3$, false
Let $n = 2$, $P(2)$: $2^4 < 3^2$
i.e. $P(2) = 16 < 9$, false
Let $n = 3$, $P(3)$: $2^5 < 3^3$
i.e. $P(3) = 32 < 27$, false
Let $n = 4$
$\therefore \quad P(4) = 2^6 < 3^4$
i.e. $P(4) = 64 < 81$, which is true.
$\therefore \quad P(n) : 2^{n+2} < 3^n$ is true for $\forall\, n > 3, n \in N$

30. **(d) :** Statement-1

Let $P(n): \frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\ldots+\frac{1}{\sqrt{n}}>\sqrt{n}$

For $n = 2$, $P(2): \frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}>\sqrt{2}$ is true

Assume $P(n)$ is true for $n = k$, *i.e.,*

$$\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\ldots+\frac{1}{\sqrt{k}}>\sqrt{k} \quad \text{...(i)}$$

For $n = k + 1$, we have to show that

$$\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}}>\sqrt{k+1} \quad \text{...(ii)}$$

By assumption step, we get

$$\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\ldots+\frac{1}{\sqrt{k}}>\sqrt{k}$$

Adding $\frac{1}{\sqrt{k+1}}$ on both sides, we get

$$\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\ldots+\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}}>\sqrt{k}+\frac{1}{\sqrt{k+1}} \quad \text{...(iii)}$$

Statement-2

For $n = k$, $\sqrt{k(k+1)}<k+1$

$\Rightarrow \sqrt{k}\sqrt{k+1}<\sqrt{k+1}\sqrt{k+1} \Rightarrow \sqrt{k}<\sqrt{k+1}$

$\because \sqrt{k+1}>\sqrt{k}$ For $k \geq 2$

$\Rightarrow 1>\frac{\sqrt{k}}{\sqrt{k+1}} \Rightarrow \sqrt{k}>\frac{k}{\sqrt{k+1}}$ [Multiplying by $\sqrt{k}$]

$\Rightarrow \sqrt{k}>\frac{(k+1)-1}{\sqrt{k+1}} \Rightarrow \sqrt{k}>\sqrt{k+1}-\frac{1}{\sqrt{k+1}}$

$$\Rightarrow \sqrt{k}+\frac{1}{\sqrt{k+1}}>\sqrt{k+1} \quad \text{...(iv)}$$

From (iii) & (iv)

$$\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}}>\sqrt{k+1}$$

hence (ii) is true for $n = k + 1$.

hence $P(n)$ is true for $n \geq 2$.

So, Statement-1 and Statement-2 are correct but Statement-2 is not a correct explanation of Statement-1.

31. **(b, c, d) :** For the system to have non-trivial solution, we have

$$\begin{vmatrix} \sin\theta & -2\cos\theta & -a \\ -1 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix}=0$$

i.e. $\sin\theta + 4\cos\theta = 3a \Rightarrow -\frac{\sqrt{17}}{3}\leq a\leq\frac{\sqrt{17}}{3}$

$\Rightarrow a$ has three integer values.

32. **(a, b, c, d) :** $\frac{1+\sin\theta}{\cos\theta}=\frac{1}{\cos\theta}+\frac{\sin\theta}{\cos\theta}=\sec\theta+\tan\theta$

Also $\frac{1+\sin\theta}{\cos\theta}=\frac{(1+\sin\theta)(1-\sin\theta)}{\cos\theta(1-\sin\theta)}=\frac{\cos^2\theta}{\cos\theta\,(1-\sin\theta)}=\frac{\cos\theta}{1-\sin\theta}$

Again $\frac{1+\sin\theta}{\cos\theta}=\frac{\left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right)^2}{\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}}$

$$=\frac{\cos\frac{\theta}{2}+\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}=\frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}}=\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$$

33. **(a, b, c, d) :** As $\tan(2\pi - \theta) > 0$ and $-1<\sin\theta<-\frac{\sqrt{3}}{2}$,

Hence, $\frac{3\pi}{2}<\theta<\frac{5\pi}{3}$

Now $2\cos\theta\,(1-\sin\varphi)=\sin^2\theta\left(\tan\frac{\theta}{2}+\cot\frac{\theta}{2}\right)\cos\varphi-1$

$\Rightarrow$ $2\cos\theta(1 - \sin\varphi) = 2\sin\theta\cos\varphi - 1$

$\Rightarrow$ $2\cos\theta + 1 = 2\sin(\theta + \varphi)$

As $\theta\in\left(\frac{3\pi}{2},\frac{5\pi}{3}\right), 1<2\sin(\theta+\varphi)<2$

As $\theta+\phi\in\left(\frac{\pi}{6},\frac{5\pi}{6}\right)$ or $(\theta+\varphi)\in\left(\frac{13\pi}{6},\frac{17\pi}{6}\right)$

We have $\varphi\in\left(-\frac{3\pi}{2},-\frac{2\pi}{3}\right)\cup\left(\frac{2\pi}{3},\frac{7\pi}{6}\right)$

34. **(b,c) :** For $n = 1$, we have

$n-\frac{4^n}{3}-\frac{1}{3}=1-\frac{4}{3}-\frac{1}{3}=-\frac{2}{3}$

$n+\frac{4^{-n}}{3}-\frac{1}{3}=1+\frac{4^{-1}}{3}-\frac{1}{3}=\frac{3}{4}$

$\frac{1}{3}\left(\frac{1-4^n}{4^n}\right)+n=\frac{1}{3}\left(\frac{1-4}{4}\right)+1=\frac{3}{4}$

$n-\frac{4^{-n}}{3}+\frac{1}{3}=1-\frac{4^{-1}}{3}+\frac{1}{3}=\frac{5}{4}$

Also, for $n = 2$, we have

$n+\frac{4^{-n}}{3}-\frac{1}{3}=2+\frac{1}{48}-\frac{1}{3}=\frac{27}{16}$ and $\frac{3}{4}+\frac{15}{16}=\frac{27}{16}$

35. **(c, d) :** For $n = 1$, we have

$n(n^2 - 1) = 0$, which is divisible by 24 and 6.

For $n = 3$, we have

$n(n^2 - 1) = 3 \times (9 - 1) = 24$, which is divisible by 24 and 6.

For $n = 5$, we have

$n(n^2 - 1) = 3 \times (25 - 1) = 72$, which is divisible by 24 and 6.

36. **(c, d) :** $\sum_{m=1}^{6}\frac{1}{\sin\left(\theta+\frac{(m-1)\pi}{4}\right)}\cdot\frac{1}{\sin\left(\theta+\frac{m\pi}{4}\right)}=4\sqrt{2}$

$$\Rightarrow \sum_{m=1}^{6}\frac{\sin\left(\theta+\frac{m\pi}{4}-\theta-(m-1)\frac{\pi}{4}\right)}{\left\{\sin\left(\theta+\frac{(m-1)\pi}{4}\right)\sin\left(\theta+\frac{m\pi}{4}\right)\right\}}\cdot\frac{1}{\sin\frac{\pi}{4}}=4\sqrt{2}$$

$$\Rightarrow \sqrt{2}\left\{\cot\theta-\cot\left(\theta+\frac{\pi}{4}\right)+\cot\left(\theta+\frac{\pi}{4}\right)-\ldots.-\cot\left(\theta+\frac{3\pi}{2}\right)\right\}=4\sqrt{2}$$

$$\Rightarrow \sqrt{2}\left\{\cot\theta-\cot\left(\theta+\frac{3\pi}{2}\right)\right\}=4\sqrt{2}$$

$\Rightarrow$ $\cot\theta + \tan\theta = 4$ giving $\tan\theta=2\pm\sqrt{3}$

$\therefore$ $\theta=\frac{\pi}{12},\frac{5\pi}{12}$

37. (a, b) : $\tan^2\theta - \sin^2\theta = \sin^2\theta\left\{\frac{1}{\cos^2\theta} - 1\right\}$

$= \sin^2\theta\,\{\sec^2\theta - 1\} = \sin^2\theta\tan^2\theta$

and $\sec^2\theta\,\text{cosec}^2\,\theta = \frac{1}{\sin^2\theta\cos^2\theta}$

$= \frac{\sin^2\theta + \cos^2\theta}{\sin^2\theta\cos^2\theta} = \sec^2\theta + \text{cosec}^2\,\theta$

$[\because \text{cosec}^2\,\theta + \cot^2\theta \neq \text{cosec}^2\,\theta\cot^2\theta]$

38. (a, b) : $f(\cos 4\theta) = \frac{2}{2 - \sec^2\theta}$

Let $\cos 4\theta = t$

$\Rightarrow\quad 2\cos^2 2\theta - 1 = t \Rightarrow \cos^2 2\theta = \frac{t+1}{2}$

For $t = \frac{1}{3}$ we have $\cos^2 2\theta = \frac{2}{3}$

$\cos 2\theta = \sqrt{\frac{2}{3}}$ or $\cos 2\theta = -\sqrt{\frac{2}{3}}$

$f(\cos 4\theta) = \frac{2}{2 - \frac{1}{\cos^2\theta}} = \frac{2\cos^2\theta}{2\cos^2\theta - 1} = \frac{1 + \cos 2\theta}{\cos 2\theta} = 1 + \frac{1}{\cos 2\theta}$

Hence $f\left(\frac{1}{3}\right) = 1 + \sqrt{\frac{3}{2}}$ or $1 - \sqrt{\frac{3}{2}}$

39. (a) : (A) – (q, r), (B) – (r), (C) – (p, s), (D) – (p)

(A) We know that range of both functions $\sin\theta$ and $\cos\theta$ is $[-1, 1]$, therefore, $0 \le \sin^2\theta \le 1$ and also $0 \le \cos^2\theta \le 1$.

(B) $\frac{1}{\sin(90^\circ + \theta)\sin(90^\circ - \theta)\sec^4\theta}$

$= \frac{1}{\cos\theta\cos\theta\sec^4\theta} = \frac{\cos^4\theta}{\cos^2\theta} = \cos^2\theta$

(C) Let $\tan\theta = x$, then

$\tan\theta - \tan^2\theta = x - x^2 = -(x^2 - x)$

$= \frac{1}{4} - (x^2 - x + \frac{1}{4}) = \frac{1}{4} - \left(x - \frac{1}{2}\right)^2 \le \frac{1}{4}$ for all $x \in R$

Hence, maximum value of $\tan\theta - \tan^2\theta$ is $\frac{1}{4}$.

Note : $\frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots = \frac{\frac{1}{8}}{1 - \frac{1}{2}} = \frac{1}{4}$.

(D) $\sin^6\theta + \cos^6\theta$

$= (\sin^2\theta + \cos^2\theta)^3 - 3\sin^2\theta\cos^2\theta\,(\sin^2\theta + \cos^2\theta)$

$= 1 - 3\sin^2\theta\cos^2\theta = 1 - \frac{3}{4}(\sin 2\theta)^2$

As $0 \le \sin^2 2\theta \le 1$,

Therefore, $0 \ge -\frac{3}{4}\sin^2 2\theta \ge -\frac{3}{4}$

$\Leftrightarrow\quad 1 \ge 1 - \frac{3}{4}\sin^2 2\theta \ge 1 - \frac{3}{4}$

$\Leftrightarrow\quad 1 \ge \sin^6\theta + \cos^6\theta \ge \frac{1}{4}$

Hence, minimum value of $\sin^6\theta + \cos^6\theta$ is $\frac{1}{4}$.

40. (a) : (A) – (r), (B) – (r), (C) – (s), (D) – (p)

(A) $\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14} = \frac{1}{8}$

$\Rightarrow\quad \sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}\sin\frac{7\pi}{14} = \frac{1}{8}\qquad \left(\because \sin\frac{7\pi}{14} = \sin\frac{\pi}{2} = 1\right)$

$\Rightarrow\quad \frac{1}{2}\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}\sin\frac{7\pi}{14} = \frac{1}{16}$.

(B) We know that

$\sin A\sin(60^\circ - A)\sin(60^\circ + A) = \frac{1}{4}\sin 3A$

Put $A = 10^\circ$ to obtain

$\sin 10^\circ\sin 50^\circ\sin 70^\circ = \frac{1}{4}\sin 30^\circ$

$\Rightarrow\quad \sin 10^\circ\sin 50^\circ\sin 70^\circ = \frac{1}{4}\times\frac{1}{2}$

$\Rightarrow\quad \sin 10^\circ\sin 30^\circ\sin 50^\circ\sin 70^\circ$

$= \frac{1}{8}\sin 30^\circ = \frac{1}{8}\times\frac{1}{2} = \frac{1}{16}$.

(C) $\sqrt{3}\,(\cos 20^\circ\cos 40^\circ\cos 60^\circ\cos 80^\circ)$

$= \sqrt{3}\,(\cos(90^\circ - 70^\circ)\cos(90^\circ - 50^\circ) \times \cos(90^\circ - 30^\circ)\cos(90^\circ - 10^\circ))$

$= \sqrt{3}\,(\sin 70^\circ\sin 50^\circ\sin 30^\circ\sin 10^\circ) = \frac{\sqrt{3}}{16}$.

(D) $2\sin 6^\circ\sin 42^\circ\sin 66^\circ\sin 78^\circ$

$= 2\,(\cos 84^\circ\cos 48^\circ\cos 24^\circ\cos 12^\circ)\ (\because \sin\theta = \cos(90^\circ - \theta))$

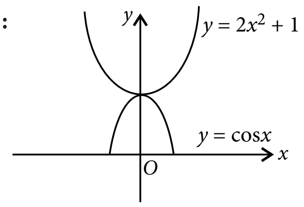
$= 2\,(-\cos 12^\circ\cos 24^\circ\cos 48^\circ\cos 96^\circ)$

$(\because \cos 84^\circ = -\cos(180^\circ - 84^\circ) = -\cos 96^\circ)$

$= 2\left(-\frac{\sin 192^\circ}{2^4\sin 12^\circ}\right)\quad \left(\because \cos\theta\cos 2\theta\cos 4\theta\cos 8\theta = \frac{\sin 16\theta}{16\sin\theta}\right)$

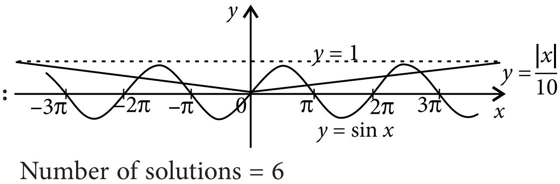
$= 2\left(-\frac{\sin(180^\circ + 12^\circ)}{16\sin 12^\circ}\right) = \frac{2}{16} = \frac{1}{8}$

41. (b) :



Only one point of intersection

42. (b) :



$\therefore\quad$ Number of solutions = 6

43. (b) **44. (c)** **45. (b)**

(43 – 45) : Clearly θ_1 and θ_0 lie on $\frac{\cos\theta}{\cos\theta_2} + \frac{\sin\theta}{\sin\theta_2} = 1$

$\Rightarrow\quad \left(\frac{\sin\theta}{\sin\theta_2}\right)^2 = \left(1 - \frac{\cos\theta}{\cos\theta_2}\right)^2$

$\Rightarrow\quad \frac{\sin^2\theta}{\sin^2\theta_2} = 1 + \frac{\cos^2\theta}{\cos^2\theta_2} - \frac{2\cos\theta}{\cos\theta_2}$

$$\Rightarrow \frac{1-\cos^2\theta}{\sin^2\theta_2} = 1 + \frac{\cos^2\theta}{\cos^2\theta_2} - \frac{2\cos\theta}{\cos\theta_2}$$

$$\Rightarrow \left(\frac{1}{\cos^2\theta_2} + \frac{1}{\sin^2\theta_2}\right)\cos^2\theta - \frac{2}{\cos\theta_2}\cos\theta + 1 - \frac{1}{\sin^2\theta_2} = 0$$

which is a quadratic equation in $\cos\theta$.
It has two roots $\cos\theta_0$, $\cos\theta_1$

$$\Rightarrow \cos\theta_1 . \cos\theta_0 = \frac{1-\frac{1}{\sin^2\theta_2}}{\frac{1}{\cos^2\theta_2}+\frac{1}{\sin^2\theta_2}}$$

$$= \frac{-\cos^2\theta_2}{\frac{1}{\cos^2\theta_2}} = -\cos^4\theta_2 \qquad \text{....(i)}$$

Similarly, $\left(\frac{1}{\cos^2\theta_2} + \frac{1}{\sin^2\theta_2}\right)\sin^2\theta$

$$-\frac{2}{\sin\theta_2}\sin\theta + 1 - \frac{1}{\cos^2\theta_2} = 0$$

$\Rightarrow \quad \sin\theta_1 \sin\theta_0 = -\sin^4\theta_2$ ...(ii)

$\therefore$ From (i) and (ii)

$$\frac{\cos\theta_1\cos\theta_0}{\cos^2\theta_2} + \frac{\sin\theta_1 . \sin\theta_0}{\sin^2\theta_2} = -\cos^2\theta_2 - \sin^2\theta_2 = -1$$

46. **(2) :** $\left[(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})(\sqrt{2}-1)+1\right]^2 = \left[\sqrt{2}\right]^2 = 2$

47. **(4) :** $|x| + |y| = 2 \Rightarrow |x|, |y| \in [0, 2]$

Also $\sin\left(\frac{\pi x^2}{3}\right) = 1 \Rightarrow \frac{\pi x^2}{3} = (4n+1)\frac{\pi}{2} \Rightarrow x^2 = (4n+1)\frac{3}{2}$

$\because \quad |x| \in [0, 2]$, then only possible value of $x^2 = 3/2$

$\therefore |x| = \sqrt{\frac{3}{2}}, \ |y| = 2 - \sqrt{\frac{3}{2}}$

Hence, total number of ordered pairs is 4.

48. **(8) :** $\frac{\sec^4\alpha}{\tan^2\beta} + \frac{\sec^4\beta}{\tan^2\alpha}$

Let $\tan^2\alpha = a$, $\tan^2\beta = b$

$$\frac{(1+a)^2}{b} + \frac{(1+b)^2}{a} = \frac{a^2+1}{b} + \frac{b^2+1}{a} + 2\left(\frac{a}{b}+\frac{b}{a}\right)$$

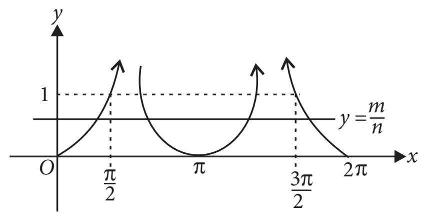
Since $\frac{\frac{a}{b}+\frac{b}{a}}{2} \ge 1$

$$\Rightarrow \frac{\frac{a^2+1}{b}+\frac{b^2+1}{a}}{2} \ge \sqrt{\left(a+\frac{1}{a}\right)\left(b+\frac{1}{b}\right)}$$

$$a + \frac{1}{a} \ge 2 \ ; \ b + \frac{1}{b} \ge 2$$

$$\therefore \ \text{Min}\left(\frac{\sec^4\alpha}{\tan^2\beta} + \frac{\sec^4\beta}{\tan^2\alpha}\right) = 4 + 4 = 8$$

49. **(4) :** $|\tan x| = \frac{m}{n}; \ 0 < \frac{m}{n} < 1$



50. **(8) :** $(\tan 20° + \tan 80°) - (\tan 60° - \tan 40°)$

$$= \frac{\sin 100°}{\cos 20° \cos 80°} - \frac{\sin 20°}{\cos 60° \cos 40°}$$

$$= \frac{\sin 80° \cos 60° \cos 40° - \sin 20° \cos 20° \cos 80°}{\cos 20° \cos 40° \cos 60° \cos 80°}$$

$$= \frac{\frac{1}{2}\sin 40°}{\frac{1}{16}} = 8\sin 40°$$. Then $\lambda = 8$

JEE MATHEMATICS

DPS-3

DAILY PRACTICE SHEET

Class XI

Complex Numbers, Quadratic Equations and Linear Inequalities

SOLUTIONS

1. **(a) :** $(1+i\sqrt{3})^n+(1-i\sqrt{3})^n$

$$=2^n\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^n+2^n\left(\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)^n$$

$$=2^n\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^n+2^n\left(\cos\frac{\pi}{3}-i\sin\frac{\pi}{3}\right)^n$$

$$=2^n\left(\cos\frac{n\pi}{3}+i\sin\frac{n\pi}{3}+\cos\frac{n\pi}{3}-i\sin\frac{n\pi}{3}\right)$$

(Using De-moivre's theorem)

$$=2^{n+1}\cos\frac{n\pi}{3}$$

2. **(c) :** $(1 + i)(1 + 2i)(1 + 3i) \ldots. (1 + ni) = x + iy$ ...(i)

$\Rightarrow \overline{(1+i)(1+2i)(1+3i)\ldots.(1+ni)}=\overline{(x+iy)}$

$\Rightarrow (1 - i)(1 - 2i)(1 - 3i) \ldots (1 - ni) = x - iy$ ... (ii)

Multiplying (i) and (ii), we get

$2 \cdot 5 \cdot 10 \ldots. (1 + n^2) = x^2 + y^2$

3. **(d) :** $2\left(\frac{1}{8!}+\frac{1}{2!6!}\right)+\frac{1}{4!4!}=\frac{2^a}{b!}$ ... (i)

Consider

$$2\left(\frac{1}{8!}+\frac{1}{2!6!}\right)+\frac{1}{4!4!}=\frac{1}{8!}\left[\frac{2!\,8!}{0!\,8!}+\frac{2!\,8!}{2!\,6!}+\frac{8!}{4!\,4!}\right]$$

$$=\frac{1}{8!}\left[2\cdot{}^8C_0+2\cdot{}^8C_2+{}^8C_4\right]$$

$$=\frac{1}{8!}\left[{}^8C_0+{}^8C_8+{}^8C_2+{}^8C_6+{}^8C_4\right]$$

$$=\frac{1}{8!}\left[{}^8C_0+{}^8C_2+{}^8C_4+{}^8C_6+{}^8C_8\right]=\frac{1}{8!}\left[2^{8-1}\right]$$ ...(ii)

Equating (i) and (ii), we have

$$\frac{2^7}{8!}=\frac{2^a}{b!}\ \Rightarrow\ a=7,\ b=8$$

Again, $x = 3 + 5i$

$\Rightarrow x - 3 = 5i \ \Rightarrow\ x^2 - 6x + 34 = 0$ ... (iii)

$\therefore\ x^3 - 8x^2 + 46x - 60 = x(x^2 - 6x + 34) - 2(x^2 - 6x + 34) + 8$

$= 8 = b$ (using (iii))

4. **(c) :** Given, $z = x + iy$ and $\arg\left(\frac{z-1}{z+1}\right)=\frac{\pi}{2}$ ...(i)

$$\therefore\ \frac{z-1}{z+1}=\frac{x+iy-1}{x+iy+1}\times\frac{(x+1)-iy}{(x+1)-iy}=\frac{\{(x^2-1)+y^2\}+i\{2y\}}{(x+1)^2+y^2}$$

Now,

$$\arg\left(\frac{z-1}{z+1}\right)=\arg\left(\frac{(x^2+y^2-1)}{(x+1)^2+y^2}+i\left(\frac{2y}{(x+1)^2+y^2}\right)\right)$$

$$\Rightarrow\ \frac{\pi}{2}=\tan^{-1}\left(\frac{2y}{x^2+y^2-1}\right)$$ [Using (i)]

$$\Rightarrow\ \tan\frac{\pi}{2}=\frac{2y}{x^2+y^2-1}\Rightarrow\frac{1}{0}=\frac{2y}{x^2+y^2-1}$$

$\Rightarrow x^2 + y^2 = 1$

5. **(d) :** Let $z=\sin\frac{\pi}{5}+i\left(1-\cos\frac{\pi}{5}\right)$

Putting $\sin\frac{\pi}{5}=r\cos\theta$ and $1-\cos\frac{\pi}{5}=r\sin\theta$ ...(i)

$$\therefore\ \tan\theta=\frac{1-\cos\frac{\pi}{5}}{\sin\frac{\pi}{5}}=\frac{2\sin^2\frac{\pi}{10}}{2\sin\frac{\pi}{10}\cos\frac{\pi}{10}}$$

$$\Rightarrow\ \tan\theta=\tan\frac{\pi}{10}\Rightarrow\theta=\frac{\pi}{10}$$

6. **(c) :** Let $z = x + iy$

$\therefore\ z - 1 = (x - 1) + iy,\ z + 1 = (x + 1) + iy$

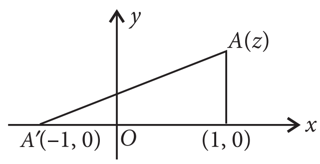
When $|z - 1| < |z + 1|$ (or $x > 0$)

$\Rightarrow |z| = |z - 1|$

$\Rightarrow x^2 + y^2 = (x - 1)^2 + y^2$

$\Rightarrow (x - 1)^2 - x^2 = 0$

$\Rightarrow\ x = 1/2$



Now, $z+\bar{z}=2x=2\left(\frac{1}{2}\right)=1=\omega^3$

Again, when $|z - 1| > |z + 1|$ (or $x < 0$)

$\therefore\ |z| = |z + 1|\ \Rightarrow\ x^2 + y^2 = (x + 1)^2 + y^2 \Rightarrow x = -1/2$

$$\therefore\ z+\bar{z}=2x=2\left(-\frac{1}{2}\right)=-1=i^2$$

Thus, $z+\bar{z}=\omega^3$ or i^2

7. **(a) :** $\cos A=\frac{1}{\sqrt{2}}\Rightarrow A=\frac{\pi}{4}$

$$\therefore\ \frac{z_3-z_1}{z_2-z_1}=\left|\frac{z_3-z_1}{z_2-z_1}\right|\text{cis}\,(\pi/4)$$

$$\Rightarrow\ \left(\frac{z_3-z_1}{z_2-z_1}\right)^4=\left(\frac{\sqrt{2}}{\sqrt{3}+1}\right)^4e^{i\pi}$$

$$\Rightarrow\ \left(\frac{z_3-z_1}{z_2-z_1}\right)^4=-\left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^4$$

8. **(c) :** $S=\sum_{n=0}^{100}(i)^{\lfloor n}$

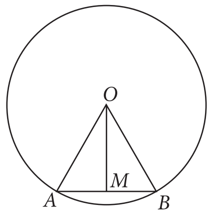
$$S=(i)^{\lfloor 0}+(i)^{\lfloor 1}+(i)^{\lfloor 2}+\ldots$$

$$=i+i+i^2+i^6+i^{24}+(i)^{\lfloor 5}+(i)^{\lfloor 6}+\ldots+(i)^{\lfloor 100}=95+2i$$

9. (a) : $A(5, -4)$, $B(-3, 2)$ subtends an angle $\dfrac{\pi}{4}$ at $C(z)$ on the circle.

Hence $\dfrac{\pi}{2}$ at centre.

$OM \perp AB \therefore AM = \dfrac{AB}{2}$



$= \dfrac{\sqrt{64+36}}{2} = \dfrac{10}{2} = 5$

Radius $= \sqrt{25+25} = \sqrt{50} = 5\sqrt{2}$

10. (a) : $z + \dfrac{1}{z} = 2\cos 15°$

$\Rightarrow z^2 - 2z\cos15° + 1 = 0 \Rightarrow z = \cos15° \pm i\sin15°$

$\therefore z = \cos15° + i\sin15°$ or $z = \cos15° - i\sin15°$

Consider $z = \cos15° + i\sin15°$

$$\therefore z^{2017} + \frac{1}{z^{2017}} = (\cos 15° + i\sin 15°)^{2017} + \frac{1}{(\cos 15° + i \sin 15°)^{2017}}$$

$= [\cos(15 \times 2017)° + i\sin(15 \times 2017)°] + [\cos(15 \times 2017)° - i\sin(15 \times 2017)°]$

(Using $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$)

$= 2\cos15°$ $[\because 15 \times 2017° = 30225° = (360 \times 84)° + 15°$ and $\cos(2n\pi + \theta) = \cos\theta]$

$$= \frac{\sqrt{3}+1}{\sqrt{2}} \left[\because \cos 15° = \cos(45° - 30°) = \frac{\sqrt{3}+1}{2\sqrt{2}}\right]$$

$$\therefore z^{2017} + \frac{1}{z^{2017}} = \frac{\sqrt{3}+1}{\sqrt{2}}$$

11. (a) : $|3z + 9 - 7i| = |3z + 6 - 3i + 3 - 4i| \le |3(z + 2 - i)| + |3 - 4i|$

$= 3|z + 2 - i| + \sqrt{3^2 + 4^2} = 3(5) + 5 = 20$

12. (c) : $Z_r = cis\left(\dfrac{\pi}{2^r}\right)$

$$\therefore Z_1 \cdot Z_2 \ldots\infty = cis\left(\frac{\pi}{2} + \frac{\pi}{2^2} + \ldots\infty\right) = cis\,\pi = -1$$

13. (d) : In the equation $x^2 + 2x + 3 = 0$, both the roots are imaginary.

Since $a, b, c \in R$, we have $\dfrac{a}{1} = \dfrac{b}{2} = \dfrac{c}{3}$

Hence, $a : b : c :: 1 : 2 : 3$

14. (c) : We have, $|x^2 - 1| = \begin{cases} x^2 - 1 & \forall\ x \le -1 \text{ or } x \ge 1 \\ -(x^2 - 1) & \forall\ -1 < x < 1 \end{cases}$

and for $|x^2 - x - 2| = \begin{cases} x^2 - x - 2 & \forall\ x \le -1 \text{ or } x \ge 2 \\ -(x^2 - x - 2) & \forall\ -1 < x < 2 \end{cases}$

$\therefore$ Set of values of x are -1, 1 and 2 for which four intervals are possible like $x \le -1$, $-1 < x < 1$, $1 \le x < 2$ and $x \ge 2$

Case I : When $x \le -1$ then from $2|x^2 - 1| - |x^2 - x - 2| = 0$, we have

$2(x^2 - 1) - (x^2 - x - 2) = 0 \Rightarrow x(x + 1) = 0$

$\therefore x = 0, -1$ (rejecting 0 as $0 \notin (-\infty, -1]$)

So $x = -1$ is the only solution.

Case II : When $-1 < x < 1$, we have from the equation

$-2(x^2 - 1) + x^2 - x - 2 = 0 \Rightarrow x(x + 1) = 0$

$\therefore x = 0$, $x = -1$ but $x = 0$ is the only solution as $-1 < x < 1$

Case III : When $1 \le x < 2$, then we have from the given equation

$2(x^2 - 1) + x^2 - x - 2 = 0 \Rightarrow 3x^2 - x - 4 = 0$

$\Rightarrow x = -1$ or $x = 4/3$

but $x = -1 \notin [1, 2)$ so $x = 4/3$ is the only solution in this case.

Case IV : When $x \ge 2$ then from given equation, we have

$2(x^2 - 1) - (x^2 - x - 2) = 0 \Rightarrow x^2 + x = 0$

$\Rightarrow x(x + 1) = 0$

$\Rightarrow x = 0$ or $x = -1$ but these values $\notin [2, \infty)$. So in this case there is no solution.

Now, from above four cases, we note that the values of $x = \{-1, 0, 4/3\}$ satisfies the equation $2|x^2 - 1| - |x^2 - x - 2| = 0$

$\therefore$ Thus, required values of x are $x = -1, 0, 4/3$

15. (a) : $\log_4\left\{\log_2(\sqrt{x+8} - \sqrt{x})\right\} = 0$

$\Rightarrow 4^0 = \log_2\left(\sqrt{x+8} - \sqrt{x}\right) \Rightarrow 2^1 = \sqrt{x+8} - \sqrt{x}$

$\Rightarrow 4 = x + 8 + x - 2\sqrt{x^2 + 8x}$

$\Rightarrow 2\sqrt{x^2 + 8x} = 2x + 4 \Rightarrow x^2 + 8x = x^2 + 4 + 4x$

$\Rightarrow 4x = 4 \Rightarrow x = 1$

16. (c) : We have, $x^2 + bx - 1 = 0$ ...(i)

and $x^2 + x + b = 0$ ...(ii)

On subtracting (i) from (ii), we get

$x(1 - b) + 1 + b = 0 \Rightarrow x = \dfrac{b+1}{b-1}$.

On putting value of x in (ii), we get

$$\left(\frac{b+1}{b-1}\right)^2 + \left(\frac{b+1}{b-1}\right) + b = 0$$

$\Rightarrow (b + 1)^2 + (b + 1)(b - 1) + b(b - 1)^2 = 0$

$\Rightarrow b^3 + 3b = 0 \Rightarrow b(b^2 + 3) = 0$

But $b \ne 0$, $\therefore b^2 = -3$

$\Rightarrow b = \pm\sqrt{3}i \Rightarrow |b| = \sqrt{3}$

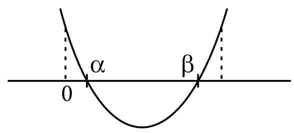
17. (b) : Let roots be am and bm

$\therefore \quad am + bm = -\dfrac{q}{p}$ ◎ $am \cdot bm = \dfrac{r}{p}$

$\Rightarrow (a + b)^2 m^2 = \dfrac{q^2}{p^2}$ & $abm^2 = \dfrac{r}{p}$

On dividing, we get $\dfrac{ab}{(a+b)^2} = \dfrac{r}{p} \cdot \dfrac{p^2}{q^2} = \dfrac{pr}{q^2}$.

18. (c) : $f(0) \cdot f(1) > 0$



$\Rightarrow k \cdot (1 - 3 + k) > 0$

$\Rightarrow k(k - 2) > 0$

$\Rightarrow k < 0$ or $k > 2$

$\therefore$ Number of values of k is infinite.

19. (c) : Only option (c) is always false.

20. (a) : $ax^2 + bx + c = 0$ has two complex conjugate roots only, if all the coefficients are real. If all the coefficients are not real, then it is not necessary that both the roots are imaginary.

Hence, statement 1 is true.

Now, equation $x^2 - 3x + 4 = 0$ has two complex conjugate roots. If $ax^2 + bx + c = 0$ has all coefficients real, then there will be two common roots. But if there is only one root common, then atleast one of a, b, c must be non-real.

Thus, both the statements are true and statement 2 is correct explanation of statement 1.

21. (b) : We have, $(\cos\theta + i\sin\theta)^{3/5} = (\cos 3\theta + i\sin 3\theta)^{1/5}$

$= [\cos(2r\pi + 3\theta) + i\sin(2r\pi + 3\theta)]^{1/5}$, where $r = 0, 1, 2, 3, 4$

$= e^{i\left(\frac{2r\pi + 3\theta}{5}\right)}$, $r = 0, 1, 2, 3, 4$

Hence, product of all values of $(\cos\theta + i\sin\theta)^{3/5}$ is

$$e^{i\frac{3\theta}{5}}e^{i\left(\frac{2\pi+3\theta}{5}\right)}e^{i\left(\frac{4\pi+3\theta}{2}\right)}e^{i\left(\frac{6\pi+3\theta}{5}\right)}e^{i\left(\frac{8\pi+3\theta}{5}\right)}$$

$= e^{i3\theta + 4\pi i} = e^{4i\pi}e^{i3\theta} = \cos 3\theta + i \sin 3\theta$

Also, product of roots of the equation $x^5 - 1 = 0$ is 1. Hence, statement 2 is true, but it is not a correct explanation of statement 1.

22. (d) : Since α is a root of $a^2x^2 + bx + c = 0$

$\Rightarrow a^2\alpha^2 + b\alpha + c = 0$

$\therefore\ b\alpha + c = -a^2\alpha^2$ ...(i)

β is a root of $a^2x^2 - bx - c = 0$

$\Rightarrow a^2\beta^2 - b\beta - c = 0$

$\Rightarrow a^2\beta^2 = b\beta + c$ ...(ii)

Now, $f(x) = a^2x^2 + 2bx + 2c$

$\therefore\ f(\alpha) = a^2\alpha^2 + 2(b\alpha + c)$

$= a^2\alpha^2 + 2(-\alpha^2a^2) = -a^2\alpha^2 < 0$ (From (i))

Similarly, $f(\beta) = 3a^2\beta^2 > 0$

Now $f(\alpha)$ and $f(\beta)$ are of opposite sign and $0 < \alpha < \beta$ (given)

$\therefore\ \exists$ exactly one real value between α and β say γ such that $f(\gamma) = 0$.

23. (a) : As given, $\alpha+\beta = p,\ \alpha\beta = q,\ \alpha'+\beta' = p',\ \alpha'\beta' = q'$

Now, $(\alpha-\alpha')^2+(\beta-\alpha')^2+(\alpha-\beta')^2+(\beta-\beta')^2$

$= 2(\alpha^2+\beta^2)+2(\alpha'^2+\beta'^2)-2\alpha'(\alpha+\beta)-2\beta'(\alpha+\beta)$

$= 2\left\{(\alpha+\beta)^2-2\alpha\beta+(\alpha'+\beta')^2-2\alpha'\beta'-(\alpha+\beta)(\alpha'+\beta')\right\}$

$= 2\{p^2-2q+p'^2-2q'-pp'\}$

24. (c) : $\alpha^2 - a\alpha = \beta^2 - a\beta = -(a+b)$

$\Rightarrow\ \dfrac{1}{\alpha^2 - a\alpha} + \dfrac{1}{\beta^2 - a\beta} + \dfrac{2}{a+b} = 0$

25. (b) : Given equation is

$p\left(\dfrac{x}{z}\right)^2 - (q-r)\left(\dfrac{x}{z}\right) - p = 0$

which is quadratic in $\left(\dfrac{x}{z}\right)$,

$D = (q-r)^2 + 4p^2 = +ve$ quantity > 0

Now, D can not be equal to zero, so can't have equal roots. Again, $D > 0$ means both roots are real but distinct.

26. (b) : Given, $4^{-x+(1/2)} - 7\cdot(2^{-x}) - 4 < 0$

$\Rightarrow\ 2\cdot 4^{-x} - 7\cdot(2^{-x}) - 4 < 0$

$\Rightarrow\ 2\cdot(2^{-x})^2 - 7\cdot(2^{-x}) - 4 < 0 \Rightarrow\ (2^{-x} - 4)(2\cdot 2^{-x} + 1) < 0$

$\Rightarrow\ (1 - 4\cdot 2^x)(2 + 1\cdot 2^x) < 0 \Rightarrow\ (1/4 - 2^x)(2 + 2^x) < 0$

$\Rightarrow\ (2^x - 1/4)(2^x + 2) > 0$

So, $2^x > 1/4$ or $2^x + 2 < 0$

But $2^x + 2 < 0$ is impossible $\forall\ x \in R$, as $2^x > 0\ \forall\ x \in R$.

Thus, $2^x > 1/4 \Rightarrow\ x > -2$. Thus, $x \in (-2, \infty)$

27. (a) : Case I : $|x-1| = x - 1,\ x \geq 1$

$\therefore\ \dfrac{|x-1|}{x+2} < 1 \Rightarrow\ \dfrac{x-1}{x+2} < 1 \Rightarrow\ \dfrac{x-1}{x+2} - 1 < 0 \Rightarrow\ \dfrac{-3}{x+2} < 0$

$\Rightarrow\ \dfrac{3}{x+2} > 0$ which is possible when $x > -2$

But $x \geq 1$ so common solution is $x \geq 1$ or $x \in [1, \infty)$ ...(i)

Case II : $|x-1| = -x + 1,\ x < 1$

$\therefore\ \dfrac{|x-1|}{x+2} < 1 \Rightarrow\ \dfrac{-x+1}{x+2} < 1 \Rightarrow\ \dfrac{x-1}{x+2} + 1 > 0 \Rightarrow\ \dfrac{2x+1}{x+2} > 0$

$\Rightarrow\ x < -2$ or $x > -\dfrac{1}{2}$

But $x < 1$ so $x \in (-\infty, -2) \cup \left(-\dfrac{1}{2}, 1\right)$ ...(ii)

$\therefore$ From (i) and (ii), x lies in the interval

$(-\infty, -2) \cup \left(-\dfrac{1}{2}, 1\right) \cup [1, \infty)$ or $x \in (-\infty, -2) \cup \left(-\dfrac{1}{2}, \infty\right)$

28. (c) : $|r| < 1 \Rightarrow r > -1$ or $r < 1$

$a = 5(1 - r) \Rightarrow\ a = 5 - 5r \Rightarrow\ r = \dfrac{5-a}{5}$

$\Rightarrow\ \dfrac{5-a}{5} > -1 \Rightarrow\ 5 - a > -5 \Rightarrow\ a < 10$ ...(i)

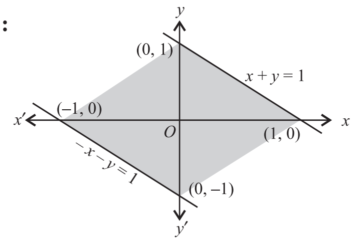
and $\dfrac{5-a}{5} < 1 \Rightarrow\ 5 - a < 5 \Rightarrow\ a > 0$ ...(ii)

From (i) and (ii), $0 < a < 10$

29. (b) : We have, $2x - 5 \leq \dfrac{4x-7}{3}$

$\Rightarrow 6x - 15 \leq 4x - 7 \Rightarrow\ x \leq 4 \Rightarrow\ x \in (-\infty, 4]$

30. (a) :



Pair of points are $\{(1, 0), (0, 1)\}$ and $\{(-1, 0), (0, -1)\}$

31. (c, d) : Let $y = \dfrac{(x-a)(x-b)}{(x-c)}$

$\Rightarrow (x - c)y = x^2 - (a + b)x + ab$

$\Rightarrow x^2 - (a + b + y)x + ab + cy = 0$

Here, $\Delta = (a + b + y)^2 - 4(ab + cy)$

$= y^2 + 2y(a + b - 2c) + (a - b)^2$

Since x is real and y assumes all real values,

$\therefore\ \Delta \geq 0$ for all real values of y

$\Rightarrow y^2 + 2y(a + b - 2c) + (a - b)^2 \geq 0$

Now, we know that the sign of a quadratic is same as that of coeff. of y^2 provided its discriminant

$B^2 - 4AC < 0$

This will be so if, $4(a + b - 2c)^2 - 4(a - b)^2 < 0$

$\Rightarrow 4(a + b - 2c + a - b)(a + b - 2c - a + b) < 0$

$\Rightarrow 16(a - c)(b - c) < 0$

$\Rightarrow 16(c - a)(c - b) < 0$ ... (i)

Now, If $a < b$ then from inequation (i), we get

$c \in (a, b) \Rightarrow\ a < c < b$

If $a > b$ then from inequation (i), we get $c \in (b, a)$

$\Rightarrow b < c < a$ or $a > c > b$

32. (a, d) : We have, $\alpha + \beta = -p$, $\alpha\beta = q$, $\alpha^4 + \beta^4 = r$ and $\alpha^4\beta^4 = s$

Therefore, $\alpha^2+\beta^2 = (\alpha+\beta)^2 - 2\alpha\beta = p^2 - 2q$, so that

$r = \alpha^4+\beta^4 = (\alpha^2+\beta^2)^2 - 2\alpha^2\beta^2 = (p^2-2q)^2 - 2q^2$

i.e., $(p^2)^2 - 4q(p^2) + 2q^2 - r = 0$

This shows that p^2 is one root of $x^2 - 4qx + 2q^2 - r = 0$.

If its other root is γ, we have $\gamma + p^2 = 4q$, *i.e.*, $\gamma = 4q - p^2$.

Further the discriminant of this quadratic equation is

$(4q)^2 - 4(2q^2 - r) = 8q^2 + 4[(p^2-2q)^2 - 2q^2]$

$= 4(p^2 - 2q)^2 \geq 0$

So that both roots, p^2 and $-p^2 + 4q$ are real. Since α and β are real $p^2 - 4q \geq 0$, *i.e.*, $-p^2 + 4q \leq 0$. Thus the roots of $x^2 - 4qx + 2q^2 - r = 0$ are positive and negative.

33. (c, d) : $w = \dfrac{\sqrt{3}+i}{2} = \dfrac{\sqrt{3}}{2} + i\dfrac{1}{2} = e^{i\pi/6}$

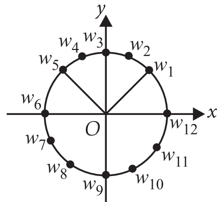
$w^n = e^{\frac{in\pi}{6}}$

For z_1, we have $\cos\frac{n\pi}{6} > \frac{1}{2}$ and

For z_2, we have $\cos\frac{n\pi}{6} < -\frac{1}{2}$

$z_1 = \frac{\sqrt{3}+i}{2}, \frac{\sqrt{3}-i}{2}, 1$

$z_2 = -1, \frac{-\sqrt{3}+i}{2}, \frac{-\sqrt{3}-i}{2} \quad \therefore \quad \angle z_1Oz_2 = \frac{5\pi}{6}, \frac{2\pi}{3}$



$z_1 \in \{w_1, w_{11}, w_{12}\}$

$z_2 \in \{w_5, w_6, w_7\}$

34. (b, c) : Since, a, b, c, are in G.P

$\Rightarrow b^2 = ac$

$\Rightarrow \; ax^2 + \sqrt{ac}x + c = 0$

$\Rightarrow \; x = \frac{-\sqrt{ac} \pm \sqrt{(\sqrt{ac})^2 - 4ac}}{2(a)}$

$x = \frac{-\sqrt{ac} \pm \sqrt{-3ac}}{2a}; \; x = \frac{-\sqrt{ac} \pm i\sqrt{3ac}}{2a}$

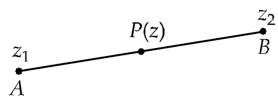
$\therefore$ Roots are imaginary.

$\frac{\alpha}{\beta} = \frac{-\sqrt{ac} + i\sqrt{3ac}}{-\sqrt{ac} - i\sqrt{3ac}} = \frac{(i\sqrt{3ac} - \sqrt{ac})^2}{ac + 3ac}$

$= \frac{-3ac + ac - 2i\sqrt{3}ac}{4ac} = \frac{-2 - 2i\sqrt{3}}{4}$

$= \frac{-1 - i\sqrt{3}}{2} = \omega^2 = \frac{1}{\omega}$

35. (a, c, d) : The given statement implies that z is on the line segment joining A and B.



$z = (1 - t)z_1 + tz_2$

Now z_1, z, z_2 are collinear, so $\text{Arg}\left(\frac{z - z_1}{z_2 - z_1}\right) = 0$ giving $\text{Arg}(z - z_1) = \text{Arg}(z_2 - z_1)$

Also, $\frac{z - z_1}{\bar{z} - \bar{z}_1} = \frac{z_2 - z_1}{\bar{z}_2 - \bar{z}_1} =$ complex slope of the line which can be rearranged as $(z - z_1)(\bar{z}_2 - \bar{z}_1) - (z_2 - z_1)(\bar{z} - \bar{z}_1) = 0$

i.e. $\begin{vmatrix} z - z_1 & \bar{z} - \bar{z}_1 \\ z_2 - z_1 & \bar{z}_2 - \bar{z}_1 \end{vmatrix} = 0$

36. (a, c) : $z = \sin\theta - i\cos\theta = \cos\left(\frac{\pi}{2} - \theta\right) - i\sin\left(\frac{\pi}{2} - \theta\right)$

$\therefore \; z^n = \cos\left(\frac{n\pi}{2} - n\theta\right) - i\sin\left(\frac{n\pi}{2} - n\theta\right)$

$z^{-n} = \cos\left(\frac{n\pi}{2} - n\theta\right) + i\sin\left(\frac{n\pi}{2} - n\theta\right)$

$\Rightarrow \; z^n + z^{-n} = 2\cos\left(\frac{n\pi}{2} - n\theta\right)$

and $z^n - z^{-n} = -2i\sin\left(\frac{n\pi}{2} - n\theta\right) = 2i\sin\left(n\theta - \frac{n\pi}{2}\right)$

37. (a, b, c) : $a + b - 2c = (a - c) + (b - c) > 0$

$\Rightarrow$ mouth opens upwards

Now $x = 1$ is a root.

Hence both roots are rational.

Vertex $= -\frac{(b + c - 2a)}{2(a + b - 2c)} > 0 \Rightarrow$ vertex > 0

38. (a, c) : $(x - \sqrt{3})(x\sqrt{3} - 1) < 0$

$\Rightarrow x$ lies between $\frac{1}{\sqrt{3}}$ and $\sqrt{3}$

$\Rightarrow$ Both $\tan A$ and $\tan B$ lie between $\frac{1}{\sqrt{3}}$ and $\sqrt{3}$.

$\Rightarrow$ Both A and B lie between 30° and 60°.

$\Rightarrow 60° < C < 120°$

$\Rightarrow \; -\frac{1}{2} < \frac{a^2 + b^2 - c^2}{2ab} < \frac{1}{2}$

39. (a) : (A)-(p, q, r), (B)-(p, q, r), (C)-(p, q), (D)-(p, q, s)

(A) Consider the polynomial

$P(x) = ax^3 + bx^2 + cx$

and use Rolle's theorem on [0, 1].

(B) Consider the polynomial

$P(x) = \frac{1}{3}ax^3 + \frac{1}{2}bx^2 + cx$

and use Rolle's theorem on [0, 1].

(C) Let $P(x) = ax^2 + bx + c$

Note : $P(-1)P(1) < 0$

$\Rightarrow P(x)$ vanishes at least once in $(-1, 1)$

$\Rightarrow P(x) = 0$ has real roots.

(D) Let $P(x) = ax^2 + bx + c$ and

$Q(x) = x^2 + \frac{b}{a}x + \frac{c}{a}$

Since, $1 + \frac{|b|}{a} + \frac{c}{a} < 0$, we get $Q(-1)$ or $Q(1) < 0$

Since $Q(x) \to \infty$ as $x \to \pm\infty$, we get $Q(x) = 0$ or $P(x) = 0$ has a root in $(-\infty, -1)$ or $(1, \infty)$.

40. (a) : (A) - (s), (B) - (r), (C) - (p), (D) - (q)

(A) $|z - 1|^2 + |z + 1|^2 = 4 \;\Rightarrow\; x^2 + y^2 = 1$

$\Rightarrow r = 1$

(B) Extremities of diameter are (0, 2) and (0, –2)

$\Rightarrow$ radius = 2

(C) $z\bar{z} - z(1 - 2i) - \bar{z}(1 + 2i) - 4 = 0$

$\Rightarrow x^2 + y^2 - 2x - 4y - 4 = 0$

$\therefore$ Centre (1, 2), $r = 3$

(D) $|z + 3| = 2|z - 3| \;\Rightarrow\; x^2 + y^2 - 10x + 9 = 0$

$\Rightarrow r = 4$

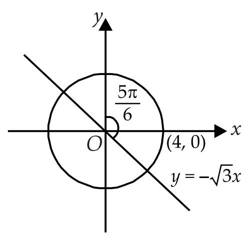
41. (b) : The curve S_1 is $x^2 + y^2 < 16$

S_2 turns out to be $\sqrt{3}x + y > 0$

And S_3 is $x > 0$

Area $= \frac{1}{2}r^2\theta$

$= \frac{1}{2} \times 16 \times \frac{5\pi}{6} = \frac{20\pi}{3}$



42. (c) : $\underset{z \in S}{\text{Min}}\,|1 - 3i - z|$ denotes the length of perpendicular from (1, –3) to the line $\sqrt{3}x + y = 0$.

And this is $\left|\frac{\sqrt{3} - 3}{\sqrt{3 + 1}}\right| = \left|\frac{\sqrt{3} - 3}{2}\right| = \frac{3 - \sqrt{3}}{2}$

43. (a) : $f(x) = x^2 - mx + 1$ and $\alpha < \beta$ are the roots of $f(x) = 0$.

Now, $\alpha < \beta < 1$ implies that $f(1)$ and the coefficients of x^2

have the same sign. This gives $1 - m + 1 = f(1) > 0$

$\Rightarrow m < 2$ ....(i)

Also, discriminant is $m^2 - 4 \geq 0$.

Therefore, $m \leq -2$ or $m \geq 2$ ....(ii)

From (i) and (ii), $m \leq -2$.

Also, note that if $m = -2$, the roots are $-1, -1$.

44. **(d) :** Let $f(x) = x^2 - 6mx + 9m^2 - 2m + 2$.

Let $\beta > \alpha > 3$ be the roots of $f(x) = 0$.

Then $6 < \alpha + \beta = 6m$ and hence $m > 1$.

Also, $9m^2 - 2m + 2 = \alpha\beta > 9$

Therefore, $9m^2 - 2m - 7 > 0$

$\Rightarrow (9m + 7)(m - 1) > 0$

This gives, $m < \frac{-7}{9}$ or $m > 1$

Also, $f(3)$ and the coefficient of x^2 have the same sign.

Therefore, $f(3) > 0$. This gives

$9 - 18m + 9m^2 - 2m + 2 > 0$

$\Rightarrow 9m^2 - 20m + 11 > 0$

$\Rightarrow (9m - 11)(m - 1) > 0$

$\Rightarrow m < 1$ or $m > \frac{11}{9}$, we get $\frac{11}{9} < m$

45. **(d) :** Let α, β, where $\alpha \leq \beta$, be the roots of $4x^2 - 2x + m = 0$.

Then $-1 < \alpha, \beta < 1$ and $\alpha + \beta = \frac{1}{2}, \alpha\beta = \frac{m}{4}$

Now, $f(-1)$ and the coefficient of x^2 have the same sign.

Therefore, $f(-1) > 0$ and hence $4 + 2 + m > 0$.

That is $m > -6$ ... (i)

Also, $f(1) > 0 \Rightarrow 4 - 2 + m > 0$.

This implies $m > -2$ ... (ii)

The discriminant is $4 - 16m \geq 0$.

Therefore, $m \leq 1/4$ ... (iii)

From (i), (ii) and (iii), we get $-2 < m \leq \frac{1}{4}$

If $m = 1/4$, then the given equation is

$4x^2 - 2x + \frac{1}{4} = 0$ or $16x^2 - 8x + 1 = 0$

Therefore, the roots are $1/4, 1/4 \in (-1, 1)$.

If the roots are distinct, then $-2 < m \leq 1/4$.

46. **(1) :** Let $\cot^{-1}p = \theta$, then $\cot\theta = p$

Now,

$$e^{2mi\theta}\cdot\left(\frac{i\cot\theta + 1}{i\cot\theta - 1}\right)^m = e^{2mi\theta}\left[\frac{i(\cot\theta - i)}{i(\cot\theta + i)}\right]^m$$

$$= e^{2mi\theta}\left[\frac{\cos\theta - i\sin\theta}{\cos\theta + i\sin\theta}\right]^m$$

$$= e^{2mi\theta}\left(\frac{e^{-i\theta}}{e^{i\theta}}\right)^m = e^{2mi\theta}(e^{-2i\theta})^m$$

$$= e^{2mi\theta}e^{-2mi\theta} = e^0 = 1$$

47. **(7) :** For the equation $px^2 + qx + 1 = 0$, to have real roots $D \geq 0$

$\Rightarrow q^2 \geq 4p$

If $p = 1$ then $q^2 \geq 4 \Rightarrow q = 2, 3, 4$

If $p = 2$ then $q^2 \geq 8 \Rightarrow q = 3, 4$

If $p = 3$ then $q^2 \geq 12 \Rightarrow q = 4$

If $p = 4$ then $q^2 \geq 16 \Rightarrow q = 4$

$\therefore$ Number of required equations = 7.

48. **(1) :** $|z - 3i| = 3 \Rightarrow |re^{i\theta} - 3i| = 3$

$|r\cos\theta + i(r\sin\theta - 3)| = 3$

$$\Rightarrow r^2\cos^2\theta + (r\sin\theta - 3)^2 = 9 \Rightarrow \sin\theta = \frac{r}{6}$$

$$\therefore \left|\cot\theta - \frac{6}{re^{i\theta}}\right| = 1$$

49. **(0) :** Since, cubic polynomial is divisible by both $x^2 + ax + b$ and $x^2 + bx + a$.

Therefore, $x^2 + ax + b$ and $x^2 + bx + a$ must have a common root.

Subtracting $x^2 + ax + b = 0$ and $x^2 + bx + a = 0$, we get

$x(a - b) = (a - b) \Rightarrow x = 1$

Therefore, roots for $x^2 + ax + b = 0$ are 1 and α

$\Rightarrow 1 \cdot \alpha = b \Rightarrow \alpha = b$

And roots for $x^2 + bx + a = 0$ are 1 and β

$\Rightarrow 1 \cdot \beta = a \Rightarrow \beta = a$

$\Rightarrow$ roots of cubic be $1, a, b$

$\Rightarrow 1 \times a \times b = -72$

and $a + b + 1 = 0$ (from $x^2 + ax + b = 0$ put $x = 1$)

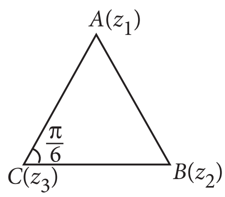
$$\Rightarrow a - \frac{72}{a} = -1 \Rightarrow a^2 + a - 72 = 0$$

$\Rightarrow (a + 9)(a - 8) = 0 \Rightarrow a = -9, 8$

Therefore, roots are $1, -9, 8$.

50. **(3) :** $|z_1 - z_2| = \sqrt{25 - 12\sqrt{3}}$

$|z_2 - z_3| = x, |z_1 - z_3| = \frac{3x}{4}$



$$\Rightarrow |z_1 - z_2|^2 = x^2 + \frac{9x^2}{16} - \frac{3x^2}{2}\cos\frac{\pi}{6} = 25 - 12\sqrt{3}$$

$\Rightarrow x = 4$

$$\therefore \text{ Area } = \frac{1}{2}\cdot 4 \cdot 3 \cdot \frac{1}{2} = 3 \text{ sq. units}$$

JEE MATHEMATICS

Class XI

DPS–4

DAILY PRACTICE SHEET

Permutations and Combinations

SOLUTIONS

1. **(b) :** In 10-digit number, each place can be filled by either 1 or 2.

$\therefore$ There are equal no. of 2 ways at each place.

$\therefore$ Required ways = 2 × 2 ×..... upto ten times = 2^{10}.

2. **(b) :** Number of ways to choose the first flag = 5

Number of ways to choose the second flag from the rest of four flags = 4

Hence, by FPC total number of ways = 5 × 4 = 20

3. **(c) :** The number of newspapers is $\dfrac{300\times 5}{60} = 25$.

4. **(d) :** (i) If $G < 4$ then $B \geq 8$ (ii) If $G \geq 4$ then $B < 8$

$\therefore$ (d) is true.

5. **(a) :** (i) Number of 3 digit nos. = 7 × 6 × 5 = 210

(ii) Number of 2 digit nos. = 7 × 6 = 42

(iii) Number of 1 digit nos. = 7

$\therefore$ Total number of such nos. = 210 + 42 + 7 = 259

6. **(b) :** We have, $T_r = (r^2+1+r-r)\lfloor r = (r^2+r)\lfloor r - (r-1)\lfloor r$

$\Rightarrow T_r = r\lfloor r+1 - (r-1)\lfloor r$

$\therefore T_1 = 1\lfloor 2 - 0$

$T_2 = 2\lfloor 3 - 1\lfloor 2$

$T_3 = 3\lfloor 4 - 2\lfloor 3$

. .

. .

. .

$T_{10} = 10\lfloor 11 - 9\lfloor 10$

$$\therefore \sum_{r=1}^{10}(r^2+1)\lfloor r = 10\lfloor 11$$

7. **(c) :** In a nine digit numbers, there are four even places for the four odd digits 3, 3, 5, 5.

$\therefore$ Required number of ways $= \dfrac{4!}{2!2!}\cdot\dfrac{5!}{2!3!} = 60$

8. **(c) :** $\dfrac{1}{2^n}\left({}^{2n}P_n\right) = \dfrac{1}{2^n}\left(\dfrac{(2n)!}{n!}\right)$

$$= \frac{(2n)(2n-1)(2n-2).......3\cdot 2\cdot 1}{2^n \times n!}$$

$$= \frac{2^n(n(n-1)........3\cdot 2\cdot 1)(1\cdot 3\cdot 5......(2n-1))}{2^n(n!)}$$

$$= \frac{2^n \times (n!)(1\cdot 3\cdot 5.......(2n-1))}{2^n\cdot(n!)} = 1\times 3\times 5\times ......\times(2n-1)$$

9. **(a) :** $\because \lfloor 17 = (1)\cdot(2)\cdot(3)...(9)\cdot(10)\cdot(11)\cdot(12)....(17)$

$\therefore$ It must be divisible by 9 and 11.

For divisibility by 9, sum of all digits must be a multiple of 9.

$\therefore$ $3+5+5+6+x+y+4+2+8+9+6 = 48+x+y$ must be a multiple of 9 for which

$x + y = 6$ or 15

For divisibility by 11, we know that difference of sum of odd place digits and sum of even place digits must be either zero or a multiple of 11.

Let $a = 3+5+x+4+8+9 = 29+x$

and $b = 5+6+y+2+0+6 = 19+y$

$a - b = 10 + x - y$

$\therefore$ $x - y = 1$ is possible only

But, $x + y = 6$ and $x - y = 1$ can't give integral value.

Hence, $x + y = 15$ only.

10. **(b) :** We can have the following distributions :

	B_1	B_2	B_3	
Case I	1	1	3	$\rightarrow \dfrac{\lfloor 5}{\lfloor 1\lfloor 1\lfloor 3}\cdot\dfrac{1}{\lfloor 2} = 10$
Case II	2	2	1	$\rightarrow \dfrac{\lfloor 5}{\lfloor 2\lfloor 2\lfloor 1}\cdot\dfrac{1}{\lfloor 2} = 15$

Total number of ways $= 25\times\lfloor 3 = 25\times 6 = 150$

11. **(d) :** The number of words starting from E are = 5! = 120

The number of words starting from H are = 5! = 120

The number of words starting from ME are = 4! = 24

The number of words starting from MH are = 4! = 24

The number of words starting from MOE are = 3! = 6

The number of words starting from MOH are = 3! = 6

The number of words starting from MOR are = 3! = 6

The number of words starting from MOTE are = 2! = 2

The number of words starting from MOTHER are = 1! = 1

Hence, rank of the word MOTHER

= 2(120) + 2 (24) + 3(6) + 2 + 1 = 309

12. **(c) :** $f(n) = {}^{8-n}P_{n-4}$, $4 \leq n \leq 6$

Putting n = 4, 5, 6, we get

$f(4) = {}^{8-4}P_{4-4} = {}^4P_0 = 1$

$f(5) = {}^{8-5}P_{5-4} = {}^3P_1 = \dfrac{3!}{2!} = 3$

$f(6) = {}^{8-6}P_{6-4} = {}^2P_2 = 2! = 2$

$\therefore$ Range = {1, 2, 3}.

13. **(c) :** There are two possibilities :

(1) The digits used are 1, 1, 1, 1, 1, 2, 3.

The number of numbers formed = $\frac{\lfloor 7}{\lfloor 5} = 42$

(2) The digits used are 1, 1, 1, 1, 2, 2, 2

The number of numbers formed = $\frac{\lfloor 7}{\lfloor 4 \lfloor 3} = \frac{7 \cdot 6 \cdot 5}{6} = 35$

$\therefore$ The total number of numbers = 42 + 35 = 77.

14. **(b) :** First we fix the alternate position of girls and they arrange in 4! ways and in the five places five boys can be arranged in $^5P_5$ ways.

$\therefore$ Total number of ways = 4! × $^5P_5$ = 4! × 5!

15. **(b) :** M, EEE, D, I, T, RR, AA, NN

R– – E

Two empty places can be filled with identical letters. [EE, AA, NN] in 3 ways. Two empty place can be filled with distinct letters [M, E, D, I, T, R, A, N] in $^8P_2$ ways.

$\therefore$ Number of words formed = 3 + $^8P_2$ = 59.

16. **(a) :** We can think of three packets. One consisting of three boys of class X, other consisting of four boys of class XI and last one consisting of five boys of class XII. These packets can be arranged in 3! ways and contents of these packets can be further arranged in 3!4! and 5! ways, respectively. Hence, the total number of ways is 3! × 3! × 4! × 5!.

17. **(d) :** A flight of stairs has 10 steps. A person can go up the steps one at a time, two at a time or any combination of 1s and 2s.

Let $x + 2y = 10$

where x is the number of times he takes single steps and y is the number of times he takes two steps.

Cases	Total no. of ways
I : When $x = 0$ and $y = 5$	$\frac{5!}{5!} = 1(2\,2\,2\,2\,2)$
II : When $x = 2$ and $y = 4$	$\frac{6!}{2!\,4!} = 15(1\,1\,2\,2\,2\,2)$
III : When $x = 4$ and $y = 3$	$\frac{7!}{4!3!} = 35(1\,1\,1\,1\,2\,2\,2)$
IV : When $x = 6$ and $y = 2$	$\frac{8!}{2!6!} = 28(1\,1\,1\,1\,1\,1\,2\,2)$
V : When $x = 8$ and $y = 1$	$\frac{9!}{8!} = 9(1\,1\,1\,1\,1\,1\,1\,1\,2)$
VI : When $x = 10$ and $y = 0$	$1(1\,1\,1\,1\,1\,1\,1\,1\,1\,1)$

Hence, total no. of ways= 1+ 15 + 35 + 28 + 9 + 1 = 89.

18. **(c) :** The number can be formed by the figures 4, 5, 6, 7, 8 which is greater than 56000 in two cases.

Case - I : Let the ten thousand place digit be greater than 5. The number of numbers = $^3C_1 \times {}^4C_1 \times {}^3C_1 \times {}^2C_1 \times {}^1C_1$

= 3 × 4 × 3 × 2 × 1 = 72

Case II : Let the ten thousand digit number be 5 and thousand digit number be either 6 or greater than 6.

Then, the number of numbers = $^3C_1 \times {}^3C_1 \times {}^2C_1 \times {}^1C_1$

= 3 × 3 × 2 × 1 = 18

$\therefore$ Required number of ways = 72 + 18 = 90

19. **(b) :** $^nC_2 - n = 54 \Rightarrow \frac{n(n-1)}{2} - n = 54$

$\Rightarrow n^2 - 3n - 108 = 0 \Rightarrow n = 12$

20. **(a) :** $^nC_3 + {}^nC_4 > {}^{n+1}C_3$

$\Rightarrow {}^{n+1}C_4 > {}^{n+1}C_3 \quad (\because {}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1})$

$\Rightarrow \frac{^{n+1}C_4}{^{n+1}C_3} > 1 \Rightarrow \frac{n-2}{4} > 1 \Rightarrow n > 6$

21. **(a) :** $^nC_1 + 2\,{}^nC_2 + ... + n\,{}^nC_n = 2n^2$

$\Rightarrow \sum_{k=1}^{n} k\,{}^nC_k = 2n^2 \Rightarrow \sum_{k=1}^{n} \frac{k(n!)}{k!(n-k)!} = 2n^2$

$\Rightarrow \sum_{k=1}^{n} \frac{k \cdot n(n-1)!}{k(k-1)!\,(n-k)!} = 2n^2 \Rightarrow n\sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!\,(n-k)!} = 2n^2$

$\Rightarrow \sum_{k=1}^{n} {}^{n-1}C_{k-1} = 2n$

$\Rightarrow 2^{n-1} = 2n$

$n = 4$ satisfy the above equality

Hence, $n = 4$.

22. **(b) :** We have to find the number of integral solutions of $x_1 + x_2 + x_3 + x_4 + x_5 = 6$ and that equals $^{5+6-1}C_{5-1} = {}^{10}C_4$

Thus, Statement-1 is false.

Number of different ways of arranging 6A's and 4B's in a row $= \frac{\lfloor 10}{\lfloor 6 \times \lfloor 4} = {}^{10}C_4 =$ Number of different ways the child can buy the six ice-creams.

$\therefore$ Statement-2 is true.

23. **(d) :** The number of divisors of ab^2c^2de

= (1 + 1) (2 + 1) (2 + 1) (1 + 1) (1 + 1) – 1

= 2 · 3 · 3 · 2 · 2 – 1 = 71

24. **(b) :** There are 9 balls in the box out of which three balls can be drawn in $^9C_3$ ways. But these ways also include those ways in which 3 black balls are drawn. Total number of ways when black ball is not drawn is $^6C_3$.

Hence, the number of ways in which atleast one black ball is included = Total number of ways of drawing 3 balls from 9 balls – Number of ways of drawing 3 balls from 6 balls = $^9C_3 - {}^6C_3$ = 84 – 20 = 64

25. **(c) :** Using the principle of inclusion and exclusion, we have the number of ways in which card number 1 be placed in envelope number 2

$= 5! - \{^4C_1 \cdot 4! - {}^4C_2 \cdot 3! + {}^4C_3 \cdot 2! - {}^4C_4 \cdot 1!\}$

= 120 – {96 – 36 + 8 – 1} = 53

26. **(c) :** $x_1 + x_2 + x_3 + x_4 = 10$

The number of positive integral solution is $^{6+4-1}C_{4-1} = {}^9C_3$

It is the same as the number of ways of choosing any 3 balls from 9 different places.

27. **(b) :** Required number of ways

= Coefficient of x^{16} in $(x^3 + x^4 + x^5 + ... + x^{16})^4$

= Coefficient of x^{16} in $x^{12}(1 + x + x^2 + ... + x^{12})^4$

= Coefficient of x^4 in $(1 - x^{13})^4 (1 - x)^{-4}$

= Coefficient of x^4 in $(1 - 13x^5 + ...)$

$\times \left[1 + 4x + ... + \frac{(r+1)(r+2)(r+3)}{3!}x^r\right]$

$= \frac{(4+1)\,(4+2)\,(4+3)}{3!} = 35$

28. **(d) :** We have, $^{32}P_6 = k\,(^{32}C_6)$

$\Rightarrow \dfrac{32!}{(32-6)!} = k\dfrac{32!}{6!(32-6)!} \Rightarrow k = 6! \Rightarrow k = 720$

29. **(b) :** Since, each student receive atleast one toy. Then, firstly we give each student one toy and the remaining 7 toys can be distributed in three students in $^{7-1}C_{3-1} = {}^6C_2$ ways.
Hence, statement-1 is false and statement-2 is true.

30. **(a) :** Here $T_n = {}^nC_3$
Now $T_{n+1} - T_n = {}^{n+1}C_3 - {}^nC_3 \Rightarrow 10 = {}^nC_2$

$\Rightarrow \dfrac{n(n-1)}{2} = 10 \Rightarrow n(n-1) = 20$

$\Rightarrow n^2 - n - 20 = 0 \Rightarrow n^2 - 5n + 4n - 20 = 0$

$\Rightarrow n(n-5) + 4(n-5) = 0$

$\Rightarrow (n-5)(n+4) = 0$

$\Rightarrow n = 5 \qquad (\because n \neq -4)$

31. **(a, b, d) :** (a) $^{21-1}C_{5-1} = 4845$
(b) Coefficient of x^{21} in $(x^2 + x^3 + ...+ x^{13})^5$
$= {}^{15}C_{11} = 1365$
(c) $^{24}C_4 = 10626$
(d) $^{16}C_1 \times {}^7C_2 = 336$

32. **(c, d) :** Required number = Number of selections of one or more out of three 25 paise coins and two 50 paise coins
$= 4 \times 3 - 1 = 11 = {}^{12}P_1 - 1$

33. **(a,c) :** Total number of ways = $\sum_{r=1}^{n} {}^{2n}C_{n+r}$

$= {}^{2n}C_{n+1} + ... + {}^{2n}C_{2n} = \dfrac{1}{2}(2^{2n} - {}^{2n}C_n)$

$= \dfrac{1}{2}\left(2^{2n} - \dfrac{2n!}{n!n!}\right)$

34. **(a, c) :** The required number of selections of 4 letters = coeff. of x^4 in the expansion of $(x^0 + x^1)(x^0 + x^1)(x^0 + x^1 + x^2)(x^0 + x^1 + x^2)(x^0 + x^1)$ = coeff. of x^4 in $(1 + x)^3(1 + x + x^2)^2 = 18$

35. **(b, d) :** In 1st round all the integers, which leaves the remainder 1 when divided by 15, will be marked Last number of this category is 991
Next number to be marked is (991+15 – 1000) = 6
Again, second round of integers which leaves the remainder '6' when divided by 15 will be marked.
Last number of this category is 996.
Next number to be marked is (996+15-1000) = 11
Thus, third round of integers which leaves the remainder 11 when divided by 15, will be marked .
Last number of this category is 986.
Next number to be marked is 986 + 15 – 1000 = 1 which is already been marked.
Thus, unmarked numbers are = 1000 – (67 + 67 + 66)
= 800

36. **(a, c) :** The number of times the digit 3 will be written in the unit place is 100. Similarly in the tens place and hundred place.
$\therefore$ Required no. of times digit '3' will be written is 300.

37. **(a, c) :** $r - 1 \geq 0, r \leq n + 1 \Rightarrow 1 \leq r \leq n + 1$

$\Rightarrow \dfrac{1}{n+1} \leq \dfrac{r}{n+1} \leq 1$

Also $k^2 - 8 = \dfrac{{}^nC_{r-1}}{{}^{n+1}C_r} = \dfrac{r}{n+1}$

$\therefore \dfrac{1}{n+1} \leq k^2 - 8 \leq 1 \Rightarrow 8 < k^2 \leq 9$

$\Rightarrow k \in [-3, -2\sqrt{2})$ or $k \in (2\sqrt{2}, 3]$

38. **(c, d) :** Divide 8 objects in 7 packs and distribute them.

$1 \times \dfrac{7!}{3!} + {}^3C_1 \times \dfrac{7!}{4!} + {}^3C_2 \times \dfrac{7!}{5!} = 1596$

39. **(b) : (A) – (q,s), (B) – (q,s), (C) – (p), (D) – (r)**
(A) Non-negative integral solutions of
$R + B + G + Y = 8$ is $^{8+4-1}C_{4-1} = {}^{11}C_3$
(B) $D_1 + D_2 + D_3 + D_4 = 9, D_1 \neq 0$
$D_1 - 1 + D_2 + D_3 + D_4 = 8$
$^{8+4-1}C_{4-1} = {}^{11}C_3$
(C) Select '3' places out of 21 places = $^{21}C_3$
(D) $\left[\dfrac{17}{3}\right] + \left[\dfrac{17}{3^2}\right] = 5 + 1 = 6$

40. **(a) : (A) – (s), (B) – (p), (C) – (q), (D) – (r)**
(A) Required number of ways = $^{8-1}C_{3-1}$
$= {}^7C_2 = \dfrac{7!}{2!5!} = \dfrac{7.6}{2.1} = 21$
(B) The number of choices available to him
$^5C_4 \times {}^8C_6 + {}^5C_5 \times {}^8C_5 = 5 \times \dfrac{8 \times 7}{2} + 1 \times \dfrac{8 \times 7 \times 6}{3 \times 2}$
$= 5 \times 4 \times 7 + 8 \times 7 = 140 + 56 = 196$
(C) The number of ways to sit men = 5!
and the number of ways to sit women = $^6P_5$
Total number of ways = $^6P_5 \times 5! = 6! \times 5!$
(D) $^nC_{r+1} + {}^nC_{r-1} + 2\,{}^nC_r = {}^nC_{r+1} + {}^nC_r + {}^nC_{r-1} + {}^nC_r$
$= {}^{n+1}C_{r+1} + {}^{n+1}C_r = {}^{n+2}C_{r+1}$

41. **(c) :** Considering CC as single object, U, CC, E can be arranged in 3! ways.
× U × CC × E ×
Now, the three S are to be placed in four available places. Hence, required number of ways = $^4P_3 \times 3! = 144$.

42. **(d) :** The alphabetic order is C, E, S, U. The number of words beginning with C is $\dfrac{6!}{3!} = 120$ and number of words beginning with E is $\dfrac{6!}{2!\,3!} = 60$

Number of words beginning with SC, $\dfrac{5!}{2!} = 60$,

Number of words beginning with SE, $\dfrac{5!}{2!2!} = 30$ and number of words beginning with SS, $\dfrac{5!}{2!} = 60$

After which the next word is SUCCESS.
Thus, the rank of SUCCESS is
120 + 60 + 60 + 30 + 60 + 1 = 331

43. **(b) :** Required number of ways = $^5P_4$

44. **(a) :** Each prize can be given in 5 ways.
$\therefore$ Required number of ways = 5·5·5·5 = 625

45. **(b) :** All prizes can be given to one student in 5 ways
$\Rightarrow$ Required number of ways = $5^4 - 5 = 625 - 5 = 620$

46. **(3) :** We know that, ${}^nC_r = \frac{{}^nP_r}{r!}$

$\therefore \quad 120 = \frac{720}{r!} \Rightarrow r! = \frac{720}{120} = 6 = 3!$

Hence, $r = 3$.

47. **(1) :** 1! + 2! + 3! + 4!= 33

5! = 120, 6! = 720, 7! = 5040, 8! = 40320, 9! = 362880

After 9! the other numbers will be divisible by 100 hence won't make any contribution in tens place.

48. **(3) :** 1! +2! + 3! + 4! = 33

5! = 120, 6! = 720, 7! = 5040

8! = 40320, 9! = 362880

Thus, the unit digit of 1! + 2! + …. + 9! = 3

Also note that $n!$ is divisible by 100 for all $n \geq 10$.

$\therefore$ Unit digit of 10! + 11! + ….. + 49! = 0

$\therefore$ Unit digit of 1! + 2! + ……+ 49! = 3.

49. **(5) :** Last two digits are 25, 75 remaining places can be filled in ${}^5P_2$ ways. Hence four digit numbers formed are = 40 = $8m$

$\Rightarrow \quad m = 5$

50. **(9) :** The number of ways in which all the letters are in wrong envelopes

$$= 4!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right)$$

$$= 4!\left(\frac{1}{2} - \frac{1}{6} + \frac{1}{24}\right) = 12 - 4 + 1 = 9$$

JEE MATHEMATICS — Class XI

DPS-5

DAILY PRACTICE SHEET

Binomial Theorem

SOLUTIONS

1. **(d):** $\frac{1}{\sqrt{(3x+1)}}\left[\left(\frac{1+\sqrt{3x+1}}{2}\right)^7-\left(\frac{1-\sqrt{3x+1}}{2}\right)^7\right]$

$=\frac{1}{2^7\sqrt{(3x+1)}}\left[\left(1+\sqrt{3x+1}\right)^7-\left(1-\sqrt{3x+1}\right)^7\right]$ ...(i)

Now, $\left(1+\sqrt{3x+1}\right)^7-\left(1-\sqrt{3x+1}\right)^7$

$=\left[{}^7C_0+{}^7C_1\sqrt{(3x+1)}+{}^7C_2\left(\sqrt{3x+1}\right)^2+....+{}^7C_7\left(\sqrt{3x+1}\right)^7\right]$

$-\left[{}^7C_0-{}^7C_1\sqrt{3x+1}+{}^7C_2\left(\sqrt{(3x+1)}\right)^2-{}^7C_7\left(\sqrt{3x+1}\right)^7\right]$

$=2\left[{}^7C_1\left(\sqrt{3x+1}\right)+{}^7C_3\left(\sqrt{3x+1}\right)^3+{}^7C_5\left(\sqrt{3x+1}\right)^5+{}^7C_7\left(\sqrt{3x+1}\right)^7\right]$

$=2\sqrt{3x+1}\times\left[7+35(3x+1)+21(3x+1)^2+(3x+1)^3\right]$

Now, putting above value in (i), so the given expression becomes $\frac{1}{2^6}[42+105x+21(3x+1)^2+(3x+1)^3]$

$\therefore$ Degree of a polynomial is the highest power of x.
So, degree of given expression is 3.

2. **(b) :** Let $1+\frac{1}{1!}\cdot\left(\frac{1}{4}\right)+\frac{1\cdot 3}{2!}\left(\frac{1}{4}\right)^2+\frac{1\cdot 3\cdot 5}{3!}\left(\frac{1}{4}\right)^3+....$

$=1+nx+\frac{n(n-1)}{2!}x^2+...=(1+x)^n$

$\therefore\quad nx=\frac{1}{1!}\cdot\frac{1}{4}\Rightarrow nx=\frac{1}{4},\ \frac{n(n-1)}{2!}x^2=\frac{1\cdot 3}{2\cdot 1}\cdot\left(\frac{1}{4}\right)^2$

$\Rightarrow\quad\frac{nx(nx-x)}{2}=\frac{1.3}{2.1}\cdot\frac{1}{16}\Rightarrow\frac{1}{4}\left(\frac{1}{4}-x\right)=\frac{3}{16}\Rightarrow x=-\frac{1}{2}$

Again $n\left(-\frac{1}{2}\right)=\frac{1}{4}\Rightarrow n=-\frac{1}{2}$

$\therefore\quad$ Sum $=(1+x)^n=\left(1-\frac{1}{2}\right)^{-1/2}=\frac{1}{\sqrt{1/2}}=\sqrt{2}$

3. **(c) :** $(1+ax)^n=1+{}^nC_1(ax)+{}^nC_2(ax)^2+....+a^nx^n$

On comparing coefficients, ${}^nC_1a=6, {}^nC_2a^2=\frac{27}{2}$ we get

$\Rightarrow\quad an=6$ ... (i) and $n(n-1)a^2=27$ ... (ii)

From (ii), $(n^2-n)a^2=27$

$\Rightarrow\quad n^2a^2-na^2=27\Rightarrow 36-na^2=27\Rightarrow na^2=9$ ...(iii)

Dividing (iii) by (i), $\frac{na^2}{an}=\frac{9}{6}\Rightarrow a=\frac{3}{2}$

Putting the value of $a=\frac{3}{2}$ in (i), $n=4$.

4. **(a) :** $1+\frac{1}{2}\cdot n+\frac{1}{3}\cdot\frac{n(n-1)}{2!}+.....+\frac{1}{n+1}\cdot 1$

$=\frac{1}{n+1}\left[(n+1)+\frac{(n+1)n}{2!}+\frac{(n+1)n(n-1)}{3!}+.....+1\right]$

$=\frac{1}{n+1}[{}^{n+1}C_1+{}^{n+1}C_2+{}^{n+1}C_3+.....+{}^{n+1}C_{n+1}]$

$=\frac{1}{n+1}[{}^{n+1}C_0+{}^{n+1}C_1+{}^{n+1}C_2+.....+{}^{n+1}C_{n+1}-1]$ [${}^{n+1}C_0=1$]

$=\frac{1}{n+1}[2^{n+1}-1]$

5. **(c) :** $\sum_{r=0}^{n}(r+1)\,{}^nC_r=\sum_{r=0}^{n}r\cdot{}^nC_r+\sum_{r=0}^{n}{}^nC_r$

$=\sum_{r=1}^{n}r\cdot\frac{n}{r}\cdot{}^{n-1}C_{r-1}+\sum_{r=0}^{n}{}^nC_r=2^{n-1}(n+2)$

Thus, Statement-1 is true.

Again $\sum_{r=0}^{n}(r+1)\,{}^nC_rx^r=\sum_{r=0}^{n}r\cdot{}^nC_rx^r+\sum_{r=0}^{n}{}^nC_rx^r$

$=n\sum_{r=1}^{n}{}^{n-1}C_{r-1}x^r+\sum_{r=0}^{n}{}^nC_rx^r=nx(1+x)^{n-1}+(1+x)^n$

Substitute $x=1$ in the above identity to get

$\sum_{r=0}^{n}(r+1)\,{}^nC_r=n\cdot 2^{n-1}+2^n$

Statement-2 is also true and explains Statement-1 also.

6. **(b) :** $P=\sum_{r=0}^{5}c_{2r}={}^{10}C_0+{}^{10}C_2+{}^{10}C_4+....+{}^{10}C_{10}=\frac{2^{10}}{2}=2^9$

$Q=\sum_{r=0}^{3}d_{2r+1}=d_1+d_3+d_5+d_7$

$={}^7C_1+{}^7C_3+{}^7C_5+{}^7C_7=\frac{2^7}{2}=2^6$

$\therefore\quad\frac{P}{Q}=2^3=8$

7. **(c) :** $7^{2n}+2^{3n-3}\cdot 3^{n-1}=49^n+24^{n-1}=(50-1)^n+(25-1)^{n-1}$

= Multiple of 50 + $(-1)^n$ + multiple of 25 + $(-1)^{n-1}$

= a multiple of 25, since $(-1)^n+(-1)^{n-1}=0$.

8. **(b) :** We make use of the fact that

$\binom{n}{1},\binom{n}{2},\binom{n}{3},...,\binom{n}{n-1}$ are divisible by n for any prime n.

Let $f(n)=\frac{n^7}{7}+\frac{n^5}{5}+\frac{2n^3}{3}-\frac{n}{105}$, $f(1)=\frac{1}{7}+\frac{1}{5}+\frac{2}{3}-\frac{1}{105}=1$

$$f(n+1)-f(n)=\frac{(n+1)^7-n^7}{7}+\frac{(n+1)^5-n^5}{5}+\frac{2}{3}((n+1)^3-n^3)-\frac{1}{105}$$

= a natural number + $f(1)$, since 3, 5, 7 are primes.

Thus, $f(1) = 1$, $f(2) - f(1)$ = a natural number, $f(3) - f(2)$ = a natural number, $f(n) - f(n-1)$ = a natural number.

Adding these, $f(n)$ = a natural number.

9. **(d) :** $\frac{\binom{n}{r}}{\binom{r+3}{r}}=\frac{n!}{(n-r)!r!}\cdot\frac{6}{(r+3)(r+2)(r+1)}$

$$=\frac{6\cdot n!}{(n-r)!(r+3)!}=\frac{6}{(n+1)(n+2)(n+3)}\cdot\binom{n+3}{r+3}$$

$\therefore$ The given series is

$$\frac{6}{(n+1)(n+2)(n+3)}\cdot\left[\binom{n+3}{3}-\binom{n+3}{4}+\binom{n+3}{5}-...\right]$$

$$=\frac{6}{(n+1)(n+2)(n+3)}\cdot\left[\binom{n+3}{0}-\binom{n+3}{1}+\binom{n+3}{2}\right]$$

$[\because C_0 - C_1 + C_2 ... = 0]$

$$=\frac{6}{(n+1)(n+2)(n+3)}\cdot\left[1-(n+3)+\frac{(n+3)(n+2)}{2}\right]$$

$$=\frac{6}{(n+1)(n+2)(n+3)}\cdot\frac{(n+1)(n+2)}{2}=\frac{3}{n+3}$$

10. **(d) :** $\left(\sqrt{2}+1\right)^n=[x]+f$. Let $\left(\sqrt{2}-1\right)^n=f'$, $0<f'<1$.

$[x]+f-f'=\left(\sqrt{2}+1\right)^n-\left(\sqrt{2}-1\right)^n$

$=2\left[\binom{n}{1}2^{\frac{n-1}{2}}+\binom{n}{3}2^{\frac{n-3}{2}}+...\right]$, which is an even integer.

$\therefore$ $[x]$ is an even integer and $f'=f$

But $xf'=\left(\sqrt{2}+1\right)^n\left(\sqrt{2}-1\right)^n=1$, or $([x]+f)f=1$

$\frac{1}{f}-f=\frac{1-f^2}{f}=[x]$, an even integer.

11. **(c) :** $(1-3x+3x^2-x^3)^{2n}=((1-x)^3)^{2n}=(1-x)^{6n}$

Concept : Index = $6n$ which is even so most middle term is $\left(\frac{6n}{2}+1\right)^{\text{th}}$ *i.e.*, $(3n+1)^{\text{th}}$ term is middle term,

$T_{3n+1}={}^{6n}C_{3n}(-x)^{3n}=\frac{6n!}{3n!\,3n!}(-x)^{3n}$

12. **(a) :** T_{r+1} in $\left(ax^2+\frac{1}{bx}\right)^{13}$

$$={}^{13}C_r(ax^2)^{13-r}\left(\frac{1}{bx}\right)^r={}^{13}C_r\cdot\frac{a^{13-r}}{b^r}\cdot x^{26-3r}$$

For coefficient of x^8, we have $26-3r=8 \Rightarrow r=6$

$\Rightarrow$ Coefficient of $x^8={}^{13}C_6\frac{a^7}{b^6}$

T_{r+1} in $\left(ax-\frac{1}{bx^2}\right)^{13}={}^{13}C_r(ax)^{13-r}\left(-\frac{1}{bx^2}\right)^r$

$={}^{13}C_r\frac{a^{13-r}}{b^r}x^{13-3r}(-1)^r$

For coefficient of x^{-8}, we have $13-3r=-8 \Rightarrow r=7$

$\therefore$ Coefficient of $x^{-8}=-{}^{13}C_7\frac{a^6}{b^7}$

Now, ${}^{13}C_6\frac{a^7}{b^6}=-{}^{13}C_7\frac{a^6}{b^7}\Rightarrow ab=-1\Rightarrow ab+1=0$

13. **(a) :** $(1-x+x^2-x^3)^4=(1-x)^4(1+x^2)^4$

$=[{}^4C_0(-x)^0+{}^4C_1(-x)+{}^4C_2(-x)^2+{}^4C_3(-x)^3+{}^4C_4(-x)^4]$
$\times[{}^4C_0(x)^0+{}^4C_1(x^2)+{}^4C_2(x^2)^2+{}^4C_3(x^2)^3+{}^4C_4(x^2)^4]$

$\therefore$ Coefficient of $x^4={}^4C_0\cdot{}^4C_2+{}^4C_2\cdot{}^4C_1+{}^4C_4\cdot{}^4C_0=31$

14. **(b) :** $\left(1+\frac{x}{2}\right)^{10}=1+{}^{10}C_1.\frac{1}{2}.x+{}^{10}C_2.\frac{1}{4}.x^2+{}^{10}C_3.\frac{1}{8}.x^3+{}^{10}C_4.\frac{1}{16}.x^4+....$

By inspection, ${}^{10}C_3.\frac{1}{8}$ is the highest coefficient.

$\therefore$ Power of x is 3.

15. **(a) :** The general term in second bracket is ${}^8C_r(2x^2)^{8-r}\left(-\frac{1}{x}\right)^r$

$\therefore$ General term for combined is,

$${}^8C_r(2x^2)^{8-r}\left(-\frac{1}{x}\right)^r-\frac{1}{x}\,{}^8C_r(2x^2)^{8-r}\cdot\left(\frac{-1}{x}\right)^r+3x^5\,{}^8C_r(2x^2)^{8-r}\left(\frac{-1}{x}\right)^r$$

$={}^8C_r2^{8-r}(-1)^rx^{16-3r}-{}^8C_r2^{8-r}(-1)^rx^{15-3r}+3\cdot{}^8C_r2^{8-r}(-1)^rx^{21-3r}$

For independent term,

$16-3r=0$, $15-3r=0\Rightarrow r=5$, $21-3r=0\Rightarrow r=7$

$r=5$, $r=7$ is in 2nd term and 3rd term respectively.

$\therefore$ Term independent of x

$=-{}^8C_5\,2^3(-1)^5-3\cdot{}^8C_7\cdot2=448-(6\times8)=400$

16. **(a) :** Let $S=(1+x)^{2016}+x(1+x)^{2015}+x^2(1+x)^{2014}+....+x^{2015}(1+x)+x^{2016}$ ...(i)

$\left(\frac{x}{1+x}\right)S=x(1+x)^{2015}+x^2(1+x)^{2014}+...+x^{2016}+\frac{x^{2017}}{1+x}$ ...(ii)

Subtracting (ii) from (i), we get

$\frac{S}{1+x}=(1+x)^{2016}-\frac{x^{2017}}{1+x}\Rightarrow S=(1+x)^{2017}-x^{2017}$

a_{17} = coefficient of $x^{17}={}^{2017}C_{17}=\frac{2017!}{17!2000!}$

17. **(c) :** By Binomial theorem,

$(1-2\sqrt{x})^{50}={}^{50}C_0-{}^{50}C_1(2\sqrt{x})+{}^{50}C_2(2\sqrt{x})^2+..........+{}^{50}C_{50}(2\sqrt{x})^{50}$

$(1+2\sqrt{x})^{50}={}^{50}C_0+{}^{50}C_1(2\sqrt{x})+{}^{50}C_2(2\sqrt{x})^2+..........+{}^{50}C_{50}(2\sqrt{x})^{50}$

On addition, we get

$$(1+2\sqrt{x})^{50}+(1-2\sqrt{x})^{50}=2({}^{50}C_0+{}^{50}C_2\,(2\sqrt{x})^2 + {}^{50}C_4(2\sqrt{x})^4+.......+{}^{50}C_{50}\,(2\sqrt{x})^{50})$$

Set $x = 1$ to obtain

$3^{50} + 1 = 2$ (sum of coefficients of integral powers of x)

$\therefore$ Sum of coeff. of integral powers of $x=\frac{1}{2}(3^{50}+1)$

18. **(b) :** Given expansion is, $\left(x\cos\alpha+\frac{\sin\alpha}{x}\right)^{10}$

General term $= {}^{10}C_r\, x^{10-r}\,(\cos\alpha)^{10-r}\cdot(\sin\alpha)^r\cdot x^{-r}$

$= {}^{10}C_r\, x^{10-2r}\,(\cos\alpha)^{10-r}\cdot(\sin\alpha)^r$

For the term independent of x, put $10 - 2r = 0 \Rightarrow r = 5$

$\therefore$ The term independent of x is,

${}^{10}C_5\,(\cos\alpha)^5\,(\sin\alpha)^5 = \frac{1}{2^5}\cdot{}^{10}C_5(\sin 2\alpha)^5$

So, the greatest value of the independent term is $\frac{1}{2^5}\cdot{}^{10}C_5$

19. **(d) :** $C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots = (1 + x)^{2n}$

Differentiate to get $1\cdot C_1 + 2\cdot C_2x + 3\cdot C_3x^2 + \ldots = 2n(1 + x)^{2n-1}$

Further, $C_0x^{2n} - C_1x^{2n-1} + C_2x^{2n-2} - \ldots = (x - 1)^{2n}$

Multiplying above two and considering the coefficients of x^{2n-1}, we get the desired series as the coefficient of x^{2n-1} in

$-2n(1 + x)^{2n-1}(1 - x)^{2n} = 2n(x - 1)(1 - x^2)^{2n-1}$

or the coefficient of x^{2n-2} in $2n(1 - x^2)^{2n-1}$

$$= 2n(-1)^{n-1}\binom{2n-1}{n-1} = (-1)^{n-1}\frac{(2n)!}{n!(n-1)!}.$$

20. **(b) :** t_7 in $\left[\sqrt[6]{3}\,\sqrt{2}+\frac{1}{\sqrt[3]{3}}\right]^n = {}^nC_6(2^{1/2}\,3^{1/6})^{n-6}\left[\frac{1}{3^{1/3}}\right]^6$

7^{th} term from the end in $\left[\sqrt[6]{3}\,\sqrt{2}+\frac{1}{\sqrt[3]{3}}\right]^n$

$$=t_7 \text{ in } \left[\frac{1}{\sqrt[3]{3}}+\sqrt[6]{3}\,\sqrt{2}\right]^n = {}^nC_6\left(\frac{1}{3^{1/3}}\right)^{n-6}\left(3^{1/6}\,2^{1/2}\right)^6$$

$$\therefore \frac{{}^nC_6[2^{1/2}\,3^{1/6}]^{n-6}\left[\frac{1}{3^{1/3}}\right]^6}{{}^nC_6\left[\frac{1}{3^{1/3}}\right]^{n-6}[2^{1/2}\,3^{1/6}]^6}=\frac{1}{6}$$

$$\Rightarrow \frac{\left[2^{\frac{n-6}{2}}\,3^{\frac{n-6}{6}}\right][3^{-2}]}{3^{-\left(\frac{n-6}{3}\right)}\,2^3\cdot 3^1}=\frac{1}{6} \Rightarrow \frac{2^{\frac{n-6}{2}}}{3^{\frac{6-n}{2}}}\cdot\frac{1}{2^3\cdot 3^3}=\frac{1}{6}$$

$$\Rightarrow \frac{2^{\frac{n-6}{2}}}{3^{\frac{6-n}{2}}}=\frac{2^3\cdot 3^3}{6} \Rightarrow 2^{\frac{n-6}{2}}\cdot 3^{\frac{n-6}{2}}=2^2\times 3^2$$

Comparing the powers of 2 and 3, we get

$$\frac{n-6}{2}=2 \Rightarrow n=10$$

21. **(c) :** Given expansion is $(2 - 3x^3)^{20}$

$\therefore$ $t_{r+1} = {}^{20}C_r\,2^{20-r}\,(-3x^3)^r$

$\therefore$ Putting $r = 9, 10$, we get $t_{10} = {}^{20}C_9\,2^{11}\,(-3x^3)^9$,

$t_{11} = {}^{20}C_{10}\,2^{10}\,(-3x^3)^{10}$

$$\because \frac{t_{10}}{t_{11}}=\frac{45}{22} \Rightarrow \frac{10}{11}\times\left(\frac{2}{1}\right)\frac{1}{-3x^3}=\frac{45}{22} \quad \left(\because \frac{{}^{20}C_9}{{}^{20}C_{10}}=\frac{10}{11}\right)$$

$$\Rightarrow x^3=\frac{-8}{27} \Rightarrow x=\frac{-2}{3}$$

22. **(a) :** $\left(1+\frac{2x}{3}\right)^{3/2}(32+5x)^{-1/5}$

$$=\left[1+\frac{3}{2}\left(\frac{2x}{3}\right)\right](32)^{-1/5}\left(1+\frac{5}{32}x\right)^{-1/5}$$

(neglect higher powers of x)

$$=[1+x]2^{-1}\left[1-\frac{1}{5}\left(\frac{5}{32}\right)x\right]$$ (neglect higher powers of x)

$$=\frac{1}{2}(1+x)\left(1-\frac{x}{32}\right)=\frac{(1+x)(32-x)}{64}=\frac{32+31x}{64}$$

(neglect x^2 term)

23. **(d) :** We have, $5! = 120$

$\therefore$ If $n \geq 5$, the digit at unit place in $n!$ is 0

Also $7^{291} = (7^4)^{72}\cdot(7)^3 = (7^4)^{72}\cdot(343)$

The digit at unit place in $(7^4)^{72} = 1$

i.e., in 7^{291} is 3

The digit at unit place in $583! + 7^{291} = 0 + 3 = 3$

24. **(d) :** Let $ab = x, bc = y, ca = z$

Since general term is $t_{r+1}=\frac{\lfloor 9}{\lfloor\alpha_1\,\lfloor\alpha_2\,\lfloor\alpha_3}\,x^{\alpha_1}\,y^{\alpha_2}\,z^{\alpha_3}$

$$=\frac{\lfloor 9}{\lfloor\alpha_1\,\lfloor\alpha_2\,\lfloor\alpha_3}\,a^{\alpha_1}\,b^{\alpha_1}\,b^{\alpha_2}\,c^{\alpha_2}\,c^{\alpha_3}\,a^{\alpha_3}$$

$$=\frac{\lfloor 9}{\lfloor\alpha_1\,\lfloor\alpha_2\,\lfloor\alpha_3}\,a^{\alpha_1+\alpha_3}\,b^{\alpha_1+\alpha_2}\,c^{\alpha_2+\alpha_3}$$

$\therefore$ Coefficient of $a^5b^6c^7$ is obtained by substitution

$\alpha_1 + \alpha_3 = 5, \alpha_1 + \alpha_2 = 6, \alpha_2 + \alpha_3 = 7.$

$\Rightarrow \alpha_1 = 2, \alpha_2 = 4, \alpha_3 = 3$

$\therefore$ Coefficient $=\frac{\lfloor 9}{\lfloor 2\,\lfloor 4\,\lfloor 3}=1260$

25. **(c) :** $(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+....$

$\therefore$ $(1 + 2x + 3x^2 + 4x^3 + ...)^{3/2} = ((1-x)^{-2})^{\frac{3}{2}}=(1-x)^{-3}$

$$=1+\frac{3x}{1!}+\frac{3\cdot 4}{2!}x^2+\frac{3\cdot 4\cdot 5}{3!}x^3+\frac{3\cdot 4\cdot 5\cdot 6}{4!}x^4+\frac{3\cdot 4\cdot 5\cdot 6\cdot 7}{5!}x^5+...$$

$\therefore$ Coefficient of $x^5=\frac{3\cdot 4\cdot 5\cdot 6\cdot 7}{5!}=21$

26. **(b) :** Coefficient of $(r-1)^{\text{th}}$, r^{th} and $(r+1)^{\text{th}}$ term in expansion of $(1 + x)^n$ are ${}^nC_{r-2}$, ${}^nC_{r-1}$, ${}^nC_r$ respectively.

$\Rightarrow$ ${}^nC_{r-2} : {}^nC_{r-1} : {}^nC_r = 1 : 3 : 5$

$\Rightarrow$ $\frac{{}^nC_{r-2}}{{}^nC_{r-1}}=\frac{1}{3}$ and $\frac{{}^nC_{r-1}}{{}^nC_r}=\frac{3}{5}$

$\Rightarrow$ $\frac{r-1}{n-r+2}=\frac{1}{3}$ and $\frac{r}{n-r+1}=\frac{3}{5}$

$\Rightarrow$ $3r - 3 = n - r + 2$ and $5r = 3n - 3r + 3$

$\Rightarrow$ $4r - n = 5$....(i) and $8r - 3n = 3$....(ii)

Solving (i) and (ii), we get $r = 3$ and $n = 7$

27. **(d) :** The given expression $= \sum_{r=1}^{10} A_r (B_{10} B_r - C_{10} A_r)$

$\therefore \quad \sum_{r=1}^{10} A_r B_r$ = coefficient of x^{20} in the expansion of

$\{(1 + x)^{10}(x + 1)^{20}\} - 1 = C_{20} - 1 = C_{10} - 1$ and

$\sum_{r=1}^{10} (A_r)^2$ = coefficient of x^{10} in $\{(1 + x)^{10} (x + 1)^{10}\} - 1$

$= B_{10} - 1$

We have the given expression $= B_{10}(C_{10} - 1) - C_{10}(B_{10} - 1)$

$= C_{10} - B_{10}$

28. **(a) :** $450 = 2^1 \times 5^2 \times 3^2$

$\therefore$ Sum of all positive divisors

$= (1 + 2)(1 + 3 + 9)(1 + 5 + 25) = 1209$

29. **(a) :** Given expansion $= (x + A)^n$

$\therefore \quad T_{r+1} = {}^nC_r x^{n-r} A^r$

Putting $r = 2, 3, 4$

$T_3 = {}^nC_2\, x^{n-2} A^2 = 84$

$T_4 = {}^nC_3\, x^{n-3} A^3 = 280$

$T_5 = {}^nC_4\, x^{n-4} A^4 = 560$

Now, $\frac{T_3}{T_4} = \frac{84}{280} = \frac{{}^nC_2 x^{n-2} A^2}{{}^nC_3 x^{n-3} A^3}$

$\Rightarrow \quad \frac{84}{280} = \frac{3}{n-2}\frac{x}{A} \Rightarrow \quad \frac{3}{10} = \frac{3}{n-2}\frac{x}{A}$ ...(i)

and $\frac{T_5}{T_4} = \frac{n-3}{4} \cdot \frac{A}{x} = \frac{2}{1}$ ...(ii)

Now, multiplying (i) and (ii), we get $n = 7$

Now, putting $n = 7$ in T_3 and T_4, we get

$T_3 = {}^7C_2\, x^5 A^2 = 84$

$\Rightarrow \quad \frac{7}{2} \cdot \frac{6}{1}\, x^5 A^2 = 84 \;\Rightarrow\; x^5 A^2 = 4$

and $T_4 = {}^7C_3\, x^4 A^3 = 280$

$\Rightarrow \quad \frac{7}{3} \cdot \frac{6}{2} \cdot \frac{5}{1}\, x^4 A^3 = 280$

$\Rightarrow \quad 35x^4A^3 = 280 \;\Rightarrow\; x^4A^3 = 8$

Now, $\frac{(x^5A^2)^3}{(x^4A^3)^2} = \frac{4^3}{8^2}$

$\Rightarrow \quad x^7 = 1 \;\Rightarrow x = 1$

Now $2x = A$

$\therefore \quad A = 2$

30. **(a) :** As sum of the coefficients in the expansion

$(x - 2y + 3z)^n$ is 128 $\quad \therefore \quad (1 - 2 + 3)^n = 128$

$\Rightarrow \quad n = 7$, which is odd.

$\therefore$ Greatest coefficient are ${}^7C_{\frac{7-1}{2}}$ & ${}^7C_{\frac{7+1}{2}}$ *i.e.*, ${}^7C_3$ & ${}^7C_4$ whose value is 35.

31. **(a, c) :** Since, t_4 is numerically the greatest term,

$|t_3| < |t_4|$ and $|t_5| < |t_4|$

$\Rightarrow \left|\frac{t_3}{t_4}\right| < 1$ and $\left|\frac{t_5}{t_4}\right| < 1$

But $\frac{t_3}{t_4} = \frac{{}^{10}C_2(2^8)\left(\frac{3x}{8}\right)^2}{{}^{10}C_3(2^7)\left(\frac{3x}{8}\right)^3} = \frac{2}{x}$ and $\frac{t_5}{t_4} = \frac{{}^{10}C_4(2^6)\left(\frac{3x}{8}\right)^4}{{}^{10}C_3(2^7)\left(\frac{3x}{8}\right)^3} = \frac{21x}{64}$

Now, $\left|\frac{t_3}{t_4}\right| < 1 \Rightarrow \left|\frac{2}{x}\right| < 1$

32. **(c, d) :** $C_0^2 + 3C_1^2 + 5C_2^2 + \ldots + (2n + 1)C_n^2$

$\Rightarrow \quad (C_0^2 + C_1^2 + C_2^2 + \ldots + C_n^2) + 2(C_1^2 + 2C_2^2 + \ldots + nC_n^2)$

$= {}^{2n}C_n + 2(C_1^2 + 2C_2^2 + \ldots + nC_n^2)$ ...(i)

$(1 + x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots + C_nx^n$

Differentiating both sides w.r. to x, we get

$n(1 + x)^{n-1} = C_1 + 2C_2x + 3C_3x^2 + \ldots + nC_nx^{n-1}$ ...(ii)

and $(x + 1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \ldots + C_n$

Multiplying eqns. (ii) and (iii), we get

$n(1 + x)^{2n-1} = (C_1 + 2C_2x + 3C_3x^2 + \ldots + nC_nx^{n-1})$

$\times (C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \ldots + C_n)$

Comparing the coefficients of x^{n-1} in both sides, then

$C_1^2 + 2C_2^2 + 3C_3^2 + \ldots + nC_n{}^2 = n\, {}^{2n-1}C_{n-1}$ ...(iv)

From (i) and (iv), we get

$C_0^2 + 3C_1^2 + 5C_2^2 + \ldots + (2n + 1)C_n^2 = {}^{2n}C_n + 2n \cdot {}^{2n-1}C_{n-1}$

$= \frac{2n}{n} \cdot {}^{2n-1}C_{n-1} + 2n \cdot {}^{2n-1}C_{n-1}$

$= 2(n + 1) \cdot {}^{2n-1}C_{n-1} = 2(n + 1)\, {}^{2n-1}C_n$

$= {}^{2n-1}C_{n-1} + (2n + 1) \cdot {}^{2n-1}C_{n-1}$

$= {}^{2n-1}C_n + (2n + 1) \cdot {}^{2n-1}C_{n-1}$

33. **(a, b, c) :** We have,

$(x + a)^n = {}^nC_0x^n + {}^nC_1x^{n-1}a + {}^nC_2x^{n-2}a^2 + \ldots + {}^nC_na^n$

$= [{}^nC_0x^n + {}^nC_2x^{n-2}a^2 + \ldots] + [{}^nC_1x^{n-1}a + {}^nC_3x^{n-3}a^3 + \ldots]$

or $\quad (x + a)^n = P + Q$ ...(i)

Similarly, $(x - a)^n = P - Q$ ...(ii)

Multiplying (i) and (ii), we get $P^2 - Q^2 = (x^2 - a^2)^n$

Squaring (i) and (ii) and then subtracting (ii) from (i), we get

$4PQ = (x + a)^{2n} - (x - a)^{2n}$

Squaring (i) and (ii) and then adding, we get

$2(P^2 + Q^2) = (x + a)^{2n} + (x - a)^{2n}$

34. **(a, b, c) :** We can write S as

$S = a_1 + \frac{1}{2}a_2 + \frac{1}{3}a_3 + \ldots + \frac{1}{r}a_r + \ldots + \frac{1}{n}a_n$ where

$a_r = (-1)^{r-1}(C_r - C_{r+1} + \ldots + (-1)^{n-r}C_n)$

$= (-1)^{n-1}(C_0 - C_1 + \ldots + (-1)^{n-r}C_{n-r})$ [Use $C_r = C_{n-r}$]

But for $k \geq 0$, $C_0 - C_1 + C_2 - \ldots + (-1)^kC^k$

= coefficient of x^k in $(C_0 - C_1x + C_2x^2 - \ldots + (-1)^nC_nx^n)$

$(1 + x + x^2 + \ldots)$

= coefficient of x^k in $(1 - x)^n(1 - x)^{-1}$

= coefficient of x^k in $(1 - x)^{n-1} = (-1)^k({}^{n-1}C_k)$

Thus, $a_r = (-1)^{n-1}(-1)^{n-r}({}^{n-1}C_{n-r}) = (-1)^{r-1}({}^{n-1}C_{r-1})$

Now,

$S = {}^{n-1}C_0 - \frac{1}{2}({}^{n-1}C_1) + \frac{1}{3}({}^{n-1}C_2) - \ldots + \frac{(-1)^{n-1}}{n}({}^{n-1}C_{n-1})$

$= \int_0^1 ({}^{n-1}C_0 - {}^{n-1}C_1x + {}^{n-1}C_2x^2 - \ldots + (-1)^{n-1}({}^{n-1}C_{n-1})x^{n-1})dx$

$= \int_0^1 (1 - x)^{n-1}dx = -\frac{1}{n}(1 - x)^n \Big|_0^1 = \frac{1}{n}$

35. **(a, d) :** $3^{4n} = 81^n = (1+80)^n = 1 + 80\lambda, \lambda \in N$

$\therefore \quad 3^{3^{4n}} = 3^{1+80\lambda} = 3 \cdot 3^{80\lambda}$

$= 3 \cdot (9)^{40\lambda} = 3(10-1)^{40\lambda}$

$= 3(1+10\mu) = 3 + 30\mu$

$\therefore$ Last digit of $3^{3^{4n}} + 1$ is 4.

36. **(b, c) :** $\dfrac{^nC_r}{^{r+2}C_r} = \dfrac{n!}{r!(n-r)!}\dfrac{r!2!}{(r+2)!} = \dfrac{2(n!)}{(r+2)!(n-r)!}$

$= \dfrac{2}{(n+1)(n+2)}\dfrac{(n+2)!}{(r+2)![(n+2)-(r+2)]!}$

$= \dfrac{2}{(n+1)(n+2)}\,{}^{n+2}C_{r+2}$

Thus, $S = \sum_{r=0}^{n}(-2)^r\left(\dfrac{^nC_r}{^{r+2}C_r}\right)$

$= \dfrac{2}{(n+1)(n+2)}\sum_{r=0}^{n}(-2)^r.{}^{n+2}C_{r+2}$

$= \dfrac{2}{(n+1)(n+2)}\sum_{s=0}^{n+2}(-2)^{s-2}.{}^{n+2}C_s$ [Putting $r + 2 = s$]

$= \dfrac{2}{4(n+1)(n+2)}\sum_{s=0}^{n+2}{}^{n+2}C_s(-2)^s$

$= \dfrac{1}{2(n+1)(n+2)}\times$

$\left[\sum_{s=0}^{n+2}{}^{n+2}C_s(-2)^s - {}^{n+2}C_0(2)^0 - {}^{n+2}C_1(-2)^1\right]$

$= \dfrac{1}{2(n+1)(n+2)}\left[(1-2)^{n+2} - 1 + 2(n+2)\right]$

$= \dfrac{1}{2(n+1)(n+2)}[2n+3+(-1)^n]$

But $2n + 3 + (-1)^n = \begin{cases} 2(n+2) \text{ if } n \text{ is even,} \\ 2(n+1) \text{ if } n \text{ is odd} \end{cases}$

Thus, $S = \begin{cases} \dfrac{1}{n+1} \text{ if } n \text{ is even,} \\ \dfrac{1}{n+2} \text{ if } n \text{ is odd} \end{cases}$

37. **(b, c) :** $\left(\sqrt[3]{4} + \dfrac{1}{\sqrt[4]{6}}\right)^{20}$

$= (4^{1/3} + 6^{-1/4})^{20} = (2^{2/3} + 6^{-1/4})^{20}$

$T_{r+1} = {}^{20}C_r\,(2^{2/3})^{20-r}(6^{-1/4})^r = {}^{20}C_r\,2^{(160-11r)/12} \cdot 3^{-r/4}$

$\therefore$ For $r = 8, 20$; T_{r+1} is rational.

$\therefore$ Only two terms are rational.

So, 21 – 2 = 19 terms are irrational.

38. **(a, b, c) :** $(101)^{50} - (99)^{50}$

$= (100+1)^{50} - (100-1)^{50}$

$= 2\{{}^{50}C_1(100)^{49} + {}^{50}C_3(100)^{47} + {}^{50}C_5(100)^{45} + ...\}$

$= (100)^{50} + 2\{{}^{50}C_3(100)^{47} + {}^{50}C_5(100)^{45} + ...\} > (100)^{50}$

$\Rightarrow \quad (101)^{50} - (99)^{50} > (100)^{50}$

or $\quad (101)^{50} - (100)^{50} > (99)^{50}$

Also, $\left(\dfrac{1001}{1000}\right)^{999} = \left(1 + \dfrac{1}{1000}\right)^{999}$

$= 1 + {}^{999}C_1\left(\dfrac{1}{1000}\right) + {}^{999}C_2\left(\dfrac{1}{1000}\right)^2 + ... < 1 + 1 + 1 + ... + 1$

$= 1000$

$\therefore \left(\dfrac{1001}{1000}\right)^{999} < (1000)$

$\Rightarrow \quad (1001)^{999} < (1000)^{1000}$

39. **(a) : (A) – (s), (B) – (q), (C) – (p), (D) – (r)**

(A) $17^{20} = (18-1)^{20} = 1 - 20 \cdot 18 + \text{multiple of } 18^2$

$= -35 + \text{multiple of } 9^2 = 46 + \text{multiple of } 81.$

(B) $X = x + 3, Y = y + 2, Z = z + 1, U = u \Rightarrow X + Y + Z + U = 9.$

The number of positive integer solutions is $\binom{8}{3} = 56.$

(C) $(1 + (x + 2y))^5$

$= 1 + \binom{5}{1}(x+2y) + \binom{5}{2}(x+2y)^2 + \binom{5}{3}(x+2y)^3 + ...$

The coefficient of x^2y is $\binom{5}{3} \cdot 3 \cdot 2 = 60.$

(D) $\sum_{r=1}^{10} r \cdot \dfrac{C_r}{C_{r-1}} = \sum_{r=1}^{10} r \cdot \dfrac{(10-r+1)}{r} = \sum_{r=1}^{10}(11-r)$

$= 10 + 9 + 8 + ... + 2 + 1 = \dfrac{10 \cdot 11}{2} = 55.$

40. **(b) : (A) – (r), (B) – (s), (C) – (q), (D) – (p)**

(A) Coefficient of x^{10} in

$(1+x)^{10} + (1+x)^{11} + ... + (1+x)^{20} = (1+x)^{10}\,\dfrac{[(1+x)^{11} - 1]}{x}$

= coefficient of x^{11} in $(1+x)^{21}$ which is $\binom{21}{11} = \binom{21}{10}.$

(B) $\binom{10}{10} + \binom{11}{10} + \binom{12}{10} + ... + \binom{20}{10} + \binom{10}{10} + \binom{11}{10} + \binom{12}{10}$

$+ ... + \binom{19}{10} + \binom{10}{10} + \binom{11}{10} + ... + ... + \binom{18}{10} + \binom{10}{10} ....$

$= \binom{21}{11} + \binom{20}{11} + \binom{19}{11} + ... + \binom{11}{11} = \binom{22}{12} = \binom{22}{10}$

(C) $(1+x)^{10} = \binom{10}{0} + \binom{10}{1}x + \binom{10}{2}x^2 + ...$

$(x+1)^{10} = \binom{10}{0}x^{10} + \binom{10}{1}x^9 + \binom{10}{2}x^8 + ...$

Multiplying and considering the coefficient of x^9, we get

$\binom{10}{0}\binom{10}{1} + \binom{10}{1}\binom{10}{2} + ...$ = coefficient of x^9 in $(1+x)^{20}$ which is $\binom{20}{9}.$

(D) The number of non negative solutions is $\binom{10+10-1}{10-1} = \binom{19}{9}.$

41. **(c) :** Setting $x = i$ in the given expansion and taking the real parts, we get the sum = Re $(1+i)^{16}$ = Re $(2i)^8 = 2^8$.

42. **(d) :** $C_0 + C_2 + C_4 \ldots = 2^{15}$

$C_0 - C_2 + C_4 - \ldots = 2^8$

Adding, $C_0 + C_4 + C_8 + \ldots = (2^{15} + 2^8)/2 = 2^7(2^7 + 1).$

43. **(b) :** Setting $x = 1, \omega, \omega^2$ in turn and adding, we get

$3(C_0 + C_3 + C_6 + C_9 + C_{12} + C_{15})$

$= (1+1)^{16} + (1+\omega)^{16} + (1+\omega^2)^{16}$

$= 2^{16} + (-\omega^2)^{16} + (-\omega)^{16} = 2^{16} + \omega^2 + \omega$

$= 2^{16} - 1.$

44 **(c) :** $(x_1 + x_2 + x_3 + ... + x_n)^4$

$= \sum \dfrac{4!}{\lambda_1!\lambda_2!\lambda_3!...\lambda_n!}(x_1)^{\lambda_1}(x_2)^{\lambda_2}(x_3)^{\lambda_3}(x_4)^{\lambda_4} ... (x_n)^{\lambda_n}$

where, $\lambda_1 + \lambda_2 + \lambda_3 + ... + \lambda_n = 4$ and $\lambda_1, \lambda_2, ..., \lambda_n$ are whole numbers *i.e.*, $\lambda_1, \geq 0$.

Thus, number of distinct terms in the expansion of $(x_1 + x_2 + x_3 + ... + x_n)^4$

= number of non-negative integral solutions of $\lambda_1 + \lambda_2 + \lambda_3 + ... + \lambda_n = 4$

$= {}^{4+n-1}C_{n-1} = {}^{n+3}C_{n-1} = {}^{n+3}C_4$

45. **(c) :** $(1 + x - y + z)^9$

$$= \Sigma \frac{9!}{\lambda_1!\lambda_2!\lambda_3!\lambda_4!}(1)^{\lambda_1}(x)^{\lambda_2}(-y)^{\lambda_3}(z)^{\lambda_4}$$

$$= \Sigma \frac{9!}{\lambda_1!\lambda_2!\lambda_3!\lambda_4!} \cdot x^{\lambda_2} \cdot y^{\lambda_3} \cdot z^{\lambda_4}(-1)^{\lambda_3}$$

Here, $\lambda_2 = 3, \lambda_3 = 4, \lambda_4 = 1$

and $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 9 \Rightarrow \lambda_1 = 1$

$\therefore$ Required coefficient $= \frac{9!}{1!3!4!1!}(-1)^4$

$= \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5}{1 \cdot 6 \cdot 1} \cdot 1$

$= 9 . 8 . 7 . 5 = 2520$

46. **(4) :** $(1 - x)^5 (1 + x + x^2 + x^3)^4$

$= (1 - x)[(1 - x)(1 + x + x^2 + x^3)]^4$

$= (1 - x)(1 - x^4)^4$

$= (1 - x)(1 - 4x^4 + 6x^8 - 4x^{12} + x^{16})$

$\therefore$ Coefficient of x^{13} is 4.

47. **(2) :** Coeff. of x^n in $(1 + x)^{2n}$ is ${}^{2n}C_n = A$ & Coeff.of x^n in $(1 + x)^{2n-1}$ is ${}^{2n-1}C_n = B$

$$\therefore \frac{A}{B} = \frac{{}^{2n}C_n}{{}^{2n-1}C_n} = \frac{\lfloor 2n}{\lfloor n \lfloor n} \cdot \frac{\lfloor n \lfloor 2n-1-n}{\lfloor 2n-1} = \frac{2n}{n} = 2$$

48. **(7) :** $3^1 = 3 \Rightarrow$ unit's place = 3

$3^2 = 9 \Rightarrow$ unit's place = 9

$3^3 = 27 \Rightarrow$ unit's place = 7

$3^4 = 81 \Rightarrow$ unit's place = 1

$3^5 = 243 \Rightarrow$ unit's place = 3

Continuing this process $3^{183} = (3^4)^{45} \cdot 3^3$

$\therefore$ Unit's place = $1 \times 7 = 7$

Unit's place of 183! = 0

Hence, unit's place of $183! + 3^{183} = 0 + 7 = 7$.

49. **(7) :** In expansion of $\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9$, the general term is

$$T_{r+1} = {}^9C_r \left(\frac{3}{2}x^2\right)^{9-r}\left(-\frac{1}{3x}\right)^r$$

$$= (-1)^r\, {}^9C_r \frac{3^{9-2r}}{2^{9-r}} x^{18-3r}$$

Now, $x^0 = x^{18-3r}$ (for independent term)

$\therefore$ $18 - 3r = 0 \Rightarrow r = 6$

$\therefore$ $T_{r+1} = T_7$ is the term independent of x

50. **(1) :** Sum of the coefficients in expansion of $(ax^3 - 3x + 2)^{25}$ = Sum of coefficients in $(x^2 - ay^2)^{25}$

$\therefore$ Putting $x = y = 1$, we get

$(a - 1)^{25} = (1 - a)^{25} \Rightarrow 2(a - 1)^{25} = 0 \Rightarrow a = 1$

JEE MATHEMATICS

DPS-6

DAILY PRACTICE SHEET

Class XI

Sequences and Series

SOLUTIONS

1. **(a) :** Let A = first term and d = common difference

Then, $S_p = \frac{p}{2}[2A+(p-1)d] = a$

$\therefore \frac{2a}{p} = 2A+(p-1)d$ ....(i)

Similarly, $\frac{2b}{q} = 2A+(q-1)d$ ...(ii)

$\frac{2c}{r} = 2A+(r-1)d$ ...(iii)

Multiplying (i), (ii) and (iii) by $q-r$, $r-p$ and $p-q$ respectively and adding, we get

$\Sigma\frac{a}{p}(q-r) = 0$

2. **(c) :** Let $t_1, t_2, t_3, ..., t_{12}$ be in A.P.

Then, $\frac{6}{2}(t_1+t_6) = 5\cdot\frac{6}{2}(t_7+t_{12})$

$\Rightarrow t_1+(t_1+5d) = 5[(t_1+6d)+(t_1+11d)]$

(where d is common difference)

$\Rightarrow t_1 + 10d = 0 \Rightarrow t_{11} = 0$

3. **(c) :** Let d be the common difference of the given A.P.

$\therefore \sum_{r=1}^{2014}\frac{1}{a_r a_{r+1}} = \frac{1}{d}\left[\frac{d}{a_1a_2}+\frac{d}{a_2a_3}+....+\frac{d}{a_{2014}a_{2015}}\right]$

$= \frac{1}{d}\left[\frac{a_2-a_1}{a_1a_2}+\frac{a_3-a_2}{a_2a_3}+....+\frac{a_{2015}-a_{2014}}{a_{2014}\cdot a_{2015}}\right]$

$= \frac{1}{d}\left[\left(\frac{1}{a_1}-\frac{1}{a_2}\right)+\left(\frac{1}{a_2}-\frac{1}{a_3}\right)+....+\left(\frac{1}{a_{2014}}-\frac{1}{a_{2015}}\right)\right]$

$= \frac{1}{d}\left[\frac{1}{a_1}-\frac{1}{a_{2015}}\right] = \frac{1}{d}\left[\frac{a_{2015}-a_1}{a_1a_{2015}}\right]$

$= \frac{1}{d}\left[\frac{a_1+(2015-1)d-a_1}{a_1a_{2015}}\right]$

$= \frac{2014}{a_1a_{2015}} = \frac{2014}{2013}$ (Given)

$\therefore a_1a_{2015} = 2013$

Also, $a_{101}+a_{305}+a_{509}+a_{1507}+a_{1711}+a_{1915} = 6042$

$\Rightarrow (a_1+100d)+(a_1+304d)+(a_1+508d)+(a_{2015}-508d) + (a_{2015}-304d)+(a_{2015}-100d) = 6042$

$\Rightarrow 3(a_1+a_{2015}) = 6042$

$\Rightarrow a_1+a_{2015} = 2014$

$\therefore$ Equation having roots a_1 and a_{2015} is

$x^2 - 2014x + 2013 = 0.$

4. **(a) :** Let the numbers in A.P. are $a-3d, a-d, a+d, a+3d$.

$\therefore (a-3d)+(a-d)+(a+d)+(a+3d) = 6 \Rightarrow a = \frac{3}{2}$

and, $(a-3d)(a+3d) = 10(a-d)(a+d)$

$\Rightarrow a^2 - 9d^2 = 10(a^2-d^2) \Rightarrow d^2 = 9a^2 \Rightarrow d = \pm\frac{9}{2}$

$\therefore$ Numbers in A.P. are 15, 6, –3, –12 or –12, –3, 6, 15

$\therefore$ Sum of their numerical values

$= |15|+|6|+|-3|+|-12| = 36$

5. **(b) :** We have, $2\log_9(3^{1-x}+2) = 1+\log_3(4\times3^x-1)$

$\Rightarrow 2.\frac{1}{2}\log_3(3^{1-x}+2) = \log_3 3+\log_3(4\times3^x-1)$

$\Rightarrow \frac{3}{3^x}+2 = 12\times3^x-3$

$\Rightarrow 12y^2-5y-3 = 0$ [where $3^x = y$]

$\Rightarrow y = -\frac{1}{3}, \frac{3}{4} \Rightarrow 3^x = -\frac{1}{3}, \frac{3}{4}$

But, $3^x \neq -\frac{1}{3}$

$\therefore 3^x = \frac{3}{4} \Rightarrow x = \log_3\left(\frac{3}{4}\right) = 1-\log_3 4$

$\because 1 < \log_3 4 < 2 \quad \therefore -1 < x < 0$

$\Rightarrow [x] = -1$

6. **(d) :** Since the given numbers are in A.P.

$\therefore 28 = 3^{2\sin2\theta-1}+3^{4-2\sin2\theta}$

$\Rightarrow 28 = \frac{9^{\sin2\theta}}{3}+\frac{81}{9^{\sin2\theta}}$

$\Rightarrow x^2-84x+243 = 0$, where $x = 9^{\sin 2\theta}$

$\Rightarrow (x-81)(x-3) = 0 \therefore x = 81$ or 3

$\therefore x = 9^{\sin2\theta} = 81, 3$ or $9^2, 9^{1/2}$

$\therefore \sin2\theta = 2$ or 1/2

Since $\sin2\theta$ cannot be greater than 1 so we choose $\sin 2\theta = 1/2$

Hence the terms in A.P. are 3^0, 14, 27 *i.e.* 1, 14, 27.

$\therefore T_5 = a+4d = 1+4\cdot13 = 53$

7. **(c) :** Let d = common difference

Now, $a_1^2-a_2^2+a_3^2-a_4^2+...+a_{2n-1}^2-a_{2n}^2$

$= (-d)[a_1+a_2 ... + a_{2n}] = (-d)\left(\frac{2n}{2}\right)[a_1+a_{2n}]$

$= -nd(a_1+a_{2n})$

$= -n\left[\frac{a_{2n}-a_1}{2n-1}\right](a_1+a_{2n})$ $\left[\begin{array}{l}\text{as } a_{2n} = a_1+(2n-1)d \\ \therefore d = \frac{a_{2n}-a_1}{2n-1}\end{array}\right]$

$= \frac{-n}{2n-1}(a_{2n}^2-a_1^2) = \frac{n}{2n-1}(a_1^2-a_{2n}^2)$

8. **(c) :** Let r be the common ratio.

$\therefore x = \frac{y}{r}, z = yr$

Now, $a+c = x-\frac{1}{x}+z-\frac{1}{z} = \frac{y}{r}-\frac{r}{y}+yr-\frac{1}{yr}$

$$= y\left(\frac{1}{r}+r\right)-\frac{1}{y}\left(r+\frac{1}{r}\right)=\left(y-\frac{1}{y}\right)\left(r+\frac{1}{r}\right)=b\left(r+\frac{1}{r}\right)$$

$\Rightarrow \quad \frac{a}{b}+\frac{c}{b}=r+\frac{1}{r}$

But, $r=\frac{y}{x}=\frac{z}{y}$ also. $\therefore\ \frac{a}{b}+\frac{c}{b}=\frac{y}{z}+\frac{y}{x}$

9. **(b) :** As $S_\infty = \frac{a}{1-r}$; a = first term, r = common ratio

$\therefore$ According to question,

$$S_r = \frac{r}{1-\frac{1}{r+1}} = r+1$$

$\sum_{r=1}^{n} S_r = 2015 \Rightarrow S_1+S_2+S_3+....+S_n = 2015$

$\Rightarrow \quad 2 + 3 + 4 + ... + (n + 1) = 2015$

$\Rightarrow \quad \frac{n}{2}\{2\times 2+(n-1)\cdot 1\} = 2015 \Rightarrow n(n + 3) = 4030$

$\Rightarrow \quad n^2 + 3n - 4030 = 0$

$\Rightarrow \quad n = 62$ or -65 But $n \nless 0$

$\Rightarrow \quad n = 62$

10. **(a) :** Let three numbers in A.P. be $a - d, a, a + d$.

$\therefore \quad (a - d) + a + (a + d) = 21$

$\Rightarrow \quad a = 7$

Also, $2 + (a - d), 2 + a, 14 + (a + d)$ are in G.P.

$\Rightarrow \quad (2 + a)^2 = (2 + a - d)(14 + a + d)$

$\Rightarrow \quad 81 = (9 - d)(21 + d) \qquad [\because a=7]$

$\Rightarrow \quad d^2 + 12d - 108 = 0$

$\Rightarrow \quad d = 6$ or -18

$\therefore \quad d = 6$ ($\because d = -18$ makes 3rd no. negative)

$\therefore \because$ Three numbers are 1, 7, 13.

11. **(b) :** Let $a, ar, ar^2, ..., ar^{n-1}$ be n terms of G.P.

$\therefore \quad S = a + ar + ar^2 + ar^3 + ... + ar^{n-1}$ *i.e.* n terms

$\Rightarrow \quad S = \frac{a(1-r^n)}{(1-r)} \qquad ...(i)$

$\therefore \quad P$ = product $= a \cdot ar \cdot ar^2 .... ar^{n-1}$

$= a^n r^{1+2+3+4+5+....+n-1} = a^n \cdot r^{n(n-1)/2}$

$\therefore\ P^2 = a^{2n} r^{n(n-1)} \qquad ...(ii)$

$R = \frac{1}{a}+\frac{1}{ar}+\frac{1}{ar^2}+\frac{1}{ar^3}+....+\frac{1}{ar^{n-1}} \qquad$ (n terms)

$\therefore\ R = \frac{1}{a}\frac{\left(1-\frac{1}{r^n}\right)}{1-1/r} = \frac{(r^n-1)}{(r-1)}\cdot\frac{1}{ar^{n-1}} \qquad ..... (iii)$

$\therefore\ \frac{S}{R} = a\cdot\frac{(1-r^n)}{1-r}\cdot\frac{(r-1)}{(r^n-1)}\cdot ar^{n-1} = a^2 r^{(n-1)} \qquad$ (by (i) and (ii))

$\therefore\ [S/R]^n = a^{2n} r^{n(n-1)} = P^2 \qquad$ (by (ii))

12. **(c) :** Let three numbers in G.P. be a, ar, ar^2, then according to question, $a + ar + ar^2 = x\cdot ar$

or, $\quad r^2 + r(1 - x) + 1 = 0$, r is real

$\Delta > 0$ *i.e.*, $(1-x)^2 - 4 > 0$

$\Rightarrow \quad x^2 - 2x - 3 > 0 \quad \Rightarrow \quad (x + 1)(x - 3) > 0$

$\Rightarrow \quad x < -1$ or $x > 3$

13. **(d) :** Let the G.P. is a, ar, ar^2, ...

Given, $T_n = T_{n+1} + T_{n+2}$

$\therefore \quad a = a(r + r^2)$

or $\quad 1 = r + r^2$

$\therefore \quad r^2 + r - 1 = 0$

$\Rightarrow \quad r = \frac{-1\pm\sqrt{5}}{2} \Rightarrow r = \frac{-1+\sqrt{5}}{2}$

($r = \frac{-1-\sqrt{5}}{2}$ is rejected as terms are positive)

Now, $r = 2\cdot\frac{\sqrt{5}-1}{4} = 2\sin 18°$

14. **(d) :** Given, $a, b, c \in$ G.P. $\therefore\ b^2 = ac$

and $\log\left(\frac{5c}{a}\right), \log\left(\frac{3b}{5c}\right), \log\left(\frac{a}{3b}\right)$ are in A.P.

$\Rightarrow \quad 2\log\left(\frac{3b}{5c}\right) = \log\left(\frac{5c}{a}\right)+\log\left(\frac{a}{3b}\right) = \log\left(\frac{5c}{a}\cdot\frac{a}{3b}\right)$

$\Rightarrow \quad \left(\frac{3b}{5c}\right)^2 = \frac{5c}{a}\cdot\frac{a}{3b} = \frac{5c}{3b}$

or $\quad (3b)^3 = (5c)^3$ or $\frac{b}{c}=\frac{5}{3}$ or $c = \frac{3b}{5} \qquad ...(i)$

or $\quad \frac{b^2}{c^2} = \frac{25}{9}$ or $\frac{ac}{c^2} = \frac{25}{9}$ ($\because a, b, c \in$ G.P.)

$\Rightarrow \quad \frac{a}{c} = \frac{25}{9}$ or $a = \frac{25}{9}c = \frac{25}{9}\times\frac{3b}{5} = \frac{5}{3}b \qquad ...(ii)$

$\therefore \quad$ Numbers are $\frac{5}{3}b, b, \frac{3}{5}b$ or 25, 15, 9 (Using (i) and (ii))

which do not form any triangle as sum of two sides is less than the third side.

15. **(c) :** As $b_1, b_2, ..., b_n \in$ H.P.

$\therefore \quad \frac{1}{b_1}, \frac{1}{b_2}, ..., \frac{1}{b_n} \in$ A.P.

$\therefore \quad d = \frac{b_1-b_2}{b_1b_2} = \frac{b_2-b_3}{b_2b_3} = ... = \frac{b_{n-1}-b_n}{b_{n-1}b_n}$

$\Rightarrow \quad d = \frac{b_1-b_n}{b_1b_2+b_2b_3+...+b_{n-1}b_n}$

$$= \frac{b_1b_n\left(\frac{1}{b_n}-\frac{1}{b_1}\right)}{b_1b_2+b_2b_3+...+b_{n-1}b_n} = \frac{b_1b_n\left(\frac{1}{b_1}+(n-1)d-\frac{1}{b_1}\right)}{b_1b_2+b_2b_3+...+b_{n-1}b_n}$$

$\Rightarrow\ b_1b_2 + b_2b_3 + ... + b_{n-1}b_n = (n-1)b_1b_n$

16. **(c) :** As $a_1, a_2, a_3, ..., a_n$ are in H.P.

$\therefore \quad \frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, ..., \frac{1}{a_n}$ are in A.P.

Multiply each term by $(a_1 + a_2 + a_3 + ... + a_n)$, then subtracting 1 from each term, we get

$\frac{a_2+a_3+....+a_n}{a_1}, \frac{a_1+a_3+....+a_n}{a_2}, ..., \frac{a_1+a_2+....+a_{n-1}}{a_n}$

are in A.P.

$\therefore \quad \frac{a_1}{a_2+a_3+....+a_n}, \frac{a_2}{a_1+a_3+...+a_n}, ...$

$\frac{a_n}{a_1+a_2+....+a_{n-1}}$ are in H.P.

17. **(a) :** As, $a_1, a_2, a_3, a_4, a_5 \in$ H.P.,

$\therefore \quad \frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \frac{1}{a_4}, \frac{1}{a_5} \in$ A.P.

$\therefore \quad d = \frac{1}{a_2}-\frac{1}{a_1} = \frac{1}{a_3}-\frac{1}{a_2} = \frac{1}{a_4}-\frac{1}{a_3} = \frac{1}{a_5}-\frac{1}{a_4}$

$\Rightarrow \quad a_1 - a_2 = da_1a_2,\ a_2 - a_3 = da_2a_3,\ a_3 - a_4 = da_3a_4,$

$a_4 - a_5 = da_4a_5$

Now, by adding all above relations, we get

$a_1 - a_5 = d[a_1a_2 + a_2a_3 + a_3a_4 + a_4a_5]$

$$\Rightarrow \quad \frac{a_1 - a_5}{a_1a_5} = \frac{d[a_1a_2 + a_2a_3 + a_3a_4 + a_4a_5]}{a_1a_5}$$

(Dividing both sides by a_1a_5)

$$\Rightarrow \quad \frac{1}{a_5} - \frac{1}{a_1} = \frac{d[a_1a_2 + a_2a_3 + a_3a_4 + a_4a_5]}{a_1a_5}$$

$$\Rightarrow \quad \frac{1}{a_1} + 4d - \frac{1}{a_1} = \frac{d[a_1a_2 + a_2a_3 + a_3a_4 + a_4a_5]}{a_1a_5}$$

$\Rightarrow \quad 4a_1a_5 = a_1a_2 + a_2a_3 + a_3a_4 + a_4a_5$

$$\Rightarrow \quad 4a_1a_5 = \sum_{j=1}^{4} a_j a_{j+1}$$

which is the root of equation $x^2 - 6x + 8 = 0$.

18. **(c) :** As h_1, h_2, h_3 ... are in H.P.

$\therefore \quad \frac{1}{h_1}, \frac{1}{h_2}, \frac{1}{h_3}, \ldots\ldots$ are in A.P.

$\therefore \quad \frac{1}{h_{r+1}} - \frac{1}{h_r}$ = common difference = d(say) …(i)

Also, $\frac{1}{h_{r+1}} = \frac{1}{h_1} + (r+1-1)d \Rightarrow \frac{1}{h_{r+1}} - \frac{1}{h_1} = rd$ …(ii)

$$\text{Now, } \sum_{r=1}^{2014} i^{2r}\left(\frac{h_r + h_{r+1}}{h_r - h_{r+1}}\right) = \sum_{r=1}^{2014} (-1)^r \left(\frac{\frac{1}{h_{r+1}} + \frac{1}{h_r}}{\frac{1}{h_{r+1}} - \frac{1}{h_r}}\right)$$

$$= \sum_{r=1}^{2014} \frac{(-1)^r}{d}\left(\frac{1}{h_{r+1}} + \frac{1}{h_r}\right) \quad \text{[using (i)]}$$

$$= \frac{1}{d}\left[-\left(\frac{1}{h_2} + \frac{1}{h_1}\right) + \left(\frac{1}{h_3} + \frac{1}{h_2}\right) - \left(\frac{1}{h_4} + \frac{1}{h_3}\right) + \ldots\ldots + \left(\frac{1}{h_{2015}} + \frac{1}{h_{2014}}\right)\right]$$

$$= \frac{1}{d}\left[\frac{1}{h_{2015}} - \frac{1}{h_1}\right] = \frac{1}{d} \cdot 2014\, d \quad \text{[Putting } r = 2014 \text{ in (ii)]}$$

$= 2014$

19. **(c) :** Given, a, b, c are in G.P.

$\Rightarrow \frac{b}{a} = \frac{c}{b} \quad \Rightarrow \frac{b}{b^{y/x}} = \frac{b^{y/z}}{b}$ $\quad [\because\ a^x = b^y = c^z]$

$\Rightarrow b^{1-\frac{y}{x}} = b^{\frac{y}{z}-1} \quad \Rightarrow \frac{1}{y} - \frac{1}{x} = \frac{1}{z} - \frac{1}{y}$

$\Rightarrow 2 \cdot \frac{1}{y} = \frac{1}{x} + \frac{1}{z}$

$\therefore \quad x, y, z$ are in H.P.

20. **(c) :** As $a, b, c \in$ H.P. and G.M. > H.M.

$\Rightarrow \sqrt{ac} > b \Rightarrow ac > b^2$

Consider $D = 4b^2 - 4ac = 4(b^2 - ac) < 0$ ...(i)

$\therefore$ Roots are imaginary and conjugate of each other.

Now, roots are imaginary and conjugate of each other, we know that $z + \bar{z}$ and $z\bar{z}$ is purely real, so our choice (c) is correct.

21. **(a) :** Sum of n A.M.s' between any 2 numbers is n times the single mean between the 2 numbers.

$\therefore \quad a + b + c = 3\left(\frac{\alpha + \beta}{2}\right) \Rightarrow \frac{9}{2} = \frac{3}{2}(\alpha + \beta) \ \therefore \alpha + \beta = 3$ …(i)

and, $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 3\left(\frac{\frac{1}{\alpha} + \frac{1}{\beta}}{2}\right) \Rightarrow \frac{9}{4} = \frac{3}{2}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)$

$\Rightarrow \frac{\alpha + \beta}{\alpha\beta} = \frac{3}{2} \ \therefore \alpha\beta = 2$ (using (i) ...(ii)

From (i) and (ii) $\alpha = 1, \beta = 2$ or $\alpha = 2, \beta = 1$

But, $2\alpha < 3\beta$ (given) $\quad \therefore \quad \alpha = 1, \beta = 2$

$\therefore \quad 1 + 10(\beta + 100\alpha) = 1 + 10(2 + 100) = 1021$

22. **(b) :** Given, $a + b = 2A$ ...(i)

and $a, p, q, b \in$ G.P.

$\therefore \quad p^2 = aq$ and $q^2 = pb$

$\Rightarrow \quad p^3 = apq$ and $q^3 = bpq$

By adding, we get

$p^3 + q^3 = apq + bpq = pq(a + b) = 2Apq$ (from (i))

23. **(b) :** As each root of the equation is 1

$\therefore \quad (x - 1)^4 = x^4 - 4x^3 + ax^2 + bx + 1$

$\Rightarrow \quad x^4 - 4x^3 + 6x^2 - 4x + 1 = x^4 - 4x^3 + ax^2 + bx + 1$

$\Rightarrow \quad a = 6, b = -4$

Again, let $\alpha, \beta, \gamma, \delta$ are the roots (each > 0) of the equation

$\therefore \quad \alpha + \beta + \gamma + \delta = 4$ and $\alpha\beta\gamma\delta = 1$

$\Rightarrow \quad \frac{\alpha + \beta + \gamma + \delta}{4} = 1 \ \& \ (\alpha\beta\gamma\delta)^{1/4} = 1$

$\Rightarrow$ A.M. of roots = G.M. of roots

$\Rightarrow$ All the roots are equal

Now Statement-1 and Statement-2 both are true but Statement-2 is not correct explanation of Statement-1. So choice (b) is correct.

24. **(d) :** A.M. $\geq$ G.M. as a, b, c are distinct real numbers

$\therefore \quad \frac{a^2 + b^2}{2} > \sqrt{a^2b^2}$ *i.e.,* $a^2 + b^2 > 2ab$ ...(i)

Similarly, $b^2 + c^2 > 2bc$ ...(ii)

and $c^2 + a^2 > 2ac$ ...(iii)

Now by adding (i), (ii) (iii), we get

$2(a^2 + b^2 + c^2) > 2(ab + bc + ca)$

$\Rightarrow \quad ab + bc + ca < 1 \ (\because a^2 + b^2 + c^2 = 1)$

$\therefore$ Statement-1 is false and Statement-2 is true.

25. **(a) :** If we write terms of all brackets in reverse order then it becomes (1) + (4 + 3 + 2) + (9 + 8 + 7 + 6 + 5) + … . Here, 1st term of each bracket are in the form $1^2, 2^2, 3^2$, ... and number of terms in brackets are 1, 3, 5, ... *i.e.*, in $(2n - 1)$ form.

$\therefore$ Required sum of all the terms in the n^{th} bracket of given series.

$$= \frac{2n-1}{2}\{2 \cdot n^2 + (2n-1-1)(-1)\} = (2n-1)(n^2 - n + 1)$$

$= 2n^3 - 3n^2 + 3n - 1 = (n - 1)^3 + n^3$

26. **(d) :** $S = \frac{1}{1 - (1/2)} = 2$

$$S_n = \frac{1 - (1/2)^n}{1 - (1/2)} = 2 - \frac{1}{2^{n-1}}$$

$$S - S_n = \frac{1}{2^{n-1}} < \frac{1}{1000} \ \text{ or } \ 2^{n-1} \geq 1000$$

Now $2^{10} = 32 \times 32 = 1024 > 1000$

$\therefore \quad n - 1 \geq 10$ or $n \geq 11$

Hence, the least value of n is 11.

27. **(c) :** We have

$$\frac{1}{k}(1 + 2 + 3 + \ldots\ldots + k) = \frac{1}{k}\frac{k(k+1)}{2} = \frac{k+1}{2}$$

Thus, $S = \frac{1}{2}[2 + 3 + 4 + \ldots\ldots + 21] = \frac{10}{2}(2 + 21) = 115$

28. **(c) :** We have $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+$........ upto ∞

$=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}$........upto ∞

$-\frac{1}{2^2}\left[1+\frac{1}{2^2}+\frac{1}{3^2}+......\right]=\frac{\pi^2}{6}-\frac{1}{4}\left(\frac{\pi^2}{6}\right)=\frac{\pi^2}{8}$

29. **(a) :** We can write S as

$S = (1 - 2)(1 + 2) + (3 - 4)(3 + 4) + ... + (2001 - 2002)(2001 + 2002) + 2003^2$

$= -[1 + 2 + 3 + 4 + ... + 2002] + 2003^2$

$=-\frac{1}{2}(2002)(2003)+2003^2=2007006$

30. **(d) :** We can write the sum upto $(2n + 1)$ terms as

$[a + (a + d)](-d) + [(a + 2d) + (a + 3d)](-d) + ... [(a + (2n - 2)d) + (a + (2n - 1)d](-d) + (a + 2nd)^2$

$= (-d)[a + (a + d) + (a + 2d) + ... + a + (2n - 1)d] + (a + 2nd)^2$

$=(-d)\frac{2n}{2}\{a+a+(2n-1)d\}+(a+2nd)^2$

$= -2nad - n(2n - 1)d^2 + a^2 + 4n(ad) + 4n^2d^2$

$= a^2 + 2nad + n(2n + 1)d^2$

31. **(b, d) :** Let the sides be a, ar, ar^2. If $r > 1$, then $(ar^2)^2 = (a)^2 + (ar)^2$ (since in this case (ar^2) will be the hypotenuse *i.e.*, the largest side)

$\Rightarrow r^4=1+r^2 \Rightarrow r^2=\frac{1+\sqrt{5}}{2},\left(\because r^2=\frac{1-\sqrt{5}}{2}\text{ is not possible}\right)$

$\Rightarrow r=\sqrt{\frac{1+\sqrt{5}}{2}}$

If $0 < r < 1$ then a is the largest side

$\therefore a^2=(ar)^2+(ar^2)^2 \quad \therefore r=\sqrt{\frac{\sqrt{5}-1}{2}}$

32. **(a, b, c, d) :** On rationalizing each term, we get series

$\frac{\sqrt{7}-\sqrt{3}}{7-3}+\frac{\sqrt{11}-\sqrt{7}}{11-7}+\frac{\sqrt{15}-\sqrt{11}}{15-11}+.....$ up to n terms

$=\frac{1}{4}\left[\sqrt{3+4n}-\sqrt{3}\right]$ which is equal to $\frac{n}{\sqrt{3+4n}+\sqrt{3}}$

$\therefore$ (a) and (b) are correct

(c) $\frac{n}{\sqrt{3+4n}+\sqrt{3}}<n$

(d) $\frac{n}{\sqrt{3+4n}+\sqrt{3}}<\frac{n}{\sqrt{4n}}=\frac{\sqrt{n}}{2}$

33. **(a, b, c, d) :** We have $b_3 > 4b_2 - 3b_1$

$\Rightarrow b_1r^2 > 4b_1r - 3b_1$

$\Rightarrow r^2>4r-3 \quad [\because b_1>0]$

$\Rightarrow r^2-4r+3>0 \Rightarrow (r-3)(r-1)>0$

$\Rightarrow r>3$ or $r<1$

And $r = 3.5$ and $r = 5.2$ are both greater than 3.

34. **(b, c) :** Let $x = t_p = a + (p - 1)d$, $y = t_q = a + (q - 1)d$ & $z = t_r = a + (r - 1)d$ where $a = 1^{st}$ term and d = common difference

Then, $\frac{x-y}{y-z}=\frac{p-q}{q-r},\ \frac{z-x}{x-y}=\frac{r-p}{p-q}$

$\because$ p, q, r are three distinct integers

$\therefore \frac{x-y}{y-z},\frac{z-x}{x-y}$ both are rational and so, their sum is also rational.

$[\pi] = 3, \{e\} = e - 2, \text{Re}(\omega)=\text{Re}\left(\frac{-1\pm\sqrt{3}i}{2}\right)=-\frac{1}{2}$

Also, $\log_2 3$ is irrational

Hence, $\frac{x-y}{y-z}+\frac{z-x}{x-y}$ can't be $\{e\}$ and $\log_2 3$ which are irrational.

35. **(b, d) :** Let A, a_1, a_2, B be in A. P.

$\therefore a_1=A+\frac{B-A}{3}=\frac{2A+B}{3}$

$\therefore a_2=A+2\cdot\frac{B-A}{3}=\frac{A+2B}{3}$

Also, A, g_1, g_2, B are in G.P.

$\therefore \left(\frac{B}{A}\right)^{1/3}=r$

$\therefore g_1=Ar=A(B/A)^{1/3}$

$\therefore g_2=Ar^2=A(B/A)^{2/3}$

$\therefore g_1g_2=A^2(B/A)=AB$

Also, A, h_1, h_2, B are in H.P.

$\therefore \frac{1}{A},\frac{1}{h_1},\frac{1}{h_2},\frac{1}{B}$ are in A.P.

$\therefore \frac{1}{h_1}=\frac{1}{A}+\frac{1}{3}\left(\frac{1}{B}-\frac{1}{A}\right)=\frac{1}{A}+\frac{A-B}{3AB}$

$\Rightarrow \frac{1}{h_1}=\frac{3B+A-B}{3AB}=\frac{A+2B}{3AB} \Rightarrow h_1=\frac{3AB}{A+2B}$ and

$\frac{1}{h_2}=\frac{1}{A}+\frac{2}{3}\left(\frac{1}{B}-\frac{1}{A}\right)=\frac{3B+2(A-B)}{3AB}$

$\Rightarrow \frac{1}{h_2}=\frac{2A+B}{3AB} \quad \therefore h_2=\frac{3AB}{2A+B}$

Obviously, $g_1g_2 = AB = a_1h_2 = a_2h_1$

36. **(a, d) :** Let the four numbers be $a, a + d, a + 2d, (a + d)\cdot r^2$.

where d is the common difference of A.P. and r is common ratio of the G.P.

$a = 6$ and $r=\frac{1}{2}$ (given)

$a + d, a + 2d, (a + d)r^2$ are in G.P.

$\Rightarrow (a + 2d)^2 = (a + d)^2 r^2 \Rightarrow (6+2d)^2=(6+d)^2\cdot\frac{1}{4}$

$\Rightarrow 6+2d=(6+d)\cdot\frac{1}{2}$

$\Rightarrow d = -2 \quad \therefore$ The four numbers are 6, 4, 2, 1.

37. **(b, c, d) :** $h_2=\frac{2h_1h_3}{h_1+h_3} \Rightarrow h_1h_2+h_2h_3=2h_1h_3$ ...(i)

$h_3=\frac{2h_1h_5}{h_1+h_5} \Rightarrow h_1h_3+h_3h_5=2h_1h_5$ ...(ii)

$h_4=\frac{2h_3h_5}{h_3+h_5} \Rightarrow h_3h_4+h_4h_5=2h_3h_5$ ...(iii)

Adding (i) and (iii), we get

$h_1h_2+h_2h_3+h_3h_4+h_4h_5=2(h_1h_3+h_3h_5)=2(2h_1h_5)$ [Using (ii)]

$\Rightarrow \sum_{r=1}^{4}h_r h_{r+1}=4h_1h_5=kh_xh_y$ (given)

$\therefore k = 4\ ;\ x = 1, y = 5$ or $x = 5, y = 1$

38. **(a, b, c) :** $A_1 = a + \frac{1}{3}(b-a),\ A_2 = a + \frac{2}{3}(b-a)$

$\Rightarrow A_1 + A_2 = a + b$

Similarly, $G_1 = a\left(\frac{b}{a}\right)^{1/3},\ G_2 = a\left(\frac{b}{a}\right)^{2/3} \Rightarrow G_1G_2 = ab$

and $\frac{1}{H_1} = \frac{1}{a} + \frac{1}{3}\left(\frac{1}{b} - \frac{1}{a}\right),\ \frac{1}{H_2} = \frac{1}{a} + \frac{2}{3}\left(\frac{1}{b} - \frac{1}{a}\right)$

Now, $\frac{1}{H_1} + \frac{1}{H_2} = \frac{1}{a} + \frac{1}{b}$

$\Rightarrow \frac{H_1 + H_2}{H_1H_2} = \frac{a+b}{ab} = \frac{A_1 + A_2}{G_1G_2}$

$\Rightarrow \frac{G_1G_2}{H_1H_2} = \frac{A_1 + A_2}{H_1 + H_2}$

Now, $H_1 + H_2 = \frac{3ab}{a+2b} + \frac{3ab}{2a+b} = \frac{9ab(a+b)}{(a+2b)(2a+b)}$

$\Rightarrow \frac{A_1 + A_2}{H_1 + H_2} = \frac{(a+2b)(2a+b)}{9ab}$

Thus, $\frac{G_1G_2}{H_1H_2} - \frac{5}{9} = \frac{2}{9}\left(\frac{a}{b} + \frac{b}{a}\right)$

39. **(c) : (A)-(s), (B)-(r), (C)-(q), (D)–(p)**

(A) $5^{2+4+6+ \,........\, + 2x} = (25)^{28}$

$\Rightarrow 5^{x(x+1)} = 5^{56} \Rightarrow x^2 + x - 56 = 0 \Rightarrow x = 7$ as $x > 0$

(B) $2\log_5 x = \log_{\sqrt{5}}\left(\frac{1/4}{1-1/2}\right)\log_5(0.2)$

$= \log_{\sqrt{5}}\left(\frac{1}{2}\right)\log_5\left(\frac{1}{5}\right) = -\frac{\log_5\left(\frac{1}{2}\right)}{\log_5\sqrt{5}} = \log_5 4 \Rightarrow x = 2$

(C) $\log x = \log_{2.5}\left(\frac{1/3}{1-1/3}\right)\log(0.16)$

$= \log_{5/2}(1/2)\log(2/5)^2 = \log 4 \Rightarrow x = 4$

(D) $3^x\frac{(1/3)}{1-1/3} = \frac{2(5^2)}{1-1/5} \Rightarrow \frac{1}{2}(3^x) = \frac{1}{2}(5^3)$

$\Rightarrow x = 3\log_3 5$

40. **(b) : (A)-(r), (B)-(p), (C)-(s), (D)-(q)**

(A) $F(n+1) = \frac{2F(n)+1}{2} = F(n) + \frac{1}{2}$

$\therefore F(1), F(2), F(3), ...$ is an AP with common difference 1/2.

(B) $a_1 + 2d + a_1 + 4d + a_1 + 10d + a_1 + 16d + a_1 + 18d$

$= 5a_1 + 50d = 5(a_1 + 10d) = 10$

i.e. $a_1 + 10d = 2$

Now, $\sum_{i=1}^{21} a_i = \frac{21}{2}[2a_1 + 20d] = 21(a_1 + 10d) = 42$

(C) $S = 1 + 5 + 13 + 29 + ... + t_{10}$

$S = 1 + 5 + 13 + ... + t_9 + t_{10}$

Subtracting

$t_{10} = 1 + 4 + 8 + 16 + ...$ upto 10 terms

$= 1 + (4 + 8 + 16 + ...$ upto 9 terms)

$= 2045$

(D) Sum of all two digit numbers $= \frac{90}{2}(10 + 99) = (45)(109)$

Sum of all two digit numbers each is divisible by 2

$= \frac{45}{2}(10 + 98) = (45)(54)$

Sum of all two digit numbers each is divisible by 3

$= \frac{30}{2}(12 + 99) = 15(111)$

Sum of all two digit numbers each is divisible by 6

$= \frac{15}{2}(12 + 96) = 15(54)$

The required sum is

$45(109) + 15(54) - (45)(54) - 15(111) = 1620$

41. **(d) :** As A.M. ≥ G.M.

$\frac{\frac{b}{c} + \frac{c}{b} + \frac{c}{a} + \frac{a}{c} + \frac{a}{b} + \frac{b}{a}}{6} \ge \left(\frac{c}{b} \times \frac{b}{c} \times \frac{c}{a} \times \frac{a}{c} \times \frac{a}{b} \times \frac{b}{a}\right)^{\frac{1}{6}}$

or, $\frac{b^2 + c^2}{bc} + \frac{c^2 + a^2}{ca} + \frac{a^2 + b^2}{ab} \ge 6$

or, $a(b^2 + c^2) + b(c^2 + a^2) + c(a^2 + b^2) \ge 6abc$

$\therefore$ Minimum value of

$a(b^2 + c^2) + b(c^2 + a^2) + c(a^2 + b^2)$ is $6abc$

According to question, $\lambda abc = 6abc$

$\therefore \lambda = 6$

42. **(a) :** As a, b, c, d, e, f are positive real numbers

$\therefore (a + f)(b + e)(c + d) > 0$

So, $x > 0$ ...(i)

As A.M. ≥ G.M.

$\frac{(a+f) + (b+e) + (c+d)}{3} \ge [(a+f)(b+e)(c+d)]^{1/3}$

$\Rightarrow \sqrt[3]{x} \le \frac{3}{3} \Rightarrow x \le 1$ ...(ii)

From (i) and (ii), $0 < x \le 1$

43. **(a) :** $x_2 - x_1 = \frac{1}{x_4} - \frac{1}{x_3}$

$\Rightarrow (x_2 - x_1)^2 = \frac{(x_3 - x_4)^2}{(x_4x_3)^2} \Rightarrow \frac{(x_2 - x_1)^2}{(x_4 - x_3)^2} = \frac{1}{(x_3x_4)^2}$

$\Rightarrow \frac{(x_2 + x_1)^2 - 4x_1x_2}{(x_3 + x_4)^2 - 4x_4x_3} = \frac{1}{(x_3x_4)^2}$

On putting values, we get $\frac{b^2 - 4ac}{q^2 - 4pr} = \frac{a^2}{r^2}$

44. **(b) :** $\frac{x_2}{x_1} = \frac{x_4}{x_3} \Rightarrow \frac{x_2 + x_1}{x_1} = \frac{x_4 + x_3}{x_3}$

$\Rightarrow \frac{(x_1 + x_2)^2}{(x_3 + x_4)^2} = \frac{x_1^2}{x_3^2} = \frac{x_1x_2}{x_3x_4}$

On putting values, we get $q^2 = pr$

45. **(a) :** $x_2 = x_1r, x_3 = x_1r^2, x_4 = x_1r^3$

$x_1x_2 = x_1^2r = c/a$ ...(i)

$x_3x_4 = x_1^2r^5 = r/p$ ...(ii)

Dividing (ii) by (i), we get $r^4 = \frac{ra}{pc} \Rightarrow \left(\frac{ra}{pc}\right)^{1/4} = r'$

46. **(3) :** Let common ratio is $\frac{1}{2^b}$ and $S_\infty = \frac{a}{1-r} = \frac{\frac{1}{2^a}}{1-\frac{1}{2^b}} = \frac{1}{7}$

$\Rightarrow$ $b = 3$ & $a = b$ hence $a = 3$

47. **(2) :** $\sum_{k=1}^{n} \tan^{-1}\left(\frac{2k}{1+(k^2+k+1)(k^2-k+1)}\right)$

$= \sum_{k=1}^{n} \tan^{-1}\left(\frac{(k^2+k+1)-(k^2-k+1)}{1+(k^2+k+1)(k^2-k+1)}\right)$

$= \sum_{k=1}^{n} [\tan^{-1}(k^2+k+1) - \tan^{-1}(k^2-k+1)]$

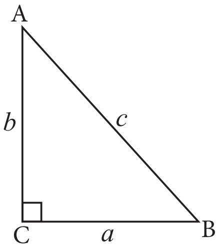
$= \tan^{-1}(n^2+n+1) - \tan^{-1} 1$

When $n \to \infty$, then,

$\sum_{k=1}^{\infty} \tan^{-1} \frac{2k}{2+k^2+k^4} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$

Hence, $\frac{8}{\pi} \times \frac{\pi}{4} = 2$

48. **(7) :** Let a, b, c are the sides of triangle.
Here $\angle C = 90°, A + B = 90°$ $c^2 = a^2 + b^2$
Since a, b, c are in A.P. $\therefore$ $2b = a + c$
Since, $c = 2b - a$ & $c^2 = a^2 + b^2$



$\Rightarrow$ $(2b-a)^2 = a^2 + b^2$ or $\frac{b}{a} = \frac{4}{3}$

$\Rightarrow$ $\frac{\sin B}{\sin A} = \frac{4}{3}$ or $\frac{\sin B + \sin A}{\sin B - \sin A} = \frac{7}{1}$

or $\tan\frac{B-A}{2} = \frac{1}{7} \Rightarrow \cos\frac{A-B}{2} = \frac{7}{5\sqrt{2}}$

Also $5(\sin A + \sin B) = 5\sqrt{2}\cos\frac{A-B}{2} = 7$

49. **(9) :** Let h be the total height.
According to question, total distance

$= h + 2 \times \frac{2}{3}h + 2 \times \left(\frac{2}{3}\right)^2 h + 2 \times \left(\frac{2}{3}\right)^3 h + \ldots$ upto infinity

$= h + 2 \times \frac{2}{3}h(3) = 5h = 4500$ cm

$\Rightarrow$ $10h = 9000$ cm $= 9$ deca metres

50. **(5) :** Let last term is T_n.

$T_n = \frac{12}{23} + (n-1)\left(\frac{-14}{115}\right) = \frac{74-14n}{115}$

Hence, for $n = 5$ last positive term is obtained

DPS-7
DAILY PRACTICE SHEET

Straight Lines

SOLUTIONS

1. **(a) :** Let the slope of BA passing through the points $B(2, 3)$ and $A(-2, 1)$ is m_1.

Then, $m_1 = \left|\frac{1-3}{-2-2}\right| = \left|\frac{-2}{-4}\right| = \frac{1}{2}$

and, the slope of BC passing through the points $B(2, 3)$ and $C(-2, -4)$ is m_2, then

$$m_2 = \left|\frac{-4-3}{-2-2}\right| = \left|\frac{-7}{-4}\right| = \frac{7}{4}$$

So, the angle between BA and BC is

$$\theta = \tan^{-1}\left[\left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|\right]$$

$$\Rightarrow \quad \theta = \tan^{-1}\left[\left|\frac{\frac{7}{4} - \frac{1}{2}}{1 + \frac{7}{4} \times \frac{1}{2}}\right|\right] = \tan^{-1}\left(\frac{2}{3}\right)$$

2. **(a) :** Let $A(-a, -b)$, $B(0, 0)$, $C(a, b)$, $D(a^2, ab)$

Slope of $AB = \frac{b}{a}$, Slope of $AC = \frac{2b}{2a} = \frac{b}{a}$

Slope of $AD = \frac{ab + b}{a^2 + a} = \frac{b(a+1)}{a(a+1)} = \frac{b}{a}$

All the four points lie on a line with slope $\frac{b}{a}$.

3. **(a) :** Two point slope form is $\frac{y_2 - y_1}{x_2 - x_1} = m$

$\Rightarrow \quad \frac{a-4}{-2-3} = \frac{-2}{5} \Rightarrow \frac{a-4}{-5} = \frac{-2}{5} \Rightarrow a - 4 = 2 \Rightarrow a = 6$

4. **(b) :** If the pair of lines $ax^2 + 2hxy + by^2 = 0$ has slopes m_1 and m_2, then

$$m_1 + m_2 = -\frac{2h}{b}, \; m_1 m_2 = \frac{a}{b}$$

$\therefore \quad (m_1 - m_2)^2 = (m_1 + m_2)^2 - 4m_1 m_2 = \frac{4(h^2 - ab)}{b^2}$

Here $a = \tan^2\theta + \cos^2\theta$, $h = -\tan\theta$, $b = \sin^2\theta$

$\therefore \quad (m_1 - m_2)^2 = \frac{4}{\sin^4\theta}[\tan^2\theta - (\tan^2\theta + \cos^2\theta)\sin^2\theta]$

$$= \frac{4}{\sin^2\theta}\left[\frac{1}{\cos^2\theta} - \tan^2\theta - \cos^2\theta\right]$$

$$= \frac{4}{\sin^2\theta}(1 - \cos^2\theta) = 4$$

$\therefore \quad |m_1 - m_2| = 2.$

5. **(c) :** Here, $x_1 = a$, $y_1 = b$ and $x_2 = -a$, $y_2 = -b$

Equation of the line through the given points is

$$y - b = \frac{-b-a}{-a-a}(x - a)$$

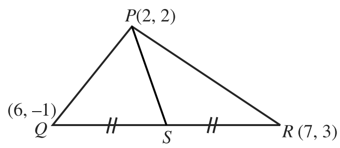
$\Rightarrow \quad y - b = \frac{-b-a}{-2a}(x - a) \quad \Rightarrow \quad \frac{x}{a} - \frac{y}{b} = 0$

6. **(d) :** Slope of $AB = \frac{3 - \frac{8}{3}}{1 - 0} = \frac{1}{3}$,

Slope of $BC = \frac{30 - 3}{82 - 1} = \frac{1}{3}$

So, the points A, B, C are collinear.

7. **(a) :** S is $\left(\frac{13}{2}, 1\right)$



Slope of $PS = -2/9$

The equation of the line passing through $(1, -1)$ and parallel to PS is

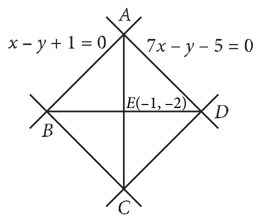
$$y + 1 = -\frac{2}{9}(x - 1)$$

$\Rightarrow \quad 2x + 9y + 9 - 2 = 0 \Rightarrow 2x + 9y + 7 = 0$

8. **(c) :** Coordinates of $A = (1, 2)$

$\therefore \quad$ Slope of $AE = 2$

$\Rightarrow \quad$ Slope of $BD = -\frac{1}{2}$



$\Rightarrow \quad$ Equation of BD is

$$\frac{y+2}{x+1} = -\frac{1}{2}$$

$\Rightarrow \quad x + 2y + 5 = 0$

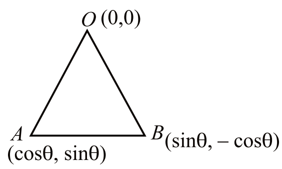
$\therefore \quad$ Co-ordinates of $D = \left(\frac{1}{3}, \frac{-8}{3}\right)$

9. **(c) :** Let OAB is the triangle and G be the centroid.

Since $G \equiv \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$

$$\equiv \left(\frac{0 + \cos\theta + \sin\theta}{3}, \frac{0 + \sin\theta - \cos\theta}{3}\right)$$

$$\equiv \left(\frac{\cos\theta + \sin\theta}{3}, \frac{\sin\theta - \cos\theta}{3}\right)$$



Since G lies on the line $y = 2x$

$\therefore \quad \frac{\sin\theta - \cos\theta}{3} = \frac{2(\sin\theta + \cos\theta)}{3}$

$\Rightarrow \quad \sin\theta - \cos\theta = 2\sin\theta + 2\cos\theta \quad \Rightarrow \quad \theta = \tan^{-1}(-3)$

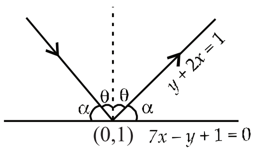
10. **(b) :** Equation of line passing through (2, 0) and perpendicular to $ax + by + c = 0$ is

$$y - 0 = \frac{b}{a}(x-2)$$

$\Rightarrow \quad ay = bx - 2b \Rightarrow \quad ay - bx + 2b = 0$

11. **(c) :** Let slope of incident ray be m.

Now angle of incidence = angle of reflection



$\therefore \quad \left|\frac{m-7}{1+7m}\right| = \left|\frac{-2-7}{1-14}\right| = \frac{9}{13}$

$\Rightarrow \quad \frac{m-7}{1+7m} = \frac{9}{13}$ or $\frac{m-7}{1+7m} = -\frac{9}{13}$

$\Rightarrow \quad 13m - 91 = 9 + 63m$ or $13m - 91 = -9 - 63m$

$\Rightarrow \quad 50m = -100$ or $76m = 82$

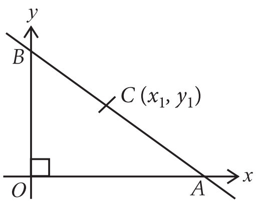
$\Rightarrow \quad m = -2$ or $m = \frac{41}{38}$

$\therefore$ Equations of incident line at (0, 1) are

$$y - 1 = -2(x-0) \text{ or } y - 1 = \frac{41}{38}(x-0)$$

i.e., $2x + y - 1 = 0$ or $38y - 38 - 41x = 0$

12. **(a) :** C is the mid point of AB

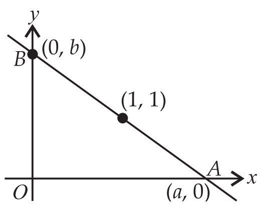


$\therefore$ Coordinates of A and B are $(2x_1, 0)$ and $(0, 2y_1)$ respectively.

$\therefore$ Equation of AB is $\frac{x}{2x_1} + \frac{y}{2y_1} = 1$

$\Rightarrow \quad \frac{x}{x_1} + \frac{y}{y_1} = 2$

13. **(d) :** It is given that (1, 1) is the midpoint of line AB.



$\Rightarrow \quad 1 = \frac{a+0}{2}, 1 = \frac{0+b}{2}$

$\Rightarrow \quad a = 2$ and $b = 2$

Equation of line AB is

$\frac{x}{2} + \frac{y}{2} = 1 \Rightarrow x + y = 2$

14. **(a) :** Since bisectors are perpendicular to each other.

Given, one bisector is $x + 2y - 5 = 0$ then other bisector is $2x - y + \lambda = 0$

According to alternative $\lambda = 0$

then, other bisector is $2x - y = 0$

15. **(b) :** $d(x, y) = \max\{|x|, |y|\}$ ...(i)

but $d(x, y) = a$ ...(ii)

From (i) and (ii), $a = \max\{|x|, |y|\}$

if $|x| > |y|$, then $a = |x|$

$\therefore \quad x = +a$

and if $|y| > |x|$, then $a = |y|$

$\therefore \quad y = +a$

Therefore, locus represents a straight line.

16. **(d) :** If 3, 4 are intercepts of a line $L = 0$, then

equation of line in intercept form is $\frac{x}{3} + \frac{y}{4} = 1$

$\therefore \quad \perp$ distance from the origin $= \left|\frac{\frac{0}{3} + \frac{0}{4} - 1}{\sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{1}{4}\right)^2}}\right| = \frac{12}{5}$

17. **(c) :** Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two points at equal distance λ from $(0, \sqrt{3})$

Then, $(x_1 - 0)^2 + (y_1 - \sqrt{3})^2 = \lambda^2$ ...(i)

and $(x_2 - 0)^2 + (y_2 - \sqrt{3})^2 = \lambda^2$ ...(ii)

Subtracting (ii) from (i), we get

$$x_1^2 - x_2^2 + (y_1 - \sqrt{3})^2 - (y_2 - \sqrt{3})^2 = 0$$

$\Rightarrow \quad x_1^2 - x_2^2 + (y_1^2 - y_2^2) - 2\sqrt{3}(y_1 - y_2) = 0$

On comparing rational and irrational parts on both sides

$y_1 - y_2 = 0 \Rightarrow y_1 = y_2 = \alpha$ (say)

and $x_1^2 - x_2^2 + (y_1^2 - y_2^2) = 0$ or $x_1^2 = x_2^2 = \beta^2$

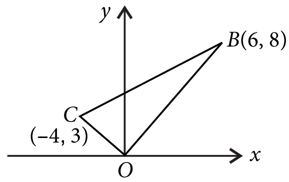
$\therefore \quad x_1 = x_2 = \pm\beta$ (say)

Hence, points are $(\pm\beta, \alpha)$

If $\beta = 0$, then $(0, \alpha)$

$\therefore \quad n \le 2$

18. **(c) :** Slope of $OB = \frac{8}{6}$



Slope of $OC = -\frac{3}{4}$

$\therefore \quad \angle BOC = \frac{\pi}{2}$

ΔOBC is right angled at O.

Circumcentre = mid point of hypotenuse $= \left(1, \frac{11}{2}\right)$

Orthocentre = vertex $O(0, 0)$.

Distance $= \sqrt{\left(1 + \frac{121}{4}\right)} = \frac{5\sqrt{5}}{2}$ unit

19. **(d) :** We have,

$$\begin{vmatrix} k & 2k & 1 \\ 3k & 3k & 1 \\ 3 & 1 & 1 \end{vmatrix} = 0 \quad \Rightarrow \quad k = \frac{-1}{3}$$

So, the line passes through (–1, –1) and (3, 1)

$\therefore$ Equation of line is, $\frac{y+1}{x+1} = \frac{1}{2} \Rightarrow x - 2y - 1 = 0$

$\therefore$ Distance of line from origin $= \left|\frac{0-0-1}{\sqrt{1+4}}\right| = \frac{1}{\sqrt{5}}$ unit

20. **(b) :** $\tan\alpha = \frac{3}{4} \Rightarrow \cos\alpha = \pm\frac{4}{5}, \sin\alpha = \pm\frac{3}{5}$

Equation : $x\cos\alpha + y\sin\alpha = p$

$\Rightarrow \quad x\left(\pm\frac{4}{5}\right) + y\left(\pm\frac{3}{5}\right) = 5 \Rightarrow \pm\left(\frac{4}{5}x + \frac{3}{5}y\right) = 5$

$\Rightarrow \quad 4x + 3y \pm 25 = 0$

21. **(c) :** Equation of the line $\perp$ to $3x + y - 3 = 0$ is given by $x - 3y + k = 0$

$\therefore \quad k = 4$ as (2, 2) lies on the line $x - 3y + k = 0$

Now, $x - 3y = -4$

$\therefore \quad \frac{x}{-4} + \frac{y}{4/3} = 1$

$\therefore \quad x$-intercept is – 4.

22. **(c) :** Length of $\perp$ from $O(0, 0)$ to $4x + 3y = 10$ is

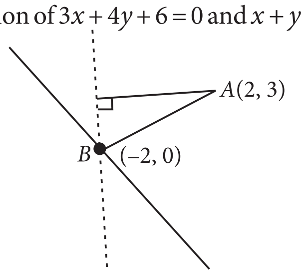
$$p_1 = \frac{|4(0) + 3(0) - 10|}{\sqrt{4^2 + 3^2}} = \frac{10}{5} = 2$$

Length of $\perp$ from $O(0, 0)$ to $8x + 6y + 5 = 0$ is

$$p_2 = \frac{|8(0)+6(0)+5|}{\sqrt{8^2+6^2}} = \frac{5}{10} = \frac{1}{2}$$

Lines are parallel to each other $\Rightarrow$ ratio will be 4 : 1 or 1 : 4.

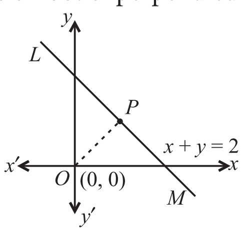
23. **(b) :** The family of lines $3x + 4y + 6 + \lambda(x + y + 2)$ are concurrent at the intersection of $3x + 4y + 6 = 0$ and $x + y + 2 = 0$ *i.e.*, at $(-2, 0)$



$\therefore$ The line through $(-2, 0)$ lying at the greatest distance from $A(2,3)$ is the line through $B(-2, 0)$ perpendicular to AB. Hence, its equation is

$$y - 0 = -\frac{4}{3}(x+2) \ \ i.e., 4x + 3y + 8 = 0$$

24. **(c) :** Let coordinates of foot of perpendicular P are (h, k).



$\therefore$ Slope of $OP = k/h$

Slope of given line = -1

$\because$ $LM \perp OP$

$\therefore$ $\frac{k}{h} \times -1 = -1 \Rightarrow k = h$ ...(i)

Also point P lies on the given line

$\therefore$ $k + h = 2$ ....(ii)

Solving (i) and (ii), we get $h = 1, k = 1$

So, coordinates of foot of perpendicular are (1, 1).

25. **(a) :** The algebraic perpendicular distance from (2, 1) to the line $3x - 2y + 1 = 0$ is $\frac{3(2)-2(1)+1}{\sqrt{(3)^2+(-2)^2}} = \frac{5}{\sqrt{13}} = L_1$ (say) and the algebraic perpendicular distance from $(-3, 5)$ to the line $3x - 2y + 1 = 0$ is $\frac{3(-3)-2(5)+1}{\sqrt{(3)^2+(-2)^2}} = \frac{18}{\sqrt{13}} = L_2$ (say)

Here, $\frac{L_1}{L_2} < 0$

$\therefore$ Given points lie on opposite side of the line $3x - 2y + 1 = 0$

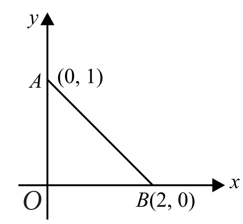
26. **(d) :** Equation of AB is $y - 1 = \frac{0-1}{2-0}(x - 0)$

or $x + 2y - 2 = 0$

$|PA - PB| \le |AB|$ thus $|PA - PB|$ to be maximum then A, B and P must be collinear.

Hence, solving $x + 2y - 2 = 0$ and $4x + 3y + 9 = 0$

we get, $P\left(-\frac{24}{5}, \frac{17}{5}\right)$.



27. **(c) :** Given family of lines $ax + by + c = 0$

also $3a + 2b + 4c = 0 \Rightarrow a = -\frac{2}{3}b - \frac{4}{3}c$

$\therefore ax + by + c = 0 \Rightarrow \left(\frac{-2}{3}b - \frac{4}{3}c\right)x + by + c = 0$

$\Rightarrow \left(-\frac{2}{3}x + y\right)b + \left(-\frac{4}{3}x + 1\right)c = 0$

$\Rightarrow -\frac{2}{3}x + y + \left(-\frac{4}{3}x + 1\right)\frac{c}{b} = 0$

$\Rightarrow -\frac{2}{3}x + y + \lambda\left(-\frac{4}{3}x + 1\right) = 0$, where $\lambda = \frac{c}{b}$

This is family of lines which is passing through point of intersection of $-\frac{2}{3}x + y = 0$ and $-\frac{4}{3}x + 1 = 0$

Solving for x and y, we get $x = \frac{3}{4}, y = \frac{1}{2}$

So, point of concurrence of lines is $\left(\frac{3}{4}, \frac{1}{2}\right)$

28. **(b) :** The family of lines

$(x + y - 1) + \lambda(2x + 3y - 5) = 0$ passes through a point such that

$x + y - 1 = 0$, $2x + 3y - 5 = 0$

i.e., $(-2, 3)$ and family of lines

$(3x + 2y - 4) + \mu(x + 2y - 6) = 0$

passes through a point such that

$3x + 2y - 4 = 0$ and $x + 2y - 6 = 0$ *i.e.*, $(-1, 7/2)$

$\therefore$ Equation of the straight line that belongs to both the families passes through $(-2, 3)$ and $(-1, 7/2)$ is

$$y - 3 = \frac{\frac{7}{2} - 3}{-1+2}(x+2)$$

$$\Rightarrow y - 3 = \frac{x+2}{2} \Rightarrow x - 2y + 8 = 0$$

29. **(b) :** We have, $\left|\frac{a(4-3+4)+b(2+6-3)}{\sqrt{(2a+b)^2+(a-2b)^2}}\right| = \sqrt{10}$

$\Rightarrow$ $25(a + b)^2 = 10(5a^2 + 5b^2)$

$\Rightarrow$ $25(a - b)^2 = 0 \Rightarrow a = b$ only line is $3x - y + 1 = 0$

30. **(d) :** Equations can be written as

$x + (\operatorname{cosec}\theta - \cot\theta)y = a$ & $x - (\operatorname{cosec}\theta + \cot\theta)y + a = 0$

i.e., $(\operatorname{cosec}\theta - \cot\theta)y = a - x$ & $(\operatorname{cosec}\theta + \cot\theta)y = x + a$

On multiplying, $y^2 = a^2 - x^2 \Rightarrow x^2 + y^2 = a^2$

31. **(a, c) :** Co-ordinates of point P are $\left(\frac{3\lambda-5}{\lambda+1}, \frac{5\lambda+1}{\lambda+1}\right)$

$$\text{Area of } \Delta PQR = \frac{1}{2}\cdot\frac{1}{\lambda+1}\begin{vmatrix} 3\lambda-5 & 5\lambda+1 & \lambda+1 \\ 1 & 5 & 1 \\ 7 & 2 & 1 \end{vmatrix} = \pm 2$$

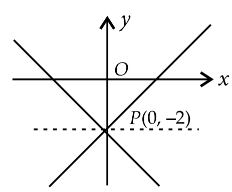
$\therefore \lambda = \frac{19}{5}, 23.$

32. **(b, c) :** $x^2 - y^2 - 4y - 4 = 0$

$\Rightarrow$ $x^2 - (y + 2)^2 = 0$

$\therefore$ Lines are $x + y + 2 = 0$ and $x - y - 2 = 0$

$\therefore$ Angle bisectors are $x = 0$ and $y = -2$



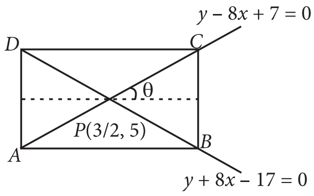
33. **(a, b, c) :** $2|a| < 6 \Rightarrow |a| < 3$

34. **(a, c) :** (a) is correct since slope $= \frac{1}{\sqrt{3}} = \tan 30°$ and also the line given in choice (a) cuts x-axis at $A(-5\sqrt{3}, 0)$ and y-axis at (0, 5) with $AB = 10$.
(b) is incorrect since the line given in choice (b) is not inclined at 30° (Indeed 150°).
Similarly choice (c) is correct and (d) is wrong.

35. **(a, b, c, d) :** The two lines will be identical if there exists some real number k, such that
$b^3 - c^3 = k(b - c)$, $c^3 - a^3 = k(c - a)$
and $a^3 - b^3 = k(a - b)$
$\Rightarrow$ $b - c = 0$ or $b^2 + c^2 + bc = k$,
$c - a = 0$ or $c^2 + a^2 + ca = k$
and $a - b = 0$ or $a^2 + b^2 + ab = k$
i.e., $b = c$ or $c = a$ or $a = b$
Next $b^2 + c^2 + bc = c^2 + a^2 + ca \Rightarrow b^2 - a^2 = c(a - b)$
Hence, $a = b$ or $a + b + c = 0$

36. **(a, c)** : Equation of bisectors $\Rightarrow \frac{x^2 - y^2}{1-(-1)} = \frac{xy}{2}$
Put $y = mx$
Then, $x^2 - m^2x^2 = x\, mx \Rightarrow m^2 + m - 1 = 0$
$\therefore \quad m = \frac{-1 \pm \sqrt{5}}{2}$

37. **(a, c) :** The intersection point of the given diagonals is
$P \equiv \left(\frac{3}{2}, 5\right)$



Equation of angular bisectors of the diagonals are
$$\frac{y + 8x - 17}{\sqrt{65}} = \pm \frac{y - 8x + 7}{\sqrt{65}}$$
$\Rightarrow x = \frac{3}{2}$ and $y = 5$
Let length of BC be a and that of CD be b
Then, $\tan\theta = \frac{a/2}{b/2} = \frac{a}{b} = 8$
Also, $ab = 8 \Rightarrow a = 8, b = 1$
So, equations of sides are $y = 1$, $y = 9$, $x = 1$ and $x = 2$.

38. **(a, c) :** Shift (1, –1) at (0, 0), then
$2(x - 1)^2 + 5(x - 1)(y + 1) + 3(y + 1)^2 = 0$
or $2x^2 + 5xy + 3y^2 + x + y = 0$

39. **(c) : (A) – (p), (B) – (r), (C) – (q), (D) – (s)**
(A) Using bisector equation, we get equation of side BC as $7y - 5x = 0$
(B) We have, $L_1 + \lambda L_2 = 0$
Thus, line farthest from (1, –3) is $15y - 6x = 7$
(C) Using bisector equation, we get angle bisector of A as $y = 1$
(D) Equation of BD is $2y + x = 17$

40. **(a) : (A) – (p,q,r), (B) – (q, r), (C) – (s), (D) – (q, r)**
(A) $a = b \neq 0 \Rightarrow |x| + |y| = \frac{6}{|a|}$
represents a square (4 straight lines)
(B) $ab \neq 0, a \neq b \Rightarrow a|x| + b|y| = 6$
represents a rhombus (4 straight lines)
(C) $|a| + |b| = 0 \Rightarrow a = b = 0$
$\therefore$ Equation is meaningless
(D) $a^2 + b^2 = 13$, $ab = 6 \Rightarrow a = 3, b = 2$
$a > b$, $a, b > 0$
$\therefore$ It represents a rhombus (4 straight lines)

41. **(d) :** $A \equiv (1, -2)$, $B \equiv (\alpha, -2\alpha)$, $C \equiv (\beta, \beta - 3)$
$1 + \alpha + \beta = 6$, $-2 - 2\alpha + \beta - 3 = 9$
$\Rightarrow \alpha = -3$, $\beta = 8$, $B(-3, 6)$, $C(8, 5)$
Equation of BC is $x + 11y = 63$
$\therefore \quad p + q = 63 + 11 = 74$

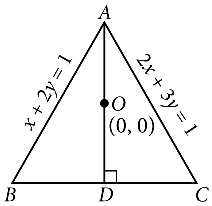
42. **(b) :** Slope of $BP = -1$
$\Rightarrow \frac{3 + 2\alpha}{2 - \alpha} = -1 \Rightarrow \alpha = -5$
Slope of $CP = 1/2 \Rightarrow \frac{\beta - 6}{\beta - 2} = \frac{1}{2} \Rightarrow \beta = 10$
$\therefore \quad B(-5, 10)$, $C(10, 7)$
Equation of $BC : x + 5y = 45$
$\therefore \quad p + q = 50.$

43. **(d) :** Pair of straight lines
$6x^2 + xy - y^2 = 0 \Rightarrow (3x + y)(2x - y) = 0$
$\Rightarrow y = 2x$ is the lines which coincides
Putting $y = 2x$ in L_2, we get $a = 4$

44. **(c) :** $p_1 = 6 \times 8^2 + 8 \times 4 - 4^2 = 400$
$p_2 = 3 \times 8^2 - 4 \times 8 \times 4 + 4^2 = 80$
$|p_1^2 - p_2^2| = \sqrt{400^2 - 80^2} = \sqrt{480 \times 320} = 40$

45. **(c) :** Using centroid formula, $G \equiv \left(\frac{1}{3}, \frac{-1}{3}\right)$

46. **(4) :** Solving $2x + 3y = 1$
$x + 2y = 1$, $A = (-1, 1)$
Orthocentre = (0,0)



$\Rightarrow$ slope of altitude $AD = -1$
Equation of BC is $x - y = k$
Solving, $x - y = k$ and
$x + 2y = 1$, we get $B = \left(\frac{1 + 2k}{3}, \frac{1 - k}{3}\right)$
Slope of $OB = \frac{1 - k}{1 + 2k}$, slope of $AC = -2/3$
$\therefore \quad \frac{1 - k}{1 + 2k} = \frac{3}{2} \Rightarrow k = \frac{-1}{8}$
Equation of BC is $x - y + \frac{1}{8} = 0$
$\Rightarrow -8x + 8y - 1 = 0 \Rightarrow a = -8, b = 8$

47. **(1) :** Centroid, $G \equiv \left(\frac{-16}{3}, 2\right)$ and Incentre, $I \equiv (-1, 0)$.
Thus, $GI = \frac{\sqrt{205}}{3}$

48. **(2) :** $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$
Thus, $-36\lambda + 200 - 64\lambda - 75 + 75 = 0 \Rightarrow \lambda = 2$

49. **(2) :** Transformed equation is
$(y + 2)^2 - 8(x + 1) - 4(y + 2) + 12 = 0$
$\Rightarrow y^2 = 8x \Rightarrow y^2 = 4ax \Rightarrow a = 2$

50. **(5) :** Let AC and BD intersect at P
$$AP = \frac{12 + 6 + 2}{\sqrt{16 + 9}} = 4$$
Area of $\Delta ABD = AP \times BP = \frac{24}{2} = 12 \Rightarrow BP = 3$
$AB = \sqrt{AP^2 + BP^2} = 5$

JEE MATHEMATICS

DPS-8

DAILY PRACTICE SHEET

Class XI

Circle and Parabola

SOLUTIONS

1. **(a) :** A line parallel to $3x + 4y = 0$ is $3x + 4y + k = 0$
Centre and radius of circle $x^2 + y^2 = 9$ are (0, 0) and 3 units respectively. Since line is a tangent to circle, then

$$\frac{0+0+k}{\sqrt{3^2+4^2}} = \pm 3 \Rightarrow k = \pm 15$$

For tangent to touch the circle in 1st quadrant, $k = -15$

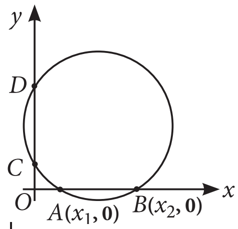
$\therefore$ Tangent line is $3x + 4y = 15$

2. **(a) :** As the circle meet x-axis at two points so put $y = 0$ in the equation of circle.

$x^2 - 16x - 36 = 0$

Let two roots of the equation are x_1 and x_2.

$\Rightarrow$ $x_1 + x_2 = 16$ and $x_1x_2 = -36$



Now $x_1 - x_2 = \sqrt{(x_1+x_2)^2 - 4x_1x_2}$

$= \sqrt{(16)^2 + 4 \times 36}$

$= \sqrt{400}$

$= 20$ = intercept on x-axis $= AB = |x_2 - x_1|$

3. **(a) :** The equation of the circle is

$x^2 + y^2 - 2x - 4y + 3 = 0$ .... (i)

and the equation of the line is $x + y = 5$ .... (ii)

We know that a line touches a circle if it intersects the circle in two coincident points. Therefore, we solve the two equations for their points of intersection. Substituting the value of y from (ii) in (i), we get

$x^2 + (5 - x)^2 - 2x - 4(5 - x) + 3 = 0$

$\Rightarrow$ $x^2 + 25 - 10x + x^2 - 2x - 20 + 4x + 3 = 0$

$\Rightarrow$ $x^2 - 4x + 4 = 0$

$\Rightarrow$ $(x - 2)^2 = 0$

i.e. $x = 2, 2$

So, the roots are equal. Thus, the given line meets the circle in two coincident points. Hence, the line touches the circle.

Finally, when $x = 2$, from (ii), we get $y = 3$.

Therefore, the point of contact is (2, 3).

4. **(a) :** Given equation is $x \cos\theta + y \sin\theta + g \cos\theta + f \sin\theta - k = 0$ this line will touch the circle if distance from $(-g, -f)$ to the line will be equal to the radius of circle $\sqrt{g^2 + f^2 - c}$

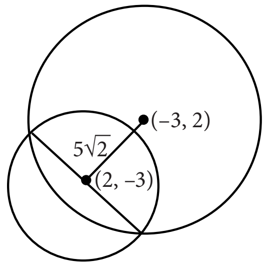
$$\Rightarrow \sqrt{g^2+f^2-c} = \frac{|-g\cos\theta - f\sin\theta + g\cos\theta + f\sin\theta - k|}{\sqrt{\sin^2\theta + \cos^2\theta}}$$

$\Rightarrow$ $g^2 + f^2 - c = k^2$

$\Rightarrow$ $g^2 + f^2 = k^2 + c$

5. **(b) :** The circle is

$x^2 + y^2 - 4x + 6y - 12 = 0$

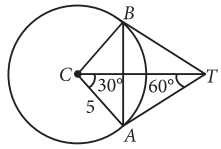


$\Rightarrow$ $(x - 2)^2 + (y + 3)^2 = 25 = 5^2$

Distance between two centres (−3, 2) and (2, − 3) is $5\sqrt{2}$.

Now, radius of $s = \sqrt{(5\sqrt{2})^2 + 5^2} = \sqrt{50+25} = 5\sqrt{3}$

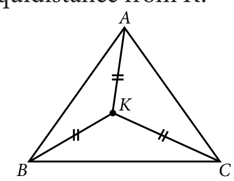
6. **(d) :** Let C be the centre and T be the point of intersection of the tangents



$\angle ATC = 60°$, $\angle ACT = 30°$

$AB = 2\ CA \sin 30° = 5$

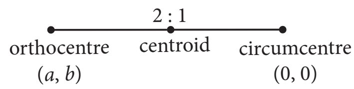
7. **(a) :** By observation, if K is a point inside the triangle, we see that A, B, C are equidistance from K.



$\therefore$ K is circumcentre of the triangle ABC

$\therefore$ K is origin.

Now, orthocentre, centroid and circumcentre lies in a straight line and centroid divides the orthocentre and circumcentre in the ratio 2 : 1.



$$\therefore \frac{a}{3} = \frac{5 + 13\cos\theta + 13\sin\theta}{3}$$

$\Rightarrow$ $a = 5 + 13 \cos\theta + 13 \sin\theta$

and $b = 12 + 13 \sin\theta - 13 \cos\theta$

$$\Rightarrow \frac{a+b-17}{26} = \sin\theta$$

and $\dfrac{a-b+7}{26} = \cos\theta$

$\Rightarrow \quad \sin^2\theta + \cos^2\theta = \left(\dfrac{a+b-17}{26}\right)^2 + \left(\dfrac{a-b+7}{26}\right)^2$

$\Rightarrow \quad (a + b - 17)^2 + (a - b + 7)^2 = (26)^2$

$\therefore$ Locus of orthocentre is

$(x + y - 17)^2 + (x - y + 7)^2 = (26)^2$.

8. **(a) :** Let $P(h, k)$ be a point on the circle $15x^2 + 15y^2 - 48x + 64y = 0$. Then the lengths PT_1 and PT_2 of the tangents from $P(h, k)$ to $5x^2 + 5y^2 - 24x + 32y + 75 = 0$, $5x^2 + 5y^2 - 48x + 64y + 300 = 0$ are respectively

$$PT_1 = \sqrt{h^2 + k^2 - \frac{24}{5}h + \frac{32}{5}k + 15}$$

$$\& \ PT_2 = \sqrt{h^2 + k^2 - \frac{48}{5}h + \frac{64}{5}k + 60}$$

Since (h, k) lies on $15x^2 + 15y^2 - 48x + 64y = 0$

$$\therefore \ h^2 + k^2 = \frac{48}{15}h - \frac{64}{15}k$$

$$\therefore \ PT_1 = \sqrt{\frac{48}{15}h - \frac{64}{15}k - \frac{24}{5}h + \frac{32}{5}k + 15}$$

$$= \sqrt{\frac{32}{15}k - \frac{24}{15}h + 15}$$

$$PT_2 = \sqrt{\frac{48}{15}h - \frac{64}{15}k - \frac{48}{5}h + \frac{64}{5}k + 60}$$

$$= \sqrt{\frac{-96}{15}h + \frac{128}{15}k + 60} = 2\sqrt{\frac{-24}{15}h + \frac{32}{15}k + 15} = 2(PT_1)$$

$$\therefore \ PT_1 : PT_2 = 1 : 2$$

9. **(a) :** The centres of the two circles are C_1 $(a/2, 0)$ and C_2 $(0, 0)$, and their radii are $\dfrac{|a|}{2}$ and c. So, the two circles will touch each other if C_1C_2 = sum or difference of radii

$$\Rightarrow \sqrt{\left(\frac{a}{2} - 0\right)^2 + (0-0)^2} = \left|c \pm \frac{|a|}{2}\right|$$

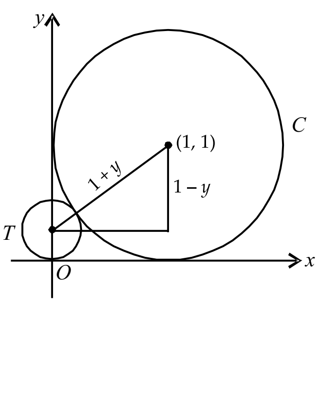
$$\Rightarrow \frac{|a|}{2} = \left|c \pm \frac{|a|}{2}\right| \ \Rightarrow \ c \pm \frac{|a|}{2} = \frac{|a|}{2}$$

$$\Rightarrow c - \frac{|a|}{2} = \frac{|a|}{2} \text{ and } c + \frac{|a|}{2} = \frac{|a|}{2}$$

$$\Rightarrow c = |a| \text{ or } c = 0 \Rightarrow c = |a|$$

10. **(c) :** We have by Pythagoras theorem

$(1 + y)^2 = (1 - y)^2 + 1 \Rightarrow 4y = 1 \quad \Rightarrow \quad y = 1/4$



11. **(b) :** The centre of the circle lies on x-axis. Let a be the radius of the circle. Then, co-ordinates of the centre are $(a, 0)$.

The circle passes through $(3, 4)$.

$$\therefore \quad \sqrt{(a-3)^2 + (0-4)^2} = a \Rightarrow 6a - 25 = 0 \Rightarrow a = \frac{25}{6}$$

So, equation of the circle is $(x - a)^2 + (y - 0)^2 = a^2$

$\Rightarrow \quad x^2 + y^2 - 2ax = 0 \ \Rightarrow \ 3(x^2 + y^2) - 25x = 0$

12. **(a) :** ΔPQX and ΔPRX are similar

So, $\dfrac{PQ}{PR} = \dfrac{PX}{RX}$

ΔPRX and ΔRXS are similar

So, $\dfrac{PR}{RS} = \dfrac{PX}{RX} \quad \therefore \quad \dfrac{PQ}{PR} = \dfrac{PR}{RS}$

$\Rightarrow (PR)^2 = PQ \cdot RS \Rightarrow (2r)^2 = PQ \cdot RS$

13. **(a) :** Let the equation of the circle through (a, b) be

$x^2 + y^2 + 2gx + 2fy + c = 0$ ...(i)

$a^2 + b^2 + 2ag + 2bf + c = 0$ ...(ii)

Since, the circle $x^2 + y^2 = p^2$ cuts the circle (i) orthogonally, therefore $2g \cdot 0 + 2f \cdot 0 = c - p^2 \Rightarrow c = p^2$

Substituting the value of c in (ii), we get

$a^2 + b^2 + 2ag + 2fb + p^2 = 0$

Hence, the locus of $(-g, -f)$ is

$a^2 + b^2 - 2ax - 2by + p^2 = 0$

14. **(b) :**

2 : 3
(5, 0) ——— $P(x, y)$ ——— $(10 \cos\theta, 10 \sin\theta)$

$$x = \frac{20\cos\theta + 15}{5}, \ y = \frac{20\sin\theta + 0}{5}$$

$\Rightarrow \quad 5x - 15 = 20 \cos\theta, 5y = 20 \sin\theta$

$$\Rightarrow \quad \left[\frac{5(x-3)}{20}\right]^2 + \left(\frac{y}{4}\right)^2 = 1$$

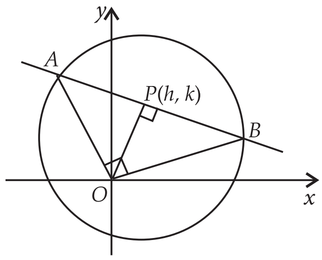
$\Rightarrow \quad (x - 3)^2 + y^2 = 16$ is the required equation of circle.

15. **(b) :** Centre of the circle $(2, 1)$; $r = \sqrt{25} = 5$

Distance of $(10, 7)$ from $(2, 1)$ is 10 units hence required distances are 5, 15 respectively.

16. **(a) :** AB is a variable chord such that $\angle AOB = \pi/2$

Let $P(h, k)$ be the foot of the perpendicular drawn from origin upon AB. Equation of the chord AB is



$$y - k = \frac{-h}{k}(x - h)$$

i.e. $hx + ky = h^2 + k^2$ ...(i)

Equation of the pair of straight lines passing through the origin and the intersection point of the given circle

$x^2 + y^2 + 2gx + 2fy + c = 0$ ....(ii)

and the variable chord AB is

$$x^2 + y^2 + 2(gx + fy)\left(\frac{hx + ky}{h^2 + k^2}\right) + c\left(\frac{hx + ky}{h^2 + k^2}\right)^2 = 0 \qquad ....(iii)$$

If equation (iii) represents a pair of perpendicular lines, then we have coeff. of x^2 + coeff. of $y^2 = 0$ *i.e.*

$$\left(1 + \frac{2gh}{h^2 + k^2} + \frac{ch^2}{(h^2 + k^2)^2}\right) + \left(1 + \frac{2fk}{h^2 + k^2} + \frac{ck^2}{(h^2 + k^2)^2}\right) = 0$$

Putting (x, y) in place of (h, k) gives the equation of the required locus as $x^2+y^2+gx+fy+\frac{c}{2}=0$

17. **(b) :** Let Δ_1 = Area of triangle ABC

$$=\frac{1}{2}\begin{vmatrix} at_1^2 & 2at_1 & 1 \\ at_2^2 & 2at_2 & 1 \\ at_3^2 & 2at_3 & 1 \end{vmatrix}=\frac{1}{2}(2a^2)\begin{vmatrix} t_1^2 & t_1 & 1 \\ t_2^2 & t_2 & 1 \\ t_3^2 & t_3 & 1 \end{vmatrix}$$

$= a^2(t_1 - t_2)(t_2 - t_3)(t_1 - t_3)$

Again area of triangle when tangents intersect at L, M, N then $L(at_1t_2, a(t_1 + t_2)), M(at_2t_3, a(t_2 + t_3)), N(at_1t_3, a(t_1 + t_3))$

$$\Delta_2 = \text{Area of triangle } LMN = \frac{1}{2}\begin{vmatrix} at_1t_2 & a(t_1+t_2) & 1 \\ at_2t_3 & a(t_2+t_3) & 1 \\ at_1t_3 & a(t_1+t_3) & 1 \end{vmatrix}$$

$$=\frac{1}{2}a^2\begin{vmatrix} t_1t_2 & t_1+t_2 & 1 \\ t_2t_3 & t_2+t_3 & 1 \\ t_3t_1 & t_3+t_1 & 1 \end{vmatrix}=\frac{1}{2}a^2(t_1-t_2)(t_2-t_3)(t_3-t_1)$$

$\Rightarrow$ Area of triangle $LMN = \frac{1}{2}$ Area of triangle ABC

$$\Rightarrow \frac{2}{1}=\frac{\text{Area of triangle } ABC \text{ w.r.t. vertices}}{\text{Area of triangle formed by tangents at } ABC}$$

$$\Rightarrow \frac{\Delta_1}{\Delta_2}=\frac{2}{1} \Rightarrow \Delta_1 : \Delta_2 = 2 : 1$$

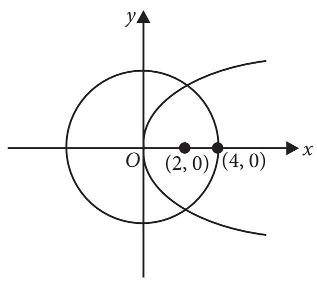
18. **(c) :** From the property of parabola, the perpendicular tangents intersect at the directrix. The equation of directrix is $x = -1$, hence this is the locus of point P.

19. **(c) :** Axis of the parabola is the line passing through the focus (0, 0) and perpendicular to the directrix $x = 2$. So, its equation is $y = 0$ *i.e.*, x-axis. Axis and directrix intersect at (2, 0).
Vertex is the mid point of focus (0, 0) and (2, 0).
Hence, the co-ordinates of vertex are (1, 0).

20. **(c) :** For parabola $y^2 = 4ax$ the line $y = mx + c$ will be tangent if $c = a/m$.
Equation of tangent to the parabola $y^2 = 8x$ is

$y = mx + \frac{2}{m}$

Now line to be chord of the circle $x^2 + y^2 = 16$ if distance from (0, 0) to the line will be less than the radius of the circle



$$\therefore \frac{\frac{2}{m}}{\sqrt{1+m^2}}<4$$

$\Rightarrow 1 < 4m^2(1 + m^2)$

$\Rightarrow 4m^4 + 4m^2 - 1 > 0$

$\Rightarrow m^4 + m^2 - 1/4 > 0$

$$\Rightarrow \left(m^2+\frac{1}{2}\right)^2-\frac{1}{2}>0$$

$$\Rightarrow \left(\left(m^2+\frac{1}{2}\right)+\frac{1}{\sqrt{2}}\right)\left(\left(m^2+\frac{1}{2}\right)-\frac{1}{\sqrt{2}}\right)>0$$

$$\Rightarrow m^2>-\frac{1}{2}+\frac{\sqrt{2}}{2} \text{ or } m^2>\frac{\sqrt{2}-1}{2}$$

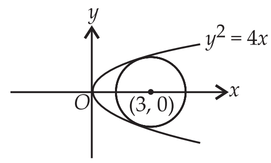
$$\Rightarrow |m|>\sqrt{\frac{\sqrt{2}-1}{2}}$$

$$\Rightarrow m\in\left(-\infty,-\sqrt{\frac{\sqrt{2}-1}{2}}\right)\cup\left(\sqrt{\frac{\sqrt{2}-1}{2}},\infty\right)$$

21. **(b) :** Given equation of parabola is
$x^2 + 2y = 8x - 7 \Rightarrow x^2 - 8x + 2y + 7 = 0$
$\Rightarrow x^2 - 8x + 16 + 2y + 7 - 16 = 0$

$\Rightarrow (x-4)^2 = -2\left(y-\frac{9}{2}\right) \therefore$ Vertex is at $\left(4, \frac{9}{2}\right)$.

22. **(b) :** To find common points of the 2 curves,
We have, $y^2 = 4x$
and $x^2 + y^2 - 6x + 1 = 0$



$\therefore x^2 + 4x - 6x + 1 = 0$
$\Rightarrow x^2 - 2x + 1 = 0$
$\Rightarrow (x - 1)^2 = 0$
Putting $x = 1$ in either equation, we get $y = \pm 2$.
So, the curves touch each other at (1, 2) and (1, –2).

23. **(c) :** Any tangent to the parabola $y^2 = 4x$ is

$y = mx+\frac{1}{m}$ $(\because a = 1)$ ...(i)

It is a tangent to the circle
$(x - 3)2 + y^2 = 9$ if the length of the perpendicular from the centre (3, 0) upon (i) is equal to the radius of the circle *i.e.,* 3

$$\therefore \frac{3m+\frac{1}{m}-0}{\sqrt{1+m^2}}=3$$

$$\Rightarrow 9m^2+\frac{1}{m^2}+6=9(1+m^2)$$

$$\Rightarrow \frac{1}{m^2}+6=9 \Rightarrow \frac{1}{m^2}=3$$

$$\Rightarrow m^2=\frac{1}{3}$$

$\Rightarrow m = \pm\frac{1}{\sqrt{3}}$. But m > 0 $\therefore m=\frac{1}{\sqrt{3}}$

Thus, the equation of the required line

$y=\frac{1}{\sqrt{3}}x+\sqrt{3}$ *i.e.* $\sqrt{3}y = x + 3$

24. **(a) :** The given parabola is $y^2 = kx-6=k\left(x-\frac{6}{k}\right)$

$\Rightarrow y^2=4\left(\frac{k}{4}\right)X$, where $X=x-\frac{6}{k}$

$\Rightarrow$ Focus of parabola is $\left(\frac{k}{4},0\right)$

$\Rightarrow$ Equation of directrix is $X=-\frac{k}{4}$

$\Rightarrow x-\frac{6}{k}=-\frac{k}{4} \Rightarrow x=\frac{6}{k}-\frac{k}{4}$, which coincide with the line $2x - 1 = 0$ *i.e.,* $x=\frac{1}{2}$

$\therefore \quad \frac{6}{k}-\frac{k}{4}=\frac{1}{2} \Rightarrow k^2 + 2k - 24 = 0$

$\Rightarrow \quad (k + 6)(k - 4) = 0 \Rightarrow k = -6 \quad \text{or } k = 4$

25. **(b) :** For $y^2 = 4ax$, the equation of tangent is $y = mx + \frac{a}{m}$

For $y^2 = 4(8)x$, the equation of tangent is $y = mx + \frac{8}{m}$

Now, using $x^2 = 108y$

$\therefore \quad x^2 = 108\left(mx+\frac{8}{m}\right)$

$\Rightarrow \quad mx^2 - m^2 \times 108x - 108 \times 8 = 0$

Now, $D = 0 \Rightarrow m^4\,(108 \times 108) = -4 \times 8 \times 108m$

$\Rightarrow \quad m^3 = -\frac{8}{27}=\left(-\frac{2}{3}\right)^3$

$\Rightarrow \quad m = -2/3$

$\therefore$ Required equation is $y = mx + \frac{8}{m}$

$\Rightarrow \quad y = -\left(\frac{2x}{3}+\frac{8\times 3}{2}\right) \Rightarrow 3y + 2x + 36 = 0$

26. **(c) :** The line $y = mx + a$ meets the parabola $y^2 = 4ax$ in two points whose abscissa are x_1 and x_2.

If x_1 and x_2 are roots of the quadratic equation $ax^2 + bx + c = 0$ then

$x_1 + x_2 = \frac{-b}{a}$ and $x_1\,x_2 = \frac{c}{a}$.

x_1, x_2 are the roots of the equation $(mx + a)^2 = 4ax$.

$\Rightarrow \quad m^2x^2 + 2a(m - 2)\,x + a^2 = 0$

$\Rightarrow \quad x_1 + x_2 = -2a(m - 2)/m^2$

$\Rightarrow \quad x_1 + x_2 = 0$ if $m = 2$

27. **(c) :** We have, $y^2 - kx + 8 = 0 \Rightarrow y^2 = k\left(x-\frac{8}{k}\right)$

$\therefore$ Directrix, $x - \frac{8}{k} = -\frac{k}{4} \Rightarrow x - \left(\frac{8}{k}-\frac{k}{4}\right) = 0$

But equation of directrix is $x - 1 = 0$.

Hence on comparing, we get

$\frac{8}{k}-\frac{k}{4}=1 \Rightarrow k^2 + 4k - 32 = 0$

$\Rightarrow \quad (k - 4)(k + 8) = 0 \Rightarrow k = 4, -8$

28. **(b) :** $x + y = 1 \Rightarrow y = -x + 1$

Using it to the parabola $y^2 = kx$, we get

$x^2 - 2x + 1 = kx \Rightarrow x^2 - x(2 + k) + 1 = 0$

Now, $D = 0$

$\Rightarrow \quad (2 + k)^2 = 4 \Rightarrow 2 + k = \pm 2 \Rightarrow k = 0, -4$

29. **(a) :** Let a tangent to the parabola be

$y = mx + \frac{\sqrt{5}}{m}\ (m \neq 0)$

As it is a tangent to the circle $x^2 + y^2 = 5/2$, we have

$\left(\frac{\sqrt{5}}{m}\right)=\frac{\sqrt{5}}{\sqrt{2}}\sqrt{1+m^2} \Rightarrow \frac{2}{m^2} = 1 + m^2$

which gives $m^4 + m^2 - 2 = 0 \Rightarrow (m^2 - 1)(m^2 + 2) = 0$

As $m \in R$, $m^2 = 1 \Rightarrow m = \pm 1$

Also, $m = \pm 1$ does satisfy $m^4 - 3m^2 + 2 = 0$

Hence common tangents are

$y = x + \sqrt{5}$ and $y = -x - \sqrt{5}$

30. **(d) :** Normal at t_1 is $y = -t_1x + 2at_1 + at_1^3$

The point t_2 lies on it.

$\therefore \quad 2at_2 = -t_1 \cdot at_2^2 + 2at_1 + at_1^3$

$\Rightarrow \quad 2a(t_2 - t_1) = at_1(t_1^2 - t_2^2)$.

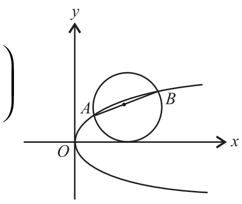
$\Rightarrow \quad -\frac{2}{t_1}=t_1+t_2 \Rightarrow t_2=-t_1-\frac{2}{t_1}$.

31. **(a, c) :** We must have $\lambda^2 + (\lambda + 1)^2 < 1$

$\Rightarrow \quad 2\lambda^2 + 2\lambda < 0 \Rightarrow \lambda(\lambda + 1) < 0 \Rightarrow -1 < \lambda < 0$

$\Rightarrow$ Choice (a), (c) are correct as $\frac{-1}{2} \in (-1,0)$

32. **(c, d) :** Let $A \equiv (t_1^2, 2t_1)$ and $B \equiv (t_2^2, 2t_2)$



The centre of the circle $= \left(\frac{t_1^2+t_2^2}{2}, t_1+t_2\right)$

As the circle touches the x-axis thus

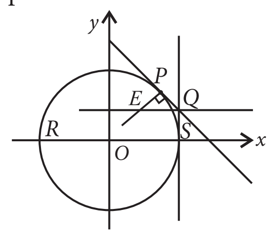
$t_1 + t_2 = \pm r$

Slope of $AB = \frac{2}{t_1+t_2}=\pm\frac{2}{r}$

33. **(a, c) :** If $P(\cos\theta, \sin\theta)$ be the variable point.

The tangent at P is $x\cos\theta + y\sin\theta = 1$

Tangent at S is $x = 1$.



Thus Q is $\left(1, \frac{1-\cos\theta}{\sin\theta}\right)$

A line through Q parallel to SR is

$y=\frac{1-\cos\theta}{\sin\theta}=\tan\frac{\theta}{2}$

The normal at P is $y = x\tan\theta = \frac{2\tan(\theta/2)}{1-\tan^2(\theta/2)}x$

We have, $h=\frac{1-\tan^2(\theta/2)}{2},\ k=\tan\frac{\theta}{2}$

The locus is $h=\frac{1-k^2}{2} \Rightarrow k^2 = 1-2h$ *i.e.*, $y^2 = 1 - 2x$

34. **(b, c):** Let normal $y = mx - 2am - am^3$ meet the parabola $y^2 = 4ax$ at P and Q. Then, the joint equation of the lines OP and OQ, where O is the vertex (0, 0) of the parabola is

$y^2 = 4ax\left(\frac{mx-y}{2am+am^3}\right)$

$\Rightarrow \quad (2m + m^3)y^2 = 4mx^2 - 4xy$

$\Rightarrow \quad 4mx^2 - (2m + m^3)y^2 - 4xy = 0$

which are at right angles if sum of the co-efficient of x^2 and y^2 is zero.

$\Rightarrow \quad 4m - 2m - m^3 = 0 \Rightarrow m = \pm\sqrt{2}$

35. **(b, c) :** Let $x - 3y + \lambda = 0$ touch the circle at (x_1, y_1). Then $xx_1 + yy_1 - 2(x + x_1) + y + y_1 - 5 = 0$ and $x - 3y + \lambda$ are identical.

Hence, comparing these

$\frac{x_1-2}{1}=\frac{y_1+1}{-3}=\frac{-2x_1+y_1-5}{\lambda}$

$=\frac{2(x_1-2)-1(y_1+1)+(-2x_1+y_1-5)}{2\times 1-1\times(-3)+\lambda}$

$\therefore \quad \frac{x_1-2}{1}=\frac{y_1+1}{-3}=\frac{-10}{5+\lambda}$

$\Rightarrow \quad x_1=\frac{-10}{5+\lambda}+2=\frac{2\lambda}{5+\lambda},\ y_1=\frac{30}{5+\lambda}-1=\frac{25-\lambda}{5+\lambda}$

(x_1, y_1) is on the circle,

So, $\left(\frac{2\lambda}{5+\lambda}\right)^2+\left(\frac{25-\lambda}{5+\lambda}\right)^2-4\left(\frac{2\lambda}{5+\lambda}\right)+2.\frac{25-\lambda}{5+\lambda}-5=0$

On simplification, $\lambda^2+10\lambda-75=0 \Rightarrow \lambda=5,-15$

$\therefore \quad x_1=\frac{10}{10}=1, y_1=\frac{20}{10}=2$

or $x_1=\frac{-30}{-10}=3, y_1=\frac{25+15}{5-15}=-4$

So, $(x_1, y_1)=(1, 2)$ or $(3, -4)$

36. (a, c, d) : Let a general point on parabola be $P(t^2, 2t)$.
The shortest distance falls along common normal. And the normal to the circle passes through the centre.
Equating slopes of normal, we have, at P

$$\frac{2t-8}{t^2-2}=-t \;\Rightarrow\; 2t-8=-t^3+2t \;\Rightarrow\; t=2$$

Thus P is (4, 4). Equation of normal is $y=-2x+12$

i.e., $\frac{x}{6}+\frac{y}{12}=1$. Slope of tangent at $Q=\frac{1}{2}$

$$\frac{SQ}{QP}=\frac{2}{2\sqrt{5}-2}=\frac{1}{\sqrt{5}-1}=\frac{\sqrt{5}+1}{4}$$

37. (b, c) : The circles $x^2+y^2-2x-15=0$ and $x^2+y^2-1=0$ has the radical axis $-2x-14=0$. $\therefore\; x+7=0$
Hence any circle orthogonal to them has its centre as $(-7, -\beta)$
Let its equation be

$S: x^2+y^2+14x+2\beta y+\lambda=0$

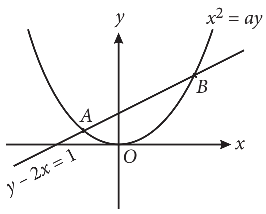
Orthogonality with 2nd circle gives

$\lambda-1=0. \;\therefore\; \lambda=1$

Again, (0, 1) lies on $S \Rightarrow 1+2\beta+1=0. \;\therefore\; \beta=-1$
Equation is $x^2+y^2+14x-2y+1=0$

Radius $=\sqrt{7^2+1^2-1}=7$

38. (a, b) : Let $A(x_1, y_1)$, $B(x_2, y_2)$ be the points of intersection.



On solving, $x^2=a(2x+1)$

$\Rightarrow \; x^2-2ax-a=0$

$\therefore \; x_1+x_2=2a, x_1x_2=-a$

$\because \; AB=\sqrt{40}$

$\therefore \; \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}=\sqrt{40}$

$\Rightarrow \; \sqrt{(x_2-x_1)^2+\{2(x_2-x_1)\}^2}=\sqrt{40}$

$\Rightarrow \; 5\{(x_2-x_1)^2\}=40 \Rightarrow (x_1+x_2)^2-4x_1x_2=8$

$\Rightarrow \; 4a^2+4a=8 \Rightarrow a^2+a-2=0 \Rightarrow a=1,-2$

39. (b) : (A)-(q), (B)-(p), (C)-(s), (D)-(r)

(A) $x=at^2, y=2at \Rightarrow \frac{x}{a}=t^2=\frac{y^2}{4a^2}$

$\Rightarrow \frac{x}{a}=\frac{y^2}{4a^2} \Rightarrow y^2=4ax$

(B) Locus of perpendicular tangents means equation of director circle.
Director circle of $x^2+y^2=a^2$ is $x^2+y^2=a^2+a^2=2a^2$

(C) $x\cos\theta=y\cot\theta=a \Rightarrow x=a\sec\theta, y=a\tan\theta$

$\Rightarrow \sec^2\theta-\tan^2\theta=1 \Rightarrow \left(\frac{x}{a}\right)^2-\left(\frac{y}{a}\right)^2=1$

$\Rightarrow x^2-y^2=a^2$

(D) $S \equiv x^2+y^2-2ax=0$

If midpoint is (x_1, y_1), then equation of chord with mid point (x_1, y_1) is $S_1=T$.

$\Rightarrow x_1^2+y_1^2, -2ax_1=xx_1+yy_1-a(x+x_1)$

$\Rightarrow x^2+y^2=ax$

40. (d) : (A)-(q), (B)-(p), (C)-(s), (D)-(r)

(A) $y=\frac{3}{4}x+\frac{5}{4}=mx+c$

$\therefore \; m=3/4, c=5/4 \Rightarrow a=15/16$

Since $c=\frac{a}{m}$

$\therefore \frac{5}{4}=\frac{a}{3/4} \Rightarrow a=15/16$

(B) $y=\frac{2}{3}x+2=mx+c \;\therefore\; m=\frac{2}{3},\; c=2$

$c=\frac{a}{m} \Rightarrow 2=\frac{a}{2/3} \;\therefore\; a=\frac{4}{3}$

(C) Let $P(x_1, y_1)$ be a point on the parabola $y^2=4ax$ and S is its focus, then

$SP=x_1+a$

Also $4a=8 \Rightarrow a=2$ and $SP=8$

$\Rightarrow \; 8=x_1+2 \Rightarrow x_1=6$

(D) $y^2=4ax$ passes through $(2, -6)$

$\therefore \; 36=8a \;\Rightarrow\; a=\frac{9}{2}$

So, latus rectum $=4a=\frac{4\times 9}{2}=18$

41. (d) : If A, B, C are given by t_1, t_2, t_3, then

$t_1+t_2=3, t_1t_2=2$

$\Rightarrow \; t_1=1, t_2=2 \Rightarrow A(1, 2), B(4, 4)$

$\therefore \; AB=\sqrt{13}$.

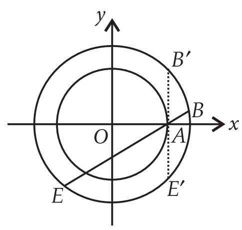
42. (d) : D is $(t_1t_2, t_1+t_2) \Rightarrow D\,(2, 3)$, C is $(9, -6)$

$\therefore \; CD=\sqrt{130}$

43. (c) : OA and EB meets at one point A. Let B be the point $(r_2\cos\theta, r_2\sin\theta)$.
Then max. BE = diameter of $C_2=2r_2$
Again BE will be minimum when BE is tangent to the circle C_1 at A.

$\Rightarrow BE=B'E'$ (in particular) $=2\sqrt{r_2^2-r_1^2}$.

44. (a) : $OA^2+OB^2+BE^2=r_1^2+r_2^2+BE^2$



For lower bound, $(OA^2+OB^2+BE^2)$

$=r_1^2+r_2^2+4(r_2^2-r_1^2)=5r_2^2-3r_1^2$

For upper bound, $(OA^2+OB^2+BE^2)$

$=r_1^2+r_2^2+4r_2^2=5r_2^2+r_1^2$

45. (c) : If $M(h, k)$ is the midpoint, then

$h=\frac{r_1+r_2\cos\theta}{2}, k=\frac{r_2\sin\theta}{2}$

On eliminating θ, we easily get $\left(\frac{2h-r_1}{r_2}\right)^2+\left(\frac{2k}{r_2}\right)^2=1$

$\Rightarrow$ Locus of (h, k) is $\left(x-\frac{r_1}{2}\right)^2+y^2=\frac{r_2^2}{4}$

46. **(3) :** The equation of tangent in slope form is $y = mx + a/m$
Now, comparing this eqn. with the given
$x + y = k$ *i.e.*, $y = -x + k$ we get $m = -1$ and $k = a/m$
Here, $a = 3$ since $y^2 = 12x$, So, $k = 3/-1 = -3$

47. **(6) :** If $(a, 0)$ is the centre C and P is $(2, -2)$ then $\angle COP = 45°$ since the equation of OP is $x + y = 0$.

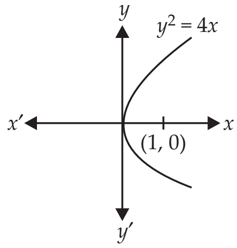
$\therefore\ OP = 2\sqrt{2} = CP$. Hence $OC = 4$

$\therefore\ \alpha = OB = OC + CB = 4 + 2\sqrt{2}$

$\Rightarrow [\alpha] = 6$

48. **(8) :** The equation of any normal to the parabola
$y^2 = -8x$ is $y = mx + 4m + 2m^3$ ...(i)
But, the given normal is $2x + y + k = 0 \Rightarrow y = -2x - k$ ...(ii)
Comparing eqn. (i) and (ii), we get
$m = -2$ and $-4m - 2m^3 = k$
$\Rightarrow\ k = 8 + 16 = 24$

49. **(1) :** Given curve is $y^2 = 4x$.



Also, point (1, 0) is the focus of the parabola. It is clear from the graph that only one normal is possible.

50. **(0) :** We note that PQ is the chord of contact of the tangents from the origin to the circle
$x^2 + y^2 - 6x + 4y + 8 = 0$ ... (i)
Equation of PQ is $3x - 2y - 8 = 0$ ... (ii)
Equation of a circle passing through the intersection of (i) and (ii) is $x^2 + y^2 - 6x + 4y + 8 + \lambda(3x - 2y - 8) = 0$ .... (iii)
If this represents the circumcircle of the triangle OPQ, it passes through $O(0, 0)$ so from (iii), $\lambda = 1$, then equation (iii) becomes $x^2 + y^2 - 3x + 2y = 0$ so that the required coordinates of the centre are $\left(\frac{3}{2},-1\right)$.

Sum of the coordinates $= \frac{3}{2} - 1 = \frac{1}{2}$ $\therefore$ $\left[\frac{1}{2}\right] = 0$

JEE MATHEMATICS

Class XI

DPS-9

DAILY PRACTICE SHEET

Ellipse and Hyperbola

SOLUTIONS

1. **(a) :** Let $P(x, y)$ be any point on the ellipse

$$\therefore \frac{\sqrt{(x-1)^2+(y+1)^2}}{\frac{x-y-3}{\sqrt{2}}} = e = \frac{1}{2}$$

$$\Rightarrow x^2 - 2x + 1 + y^2 + 1 + 2y = \frac{(x-y-3)^2}{8}$$

$$\Rightarrow 8x^2 + 8y^2 - 16x + 16y + 16 = x^2 + y^2 + 9 - 6x + 6y - 2xy$$

$$\Rightarrow 7x^2 + 7y^2 + 2xy - 10x + 10y + 7 = 0$$

2. **(d) :** Given equation of line is $16x + 9y = 25$ ..(i)

and ellipse is $16x^2 + 9y^2 = 144$ ...(ii)

Let (h, k) be middle point of the chord intercepted on the line $16x + 9y = 25$, the equation of chord is

$T = S'$

$\Rightarrow 16xh + 9yk - 144 = 16h^2 + 9k^2 - 144$

$\Rightarrow 16hx + 9ky = 16h^2 + 9k^2$ ...(iii)

Now (i) & (ii) represent the same line

$16hx + 9ky = 16h^2 + 9k^2$

$\therefore\ 16x + 9y = 25$

$$\Rightarrow \frac{16h}{16} = \frac{9k}{9} = \frac{16h^2+9k^2}{25} = \lambda \text{ (say)}$$

$$\Rightarrow h = k = \frac{16h^2+9k^2}{25} = \lambda$$

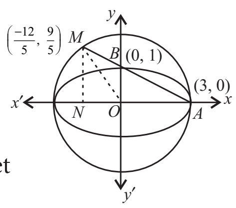
$\Rightarrow h = \lambda = k$ and $16h^2 + 9k^2 = 25\lambda$

$\Rightarrow 25\lambda^2 = 25\lambda \Rightarrow \lambda = 0, \lambda = 1$ $(\because \lambda \neq 0)$

$\Rightarrow h = 1 = k$ is the required middle point on the given chord.

3. **(d) :** The ellipse is $\frac{x^2}{9} + \frac{y^2}{1} = 1$

Its auxiliary circle is $x^2 + y^2 = 9$ ...(i)



The equation of AB is

$\frac{x}{3} + \frac{y}{1} = 1 \Rightarrow x + 3y = 3$ ...(ii)

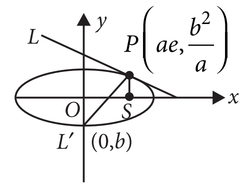
Eliminating x from (i) and (ii), we get

$$M\left(-\frac{12}{5}, \frac{9}{5}\right)$$

$$\therefore \text{ Area of triangle } AOM = \frac{1}{2}\cdot AO\cdot MN = \frac{1}{2}\cdot 3\cdot\frac{9}{5} = \frac{27}{10}$$

4. **(d) :** Let equation of ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $P\left(ae, \frac{b^2}{a}\right)$ be one end of a latus rectum through $S(ae, 0)$

Now equation of normal at L is



$$\frac{a^2x}{ae} - \frac{b^2y}{b^2/a} = a^2 - b^2$$

$$\Rightarrow \frac{ax}{e} - ay = a^2e^2 \Rightarrow \frac{x}{e} - y = ae^2$$

Now this normal passes through $L'(0, b)$

$\therefore\ -b = ae^2 \Rightarrow\ b^2 = a^2e^4$

$\therefore\ a^2(1 - e^2) = a^2e^4 \Rightarrow e^4 + e^2 - 1 = 0$

5. **(d) :** Equation of any tangent to the ellipse is

$y = mx + \sqrt{a^2m^2 + b^2}$ ...(i)

Equation of the line through the centre (0, 0) and perpendicular to (i) is

$x + my = 0$ ...(ii)

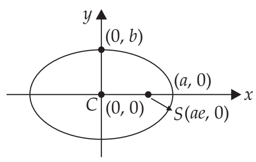
Eliminating m from (i) and (ii), we get the required locus of the foot of the perpendicular as

$$y = -\frac{x^2}{y} + \sqrt{a^2\frac{x^2}{y^2} + b^2}$$

$\Rightarrow (x^2 + y^2)^2 = a^2x^2 + b^2y^2$

which is fourth degree equation in x and y does not represent a circle, an ellipse or a hyperbola.

6. **(b) :** We have $\frac{x^2}{16} + \frac{y^2}{9} = 1$



$\Rightarrow a = 4, b = 3$

and $e = \sqrt{1 - \frac{9}{16}} = \sqrt{\frac{7}{16}} = \frac{\sqrt{7}}{4}$

$\therefore\ ae = \sqrt{7} \Rightarrow CS = ae = \sqrt{7}$

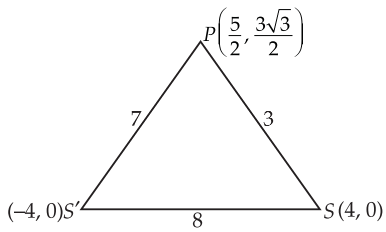
Length of semi-minor axis is $b = 3$ $\therefore\ \frac{CS}{b} = \frac{\sqrt{7}}{3} = \sqrt{7} : 3.$

7. **(a) :** We have $e = \sqrt{1 - \frac{9}{25}} = \frac{4}{5}$

$\therefore\ S(4, 0), S'(-4, 0)$

$P\left(\frac{\pi}{3}\right)$ corresponds to $\left(5\cos\frac{\pi}{3}, 3\sin\frac{\pi}{3}\right)$ *i.e.*, $\left(\frac{5}{2}, \frac{3\sqrt{3}}{2}\right)$

$PS = 3, PS' = 7$ & $SS' = 8$



$\therefore$ Incentre is given by

$$\left\{\frac{8\left(\frac{5}{2}\right) + 3(-4) + 7(4)}{8+3+7}, \frac{8\left(\frac{3\sqrt{3}}{2}\right) + 3(0) + 7(0)}{8+3+7}\right\}$$

$$i.e., \left(2, \frac{2}{\sqrt{3}}\right) i.e., \left(2, \frac{2}{\sqrt{2^2-1}}\right) \Rightarrow \alpha = 2$$

8. **(d) :** $mx - y + c = 0$ is a normal to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\Leftrightarrow \frac{a^2}{m^2} + \frac{b^2}{1} = \frac{(a^2 - b^2)^2}{c^2} \Rightarrow (a^2 + b^2m^2)c^2 = m^2(a^2 - b^2)^2$$

$$\Rightarrow c^2 = \frac{(a^2 - b^2)^2 m^2}{a^2 + b^2 m^2}$$

9. **(c) :** Let the required ellipse be $\frac{x^2}{\alpha} + \frac{y^2}{\beta} = 1$

It passes through (0, 4) $\Rightarrow \frac{16}{\beta} = 1 \therefore \beta = 16$

It passes through (3, 2) $\Rightarrow \frac{9}{\alpha} + \frac{4}{\beta} = 1 \Rightarrow \alpha = 12$

Also $\alpha = \beta(1 - e^2) \Rightarrow 12 = 16(1 - e^2)$

$$\Rightarrow \frac{3}{4} = 1 - e^2 \Rightarrow e^2 = \frac{1}{4} \therefore e = \frac{1}{2}$$

10. **(d) :** Length of minor axis = Distance between the foci

$\therefore 2b = 2c$ or $b = c$

$\therefore a^2 = b^2 + c^2 = b^2 + b^2 \Rightarrow a^2 = 2b^2 \therefore a = \sqrt{2}b$

Eccentricity of ellipse, $e = \frac{c}{a}$ or $e = \frac{b}{\sqrt{2}b} = \frac{1}{\sqrt{2}}$

11. **(d) :** The equation can be written as

$4(x - 2)^2 + 5(y - 3)^2 = 1$

$$\Rightarrow \frac{(x-2)^2}{\frac{1}{4}} + \frac{(y-3)^2}{\frac{1}{5}} = 1$$

Length of major axis $= 2 \times \frac{1}{2} = 1 = 2a$

Length of minor axis $= 2 \times \frac{1}{\sqrt{5}} = \frac{2}{\sqrt{5}} = 2b$

$$e^2 = 1 - \frac{b^2}{a^2} = \sqrt{1 - \left(\frac{2b}{2a}\right)^2} = \sqrt{1 - \left(\frac{2}{\sqrt{5}}\right)^2} = \frac{1}{\sqrt{5}} \therefore e = \frac{1}{\sqrt{5}}$$

$$\text{Centre} = \left(\frac{hf - bg}{ab - h^2}, \frac{hg - af}{ab - h^2}\right) = (2, 3)$$

12. **(b) :** Let the equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\therefore \frac{2b^2}{a} = 4$ *i.e.* $b^2 = 2a$ …(i)

Also, given that distance between focus and its nearest vertex is 3/2.

i.e., $a - ae = \frac{3}{2} \Rightarrow 1 - e = \frac{3}{2a}$

$\Rightarrow e = 1 - \frac{3}{2a} = \frac{2a - 3}{2a}$ …(ii)

$$\Rightarrow e^2 = \frac{4a^2 - 12a + 9}{4a^2} \Rightarrow 1 - \frac{b^2}{a^2} = \frac{4a^2 - 12a + 9}{4a^2}$$

$$\Rightarrow 1 - \frac{2}{a} = \frac{4a^2 - 12a + 9}{4a^2} \quad \text{[by (i)]}$$

$\Rightarrow 4a(a - 2) = 4a^2 - 12a + 9$

$\Rightarrow 4a = 9 \therefore a = \frac{9}{4}$ …(iii)

From (ii) and (iii), we get $e = 1/3$.

13. **(d) :** Equation of ellipse is $9x^2 + 16y^2 = 144$

$$\Rightarrow \frac{x^2}{\frac{144}{9}} + \frac{y^2}{\frac{144}{16}} = 1 \Rightarrow \frac{x^2}{(4)^2} + \frac{y^2}{(3)^2} = 1 \Rightarrow a = 4, b = 3.$$

Sum of focal distances from a point $= 2a = 2 \times 4 = 8$

14. **(d) :** The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. It passes through (2, 2) and (3, 1).

$$\frac{4}{a^2} + \frac{4}{b^2} = 1, \frac{9}{a^2} + \frac{1}{b^2} = 1, \frac{1}{b^2} = 1 - \frac{9}{a^2}$$

$$\therefore \frac{4}{a^2} + 4\left(1 - \frac{9}{a^2}\right) = 1 \Rightarrow \frac{1}{a^2} = \frac{3}{32}, \frac{1}{b^2} = \frac{5}{32}$$

$$b^2 = a^2(1 - e^2) \Rightarrow \frac{32}{5} = \frac{32}{3}(1 - e^2) \Rightarrow 1 - e^2 = \frac{3}{5}$$

$$\Rightarrow e^2 = \frac{2}{5}, e = \sqrt{\frac{2}{5}}.$$

15. **(d) :** Fact : Locus of the point of intersection of perpendicular tangents to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is the director circle whose equation is given by $x^2 + y^2 = a^2 + b^2$

$\Rightarrow$ Equation of director circle of the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ is the director circle whose equation is given by

$x^2 + y^2 = 25 + 16$, now $(5, \lambda)$ lies on it

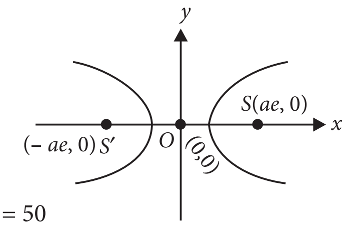
$\therefore 25 + \lambda^2 = 25 + 16 \Rightarrow \lambda = \pm 4$

$\Rightarrow$ Statement-1 is correct and statement-2 is locus of point of intersection of $\perp$ tangents to $\frac{x^2}{25} + \frac{y^2}{16} = 1$ is $x^2 + y^2 = 41$ is also correct and correct explanation of statement-1

16. **(c) :** $\frac{x^2}{(2 - r)} + \frac{y^2}{(r - 5)} = -3$ is an ellipse

if $2 - r < 0, r - 5 < 0 \Rightarrow 2 < r < 5$

17. **(b) :** Given $2ae = 20$ and $e = \sqrt{2}$



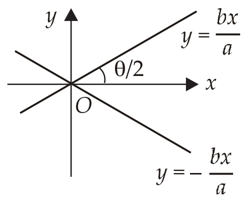
$$\therefore a = \frac{20}{2e} = 5\sqrt{2}$$

Again $b^2 = a^2(e^2 - 1)$

$b^2 = 25 \times 2\,(2 - 1) = 50$

$\therefore$ Equation of hyperbola $x^2 - y^2 = 50$

18. **(d) :** $\tan\frac{\theta}{2} = \frac{b}{a}$



$$e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \tan^2\frac{\theta}{2}}$$

$$= \sec\frac{\theta}{2}.$$

19. **(a) :** A point on $x^2 + y^2 = a^2$ is $(a\cos\theta, a\sin\theta)$. The chord of contact of tangents from it to the hyperbola is $S_1 = 0$.

$x\cos\theta - y\sin\theta = a$ ...(i)

If $P(x_1, y_1)$ is the midpoint of the chord of the hyperbola, $S_1 = S_{11}$

$xx_1 - yy_1 = x_1^2 - y_1^2$ ...(ii)

(i) and (ii) are same. $\frac{\cos\theta}{x_1} = \frac{\sin\theta}{y_1} = \frac{a}{x_1^2 - y_1^2}$,

or $\frac{1}{\sqrt{x_1^2 + y_1^2}} = \frac{a}{x_1^2 - y_1^2}$

$\therefore$ The locus of P is $a^2(x^2 + y^2) = (x^2 - y^2)^2$.

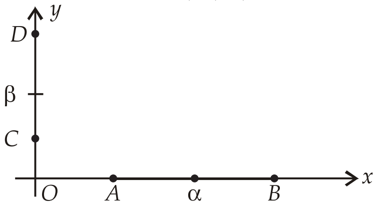
20. (c) : We have $2b = ae$ and $\frac{2b^2}{a} = 8$

Also, we have $b^2 = a^2(e^2 - 1)$

Now, eliminating a and b from these equations, we get

$$\frac{e^2}{4} = e^2 - 1 \;\Rightarrow\; 4 = 3e^2 \;\therefore\; e = \frac{2}{\sqrt{3}}$$

21. (c) : The ends of the rods are A, B, C, D



$AB = 2a, CD = 2b$ Since A, B, C, D are concyclic,

$OA \cdot OB = OC \cdot OD$ ...(i)

The centre (x, y) of the circle is on the perpendicular bisectors of AB and CD.

$$x = \frac{A + B}{2} = \alpha,\; y = \frac{C + D}{2} = \beta$$

$OA = \alpha - a,\; OB = \alpha + a,\; OC = \beta - b,\; OD = \beta + b$

Now, (i) $\Rightarrow \alpha^2 - a^2 = \beta^2 - b^2$, or $x^2 - y^2 = a^2 - b^2$, a rectangular hyperbola.

22. (b) : The given equation can be written as

$9(x^2 + 2x) - 16(y^2 - 2y) = 151$

or, $9(x^2 + 2x + 1) - 16(y^2 - 2y + 1)$

$= 151 + 9 - 16 = 144.$

$$\frac{(x+1)^2}{16} - \frac{(y-1)^2}{9} = 1 \text{ or } \frac{X^2}{16} - \frac{Y^2}{9} = 1$$

where $X = x + 1$, $Y = y - 1$, $a^2 = 16$, $b^2 = 9$.

Centre is $X = 0$, $Y = 0$. *i.e.* $(-1, 1)$.

$b^2 = a^2(e^2 - 1)$. $\therefore$ $e = 5/4$.

Foci are $X = \pm\, ae$, $Y = 0$.

or, $x + 1 = \pm 4 \cdot 5/4$, $y - 1 = 0$

or, $(4, 1)$ and $(-6, 1)$.

23. (b) : The normal at $P\left(ct, \frac{c}{t}\right)$ is $t^2 x - y = c\left(t^3 - \frac{1}{t}\right)$

It meets the curve at $P_1\left(ct_1, \frac{c}{t_1}\right)$

$$\therefore\quad t^2 t_1 - \frac{1}{t_1} = t^3 - \frac{1}{t}, t^2\,(t_1 - t) + \frac{1}{t} - \frac{1}{t_1} = 0,$$

$$t^2 + \frac{1}{tt_1} = 0 \Rightarrow t_1 = -\frac{1}{t^3}$$

The normal at P_1 meets the curve $P_2\left(ct_2, \frac{c}{t_2}\right)$, $t_2 = -\frac{1}{t_1^{\,3}}$

$$\therefore\quad t_2 = -t_1^{-3} = -\left(\frac{-1}{t^3}\right)^{-3} = t^9 \quad\therefore\quad P_2 = \left(ct^9, \frac{c}{t^9}\right).$$

24. (a) : The point $\left(ct, \frac{c}{t}\right)$ lies on $x^2 + y^2 = a^2$

$\therefore\; c^2t^2 + \frac{c^2}{t^2} = a^2$ or $c^2t^4 - a^2t^2 + c^2 = 1$

If t_1, t_2, t_3, t_4 are the roots, then

$t_1 + t_2 + t_3 + t_4 = 0,\; t_1t_2t_3t_4 = \frac{c^2}{c^2} = 1,$ ....(i)

$t_1t_2t_3 + t_2t_3t_4 + t_3t_4t_1 + t_4t_1t_2 = 0$ ....(ii)

(i), (ii) $\Rightarrow \frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + \frac{1}{t_4} = 0$ ....(iii)

$\therefore\; x_1 + x_2 + x_3 + x_4 = c(t_1 + t_2 + t_3 + t_4) = 0$

25. (c) : $\frac{x^2}{36} - \frac{y^2}{k^2} = 1$, , is a hyperbola $\Rightarrow$ $k^2 > 0$

Now, $\frac{y^2}{k^2} = \frac{x^2}{36} - 1 = \frac{x^2 - 36}{36} \;\Rightarrow\; k^2 = \frac{36y^2}{x^2 - 36}$

Now for $k^2 > 0 \Rightarrow x^2 > 36$

This is true only for point (10, 4)

$\therefore$ (10, 4) lies on given hyperbola.

26. (b) : $PQ = OP \;\Rightarrow\; PQ^2 = OP^2$.

$\therefore\; 4a^2\sec^2\theta = a^2\sec^2\theta + b^2\tan^2\theta$

We have $\frac{b^2}{a^2} = 3\operatorname{cosec}^2\theta \Rightarrow e^2 - 1 = 3\operatorname{cosec}^2\theta$

$$\Rightarrow\; \frac{e^2 - 1}{3} = \operatorname{cosec}^2\theta > 1$$

27. (c) : Tangents at (h, k) is $\frac{hx}{a^2} - \frac{ky}{b^2} = 1$ ....(i)

Eliminating x from (i) and $x^2 + y^2 = a^2$, we get

$$\frac{a^4}{h^2}\left(1 + \frac{ky}{b^2}\right)^2 + y^2 = a^2$$

$$\Rightarrow (b^2 + ky)^2 + \frac{h^2 b^4}{a^4} y^2 = \frac{h^2 b^4}{a^2}$$

$$\left(k^2 + \frac{h^2 b^4}{a^4}\right) y^2 + 2b^2 ky + b^4\left(1 - \frac{h^2}{a^2}\right) = 0$$

But $\frac{h^2}{a^2} - \frac{k^2}{b^2} = 1$

$$\therefore\; \left(k^2 + \frac{h^2 b^4}{a^4}\right) y^2 + 2b^2 ky - k^2 b^2 = 0$$

If y_1 and y_2 are the roots, then $\frac{y_1 + y_2}{y_1 y_2} = \frac{2b^2 k}{k^2 b^2} = \frac{2}{k}$

$\therefore\; k = \frac{2y_1y_2}{y_1 + y_2} \;\Rightarrow\; y_1, k, y_2$ are in H.P.

28. (b) : The hyperbola $x^2 - 2y^2 - 2\sqrt{2}\,x - 4\sqrt{2}\,y - 6 = 0$ can be reduced to $\frac{(x - \sqrt{2})^2}{4} - \frac{(y + \sqrt{2})^2}{2} = 1$

We have $a = 2,\; b = \sqrt{2} \;\Rightarrow\; e = \sqrt{\frac{b^2}{a^2} + 1} = \sqrt{\frac{3}{2}}$

The area of the triangle $ABC = \frac{1}{2} a(e - 1) \cdot \frac{b^2}{a}$

$$= \frac{b^2(e - 1)}{2} = \frac{2\left(\sqrt{\frac{3}{2}} - 1\right)}{2} = \sqrt{\frac{3}{2}} - 1$$

29. (c) : Tangent at $(-1, 1)$ is $S_1 = 0 \Rightarrow -3x - 4y + 1 = 0$

It meets x-axis at $A\left(\frac{1}{3}, 0\right)$ and y-axis at $B\left(0, \frac{1}{4}\right)$

$$\Delta OAB = \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{24}.$$

30. **(d) :** If the line $y = mx + c$ touches the ellipse $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
$\Rightarrow b^2x^2 - a^2(mx + c)^2 - a^2b^2 = 0$
$\Rightarrow x^2(b^2 - a^2m^2) - 2a^2mcx - (a^2b^2 + a^2c^2) = 0$ whose roots are equal (*i.e.* $D = 0$)
$\Rightarrow 4a^4m^2c^2 = -4a^2(b^2 + c^2)(b^2 - a^2m^2)$
$\Rightarrow c^2 = a^2m^2 - b^2$ (Statement-2 which is given)
Now using $a^2 = 5, b^2 = 9, m = 3$
$\therefore\ \lambda^2 = a^2m^2 - b^2 \quad \Rightarrow \quad \lambda = \pm\sqrt{45-9}$
$\lambda = \pm 6 \quad \therefore \quad \lambda \neq 45$
$\Rightarrow$ Statement-1 is false
$\therefore$ Statement-1 is false and statement-2 is true.

31. **(b, c) :** $\cos B + \cos C = 4\sin^2(A/2)$
We have $2\cos\frac{B+C}{2}\cos\frac{B-C}{2} = 4\sin^2\frac{A}{2}$
$\Rightarrow \quad \sin\frac{A}{2}\cos\frac{B-C}{2} = 2\sin^2\frac{A}{2}$
$\Rightarrow \quad \cos\frac{B-C}{2} = 2\sin\frac{A}{2} \qquad \left(\because\ \sin\frac{A}{2} \neq 0\right)$
Multiplying with $\cos(A/2) \neq 0$, we have
$\cos\frac{A}{2}\cos\frac{B-C}{2} = 2\sin\frac{A}{2}\cos\frac{A}{2}$
$\Rightarrow \quad \cos\left(\frac{\pi-(B+C)}{2}\right)\cos\left(\frac{B-C}{2}\right) = \sin A$
$\Rightarrow \quad 2\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right) = 2\sin A$
$\Rightarrow \sin B + \sin C = 2\sin A \ \Rightarrow \ b + c = 2a$
Given 'a' is fixed we have $b + c$ = constant, which means that the locus of A is an ellipse.

32. **(a, b) :** As slope of tangent = 2
Equation of tangent $y = 2x \pm \sqrt{36-4} \quad \Rightarrow \quad y = 2x \pm 4\sqrt{2}$
$\frac{x}{2\sqrt{2}} - \frac{y}{4\sqrt{2}} = 1$ and $-\frac{x}{2\sqrt{2}} + \frac{y}{4\sqrt{2}} = 1$
Compare with $\frac{xx_1}{9} - \frac{yy_1}{4} = 1$, we have point of contact as
$\left(\frac{9}{2\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{-9}{2\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

33. **(a,b) :** The equation of the ellipse can be written as $\frac{x^2}{1/4} + \frac{y^2}{1/9} = 1$
where $a^2 = 1/4$, $b^2 = 1/9$. If e is the eccentricity of the ellipse, then $e^2 = (a^2 - b^2)/a^2$
$\Rightarrow e^2 = 5/9 \Rightarrow \quad e = \sqrt{5}/3.$
The co-ordinates of the foci are $(\pm ae, 0) = (\pm\sqrt{5}/6, 0)$

34. **(a, b, c, d) :** From given equation
$16(x - 1)^2 - 9(y - 2)^2 = 144$
$\Rightarrow \frac{(x-1)^2}{9} - \frac{(y-2)^2}{16} = 1$ or $\frac{(x-1)^2}{3^2} - \frac{(y-2)^2}{4^2} = 1$
$\therefore$ Length of major axis = $2 \times 4 = 8$
Centre : $x - 1 = 0$ and $y - 2 = 0 \ \therefore$ Centre (1, 2)
Length of minor axis = $2 \times 3 = 6$
Eccentricity : $e^2 = \frac{b^2}{a^2} + 1 = \frac{a^2 + b^2}{a^2}$
$e^2 = \frac{25}{9} \therefore e = \frac{5}{3}$

35. **(b, c) :** The normals at $\left(t, \frac{1}{t}\right)$ is $xt^2 - y = t^3 - \frac{1}{t}$
Slope $= t^2 > 0 \Rightarrow -\frac{a}{b} > 0 \ \therefore \ a > 0, b < 0$ or $a < 0, b > 0$.

36. **(a, c) :** Equation of asymptotes of the given hyperbola differ with the constant only
$\therefore$ Equation of asymptotes is given by
$2x^2 + 5xy + 2y^2 + 4x + 5y + \lambda = 0$, will represent a pair of straight lines if $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$
$\therefore$ Here $a = 2 = b,\ h = \frac{5}{2},\ g = 2,\ f = \frac{5}{2},\ c = \lambda$
$2(2\lambda) + 2(2)\left(\frac{5}{2}\right)\left(\frac{5}{2}\right) - 2\left(\frac{25}{4}\right) - 2(4) - \lambda\left(\frac{25}{4}\right) = 0$
$\Rightarrow \quad 4\lambda + 25 - \frac{25}{2} - 8 - \frac{25}{4}\lambda = 0 \Rightarrow \ -\frac{9\lambda}{4} + \frac{9}{2} = 0 \Rightarrow \lambda = 2$
$\therefore$ Combined equation of asymptotes of the hyperbola is
$2x^2 + 5xy + 2y^2 + 4x + 5y + 2 = 0$

37. **(a, d) :** Given $9x^2 - 5y^2 = 45$ & $y = 2x + \lambda$
$\Rightarrow 9x^2 - 5(2x + \lambda)^2 - 45 = 0$
$\Rightarrow (9 - 20)x^2 - 20\lambda x - (5\lambda^2 + 45) = 0$
$\Rightarrow 11x^2 + 20\lambda x + (5\lambda^2 + 45) = 0$ ...(i)
The line will touch the hyperbola if both roots of (i) coincide
$\therefore\ D = 0 \Rightarrow\ 400\lambda^2 = 4 \times 11(5\lambda^2 + 45)$
$\Rightarrow (100 - 55)\lambda^2 = 11 \times 45$
$\Rightarrow \lambda^2 = 11 \quad \therefore\ \lambda = \pm\sqrt{11}$

38. **(a, b, c) :** We have $\frac{(x-1)^2}{3} - \frac{(y+2)^2}{16} = 1$
Here, $a = \sqrt{3}, b = 4$
$\therefore$ Length of transverse axis = $2a = 2\sqrt{3}$
Length of latus rectum = $\frac{2b^2}{a} = \frac{2(16)}{\sqrt{3}} = \frac{32}{\sqrt{3}}$
$\Rightarrow e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{\frac{19}{3}}$
$\therefore$ Equation of directrices are given by
$x - 1 = \pm\frac{a}{e}$ *i.e.* $x = 1 \pm \frac{3}{\sqrt{19}}$

39. **(b) : (A)-(r) , (B)-(s) , (C)-(q) , (D)-(p)**
The equation of the given ellipse can be written as $x^2/a^2 + y^2/b^2 = 1$ where $a^2 = 5$, $b^2 = 4$
$\Rightarrow \quad e^2 = \frac{a^2 - b^2}{a^2} = \frac{5-4}{5} = \frac{1}{5}$
Directrix is $x = \frac{a}{e} = \frac{\sqrt{5}}{1/\sqrt{5}} = 5$
A latus rectum is $x = ae = \sqrt{5} \times \frac{1}{\sqrt{5}} = 1$
Tangent at an extremity $(\sqrt{5},\ 0)$ of the major axis is $x = \sqrt{5}$
Minor axis is $x = 0$.

40. **(a) : (A)-(s), (B)-(r) , (C)-(q) , (D)-(p)**
Here $L = \left(1, \frac{1}{\sqrt{2}}\right)$, The tangent at L is $x + \sqrt{2}y = 2$
The normal at L is $\sqrt{2}x - y = \frac{1}{\sqrt{2}}\ \therefore\ T = (2, 0), N = (1/2, 0),$
$M = \left(-2, \frac{-5}{\sqrt{2}}\right)$

(A) Area of $\Delta LNT = \frac{3}{4\sqrt{2}}$

(B) Circum-radius $= \frac{NT}{2} = \frac{3}{4}$

(C) $LN : LT = \frac{1}{\sqrt{2}}$

(D) $\frac{LN}{NM} = \frac{1}{5}$

41. (b) : Let $P(a\cos\theta, \sin\theta)$ be a point on the ellipse. The radius of the largest circle is the minimum value of PQ.

$f(\theta) = PQ^2 = (a\cos\theta - h)^2 + \sin^2\theta, f'(\theta) = -2\sin\theta\,[(a^2-1)\cos\theta - ah]$

$f'(\theta) = 0 \Rightarrow \theta = 0,\ \cos\theta = \frac{ah}{a^2-1}$

$\Rightarrow f(\theta)$ is minimum if $\cos\theta = \frac{ah}{a^2-1}$

Provided $h \le a - \frac{1}{a}$ and $\theta = 0$ provided $h > a - \frac{1}{a}$

$a = 2, h = 1 \Rightarrow \cos\theta = \frac{2}{3}, PQ^2 = \left(\frac{4}{3}-1\right)^2 + 1 - \frac{4}{9} = \frac{2}{3}$

$\therefore\ r = \sqrt{\frac{2}{3}}.$

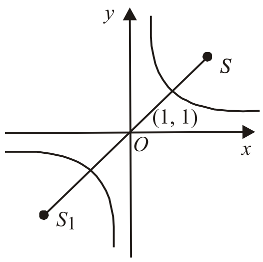
42. (a) : $\cos\theta = \frac{ah}{a^2-1} = \frac{3}{4}, PQ^2 = \left(\frac{9}{4}-2\right)^2 + 1 - \frac{9}{16} = \frac{1}{2}$

$\therefore\ r = \frac{1}{\sqrt{2}}.$

43. (c) : Transverse axis $= 2\sqrt{2} = 2a$

Conjugate axis $= 2\sqrt{2} = 2b$

Latus-rectum $= 2\cdot\frac{b^2}{a} = 2a = 2\sqrt{2}.$



44. (b) : $OS = \sqrt{2}a = 2 \Rightarrow S = (\sqrt{2}, \sqrt{2}).$

45. (d) : Since length of latus rectum $= 2a = 2\sqrt{2}$

$\therefore\ \Delta OLL_1 = \frac{1}{2}(2)2\sqrt{2} = 2\sqrt{2}$

46. (4) : The second curve is the image of the first curve in the line $x = y$. The tangent to $\frac{x^2}{3} - y^2 = 1$ is $y = mx \pm \sqrt{3m^2-1}$ ...(i)

The tangent to $\frac{y^2}{3} - x^2 = 1$ is $x = my \pm \sqrt{3m^2-1}$ ...(ii)

(i), (ii) are identical if $\frac{1}{m} = m$ or $m = \pm 1$ (i) gives $y = x + \sqrt{2}$

which meets the first curve at $A\left(\frac{-3}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right).$

and the second curve at $B\left(\frac{1}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$

$\therefore\ AB = \sqrt{(2\sqrt{2})^2 + (2\sqrt{2})^2} = 4.$

47. (4) : Equation of normal of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at $(a\cos\theta, b\sin\theta)$ is $\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$. Let normal is passing through a given point (h, k), the $\frac{ah}{\cos\theta} - \frac{bk}{\sin\theta} = a^2 - b^2$

$\Rightarrow 2ah\left(1 + \tan^2\frac{\theta}{2}\right)\tan\frac{\theta}{2} - bk\left(1 - \tan^4\frac{\theta}{2}\right)$

$= 2(a^2 - b^2)\tan\frac{\theta}{2}\left(1 - \tan^2\frac{\theta}{2}\right)$

which is a four degree polynomial in $\tan\frac{\theta}{2}$, therefore, it may have four real roots.

48. (4) : We have, $ae = \sqrt{7}$

$\therefore\ r = \sqrt{(ae)^2 + b^2} = \sqrt{7+9} = 4$

49. (3) : Foci of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is given by $(\pm ae, 0)$ where $e^2 = 1 + \frac{b^2}{a^2}$

Now, in the given hyperbola, $e = \sqrt{1 + \frac{3}{a^2}}$

Thus, $ae = \sqrt{a^2+3}$

For the ellipse, $e^2 = 1 - 4/16 \Rightarrow e = \frac{\sqrt{3}}{2}$

$\therefore$ Foci $(\pm 4\sqrt{3}/2, 0)$ *i.e.* $(\pm 2\sqrt{3}, 0)$

Both foci coincide

$\therefore\ \sqrt{a^2+3} = 2\sqrt{3} \Rightarrow a^2 + 3 = 12 \Rightarrow a = \pm 3 \Rightarrow |a| = 3$

50. (5) : Equation of ellipse is $9x^2 + 16y^2 = 144$ or $\frac{x^2}{16} + \frac{y^2}{9} = 1$, Comparing this with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then we get $a^2 = 16, b^2 = 9$ and comparing the line with $y = mx + c$

$\therefore\ m = 1$ and $c = \lambda$, if the line $y = x + \lambda$ touches ellipse $9x^2 + 16y^2 = 144$, then $c^2 = a^2m^2 + b^2$

$\Rightarrow \lambda = \pm 5$, so numerical value of $\lambda = 5$

JEE MATHEMATICS | Class XI

DPS-10

DAILY PRACTICE SHEET

Limits and Derivatives

SOLUTIONS

1. **(d) :** $\lim_{n\to\infty}\left\{\frac{1+2+3+...+n}{n+2}-\frac{n}{2}\right\}$

$=\lim_{n\to\infty}\left\{\frac{n(n+1)}{2(n+2)}-\frac{n}{2}\right\}=\lim_{n\to\infty}\frac{-n}{2(n+2)}=-\lim_{n\to\infty}\frac{1}{2\left(1+\frac{2}{n}\right)}=-\frac{1}{2}$

2. **(d) :** $\lim_{x\to 0}\frac{(1+2x)^{10}-1}{x}$

$=\lim_{x\to 0}\frac{1+10\cdot(2x)+45\cdot(2x)^2+.....-1}{x}$

$=\lim_{x\to 0}\left(20+\text{multiple of } x\right)=20$

3. **(c) :** $\lim_{n\to\infty}[n]=\lim_{n\to\infty}n=\infty$ ($\because$ for $x\in I,[x]=x$)

$\therefore$ Limit does not exist.

4. **(b) :** Let $L=\lim_{x\to\infty}\left(1+\frac{a}{x}+\frac{b}{x^2}\right)^{2x}=e^2$

Since, L is 1^∞ form $L=\exp\lim_{x\to\infty}\left(\frac{a}{x}+\frac{b}{x^2}\right)2x$

$\Rightarrow\ e^2=\exp\lim_{x\to\infty}2\left(a+\frac{b}{x}\right)=e^{2a}\Rightarrow a=1, b\in R$

5. **(b) :** By L.H. rule $L=\lim_{x\to 1}\sum_{k=1}^{25}kx^{k-1}=\sum_{k=1}^{25}k=25\times\frac{26}{2}=325$

6. **(b) :** $\lim_{n\to\infty}\frac{1^2+2^2+...+n^2}{4n^3+6n^2-5n+1}$

$=\lim_{n\to\infty}\frac{\frac{n(n+1)(2n+1)}{6}}{4n^3+6n^2-5n+1}=\frac{1}{6}\lim_{n\to\infty}\left[\frac{2n^3+3n^2+n}{4n^3+6n^2-5n+1}\right]$

$=\frac{1}{6}\lim_{n\to\infty}\left[\frac{2+\frac{3}{n}+\frac{1}{n^2}}{4+\frac{6}{n}-\frac{5}{n^2}+\frac{1}{n^3}}\right]=\frac{1}{6}\left[\frac{2+0+0}{4+0-0+0}\right]=\frac{1}{6}\times\frac{2}{4}=\frac{1}{12}$

7. **(a) :** Given limit is in 1^∞ form

$\therefore\ L=\exp\lim_{x\to 0}\frac{a_1^x+a_2^x+...+a_n^x-n}{x}$

$\left(\text{Substituting }\frac{1}{x}=y\text{ as }x\to\infty, y\to 0,\text{after that replace } y \text{ by } x\right)$

$=\exp\lim_{x\to 0}\left(a_1^x\ln a_1+a_2^x\ln a_2+...+a_n^x\ln a_n\right)$ (Using L.H. Rule)

$=\exp\ln(a_1a_2.....a_n)=a_1a_2.....a_n$.

8. **(b) :** Let $L=\lim_{x\to 0}x^{\frac{1}{x}}+\lim_{x\to\infty}x^{\frac{1}{x}}=L_1+L_2$

$L_1=\lim_{x\to 0}x^{1/x}=\exp\lim_{x\to 0}\frac{\ln x}{x}=e^{-\infty}=0$

$L_2=\lim_{x\to\infty}x^{1/x}=\exp\lim_{x\to\infty}\frac{\ln x}{x}$ (Using L.H. Rule)

$=\exp\lim_{x\to\infty}\frac{1}{x}=e^0=1\ \therefore\ L=L_1+L_2=0+1=1.$

9. **(c) :** $\lim_{n\to\infty}{}^nC_r\left(\frac{m}{n}\right)^r\left(1-\frac{m}{n}\right)^{n-r}$

$=\lim_{n\to\infty}\frac{n(n-1)(n-2)....(n-r+1)}{r!}\cdot\frac{m^r}{n^r}\times\left[\left(1-\frac{m}{n}\right)^{-\frac{n}{m}}\right]^{-\frac{(n-r)\times m}{n}}$

$=\lim_{n\to\infty}\frac{n(n-1)(n-2)....(n-r+1)}{n^r\ \ r!}(m^r)\times\lim_{n\to\infty}\left[\left(1-\frac{m}{n}\right)^{-\frac{n}{m}}\right]^{\left(1-\frac{r}{n}\right)(-m)}$

$=\lim_{n\to\infty}\frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)....\left(1-\left(\frac{r-1}{n}\right)\right)}{r!}m^r\cdot e^{(1-0)(-m)}$

$=\frac{(1-0)(1-0)...(1-0)}{r!}m^r\ \ e^{-m}=\frac{e^{-m}\ \ m^r}{r!}$

10. **(c) :** $\lim_{x\to\infty}\frac{(x+1)^{10}+(x+2^2)^{10}+(x+3^2)^{10}+...+(x+10^2)^{10}}{x^{10}+(100)^{10}}$

$=\lim_{x\to\infty}\frac{x^{10}\left[\left(1+\frac{1}{x}\right)^{10}+\left(1+\frac{4}{x}\right)^{10}+...+\left(1+\frac{100}{x}\right)^{10}\right]}{x^{10}\left(1+\left(\frac{100}{x}\right)^{10}\right)}$

$=1+1+1+.....10\text{ times}=10$

11. **(c) :** $L=\lim_{x\to 0}\left(1^{\text{cosec}^2x}+2^{\text{cosec}^2x}+.....+n^{\text{cosec}^2x}\right)^{\sin^2x}$

$=\lim_{x\to 0}\left(\left(\frac{1}{n}\right)^{\text{cosec}^2x}+\left(\frac{2}{n}\right)^{\text{cosec}^2x}+.....+\left(\frac{n-1}{n}\right)^{\text{cosec}^2x}+1\right)^{\sin^2x}\cdot n$

$=(0+0+.....+0+1)^0\ n=n.$

12. **(d) :** L is 1^∞ form.

$$L = \exp \lim_{x\to 0} \frac{\cos x + a\sin bx - 1}{x}$$

$$= \exp \lim_{x\to 0}\left(\frac{a\sin bx}{x} - \frac{2\sin^2\frac{x}{2}}{x}\right) = e^{ab}.$$

13. **(b) :** $\lim_{x\to\pi/2} (\sec x)^{\pi/2 - x}$

Put $x = y + \frac{\pi}{2}$. If $x \to \frac{\pi}{2}$ then $y \to 0$

$$\lim_{y\to 0}\left(\sec\left(y+\frac{\pi}{2}\right)\right)^{\frac{\pi}{2}-y-\frac{\pi}{2}} = \lim_{y\to 0}(-\cos y)^{-y} = \lim_{y\to 0}\frac{(-1)^{-y}}{(\cos y)^y} = 1.$$

14. **(c) :** Let $L = \lim_{x\to 0}\left[(\sin x)^{\frac{1}{x}} + \left(\frac{1}{x}\right)^{\sin x}\right] = L_1 + L_2$

$$L_1 = \lim_{x\to 0}(\sin x)^{1/x} = \exp\lim_{x\to 0}\frac{\ln(\sin x)}{x} = e^{-\infty} = 0$$

$$L_2 = \lim_{x\to 0}\left(\frac{1}{x}\right)^{\sin x} = \exp\lim_{x\to 0}\sin x(-\ln x) = \exp\lim_{x\to 0}\frac{-\ln x}{\operatorname{cosec} x}$$

$$= \exp\lim_{x\to 0}\left(-\frac{1}{x}\times\frac{1}{(-\operatorname{cosec} x\cot x)}\right)$$

$$= \exp\lim_{x\to 0}\frac{\sin x}{x}\cdot\tan x = e^0 = 1 \qquad \therefore\ L = L_1 + L_2 = 0 + 1 = 1$$

15. **(b) :** For $x \in (-\delta, \delta)$, $\sin x < x$ or $\lim_{x\to 0}\frac{\sin x}{x} = 1^-$

or $\left[\lim_{x\to 0}\frac{\sin x}{x}\right] = 0$

Also, $x \in (-\delta, \delta)$, $\tan x > x$. But from this, nothing can be said about the relation between $\sin x$ and x.

Hence, both the statements are true but statement 2 is not the correct explanation of statement 1.

16. **(a) :** $\{(1+x)^{2/x}\} = (1+x)^{2/x} - [(1+x)^{2/x}]$

Now, $\lim_{x\to 0}(1+x)^{2/x} = e^2$

or $\lim_{x\to 0}\{(1+x)^{2/x}\} = e^2 - [e^2] = e^2 - 7$

17. **(a) :** $\lim_{x\to 0}\left[\frac{1}{x^2} + \frac{2013^x - 1}{e^x - 1}\right] = \lim_{x\to 0}\left[\frac{1}{x^2} + \frac{2013^x - 1}{x}\cdot\frac{1}{\frac{e^x - 1}{x}}\right]$

$$= \lim_{x\to 0}\frac{1}{x^2} + \lim_{x\to 0}\frac{2013^x - 1}{x}\cdot\frac{1}{\lim_{x\to 0}\frac{e^x - 1}{x}}$$

$= \lim_{x\to 0}\frac{1}{x^2} + \log_e 2013$, which approaches to $+\infty$

18. **(b) :** $\lim_{x\to 0}\frac{\log(1+2x)}{x} = \lim_{x\to 0}\frac{\log(1+2x)}{2x}\times 2$

$$= \lim_{x\to 0}\frac{\log(1+2x)}{2x}\times\lim_{x\to 0} 2 = 2$$

19. **(a) :** $\lim_{x\to 0^+} x\left(\frac{e^{|x|+[x]} - 2}{|x| + [x]}\right)$

$$= \lim_{x\to 0^+} x\left(\frac{e^{x+0} - 2}{x+0}\right) = \lim_{x\to 0^+}(e^x - 2) = 1 - 2 = -1$$

$$\lim_{x\to 0^-} x\left(\frac{e^{|x|+[x]} - 2}{|x|+[x]}\right) = \lim_{x\to 0^-} x\left(\frac{e^{-x-1} - 2}{-x-1}\right) = 0$$

20. **(b) :** $\lim_{n\to\infty} 5^{-2\log_5 \frac{1}{4}\frac{\left(1-\left(\frac{1}{2}\right)^n\right)}{1-1/2}}$

$$= \lim_{n\to\infty} 5^{\log_5\left[\frac{1}{4}\left(1-\left(\frac{1}{2}\right)^n\right)\times\frac{2}{1}\right]^{-2}} \quad \left\{\begin{array}{l} S_n \text{ of G.P} = \frac{a(1-r^n)}{1-r} \text{ if } r<1 \\ \text{Also, } a^{\log_a x} = x \end{array}\right\}$$

$$= \lim_{n\to\infty}\left(\frac{1}{4}\times\frac{2}{1}\right)^{-2} = \left(\frac{1}{2}\right)^{-2} = 2^2 = 4$$

21. **(b) :** $f(x) = (x-2)(x-4)(x-6)\ldots(x-2n)$.

Differentiating both sides

$f'(x) = (x-4)(x-6)\ldots(x-2n) + (x-2)(x-6)\ldots(x-2n) + \ldots + (x-2)(x-4)(x-6)\ldots(x-(2n-1))$

$\Rightarrow f'(2) = (-2)(-4)\ldots(2-2n)$

$= (-2)^{n-1}\{1\cdot 2\ldots(n-1)\} = (-2)^{n-1}(n-1)!$

22. **(c) :** $y = \sqrt{f(x) + \sqrt{f(x) + \sqrt{f(x) + \ldots\ldots\infty}}}$

$\Rightarrow y = \sqrt{f(x) + y} \quad \Rightarrow y^2 - y = f(x)$

$\Rightarrow (2y-1)\frac{dy}{dx} = f'(x) \quad \therefore \frac{dy}{dx} = \frac{f'(x)}{2y-1}$

23. **(c) :** $g(x) = \frac{(x-1)^n}{\log\cos^m(x-1)}$,

$$p = \lim_{x\to 1^+} g(x) = \lim_{x\to 1^+}\frac{(x-1)^n}{\log\cos^m(x-1)}$$

Set $x = 1 + h$, then $h > 0$. Also, the limit reduces to

$$p = \lim_{h\to 0}\frac{h^n}{\log\cos^m h} = \lim_{h\to 0}\frac{h^n}{m\log(\cos h)}$$

$$= \frac{1}{m}\lim_{h\to 0}\frac{h^n}{\log[1-(1-\cos h)]}$$

$$= \frac{1}{m}\cdot\lim_{h\to 0}\frac{h^n}{\frac{\log[1-(1-\cos h)]}{(1-\cos h)}\cdot(1-\cos h)}$$

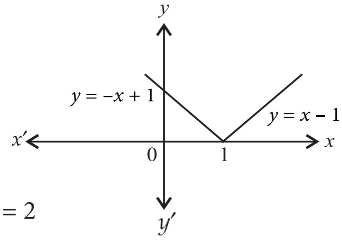
$$= \frac{1}{m}\cdot\lim_{h\to 0} h^{n-2}\cdot\frac{1}{\frac{\log[1-(1-\cos h)]}{1-\cos h}}\cdot\frac{h^2}{1-\cos h}$$

$$= \frac{1}{m}\lim_{h\to 0} h^{n-2}\cdot\frac{1}{\frac{\log[1-(1-\cos h)]}{1-\cos h}}\cdot\frac{h^2}{2\sin^2\frac{h}{2}}$$

$$= \frac{1}{m}\left(\lim_{h\to 0} h^{n-2}\right)\left(\frac{1}{\lim_{h\to 0}\frac{\log[1-(1-\cos h)]}{1-\cos h}}\right)\cdot 2\left\{\lim_{h\to 0}\left(\frac{h/2}{\sin\frac{h}{2}}\right)^2\right\}$$

$= \frac{2}{m}\cdot\left(\lim_{h\to 0} h^{n-2}\right)\cdot(-1)(1) = -\frac{2}{m}\left(\lim_{h\to 0} h^{n-2}\right)$ ....(A)

As p is the left hand derivative of $|x-1|$ at $x = 1$ we have $p = -1$, as can be seen from the graph of $y = |x-1|$ using $p = -1$ in (A), we have



$-1 = -\frac{2}{m}\left(\lim_{h\to 0} h^{n-2}\right)$

$\Rightarrow 1 = \frac{2}{m}\left(\lim_{h\to 0} h^{n-2}\right)$

For the above to be satisfied, $n = 2$

which gives, $1 = \frac{2}{m}\cdot 1 \Rightarrow m = 2$

Thus, $m = 2$ and $n = 2$.

24. **(b) :** $f(x) = \log\left(\frac{1+x}{1-x}\right)$, $g(x) = \frac{3x+x^3}{1+3x^2}$

$\therefore \quad fog(x) = f(g(x))$

$= f\left(\frac{3x+x^3}{1+3x^2}\right) = \log\left(\frac{1+\frac{3x+x^3}{1+3x^2}}{1-\frac{3x+x^3}{1+3x^2}}\right)$

$= \log\left(\frac{1+x}{1-x}\right)^3 = 3\log\left(\frac{1+x}{1-x}\right) = 3f(x)$

$\therefore \quad \frac{d}{dx}(f(g(x))) = 3f'(x)$

25. **(c) :** $y = \sin x + e^x$ or $\frac{dx}{dy} = \cos x + e^x$

or $\quad \frac{dx}{dy} = (\cos x + e^x)^{-1}$ …(i)

Again, $\frac{d^2x}{dy^2} = -\left(\cos x + e^x\right)^{-2}\left(-\sin x + e^x\right)\frac{dx}{dy}$

Substituting the value of $\frac{dx}{dy}$ from (i),

$$\frac{d^2x}{dy^2} = \frac{\left(\sin x - e^x\right)}{\left(\cos x + e^x\right)^2}\left(\cos x + e^x\right)^{-1} = \frac{\sin x - e^x}{\left(\cos x + e^x\right)^3}$$

26. **(a) :** Let $y^{y^{y^{\cdot^{\cdot^{\cdot\infty}}}}} = \log_e(x + \log_e(x + ...)) = v$

$\therefore \quad y^v = \log_e(x+v) = v \quad \therefore \quad y = v^{1/v}$ and $x = e^v - v$

$\Rightarrow \frac{dy}{dv} = v^{1/v}\left(\frac{d}{dx}\left(\frac{1}{v}\log_e v\right)\right)$ and $\frac{dx}{dv} = e^v - 1$

$= v^{1/v}\left(\frac{1}{v}\cdot\frac{1}{v} - \frac{1}{v^2}\log_e v\right) = v^{\frac{1}{v}-2}\left(1-\log_e v\right)$

$\therefore \quad \frac{dy}{dx} = \frac{v^{\frac{1}{v}-2}\left(1-\log_e v\right)}{e^v - 1}$

Now, $v^{1/v} = \sqrt{2}$ & $e^v - v = e^2 - 2$

$\Rightarrow \quad v = 2$ or $v = 4$, which is not valid

$\therefore \quad v = 2$, $e^v - v = e^2 - 2 = x$, given so true

$\therefore \quad \left(\frac{dy}{dx}\right)_{(e-2),\sqrt{2}} = \left(\frac{dy}{dx}\right)_{v=2} = \frac{1-\log 2}{2\sqrt{2}(e^2-1)}$

27. **(b) :** We have $\frac{f(2x+2y)}{f(2x-2y)} = \frac{\sin(x+y)}{\sin(x-y)}$

or $\quad \frac{f(\alpha)}{\sin\frac{\alpha}{2}} = \frac{f(\beta)}{\sin\frac{\beta}{2}} = K$ or $f(x) = K\sin\frac{x}{2}$

$\therefore \quad f'(x) = \frac{K}{2}\cos\frac{x}{2}$ and $f''(x) = \frac{-K}{4}\sin\frac{x}{2}$

or $\quad 4f''(x) + f(x) = 0$

28. **(a) :** $y = x^2 + \frac{1}{y}$ or $y^2 = x^2y + 1$

$\Rightarrow \quad 2y\frac{dy}{dx} = y\cdot 2x + x^2\frac{dy}{dx} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{2xy}{2y - x^2}.$

29. **(b) :** Given, $f(x) = \sqrt{x+2\sqrt{2x-4}} + \sqrt{x-2\sqrt{2x-4}}$

Let $f(x) = I_1 + I_2$

where, $I_1 = \sqrt{x+2\sqrt{2x-4}} = \sqrt{(x-2)+2+2\sqrt{2(x-2)}}$

$= \sqrt{\left(\sqrt{x-2}\right)^2 + \left(\sqrt{2}\right)^2 + 2\sqrt{2}\sqrt{x-2}}$

$= \sqrt{\left(\sqrt{x-2}+\sqrt{2}\right)^2} = \sqrt{x-2} + \sqrt{2}$

Similarly, $I_2 = \sqrt{x-2\sqrt{2x-4}} = \left|\sqrt{x-2} - \sqrt{2}\right|$

$\Rightarrow \quad f(x) = \left(\sqrt{x-2}+\sqrt{2}\right) + \left|\sqrt{x-2} - \sqrt{2}\right|$

$$\Rightarrow \quad f(x) = \begin{cases} \frac{4}{\sqrt{2}} & \text{if } \sqrt{x-2} - \sqrt{2} < 0 \Rightarrow x < 4 \\ \sqrt{x-2} & \text{if } \sqrt{x-2} \ge \sqrt{2} \Rightarrow x \ge 4 \\ 2\sqrt{2} & \text{if } 2 \le x < 4 \end{cases}$$

Now, $f'(x) = \frac{1}{2\sqrt{x-2}}$ if $4 \le x < \infty$

$= 0$ if $2 \le x < 4$

But when $x = 4$, $f'(4) = \frac{1}{0}$ is not defined

$\therefore \quad f(x)$ is differentiable on $(2, 4) \cup (4, \infty)$

i.e. $x \in (2, \infty) - \{4\}$.

30. **(a) :** Since $|f(x) - f(y)| \le |x-y|^3$, where $x \ne y$, we have

$\left|\frac{f(x)-f(y)}{x-y}\right| \le |x-y|^2$

Taking limit as $y \to x$, we get

$\lim_{y\to x}\left|\frac{f(x)-f(y)}{x-y}\right| \le \lim_{y\to x}|x-y|^2 \le \left|\lim_{y\to x}(x-y)^2\right|$

or $\quad |f'(x)| \le 0 = 0$ $\quad$ ($|f'(x)| \ge 0$)

$\therefore \quad f'(x) = 0$

or $\quad f(x) = c$ (constant)

31. **(a, c) :** Since, $(1-x)^{1/2} = 1 - \frac{1}{2}x + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{\lfloor 2}x^2$

$- \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{\lfloor 3}x^3 + ...$

We have $L = \lim_{x \to 0} \dfrac{a-(a^2-x^2)^{1/2}-\dfrac{x^2}{4}}{x^4}$

$$= \lim_{x \to 0} \frac{a-a\left\{1-\dfrac{x^2}{a^2}\right\}^{1/2}-\dfrac{x^2}{4}}{x^4} \quad (\sqrt{a^2}=a \ \because \ a>0)$$

$$= \lim_{x \to 0} \frac{a-a\left\{1-\dfrac{1}{2}\cdot\dfrac{x^2}{a^2}-\dfrac{1}{8}\dfrac{x^4}{a^4}+.....\right\}-\dfrac{x^2}{4}}{x^4}$$

$$= \lim_{x \to 0} \frac{x^2\left(\dfrac{1}{2a}-\dfrac{1}{4}\right)+\dfrac{x^4}{8a^3}+.....}{x^4}$$

As the limit exists we have $\dfrac{1}{2a}=\dfrac{1}{4} \Rightarrow a=2,$ and then the limit is $\dfrac{1}{8a^3}=\dfrac{1}{8}\cdot\dfrac{1}{2^3}=\dfrac{1}{64}$.

32. **(b, c) :** $y=\dfrac{(x^2+1)^2-3x^2}{x^2+\sqrt{3}x+1}=\dfrac{(x^2+1+\sqrt{3}x)(x^2+1-\sqrt{3}x)}{x^2+1+\sqrt{3}x}$

$\dfrac{dy}{dx}=2x-\sqrt{3}$ or $a=2$ and $b=-\sqrt{3}$

$a-b=2+\sqrt{3}=\tan\dfrac{5\pi}{12}=\cot\dfrac{\pi}{12}$

33. **(b, c) :** $\lim_{x \to 0}\left(1+ax+bx^2\right)^{2/x}=e^3$

$\Rightarrow e^{\lim_{x \to 0} \frac{2}{x}(ax+bx^2)}=e^3$

$\Rightarrow \quad e^{2a}=e^3 \Rightarrow 2a=3 \quad \Rightarrow \quad a=\dfrac{3}{2}$

Since, limit value e^{2a} does not involve b.

$\therefore \quad b$ can have any value.

Thus, $a=\dfrac{3}{2}$ and $b \in R$.

34. **(a, b, c) :** $L=\lim_{x \to a}\dfrac{|2\sin x-1|}{2\sin x-1}$

For $a=\pi/6$,

$$\text{L.H.L} = \lim_{x \to \frac{\pi^-}{6}} \frac{1-2\sin x}{2\sin x-1}=-1$$

$$\text{R.H.L} = \lim_{x \to \frac{\pi^+}{6}} \frac{2\sin x-1}{2\sin x-1}=1$$

Hence, the limit does not exist.

For $a=\pi$, $\lim_{x \to \pi}\dfrac{1-2\sin x}{2\sin x-1}=-1$

(as in neighborhood of π, $\sin x$ is less than 1/2).

For $a=\dfrac{\pi}{2}$, $\lim_{x \to \pi/2}\dfrac{2\sin x-1}{2\sin x-1}=1$

(as in neighborhood of $\pi/2$ $\sin x$ approaches 1).

35. **(a,b,d) :** $f(x)=\lim_{n\to\infty}\dfrac{\log(2+x)-x^{2n}\sin x}{1+x^{2n}}$

$f(1)=\lim_{n\to\infty}\dfrac{\log 3-\sin 1}{2}=\dfrac{1}{2}(\log 3-\sin 1)$

Now, $\lim_{n\to\infty} x^{2n}=\begin{cases}0 & \text{if} \quad x^2<1\\ \infty & \text{if} \quad x^2>1\end{cases}$ $\quad\therefore\quad$ For $x^2<1$, we have

$f(x)=\lim_{n\to\infty}\dfrac{\log(2+x)-x^{2n}\sin x}{1+x^{2n}}=\log(2+x)$ for $x^2>1$

$$f(x)=\lim_{n\to\infty}\left(\frac{\dfrac{\log(2+x)}{x^{2n}}-\sin x}{1+\dfrac{1}{x^{2n}}}\right)=-\sin x$$

$$\therefore \quad f(x)=\begin{cases}\log(2+x), & x^2<1\\ \dfrac{1}{2}(\log 3-\sin 1), & x=1\\ -\sin x, & x^2>1\end{cases}$$

$\therefore \quad \lim_{x\to 1^-} f(x)=\lim_{h\to 0}\log(2+1-h)=\log 3$

$\lim_{x\to 1^+} f(x)=\lim_{h\to 0}(-\sin(1+h))=-\sin 1$

Clearly both limits exist and $\lim_{x\to 1^-} f(x)\neq \lim_{x\to 1^+} f(x)$

36. **(b, d) :** $x=\dfrac{1+\log_e t}{t^2}; y=\dfrac{3+2\log_e t}{t}$

$$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{\dfrac{t\left(\dfrac{2}{t}\right)-(3+2\log_e t)}{t^2}}{\dfrac{t^2\left(\dfrac{1}{t}\right)-(1+\log_e t)2t}{t^4}}=\left(\frac{-1-2\log_e t}{-1-2\log_e t}\right)t=t$$

Eliminating $\log_e t$ term from y, get

$y=\dfrac{1+2t^2x}{t}=\dfrac{1+2(y')^2x}{y'}$

or $\quad yy'=1+2x(y')^2$ (Differentiating w.r.t x)

or $\quad yy''+(y')^2=4xyy'\cdot y''+2(y')^2$

or $\quad yy''=4xy'\cdot y''+(y')^2$

37. **(a, c) :** $\dfrac{dy}{dx}=\dfrac{e^{\sqrt{x}}}{2\sqrt{x}}-\dfrac{e^{-\sqrt{x}}}{2\sqrt{x}}=\dfrac{e^{\sqrt{x}}-e^{-\sqrt{x}}}{2\sqrt{x}}$

$$=\frac{\sqrt{\left(e^{\sqrt{x}}+e^{-\sqrt{x}}\right)^2-4}}{2\sqrt{x}}=\frac{\sqrt{y^2-4}}{2\sqrt{x}}$$

38. **(a, c) :** Since $x^2>0$ and limit equals 2, $f(x)$ must be a positive quantity.

Also, since $\lim_{x\to 0}\dfrac{f(x)}{x^2}=2$, denominator $\to$ zero and limit is finite.

Therefore, $f(x)$ must be approaching zero or $\lim_{x\to 0}[f(x)]=0^+$.

Hence, $\lim_{x\to 0}[f(x)]=0^+$.

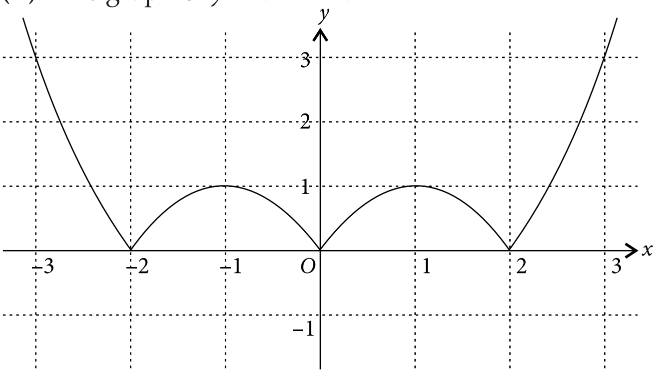
$$\lim_{x\to0^+}\left[\frac{f(x)}{x}\right]=\lim_{x\to0^+}\left[x\frac{f(x)}{x^2}\right]=0 \text{ and } \lim_{x\to0^-}\left[\frac{f(x)}{x}\right]$$

$$=\lim_{x\to0^-}\left[x\frac{f(x)}{x^2}\right]=-1$$

Hence, $\lim_{x\to0}\left[\frac{f(x)}{x}\right]$ does not exist.

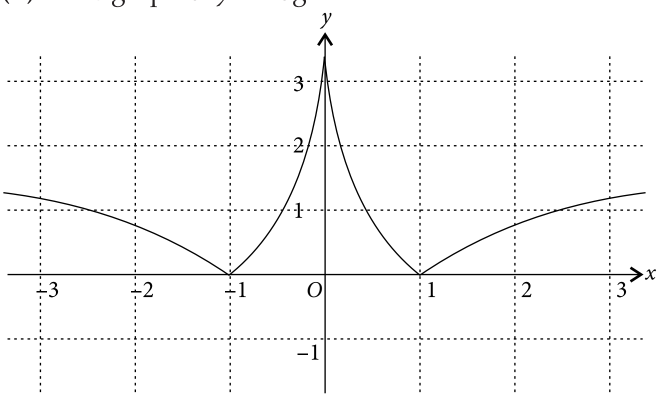
39. **(a) : (A) – (p, q, r), (B) – (q,s), (C) – (q, r), (D) – (r)**

(A) The graph of $y = |x^2 - 2|x||$:



From the graph, dy/dx is negative for (p), (q), (r)

(B) The graph of $y = |\log|x||$:

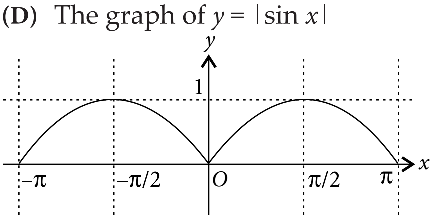


From the graph, dy/dx is negative for (q), (s).

(C) $y = x[x/2] = \begin{cases} -x, & -4 \le x < -2 \\ -x, & -2 \le x < 0 \\ 0, & 0 \le x < 2 \\ x, & 2 \le x < 4 \end{cases}$

Hence, dy/dx is negative for (q), (r)

(D) The graph of $y = |\sin x|$



From the graph, dy/dx is negative for (r).

40. **(c) : (A) – (s), (B) – (p), (C) – (q), (D) – (r)**

(A) $\lim_{x\to0}\frac{2e^{2x}-2}{2x}=\lim_{x\to0}\frac{e^{2x}-1}{x}=2$ [Using L. H. rule]

(B) $\exp\lim_{x\to0}\left(\frac{\cos x-1}{x^2}\right)=\exp\lim_{x\to0}-\frac{\sin x}{2x}$ [By L.H. Rule]

$=\frac{1}{\sqrt{e}}$

(C) $L=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{[(2r-1)x]}{n^2}$

$=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{(2r-1)x}{n^2}-\sum_{r=1}^{n}\frac{\{(2r-1)x\}}{n^2}$

$=\lim_{n\to\infty}\frac{x}{n^2}\sum_{r=1}^{n}(2r-1)-0=x$

(since $1+3+\ldots+(2n-1)=n^2$ and each of the terms $\frac{\{(2r-1)x\}}{n}<\frac{1}{n}$ and their sum $<\frac{n}{n^2}=\frac{1}{n}\to0$ as $n\to\infty$)

(D) $\lim_{n\to\infty}\prod_{r=3}^{n}\frac{(r+2)(r^2-2r+2^2)}{(r-2)(r^2+2r+2^2)}$

$=\lim_{n\to\infty}\prod_{r=3}^{n}\frac{r+2}{r-2}\cdot\prod_{r=3}^{n}\frac{r^2-2r+2^2}{r^2+2r+2^2}$

$=\lim_{n\to\infty}\frac{5}{1}\cdot\frac{6}{2}\cdot\frac{7}{3}\cdot\frac{8}{4}\cdot\frac{9}{5}\cdots\frac{n+2}{n-2}\times\frac{7}{19}\cdot\frac{12}{28}\cdot\frac{19}{39}\cdots\frac{n^2-2n+4}{n^2+2n+4}$

$=\lim_{n\to\infty}\frac{(n-1)n(n+1)(n+2)}{1\cdot2\cdot3\cdot4}\cdot\frac{7\times12}{(n^2+3)(n^2+2n+4)}$

$=\lim_{n\to\infty}\frac{7}{2}\frac{\left(1-\frac{1}{n}\right)\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}{\left(1+\frac{3}{n^2}\right)\left(1+\frac{2}{n}+\frac{4}{n^2}\right)}=\frac{7}{2}$

41. **(b) :** $y=\log_a x=\log_e x\cdot\log_a e$

$\therefore\ y_1=\frac{1}{x}\log_a e$

$\therefore\ y_2=\frac{(-1)}{x^2}\log_a e$

$y_3=(\log_a e)(-1)(-2)x^{-3}=(\log_a e)(-1)^2\cdot2!\,x^{-3}$

$\vdots \quad \vdots \qquad \vdots \quad \vdots \qquad \vdots$

$\therefore\ y_n=(\log_a e)\frac{(-1)^{n-1}n!}{n}x^{-n}$

$\therefore\ y_n(e)=(\log_a e)\frac{(-1)^{n-1}n!}{n}\left(\frac{1}{e}\right)^n$

42. **(a) :** $x^y=e^{x-y} \Rightarrow y\log_e x=x-y$

$\Rightarrow y=\frac{x}{1+\log_e x}\ \therefore y_1=\frac{\log_e x}{(1+\log_e x)^2}$

$\Rightarrow y_1(e)=\frac{1}{4}$

43. **(b)** **44.** **(d)** **45.** **(c)**

(43-45) : $L=\lim_{x\to0}\frac{\sin x+ae^x+be^{-x}+c\ln(1+x)}{x^3}$

$$=\lim_{x\to0}\frac{\left(x-\frac{x^3}{3!}+\ldots..\right)+a\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}\right)}{x^3}$$

$$\frac{+b\left(1-\frac{x}{1!}+\frac{x^2}{2!}-\frac{x^3}{3!}\right)+c\left(x-\frac{x^2}{2}+\frac{x^3}{3}\right)}{x^3}$$

$$= \lim_{x\to 0} \frac{(a+b)+(1+a-b+c)x+\left(\frac{a}{2}+\frac{b}{2}-\frac{c}{2}\right)x^2}{x^3}$$

$$+\frac{\left(-\frac{1}{3!}+\frac{a}{3!}-\frac{b}{3!}+\frac{c}{3}\right)x^3}{x^3}$$

or $a + b = 0$, $1 + a - b + c = 0$, $\frac{a}{2}+\frac{b}{2}-\frac{c}{2}=0$

and $L=-\frac{1}{3!}+\frac{a}{3!}-\frac{b}{3!}+\frac{c}{3}$

Solving the first three equations, we get $c = 0$, $a = -1/2$, $b = 1/2$. Then, $L = -1/3$.

Equation $ax^2 + bx + c = 0$ reduces to $x^2 - x = 0$ or $x = 0, 1$.

$||x + c| - 2a| < 4b$ reduces to

$||x| + 1| < 2$ or $-2 < |x| + 1 < 2$ or $x \in [-1, 1]$

46. **(3) :** $y=\frac{x^4-(x^2+2x+1)}{x^2-x-1}=x^2+x+1$

$\therefore \quad \frac{dy}{dx}=2x+1=ax+b$

Hence, $a = 2$ and $b = 1$

47. **(1) :** $f(x)+f(y)=f\left(\frac{x+y}{1-xy}\right)$ ...(i)

Putting $x = y = 0$, we get $f(0) = 0$

Putting $y = -x$, we get $f(x) + f(-x) = f(0)$

$\Rightarrow \quad f(-x) = -f(x)$ ...(ii)

also, $\lim_{x\to 0}\frac{f(x)}{x}=2$

Now, $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ ...(iii)

$= \lim_{h\to 0}\frac{f(x+h)+f(-x)}{h}$ [using (ii)]

$\Rightarrow \quad f'(x)=\lim_{h\to 0}\frac{f\left(\frac{x+h-x}{1-(x+h)(-x)}\right)}{h}$ [using (i)]

$\Rightarrow \quad f'(x)=\lim_{h\to 0}\left[\frac{f\left(\frac{h}{1+x(x+h)}\right)}{h}\right]$

$\Rightarrow \quad f'(x)=\lim_{h\to 0}\frac{f\left(\frac{h}{1+xh+x^2}\right)}{\left(\frac{h}{1+xh+x^2}\right)}\times\left(\frac{1}{1+xh+x^2}\right)$

$\Rightarrow \quad f'(x)=\lim_{h\to 0}\frac{f\left(\frac{h}{1+xh+x^2}\right)}{\left(\frac{h}{1+xh+x^2}\right)}\times\lim_{h\to 0}\frac{1}{1+xh+x^2}$

$\left(\text{using }\lim_{x\to 0}\frac{f(x)}{x}=2\right)$

$\Rightarrow \quad f'(x)=2\times\frac{1}{1+x^2} \Rightarrow f'(x)=\frac{2}{1+x^2}$

$\Rightarrow \quad f'(1)=\frac{2}{1+1^2}=\frac{2}{2}=1$

48. **(9) :** Let degree of $f(x)$ is n, degree of $f'(x)$ is $n - 1$, and degree of $f''(x)$ $(n - 2)$. Hence,

$n = (n - 1) + (n - 2) = 2n - 3 \quad \therefore \quad n = 3$

Hence, $f(x) = ax^3 + bx^2 + cx + d$ $(a \neq 0)$

$f'(x) = 3ax^2 + 2bx + c$

$f''(x) = 6ax + 2b$

$\therefore \quad ax^3 + bx^2 + cx + d = (3ax^2 + 2bx + c)(6ax + 2b)$

$\therefore \quad 18a^2 = a$ or $a=\frac{1}{18} \Rightarrow \frac{1}{2a}=\frac{1}{2\times\frac{1}{18}}=9$

49. **(9) :** We have, $y=\frac{\sin(x+9)}{\cos x}$ ...(i)

Differentiating (i) w.r.t. x, we get

$$\frac{dy}{dx}=\frac{\cos x\frac{d}{dx}(\sin(x+9))-\sin(x+9)\frac{d}{dx}(\cos x)}{\cos^2 x}$$

$$=\frac{\cos x(\cos(x+9))\times 1-\sin(x+9)(-\sin x)}{\cos^2 x}$$

$$=\frac{\cos x\cos(x+9)+\sin x\sin(x+9)}{\cos^2 x}$$

$$=\frac{\cos(x-x-9)}{\cos^2 x}=\frac{\cos(-9)}{\cos^2 x}=\frac{\cos 9}{\cos^2 x}$$

$$\therefore \quad \left.\frac{dy}{dx}\right|_{\text{at } x=0}=\frac{\cos 9}{(\cos 0)^2}=\frac{\cos 9}{1}=\cos 9$$

50. **(7) :** $g'(0)=b=\lim_{x\to 0}\frac{x^2+x\tan x-x\tan 2x}{x(ax+\tan x-\tan 3x)}$

$$=\lim_{x\to 0}\frac{x+\tan x-\tan 2x}{ax+\tan x-\tan 3x}$$

$$=\lim_{x\to 0}\frac{x+\left(x+\frac{x^3}{3}+\frac{2}{15}x^5+.....\infty\right)-\left(2x+\frac{8x^3}{3}+\frac{2}{15}\cdot 32x^5+.....\infty\right)}{ax+\left(x+\frac{x^3}{3}+\frac{2}{15}x^5+.....\infty\right)-\left(3x+\frac{27x^3}{3}+\frac{2}{15}243x^5+.....\infty\right)}$$

$$=\lim_{x\to 0}\frac{x^3\left(-\frac{7}{3}+\frac{-62}{15}x^2+.....\infty\right)}{(a+1-3)x+\left(\frac{1}{3}-9\right)x^3+\frac{2}{15}(-242)x^5+.....\infty}$$

b can be finite if $a + 1 - 3 = 0$. Therefore,

$a = 2$ and $b=\frac{-\frac{7}{3}}{\frac{1}{3}-9}=\left(\frac{-7}{3}\right)\left(\frac{3}{-26}\right)=\frac{7}{26}\Rightarrow 52\frac{b}{a}=7$

JEE MATHEMATICS

DPS-11

DAILY PRACTICE SHEET

Class XI

Statistics and Probability, Mathematical Reasoning

SOLUTIONS

1. **(c) :** Variance doesn't change with the change of origin.

2. **(c) : Firm A :** Number of wages earners (n_1) = 586
Mean of weeakly wages ($\bar{x}_1$) = ₹52.5
$\therefore$ Total weekly wages = 586 × ₹52.5 = ₹30765
Firm B : Number of wages earners (n_2) = 648
Mean of weeakly wages ($\bar{x}_2$) = ₹47.5
$\therefore$ Total weekly wages = 648 × ₹47.5 = ₹30780
For firm A, $\overline{x_1} = 52.5$ and $\sigma = 100$
$$\therefore \quad \text{C.V.} = \frac{100}{52.5} \times 100 = 190.47$$
For firm B, $\overline{x_2} = 47.5$ and $\sigma = 121$
$$\therefore \quad \text{C.V.} = \frac{121}{47.5} \times 100 = 254.73$$
So, firm B pays larger amount as monthly wages.

3. **(b) :** Given, $\sum x_i = 256$ ($\therefore$ $\overline{x} = 16$ and number of terms = 16)
New sum of all observations
= (256 – 16) + (3 + 4 + 5) = 240 + 12 = 252
Also, the number of observations now become 18.
Hence, new mean = $\frac{252}{18} = 14$

4. **(a) :** The mean of a, b, 8, 5, 10 is 6.
$$\Rightarrow \frac{a+b+8+5+10}{5} = 6$$
$\Rightarrow$ $a + b + 23 = 30$ $\Rightarrow$ $a + b = 7$ ...(1)
Again variance $= \frac{\sum(x_i - \bar{x})^2}{n} = 6.8$
$$\Rightarrow \frac{(a-6)^2 + (b-6)^2 + 4 + 1 + 16}{5} = 6.8$$
$\Rightarrow$ $a^2 + b^2 - 12(a + b) + 21 + 72 = 5 \times 6.8 = 34$
$\Rightarrow$ $a^2 + b^2 - 12 \times 7 + 72 + 21 = 34$
$\therefore$ $a^2 + b^2 = 25$ ...(2)
Using (1), we have
$a^2 + (7 - a)^2 = 25$ $\Rightarrow$ $a^2 + 49 - 14a + a^2 = 25$
$\Rightarrow$ $a^2 - 7a + 12 = 0$ $\therefore$ $a = 3, 4$ which gives $b = 4, 3$

5. **(a) :** Arranging data in ascending order, we get
$$x - \frac{7}{2},\ x-3,\ x - \frac{5}{2},\ x-2,\ x - \frac{1}{2},\ x + \frac{1}{2},\ x+4,\ x+5$$
Let, n = number of terms = 8 (even)
$$\therefore \text{ Median } = \frac{(4^{\text{th}} + 5^{\text{th}})\text{observation}}{2}$$
$$= \frac{x - 2 + x - \frac{1}{2}}{2} = \frac{2x - \frac{5}{2}}{2} = x - \frac{5}{4}$$

6. **(a) :** $3630 = 2^1\, 3^1\, 5^1\, 11^2 = 2^a\, 3^b\, 5^c\, 11^d$
$\therefore$ Number of odd divisors of 3630 are $(b + 1)(c + 1)(d + 1) = 2 \times 2 \times 3$
$\therefore$ Total number of odd divisors of 3630 = 12.
These divisors are the terms of the product given as
(1 + 3) (1 + 5) (1 + 11 + 121)
= (1 + 3 + 5 + 15) (1 + 11 + 121)
$\therefore$ Odd divisors are 1, 3, 5, 11, 15, 33, 55, 121, 165, 363, 605, 1815
From these divisors we need those divisor which leaves the remainder 1, when divided by 4.
$\therefore$ Those divisors are 1, 5, 33, 121, 165, 605
$\therefore$ Number of such divisors are (n = 6) in counting.
$\therefore$ Mean of such divisors =
$$\frac{\text{Sum of all odd divisors which leave remainder 1 when divided by 4}}{\text{Number of divisors of the form } (4k+1)}$$
$$= \frac{1+5+33+121+165+605}{6} = \frac{930}{6} = 155$$

7. **(a) :** Let x and y be the number of boys and girls in the class respectively.
We have, $\frac{52x + 42y}{x+y} = 50 \Rightarrow x = 4y$
$$\Rightarrow \frac{x}{y} = \frac{4}{1} \text{ and } \frac{x}{x+y} = \frac{4}{5}$$
$$\therefore \quad \text{Required percentage} = \frac{x}{x+y} \times 100 = \frac{4}{5} \times 100 = 80\%.$$

8. **(c) :** The numbers are 2, 4, 6, ..., $2n$
$$\text{Mean} = \bar{X} = \frac{2 + 4 + \ldots + 2n}{n} = \frac{2n(n+1)}{2n} = n+1$$
$$\text{Variance} = \frac{1}{n}(2^2 + 4^2 + \ldots + (2n^2)) - \bar{X}^2$$
$$= \frac{4 \cdot (n+1)(2n+1)}{6} - (n+1)^2$$
$$= \frac{(n+1)}{3}(4n + 2 - 3n - 3) = \frac{n^2 - 1}{3}.$$

9. **(b) :** In the problem $x_i = i$ $(i = 1, 2, 3, \ldots, n)$ and weights $w_i = i^2 + i$
$$\therefore \quad \text{Mean} = \frac{\Sigma w_i x_i}{\Sigma w_i}$$
$$= \frac{\sum_{i=1}^{n} i(i^2 + i)}{\sum_{i=1}^{n} i^2 + i} = \frac{\sum_{i=1}^{n} (i^3 + i^2)}{\sum_{i=1}^{n} \left(i^2 + i\right)}$$

$$= \frac{\Sigma n^3 + \Sigma n^2}{\Sigma n^2 + \Sigma n} = \frac{\frac{n^2(n+1)^2}{4} + \frac{n(n+1)(2n+1)}{6}}{\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}}$$

$$= \frac{3n^2 + 7n + 2}{4(n+2)} = \frac{(3n+1)(n+2)}{4(n+2)} = \frac{(3n+1)}{4}$$

10. **(a) :** Number of binomial coefficients ${}^{2n}C_0$, ${}^{2n}C_1$, ${}^{2n}C_2$,..., ${}^{2n}C_n$ is odd (when n is even) and middle binomial coefficient is ${}^{2n}C_{n/2}$ which is required median.

11. **(c) :** Mean $= \frac{\frac{n(n+1)}{2}}{n} = \frac{(n+1)}{2}$.

Mean deviation from mean

$$= \frac{\left\{\sum_{i=1}^{n}\left|\frac{n+1}{2} - i\right|\right\}}{n} = \frac{1}{n} \times \left[\sum_{i=1}^{n/2}\left(\frac{n+1}{2} - i\right) + \sum_{i=\frac{n}{2}+1}^{n}\left(i - \frac{n+1}{2}\right)\right]$$

$$\left[\because \frac{n+1}{2} - i < 0 \text{ for } i = \frac{n}{2}+1, \ldots.., n\right]$$

$$= \frac{1}{n} \times \left[\left\{\frac{n}{2} \times \frac{(n+1)}{2} - \frac{\frac{n}{2}\left(\frac{n}{2}+1\right)}{2}\right\} + \left\{\frac{n/2}{2}\left(\frac{3n}{2}+1\right) - \frac{n}{2}\frac{(n+1)}{2}\right\}\right]$$

$$= \frac{1}{n} \times \frac{n}{4}\left[\frac{3n}{2} + 1 - \frac{n}{2} - 1\right] = \frac{n}{4}.$$

12. **(d) :** Total number of arrangements of the letters of the word "FAVOURABLE" is $\frac{10!}{2!}$

Now, say 2A's as one letter ∴ total number of arrangement if 'AA' consider one letter is 9!

∴ Number of ways in which 'AA' never come together is $= \frac{10!}{2!} - 9!$.

∴ Required probability $= \frac{\frac{10!}{2!} - 9!}{\frac{10!}{2!}} = \frac{4}{5}$

13. **(a) :** Probability of throwing a six $= \frac{1}{6}$

$$P(A) = \frac{1}{6}, P(\bar{A}) = \frac{5}{6}, P(B) = \frac{1}{6}, P(\bar{B}) = \frac{5}{6}$$

Probability of winning A

$= P(A) + P(\bar{A})\,P(\bar{B})\,P(A) + P(\bar{A})\,P(\bar{B})\,P(\bar{A})\,P(\bar{B})\,P(A) + \ldots$

$$= \frac{1}{6} + \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6} + \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6} + \ldots$$

$$= \frac{\frac{1}{6}}{1 - \frac{25}{36}} = \frac{6}{11}$$

Probability of winning $B = 1 - \frac{6}{11} = \frac{5}{11}$

∴ Expectations of A and B are $\frac{6}{11} \times 11 =$ ₹ 6 and $\frac{5}{11} \times 11 =$ ₹ 5

14. **(a) :** P(the problem is solved by at least one person)

= 1 – P(the problem is solved by none of the persons)

$$= 1 - \left[\left(1 - \frac{1}{2}\right)\left(1 - \frac{3}{4}\right)\left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{8}\right)\right]$$

$$= 1 - \left(\frac{1}{2} \cdot \frac{1}{4} \cdot \frac{3}{4} \cdot \frac{7}{8}\right) = 1 - \frac{21}{256} = \frac{235}{256}$$

15. **(d) :** In out of 9 tickets, 5 tickets are odd number and 4 tickets are even number.

Required probability

$$= \left\{\frac{{}^5C_1}{{}^9C_1} \times \frac{{}^4C_1}{{}^8C_1} \times \frac{{}^4C_1}{{}^7C_1} + \frac{{}^4C_1}{{}^9C_1} \times \frac{{}^5C_1}{{}^8C_1} \times \frac{{}^3C_1}{{}^7C_1}\right\}$$

$$= \frac{5}{9} \times \frac{4}{8} \times \frac{4}{7} + \frac{4}{9} \times \frac{5}{8} \times \frac{3}{7} = \frac{80 + 60}{504} = \frac{140}{504} = \frac{5}{18}$$

16. **(b) :** $P(A) = 0.3$, $P(B) = 0.6$

$P(A \cap B) = 0.18$

$P(\bar{A} \cap \bar{B}) = 1 - P(A \cup B)$

$= 1 - [P(A) + P(B) - P(A \cap B)]$

$= 1 - (0.3 + 0.6 - 0.18)$

$= 1 - (0.90 - 0.18)$

$= 0.10 + 0.18$

$= 0.28$

17. **(b) :** $P(\bar{A}) = 0.2$, $P(\bar{B}) = 0.3$

∴ $P(A) = 0.8$, $P(B) = 0.7$

∴ Required probability

$= P(\bar{A})\,P(B) + P(A)\,P(\bar{B}) + P(\bar{A})P(\bar{B})$

$= 0.2 \times 0.7 + 0.8 \times 0.3 + 0.2 \times 0.3 = 0.44$

18. **(b) :** Total cases when sum of the two sides is greater than the third side are

{(1, 1, 1), (2, 2, 1), (2, 2, 2), (2, 2, 3), (3, 3, 1), ..., (3, 3, 5), (4, 4, 1), ..., (4, 4, 6), (5, 5, 1), …, (5, 5, 6), (6, 6, 1), ..., (6, 6, 6)} = 27

∴ Required probability $= \frac{1}{27}$.

19. **(d) :** Total ways = $n!$ (∵ letters different)

Favourable ways when all letters placed in the right envelope = 1

∴ Probability of all letters in right envelopes $= \frac{1}{n!}$

∴ Probability that atleast one letter is not placed in the right envelope $= 1 - \frac{1}{n!}$

20. **(b) :** Odd digits = 1, 3, 5, 7, 9

Even digits = 0, 2, 4, 6, 8

Since, odd digits at odd place and even digits at even place

Place of odd digits = 3

and place of even digits = 2

∴ Favourable ways = ${}^5P_3 \times {}^5P_2 = 1200$

Total ways = 9 × 9 × 8 × 7 × 6

∴ Required probability $= \frac{1200}{9 \times 9 \times 8 \times 7 \times 6} = \frac{25}{567}$

21. **(b) :** Number of vowels are A,O,U,E

∴ One can be chosen by ${}^4C_1$ ways

Total letters in the given word = 10

∴ Required Probability $= \frac{4}{10} = \frac{2}{5}$

22. **(c) :** This problem is based on the property of ω that sum of the consecutive powers of ω is zero. But three consecutive numbers are of the form $3m$, $3m + 1$, $3m + 2$, m being an integer.

As we have r_1, r_2, r_3 are belong to the set {1, 2, 3, 4, 5, 6}, the number of ways to choose an ordered triplet

$r_1, r_2, r_3 = (^2C_1)^3 = 8$

r_1, r_2, r_3 can be arranged among themselves in $\lfloor 3$ ways,

Thus, the probability $= \frac{8 \times \lfloor 3}{6^3} = \frac{6 \times 8}{6^3} = \frac{2}{9}$

23. **(a) :** 3 – 2 = 5, is a false statement but take two aspirins and do you understand are not a statement.

24. **(a) :** The contrapositive of the statement is "If I will come, then it is not raining".

25. **(d) :** The statement can be written as

$P \wedge \sim R \Leftrightarrow Q$

Thus, the negation is

$\sim (Q \leftrightarrow (P \wedge \sim R))$

26. **(c) :** The negation of the proposition

"If a quadrilateral is a square, then it is a rhombus" is

"A quadrilateral is a square and it is not a rhombus".

27. **(b) :** Negation of the given proposition is 2 is prime and 3 is not odd.

28. **(d) :** See the following truth table.

p	q	$\sim q$	$p \leftrightarrow \sim q$	$\sim (p \leftrightarrow \sim q)$	$p \leftrightarrow q$
T	T	F	F	T	T
T	F	T	T	F	F
F	T	F	T	F	F
F	F	T	F	T	T

As the truth table matches, we have the statement

$\sim (p \leftrightarrow \sim q)$ is equivalent to $p \leftrightarrow q$

29. **(a) :**

p	q	$\sim p$	$\sim q$	$p \wedge \sim q$	$\sim p \wedge q$	$(p \wedge \sim q) \wedge (\sim p \wedge q)$
T	T	F	F	F	F	F
T	F	F	T	T	F	F
F	T	T	F	F	T	F
F	F	T	T	F	F	F

Hence, Statement-1 is a fallacy.

p	q	$\sim p$	$\sim q$	$p \to q$	$\sim q \to \sim p$	$(p \to q) \leftrightarrow (\sim q \to \sim p)$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

Hence, Statement-2 is a tautology.

30. **(a) :** $p \wedge (\sim q)$ shows the given compound statement.

31. **(a, c) :** $P(A \cup B) \geq \frac{3}{4}$ ...(i), $\frac{1}{8} \leq P(A \cap B) \leq \frac{3}{8}$ ...(ii)

Let $P(A) + P(B) = x$

From (i), $x - P(A \cap B) \geq \frac{3}{4}$

$\Rightarrow x - \frac{3}{4} \geq P(A \cap B) \geq \frac{1}{8}$ [From (ii)]

$\Rightarrow x \geq \frac{7}{8} \Rightarrow P(A \cup B) \leq 1 \Rightarrow x - P(A \cap B) \leq 1$ [From (ii)]

$\Rightarrow x - 1 \leq P(A \cap B) \leq \frac{3}{8} \Rightarrow x \leq \frac{11}{8}$

32. **(a, b, c) : (a) :** $P(E_1) = P(E_2) = \frac{\frac{10!}{2!}}{\frac{11!}{2!2!}} = \frac{2}{11}$

(b) : $P(E_1 \cap E_2)$ = Probability of two I's are together and two B's are together

$= \frac{9!}{\frac{11!}{2!2!}} = \frac{4}{110} = \frac{2}{55}$

(c) : $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$

$= \frac{2}{11} + \frac{2}{11} - \frac{2}{55} = \frac{4}{11} - \frac{2}{55} = \frac{18}{55}$

33. **(a, b) :** Since, $\sigma_x^2 = \left\{\frac{f_1x_1^2 + f_2x_2^2}{f_1 + f_2} - \left(\frac{f_1x_1 + f_2x_2}{f_1 + f_2}\right)^2\right\}$

$= \frac{1}{(f_1 + f_2)}\left\{f_1x_1^2 + f_2x_2^2 - \frac{(f_1x_1 + f_2x_2)^2}{f_1 + f_2}\right\}$

$= \frac{1}{(f_1 + f_2)^2}(f_1^2x_1^2 + f_1f_2x_1^2 + f_2f_1x_2^2 + f_2^2x_2^2 - f_1^2x_1^2 - f_2^2x_2^2$

$-2f_1f_2x_1x_2)$

$= \frac{f_1f_2}{(f_1 + f_2)^2}(x_1 - x_2)^2$

34. **(a, c) :** Let p_1 and p_2 be the chances of happeing of the first and second events, respectively. Then, according to the given conditions, we have

$p_1 = p_2^2$

and $\frac{1 - p_1}{p_1} = \left(\frac{1 - p_2}{p_2}\right)^3$

Hence, $\frac{1 - p_2^2}{p_2^2} = \left(\frac{1 - p_2}{p_2}\right)^3 \Rightarrow p_2(1 + p_2) = (1 - p_2)^2$

$\Rightarrow 3p_2 = 1 \Rightarrow p_2 = \frac{1}{3}$

$\therefore p_1 = \frac{1}{9}$

35. **(b, c) :** Here, $P(M) = \alpha$, $P(P) = \beta$ and $P(C) = \gamma$

$\therefore$ The probability of passing in atleast one subject = 0.75 (given)

$\Rightarrow 1 - P(\overline{M}\ \overline{P}\ \overline{C}) = 0.75$

$\Rightarrow 1 - P(\overline{M})\ P(\overline{P})\ P(\overline{C}) = 0.75$

$\Rightarrow 1 - (1 - \alpha)(1 - \beta)(1 - \gamma) = 0.75$

or, $\alpha + \beta + \gamma - \alpha\beta - \beta\gamma - \gamma\alpha + \alpha\beta\gamma = \frac{3}{4}$ ...(i)

The probability of passing in atleast two subjects = 0.50 (given)

or $P(M\ P\ \overline{C}) + P(M\ \overline{P}\ C) + P(\overline{M}\ P\ C) + P(MPC) = 0.50$

$\Rightarrow P(M)\ P(P)\ P(\overline{C}) + P(M)P(\overline{P})P(C) + P(\overline{M})P(P)P(C) + P(M)P(P)P(C) = 0.50$

$\Rightarrow \alpha\beta(1 - \gamma) + \alpha(1 - \beta)\gamma + (1 - \alpha)\beta\gamma + \alpha\beta\gamma = \frac{1}{2}$

$\Rightarrow 2\alpha\beta\gamma = \alpha\beta + \beta\gamma + \gamma\alpha - \frac{1}{2}$ ...(ii)

and the probability of passing in exactly two subjects = 0.40

$\Rightarrow P(M\ P\ \overline{C}) + P(M\ \overline{P}\ C) + P(\overline{M}\ P\ C) = \frac{2}{5}$

$\Rightarrow P(M)\ P(P)\ P(\overline{C}) + P(M)P(\overline{P})P(C) + P(\overline{M})P(P)P(C) = \frac{2}{5}$

$\Rightarrow \alpha\beta(1 - \gamma) + \alpha(1 - \beta)\gamma + (1 - \alpha)\beta\gamma = \frac{2}{5}$

$\Rightarrow \quad \frac{1}{2}-\alpha\beta\gamma=\frac{2}{5}$ ...(iii)

$\therefore \quad \alpha\beta\gamma=\frac{1}{2}-\frac{2}{5}=\frac{1}{10}\ \alpha$

From Eq. (ii), $\frac{1}{5}=\alpha\beta+\beta\gamma+\gamma\alpha-\frac{1}{2}$

$\therefore \quad \alpha\beta+\beta\gamma+\gamma\alpha=\frac{7}{10}$

Substituting the value of $\alpha\beta\gamma$ and $\alpha\beta + \beta\gamma + \gamma\alpha$ in Eq. (i), then

$\therefore \quad \alpha+\beta+\gamma-\frac{7}{10}+\frac{1}{10}=\frac{3}{4}$

$\Rightarrow \quad \alpha+\beta+\gamma=\frac{6}{10}+\frac{3}{4}=\frac{12+15}{20}=\frac{27}{20}$

36. **(a, d)** : (a) $q \Rightarrow p \equiv \sim q \vee p$

and $\sim p \Rightarrow \sim q \equiv \sim(\sim p) \vee (\sim q) = p \vee \sim q = \sim q \vee p$

Thus, $q \Rightarrow p \equiv \sim p \Rightarrow \sim q$

(b) $\sim(p \Leftrightarrow q) \equiv (p \wedge \sim q) \vee (q \wedge \sim p)$

$\sim(p \Rightarrow q) \wedge \sim(q \Rightarrow p) \equiv \sim(\sim p \vee q) \wedge \sim(\sim q \vee p)$

$= (p \wedge \sim q) \wedge (q \wedge \sim p)$

So, (b) is not true.

(c) $\sim(p \Rightarrow \sim q) \equiv \sim(\sim p \vee (\sim q))$

$= p \wedge q$

So, (c) is not true.

(d) $\sim(\sim p \Rightarrow \sim q) \equiv \sim(\sim(\sim p) \vee \sim q)$

$= \sim p \wedge q$

Thus $\sim(\sim p \Rightarrow \sim q) \equiv \sim p \wedge q$

37. **(b, c, d)** : (a) $(p \Rightarrow q) \Leftrightarrow (\sim q \Rightarrow \sim p)$

$\equiv (p \Rightarrow q) \Leftrightarrow (q \vee \sim p)$

$\equiv (p \Rightarrow q) \Leftrightarrow (\sim p \vee q)$

$\equiv (p \Rightarrow q) \Leftrightarrow (p \Rightarrow q)$, which is a tautology.

(b) $\sim(\sim p) \Leftrightarrow p \equiv p \Leftrightarrow p$, which is a tautology.

(c) $p \vee (\sim p)$ is a tautology.

(d) $p \wedge (\sim p)$ is a contradiction.

38. **(a, b, c)** : We have, $P(X = x) \propto (x + 1)\left(\frac{1}{5}\right)^x$

$= k(x+1)\left(\frac{1}{5}\right)^x$

Since, $\sum_{x=0}^{\infty} P(X=x)=1$

$\Rightarrow k\left\{1+2\left(\frac{1}{5}\right)+3\left(\frac{1}{5}\right)^2+4\left(\frac{1}{5}\right)^3+\ldots\infty\right\}=1$

$\Rightarrow k\left\{\left(1-\frac{1}{5}\right)^{-2}\right\}=1 \quad \Rightarrow k\left(\frac{25}{16}\right)=1 \quad \Rightarrow k=\frac{16}{25}$

(a) : $P(X=0)=k(0+1)\left(\frac{1}{5}\right)^0=k=\frac{16}{25}$

(b) : $P(X \le 1) = 1 - P(X = 1)$

$=k+\frac{2k}{5}=\frac{7}{5}k=\frac{7}{5}\times\frac{16}{25}=\frac{112}{125}$

(c) : $P(X \ge 1) = 1 - P(X = 0)$

$= 1 - k = 1-\frac{16}{25}=\frac{9}{25}$

(d) : $E(X)=\sum_{x=0}^{\infty} xP(X=x)$

$=\sum_{x=0}^{\infty} x(x+1)\left(\frac{1}{5}\right)^x$

$E(X)=1\cdot 2\left(\frac{1}{5}\right)+2\cdot 3\left(\frac{1}{5}\right)^2+3\cdot 4\left(\frac{1}{5}\right)^3+\ldots\infty$ ...(i)

$\therefore \frac{1}{5}E(X)=1\cdot 2\left(\frac{1}{5}\right)^2+2\cdot 3\left(\frac{1}{5}\right)^3+\ldots\infty$ ...(ii)

$\frac{4}{5}E(X)=\frac{2}{5}+\frac{2}{5}\left\{\frac{2}{5}+\frac{3}{5^2}+\frac{4}{5^3}+\ldots\infty\right\}$

$=\frac{2}{5}+\frac{2}{5}\left\{\left(1+\frac{2}{5}+\frac{3}{5^2}+\frac{4}{5^3}+\ldots\infty\right)-1\right\}$

$=\frac{2}{5}\left(1+\frac{2}{5}+\frac{3}{5^2}+\frac{4}{5^3}+\ldots\infty\right) = \frac{2}{5}\left(1-\frac{1}{5}\right)^{-2} = \frac{2}{5}\times\frac{25}{16}=\frac{5}{8}$

$\therefore \quad E(X)=\frac{25}{32}$

39. **(b)** : **(A)–(r), (B)–(p), (C)–(q), (D)–(s)**

(A) $P(A \cup B) = P(A) + P(B) \Rightarrow P(B) = 0.7 - 0.4 = 0.3$

(B) $P(A \cup B) = P(A) + P(B) - P(A)\,P(B) \Rightarrow 0.7 = 0.4 + 0.6\,P(B)$

$\Rightarrow P(B) = 0.5$

(C) $A \subseteq B \Rightarrow A \cup B = B \Rightarrow P(B) = 0.7$

(D) $B \subseteq A \Rightarrow A \cup B = A \Rightarrow 0.7 = 0.4$, not possible.

40. **(a)** : **(A)–(q), (B)–(r), (C)–(s), (D)–(p)**

(A) The 3 numbers are *aaa* choosen in 6 ways. The 3 numbers are *aab* in 30 ways. The 3 numbers are *abc* in 20 ways

Probability $=\frac{20}{56}=\frac{5}{14}$

(B) $720 = 2^4 . 3^2 . 5$

The divisors of the form $4n + 2 = 2(2n + 1)$ are 2, 6, 10, 18, 30, 90

Probability $=\frac{6}{5\cdot 3\cdot 2}=\frac{1}{5}$

(C) SS stand for spade lost and spade drawn NS stand for nonspade lost and spade drawn

Probability = P(SS) + P(NS) = $\frac{1}{4}\cdot\frac{12}{51}+\frac{3}{4}\cdot\frac{13}{51}=\frac{1}{4}$

(D) $a < b < c$ are in A. P. Then a and c are both odd or both even

Probability $=\frac{2\binom{5}{2}}{\binom{10}{3}}=\frac{5\cdot 4\cdot 6}{10\cdot 9\cdot 8}=\frac{1}{6}.$

41. **(c)** : The sum of the observations is $n\bar{x}$.

The sum of new observations = $nx- - x_i + x_i'$

$\therefore$ Mean of the new observations $=\frac{n\bar{x}-x_i-x_i'}{n}$

42. **(a)** : As we known that variance is independent of change of origin, so the new variance will be same.

43. **(d)** : Let p_1 be the probability of being an answer correct from section 1, then $p_1 = 1/5$. Let p_2 be the probability of being an answer correct from section 2 ,then $p_2 = 1/15$. Hence, the required probability is $1/5 \times 1/15 = 1/75$.

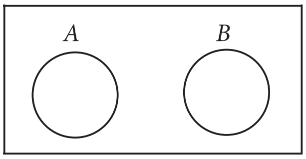
44. **(a) :** Scoring 10 marks from four questions can be done in $3 + 3 + 3 + 1 = 10$ ways, so as to answer three questions from section 2 and 1 questions from section 1 correctly. Hence, the required probability is

$$\frac{{}^{10}C_3 \times {}^{10}C_1}{{}^{20}C_4} = \frac{1}{5}\left(\frac{1}{15}\right)^3$$

45. **(b) :** To get 40 marks, he has to answer all questions correctly and its probability is $(1/5)^{10}\,(1/15)^{10}$. Hence, probability of getting a score less than 40 is

$$1-\left(\frac{1}{5}\right)^{10}\left(\frac{1}{15}\right)^{10} = 1-\left(\frac{1}{75}\right)^{10}$$

46. **(0) :** $P(A) = 0.45, P(B) = 0.35$ (events are mutually exclusive)



Hence, $P(A \cap B) = 0$

47. **(1) :** Since A, B are mutually exclusive events

$\therefore \quad P(A \cap B) = 0$

Since, $A \cup B = S \;\; \therefore \;\; P(A \cup B) = P(S) = 1$

And $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\Rightarrow \quad 1 = P(A) + 2P(A)$

$\Rightarrow \; 3P(A) = 1$

48. **(9) :** Total number of possible outcomes = 36

Let E be the event of getting a total of 5.

So, $E = \{(1, 4), (2, 3), (3, 2), (4, 1)\}$

$\therefore \quad n(E) = 4$

So, required probability, $P(E) = \frac{4}{36} = \frac{1}{9}$

49. **(1) :** In the 1st case,

$P(\text{they match}) = \frac{{}^3C_2 + {}^nC_2}{{}^{n+3}C_2} = \frac{1}{2}; \frac{6+n(n-1)}{(n+3)(n+2)} = \frac{1}{2}$

$\Rightarrow \quad 2(n^2 - n + 6) = n^2 + 5n + 6$

$\Rightarrow \quad n^2 - 7n + 6 = 0$

$\Rightarrow \quad n = 1$ or 6 ...(i)

In the 2nd case,

$$\frac{3}{n+3}\times\frac{3}{n+3}+\frac{n}{n+3}\times\frac{n}{n+3}=\frac{5}{8}$$

Solving, $n^2 - 10n + 9 = 0$

$n = 9$ or 1 ...(ii)

Form eqns. (i) and (ii), $n = 1$

50. **(8) :** Variance, $\sigma^2 = \frac{1}{n}\cdot\sum_{i=1}^{5} x_i^2 - \left(\frac{1}{n}\sum_{i=1}^{5} x_i\right)^2$

x_i	x_i^2
2	4
4	16
6	36
8	64
10	100
$\Sigma x_i = 30$	$\Sigma x_i^2 = 220$

$\therefore \quad \sigma^2 = \frac{1}{5}(220) - \left(\frac{1}{5}\times 30\right)^2 = 44 - 36 = 8$

JEE MATHEMATICS

Class XII

DPS-12

DAILY PRACTICE SHEET

Relations and Functions

SOLUTIONS

1. **(d) :** $G = \{[(a, b), (c, d)] \mid b - a = d - c\}$

For reflexive: Let $(x, y) \in R^2 \Rightarrow y - x = y - x$

$\therefore$ $[(x, y), (x, y)] \in G \ \forall \ (x, y) \in R^2$

$\therefore$ G is reflexive.

For symmetric: Let $[(a, b), (c, d)] \in G$

$\Rightarrow$ $b - a = d - c \Rightarrow d - c = b - a$

$\Rightarrow$ $[(c, d), (a, b)] \in G$

$\therefore$ G is symmetric.

For transitive: Let $[(a, b), (c, d)] \in G$ ... (i)

and $[(c, d), (x, y)] \in G$ ... (ii)

$\Rightarrow$ $b - a = d - c$ (from eq. (i))

and $d - c = y - x$ (from eq. (ii))

$\Rightarrow$ $b - a = y - x \Rightarrow [(a, b), (x, y)] \in G \ \therefore G$ is transitive.

Hence, G is an equivalence relation.

2. **(b) :** As both R_1 and R_2 are equivalence relations on A, therefore, $(x, x) \in R_1$ and also $(x, x) \in R_2$ for all $x \in A$

$\Rightarrow$ $(x, x) \in R_1 \cap R_2$ for all $x \in A$

$\Rightarrow$ $R_1 \cap R_2$ is reflexive.

Let $(x, y) \in R_1 \cap R_2$

$\Rightarrow$ $(x, y) \in R_1$ and $(x, y) \in R_2$

$\Rightarrow$ $(y, x) \in R_1$ and $(y, x) \in R_2$

$\Rightarrow$ $(y, x) \in R_1 \cap R_2$

$\therefore$ $R_1 \cap R_2$ is symmetric.

Let $(x, y) \in R_1 \cap R_2$ and $(y, z) \in R_1 \cap R_2$

$\Rightarrow$ $(x, y) \in R_1, R_2$ and $(y, z) \in R_1, R_2$

$\Rightarrow$ $(x, z) \in R_1, R_2$

$\Rightarrow$ $(x, z) \in R_1 \cap R_2$

$\Rightarrow$ $R_1 \cap R_2$ is transitive.

3. **(a) :** Since $x - x + \sqrt{2} = \sqrt{2}$ is irrational for all $x \in R$, therefore, $x\,R\,x$

$\Rightarrow$ R is reflexive.

However, R is not symmetric. Observe that $\sqrt{2}\ R\ 1$

($\because$ $\sqrt{2} - 1 + \sqrt{2} = 2\sqrt{2} - 1$ is irrational)

but $(1, \sqrt{2}) \notin R$ ($\because$ $1 - \sqrt{2} + \sqrt{2} = 1$, which is not irrational)

Again R is not transitive as $1\ R\sqrt{3}$ and $\sqrt{3}R\sqrt{2}$ but $(1, \sqrt{2}) \notin R$

4. **(a) :** We have, relation aRb is defined iff $|a - b| \le 1$

$|a - a| = 0 < 1 \Rightarrow aRa \Rightarrow R$ is reflexive.

5. **(d) :** Let each line $l \in$ set of the lines (L)

(i) Consider $l \,||\, l \Rightarrow (l, l) \in R \ \forall \ l \in L$

$\Rightarrow$ R is reflexive.

(ii) Let $l_1, l_2 \in L$ such that $(l_1, l_2) \in R$ then

$\Rightarrow$ $l_1 \,||\, l_2 \Rightarrow l_2 \,||\, l_1 \Rightarrow (l_2, l_1) \in R$

$\therefore$ R is symmetric.

(iii) Again $l_1, l_2, l_3 \in L$ such that $(l_1, l_2) \in R$ and $(l_2, l_3) \in R \Rightarrow l_1 \,||\, l_2$ and $l_2 \,||\, l_3$

$\Rightarrow$ $l_1 \,||\, l_2 \,||\, l_3$

$\Rightarrow$ $l_1 \,||\, l_3 \Rightarrow (l_1, l_3) \in R$

Hence, $(l_1, l_2) \in R$ and $(l_2, l_3) \in R$

$\Rightarrow$ $(l_1, l_3) \in R \Rightarrow R$ is transitive.

Since a relation R which is reflexive, symmetric and transitive is known as equivalence relation.

$\therefore$ Given relation is equivalence relation.

6. **(d) :** (i) Relation R be reflexive if it contains (3,3), (7, 7), (8, 8)

$\because$ $(7,8) \in R$, $(8,3) \in R$ (given)

(ii) R is symmetric if $(8,7), (3, 8) \in R$

Now $R = \{(3, 3), (7, 7), (8, 8), (8, 7), (3, 8), (7, 8), (8, 3)\}$

(iii) Relation R will be transitive if

$(3, 7)\ (7, 3) \in R \Rightarrow (3, 3) \in R$

$\therefore$ Relation R will be equivalence if (3, 3), (7, 7) (8, 8), (8, 7), (3, 8), (7, 3), (3, 7) be added.

$\therefore$ Total number of pairs to be added are 7.

7. **(a) :** (i) Reflexive : $a \in R$

$aR_1a \Rightarrow |a| = |a|$

(ii) Symmetric : $a, b \in R$

$aR_1b \Rightarrow |a| = |b| \Rightarrow |b| = |a| \Rightarrow bR_1a$

(iii) Transitive : $a, b, c \in R$

$$\left.\begin{aligned} aR_1b &\Rightarrow |a| = |b| \\ bR_1c &\Rightarrow |b| = |c| \end{aligned}\right\} \Rightarrow |a| = |c|$$

Hence, if $|a| = |c| \Rightarrow aR_1c$

$\Rightarrow$ R_1 is an equivalence relation on R.

8. **(a) :** $y - x$ = integer and $z - y$ = integer

$\Rightarrow$ $z - x$ = integer

$\therefore$ $(x, y) \in A$ and $(y, z) \in A$

$\Rightarrow$ $(x, z) \in A \Rightarrow$ Transitive

Also $(x, x) \in A$ is true $\Rightarrow$ Reflexive

As $(x, y) \in A \Rightarrow (y, x) \in A \Rightarrow$ Symmetric

Hence A is a equivalance relation.

Also, $(0, y)$ is in B but $(y, 0)$ is not in B.

$\therefore$ B is not symmetric

Hence, B is not an equivalence relation.

9. **(c) :** $f(x) + 2f(1 - x) = x^2 + 5$ ...(i)

Now, replace $x \to 1 - x$

$\therefore$ $f(1 - x) + 2f(x) = (1 - x)^2 + 5$ ...(ii)

$2 \times$ (ii) – (i), we get

$2f(1 - x) + 4f(x) - f(x) - 2f(1 - x)$

$= 2(1 - x)^2 + (5 \times 2) - x^2 - 5$

$\Rightarrow$ $3f(x) = 2(x^2 - 2x + 1) - x^2 + 5 = x^2 - 4x + 7$

$\therefore$ $f(x) = \dfrac{(x-2)^2 + 3}{3}$

Thus, $f(x)$ is neither one-one nor onto.

10. **(d) :** Number of onto functions from A to B if $n(A) = m$, $n(B) = n$ and $1 \le n \le m$ is equal to

$$\sum_{r=1}^{n} (-1)^{n-r}\ {}^{n}C_r r^m$$

Here $n = 3$, $m = 6$

$\therefore$ Number of onto functions

$$= \sum_{r=1}^{3} (-1)^{3-r}\ {}^{3}C_r\ r^6$$

$= (-1)^2\, {}^3C_1\, 1^6 + (-1)^1\, {}^3C_2\, 2^6 + (-1)^0\, {}^3C_3\, 3^6$

$= 3^6 - 3 \times 2^6 + 3 = 3(3^5 - 2^6 + 1) = 540$

11. **(d) :** In options (a), (b) and (c), we get same value of $f(x)$ for the different values of x. So, $f(x)$ is not one-to-one.

For option (d), $f(x) = \cos x$, $x \in [\pi, 2\pi)$

For each value of $x \in [\pi, 2\pi)$ we have $-1 \le \cos x < 1$

Therefore for different values of x, we get different value of $f(x)$.

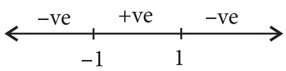
$\therefore$ $f(x)$ is one to one.

12. **(b) :** We have $f(x) = \dfrac{x}{1+x^2}$

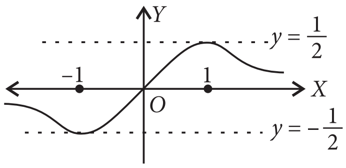
$$\therefore\quad f'(x) = \frac{(1+x^2)\cdot 1 - x\cdot 2x}{(1+x^2)^2}$$

$$= \frac{1-x^2}{(1+x^2)^2} = \frac{(1-x)(1+x)}{(1+x^2)^2}$$

The sign of $f'(x)$ is given as



Now f can be graphed as under



Clearly function is surjective but not injective, as a horizontal line meet the curve in two points.

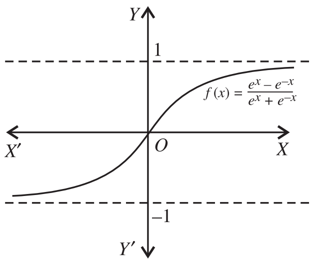
13. **(a) :** Given $f(x) = \begin{cases} 0, \text{if } x \text{ is rational} \\ x, \text{if } x \text{ is irrational} \end{cases}$

$$g(x) = \begin{cases} x, \text{if } x \text{ is rational} \\ 0, \text{if } x \text{ is irrational} \end{cases} \Rightarrow (f-g)(x) = \begin{cases} -x, \text{if } x \text{ is rational} \\ x, \text{if } x \text{ is irrational} \end{cases}$$

$\therefore$ $f - g$ is one-one and onto.

14. **(b) :** Given function is

$$f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$



To check one-one function

$$f'(x) = \frac{4}{(e^x + e^{-x})^2} > 0$$

for all $x \in R$

$\therefore$ $f(x)$ is strictly increasing function

$\therefore$ $f(x)$ is one-one (injection) function only

15. **(b) :** $\because -1 \le x \le 1 \Rightarrow 0 \le x^2 \le 1 \Rightarrow 1 \le x^2 + 1 \le 2$

$\Rightarrow$ $1 \le f(x) \le 2 \Rightarrow R_f = [1, 2] = \text{Codomain}(B)$

$\therefore$ f is surjective.

16. **(a) :** If $x = a$, where 'a' is an integer then

$$f(a) = 2a + a + \frac{1}{2}\sin 2a$$

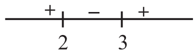
$$\text{But } f(a-h) = 2a + (a-1) + \frac{1}{2}\sin 2a$$

Thus, values between $\lim\limits_{h\to 0} f(a-h)$ and $f(a)$ are never achieved.

Also, $f'(x) = 2 + \cos 2x > 0$, *i.e.*, $f(x)$ is strictly increasing.

17. **(b) :** $f(x) = 2x^3 - 15x^2 + 36x + 1$

We have $f'(x) = 6x^2 - 30x + 36 = 6(x-2)(x-3)$



$f(0) = 1$

$f(3) = 54 - 135 + 108 + 1 = 28$

$f(2) = 16 - 60 + 72 + 1 = 29$

Then the range $= [1, 29]$

Hence, the given function is onto. But it is not one-one, as f' takes both positive and negative values.

18. **(c) :** Let $f(x) = y = (x+2)^2 - 2$

$\Rightarrow$ $(x+2)^2 = y + 2 \Rightarrow x = \pm\sqrt{y+2} - 2$

$\Rightarrow$ $x = \sqrt{y+2} - 2$ $\quad [\because x \ge -2]$

$\Rightarrow$ $f^{-1}(y) = \sqrt{y+2} - 2 \Rightarrow f^{-1}(x) = \sqrt{x+2} - 2$

19. **(b) :** $f(x) = \begin{cases} -1, \text{ when } x \text{ is rational} \\ 1, \text{ when } x \text{ is irrational} \end{cases}$

$\therefore$ $fof(1-\sqrt{3}) = f(f(1-\sqrt{3})) = f(1) = -1$

$(\because 1-\sqrt{3}$ is an irrational number$)$

20. **(d) :** Let $f(a) = f(b) \Rightarrow 2^{a(a-1)} = 2^{b(b-1)}$

$\Rightarrow$ $a(a-1) = b(b-1) \Rightarrow a^2 - b^2 - a + b = 0$

$\Rightarrow$ $(a-b)(a+b-1) = 0$

$a + b = 1$ is impossible for the domain $[1, \infty)$, since the minimum possible value is 2 for $a + b$.

Thus, $a = b$ is implied. So, f is one-one.

Let $y \in [1, \infty)$ (co-domain). Then $f(x) = y \Rightarrow y = 2^{x(x-1)}$

$\Rightarrow$ $\log_2 y = x(x-1) \Rightarrow x^2 - x - \log_2 y = 0$

$$\Rightarrow\ x = \frac{1 \pm \sqrt{1 + 4\log_2 y}}{2}$$

Thus every element of $y \in [1, \infty)$ (co-domain) has a pre-image $\dfrac{1 \pm \sqrt{1+4\log_2 y}}{2} \in [1, \infty)$ (domain). Thus, f is invertible.

Now $y = f(x) = 2^{x(x-1)}$ and $x = \dfrac{1}{2}(1 \pm \sqrt{1+4\log_2 y})$

Negative sign in above will be rejected because if $x < 1/2$, then $x \notin$ domain.

$\therefore$ $f^{-1}(y) = \dfrac{1}{2}(1 + \sqrt{1+4\log_2 y}) \Rightarrow f^{-1}(x) = \dfrac{1}{2}(1 + \sqrt{1+4\log_2 x})$.

21. **(b) :** The solution of $f(x) = f^{-1}(x)$ are given by $f(x) = x$, which gives $(x+1)^2 - 1 = x$

$\Rightarrow$ $(x+1)^2 - (x+1) = 0 \Rightarrow (x+1)x = 0$

$\therefore$ $x = -1, 0$

But as no co-domain of f is specified, nothing can be said about f being onto or not.

22. **(d) :** All functions are one-one and onto in their domain. Hence all are invertible.

23. **(b) :** Since, $g(x) = \dfrac{3x + x^3}{1 + 3x^2} = y$, (say) $\quad$ ... (i)

$$\therefore\quad f[g(x)] = f(y) = \log\left(\frac{1+y}{1-y}\right)$$

$$\Rightarrow\quad f(g(x)) = \log\left\{\frac{1+\dfrac{3x+x^3}{1+3x^2}}{1-\dfrac{3x+x^3}{1+3x^2}}\right\}$$

$$\Rightarrow\quad f(g(x)) = \log\left(\frac{1+x}{1-x}\right)^3$$

$$\therefore\quad f(g(x)) = 3\log\left(\frac{1+x}{1-x}\right) = 3f(x)$$

24. **(b) :** Since $g(x) = 3 + 4x$

$\therefore\quad g^2(x) = (gog)\,(x) = g\{g(x)\} = g(3 + 4x)$

$= 3 + 4\,(3 + 4x)$

or $\quad g^2(x) = 15 + 16x = (16 - 1) + 16x$

Now $g^3(x) = (gogog)x = g\{g^2(x)\}$

$= g(15 + 16x) = 3 + 4\,(15 + 16x)$

$= 63 + 64x = (64 - 1) + 64x$

Similarly, we get $g^n(x) = (4^n - 1) + 4^n x$

25. **(b) :** $1 \le 3x^2 + 1 \le 6 \Rightarrow x^2 \le \frac{5}{3} \Rightarrow x \in \left[-\sqrt{\frac{5}{3}}, \sqrt{\frac{5}{3}}\right]$

26. **(d) :** * is defined by $a * b = a + b + 7$

Now, $a * e = a \Rightarrow a + e + 7 = a \Rightarrow e + 7 = 0 \Rightarrow e = -7$

$\therefore\quad -7$ is the identity element. Identity element has its own inverse.

$\therefore\quad (-7)^{-1} = -7$

27. **(d) :** Set is Q_{-1} *i.e.*, $Q - \{-1\}$ and * defined by

$a * b = a + b + ab$

$\therefore\quad 4 * x = 4 + x + 4x = 3 \Rightarrow 4 + 5x = 3 \Rightarrow x = -\frac{1}{5}.$

28. **(a) :** $a * b = a^b$ for all $a, b \in N$, then the statement * is commutative in N is false.

For example, let $a = 2$, $b = 3$

$a * b = a^b = 2^3 = 8 \Rightarrow b * a = b^a = 3^2 = 9$

$\therefore\quad a * b \ne b * a$

29. **(c) :** We know very well, that number of binary operations on set S having n elements is n^{n^2}.

30. **(d) :** If the identity element is 'e', then $m * e = m = e * m$

$$\Rightarrow\quad \frac{me}{p} = m = \frac{em}{p} \Rightarrow e = p.$$

31. **(a, b) :** $f(f(x)) = x \Rightarrow f^{-1}(x) = f(x) \Rightarrow f(x) = x$

$\therefore\quad f(x)$ is one-one and onto.

32. **(a,c,d) :** $(f + 2g)(-1) = f(-1) + 2g(-1)$

$= [-1] + 2|-1| = -1 + 2 = 1$

$(f + 2g)(1) = f(1) + 2g(1) = (1) + 2(1) = 3.$

$$(gof - fog)\left(\frac{5}{3}\right) = g\left(f\left(\frac{5}{3}\right)\right) - f\left(g\left(\frac{5}{3}\right)\right)$$

$$= g\left(\left[\frac{5}{3}\right]\right) - f\left(\left|\frac{5}{3}\right|\right) = g(1) - f\left(\frac{5}{3}\right)$$

$$= |1| - \left[\frac{5}{3}\right] = 1 - 1 = 0$$

$$(gof) - (fog)\left(-\frac{5}{3}\right) = g\left(f\left(-\frac{5}{3}\right)\right) - f\left(g\left(-\frac{5}{3}\right)\right)$$

$$= g\left(\left[-\frac{5}{3}\right]\right) - f\left(\left|-\frac{5}{3}\right|\right) = g(-2) - f\left(\frac{5}{3}\right)$$

$$= |-2| - \left[\frac{5}{3}\right] = 2 - 1 = 1$$

33. **(b, c, d) :** To be an equivalence relation the relation must be reflexive, symmetric and transitive.

$T = \{(x, y) : x - y \in Z\}$

Reflexive : For $(x, x) \in T$ *i.e.* $x - x = 0 \in Z$

Symmetric : For $(x, y) \in T \Rightarrow x - y \in Z$

$\Rightarrow\quad y - x \in Z$ *i.e.* $(y, x) \in T$

Transitive : For $(x, y) \in T$ and $(y, w) \in T$

$\Rightarrow\quad x - y \in Z$ and $y - w \in Z$, giving

$x - w \in Z$ *i.e.* $(x, w) \in T$.

$\therefore\quad T$ is an equivalence relation on R.

$S = \{(x, y) : y = x + 1, 0 < x < 2\}$ is not reflexive for

$(x, x) \in S$ would imply $x = x + 1$

$\Rightarrow\quad 1 = 2$ (impossible)

Thus S is not an equivalence relation.

34. **(a, c) :** Every linear function is either strictly increasing or strictly decreasing.

If $f(x) = ax + b$, $D_f = [p, q]$, $R_f = [m, n]$

then $f(p) = m$, $f(q) = n$

if $f(x)$ is strictly increasing and $f(p) = n$,

$f(q) = m$ if $f(x)$ is strictly decreasing then

Let $f(x) = ax + b$ be the linear function which maps $[-1, 1]$ onto $[0, 2]$

$\therefore f(-1) = 0$ and $f(1) = 2$

or $f(-1) = 2$ and $f(1) = 0$

Depending upon $f(x)$ is increasing or decreasing.

$\Rightarrow -a + b = 0$ and $a + b = 2$ ... (i)

or $-a + b = 2$ and $a + b = 0$ ... (ii)

Solving (i) we get $a = 1$, $b = 1$, and solving (ii) we get $a = -1$, $b = 1$.

Thus there are only two functions either $x + 1$ or $-x + 1$.

35. **(a,b,c,d) :** $f(g(x)) = \dfrac{1}{(x-1)^2} + 1,\ g(f(x)) = \dfrac{1}{x^2}$

36. **(a,b,c) :** $f(x) = \dfrac{ax+b}{cx+d} \Rightarrow f(f(x)) = \dfrac{af(x)+b}{cf(x)+d} = \dfrac{a\left(\dfrac{ax+b}{cx+d}\right)+b}{c\left(\dfrac{ax+b}{cx+d}\right)+d}$

$$= \frac{(a^2+bc)x + b(a+d)}{c(a+d)x + bc + d^2}$$

$= x$, if $a + d = 0$, $a^2 + bc \ne 0$ and $a^2 = d^2$

37. **(a, c, d) :** As $(x, x) \in R$ for all $x \in A$. So, R is reflexive.

Also, R is anti-symmetric and transitive.

38. **(b, c) : Reflexivity :** For any $a \in N$, we have $a|a \Rightarrow aRa$

Thus, $aRa\ \forall\ a \in N$. So, R is reflexive.

Symmetry : R is not symmetric because if $a|b$, then a is divisible by b, but it may not be true that b is divisible by a.

For example, $6|2 \Rightarrow 6$ is divisible by 2.

But 2 is not divisible by 6.

Transitivity : Let $a, b, c \in N$ such that aRb and bRc.

$\Rightarrow\quad a|b$ and $b|c \Rightarrow a|c \Rightarrow aRc$

So, R is transitive relation on N.

39. (a) : (A)-(s), (B)-(r), (C)-(r), (D)-(p)

(A) $(gof)\left(\frac{1}{2}\right)=g\left[f\left(\frac{1}{2}\right)\right]=g\left(\frac{5}{4}\right)=2(1)-1=1$

(B) $(fog)\left(\frac{3}{2}\right)=f\left[g\left(\frac{3}{2}\right)\right]=f(1)=1^2+1=2$

(C) $(fogof)\left(\frac{3}{4}\right)=(fog)\left(f\left(\frac{3}{4}\right)\right)=(fog)\left(\frac{25}{16}\right)=f\left[g\left(\frac{25}{16}\right)\right]$

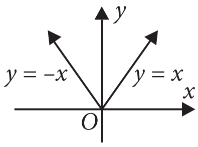
$= f(1) = 1^2 + 1 = 2$

(D) $(gofog)\left(\frac{2}{3}\right)=(gof)\left[g\left(\frac{2}{3}\right)\right]=(gof)(-1)=g[f(-1)]$

$= g(2) = 2 \times 2 - 1 = 3$

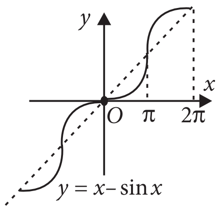
40. (a) : (A)-(p, s), (B)-(q, r), (C)-(r, s), (D)-(r, s)

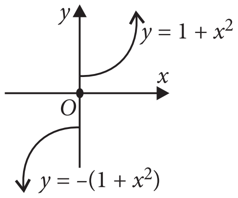
(A) $f(x)=\begin{cases} x \ , & x>0 \\ -x \ , & x<0 \\ 0 \ , & x=0 \end{cases} = \begin{cases} |x|; & x\neq 0 \\ 0; & x=0 \end{cases}$



(B) $f(x) = x - \sin x$.

(C) $f(x)=\begin{cases} 1+x^2 \ , & x>0 \\ 0 \ , & x=0 \\ -(1+x^2) \ , & x<0 \end{cases}$





(D) Its a cubic but on set of integers, hence one one & into.

41. (d) : As $f(296) = f(962)$, f is many-one.
Further the numbers whose digits increase from left to right (for example 269) have no pre-image.
Hence f is into.

42. (b) : It is easy to see that the remainder, when a positive integer is divided by 9, is the same as the sum of the digits of the number (until the sum becomes a one digit number). Thus $f(n)$ and n leave the same remainder, when divided by 9. Hence 9 divides $f(n) - n$. Further there is no reason to expect that the number is divisible by 27. The number $f(n) - n$ is not divisible by 18 also, in case $f(n) - n$ is odd. Hence 9 is the biggest number.
By the definition of f, digits of $f(x)$ are non-increasing from left to right.

43. (c) : Here, $A_1 \times A_1$
$= \{(x, y) : x, y \in A_1\}$
$= \{(1, 1), (1, 3), (1, 4), (3, 1), (3, 3), (3, 4), (4, 1), (4, 3), (4, 4)\}$

This relation is symmetric and transitive on A_1 and hence on A also. Note that it is not an equivalence relation on A.

44. (b) : As $A_1 \cup A_2 \cup A_3 = A$ and
$A_1 \cap A_2 = A_2 \cap A_3 = A_1 \cap A_3 = \phi$, therefore
$\bigcup_{i=1}^{3}(A_i \times A_i)$ defines an equivalence relation on A, where $x, y \in A$ are related iff they are in the same subset.

45. (d) : Number of symmetric relations that can be defined on a set containing n elements is $2^{n(n+1)/2}$. Hence in this case, when $n = 7$, the required number of symmetric relations $= 2^{7(7+1)/2} = 2^{28}$.

46. (3) : From the given equation, we have

$$\left(\frac{n}{2}-\left[\frac{n}{2}\right]\right)+\left(\frac{n}{3}-\left[\frac{n}{3}\right]\right)+\left(\frac{n}{5}-\left[\frac{n}{5}\right]\right)=0$$

$\Rightarrow \left\{\frac{n}{2}\right\}+\left\{\frac{n}{3}\right\}+\left\{\frac{n}{5}\right\}=0$; where $\{.\}$ is a fractional part

But each of the fraction part function is positive and their sum is zero. Hence, each of the fraction part function is zero. Consequently, each of $\frac{n}{2},\frac{n}{3},\frac{n}{5}$ is an integer. The L.C.M. of 2, 3, 5 is 30. Therefore we can take $n = 30k$, where k is an integer.
Hence the number of solutions such that $1 \le n \le 100$ is = 3 (n = 30, 60 and 90).

47. (2) : We have, $f\left(\frac{x+1}{2x-1}\right)=2x$
Put $x = 1$ on both sides, we get $f(2) = 2$

48. (1) : $f(f(x)) = (100^5 - (f(x))^{10})^{1/10}$
$= [100^5 - (100^5 - x^{10})]^{1/10} = x$

$f(f(1024)) = 1024 \Rightarrow \frac{1}{2^{10}}f(f(1024))=1$

49. (0) : $g(x)=||x-2|-2|=\begin{cases} |x|, & x\le 2 \\ 4-x, & 2<x\le 4 \\ x-4, & 4<x \end{cases}$

$g'(-1) + g'(1) + g'(3) + g'(5) = -1 + 1 - 1 + 1 = 0$

50. (2) : For $n = 2$, $1 + 2f(2) = 6f(2)$

$\Rightarrow \ f(2)=\frac{1}{4}$

For $n = 3$, $1+\frac{1}{2}+3f(3)=12f(3) \Rightarrow 9f(3)=\frac{3}{2}$

$\Rightarrow \ f(3)=\frac{1}{6}$ and so on. $\Rightarrow \ f(n)=\frac{1}{2n}$

$\therefore \ 48f(12)=48\times\frac{1}{24}=2$

JEE MATHEMATICS

DPS-13

DAILY PRACTICE SHEET

Class XII

Inverse Trigonometric Functions

SOLUTIONS

1. **(a) :** Let $f(x, y, z) = \sin^{-1}x + \sin^{-1}y + \cos^{-1}z$, it will attain the value 2π only if

$\sin^{-1}x = \sin^{-1}y = \frac{\pi}{2}$ and $\cos^{-1}z = \pi$.

It will be possible only if $x = y = 1$ and $z = -1$. Hence there is only $f(1, 1, -1) = 2\pi$

2. **(b) :** $\tan^{-1}(x + 1) + \tan^{-1}(x - 1) = \tan^{-1}\left(\frac{8}{31}\right)$

$$\frac{(x+1)+(x-1)}{1-(x+1)(x-1)} = \frac{8}{31}$$

$\Rightarrow 31x = 4(2 - x^2) \Rightarrow 4x^2 + 31x - 8 = 0$

Solving, $x = \frac{1}{4}, -8$; $x = -8$ does not satisfy the equation.

3. **(a) :** $x = \sin(2\tan^{-1} 2)$

$\Rightarrow x = \sin(2\theta)$ (Putting $\tan^{-1} 2 = \theta \Rightarrow \tan\theta = 2$)

$$\Rightarrow x = \frac{2\tan\theta}{1+\tan^2\theta} \Rightarrow x = \frac{2\cdot 2}{1+(2)^2} = \frac{4}{5}$$

and $y = \sin\left(\frac{1}{2}\tan^{-1}\frac{4}{3}\right)$

Let $\tan^{-1}\frac{4}{3} = \theta \Rightarrow \tan\theta = \frac{4}{3}$

$$\Rightarrow y = \sin\left(\frac{\theta}{2}\right) \Rightarrow y = \sqrt{\frac{1-\cos\theta}{2}} = \sqrt{\frac{1-\frac{3}{5}}{2}} = \frac{1}{\sqrt{5}}$$

Hence, $x > y$.

4. **(d) :** Here, $\frac{-\pi}{2} \le \cos^{-1}(\log_2(x^2+5x+8)) \le \frac{\pi}{2}$

$-1 \le \log_2(x^2 + 5x + 8) \le 1 \Rightarrow \frac{1}{2} \le (x^2+5x+8) \le 2.$

$$\Rightarrow x^2+5x+\frac{15}{2} \ge 0 \Rightarrow \left(x+\frac{5}{2}\right)^2 + \frac{5}{4} \ge 0$$

and $x^2 + 5x + 6 \le 0$

$\Rightarrow (x+3)(x+2) \le 0 \Rightarrow x \in [-3, -2]$.

5. **(c) :** $f(x) = \sin^{-1}(|x - 1| - 2)$

Let $y = \sin^{-1}(|x - 1| - 2) \Rightarrow \sin y = |x - 1| - 2$

Case I : When $|x - 1| = -(x - 1)$ $\therefore \sin y = -(x - 1) - 2 = -x - 1$

Now, $\sin y$ should be between -1 to 1

So, if $\sin y = -1$; $-1 = -x - 1 \Rightarrow x = 0$

If $\sin y = 1$; $1 = -x - 1 \Rightarrow x = -2$

So, domain = $[-2, 0]$

Case II : When $|x - 1| = x - 1$ $\therefore \sin y = (x - 1) - 2 = x - 3$

If $\sin y = -1$; $-1 = x - 3 \Rightarrow x = 2$

If $\sin y = 1$; $1 = x - 3 \Rightarrow x = 4$

Domain = $[2, 4]$

So, from Case I and Case II domain is $[-2, 0] \cup [2, 4]$

6. **(b) :** $f(x)$ is defined if $-1 \le \frac{x}{2} - 1 \le 1$ and $\cos x > 0$

or $0 \le x \le 4$ and $-\frac{\pi}{2} < x < \frac{\pi}{2}$ $\therefore$ $0 \le x < \frac{\pi}{2}$.

7. **(c) :** We have, $\frac{(x+1)^2}{x^3+x} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$

$\Rightarrow (x + 1)^2 = A(x^2 + 1) + (Bx + C)x$

$\Rightarrow x^2 + 2x + 1 = (A + B)x^2 + Cx + A$

Comparing coefficients of x^2, x and constant, we get

$A + B = 1$, $C = 2$, $A = 1$

Put $A = 1 \Rightarrow B = 0$

Now, $\text{cosec}^{-1}\left(\frac{1}{A}\right) + \cot^{-1}\left(\frac{1}{B}\right) + \sec^{-1} C$

$= \text{cosec}^{-1}(1) + \cot^{-1}(\infty) + \sec^{-1}(2) = \frac{\pi}{2} + 0 + \frac{\pi}{3} = \frac{5\pi}{6}$.

8. **(c) :** Let $\sin^{-1} x = \alpha$

where $-\frac{\pi}{2} < \alpha < 0$ since x is negative

$\Rightarrow x = \sin\alpha \Rightarrow x^2 = \sin^2\alpha$

$\Rightarrow \cos^2\alpha = 1 - x^2 = \cos\alpha = \pm\sqrt{1-x^2}$

Since α lies in fourth quadrant so $\cos\alpha$ is positive or $\cos\alpha = \sqrt{1-x^2}$

$\Rightarrow \cos(-\alpha) = \sqrt{1-x^2}$

($\because$ Principal value of $\cos^{-1}\alpha$ lies in $[0, \pi]$, so $-\alpha > 0$)

$\Rightarrow -\alpha = \cos^{-1}\sqrt{1-x^2} \Rightarrow \alpha = -\cos^{-1}\sqrt{1-x^2}$

$\Rightarrow \sin^{-1} x = -\cos^{-1}\sqrt{1-x^2}$

9. **(c) :** $\tan^2(\sec^{-1} 2) + \cot^2(\text{cosec}^{-1} 3)$

$= -1 + [\sec^2(\sec^{-1}(2))] + \text{cosec}^2(\text{cosec}^{-1} 3) - 1$

$= 4 - 1 + 9 - 1 = 11$

10. **(b) :** Since $\cot^{-1} x = \sin^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$

$$\Rightarrow \sin\left(\cot^{-1} x\right) = \frac{1}{\sqrt{1+x^2}}$$

11. **(c) :** $\sin^{-1}[\cos(\sin^{-1} x)] + \cos^{-1}[\sin(\cos^{-1} x)]$

$$= \sin^{-1}\left[\sin\left(\frac{\pi}{2} - \sin^{-1} x\right)\right] + \cos^{-1}\left[\cos\left(\frac{\pi}{2} - \cos^{-1} x\right)\right]$$

$$= \frac{\pi}{2} - \sin^{-1} x + \frac{\pi}{2} - \cos^{-1} x = \pi - (\sin^{-1} x + \cos^{-1} x) = \pi - \frac{\pi}{2} = \frac{\pi}{2}$$

12. **(b) :** Put $x = \tan\theta$

$$\therefore \sin\left[\tan^{-1}\frac{1-x^2}{2x} + \cos^{-1}\frac{1-x^2}{1+x^2}\right]$$

$$= \sin\left[\tan^{-1}\left(\frac{1-\tan^2\theta}{2\tan\theta}\right)+\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)\right]$$

$= \sin[\tan^{-1}(\cot 2\theta) + \cos^{-1}(\cos 2\theta)]$

$$= \sin\left[\tan^{-1}\left\{\tan\left(\frac{\pi}{2}-2\theta\right)\right\}+2\theta\right]$$

$$= \sin\left[\frac{\pi}{2}-2\theta+2\theta\right] = \sin\frac{\pi}{2} = 1$$

13. **(c) :** $\cot^{-1}(2\cdot 1^2) + \cot^{-1}(2\cdot 2^2) + \cot^{-1}(2\cdot 3^2) + \ldots$ upto ∞

$$= \sum_{n=1}^{\infty} \cot^{-1}\left[\frac{(2n+1)(2n-1)+1}{(2n+1)-(2n-1)}\right]$$

$$= \sum_{n=1}^{\infty}\{\cot^{-1}(2n-1)-\cot^{-1}(2n+1)\}$$

$= (\cot^{-1}1 - \cot^{-1}3) + (\cot^{-1}3 - \cot^{-1}5) + (\cot^{-1}5 - \cot^{-1}7) + \ldots$

$$= \cot^{-1}1-\cot^{-1}\infty = \frac{\pi}{4}-0=\frac{\pi}{4}.$$

14. **(b) :** $3\tan^{-1}\left(\frac{1}{2+\sqrt{3}}\right) = \tan^{-1}\frac{1}{x}+\tan^{-1}\frac{1}{3}$ ...(i)

Since $\frac{1}{2+\sqrt{3}} = \frac{2-\sqrt{3}}{(2+\sqrt{3})(2-\sqrt{3})} = \frac{2-\sqrt{3}}{4-3} = 2-\sqrt{3}$

$\therefore$ Equation (i) becomes $3\tan^{-1}(2-\sqrt{3}) = \tan^{-1}\frac{1}{x}+\tan^{-1}\frac{1}{3}$

$$\Rightarrow \tan^{-1}\left[\frac{3(2-\sqrt{3})-(2-\sqrt{3})^3}{1-3(2-\sqrt{3})^2}\right] = \tan^{-1}\left[\frac{\frac{1}{x}+\frac{1}{3}}{1-\frac{1}{x}\cdot\frac{1}{3}}\right]$$

$$\left[\text{For L.H.S., using } 3\tan^{-1}x = \tan^{-1}\frac{3x-x^3}{1-3x^2}\right]$$

$$\Rightarrow \tan^{-1}1 = \tan^{-1}\left(\frac{3+x}{3x-1}\right) \Rightarrow 1 = \frac{3+x}{3x-1}$$

$\Rightarrow 3 + x = 3x - 1 \Rightarrow 2x = 4 \Rightarrow x = 2$

15. **(a) :** $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi$

$\therefore \tan^{-1}y + \tan^{-1}z = \pi - \tan^{-1}x$

$$\Rightarrow \frac{y+z}{1-yz} = -x \Rightarrow x+y+z = xyz$$

16. **(a) :** $a\sin^{-1}x - b\cos^{-1}x = c$ ... (i)

Also, $\sin^{-1}x + \cos^{-1}x = \pi/2$ ... (ii)

Using (i) and (ii), we get $(-b - a)\cos^{-1}x = c - a\,\pi/2$

$$\Rightarrow \cos^{-1}x = \frac{(a\cdot\pi)/2-c}{a+b}$$

Also, $\sin^{-1}x = \frac{\pi}{2} - \frac{\frac{a\pi}{2}-c}{a+b}$ [From (ii)]

$\therefore a\sin^{-1}x + b\cos^{-1}x$

$$= \frac{a\pi}{2}-\frac{(a^2\pi)/2-ca}{a+b}+\frac{ab\cdot\pi/2-bc}{a+b}$$

$$= \frac{a(a+b)\pi-(a^2\pi-2ca)+ab\cdot\pi-2bc}{2(a+b)}$$

$$= \frac{\{2\pi ab+2c(a-b)\}}{2(a+b)} = \frac{\pi ab+c(a-b)}{a+b}$$

17. **(b) :** Given, $2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x)$

$$\Rightarrow \tan^{-1}\left(\frac{2\cos x}{1-\cos^2 x}\right) = \tan^{-1}(2\operatorname{cosec} x)$$

$$\Rightarrow \frac{2\cos x}{1-\cos^2 x} = 2\operatorname{cosec} x \Rightarrow \frac{2\cos x}{\sin^2 x} = 2\operatorname{cosec} x$$

$$\Rightarrow \sin x = \cos x \Rightarrow x = \frac{\pi}{4}.$$

18. **(d) :** Putting $p = \tan\theta$, $q = \tan\phi$, $x = \tan\psi$, we get

$$\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)-\cos^{-1}\left(\frac{1-\tan^2\phi}{1+\tan^2\phi}\right) = \tan^{-1}\left(\frac{2\tan\psi}{1-\tan^2\psi}\right)$$

$\Rightarrow \sin^{-1}(\sin 2\theta) - \cos^{-1}(\cos 2\phi) = \tan^{-1}(\tan 2\psi)$

$\Rightarrow \theta - \phi = \psi \Rightarrow \tan^{-1}p - \tan^{-1}q = \tan^{-1}x$

$$\Rightarrow \tan^{-1}\left(\frac{p-q}{1+pq}\right) = \tan^{-1}x \Rightarrow x = \frac{p-q}{1+pq}$$

19. **(a) :** $\sin\left[2\cos^{-1}\cot\left(2\tan^{-1}\frac{1}{2}\right)\right]$

$$= \sin\left[2\cos^{-1}\cot\left\{\tan^{-1}\left(\frac{2\left(\frac{1}{2}\right)}{1-\frac{1}{4}}\right)\right\}\right]$$

$$= \sin\left[2\cos^{-1}\cot\left\{\tan^{-1}\left(\frac{4}{3}\right)\right\}\right]$$

$$= \sin\left[2\cos^{-1}\cot\left(\cot^{-1}\left(\frac{3}{4}\right)\right)\right]$$

$$= \sin\left[2\cos^{-1}\left(\frac{3}{4}\right)\right] = \sin\left[\cos^{-1}\left(2\left(\frac{9}{16}\right)-1\right)\right]$$

$$= \sin\left[\cos^{-1}\left(\frac{1}{8}\right)\right] = \sin\left(\sin^{-1}\left(\sqrt{1-\frac{1}{64}}\right)\right) = \frac{3\sqrt{7}}{8}$$

20. **(d) :** $\sin^{-1}\left[\sin\left(\frac{2\pi}{3}\right)\right] = \sin^{-1}\left[\sin\left(\pi-\frac{\pi}{3}\right)\right]$

$$= \sin^{-1}\left[\sin\left(\frac{\pi}{3}\right)\right] = \frac{\pi}{3}.$$

21. **(a) :** Let, $y = \tan\left[\sin^{-1}\left\{\frac{x}{\sqrt{2}}+\frac{\sqrt{1-x^2}}{\sqrt{2}}\right\}-\sin^{-1}x\right]$

Put $x = \sin\theta$, we get

$$y = \tan\left[\sin^{-1}\left[\frac{\sin\theta}{\sqrt{2}}+\frac{\sqrt{1-\sin^2\theta}}{\sqrt{2}}\right]-\sin^{-1}(\sin\theta)\right]$$

$$= \tan\left[\sin^{-1}\left[\frac{\sin\theta+\cos\theta}{\sqrt{2}}\right]-\theta\right]$$

$$= \tan\left[\sin^{-1}\left\{\sin\theta\cos\frac{\pi}{4}+\cos\theta\sin\frac{\pi}{4}\right\}-\theta\right]$$

$= \tan\left[\sin^{-1}\left\{\sin\left(\theta+\frac{\pi}{4}\right)\right\}-\theta\right] = \tan\frac{\pi}{4} = 1$

22. (b) : $\cos\left[\cos^{-1}\left(\frac{1}{5}\right)+\cos^{-1}\left(\frac{1}{5}\right)+\sin^{-1}\left(\frac{1}{5}\right)\right]$

$= \cos\left[\cos^{-1}\left(\frac{1}{5}\right)+\frac{\pi}{2}\right]$

$= \cos\left(\frac{\pi}{2}+\cos^{-1}\frac{1}{5}\right) = -\sin\left(\cos^{-1}\frac{1}{5}\right)$

$= -\sin\left(\sin^{-1}\frac{\sqrt{24}}{5}\right) = -\frac{2\sqrt{6}}{5}$

23. (d) : $\sin^{-1}\left[\tan\left(-\frac{5\pi}{4}\right)\right] = \sin^{-1}\left(-\tan\left(\frac{5\pi}{4}\right)\right)$

($\because \tan(-\theta) = -\tan\theta$)

$= -\sin^{-1}\left(\tan\left(\pi+\frac{\pi}{4}\right)\right)$ ($\because \sin^{-1}(-x) = -\sin^{-1}x$)

$= -\sin^{-1}\left(\tan\frac{\pi}{4}\right) = -\sin^{-1}(1) = -\frac{\pi}{2}$

24. (c) : $\sin\left(\sin^{-1}\frac{2}{3}+2\cos^{-1}\frac{2}{3}\right)$

$= \sin\left(\sin^{-1}\frac{2}{3}+\cos^{-1}\frac{2}{3}+\cos^{-1}\frac{2}{3}\right)$

$= \sin\left(\frac{\pi}{2}+\cos^{-1}\frac{2}{3}\right) = \cos\left(\cos^{-1}\frac{2}{3}\right) = \frac{2}{3}.$

25. (b) : $\cot\left(\operatorname{cosec}^{-1}\frac{5}{3}+\tan^{-1}\frac{2}{3}\right) = \cot\left(\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{2}{3}\right)$

$= \cot\left(\tan^{-1}\frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{4}\cdot\frac{2}{3}}\right) = \cot\left(\tan^{-1}\frac{17}{6}\right) = \frac{6}{17}$

26. (c) : $\sqrt{1+x^2}\left[\{x\cos(\cot^{-1}x)+\sin(\cot^{-1}x)\}^2-1\right]^{1/2}$

$= \sqrt{1+x^2}\left[\left\{x\cos\left(\cos^{-1}\frac{x}{\sqrt{1+x^2}}\right)+\sin\left(\sin^{-1}\frac{1}{\sqrt{1+x^2}}\right)\right\}^2-1\right]^{1/2}$

$= \sqrt{1+x^2}\left[\left\{\frac{x^2}{\sqrt{1+x^2}}+\frac{1}{\sqrt{1+x^2}}\right\}^2-1\right]^{1/2}$

$= \sqrt{1+x^2}\left[\left\{\sqrt{1+x^2}\right\}^2-1\right]^{1/2} = \sqrt{1+x^2}\left(x^2\right)^{1/2}$

$= \sqrt{1+x^2}\cdot|x| = x\sqrt{1+x^2}$ (as $0 < x < 1$, $|x| = x$)

27. (d) : $\cos^{-1}\left(-\frac{1}{2}\right)-2\sin^{-1}\left(\frac{1}{2}\right)+3\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) - 4\tan^{-1}(-1)$

$= \pi-\cos^{-1}\left(\frac{1}{2}\right)-2\sin^{-1}\left(\frac{1}{2}\right)+3\left(\pi-\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)\right) + 4\tan^{-1}(1)$

$= \pi-\frac{\pi}{3}-\frac{\pi}{3}+3\left(\pi-\frac{\pi}{4}\right)+4\cdot\frac{\pi}{4} = \frac{\pi}{3}+3\cdot\frac{3\pi}{4}+\pi = \frac{43\pi}{12}$

28. (c) : $\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{7}{8}\right)$

$= \tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}}\right)+\tan^{-1}\frac{7}{8} = \tan^{-1}\left(\frac{5/6}{5/6}\right)+\tan^{-1}\frac{7}{8}$

$= \tan^{-1}(1)+\tan^{-1}\frac{7}{8} = \tan^{-1}\left(\frac{1+\frac{7}{8}}{1-\frac{7}{8}}\right) = \tan^{-1}(15)$

29. (c) : Given, $\sin^{-1}x+\sin^{-1}2x = \frac{\pi}{3}$

$\Rightarrow \sin^{-1}2x = \frac{\pi}{3}-\sin^{-1}x$

$\Rightarrow 2x = \sin\left(\frac{\pi}{3}-\sin^{-1}x\right)$

$\Rightarrow 2x = \sin\frac{\pi}{3}\cos\left(\sin^{-1}x\right)-\cos\frac{\pi}{3}\sin\left(\sin^{-1}x\right)$

$\Rightarrow 2x = \frac{\sqrt{3}}{2}\sqrt{1-x^2}-\frac{1}{2}x$

$\Rightarrow (5x)^2 = 3(1-x^2)$

$\Rightarrow 28x^2 = 3$

$\therefore x = \pm\frac{\sqrt{3}}{2\sqrt{7}} = \frac{1}{2}\sqrt{\frac{3}{7}}.$

30. (c) : $\because \tan^{-1}\left(\frac{2}{5}\right)+\tan^{-1}\left(\frac{3}{7}\right) = \tan^{-1}\left(\frac{\frac{2}{5}+\frac{3}{7}}{1-\frac{6}{35}}\right) = \tan^{-1}\left(\frac{29}{29}\right)$

$\Rightarrow \tan^{-1}\left(\frac{2}{5}\right)+\tan^{-1}\left(\frac{3}{7}\right) = \tan^{-1}(1) = \frac{\pi}{4}$

And for $x = 2$, $y = 5$, $\frac{y-x}{y+x} = \frac{5-2}{5+2} = \frac{3}{7}$

31. (b, c, d) : $\alpha = 3\sin^{-1}\frac{6}{11} > 3\sin^{-1}\frac{6}{12} = \frac{3\pi}{6} = \frac{\pi}{2}$

$\Rightarrow \alpha > \frac{\pi}{2}$

$\beta = 3\cos^{-1}\frac{4}{9} > 3\cos^{-1}\frac{4}{8}$ *i.e.,* $\beta > \pi$

Then, $\alpha+\beta > \frac{3\pi}{2}$

Also, $\alpha+\beta$ lies in fourth quadrant. Hence, $\cos(\alpha+\beta) > 0$

32. (a,c) : Since $\sin^{-1}x$ is defined for $|x| \le 1$ and $\sec^{-1}x$ is defined for $|x| \ge 1$, therefore, $D_f = \{x \in R : |x| = 1\} = \{-1,1\}$.

Further $f(-1) = \sin^{-1}(-1)+\sec^{-1}(-1)$

$= -\frac{\pi}{2}+\pi = \frac{\pi}{2}$

and $f(1) = \sin^{-1}(1)+\sec^{-1}(1)$

$= \frac{\pi}{2}+0 = \frac{\pi}{2}$

Hence, $R_f = \left(\frac{\pi}{2}\right)$.

33. (a,b,c) : Here, $\sin^2\left(2\tan^{-1}\sqrt{\frac{1+x}{1-x}}\right)$

$= \sin^2(2\theta)$, where $\theta = \tan^{-1}\sqrt{\frac{1+x}{1-x}}$

$$= \left(\frac{2\tan\theta}{1+\tan^2\theta}\right)^2 = \frac{4\tan^2\theta}{(1+\tan^2\theta)^2}, \text{ but } \tan^2\theta = \frac{1+x}{1-x}$$

$$= \frac{4\left(\frac{1+x}{1-x}\right)}{\left(1+\frac{1+x}{1-x}\right)^2} = 1-x^2$$

$$\Rightarrow \sin\left(2\tan^{-1}\sqrt{\frac{1+x}{1-x}}\right) = \sqrt{1-x^2}$$

$$(\because \theta = \tan^{-1}\sqrt{\frac{1+x}{1-x}},\ \therefore 0 \le \theta < \frac{\pi}{2} \Rightarrow 0 \le 2\theta < \pi \Rightarrow \sin 2\theta \not< 0)$$

Again, $\sin^2(\tan^{-1}\sqrt{\frac{1+x}{1-x}})$

$$= \sin^2\theta = \frac{1}{1+\cot^2\theta} = \frac{\tan^2\theta}{1+\tan^2\theta} = \frac{\frac{1+x}{1-x}}{1+\frac{1+x}{1-x}} = \frac{1+x}{2}.$$

Hence, the options (a), (b) and (c) are correct.

34. (a,c) : $\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{2}{9} = \tan^{-1}\left(\frac{\frac{1}{4}+\frac{2}{9}}{1-\frac{1}{4}\cdot\frac{2}{9}}\right)$

$$= \tan^{-1}\left(\frac{17}{36-2}\right) = \tan^{-1}\left(\frac{1}{2}\right)$$

$$= \frac{1}{2}\left\{2\tan^{-1}\frac{1}{2}\right\} = \frac{1}{2}\cos^{-1}\left(\frac{1-\frac{1}{4}}{1+\frac{1}{4}}\right) = \frac{1}{2}\cos^{-1}\left(\frac{3}{5}\right)$$

$$\left(\because 2\tan^{-1}x = \cos^{-1}\frac{1-x^2}{1+x^2} \text{ for } x \ge 0\right)$$

35. (b,d) : For $x > 0$, $\cot^{-1}\left(\frac{1}{x}\right) = \tan^{-1}x$ and also $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$ for all $x \in R$, therefore, when $x > 0$, then $\tan^{-1}x = \cot^{-1}\left(\frac{1}{x}\right)$ and also $\tan^{-1}x = \frac{\pi}{2} - \cot^{-1}x$.

36. (a, c) : $\cos^{-1}x = \tan^{-1}x \Rightarrow x = \cos\theta = \tan\theta$

$\Rightarrow \cos^2\theta = \sin\theta \Rightarrow \sin^2\theta + \sin\theta - 1 = 0$

$\Rightarrow \sin\theta = \frac{-1\pm\sqrt{1+4}}{2} \Rightarrow \sin\theta = \frac{\sqrt{5}-1}{2}$

$\Rightarrow x^2 = \cos^2\theta = \frac{\sqrt{5}-1}{2}$

so (a) is correct and

$\sin(\cos^{-1}x) = \sin(\theta) = \frac{\sqrt{5}-1}{2}$,

so (c) is correct and $\tan(\cos^{-1}x) = \tan\theta \ne \frac{(\sqrt{5}-1)}{2}$.

So (d) is not correct.

37. (a,b,c) : $\sum_{r=1}^{n}\tan^{-1}\left(\frac{1}{r^2+r+1}\right)$

$$= \sum_{r=1}^{n}\tan^{-1}\left(\frac{(r+1)-r}{1+(r+1)r}\right) = \sum_{r=1}^{n}\{\tan^{-1}(r+1) - \tan^{-1}r\}$$

$(\because (r+1)r > -1)$

$$= \tan^{-1}(2) - \tan^{-1}(1) + \tan^{-1}(3) - \tan^{-1}(2) + \ldots + \tan^{-1}(n+1) - \tan^{-1}n$$

$$= \tan^{-1}\left(\frac{n+1-1}{1+(n+1)}\right) = \tan^{-1}\left(\frac{n}{n+2}\right)$$

Since, $\tan^{-1}(n+1) - \tan^{-1}1 = \cot^{-1}\left(\frac{1}{n+1}\right) - \cot^{-1}1$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{1}{n+1}\right) - \left(\frac{\pi}{2} - \tan^{-1}(1)\right).$$

38. (a,b) : For D_f, $1 - 2x \ge 0$, $-1 \le \frac{3x-1}{2} \le 1$

$(\because \sin^{-1}x$ is defined for $-1 \le x \le 1)$

$\Rightarrow 2x \le 1$ and $-2 \le 3x - 1 \le 2$

$\Rightarrow x \le \frac{1}{2}$ and $-\frac{1}{3} \le x \le 1$

$\Rightarrow D_f = \left[-\frac{1}{3}, \frac{1}{2}\right]$. Hence only (a) and (b) are correct options as $\left[-\frac{1}{3}, 0\right] \subset \left[-\frac{1}{3}, \frac{1}{2}\right]$.

39. (a) : (A)-(q), (B)-(r), (C)-(p), (D)-(s)

(A) Let $\cot^{-1}x = \theta \Rightarrow \cot\theta = x \Rightarrow \sin\theta = \frac{1}{\sqrt{1+x^2}}$

Now, $\sin(\cot^{-1}x) = \sin\theta = \frac{1}{\sqrt{1+x^2}}$

(B) Let $\tan^{-1}\left(\frac{3}{4}\right) = \theta \Rightarrow \tan\theta = \frac{3}{4}$

$\therefore \cos\theta = \frac{4}{5}$

Now, $\cos\left(\tan^{-1}\left(\frac{3}{4}\right)\right) = \cos\theta = \frac{4}{5}$

(C) $\cos^{-1}\left(\frac{5}{13}\right) - \sin^{-1}\left(\frac{12}{13}\right) = \cos^{-1}(x)$

$$\sin^{-1}\left(\frac{12}{13}\right) = \cos^{-1}\left(\sqrt{1-\left(\frac{12}{13}\right)^2}\right)$$

$$= \cos^{-1}\left(\frac{\sqrt{169-144}}{169}\right) = \cos^{-1}\left(\frac{5}{13}\right)$$

$$\therefore \cos^{-1}\left(\frac{5}{13}\right) - \cos^{-1}\left(\frac{5}{13}\right) = \cos^{-1}(x)$$

$\Rightarrow 0 = \cos^{-1}x \Rightarrow x = \cos 0 = 1 \therefore x = 1$

(D) We have, $\cot(\sin^{-1}x) = \cos(\tan^{-1}\sqrt{3})$

$$\Rightarrow \cot(\sin^{-1}x) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$$

$$\Rightarrow \sin^{-1}x = \cot^{-1}\left(\frac{1}{2}\right)$$

$$\Rightarrow x = \sin\left[\cot^{-1}\left(\frac{1}{2}\right)\right] = \sin\left[\sin^{-1}\left(\frac{2}{\sqrt{5}}\right)\right]$$

$$\therefore x = \frac{2}{\sqrt{5}} \quad \left[\because \cot^{-1}\left(\frac{1}{2}\right) = \theta \Rightarrow \cot\theta = \frac{1}{2} \therefore \sin\theta = \frac{2}{\sqrt{5}}\right]$$

40. (a) : (A)-(q, r, s), (B)-(q), (C)-(r, s), (D)-(p)

(A) $(\sin^{-1}x)^2 + (\sin^{-1}y)^2 = \frac{\pi^2}{2}$

$\therefore (\sin^{-1}x)^2 = (\sin^{-1}y)^2 = \frac{\pi^2}{4}$

or $\sin^{-1}x = \pm\frac{\pi}{2}$, $\sin^{-1}y = \pm\frac{\pi}{2}$

or $x = \pm 1$ and $y = \pm 1$

$\Rightarrow x^3 + y^3 = -2, 0, 2$

(B) $(\cos^{-1}x)^2 + (\cos^{-1}y)^2 = 2\pi^2$

$\therefore (\cos^{-1}x)^2 = (\cos^{-1}y)^2 = \pi^2$

or $x = y = -1$

$\Rightarrow x^5 + y^5 = -2$

(C) $(\sin^{-1}x)^2(\cos^{-1}y)^2 = \frac{\pi^4}{4}$

$\therefore (\sin^{-1}x)^2 = \frac{\pi^2}{4}$ and $(\cos^{-1}y)^2 = \pi^2$

or $(\sin^{-1}x) = \pm\frac{\pi}{2}$ and $(\cos^{-1}y) = \pi$

or $x = \pm 1$ and $y = -1$

$\Rightarrow |x - y| = 0, 2$

(D) $|\sin^{-1}x - \sin^{-1}y| = \pi$

$\therefore \sin^{-1}x = -\frac{\pi}{2}$ and $\sin^{-1}y = \frac{\pi}{2}$

or $\sin^{-1}x = \frac{\pi}{2}$ and $\sin^{-1}y = -\frac{\pi}{2}$

$\Rightarrow x = 1$ and $y = -1 \Rightarrow xy = -1$

41. (c) **42. (a)**

(41-42) : Since, $\cos^{-1}(4x^3 - 3x) = \begin{cases} 3\cos^{-1}x - 2\pi, & -1 \le x < (-1/2) \\ 2\pi - 3\cos^{-1}x, & (-1/2) \le x \le (1/2) \\ 3\cos^{-1}x, & (1/2) < x \le 1 \end{cases}$

For $x \in \left[-1, -\frac{1}{2}\right)$, $\cos^{-1}(4x^3 - 3x) = 3\cos^{-1}x - 2\pi$

$\Rightarrow a = -2\pi$ and $b = 3 \Rightarrow a + b\pi = \pi$

For $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$, $\cos^{-1}(4x^3 - 3x) = 2\pi - 3\cos^{-1}x$

$\Rightarrow a = 2\pi$ and $b = -3$

$\Rightarrow \sin^{-1}\left(\sin\frac{a}{b}\right) = \sin^{-1}\left(\sin\frac{2\pi}{-3}\right)$

$= \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$

43. (b) : Let $\cos^{-1}x = A$, $\cos^{-1}y = B$, $\cos^{-1}z = C$ then $A + B + C = \pi$, $x = \cos A$, $y = \cos B$, $z = \cos C$

Also, $0 \le A, B, C \le \pi$

$x\sqrt{1-x^2} + y\sqrt{1-y^2} + z\sqrt{1-z^2}$

$= \frac{1}{2}\{2\cos A\sqrt{1-\cos^2 A}$

$+ 2\cos B\sqrt{1-\cos^2 B} + 2\cos C\sqrt{1-\cos^2 C}\}$

$= \frac{1}{2}\{\sin 2A + \sin 2B + \sin 2C\} = \frac{1}{2}[4\sin A\sin B\sin C]$

$= 2\sin(\cos^{-1}x)\sin(\cos^{-1}y)\sin(\cos^{-1}z)$

$= 2\sqrt{1-x^2}\sqrt{1-y^2}\sqrt{1-z^2}$

44. (c) : $x^2 + y^2 + z^2 = \cos^2 A + \cos^2 B + \cos^2 C$

$= \frac{1+\cos 2A}{2} + \frac{1+\cos 2B}{2} + \frac{1+\cos 2C}{2}$

$= \frac{3}{2} + \frac{1}{2}(\cos 2A + \cos 2B + \cos 2C)$

$= 1 - 2xyz$

45. (a) : $yz\sqrt{1-x^2} + zx\sqrt{1-y^2} + xy\sqrt{1-z^2}$

$= xyz\left\{\frac{\sqrt{1-x^2}}{x} + \frac{\sqrt{1-y^2}}{y} + \frac{\sqrt{1-z^2}}{z}\right\}$

$= xyz\left\{\frac{\sqrt{1-\cos^2 A}}{\cos A} + \frac{\sqrt{1-\cos^2 B}}{\cos B} + \frac{\sqrt{1-\cos^2 B}}{C}\right\}$

$= xyz\{\tan A + \tan B + \tan C\}$

$= xyz\{\tan A\tan B\tan C\}$

$= \sin A\sin B\sin C$ ($x = \cos A$, $y = \cos B$, $z = \cos C$)

$= \sqrt{1-x^2}\sqrt{1-y^2}\sqrt{1-z^2}$

46. (1) : We have $\frac{\sin^{-1}(\cos x) + \cos^{-1}(\sin x)}{\tan^{-1}(\cot x) + \cot^{-1}(\tan x)}$

$= \frac{\frac{\pi}{2} - \cos^{-1}(\cos x) + \frac{\pi}{2} - \sin^{-1}(\sin x)}{\frac{\pi}{2} - \cot^{-1}(\cot x) + \frac{\pi}{2} - \tan^{-1}(\tan x)} = \frac{\frac{\pi}{2} - x + \frac{\pi}{2} - x}{\frac{\pi}{2} - x + \frac{\pi}{2} - x} = \frac{\pi - 2x}{\pi - 2x} = 1$

47. (3) : $\sin^{-1}\left(\frac{x}{5}\right) + \sin^{-1}\left(\frac{4}{5}\right) = \frac{\pi}{2}$

$\Rightarrow \sin^{-1}\left(\frac{x}{5}\right) = \frac{\pi}{2} - \sin^{-1}\left(\frac{4}{5}\right)$

$\Rightarrow \frac{x}{5} = \sin\left(\frac{\pi}{2} - \sin^{-1}\left(\frac{4}{5}\right)\right)$

$\Rightarrow \frac{x}{5} = \cos\left(\sin^{-1}\frac{4}{5}\right) = \cos\left(\cos^{-1}\frac{3}{5}\right) = \frac{3}{5} \Rightarrow x = 3$

48. (8) : $\frac{\tan^{-1}(\sqrt{3}) - \sec^{-1}(-2)}{\operatorname{cosec}^{-1}(-\sqrt{2}) + \cos^{-1}\left(\frac{-1}{2}\right)} \times (-10)$

$= \frac{(\pi/3) - (2\pi/3)}{(-\pi/4) + (2\pi/3)} \times (-10) = \left(\frac{-\pi/3}{\frac{-3\pi + 8\pi}{12}}\right) \times (-10)$

$= \left(\frac{-\pi/3}{5\pi/12}\right) \times (-10) = \frac{-4}{5} \times (-10) = 8$

49. (2) : $\because -\frac{\pi}{2} \le \sin^{-1}t \le \frac{\pi}{2}$

Since $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \frac{3\pi}{2}$

$\Rightarrow \sin^{-1}x = \sin^{-1}y = \sin^{-1}z = \frac{\pi}{2}$

$\Rightarrow x = y = z = 1$

$\therefore x^9 + y^9 + z^9 - \frac{1}{x^9y^9z^9} = 3 - 1 = 2$

50. (1) : Given trigonometrical equation is

$\sin^{-1}2x + \cos^{-1}2x + 2\tan^{-1}x = \pi$

$\Rightarrow \frac{\pi}{2} + 2\tan^{-1}x = \pi \Rightarrow 2\tan^{-1}x = \frac{\pi}{2}$

$\Rightarrow \tan^{-1}x = \frac{\pi}{4} \Rightarrow x = \tan\left(\frac{\pi}{4}\right) = 1$

JEE MATHEMATICS

DPS-14

DAILY PRACTICE SHEET

Class XII

Matrices and Determinants

SOLUTIONS

1. **(b) :** $A=\begin{bmatrix}1&0&0\\2&1&0\\3&2&1\end{bmatrix}$, $Au_1=\begin{bmatrix}1\\0\\0\end{bmatrix}$, $Au_2=\begin{bmatrix}0\\1\\0\end{bmatrix}$

Let $u_1=\begin{bmatrix}a\\b\\c\end{bmatrix}$, $Au_1=\begin{bmatrix}a\\2a+b\\3a+2b+c\end{bmatrix}=\begin{bmatrix}1\\0\\0\end{bmatrix}$

$\Rightarrow \quad a=1, 2a+b=0 \Rightarrow b=-2,\ 3a+2b+c=0 \Rightarrow c=1$

Let $u_2=\begin{bmatrix}p\\q\\r\end{bmatrix}$, $Au_2=\begin{bmatrix}p\\2p+q\\3p+2q+r\end{bmatrix}=\begin{bmatrix}0\\1\\0\end{bmatrix}$

$\Rightarrow \quad p=0, 2p+q=1 \Rightarrow q=1, 3p+2q+r=0 \Rightarrow r=-2$

$\therefore \quad u_1+u_2=\begin{bmatrix}1\\-2\\1\end{bmatrix}+\begin{bmatrix}0\\1\\-2\end{bmatrix}=\begin{bmatrix}1\\-1\\-1\end{bmatrix}$

2. **(b) :** $A=\begin{bmatrix}3&2\\1&1\end{bmatrix}$

$A^2+xA+yI=\begin{bmatrix}3&2\\1&1\end{bmatrix}\begin{bmatrix}3&2\\1&1\end{bmatrix}+x\begin{bmatrix}3&2\\1&1\end{bmatrix}+y\begin{bmatrix}1&0\\0&1\end{bmatrix}=0$

$\Rightarrow \begin{bmatrix}11&8\\4&3\end{bmatrix}+\begin{bmatrix}3x&2x\\x&x\end{bmatrix}+\begin{bmatrix}y&0\\0&y\end{bmatrix}=0$

$\Rightarrow 11+3x+y=0$ ...(i) and $8+2x+0=0 \Rightarrow x=-4$

Puting value of x in (i), we get $11-12+y=0 \Rightarrow y=1$

$\therefore \quad (x, y)=(-4, 1)$

3. **(c) :** $(A+B)(A-B)=A^2-B^2$

$\Rightarrow \quad AA+BA-AB-BB=A^2-B^2$

$\Rightarrow \quad A^2+BA-AB-B^2=A^2-B^2$

$\Rightarrow \quad BA-AB=0 \Rightarrow BA=AB$

$\therefore \quad (ABA^{-1})^2=(BAA^{-1})^2 \Rightarrow (ABA^{-1})^2=(BI)^2=B^2$

4. **(d) :** $X^T=-X$, $Y^T=-Y$, $Z^T=Z$

(a) $(Y^3Z^4-Z^4Y^3)^T=(Y^3Z^4)^T-(Z^4Y^3)^T$
$=-Z^4Y^3+Y^3Z^4=Y^3Z^4-Z^4Y^3$, symmetric.

(b) $(X^{44}+Y^{44})^T=X^{44}+Y^{44}$, symmetric.

(c) $(X^4Z^3+Z^3X^4)^T=(X^4Z^3)^T+(Z^3X^4)^T$
$=Z^3X^4+X^4Z^3$, symmetric.

(d) $(X^{23}+Y^{23})^T=-X^{23}-Y^{23}=-(X^{23}+Y^{23})$, skew symmetric.

5. **(a) :** Let $B=A+A^T$

$\therefore \quad B^T=(A+A^T)^T=A^T+(A^T)^T=A^T+A \ (\because (A^T)^T=A)$

$=A+A^T=B$

$\therefore \quad A+A^T$ is symmetric

Let $B=AA^T$

$B^T=(AA^T)^T$

$=(A^T)^T A^T=AA^T=B$, symmetric.

6. **(d) :** We have, $P^T=2P+I$

We get, $P^T-2P=I$ ...(i)

Taking transpose, we get $(P^T-2P)^T=I^T$

$\Rightarrow \quad P-2P^T=I$ ... (ii)

From (i) and (ii) on eliminating P^T, we get

$-4P+P=3I \Rightarrow P=-I \Rightarrow P+I=O$

Thus, $(P+I)X=O \Rightarrow PX+X=O \Rightarrow PX=-X$

7. **(d) :** We have,

$x\begin{bmatrix}2x&2\\3&x\end{bmatrix}+2\begin{bmatrix}8&5x\\4&4x\end{bmatrix}=2\begin{bmatrix}x^2+8&24\\10&6x\end{bmatrix}$

$\Rightarrow \begin{bmatrix}2x^2+16&2x+10x\\3x+8&x^2+8x\end{bmatrix}=\begin{bmatrix}2x^2+16&48\\20&12x\end{bmatrix}$

So, $12x=48$ ...(i)

$3x+8=20$ ...(ii)

$x^2+8x=12x$ ...(iii)

From (i) and (ii), we get $x=4$

Now, from (iii), we get $x^2-4x=0$

$\Rightarrow \quad x(x-4)=0$

$\Rightarrow \quad x=4$ [as $x \neq 0$]

8. **(a) :** $\begin{bmatrix}a&b&c\\p&q&r\\u&v&w\end{bmatrix}\begin{bmatrix}1&3&3\\1&4&3\\1&3&4\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$

$p+q+r=0$

$3p+4q+3r=1$

$3p+3q+4r=0$

On solving, we get $p=-1$, $q=1$, $r=0$

9. **(d) :** $AB=B$

So, A should be unit matrix.

Now $BA=A$

So, B should be unit matrix.

Square of unit matrix is also unit matrix.

So, $A^2=A$ and $B^2=B$

So, both A and B are idempotent.

10. **(c) :** Matrix $\begin{bmatrix}a&b\\c&d\end{bmatrix}$ is commutative with the matrix $\begin{bmatrix}1&1\\0&1\end{bmatrix}$.

If two matrices A and B are commutative on product, then $AB=BA$.

$\Rightarrow \begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}1&1\\0&1\end{bmatrix}=\begin{bmatrix}1&1\\0&1\end{bmatrix}\begin{bmatrix}a&b\\c&d\end{bmatrix} \Rightarrow \begin{bmatrix}a&a+b\\c&c+d\end{bmatrix}=\begin{bmatrix}a+c&b+d\\c&d\end{bmatrix}$

On comparing corresponding elements of two matrices, we get

$a+c=a$, $a+b=b+d$ and $c+d=d \Rightarrow c=0$ and $a=d$

11. **(b) :** $\begin{vmatrix}xp+y&x&y\\yp+z&y&z\\0&xp+y&yp+z\end{vmatrix}=0$

Operating $C_1 \to C_1-p\,C_2-C_3$, we get

$$\begin{vmatrix} 0 & x & y \\ 0 & y & z \\ -(p^2x+2py+z) & xp+y & yp+z \end{vmatrix} = 0$$

$\Rightarrow$ $-(p^2x + 2py + z)(xz - y^2) = 0$

$\Rightarrow$ $p^2x + 2py + z = 0$ or $y^2 = xz$

$\Rightarrow$ $y^2 = xz$ [$\because$ p is a constant]

$\Rightarrow$ x, y, z are in G. P.

12. (c) : Given, $\begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} = 0$

Appling $R_1 \to R_1 + R_2 + R_3$, we get

$$\begin{vmatrix} \sin x + 2\cos x & \sin x + 2\cos x & \sin x + 2\cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} = 0$$

Taking common (sin x + 2 cos x) from R_1, we get

$$(\sin x + 2\cos x)\begin{vmatrix} 1 & 1 & 1 \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} = 0$$

Operating $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we get

$$(\sin x + 2\cos x)\begin{vmatrix} 1 & 0 & 0 \\ \cos x & \sin x - \cos x & 0 \\ \cos x & 0 & \sin x - \cos x \end{vmatrix} = 0$$

$\Rightarrow$ $(\sin x + 2\cos x)\,[1\,(\sin x - \cos x)^2] = 0$

$\Rightarrow$ $\sin x + 2\cos x = 0$, or $(\sin x - \cos x)^2 = 0$

$\Rightarrow$ $\tan x = -2$, or $\tan x = 1$

For $-\dfrac{\pi}{4} \le x \le \dfrac{\pi}{4}, \tan x \ne -2.$

But $\tan x = 1 \Rightarrow x = \dfrac{\pi}{4}$

Hence, only one solution exists.

13. (d) : Given, $\begin{vmatrix} \dfrac{\log x}{\log z} & \dfrac{\log y}{\log z} & 1 \\ 1 & \dfrac{\log y}{\log x} & \dfrac{\log z}{\log x} \\ \dfrac{\log x}{\log y} & 1 & \dfrac{\log z}{\log y} \end{vmatrix}$

Multiplying R_1 by log z, R_2 by log x and R_3 by log y, we get

$$\frac{1}{\log x \log y \log z}\begin{vmatrix} \log x & \log y & \log z \\ \log x & \log y & \log z \\ \log x & \log y & \log z \end{vmatrix} = 0$$

[$\because$ All rows are identical]

14. (c) : We have,

$(a^2 + b^2 + c^2)x^2 - 2(ab + bc + cd)x + b^2 + c^2 + d^2 \le 0$

$\Rightarrow$ $(ax - b)^2 + (bx - c)^2 + (cx - d)^2 \le 0$

$\Rightarrow$ $(ax - b)^2 + (bx - c)^2 + (cx - d)^2 = 0$

$\Rightarrow$ $\dfrac{b}{a} = \dfrac{c}{b} = \dfrac{d}{c} = x$

$\Rightarrow$ $b^2 = ac$ taking log both sides we get, $2\log b = \log a + \log c$,

Now, $\begin{vmatrix} 33 & 14 & \log a \\ 65 & 27 & \log b \\ 97 & 40 & \log c \end{vmatrix} = \begin{vmatrix} 130 & 54 & \log a + \log c \\ 65 & 27 & \log b \\ 97 & 40 & \log c \end{vmatrix}$

[Apply $R_1 \to R_1 + R_3$]

$\Rightarrow \begin{vmatrix} 0 & 0 & 0 \\ 65 & 27 & \log b \\ 97 & 40 & \log c \end{vmatrix} = 0$ [Apply $R_1 \to R_1 - 2R_2$]

15. (b) : $f(x) = \begin{vmatrix} x & 1+\sin x & \cos x \\ 1 & \log(1+x) & 2 \\ x^2 & 1+x^2 & 0 \end{vmatrix}$

$= x\{0 - 2(1 + x^2)\} - (1 + \sin x)(-2x^2) + \cos x\{1 + x^2 - x^2\log(1 + x)\}$

$= -2x - 2x^3 + 2x^2 + 2x^2\sin x + \cos x + x^2\cos x - x^2\cos x\log(1 + x)$

$\therefore$ Coefficient of x is -2.

16. (b) : Area of triangle $= \dfrac{1}{2}\begin{vmatrix} \dfrac{x_1}{a} & \dfrac{y_1}{a} & 1 \\ \dfrac{x_2}{b} & \dfrac{y_2}{b} & 1 \\ \dfrac{x_3}{c} & \dfrac{y_3}{c} & 1 \end{vmatrix}$

Applying $R_1 \to aR_1, R_2 \to bR_2, R_3 \to cR_3$, we get $\dfrac{1}{2abc}\begin{vmatrix} x_1 & y_1 & a \\ x_2 & y_2 & b \\ x_3 & y_3 & c \end{vmatrix}$

Applying $C_3 \to 2C_3$, we get

$$\frac{1}{4abc}\begin{vmatrix} x_1 & y_1 & 2a \\ x_2 & y_2 & 2b \\ x_3 & y_3 & 2c \end{vmatrix} = \frac{(-1)^2}{4abc}\begin{vmatrix} 2a & x_1 & y_1 \\ 2b & x_2 & y_2 \\ 2c & x_3 & y_3 \end{vmatrix} = \frac{1}{4abc} \times \frac{abc}{2} = \frac{1}{8}$$

17. (b) : If we interchange any two rows of a determinant in the set B, its value becomes -1 and hence it is in C. Likewise, for every determinant in C, there is a corresponding determinant in B. $\therefore$ B and C have the same number of elements.

18. (a) : Let $A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$. We have

$$A^2 = \begin{bmatrix} \alpha^2 + \beta\gamma & \beta(\alpha+\delta) \\ \gamma(\alpha+\delta) & \delta^2 + \beta\gamma \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\Rightarrow$ $\alpha^2 + \beta\gamma = 1 = \delta^2 + \beta\gamma$ and $\gamma(\alpha + \delta) = \beta(\alpha + \delta) = 0$

As $A \ne I$, $A \ne -I$, we have $\alpha = -\delta$

$$\det A = \begin{vmatrix} \sqrt{1-\beta\gamma} & \beta \\ \gamma & -\sqrt{1-\beta\gamma} \end{vmatrix} = -1 + \beta\gamma - \beta\gamma = -1$$

Statement-1 is therefore true.

$tr(A) = \alpha + \delta = 0$ {$\because$ $\alpha = -\delta$}

Statement-2 is false because $tr(A) = 0$

19. (d) : $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$

det $A = (a_1b_2c_3 + a_2c_1b_3 + a_3b_1c_2) - (a_1c_2b_3 + a_2b_1c_3 + a_3c_1b_2)$

If any of the terms be non-zero, then det A will be non-zero and all the elements of that term will be 1 each.

Number of non-singular matrices $= {}^6C_1 \times {}^6C_1 = 36$

We can also exhibit more than 6 matrices to pick the right choice.

20. (c) : We have, $M = \begin{vmatrix} a & l & p \\ b & m & q \\ c & n & r \end{vmatrix}$, $N = \begin{vmatrix} p & q & r \\ a & b & c \\ l & m & n \end{vmatrix}$

Then, $M' = \begin{vmatrix} a & b & c \\ l & m & n \\ p & q & r \end{vmatrix}$

Applying $R_1 \leftrightarrow R_2$ in M', we get $M' = -\begin{vmatrix} l & m & n \\ a & b & c \\ p & q & r \end{vmatrix}$

Applying $R_1 \leftrightarrow R_3$ in M', we get $M' = \begin{vmatrix} p & q & r \\ a & b & c \\ l & m & n \end{vmatrix} = N$

21. **(d) :** Each entry of A is an integer, so the cofactor of every entry is an integer and each entry of adjoint is also an integer.
Also, $\det A = \pm 1$

$$A^{-1} = \frac{1}{\det A}(\text{adj } A)$$

This means all entries in A^{-1} are integers.

22. **(c) :** Let $z = x + iy$, then $z + iz = x + iy + i(x + iy) = (x - y) + i(x + y)$ and $iz = i(x + iy) = -y + ix$
Then, the area of the triangle formed by these lines is

$$\Delta = \frac{1}{2}\begin{vmatrix} x & y & 1 \\ (x-y) & (x+y) & 1 \\ -y & x & 1 \end{vmatrix}$$

Applying $R_2 \to R_2 - (R_1 + R_3)$, $\Delta = \frac{1}{2}\begin{vmatrix} x & y & 1 \\ 0 & 0 & -1 \\ -y & x & 1 \end{vmatrix} = \frac{1}{2}(x^2 + y^2)$

$\frac{1}{2}|z|^2 = 200$ (given)

$\Rightarrow \quad |z|^2 = 400 \Rightarrow |z| = 20 \qquad \therefore \quad 3|z| = 3 \times 20 = 60$

23. **(c) :** Let $A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$, $A^2 = \begin{bmatrix} \alpha^2 + \beta\gamma & \beta(\alpha+\delta) \\ \gamma(\alpha+\delta) & \delta^2 + \beta\gamma \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

which gives $\alpha + \delta = 0$ and $\alpha^2 + \beta\gamma = 1$
So, we have $Tr(A) = 0$
$\Rightarrow \quad \det A = \alpha\delta - \beta\gamma = -\alpha^2 - \beta\gamma = -(\alpha^2 + \beta\gamma) = -1 \neq 1$
Thus Statement-1 is true but Statement-2 is false.

24. **(b) :** $AA^{-1} = I$

$\Rightarrow \quad \frac{A.(\text{adj } A)}{|A|} = I \Rightarrow A(\text{adj } A) = |A|I$

Thus, $A(\text{adj } A) = 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

25. **(c) :** $A = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$|A| = (\cos^2\alpha + \sin^2\alpha) = 1$

$\Rightarrow \text{adj } A = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\therefore \quad A^{-1} = \frac{\text{adj } A}{|A|} = \text{adj } A$

26. **(b) :** The given system of equation is inconsistent iff its determinant is zero.

$\Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{vmatrix}$

$= 1\cdot[9 - 5(-1)] - 1[4\cdot3 - (-1)\cdot3] + 1[4\cdot5 - 3\cdot3]$
$= 14 - 1\,(15) + [11] = 10 \neq 0$
Determinant of coefficient matrix is not zero. So, the system of equation is consistent.

27. **(a) :** Given system of equations are

$$\left.\begin{aligned} x + y + z &= 6 \\ x + 2y + 3z &= 10 \\ x + 2y + az &= b \end{aligned}\right\} \qquad \text{...(i)}$$

Since the system of equation has no solution, then it is inconsistant and if $D = 0$ and atleast one of D_1, D_2 or D_3 is non-zero, then given system of equation is inconsistent.

Now, $D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & a \end{vmatrix} = 0 \Rightarrow 1(2a - 6) - 1(a - 3) + 1(2 - 2) = 0$

$\Rightarrow \quad a - 3 = 0 \Rightarrow \quad a = 3 \qquad$ ...(ii)

Let $D_1 \neq 0$, then

$\begin{vmatrix} 6 & 1 & 1 \\ 10 & 2 & 3 \\ b & 2 & a \end{vmatrix} \neq 0 \Rightarrow 6(2a - 6) - 10(a - 2) + b(3 - 2) \neq 0 \Rightarrow b \neq 10$

28. **(a) :** Let $A = \begin{bmatrix} 3 & -1 & 4 \\ 1 & 2 & -3 \\ 6 & 5 & \lambda \end{bmatrix} \Rightarrow |A| = \begin{vmatrix} 3 & -1 & 4 \\ 1 & 2 & -3 \\ 6 & 5 & \lambda \end{vmatrix}$

$= 3(2\lambda + 15) + 1(\lambda + 18) + 4(5 - 12)$
$= 6\lambda + 45 + \lambda + 18 + 20 - 48 = 7\lambda + 35$
For atleast one solution
$|A| = 0 \quad$ ...(i) $\quad$ and $(\text{adj } A) \cdot B = 0 \qquad$ ...(ii)
$\Rightarrow \quad 7\lambda + 35 = 0 \Rightarrow 7\lambda = -35 \Rightarrow \lambda = -5$
Also, $\lambda = -5$ is satisfying (ii).

29. **(c) :** $a(x + 1)(x - 2) + b(x - 2)(x - 1) + c(x - 1)(x + 1) = 2x^2 + 5x + 1$
On equating, $a + b + c = 2 \qquad$ ...(i)
$-a - 3b + 0\cdot c = 5 \quad$ ...(ii) $\qquad -2a + 2b - c = 1 \qquad$ ...(iii)

$\therefore \quad D = \begin{vmatrix} 1 & 1 & 1 \\ -1 & -3 & 0 \\ -2 & 2 & -1 \end{vmatrix} = 1(3 - 0) - (1 - 0) + (-2 - 6) = -6 \neq 0$

$\therefore \quad$ A unique solution exists for above three equations.
$\therefore \quad$ Exactly one choice for each of a, b, c exists.

30. **(c) :** Since system of equations has non-trivial (non-zero) solutions, then determinant formed by coefficient of unknown is zero. So, $\begin{vmatrix} a & a & -1 \\ b & -1 & b \\ -1 & c & c \end{vmatrix} = 0$

$\begin{vmatrix} a & 0 & -a-1 \\ b & -b-1 & 0 \\ -1 & c+1 & c+1 \end{vmatrix} = 0 \qquad \begin{bmatrix} C_2 \to C_2 - C_1 \\ C_3 \to C_3 - C_1 \end{bmatrix}$

$\Rightarrow \quad a[-(b + 1)(c + 1)] - (a + 1)[b(c + 1) - (b + 1)] = 0$
$\Rightarrow \quad -a(b + 1)(c + 1) - b(a + 1)(c + 1) + (a + 1)(b + 1) = 0$

$\Rightarrow \quad \frac{-a}{a+1} - \frac{b}{b+1} + \frac{1}{c+1} = 0$

$\Rightarrow \quad \frac{1}{c+1} = \frac{a}{a+1} + \frac{b}{b+1} = \frac{a+1}{a+1} - \frac{1}{a+1} + \frac{b+1}{b+1} - \frac{1}{b+1}$

$\Rightarrow \quad \frac{1}{c+1} + \frac{1}{a+1} + \frac{1}{b+1} = 1 + 1 = 2$

31. **(a, b) :** $M^2 = N^4 \Rightarrow (M - N^2)(M + N^2) = 0$
as M and N commute.
$M - N^2 \neq 0$, so $\det(M + N^2) = 0$
$\det(M^2 + MN^2) = \det(M(M + N^2))$
$= (\det M)(\det (M + N^2)) = (\det M) \cdot 0 \equiv 0$
$(M^2 + MN^2)U = O$ for some 3×3 non-zero matrix U.

32. **(a, d) :** $A^2 + A + 2I = 0$
$A^2 + A = -2I$
$\Rightarrow \quad |A^2 + A| = |-2I|$
$\Rightarrow \quad |A|\,|A + I| = (-2)^n$
$\Rightarrow \quad |A| \neq 0$

$\Rightarrow$ 'A' is non-singular.

Hence, its inverse exists. Also multiplying the given equation both sides with A^{-1} we get $A^{-1} = -\frac{1}{2}(A+I)$

33. **(a, c, d) :** As A and B are skew hermitian

$\therefore \quad A^{\theta} = -A, B^{\theta} = -B$

$(A + B)^{\theta} = A^{\theta} + B^{\theta}$

$= -A - B = -(A + B)$ (a is true)

$(AB)^{\theta} = B^{\theta} A^{\theta} = (-B)(-A) = BA$

$= AB$ (b is false and d is true)

(c) is well-known as diagonal elements in hermitian matrix must be real.

34. **(c, d) :** $(N^TMN)^T = (N^T)M^T(N^T)^T = N^TM^TN = N^TMN$, if M is symmetric

$= -N^TMN$, if M is skew symmetric

Again $(MN - NM)^T = (MN)^T - (NM)^T$

$= N^TM^T - M^TN^T = NM - MN = -(MN - NM)$

Now, $(MN)^T = N^TM^T = NM \neq MN$ (in general)

and adj(MN) = (adj N)(adj M)

and they don't commute in general.

Thus (a) and (b) are correct and (c) and (d) are incorrect.

35. **(a, b, c, d) :** $AA^T = A^TA = I$

Also $A^T = A$, so $A^2 = I \Rightarrow A$ is an involutory matrix.

$\Rightarrow \quad |A^2| = |A|^2 = 1$ or, $|A| = \pm 1$.

But $|A| = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = (a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix}$

$= (a + b + c)(ab + bc + ca - a^2 - b^2 - c^2)$

$|A| = ab + bc + ca - a^2 - b^2 - c^2 \quad (\because a + b + c = 1)$

$\therefore \quad a^2 + b^2 + c^2 - ab - bc - ca \geq 0$

So $|A| = -1$. Hence, $a^3 + b^3 + c^3 - 3abc = 1$.

Again $a^2 + b^2 + c^2 - ab - bc - ca = 1$

$\Rightarrow$ $1 - 3(ab + bc + ca) = 1$, so $ab + bc + ca = 0$,

$\Rightarrow$ atleast one of a, b, and c is negative.

36. **(b, c, d) :** Let $P^2 = [m_{ij}]_{n \times n}$

Now, $m_{ij} = \sum_{l=1}^{n} p_{il}\ p_{lj} = \sum_{l=1}^{n} \omega^{i+l}\ \omega^{l+j} = \omega^{i+j} \sum_{l=1}^{n} \omega^{2l}$

$= \omega^{i+j} \cdot \omega^2[1 + \omega^2 + \omega^4 + \ldots + \omega^{2n-2}]$

P^2 is a null matrix when n is a multiple of 3.

Hence, $n \neq 3k \Rightarrow P^2 \neq 0$. Thus, n can be 55, 56 or 58.

37. **(a, b) :** $A = \begin{bmatrix} 0 & 1^2-2^2 & 1^2-3^2\ldots \\ 2^2-1^2 & 0 & \\ 3^2-1^2 & & 0 \\ \vdots & & \end{bmatrix}$

$\therefore$ A is skew-symmetric of even order

$\therefore$ $|A|$ is a perfect square

38. **(c, d) :** Let $M = \begin{bmatrix} p & q \\ q & r \end{bmatrix}$. If $\begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} q \\ r \end{bmatrix}$ then $p = q = r$

det $M = pr - q^2 = q^2 - q^2 = 0$

Hence M is not invertible. So choice (a) doesn't hold.

If $[q\ r] = [p\ q]$ then $p = q = r$

Again, det $M = 0$. So choice (b) doesn't hold.

Now, Let $M = \begin{bmatrix} p & 0 \\ 0 & r \end{bmatrix}$ with $pr \neq 0$

det $M = pr \neq 0$. $\therefore$ M is invertible.

So choice (c) holds.

Also, let $M = \begin{bmatrix} p & q \\ q & r \end{bmatrix} \Rightarrow$ det $M = pr - q^2$

If pr is not the square of an integer then $pr - q^2 \neq 0$

$\therefore$ det $M \neq 0$ $\therefore$ M is invertible.

So choice (d) holds.

39. **(a) : (A) – (r), (B) – (s), (C) – (q), (D) – (p)**

(A) $(I + A)^8 = {}^8C_0I + {}^8C_1IA + {}^8C_2IA^2 + \ldots\ldots + {}^8C_8IA^8$

$= {}^8C_0I + {}^8C_1A + {}^8C_2A + \ldots\ldots + {}^8C_8A$

$= I + A({}^8C_1 + {}^8C_2 + \ldots.. + {}^8C_8)$

$= I + A(2^8 - 1) \Rightarrow \lambda = 2^8 - 1$

$\lambda + 1 = 2^8 - 1 = 2^8 - 1 + 1 = 2^8 = 256$

(B) $|\text{adj}(A^{-1})| = |A^{-1}|^2 = \frac{1}{|A|^2}$

$|(\text{adj}(A^{-1}))^{-1}| = \frac{1}{|\text{adj } A^{-1}|} = |A|^2 = 2^2 = 4$

(C) $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

$\Rightarrow |B| = \begin{vmatrix} a_{11} & \lambda^{-1}a_{12} & \lambda^{-2}a_{13} \\ \lambda a_{21} & a_{22} & \lambda^{-1}a_{23} \\ \lambda^2 a_{31} & \lambda a_{32} & a_{33} \end{vmatrix}$

$= \frac{1}{\lambda^3}\begin{vmatrix} \lambda^2 a_{11} & \lambda a_{12} & a_{13} \\ \lambda^2 a_{21} & \lambda a_{22} & a_{23} \\ \lambda^2 a_{31} & \lambda a_{32} & a_{33} \end{vmatrix} = |A|$

Hence, $|A| = |B| \Rightarrow \lambda = 1$.

(D) A diagonal matrix is commutative with every square matrix, if it is a scalar matrix.

So, every diagonal element is 4.

$\therefore$ $|A| = 64$.

40. **(c) : (A) – (r,s), (B) – (p,q), (C) –(p,q), (D) – (p,q)**

(A) $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} \Rightarrow A^T = A \;\&\; AA^T = 9I \Rightarrow A^2 = 9I$

Again, cofactor matrix of $A = \begin{bmatrix} -3 & -6 & -6 \\ -6 & -3 & 6 \\ -6 & 6 & -3 \end{bmatrix} = -3A$

(B) $3B = A \Rightarrow B = (1/3)A$. But $AA^T = A^TA = 9I$

$\therefore$ A is orthogonal & involutory matrix so the matrix B is also orthogonal involutory matrix.

(C) $B = \frac{1}{3}A \Rightarrow B^T = \frac{1}{3}A^T$. Also, $BB^T = B^TB = \frac{1}{9}A^2 = I$

(D) Again $B = (1/3)A$ where $(1/3)A$ is involutory & orthogonal.

41. **(c) :** $a + b + c = p$, $ab + bc + ca = 0$, $abc = r$

$\therefore$ $a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + bc + ca)$

$= p^2 - 0 = p^2$

If $\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

$$\therefore\ \Delta^c = \begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix} = \Delta^{3-1}$$

$$\text{Now, } \Delta^2 = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

$$= \begin{vmatrix} a^2+b^2+c^2 & ab+bc+ca & ab+bc+ca \\ ab+bc+ca & a^2+b^2+c^2 & ab+bc+ca \\ ab+bc+ca & ab+bc+ca & a^2+b^2+c^2 \end{vmatrix}$$

$$= \begin{vmatrix} p^2 & 0 & 0 \\ 0 & p^2 & 0 \\ 0 & 0 & p^2 \end{vmatrix} = p^6$$

42. **(a) :** $\begin{vmatrix} A & B & C \\ B & C & A \\ C & A & B \end{vmatrix} = \begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix}$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = 49$$

43. **(c) :** $AA^\theta = kk\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix} = k^2\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = I$

$\Rightarrow\ 3k^2 = 1\ \therefore\ k = \pm\dfrac{1}{\sqrt{3}}$

44. **(b) :** $A^\theta Q^{20}A = A^\theta(ABA^\theta)Q^{19}A = IBA^\theta Q^{19}A$
$= IBA^\theta\ ABA^\theta Q^{18}A$
$= B^2A^\theta Q^{18}A = \text{..........} = B^{20}$

45. **(a) :** Let X be a square matrix of order 2, then det(adj(adj$X$))
$= (\det X)^{(n-1)^2} = 1^1 = 1$ $\qquad (\because\ |X| = 1)$

46. **(7) :** Given equation $A^2 = 8A + kI_2$ ...(i)
Here, $|A - \lambda I| = 0$

$\Rightarrow\ \begin{vmatrix} 1-\lambda & 0 \\ -1 & 7-\lambda \end{vmatrix} = 0$

$\Rightarrow\ (1-\lambda)(7-\lambda) = 0$
$\Rightarrow\ \lambda^2 - 8\lambda + 7 = 0$
$\Rightarrow\ A^2 - 8A + 7I_2 = 0$
$\Rightarrow\ A^2 = 8A - 7I_2$ ...(ii)
On comparing the coefficient of I_2 in (i) and (ii) eq. we get
$\Rightarrow\ k = -7\ \therefore\ |k| = 7$

47. **(0) :** The given system has no solution.

$\therefore\ \begin{vmatrix} 1 & 2 & -3 \\ 0 & 0 & p+2 \\ 0 & 2p+1 & 1 \end{vmatrix} = 0$

$\Rightarrow\ \begin{vmatrix} 0 & 0 & p+2 \\ 1 & 2 & -3 \\ 0 & 2p+1 & 1 \end{vmatrix} = 0$ (Apply $R_1 \Leftrightarrow R_2$)

$\Rightarrow\ (p+2)(2p+1) = 0\ \Rightarrow\ p = -2, -1/2$
$\therefore\ 2 + P = 2 - 2 = 0$

48. **(1) :** $\dfrac{|\text{adj } B|}{|C|} = \dfrac{|\text{adj}(\text{adj} A)|}{|5A|} = \dfrac{|A|^{(3-1)^2}}{5^3|A|} = \dfrac{|A|^4}{5^3|A|} = \dfrac{|A|^3}{125}$

Now, $|A| = 5$

$\therefore\ \dfrac{|\text{adj } B|}{|C|} = 1$

49. **(5) :** $(A+B)^2 = A^2 + B^2$
$\Rightarrow\ (A+B)(A+B) = A^2 + B^2$
$\Rightarrow\ A^2 + BA + AB + B^2 = A^2 + B^2$
$\Rightarrow\ AB + BA = O$

$\Rightarrow\ \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix} + \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow\ \begin{bmatrix} a-b & 2 \\ 2a-b & 3 \end{bmatrix} + \begin{bmatrix} a+2 & -a-1 \\ b-2 & -b+1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow\ \begin{bmatrix} 2a-b+2 & -a+1 \\ 2a-2 & -b+4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow\ 2a - b + 2 = 0,\ -a + 1 = 0,\ 2a - 2 = 0,\ -b + 4 = 0$
$\Rightarrow\ a = 1, b = 4$
$\therefore\ a + b = 1 + 4 = 5$

50. **(1) :** $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$

$A^2 = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 5 & -8 \\ 2 & -3 \end{bmatrix}$

$A^3 = A^2A = \begin{bmatrix} 5 & -8 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 7 & -12 \\ 3 & -5 \end{bmatrix}$

$\therefore\ A^n = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$

$\Rightarrow \det(A^n) = 1 - 4n^2 + 4n^2 = 1\ \therefore\ \det(A^{2005}) = 1$

DPS-15 DAILY PRACTICE SHEET

Continuity and Differentiability

SOLUTIONS

1. **(b) :** RHL $= \lim_{x\to -1^+} \frac{\pi-\cos^{-1}(x)}{\sqrt{x+1}}\cdot\frac{1}{\sqrt{\pi}+\sqrt{\cos^{-1}x}}$

$= \lim_{x\to -1^+} \frac{\cos^{-1}(-x)}{\sqrt{x+1}}\cdot\frac{1}{\sqrt{\pi}+\sqrt{\cos^{-1}x}}$

$= \sqrt{2}\cdot\frac{1}{\sqrt{\pi}+\sqrt{\pi}} = \frac{1}{\sqrt{2\pi}}$ [Putting $\cos^{-1}(-x) = t$]

$\therefore \quad \lambda = 2$

2. **(b) :** Since $f(x)$ is continuous at $x = 0$

So, $f(0 + h) = f(0 - h) = f(0)$ ...(i)

Now $f(x + y) = f(x) + f(y)$ ...(ii)

Keeping $x = 0,\ y = h$

$f(0 + h) = f(0) + f(h)$ ...(iii)

Keeping $x = 0, y = -h$

$f(0 - h) = f(0) + f(-h)$ ...(iv)

From (i), (iii) and (iv), $f(0) + f(h) = f(0) + f(-h)$

or $f(h) = f(-h)$ ...(v)

Now $x = k, y = h$

$f(k + h) = f(k) + f(h)$ (from (ii))

$= f(k) + f(-h)$ (from (v))

$= f(k - h) = f(k)$

$\therefore \quad f(x)$ is continuous $\forall\ k \in R$

3. **(c) :** $f(x)$ is continuous at $x = \frac{-\pi}{2}$.

$\Rightarrow \quad -a+b=-2\sin\left(\frac{-\pi}{2}\right) \Rightarrow -a+b=2$

$\Rightarrow \quad -a + b = 2$

Then, $a = -1$ and $b = 1$, option (c) is satisfied.

4. **(c) :** $\lim_{x\to 0^-} f(x) = \lim_{x\to 0} x = 0$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0} x = 0$ and $f(0) = 0$

$\therefore \quad f(x)$ is continuous at $x = 0$

5. **(a) :** $f(x) = x\left(\sqrt{x}-\sqrt{x+1}\right)$

$\Rightarrow \quad f(x) = \frac{x(x-(x+1))}{\sqrt{x}+\sqrt{x+1}} = \frac{-x}{\sqrt{x}+\sqrt{x+1}}$

$\therefore \quad$ Function is continuous in $(0, \infty)$

6. **(b) :** $f(x) = \begin{cases} (1+|\sin x|)^{a/|\sin x|}, & -\frac{\pi}{6} < x < 0 \\ b, & x = 0 \\ e^{\tan 2x/\tan 3x}, & 0 < x < \frac{\pi}{6} \end{cases}$

For $f(x)$ to be continuous at $x = 0$

$\lim_{x\to 0^-} f(x) = f(0) = \lim_{x\to 0^+} f(x)$

L.H.L. $= \lim_{x\to 0}(1+|\sin x|)^{a/|\sin x|} = e^{\lim_{x\to 0}\left\{|\sin x|\frac{a}{|\sin x|}\right\}} = e^a$

Now, R.H.L. $= \lim_{x\to 0} e^{\tan 2x/\tan 3x}$

$= \lim_{x\to 0} e^{\left(\frac{\tan 2x}{2x}\times 2x\right)\Big/\left(\frac{\tan 3x}{3x}\times 3x\right)} = \lim_{x\to 0} e^{2/3} = e^{2/3}$

Since, $f(x)$ is continuous at $x = 0$

$\therefore \quad e^a = e^{2/3} \Rightarrow a = \frac{2}{3}$ and $b = e^{2/3}$

7. **(b) :** $f(x) = \frac{e^{1/x}}{1+e^{1/x}}$

L.H.L. $= \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\frac{e^{1/x}}{1+e^{1/x}}$

$= \lim_{x\to 0^-}\frac{1}{\left(\frac{1}{e^{1/x}}\right)+1} = \frac{1}{\left(\frac{1}{e^{-\infty}}\right)+1} = \frac{1}{\infty+1} = 0$

R.H.L $= \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\frac{e^{1/x}}{1+e^{1/x}}$

$= \lim_{x\to 0^+}\frac{1}{e^{-1/x}+1} = \frac{1}{e^{-\infty}+1} = \frac{1}{0+1} = 1$

Since L.H.L. $\neq$ R.H.L.

$\therefore \quad$ Function is discontinuous at $x = 0$

8. **(c) :** $f(x) = [x]\cos\left(\frac{2x-1}{2}\right)\pi = [x]\cos\left(\pi x-\frac{\pi}{2}\right) = [x]\sin \pi x$

Let n be an integer.

$\lim_{x\to n^+} f(x) = 0,\ \lim_{x\to n^-} f(x) = 0$

Also $f(n) = 0 \Rightarrow f(x)$ is continuous for every real x.

9. **(d) :** $\lim_{x\to 0} f(x) = \lim_{x\to 0}\frac{1-\cos(1-\cos x)}{x^4}$

$= \lim_{x\to 0}\frac{1-\cos\left(2\sin^2\frac{x}{2}\right)}{x^4} = \lim_{x\to 0}\frac{2\sin^2\left(\sin^2\frac{x}{2}\right)}{x^4}$

$= 2\cdot\lim_{x\to 0}\left\{\frac{\sin\left(\sin^2\frac{x}{2}\right)}{\sin^2\frac{x}{2}}\right\}^2\cdot\frac{\sin^4\frac{x}{2}}{x^4}$

$= 2\cdot 1^2\left(\lim_{x\to 0}\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^4\cdot\frac{1}{2^4} = 2\cdot 1^4\cdot\frac{1}{2^4} = \frac{1}{8}$

$\therefore \quad$ If $f(0) = \lim_{x\to 0} f(x) = \frac{1}{8}$, then $f(x)$ will be continuous everywhere.

10. **(c)** : $f(x)=\dfrac{1+\cos 5x}{1-\cos 4x}$

$f(x)$ will be discontinuous when

$1-\cos 4x=0 \Rightarrow \cos 4x=1$

$\Rightarrow 4x=2n\pi, n=0, \pm 1, \pm 2, \ldots$

$x=\dfrac{n\pi}{2}, n=0,\pm 1,\pm 2,\ldots \Rightarrow x=0, \dfrac{\pi}{2}, \pi$ in $[0,\pi]$.

11. **(b)** : $f(0)=\lim\limits_{x\to 0}\left[\dfrac{1}{x}-\dfrac{2}{e^{2x}-1}\right]$

$=\lim\limits_{x\to 0}\dfrac{e^{2x}-1-2x}{x(e^{2x}-1)}$ $\left(\dfrac{0}{0}\text{ form}\right)$

By using L' Hospital rule, we get

$f(0)=\lim\limits_{x\to 0}\dfrac{2e^{2x}-2}{(e^{2x}-1)+2xe^{2x}}$ $\left(\dfrac{0}{0}\text{ form}\right)$

Again by using L' Hospital rule, we get

$f(0)=\lim\limits_{x\to 0}\dfrac{4e^{2x}}{4e^{2x}+4xe^{2x}}=1.$

12. **(c)** : $f(x)=[x^3-3]$

$\because$ $[x]$ is discontinuous when x is an integer

$\therefore$ $x^3-3=-1, 0, 1, 2, 3, 4$

or $x^3=2, 3, 4, 5, 6, 7$

or $x=2^{\frac{1}{3}}, 3^{\frac{1}{3}}, 4^{\frac{1}{3}}, 5^{\frac{1}{3}}, 6^{\frac{1}{3}}, 7^{\frac{1}{3}} \in (1,2)$

$\therefore$ No. of points of discontinuity is 6.

13. **(b)** : As we are concerned about differentiability at '0' in the vicinity of $\sin x$

$\therefore$ $f(x)=\log 2-\sin x$

$g(x)=f(f(x))=\log 2-\sin(\log 2-\sin x)$

As g is difference of two differentiable functions, so g is differentiable.

$g'(x)=\cos(\log 2-\sin x)\cdot\cos x$

Then, $g'(0)=\cos(\log 2)$.

14. **(b)** : $x^2+y^2=t+\dfrac{1}{t} \Rightarrow (x^2+y^2)^2=\left(t+\dfrac{1}{t}\right)^2$

$\Rightarrow x^4+y^4+2x^2y^2=t^2+\dfrac{1}{t^2}+2 \Rightarrow 2x^2y^2=2$

$\therefore$ $x^2y^2=1$

Differentiating w.r.t. x, we get $\dfrac{dy}{dx}=\dfrac{-y}{x}$

15. **(d)** : Let $L=\lim\limits_{t\to x}\dfrac{t^2f(x)-x^2f(t)}{t-x}=1$

Applying L' Hospital rule, we get

$L=\lim\limits_{t\to x}\dfrac{2tf(x)-x^2f'(t)}{1}=1$

or $2xf(x)-x^2f'(x)=1 \Rightarrow f'(x)-\dfrac{2}{x}f(x)=\dfrac{-1}{x^2}$

I.F. $=e^{\int\frac{-2}{x}dx}=e^{-2\log x}=\dfrac{1}{x^2}$

Solution is $[f(x)]\dfrac{1}{x^2}=\int\dfrac{1}{x^2}\left(-\dfrac{1}{x^2}\right)dx \Rightarrow \dfrac{f(x)}{x^2}=\dfrac{1}{3x^3}+C$

We have, $f(1)=1$

$\Rightarrow 1=\dfrac{1}{3}+C \Rightarrow C=\dfrac{2}{3} \quad \therefore \quad f(x)=\dfrac{2}{3}x^2+\dfrac{1}{3x}$

$\therefore \quad f\left(\dfrac{3}{2}\right)=\dfrac{2}{3}\times\left(\dfrac{3}{2}\right)^2+\dfrac{1}{3}\times\dfrac{2}{3}=\dfrac{31}{18}$

16. **(b)** : $f(x)=\sin\left(\dfrac{\pi}{2}[x]-x^5\right)$

$=\sin\left(\dfrac{\pi}{2}-x^5\right)$ $[\because 1<x<2 \Rightarrow [x]=1]$

$\therefore$ $f'(x)=\cos\left(\dfrac{\pi}{2}-x^5\right)(-5x^4)$

$f'\left(\sqrt[5]{\dfrac{\pi}{2}}\right)=\cos\left\{\dfrac{\pi}{2}-\left(\sqrt[5]{\dfrac{\pi}{2}}\right)^5\right\}\left\{-5\left(\sqrt[5]{\dfrac{\pi}{2}}\right)^4\right\}$

$=\cos\left(\dfrac{\pi}{2}-\dfrac{\pi}{2}\right)\left(-5\left(\dfrac{\pi}{2}\right)^{4/5}\right)=-5\left(\dfrac{\pi}{2}\right)^{4/5}$

17. **(b)** : $\lim\limits_{x\to 4}\dfrac{xf(4)-4f(x)}{x-4}$

$=\lim\limits_{x\to 4}\dfrac{xf(4)-4f(x)+4f(4)-4f(4)}{x-4}$

$=\lim\limits_{x\to 4}\dfrac{f(4)(x-4)-4(f(x)-f(4))}{x-4}=\lim\limits_{x\to 4}f(4)-4\lim\limits_{x\to 4}\dfrac{f(x)-f(4)}{x-4}$

$=6-4\times f'(4)=6-2\times 4=-2$

18. **(b)** : $(gof)(x)=\sin x|x|=\begin{cases}-\sin x^2, & x<0\\ \sin x^2, & x\geq 0\end{cases}$

$(gof)'(x)=\begin{cases}-2x\cos x^2, & x<0\\ 2x\cos x^2, & x\geq 0\end{cases}$

$L(gof)'(0)=0=R(gof)'(0)$

$\therefore$ gof is differentiable at $x=0$ and $(gof)'$ is continuous at $x=0$

$(gof)''(x)=\begin{cases}-2\cos x^2+4x^2\sin x^2, & x<0\\ 2\cos x^2-4x^2\sin x^2, & x\geq 0\end{cases}$

$(gof)''(0^+)=2$ and $(gof)''(0^-)=-2$

$\therefore$ $(gof)''(0)$ does not exist.

19. **(c)** : $f'(x)=g(x) \Rightarrow g(1)=f'(1)=4$

$\therefore$ $f^2(1)+g^2(1)=(5)^2+(4)^2=25+16=41$

20. **(a)** : $\cos^2\left(\cot^{-1}\sqrt{\dfrac{2+x}{2-x}}\right)$

$=\cos^2\left(\tan^{-1}\sqrt{\dfrac{2-x}{2+x}}\right)$ $[\because \cot^{-1}x=\tan^{-1}(1/x)]$

$=\cos^2\left(\tan^{-1}\sqrt{\dfrac{1-(x/2)}{1+(x/2)}}\right)$

$=\cos^2\left(\tan^{-1}\sqrt{\dfrac{1-\cos\theta}{1+\cos\theta}}\right)$ [Taking $x/2=\cos\theta$]

$=\cos^2(\tan^{-1}(\tan\theta/2))=\cos^2(\theta/2)$

$=\dfrac{(1+\cos\theta)}{2}=\dfrac{1+(x/2)}{2}=\dfrac{1}{4}\cdot x+\dfrac{1}{2}$

Now, $\dfrac{d}{dx}\left[\dfrac{1}{4}\cdot x+\dfrac{1}{2}\right]=\dfrac{1}{4}$

21. **(a)** : $f(x)=\sin|x|$, consider $h(x)=\sin x$, $g(x)=|x|$, then $f=hog$, h and g are continuous, thus $\sin|x|$ is continuous everywhere.

22. **(a)** : $f(x)=|x-2|+|x-5|$

$\Rightarrow f(x)=\begin{cases}7-2x, & x<2\\ 3, & 2\leq x\leq 5\\ 2x-7, & x>5\end{cases}$

Statement-1 : $f'(4) = 0$. True

Statement-2 : f is continuous in [2, 5], differentiable in (2, 5) and $f(2) = f(5)$. True

But Statement-2 is not a correct explanation for Statement-1.

23. **(b) :** Let $y = \log_{10}x$ and $z = x^2$

Taking derivative w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{x\log_e 10} \text{ ......(i)} \qquad \frac{dz}{dx} = 2x \text{ ......(ii)}$$

$$\therefore \quad \frac{dy}{dz} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}} = \frac{1}{x\times 2x\log_e 10} = \frac{\log_{10} e}{2x^2}$$

24. **(b) :** $g(x) = \{f(2f(x) + 2)\}^2$

We have on differentiation with respect to x,

$g'(x) = 2f(2f(x) + 2) \cdot f'(2f(x) + 2) \cdot 2f'(x)$

Let $x = 0$

$g'(0) = 2f(2f(0) + 2) \cdot f'(2f(0) + 2) \cdot 2f'(0)$

$= 2f(0) \cdot f'(0) \cdot 2f'(0) = (-2)(1)(2) = -4.$

25. **(b) :** We have, $x^y = \log x$ ...(i)

Taking log on both sides of (i), we get

$$y\log x = \log(\log x) \Rightarrow y = \frac{\log(\log x)}{\log x} \qquad \text{...(ii)}$$

$$\therefore \quad \frac{dy}{dx} = \frac{\log x\left(\frac{1}{\log x}\right)\left(\frac{1}{x}\right) - \log(\log x)\left(\frac{1}{x}\right)}{(\log x)^2}$$

The point where the curve cuts the x-axis is $(e, 0)$.

$$\therefore \quad \left[\frac{dy}{dx}\right]_{\text{at } (e,\,0)} = \frac{1\cdot 1\cdot \frac{1}{e} - 0}{(1)^2} = \frac{1}{e}.$$

26. **(d) :** $f(0) = 0, f(1) = \sin 1 \ \therefore\ f(0) \ne f(1)$

$\Rightarrow$ 3rd condition of Rolle's theorem is violated in [0, 1]

Also, $f(x)$ is not differentiable at $x = 0$

$\therefore$ 2nd condition of Rolle's theorem as well as LMVT are violated in [–1, 1] as $0 \in (-1, 1)$

f satisfies the condition of LMVT on [0, 1]

27. **(a) :** $f(x) = \dfrac{\sin x}{e^x}$

Since $\sin x$ and e^x both are continuous and derivable functions

$\therefore$ $f(x)$ is also continuous and derivable

Since $f(0) = f(\pi) = 0$

$\therefore$ $f(x)$ satisfies Rolle's theorem and there exist a point c such that $f'(c) = 0$

Since, $f(x) = \dfrac{\sin x}{e^x}$

Taking derivative w.r.t. 'x', $f'(x) = \dfrac{e^x\cos x - \sin x e^x}{(e^x)^2}$

Put $f'(x) = 0$, we get $\cos x - \sin x = 0$

or $\tan x = 1 \Rightarrow x = \dfrac{\pi}{4}$.

$$\therefore\ f'\left(\frac{\pi}{4}\right) = 0$$

28. **(c) :** $\because f(h) = f(0) + hf'(\theta h), [0 < \theta < 1]$

We have, $f(x) = \cos x$

$\therefore$ $\cos h = \cos 0 + h(-\sin\theta h)$

$$\Rightarrow \sin\theta h = \frac{1-\cos h}{h} \Rightarrow \theta = \frac{1}{h}\sin^{-1}\left(\frac{1-\cos h}{h}\right)$$

$$\Rightarrow \lim_{h\to 0^+}\theta = \lim_{h\to 0^+}\frac{\sin^{-1}\left(\frac{1-\cos h}{h}\right)}{h}$$

$$= \lim_{h\to 0^+}\frac{\sin^{-1}\left(\frac{1-\cos h}{h}\right)}{\frac{1-\cos h}{h}}\cdot\frac{(1-\cos h)}{h^2} = 1\cdot\frac{1}{2} = \frac{1}{2}$$

29. **(d) :** (a) $\because f(x) = |x|$ is not differentiable at $x = 0 \in (-2, 2)$

$\therefore$ For $f(x) = |x|$, Rolle's theorem is not applicable

(b) $f(x) = \tan x$ is undefined at $x = \dfrac{\pi}{2} \in [0, \pi]$

$\therefore$ For $f(x) = \tan x$, Rolle's theorem is not applicable

(c) $f(x) = 1 + (x - 2)^{2/3}$

$$\Rightarrow f'(x) = \frac{2}{3}(x-2)^{-1/3} = \frac{2}{3(x-2)^{1/3}}$$

$f'(x)$ does not exist for $x = 2 \in (1, 3)$

$\therefore$ For $f(x) = 1 + (x - 2)^{2/3}$, Rolle's theorem is not applicable

(d) If $f(x) = x(x - 2)^2$ then $f(0) = 0 = f(2)$

$f'(x) = 1\cdot(x - 2)^2 + x\cdot 2(x - 2)$, which exists for all $x \in (0, 2)$ and $f(x)$ is continuous in [0, 2]

$\therefore$ Rolle's theorem is applicable for $f(x) = x(x - 2)^2$ in $0 \le x \le 2$

30. **(a) :** Let $f(x) = a_0 x + a_1\dfrac{x^2}{2} + a_2\dfrac{x^3}{3} + a_3\dfrac{x^4}{4}$

$f'(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

$f(0) = 0 = f(1)$, $\left[\because f(1) = a_0 + \dfrac{a_1}{2} + \dfrac{a_2}{3} + \dfrac{a_3}{4} = 0\right]$

$\therefore$ $f(x)$ satisfies all conditions of Rolle's theorem.

So, $f'(c) = 0$ for some $c \in (0, 1)$

$\therefore$ $f'(x) = 0$ has a real root in [0, 1]

31. **(a, d)** : As f and g are continuous, they attain their maximum on the closed and bounded interval.

If f and g achieve their maximum (common by hypothesis) at the same points, say, $x_0 \in [0, 1]$ then $f(x_0) = g(x_0)$.

Hence x_0 is the point c we are looking for and hence $f(c) = g(c)$. So choice (a) and (d) follows.

Next consider the case when f and g attain their maximum at points α and β such that $0 \le \alpha < \beta \le 1$.

Define $h(x) = f(x) - g(x)$, $h(\alpha) = f(\alpha) - g(\alpha) > 0$

$h(\beta) = f(\beta) - g(\beta) < 0$

Hence $\exists\, c \in [0,1]$ such that $h(c) = 0$, as h is continuous.

Thus, $f(c) = g(c)$.

32. **(a, b)** : As $x = 0$ is a repeated root of $|x|(\sin|x^3 + x|)$, hence $f(x)$ is differentiable at $x = 0$

At $x = 1$, we have $\cos|x^3 - x| = 1$

Again, $f(x)$ is differentiable at $x = 1$

33. **(b, c)** : $f(x) = [x^2] - 3$ is discontinuous at

$$x = 1, \sqrt{2}, \sqrt{3}, 2 \text{ in } \left[-\frac{1}{2}, 2\right]$$

Also, $g(x) = (|x| + |4x - 7|)([x^2] - 3)$

As at these points $|x| + |4x - 7|$ doesn't vanish. Thus, g is discontinuous at $x = 1, \sqrt{2}, \sqrt{3}$ in $\left(-\dfrac{1}{2}, 2\right)$

Hence, g is not differentiable as well.
In the neighbourhood of 0, $g(x) = (|x| + |4x - 7|)(-3)$
Thus, g is non-differentiable at $x = 0$.
So, g is not differentiable at exactly four points in $\left(-\frac{1}{2}, 2\right)$

34. **(b, d)** : Around $x = 2$, $\sin(|x| - 1) = \sin(x - 1)$
and $\sin(x - 1) > 0 \Rightarrow |\sin(x - 1)| = \sin(x - 1)$
we know, $\sin(x - 1) < 2$
Hence $|\sin(x - 1) - 2| = 2 - \sin(x - 1)$
$f(x) = 2 - \sin(x - 1)$ for $x \in (2 - \delta, 2 + \delta)$
$f'(2) = -\cos 1$

35. **(a,b,c)** : $f(x)$ is continuous at $x = a$ if $\lim_{h\to 0} f(a-h) = f(a)$
We consider the continuity at $x = k\pi (k \in I)$
$\therefore \quad \lim_{h\to 0} f(k\pi + h) = f(k\pi)$ for continuity at $x = k\pi$ $(k \in I)$
Now, $\lim_{h\to 0} f(k\pi + h)$
$$= \begin{cases} \lim_{h\to 0} 0, & \text{if } k\pi + h \text{ is irrational} \\ \lim_{h\to 0} \sin|k\pi + h|, & \text{if } k\pi + h \text{ is rational} \end{cases}$$
$= 0$ [$\because \sin|k\pi| = 0$ as $k \in I$]
And $f(k\pi) = 0$ [$\because k\pi$ is irrational as $k \in I$]

36. **(b, d)** : We have at the points $x = 2n$
$f(2n) = a_n + \sin 2n\pi = a_n$
Also for the L.H.L., we have
$$\text{L.H.L.} = \lim_{h\to 0}(b_n + \cos\pi(2n - h)) = b_n + 1$$
$$\text{R.H.L.} = \lim_{h\to 0}(a_n + \sin\pi(2n + h)) = a_n$$
For continuity $b_n + 1 = a_n \Rightarrow a_n - b_n = 1$
Again at $x = 2n + 1$
$$\text{L.H.L.} = \lim_{h\to 0}(a_n + \sin(\pi(2n + 1 - h))) = a_n$$
$$\text{R.H.L.} = \lim_{h\to 0}(b_{n+1} + \cos(\pi(2n + 1 + h))) = b_{n+1} - 1$$
Also $f(2n + 1) = a_n$
For continuity $a_n = b_{n+1} - 1 \Rightarrow \quad a_{n-1} - b_n = -1$

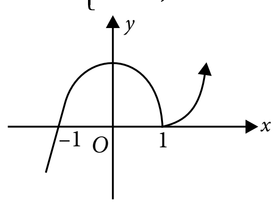
37. **(a, b, d)** : The function $u = \frac{1}{x-1}$ is discontinuous at $x = 1$
$f(x) = \frac{1}{(u+1)(u-2)}$ is discontinuous at $u = -1, 2$
i.e., at $x = 0, \frac{3}{2}$
Also we have, $\lim_{x\to 0} f(x) = \lim_{u\to -1} f(x) = \infty$
$$\lim_{x\to \frac{3}{2}} f(x) = \lim_{u\to 2} f(x) = \infty$$
$$\lim_{x\to 1} f(x) = \lim_{u\to\infty} f(x) = 0$$

38. **(b, c)** : $f(x) = x^3 + 3x + 2$, $g(f(x)) = x$, $h(g(g(x))) = x$
Now, $f'(x) = 3x^2 + 3$
Again, $g'(2) = \frac{1}{f'(0)} = \frac{1}{3}$ (As we have $f(0) = 2$)
Now, $h(g(g(x))) = x$
Differentiating with respect to x,
$$h'(g(g(x))) = \frac{1}{g'(g(x))\, g'(x)}$$
Now, to solve $g(g(x)) = 1$ we have $g(x) = f(1) = 6$
$f(6) = 236$
$$\therefore \quad h'(1) = \frac{1}{g'(6)g'(236)} = \frac{1}{\frac{1}{6}\cdot\frac{1}{111}} = 666$$
Solving, $g(g(x)) = 0$ means $g(x) = g^{-1}(0)$.
$\Rightarrow \quad g(x) = 2$
$\therefore \quad x = g^{-1}(2) = f(2) = 16$
$\therefore \quad h(0) = 16$
Again, $h(g(g(x))) = x$
Put x to $f(x)$ then $h(g(g(f(x)))) = f(x)$
$\Rightarrow \quad h(g(x)) = f(x)$
$\therefore \quad h(g(3)) = f(3) = 38$

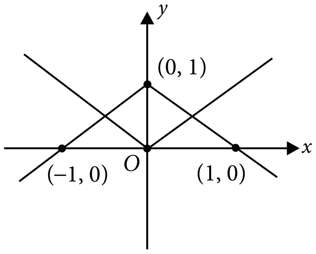
39. **(a) : (A) – (r, s), (B) – (p, q), (C) – (p, q), (D) – (p, r)**
(A) The given function is clearly continuous at all points except possibly at $x = \pm 1$
As $f(x)$ is an even function, so we need to check its continuity only at $x = 1$.
$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$$
$$\Rightarrow \lim_{x\to 1^-}(ax^2 + b) = \lim_{x\to 1^+}\frac{1}{|x|} \Rightarrow a + b = 1$$
or $\quad a = 1 - b$ ...(i)
Clearly, $f(x)$ is differentiable for all x, except possibly at $x = \pm 1$. As $f(x)$ is even function, so we need to check its differentiability at $x = 1$ only.
$$\lim_{x\to 1^-}\frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1^+}\frac{f(x) - f(1)}{x - 1}$$
$$\Rightarrow \lim_{x\to 1}\frac{ax^2 + b - 1}{x - 1} = \lim_{x\to 1^+}\frac{\frac{1}{|x|} - 1}{x - 1}$$
$$\Rightarrow \lim_{x\to 1}\frac{ax^2 - a}{x - 1} = \lim_{x\to 1}\frac{-1}{1}$$
$$\Rightarrow 2a = -1 \Rightarrow a = \frac{-1}{2} \text{ and } b = \frac{3}{2} \text{ [from (i)]}$$
(B) $f(x) = \text{sgn}(x^2 - ax + 1)$ is discontinuous then $x^2 - ax + 1 = 0$ must have only one real root. Hence, $a = \pm 2$.
(C) $f(x) = [2 + 3|n| \sin x]$, $n \in N$ has exactly 11 points of discontinuity in $x \in (0, \pi)$
$2(3)|n| - 1 = 11 \Rightarrow 6|n| = 12 \Rightarrow n = \pm 2$
($\because f(x) = [a + b\sin x]$ $0 < x < \pi$ $\forall$ $a \in I$ has $2b - 1$ points of non-differentiability)
(D) If $f(x) = ||x| - 2| + a|$ has exactly three points of non-differentiability.
$f(x)$ is non-differentiable at $x = 0$, $|x| - 2 = 0$
or $\quad x = 0, \pm 2$.
Hence, the value of a must be positive, as negative value of a allows $||x| - 2| + a = 0$ to have real roots, which gives two more points of non-differentiability.

40. **(a) : (A) – (p, q), (B) – (p, s), (C) – (r, s), (D) – (p, q)**

(A) $f(x) = (x + 1)|x - 1| = \begin{cases} x^2 - 1; & x \geq 1 \\ 1 - x^2; & x < 1 \end{cases}$



(B) $f(x) = \min\{|x|, 1 - |x|\}$



(C) $f(x) = \{x\} + 2[x] = x + [x]$ at $x = I$

L.H.L. $= I + (I - 1) = 2I - 1$

R.H.L. $= I + I = 2I = f(I)$.

$\therefore$ Not continuous hence not differentiable at integral points but increasing.

(D) $f(x) = \sqrt{\cos^2 \frac{\pi x}{2}} = \left|\cos \frac{\pi x}{2}\right|$ is continuous and differentiable

41. **(d) :** For any $x \neq 0$, $-1 \leq \sin\frac{1}{x} \leq 1$, but as $x \to 0$

$\sin\frac{1}{x}$ does not approach any particular value but oscillates between –1 and 1

42. **(a) :** $\lim\limits_{x\to 1^+} f(x) = 0 = \lim\limits_{x \to 1^-} f(x)$

$\therefore$ $f(x)$ is continuous at $x = 0$

43. **(a) :** As $x < 1$, $\lim\limits_{n\to\infty} x^{2n} = 0$

$$\text{So, } \lim_{x\to 1^-} f(x) = \frac{\log(2+1) - 0}{1+0} = \log 3$$

As $x > 1$, $\lim\limits_{n\to\infty} x^{2n} = \infty$

$$\text{So, } \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}\left[\lim_{n\to\infty} \frac{\dfrac{\log(2+x)}{x^{2n}} - \sin x}{\dfrac{1}{x^{2n}} + 1}\right] = -\sin 1$$

$$\text{Also, } f(1) = \frac{\log 3 - \sin 1}{2}$$

Clearly, $f(x)$ does not have removable discontinuity at $x = 1$.

44. **(c) :** By given fact, $\dfrac{x \sin|x|}{1 - |x|^2}$ is differentiable at $x = 0$ but it is certainly not continuous at $x = 1$ and $x = -1$

$\Rightarrow$ not differentiable at $x = 1$ and $x = -1$

45. **(b) :** $h(x) = \sin x\,|\sin x|$ as $f(x) = |x|$ and $g(x) = \sin x$

$f(g(x)) = f(\sin x) = |\sin x|$

Clearly, $h(x)$ is differentiable everywhere.

As $h(x)$ is the product of two continuous differentiable functions.

46. **(1) :** Let $u = f(\tan x)$

$$\frac{du}{dx} = f'(\tan x)\sec^2 x$$

and $v = g(\sec x)$

$$\frac{dv}{dx} = g'(\sec x)\sec x \tan x$$

$$\text{Now } \frac{du}{dv} = \frac{f'(\tan x)\sec^2 x}{g'(\sec x)\sec x \tan x}$$

$$\Rightarrow \left(\frac{du}{dv}\right)_{x=\frac{\pi}{4}} = \frac{f'(1)2}{g'(\sqrt{2})\sqrt{2}} = \frac{2\cdot 2\sqrt{2}}{4\sqrt{2}} = 1$$

47. **(1) :** $f = g^{-1}$ *i.e.* $g^{-1}(x) = f(x)$

where $g(x) = y = 2\tan^{-1}(e^x) - \dfrac{\pi}{2}$

$$\Rightarrow \tan^{-1}(e^x) = \frac{\pi}{4} + \frac{y}{2}$$

$$\Rightarrow x = \log\tan\left(\frac{\pi}{4} + \frac{y}{2}\right)$$

$$\Rightarrow g^{-1}(y) = \log\tan\left(\frac{\pi}{4} + \frac{y}{2}\right)$$

$$\Rightarrow g^{-1}(x) = f(x) = \log\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$$

$$\Rightarrow f'(x) = \frac{1}{\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)}\cdot\sec^2\left(\frac{\pi}{4} + \frac{x}{2}\right) \times \frac{1}{2}$$

$$f'(0) = \frac{1}{\tan\frac{\pi}{4}} \times \sec^2\frac{\pi}{4} \times \frac{1}{2} = 1$$

48. **(7) :** $1 = f(0) = \lim\limits_{x\to 0} f(x)$

$$= \lim_{x\to 0} \frac{x\left(1 + a\left(1 - \frac{x^2}{\underline{|2}} + ...\right)\right) - b\left(x - \frac{x^3}{\underline{|3}} + ...\right)}{x^3}$$

$$= \lim_{x\to 0} \frac{x(1+a-b) + x^3\left(\frac{-a}{2} + \frac{b}{6}\right) + x^5(\lambda) + ...}{x^3}$$

$$\Rightarrow \quad 1 + a - b = 0 \text{ and } \frac{-a}{2} + \frac{b}{6} = 1$$

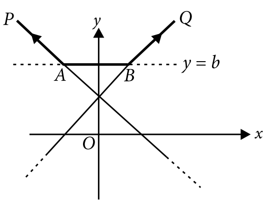
$$\Rightarrow \quad a = \frac{-5}{2}, b = \frac{-3}{2} \text{ and } 2a - 8b = 7$$

49. **(2) :** Graph of

$f(x) = \max\{a - x, a + x, b\}$

where $0 < a < b$ is given ($PABQ$).

As there are 2 steep corners (at A and B), $f(x)$ is not differentiable at these 2 points.



50. **(4) :** $y' \cdot 2\sqrt{x} = e^{\sqrt{x}} - e^{-\sqrt{x}}$

$$\Rightarrow \quad y'' \cdot 2\sqrt{x} + 2y'\frac{1}{2\sqrt{x}} = \frac{e^{\sqrt{x}} + e^{-\sqrt{x}}}{2\sqrt{x}}$$

$\Rightarrow$ $4xy'' + 2y' - y = 0$ $\quad\therefore k = 4$

JEE MATHEMATICS

DPS-16

DAILY PRACTICE SHEET

Class XII

Application of Derivatives

SOLUTIONS

1. **(a) :** $\frac{dV}{dt} = -72\pi \text{ m}^3/\text{min}, \; V_0 = 4500\pi$

$V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt} = \frac{4}{3}\pi \times 3r^2 \times \frac{dr}{dt}$ ...(i)

After 49 mins, $V = V_0 + 49 \cdot \frac{dV}{dt} = 4500\pi - 49 \times 72\pi$

$= 4500\pi - 3528\pi = 972\pi$

$\therefore \quad 972\pi = \frac{4}{3}\pi r^3 \Rightarrow r^3 = 243 \times 3 = 3^6 \Rightarrow r = 9$

From (i), $-72\pi = 4\pi \times 81 \times \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = -\frac{18}{81} = -\frac{2}{9}$

Thus, radius decreases at the rate of $\frac{2}{9}$ m/min.

2. **(d) :** Let V be the velocity.

$S \propto \sqrt[3]{V} \Rightarrow KS^3 = V$ (where K is any constant)

$\Rightarrow \frac{dS}{dt} = KS^3$ ...(i)

$\Rightarrow \frac{d^2S}{dt^2} = 3KS^2 \frac{dS}{dt} = 3KS^2(KS^3)$ (from (i))

$= 3K^2S^5$

So acceleration $\propto S^5$

3. **(a) :** We have, $16x^2 + 9y^2 = 400$ ...(i)

Given that $\frac{dy}{dt} = -\frac{dx}{dt}$ ...(ii)

Differentiating (i) w.r.t. t, we get

$16x \cdot \frac{dx}{dt} = 9y \cdot \frac{dx}{dt}$ [using (ii)] $\Rightarrow x = \frac{9y}{16}$

From (i), $16 \cdot \frac{81y^2}{16 \times 16} + 9y^2 = 400 \Rightarrow y = \pm\frac{16}{3}$

When $y = \frac{16}{3}, \; x = 3$; when $y = -\frac{16}{3}, \; x = -3$

$\therefore$ Required points are $\left(3, \frac{16}{3}\right)$ and $\left(-3, -\frac{16}{3}\right)$

4. **(d) :** Let v and r be the volume and radius of spherical iron ball respectively.

$\frac{dv}{dt} = 100\pi \text{ cm}^3/\text{min}$

$v = \frac{4}{3}\pi r^3$

$\Rightarrow \frac{dv}{dt} = \frac{4}{3}\pi 3r^2 . \frac{dr}{dt}$

$\Rightarrow \frac{dv}{dt} = \frac{4}{3}\pi 3(r + \Delta r)^2 . \frac{dr}{dt}$

$\Rightarrow 100\pi = \frac{4}{3}\pi 3(10+5)^2 \times \frac{dr}{dt}$

$\Rightarrow \frac{dr}{dt} = \frac{100\pi \times 3}{4\pi \times 225 \times 3} = \frac{1}{9}$

5. **(d) :** Slant height of cone (l) = 7 cm

$l^2 = h^2 + r^2$ ...(i)

$\Rightarrow r^2 = 7^2 - 4^2 = 33 \Rightarrow r = \sqrt{33}$ cm (when h = 4 cm)

Now, differentiating equation (i) w.r.t 't'

$\Rightarrow 0 = 2h\frac{dh}{dt} + 2r\frac{dr}{dt} \Rightarrow \frac{dr}{dt} = -\frac{h}{r}\frac{dh}{dt}$ ...(ii)

Volume of the cone, $V = \frac{1}{3}\pi r^2 h$

$\Rightarrow \frac{dV}{dt} = \frac{1}{3}\pi\left[2rh\frac{dr}{dt} + r^2\frac{dh}{dt}\right]$

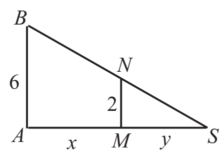
$\Rightarrow \frac{dV}{dt} = \frac{1}{3}\pi\left[2rh\left(\frac{-h}{r}\frac{dh}{dt}\right) + r^2\frac{dh}{dt}\right]$ (From (ii))

$\Rightarrow \frac{dV}{dt} = \frac{1}{3}\pi\frac{dh}{dt}\left[-2h^2 + r^2\right]$

$\Rightarrow \left[\frac{dV}{dt}\right]_{h=4} = \frac{1}{3}\pi(0.3)\left[-2 \times 4^2 + \left(\sqrt{33}\right)^2\right]$

$= \frac{\pi}{10}$ cc/sec

6. **(c) :** Given, $\frac{dx}{dt} = 6$ km/ hr



$\Delta SAB \sim \Delta SMN$

$\therefore \frac{AS}{MS} = \frac{AB}{MN} = \frac{6}{2} = 3$

$\Rightarrow \frac{x+y}{y} = 3 \Rightarrow x + y = 3y \Rightarrow x = 2y$

$\Rightarrow \frac{dx}{dt} = 2\frac{dy}{dt} \Rightarrow 6 = 2\frac{dy}{dt} \Rightarrow \frac{dy}{dt} = 3$ km/hr.

7. **(b) :** $f'(x) = \frac{\ln(e+x)^{e+x} - \ln(\pi+x)^{\pi+x}}{(\pi+x)(e+x)(\ln(e+x))^2} < 0$

Since $e < \pi$, $(e+x)^{e+x} < (\pi+x)^{\pi+x}$.

8. **(d) :** Since $f(x)f'(x) < 0$

$\therefore f(x)$ and $f'(x)$ must be of opposite sign.

(i) Let $f(x) = e^{-x} \Rightarrow f'(x) = -e^{-x}$

$\Rightarrow f(x) > 0, f'(x) < 0$ for all $x \in R$

(ii) Let $f(x) = -e^{-x} \Rightarrow f'(x) = e^{-x}$

$\Rightarrow f(x) < 0, f'(x) > 0$ for all $x \in R$

But $|f(x)| = |\pm e^{-x}| = e^{-x}$ in both cases

$\therefore \frac{d}{dx}|f(x)| = -e^{-x} < 0$ in both cases for all $x \in R$

9. **(c) :** $g'(x) = f'\left(\frac{x}{2}\right) - f'(2-x) > 0$ if $f'\left(\frac{x}{2}\right) > f'(2-x)$

$\Rightarrow \quad \frac{x}{2} < 2 - x$ [Because $f'(x)$ is a decreasing function $(f''(x) < 0)$]

$\therefore \quad x < 4 - 2x \Rightarrow x < \frac{4}{3}$. If $x \in \left(0, \frac{4}{3}\right)$, $g(x)$ increases and decreases in $\left(\frac{4}{3}, 2\right)$.

10. **(a) :** We have, $f(x) = \sin x - x$

which is decreasing function in $\left[\frac{\pi}{4}, \frac{\pi}{3}\right]$ $\quad (\because f'(x) \le 0)$

As $\quad \frac{\pi}{4} \le x \le \frac{\pi}{3}$

$$\Rightarrow \quad f\left(\frac{\pi}{4}\right) \ge f(x) \ge f\left(\frac{\pi}{3}\right) \Rightarrow \frac{1}{\sqrt{2}} - \frac{\pi}{4} \ge f(x) \ge \frac{\sqrt{3}}{2} - \frac{\pi}{3}$$

$$\therefore \quad f(A) \in \left[\frac{\sqrt{3}}{2} - \frac{\pi}{3}, \frac{1}{\sqrt{2}} - \frac{\pi}{4}\right]$$

11. **(a) :** $\lim_{x\to 0} \frac{f(x^2) - f(x)}{f(x) - f(0)} = \lim_{x\to 0} \frac{f(x^2) - f(0) - f(x) + f(0)}{f(x) - f(0)}$

$$= \lim_{x\to 0}\left\{\frac{f(x^2) - f(0)}{f(x) - f(0)} - 1\right\} = \lim_{x\to 0}\left\{x\frac{f(x^2) - f(0)}{x^2} \times \frac{x}{f(x) - f(0)} - 1\right\}$$

$$= f'(0) \times \frac{1}{f'(0)} \lim_{x\to 0}(x - 1) = -1$$

12. **(a) :** $h'(x) = \sin 2x\,[f'(\sin^2 x) - f'(\cos^2 x)]$

Since $f''(x) < 0 \therefore f'(x)$ is an decreasing function.

$x > 0 \Rightarrow f'(\sin^2 x) > f'(\cos^2 x)$

$\Rightarrow \quad \sin^2 x < \cos^2 x$ (since f' decreases)

$\Rightarrow \quad x \in \left(0, \frac{\pi}{4}\right)$

13. **(d) :** $\frac{d}{dx}\left(\frac{\tan x}{x}\right) = \frac{x\sec^2 x - \tan x}{x^2} = \frac{2x - \sin 2x}{2x^2\cos^2 x} > 0$

$\therefore \quad \frac{\tan x}{x}$ increases. $\frac{d}{dx}\left(\frac{\sin x}{x}\right) = \frac{x\cos x - \sin x}{x^2} = \frac{x - \tan x}{x^2 \sec x} < 0$

$\therefore \quad f(x) = \frac{\sin x}{x}$ decreases

$\therefore \quad \sin x < \tan x \Rightarrow f(\tan x) < f(\sin x)$

$\therefore \quad \frac{\sin(\tan x)}{\tan x} < \frac{\sin(\sin x)}{\sin x}$

$\Rightarrow \quad \cos x \sin(\tan x) < \sin(\sin x)$.

14. **(b) :** The given curve is $x^2 + 2xy - 3y^2 = 0$

Factorizing, it becomes $(x - y)(x + 3y) = 0$

Normal at (1, 1) is $x + y = \lambda$ *i.e.*, $1 + 1 = \lambda \Rightarrow \lambda = 2$

Thus the equation is $x + y = 2$

Obviously $x + 3y = 0$ doesn't have the point (1, 1) on it.

Now, $x + y = 2$ meets $x + 3y = 0$ in the point (3, –1) obtained by solving the system of linear equations. Hence the point is in the 4th quadrant.

15. **(d) :** We have, $x = 4t^2 + 3$, $y = 8t^3 - 1$

$\therefore \quad P \equiv (4t^2 + 3, 8t^3 - 1)$

Now, $\frac{dx}{dt} = 8t$ and $\frac{dy}{dt} = 24t^2$

Slope of tangent at $P = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = 3t$

Let $Q \equiv (4\lambda^2 + 3, 8\lambda^3 - 1)$. Slope of $PQ = 3t$

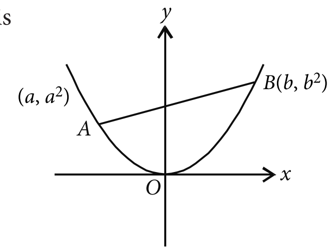
$$\Rightarrow \quad \frac{8t^3 - 8\lambda^3}{4t^2 - 4\lambda^2} = 3t \Rightarrow \frac{8(t-\lambda)(t^2 + \lambda^2 + t\lambda)}{4(t-\lambda)(t+\lambda)} = 3t$$

$\Rightarrow \quad t^2 + t\lambda - 2\lambda^2 = 0 \Rightarrow (t - \lambda)(t + 2\lambda) = 0$

$\Rightarrow \quad t = \lambda$ or $\lambda = \frac{-t}{2} \quad \therefore \; Q \equiv [t^2 + 3, -t^3 - 1]$

16. **(a) :** For $y = x^2$, any point on it is $A(a, a^2)$.

Equation of the normal at (a, a^2) is



$$y - a^2 = -\frac{1}{2a}(x - a)$$

This meets $y = x^2$ again at $B\,(b, b^2)$ (say), then

$b^2 - a^2 = (-1/2a)(b - a)$

$$\Rightarrow b = -\left(a + \frac{1}{2a}\right)$$

$\therefore \quad y = AB^2 = (a - b)^2 + (a^2 - b^2)^2$

$$= \left(2a + \frac{1}{2a}\right)^2\left(1 + \frac{1}{4a^2}\right) = \left(4a^2 + \frac{1}{4a^2} + 2\right)\left(1 + \frac{1}{4a^2}\right)$$

$$\Rightarrow y = \left(\frac{1}{x} + x + 2\right)(1 + x) \text{ where } x = \frac{1}{4a^2}$$

$$= \frac{1}{x} + 3 + 3x + x^2$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{x^2} + 3 + 2x \text{ and } \frac{d^2y}{dx^2} = \frac{2}{x^3} + 2 > 0$$

Now $\frac{dy}{dx} = 0 \Rightarrow 2x^3 + 3x^2 - 1 = 0 \Rightarrow x = \frac{1}{2}, -1$

Reject $x = -1$ as $x = \frac{1}{4a^2} > 0$

$\therefore \quad$ The two required normals are given by

$$y = \frac{x}{\sqrt{2}} + 1 \text{ and } y = \frac{-x}{\sqrt{2}} + 1$$

17. **(b) :** Slope of the tangent at $(a, b) = -\frac{a^2}{b^2}$

Tangent at (a, b) is $a^2x + b^2y = a^3 + b^3$

It passes through (p, q)

$\Rightarrow \quad a^2p + b^2q = a^3 + b^3$ ...(i)

Also, $p^3 + q^3 = a^3 + b^3$ ...(ii)

If $p = a\alpha$, $q = b\beta$, (i), (ii) $\Rightarrow (\alpha - 1)a^3 + (\beta - 1)b^3 = 0$

$(\alpha^3 - 1)a^3 + (\beta^3 - 1)b^3 = 0$

Eliminating a^3, b^3, we have

$$\frac{\alpha^3 - 1}{\alpha - 1} = \frac{\beta^3 - 1}{\beta - 1}$$

$\alpha^2 + \alpha + 1 = \beta^2 + \beta + 1 \Rightarrow \alpha + \beta + 1 = 0$

$\Rightarrow \quad \frac{p}{a} + \frac{q}{b} + 1 = 0$, $bp + aq + ab = 0$.

18. **(c) :** $x = t^2, y = t \Rightarrow \frac{dy}{dx} = \frac{1}{2t}$

Normal : $\frac{1}{2t}(y - t) + x - t^2 = 0$. It passes through (c, 0)

$\Rightarrow t^2 + \frac{1}{2} - c = 0$

t_1, t_2 are roots $\Rightarrow t_1t_2 = \frac{1}{2} - c$, slopes $m_1 = \frac{1}{2t_1}, m_2 = \frac{1}{2t_2}$

$m_1m_2 = -1 \Rightarrow \frac{1}{4t_1t_2} = -1 = 2 - 4c \Rightarrow c = \frac{3}{4}$.

19. **(a) :** Let $y = x^{1/3}$

Let $x = 8$ and $\Delta x = 0.005$

$\therefore \quad y = (8)^{1/3} = 2$

Now, $\frac{dy}{dx} = \frac{1}{3}x^{-2/3} = \frac{1}{3}.8^{-2/3} = 0.08333$

$\therefore \quad \Delta y = \frac{dy}{dx}.\Delta x = 0.0833 \times 0.005 = 0.0004167$

$\therefore \quad y - \Delta y = 2 - 0.0004167 = 1.9995$

20. **(c) :** Let $P_1(t_1, t_1^3), P_2(t_2, t_2^3), \ldots$

Tangent at P_1: $y - t_1^3 = 3t_1^2(x - t_1)$

It passes through P_2: $t_2^3 - t_1^3 = 3t_1^2(t_2 - t_1)$

$\Rightarrow \quad t_2 = -2t_1, t_3 = 4t_1, t_4 = -8t_1$

$$f(t_1) = \begin{vmatrix} t_1 & t_1^3 & 1 \\ -2t_1 & -8t_1^3 & 1 \\ 4t_1 & 64t_1^3 & 1 \end{vmatrix} = t_1^4 \begin{vmatrix} 1 & 1 & 1 \\ -2 & -8 & 1 \\ 4 & 64 & 1 \end{vmatrix}$$

$= -162t_1^4, f(t_2) = -162t_2^4$

$$\frac{\text{Area of } \Delta P_1P_2P_3}{\text{Area of } \Delta P_2P_3P_4} = \left(\frac{t_1}{t_2}\right)^4 = \left(-\frac{1}{2}\right)^4 = \frac{1}{16}$$

21. **(c) :** Equation of tangent to $y^2 = 4ax$ at $(at^2, 2at)$ is

$y \cdot 2at = 2a(x + at^2) \Rightarrow ty = x + at^2$ ...(i)

For $x^2 - y^2 = a^2$, $\frac{dy}{dx} = \frac{x}{y}$

$\therefore \quad$ At $(a \sec\theta, a \tan\theta)$, $\frac{dy}{dx} = \frac{a\sec\theta}{a\tan\theta} = \frac{1}{\sin\theta}$

$\therefore \quad$ Eqn. of normal to $x^2 - y^2 = a^2$ at $(a \sec\theta, a \tan\theta)$ is

$y - a\tan\theta = -\sin\theta(x - a\sec\theta)$

$\Rightarrow \quad y = -x\sin\theta + 2a\tan\theta$ ...(ii)

$\because \quad$ (i) & (ii) are identical

$\therefore \quad \frac{t}{1} = \frac{1}{-\sin\theta} = \frac{at^2}{2a\tan\theta} = \frac{t^2}{2\tan\theta}$

$\Rightarrow \quad t = -\text{cosec}\,\theta \quad \& \quad t = 2\tan\theta$

22. **(b) :** We have, $y^2(2 - x) = x^3$

Differentiating with respect to x

$\Rightarrow \quad (2-x)2y\frac{dy}{dx} + y^2(-1) = 3x^2$

$\Rightarrow \quad (2-x)2y\frac{dy}{dx} = 3x^2 + y^2 \Rightarrow \frac{dy}{dx} = \frac{3x^2+y^2}{4y-2xy}$

$\left[\frac{dy}{dx}\right]_{(1,1)} = \frac{3(1)^2+(1)^2}{4(1)-2(1)(1)} = \frac{4}{2} = 2$

Equation of normal at (1, 1) is

$(y-1) = \frac{1}{-2}(x-1) \Rightarrow x + 2y - 3 = 0$

$x + 2y - 3 = 0$ cuts the x-axis at (3,0)

$\therefore \quad$ Required length $= \sqrt{(3-1)^2+(1-0)^2} = \sqrt{5}$

23. **(d) :** The length of the sub-tangent, ordinate and sub-normal are in G.P. $\left(\frac{y_1}{m}, y_1, y_1m\right)$.

24. **(a) :** We have, $x^2y^2 - 2x = 4(1 - y)$

$\Rightarrow \quad x^2y^2 - 2x = 4 - 4y$

Differentiating both sides w.r.t x, we get

$2xy^2 + 2y \cdot x^2\frac{dy}{dx} - 2 = -4\frac{dy}{dx}$

$\Rightarrow \quad \frac{dy}{dx}(2yx^2+4) = 2 - 2xy^2 \Rightarrow \frac{dy}{dx} = \frac{2-2xy^2}{2yx^2+4}$

$\left.\frac{dy}{dx}\right|_{(2,-2)} = \frac{2-(2\times 2\times(-2)^2)}{(2(-2)\cdot 2^2)+4} = \frac{-14}{-12} = \frac{7}{6}$

$\therefore \quad$ Slope of tangent to the curve $= \frac{7}{6}$

Equation of tangent passes through (2, –2) is

$y + 2 = \frac{7}{6}(x-2) \Rightarrow 7x - 6y = 26$

$\therefore \quad$ Equation of tangent does not pass through (–2, –7).

25. **(a) :** We have, $f(x) = (x - p)^2 + (x - q)^2 + (x - r)^2$

$\Rightarrow \quad f'(x) = 2(x - p) + 2(x - q) + 2(x - r)$

$= 2(x - p + x - q + x - r) = 2(3x - p - q - r)$

For critical points, we have

$f'(x) = 0 \Rightarrow 2(3x - p - q - r) = 0 \Rightarrow 3x - p - q - r = 0$

$\Rightarrow \quad 3x = p + q + r \Rightarrow x = \frac{p+q+r}{3}$

So, $x = \frac{p+q+r}{3}$ is the critical point.

Now, we test the function at this point.

We have, $f''(x) = 2 \times 3 = 6$

At $x = \frac{p+q+r}{3}$, $f''\left(\frac{p+q+r}{3}\right) = 6 > 0$

$\Rightarrow \quad f(x)$ has a minimum at $x = \frac{p+q+r}{3}$ *i.e.*, $\lambda = \frac{p+q+r}{3}$

26. **(a) :** $\lim_{x\to 2}\frac{f(x)g(x)}{f'(x)g'(x)} = 1$

Using L'Hospital rule, we have

$$\lim_{x\to 2}\frac{f'(x)g(x)+f(x)g'(x)}{f''(x)g'(x)+g''(x)f'(x)} = \frac{f'(2)g(2)+g'(2)f(2)}{f''(2)g'(2)+g''(2)f'(2)} = 1$$

$\Rightarrow \quad \frac{g'(2)f(2)}{f''(2)g'(2)} = 1$

$\therefore \quad f''(2) = f(2) = +\text{ve}$

$\therefore \quad f(x)$ has minima at $x = 2$

Also $f(2) = f''(2)$.

$\therefore \quad f(x) = f''(x)$ has at least one solution in $x \in R$.

27. **(b) :** $f(x) = \sin(\tan x) - x$, $f(0) = 0$

$f'(x) = \cos(\tan x)(\tan^2 x + 1) - 1$

$= \cos(\tan x)\tan^2 x + \cos(\tan x) - 1$

$= \cos(\tan x)\tan^2 x - \frac{\tan^2 x}{2} = \tan^2 x\left(\cos(\tan x) - \frac{1}{2}\right)$

[Using expansion of $\cos x = \left[1 - \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} \ldots\infty\right]$

neglecting higher terms]

$0 \le \tan x \le 1 < \frac{\pi}{3} \Rightarrow \cos(\tan x) > \cos\frac{\pi}{3} = \frac{1}{2}$

$\therefore \quad f'(x) > 0$.

$f(0) = 0 \Rightarrow f(x) \ge 0$, $\sin(\tan x) \ge x$.

28. **(b) :** $f'(x) = 0$ at $x = 1, 2, 3, 4, 5$

The presence of even powers for $(x - 2)$ and $(x - 4)$ imply $x = 2, 4$ are inflection points.

$f'(x)$ is a polynomial $\Rightarrow f(x)$ is a polynomial tending to ∞ as $x \to \infty$

$\therefore$ $x = 5$ is minimum point since maximum and minimum occur alternatively for a polynomial $x = 3$ is maximum point.

$x = 1$ is minimum point.

$\therefore$ $f(x)$ has minimum at $x = 1, 5$ maximum at $x = 3$ and inflection points at $x = 2, 4$.

29. **(c) :** $P'(x) = 4x^3 + 3ax^2 + 2bx + c$

$P'(0) = 0 \Rightarrow c = 0$, $P'(x) = x(4x^2 + 3ax + 2b)$

$4x^2 + 3ax + 2b = 0$ has no real roots

$\therefore$ $4x^2 + 3ax + 2b > 0$

$P'(x) > 0$ in $[0, 1]$ and $P'(x) < 0$ in $[-1, 0]$

$P_{max} = \max.\{P(-1), P(1)\} = P(1)$, $P_{min} = P(0)$.

30. **(d) :** $f(x) = \ln|x| + bx^2 + ax, x \neq 0$ has extreme values at $x = -1, x = 2$.

$$f'(x) = \frac{1}{x} + 2bx + a$$

$\therefore$ $f'(-1) = 0$ and $f'(2) = 0$

$$\Rightarrow -1 - 2b + a = 0 \text{ and } \frac{1}{2} + 4b + a = 0$$

$$\Rightarrow a = \frac{1}{2} \text{ and } b = -\frac{1}{4}$$

$$f''(x) = -\frac{1}{x^2} + 2b = -\frac{1}{x^2} - \frac{1}{2} = -\left(\frac{1}{x^2} + \frac{1}{2}\right) < 0$$

for all $x \in R - \{0\}$

$\Rightarrow$ f has a local maximum at $x = -1, x = 2$

31. **(b, c, d) :** $f(x) = x\cos\frac{1}{x},\ x \geq 1$

$$f'(x) = \cos\frac{1}{x} + \frac{1}{x}\sin\frac{1}{x} \to 1 \text{ as } x \to \infty$$

$$f''(x) = -\frac{1}{x^3}\cos\frac{1}{x} < 0 \text{ for } x \geq 1$$

So $f'(x)$ decreases on $[1, \infty)$.

In the interval $[x, x + 2]$ where $x \in [1, \infty)$, $f(x)$ is continuous and differentiable.

By Lagrange's Mean value theorem, $f'(x) = \dfrac{f(x+2) - f(x)}{2}$

As $f'(x) > 1$ so, $\dfrac{f(x+2) - f(x)}{2} > 1$

$\Rightarrow$ $f(x + 2) - f(x) > 2$.

32. **(b, d) :** $xy = 1$

$$\therefore \quad y = \frac{1}{x} \Rightarrow \frac{dy}{dx} = -\frac{1}{x^2}$$

$\therefore$ Slope of normal $= x_1^2 > 0$ at any point $P(x_1, y_1)$

If $(a - 1)x - by + 4 = 0$ is a normal

then its slope $= \dfrac{a-1}{b} > 0$

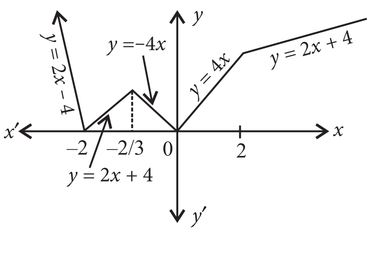
$\therefore$ $a - 1$ and b must be of same sign.

$\Rightarrow$ $a > 1, b > 0$ or $a < 1, b < 0$

33. **(a, b) :** We have

$f(x) = 2|x| + |x + 2| - ||x + 2| - 2|x||$

$$= \begin{cases} -2x - 4, & x \leq -2 \\ 2x + 4, & -2 < x \leq -2/3 \\ -4x, & -\frac{2}{3} < x \leq 0 \\ 4x, & 0 < x \leq 2 \\ 2x + 4, & x > 2 \end{cases}$$



It has local minima at $x = -2, 0$ and maxima at $-2/3$.

34. **(b, c) :** $f'(x) = (x - 1)^2(4 - x) = -x^3 + 6x^2 - 9x + 4$

On integrating, $f(x) = -\dfrac{x^4}{4} + 2x^3 - \dfrac{9x^2}{2} + 4x + C$

Putting $x = 0$, $f(0) = C$ (may or may not be zero)

$\because$ $f'(x) \geq 0$ in $(0, 3)$ $\therefore$ $f(x)$ is increasing in $(0, 3)$

$\because$ $f'(4) = 0$, $\therefore$ $x = 4$ is a critical point of $f(x)$

$\because$ $f'(x) > 0$ in $(3, 4)$ and $f'(x) < 0$ in $(4, 5)$

$\therefore$ We can't say that $f(x)$ is decreasing in $(3, 5)$.

35. **(b, d) :** We have, $4x^2 + 9y^2 = 1$ ...(i) & $8x = 9y$ ...(ii)

Differentiating (i) w.r.t. x, we get

$$8x + 18y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{4x}{9y}$$

$\Rightarrow$ Slope of tangent $= \dfrac{-4x}{9y}$. Also, slope of line (ii) $= \dfrac{8}{9}$

Since line (ii) is parallel to the tangent.

$$\therefore \quad \frac{-4x}{9y} = \frac{8}{9} \Rightarrow x = -2y$$

From (i), $4(4y^2) + 9y^2 = 1 \Rightarrow y^2 = \dfrac{1}{25} \Rightarrow y = \pm\dfrac{1}{5}$

When $y = \dfrac{1}{5}$, $x = -\dfrac{2}{5}$; when $y = -\dfrac{1}{5}$, $x = \dfrac{2}{5}$

$\therefore$ Points are $\left(-\dfrac{2}{5}, \dfrac{1}{5}\right)$ and $\left(\dfrac{2}{5}, -\dfrac{1}{5}\right)$

36. **(b, c) :** Let $g(x) = f'(x) - 2f(x)$

We have $e^{-2x}g(x) = f'(x)e^{-2x} - 2e^{-2x}f(x) = \dfrac{d}{dx}(e^{-2x}f(x))$

As $g(x) > 0$, we have $\dfrac{d}{dx}(e^{-2x}f(x)) > 0$

i.e. $e^{-2x}f(x)$ is an increasing function on R.

Now, $e^{-2x}f(x) > e^0 f(0)$ $\Rightarrow$ $e^{-2x}f(x) > 1$

Then, $f(x) > e^{2x}$

As $f'(x) > 2f(x)$ *i.e.* we have $f'(x) > 2e^{2x}$

Thus, $f'(x)$ is positive. So we can conclude that $f(x)$ is increasing on $(0, \infty)$

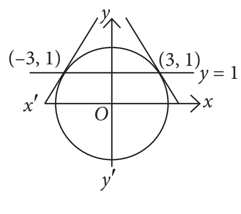
37. **(a, b) :** $|\sin x| + |\cos x| = \sqrt{1 + |\sin 2x|}$

So, $1 < |\sin x| + |\cos x| \leq \sqrt{2}$.

$y = [|\sin x| + |\cos x|] = 1$.

$x^2 + y^2 = 10$

$$\Rightarrow 2x + 2y\frac{dy}{dx} = 0$$



So, angle is either $\tan^{-1}(-3)$ or $\tan^{-1}(3)$.

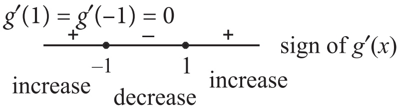
38. **(b, d) :** We have, $f(x) = x^5 - 5x + a$

$x^5 - 5x + a = 0 \Rightarrow x^5 - 5x = -a = b$ (say)

$g(x) = x^5 - 5x$

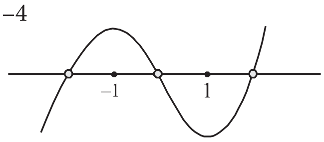
$g'(x) = 5x^4 - 5 = 5(x^4 - 1) = 5(x^2 + 1)(x - 1)(x + 1)$

$g'(1) = g'(-1) = 0$



Hence g has a maximum at $x = -1$ and a minimum at $x = 1$.

$g(-1) = 4$, $g(1) = -4$



Also, $x \to -\infty$, $g(x) \to -\infty$ and $x \to \infty$, $g(x) \to \infty$

For three real roots the line $g(x) = b$ should meet the curve at three points.

$\Rightarrow \quad -4 < b < 4 \;\Rightarrow\; -4 < -a < 4 \;\Rightarrow\; -4 < a < 4$

For one real root, $b < -4$ or $b > 4$

i.e., $-a < -4$ or $-a > 4$

i.e., $a > 4$ or $a < -4$

39. (a) : (A) – (r), (B) – (s), (C) – (q), (D) – (p)

(A) Tangent at $\left(t, \frac{1}{t}\right)$ is $y - \frac{1}{t} = \frac{-1}{t^2}(x - t)$

which meets the axes at $A(2t, 0)$ and $B\left(0, \frac{2}{t}\right)$

$\therefore \quad OA \times OB = 2t.\frac{2}{t} = 4$

(B) Tangent at $(\cos\theta, \sin\theta)$ is $x\cos\theta + y\sin\theta = 1$

which meets the axes at $A\left(\frac{1}{\cos\theta}, 0\right)$ and $B\left(0, \frac{1}{\sin\theta}\right)$

$\therefore \quad \frac{1}{OA^2} + \frac{1}{OB^2} = \cos^2\theta + \sin^2\theta = 1$

(C) The tangent at $(\cos^4\theta, \sin^4\theta)$ is $y - \sin^4\theta = -\frac{\sin^2\theta}{\cos^2\theta}(x - \cos^4\theta)$

which meets the axes at $A(\cos^2\theta, 0)$ and $B(0, \sin^2\theta)$

$\therefore \quad OA + OB = \cos^2\theta + \sin^2\theta = 1$

(D) The tangent at $(\cos^3\theta, \sin^3\theta)$ is $y - \sin^3\theta = -\frac{\sin\theta}{\cos\theta}(x - \cos^3\theta)$

which meets the axes at $A(\cos\theta, 0)$ and $B(0, \sin\theta)$

$\therefore \quad AB = \sqrt{\cos^2\theta + \sin^2\theta} = 1$

40. (b) : (A) – (s), (B) – (r), (C) – (p), (D) – (q)

Since the curve $y = ax^3 + bx^2 + cx + 5$ touches x-axis at $P(-2,0)$ then x-axis is the tangent at $(-2,0)$. The curve meets y-axis at $(0,5)$.

We have, $\frac{dy}{dx} = 3ax^2 + 2bx + c$

$\Rightarrow \left.\frac{dy}{dx}\right|_{(0,5)} = 0 + 0 + c = 3$ (given)

$\Rightarrow \quad c = 3$ ...(i)

and $\left.\frac{dy}{dx}\right|_{(-2,0)} = 0$

$\Rightarrow \quad 12a - 4b + 3 = 0$ ...(ii)

and $(-2,0)$ lies on the curve, then

$0 = -8a + 4b - 2c + 5$

$\Rightarrow \quad 0 = -8a + 4b - 1$

$\Rightarrow \quad 8a - 4b + 1 = 0$ ...(iii)

From (ii) and (iii), we get $a = -\frac{1}{2}, b = -\frac{3}{4}$.

Hence, $a = -\frac{1}{2}, b = -\frac{3}{4}$ and $c = 3$.

Also, $y'(1) = 0$

41. (a) : $f(x) = \frac{x^2 - ax + 1}{x^2 + ax + 1}$

$$f'(x) = \frac{(x^2 + ax + 1)(2x - a) - (x^2 - ax + 1)(2x + a)}{(x^2 + ax + 1)^2}$$

$$= \frac{2x^3 - ax^2 + 2ax^2 - a^2x + 2x - a - 2x^3 - ax^2 + 2ax^2 + a^2x - 2x - a}{(x^2 + ax + 1)^2}$$

$$= \frac{2ax^2 - 2a}{(x^2 + ax + 1)^2} = \frac{2a(x^2 - 1)}{(x^2 + ax + 1)^2}$$

Now,

$$f''(x) = \frac{(x^2 + ax + 1)^2\,[2a(2x)] - 2a(x^2 - 1)\,[2(x^2 + ax + 1)(2x + a)]}{(x^2 + ax + 1)^4}$$

$$= \frac{4(x^2 + ax + 1)ax - 4a(x^2 - 1)(2x + a)}{(x^2 + ax + 1)^3}$$

$$\therefore \; f''(1) = \frac{4a\,(1 + a + 1)}{(1 + a + 1)^3} = \frac{4a\,(2 + a)}{(2 + a)^3} = \frac{4a}{(2 + a)^2}$$

$$\text{And } f''(-1) = \frac{4\,(1 - a + 1)(-1)\,a}{(1 - a + 1)^3} = \frac{-4a\,(2 - a)}{(2 - a)^3} = \frac{-4a}{(2 - a)^2}$$

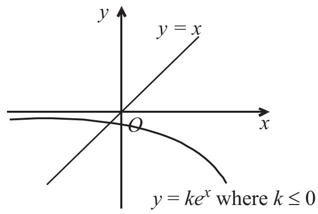
Now, $(2 + a)^2 f''(1) + (2 - a)^2 f''(-1)$

$$= (2 + a)^2 \cdot \frac{4a}{(2 + a)^2} + (2 - a)^2 \frac{(-4a)}{(2 - a)^2} = 0.$$

42. (a) : $f'(x) = 2a\frac{(x^2 - 1)}{(x^2 + ax + 1)^2} = 2a\frac{(x - 1)(x + 1)}{(x^2 + ax + 1)^2}$

It is easily seen that $f(x)$ decreases on $(-1, 1)$ and has a local minimum at $x = 1$, because the derivative changes its sign from –ve to +ve.

43. (b) : Clearly, $y = x$ meets $y = ke^x$ (when $k \le 0$) only once *i.e.* in the third quadrant.



44. (a) : $f(x) = ke^x - x \Rightarrow f'(x) = ke^x - 1$

$f'(x) = 0$

$\Rightarrow x = -\log k$

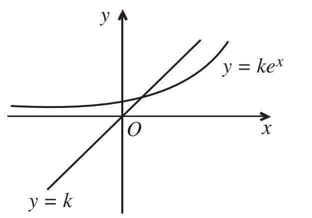
$f''(x) = ke^x > 0$

$\therefore$ Minima occurs at $x = -\log k$.

$\therefore \; f(x)_{\min} = 1 + \log k$

For only one root of $f(x)$,

$f(x)_{\min} = 0$, *i.e.*, $k = \frac{1}{e}$



45. (a) : For 2 distinct roots of $f(x) = ke^x - x$, $f(x)_{\min} < 0$

$\Rightarrow \quad 1 + \ln k < 0$ or $0 < k < \frac{1}{e} \qquad \therefore \; k \in \left(0, \frac{1}{e}\right)$

46. (1) : $g(x) = e^{f(x)}$, $g'(x) = e^{f(x)} f'(x)$

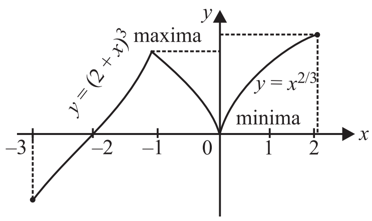
$\therefore \quad g'(x) = e^{f(x)} \times 2010(x - 2009)(x - 2010)^2(x - 2011)^3(x - 2012)^4$

g has points of inflection at $x = 2010$, $x = 2012$

g has minimum at $x = 2011$

g has maximum at $x = 2009$.

47. (2) : $f(x) = \begin{cases} (2 + x)^3, & -3 < x \le -1 \\ x^{2/3}, & -1 < x < 2 \end{cases}$



One can easily graph the function. It is an example of a piecewise continuous function.
The function has a local maxima at $x = -1$ and a local minima at $x = 0$. Thus the total number of local maxima and local minima of the function is 2.

48. **(1) :** The given curve is $\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}=1$ ...(i)

Differentiating (i) w.r.t. x, we get

$$\frac{1}{\sqrt{a}}\times\frac{1}{2\sqrt{x}}+\frac{1}{\sqrt{b}}\times\frac{1}{2\sqrt{y}}\frac{dy}{dx}=0\Rightarrow\frac{dy}{dx}=-\frac{\sqrt{by}}{\sqrt{ax}}$$

$$\left[\frac{dy}{dx}\right]_{(x_1,y_1)}=-\frac{\sqrt{by_1}}{\sqrt{ax_1}}$$

Equation of tangent is $(y-y_1)=-\frac{\sqrt{by_1}}{\sqrt{ax_1}}(x-x_1)$

$$\Rightarrow \frac{x}{\sqrt{ax_1}}+\frac{y}{\sqrt{by_1}}=\frac{y_1}{\sqrt{by_1}}+\frac{x_1}{\sqrt{ax_1}}=1\Rightarrow k=1$$

49. **(4) :** Let $g(x) = f'(x) \sin(\pi f(x))$

$g(a) = f'(a)\sin(\pi f(a)) = f'(a) \sin 0 = 0$

Similarly $g(b) = 0$

$$f'\left(\frac{a+b}{2}\right)=0$$

$$\therefore \quad g\left(\frac{a+b}{2}\right)=0$$

Also $f(\alpha) = f(\beta) = 0$

So, according to Rolle's theorem,

$g'(x) = 0$ has at least one root in (α, a) $\left(a,\frac{a+b}{2}\right),\left(\frac{a+b}{2},b\right)$ and (b, β).

i.e., equation has minimum 4 roots.

50. **(7) :** $x = t^2 + 3t - 8$ and $y = 2t^2 - 2t - 5$

$\therefore \quad \frac{dx}{dt} = 2t + 3$ and $\frac{dy}{dt} = 4t - 2$

Now, $\frac{dy}{dx}=\frac{4t-2}{2t+3}$

At the point $(2, -1)$, $t = 2$

$\therefore \quad \left[\frac{dy}{dx}\right]_{t=2}=\frac{8-2}{4+3}=\frac{6}{7}$ $\quad\therefore$ Slope of normal $= -\frac{7}{6}$

Hence, $k = 7$

JEE MATHEMATICS

DPS-17

DAILY PRACTICE SHEET

Class XII

Indefinite Integrals

SOLUTIONS

1. **(b) :** As $f''(x) - g''(x) = 0$

$\Rightarrow \frac{d}{dx}(f'(x) - g'(x)) = 0 \Rightarrow f'(x) - g'(x) = k$ constant

Let $x = 1$ we get $f'(1) - g'(1) = k \Rightarrow 4 - 2 = k \therefore k = 2$

$\therefore f'(x) - g'(x) = 2$

On integrating, we get $f(x) - g(x) = 2x + r$

Let $x = 2$ we get $f(2) - g(2) = 2\cdot 2 + r$

$\Rightarrow 9 - 3 = 4 + r \quad \therefore r = 2$

Then, $f(x) - g(x) = 2x + 2$

Hence, $f(x) - g(x)$ at $x = 4$ is equal to $2 \times 4 + 2 = 10$

2. **(c)** $I = \int \frac{(x^2-1)dx}{(x+1)^2\sqrt{x+x^2+x^3}}$

$= \int \frac{x\left(1-\frac{1}{x^2}\right)dx}{(x+1)^2\sqrt{\frac{1}{x}+1+x}}$

Put $t^2 = x+\frac{1}{x}+1$, then $2t\,dt = \left(1-\frac{1}{x^2}\right)dx$

$\therefore \quad I = 2\int\frac{dt}{t^2+1} = 2\tan^{-1}t + c = 2\tan^{-1}\sqrt{x+\frac{1}{x}+1}+c$.

3. **(b) :** $u = -f''(\theta)\sin\theta + f'(\theta)\cos\theta$

$\frac{du}{d\theta} = -f'''(\theta)\sin\theta - f''(\theta)\cos\theta + f''(\theta)\cos\theta - f'(\theta)\sin\theta$

$= -\sin\theta\,(f'''(\theta) + f'(\theta))$

$\Rightarrow \left(\frac{du}{d\theta}\right)^2 = \sin^2\theta(f'''(\theta)+f'(\theta))^2$

$v = f''(\theta)\cos\theta + f'(\theta)\sin\theta$

$\frac{dv}{d\theta} = f'''(\theta)\cos\theta - f''(\theta)\sin\theta + f''(\theta)\sin\theta + f'(\theta)\cos\theta$

$= \cos\theta\,(f'''(\theta) + f'(\theta))$

$\Rightarrow \left(\frac{dv}{d\theta}\right)^2 = \cos^2\theta(f'''(\theta)+f'(\theta))^2$

$\Rightarrow \left(\frac{du}{d\theta}\right)^2 + \left(\frac{dv}{d\theta}\right)^2 = [f'''(\theta)+f'(\theta)]^2[\sin^2\theta + \cos^2\theta]$

$= [f'''(\theta) + f'(\theta)]^2$

$\Rightarrow \left(\left(\frac{du}{d\theta}\right)^2 + \left(\frac{dv}{d\theta}\right)^2\right)^{1/2} = f'''(\theta) + f'(\theta)$

$\therefore \int\left[\left(\frac{du}{d\theta}\right)^2 + \left(\frac{dv}{d\theta}\right)^2\right]^{1/2} d\theta = \int f'''(\theta)d\theta + \int f'(\theta)\,d\theta$

$= f''(\theta) + f(\theta) + C$

4. **(d) :** Let $I = \int\frac{dx}{\sin x - \cos x + \sqrt{2}}$

$= \int\frac{dx}{\sin x\frac{\sqrt{2}}{\sqrt{2}} - \cos x\frac{\sqrt{2}}{\sqrt{2}} + \sqrt{2}}$

$= \int\frac{dx}{\sqrt{2}\left(\sin x\sin\frac{\pi}{4} - \cos x\cos\frac{\pi}{4}+1\right)}$

$= \frac{1}{\sqrt{2}}\int\frac{dx}{1-\cos\left(x+\frac{\pi}{4}\right)} = \frac{1}{\sqrt{2}}\int\frac{dx}{2\sin^2\left(\frac{x}{2}+\frac{\pi}{8}\right)}$

$= \frac{1}{2\sqrt{2}}\int\operatorname{cosec}^2\left(\frac{x}{2}+\frac{\pi}{8}\right)dx$

$= \frac{1}{2\sqrt{2}}\cdot\frac{-\cot\left(\frac{x}{2}+\frac{\pi}{8}\right)}{\frac{1}{2}} + C = \frac{-1}{\sqrt{2}}\cot\left(\frac{x}{2}+\frac{\pi}{8}\right)+C$

5. **(a) :** $\int\frac{f(x)}{\log(\sin x)}dx = \log[\log\sin x] + c$

Differentiating both sides w.r.t. x, we get

$\frac{f(x)}{\log(\sin x)} = \frac{1}{\log\sin x}\frac{d}{dx}[\log\sin x]$

$= \frac{1}{\log\sin x}\cdot\frac{1}{\sin x}\times\cos x = \frac{\cot x}{\log\sin x} \Rightarrow f(x) = \cot x$

6. **(c) :** Put $\sec x + \tan x = t$. Also $\sec x - \tan x = \frac{1}{t}$

Then $(\sec x\tan x + \sec^2 x)dx = dt$

So that $dx = \frac{dt}{\sec x\cdot t}$

$\therefore \quad I = \int\frac{\sec^2 x\,dx}{(\sec x + \tan x)^{9/2}} = \int\frac{\left(t+\frac{1}{t}\right)}{2\cdot t\cdot t^{9/2}}dt$

$= \frac{1}{2}\int t^{-9/2}dt + \frac{1}{2}\int t^{-13/2}dt$

$= -\frac{1}{2}\left[\frac{2}{7t^{7/2}} + \frac{2}{11t^{11/2}}\right] + K = -\frac{1}{t^{11/2}}\left[\frac{t^2}{7}+\frac{1}{11}\right]+K$

where K is an arbitrary constant.

Hence, $A = \frac{1}{11}$ and $B = \frac{1}{7}$

$A + B = \frac{18}{77}$

7. **(a) :** $I = \int \frac{\sqrt{\cos 2x}}{\sin x} dx + c$, Put $t = \tan x$

$\therefore \quad I = \int \frac{\sqrt{1-t^2}\, dt}{t(1+t^2)} = \int \frac{\cos^2\theta\, d\theta}{\sin\theta(1+\sin^2\theta)}$ [Putting $t = \sin\theta$]

$= \int \left(\frac{1}{\sin\theta} - \frac{2\sin\theta}{2-\cos^2\theta}\right) d\theta$

$= \ln(\operatorname{cosec}\theta - \cot\theta) + \frac{1}{\sqrt{2}} \ln\left(\frac{\sqrt{2}+\cos\theta}{\sqrt{2}-\sin\theta}\right) + c$

$= \ln\left(\frac{1-\sqrt{1-t^2}}{t}\right) + \frac{1}{\sqrt{2}} \ln\left(\frac{\sqrt{2}+\sqrt{1-t^2}}{\sqrt{2}-\sqrt{1-t^2}}\right) + c$

8. **(c) :** Let $I = \int \frac{dx}{(1+\sqrt{x})\sqrt{x}\sqrt{1-x}}$

Put $1+\sqrt{x} = t \Rightarrow \frac{1}{2\sqrt{x}} dx = dt$ $\quad \therefore \; I = \int \frac{2dt}{t\sqrt{2t-t^2}}$

Again put $t = \frac{1}{z} \Rightarrow dt = \frac{-1}{z^2} dz$

$\therefore \quad I = 2\int \frac{-\frac{1}{z^2} dz}{\frac{1}{z}\sqrt{\frac{2}{z} - \frac{1}{z^2}}} = 2\int \frac{-dz}{\sqrt{2z-1}} = -2\sqrt{2z-1} + C$

$= -2\sqrt{\frac{2}{t} - 1} + C = -2\sqrt{\frac{2-t}{t}} + C = -2\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}} + C$

9. **(d) :** $F(x) = \int \sin^2 x\, dx = \frac{1}{2}\int (1-\cos 2x) dx$

$= \frac{x}{2} - \frac{\sin 2x}{4} + c$

$F(x+\pi) - F(x) = \frac{\pi}{2} \neq 0$

$\therefore$ Statement-1 is false.

$\sin^2(x+\pi) = (-\sin x)^2 = \sin^2 x$

$\therefore$ Statement-2 is true.

10. **(c) :** Let $I = \frac{\sqrt{5+x^2}}{x^4} dx$

Put $x = \sqrt{5}\tan\theta \Rightarrow dx = \sqrt{5}\sec^2\theta\, d\theta$

$I = \int \frac{\sqrt{5+5\tan^2\theta}}{25\tan^4\theta} \cdot \sqrt{5}\sec^2\theta\, d\theta$

$= \int \sqrt{5}\, \frac{\sqrt{5}\sec\theta}{25\tan^4\theta} \cdot \sec^2\theta\, d\theta = \frac{1}{5}\int \frac{\cos\theta}{\sin^4\theta} d\theta$

Put $\sin\theta = t \quad \Rightarrow \quad \cos\theta\, d\theta = dt$

$\therefore \quad I = \frac{1}{5}\int \frac{dt}{t^4} = \frac{1}{5}\left(\frac{t^{-3}}{-3}\right) = -\frac{1}{15}\frac{1}{t^3} + C$

$= -\frac{1}{15}\frac{1}{\sin^3\theta} + C$

$= -\frac{1}{15}\left(\sqrt{\frac{x^2+5}{x^2}}\right)^3 + C = -\frac{1}{15}\left(1+\frac{5}{x^2}\right)^{3/2} + C$

11. **(c) :** L.H.S $= \int \frac{\cos 8x + 1}{\tan 2x - \cot 2x} dx$

$= \int \frac{2\cos^2 4x}{\left(\frac{\sin^2 2x - \cos^2 2x}{\sin 2x \cos 2x}\right)} dx = -\int \frac{\cos^2 4x(2\sin 2x\cos 2x)}{(\cos^2 2x - \sin^2 2x)} dx$

$= -\int \frac{\cos^2 4x \times \sin 4x}{\cos 4x} dx = -\frac{1}{2}\int 2\sin 4x\cos 4x\, dx$

$= -\frac{1}{2}\int \sin 8x\, dx = \frac{1}{2} \times \frac{\cos 8x}{8} + c$

Now $\frac{1}{2}\frac{\cos 8x}{8} + c = a\cos 8x + c$

$\therefore \quad a = \frac{1}{16}$

12. **(b) :** Let $I = \int \frac{f(x)g'(x) - f'(x)g(x)}{f(x)\cdot g(x)} [\log(g(x)) - \log(f(x))] dx$

$= \int \frac{f(x)g'(x) - f'(x)g(x)}{f(x)g(x)} \cdot \left[\log\left(\frac{g(x)}{f(x)}\right)\right] dx$

Put $\log\left(\frac{g(x)}{f(x)}\right) = t$

$\frac{f(x)}{g(x)} \cdot \left[\frac{f(x)\cdot g'(x) - g(x)f'(x)}{(f(x))^2}\right] dx = dt$

$\Rightarrow \quad \frac{f(x)g'(x) - g(x)f'(x)}{f(x)g(x)} dx = dt$

$\therefore \quad I = \int t\, dt = \frac{t^2}{2} + C = \frac{1}{2}\left[\log\left(\frac{g(x)}{f(x)}\right)\right]^2 + C$

13. **(c) :** Given integral can be written as

$I = \int \frac{\cot^{n-1} x}{\sin^2 x} dx = \int \cot^{n-1} x \operatorname{cosec}^2 x\, dx$

Put $\cot x = t \quad \Rightarrow \quad -\operatorname{cosec}^2 x\, dx = dt$

$\therefore \quad I = -\int t^{n-1} dt = \frac{-t^n}{n} + C = \frac{-\cot^n x}{n} + C$

14. **(b) :** $\int \frac{1}{1+x^4} dx = \frac{1}{2}\int \frac{(x^2+1)-(x^2-1)}{x^4+1} dx$

$= \frac{1}{2}\int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}} dx - \frac{1}{2}\int \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}} dx$

$= \frac{1}{2}\int \frac{\left(1+\frac{1}{x^2}\right)}{\left(x-\frac{1}{x}\right)^2 + (\sqrt{2})^2} dx - \frac{1}{2}\int \frac{\left(1-\frac{1}{x^2}\right)}{\left(x+\frac{1}{x}\right)^2 - (\sqrt{2})^2} dx$

$= \frac{1}{2\sqrt{2}} \tan^{-1}\left(\frac{x-\frac{1}{x}}{\sqrt{2}}\right) - \frac{1}{2\cdot 2\sqrt{2}} \log\left|\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right| + C$

$= \frac{1}{2\sqrt{2}} \tan^{-1}\left(\frac{x^2-1}{\sqrt{2}x}\right) - \frac{1}{4\sqrt{2}} \log\left|\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right| + C$

15. **(b)** : Let $I=\int\frac{1}{\sqrt{7-x^2}}dx \Rightarrow I=\sin^{-1}\left(\frac{x}{\sqrt{7}}\right)+c$

16 **(c)** : $J-I=\int\left(\frac{-e^x}{e^{4x}+e^{2x}+1}+\frac{e^{-x}}{e^{-4x}+e^{-2x}+1}\right)dx$

$$=\int\left(\frac{-e^x}{e^{4x}+e^{2x}+1}+\frac{e^{3x}}{e^{4x}+e^{2x}+1}\right)dx=\int\frac{e^x(e^{2x}-1)}{e^{4x}+e^{2x}+1}dx$$

Put $t=e^x$, then

$$J-I=\int\frac{t^2-1}{t^4+t^2+1}dt=\int\frac{\left(1-\frac{1}{t^2}\right)dt}{\left(t+\frac{1}{t}\right)^2-1}=\frac{1}{2}\log\left|\frac{t+\frac{1}{t}-1}{t+\frac{1}{t}+1}\right|+C$$

$\therefore\ J-I=\frac{1}{2}\log\left|\frac{e^{2x}-e^x+1}{e^{2x}+e^x+1}\right|+C$

17. **(c)** : $\int\sqrt{1+\operatorname{cosec}x}\,dx=\int\sqrt{1+\frac{1}{\sin x}}dx=\int\sqrt{\frac{1+\sin x}{\sin x}}dx$

Put $\sin x=y \Rightarrow dy=\cos x\,dx \Rightarrow dx=\frac{dy}{\sqrt{\cos^2 x}}=\frac{dy}{\sqrt{1-y^2}}$

$\therefore\ \int\sqrt{1+\operatorname{cosec}x}\,dx=\int\sqrt{\frac{1+y}{y}}.\frac{1}{\sqrt{1-y^2}}dy$

$$=\int\frac{1}{\sqrt{y-y^2}}dy=\int\frac{dy}{\sqrt{\left(\frac{1}{2}\right)^2-\left(y-\frac{1}{2}\right)^2}}=\sin^{-1}\left(\frac{y-1/2}{1/2}\right)+C$$

$=\sin^{-1}(2y-1)+C=\sin^{-1}(2\sin x-1)+C$

18. **(d)** : Let $I=\int\frac{dx}{4\sin^2 x+3\cos^2 x}$

Divide numerator and denominator by $\cos^2 x$

$I=\int\frac{dx/\cos^2 x}{4\tan^2 x+3}=\int\frac{\sec^2 x\,dx}{3+4\tan^2 x}$

Put $\tan x=t \Rightarrow \sec^2 x\,dx=dt$

$\therefore\ I=\int\frac{dt}{3+4t^2}=\int\frac{\frac{dt}{4}}{\frac{3}{4}+t^2}$

$$=\frac{1}{4}\int\frac{dt}{\left(\frac{\sqrt{3}}{2}\right)^2+t^2}=\frac{1}{4}\times\frac{2}{\sqrt{3}}\tan^{-1}\frac{2t}{\sqrt{3}}+C=\frac{1}{2\sqrt{3}}\tan^{-1}\left[\frac{2(\tan x)}{\sqrt{3}}\right]+C.$$

19. **(a)** : Let $I=\int\frac{x^2+4}{x^4+16}dx$

$$\Rightarrow I=\int\frac{1+\frac{4}{x^2}}{x^2+\frac{16}{x^2}}dx=\int\frac{1+\frac{4}{x^2}}{\left(x-\frac{4}{x}\right)^2+8}dx$$

Put $x-\frac{4}{x}=t \Rightarrow \left(1+\frac{4}{x^2}\right)dx=dt$

$\therefore\ I=\int\frac{dt}{t^2+8}=\int\frac{dt}{t^2+(2\sqrt{2})^2}$

$$=\frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{t}{2\sqrt{2}}\right)+C=\frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x^2-4}{2\sqrt{2}x}\right)+C$$

20. **(b)** : Let $I=\int\frac{x+1}{x(1+xe^x)}dx=\int\frac{e^x(x+1)}{xe^x(1+xe^x)}dx$

Put $xe^x=t \Rightarrow (x+1)\,e^x dx=dt$

$\therefore\ I=\int\frac{dt}{t(1+t)}=\int\frac{1}{t}dt-\int\frac{1}{(t+1)}dt$

$$=\log|t|-\log|t+1|+C=\log\left|\frac{t}{t+1}\right|+C=\log\left|\frac{xe^x}{1+xe^x}\right|+C$$

21. **(b)** : Let $I=\int\left[\frac{4e^x-25}{2e^x-5}\right]dx$

Use : Numerator = C(Denominator) + $D\left[\frac{d}{dx}\text{[Denominator]}\right]$

$4e^x-25 = C(2e^x-5)+D(2e^x)=2(C+D)e^x-5C$

$\Rightarrow\ 2(C+D)=4$ and $-5C=-25 \Rightarrow C=5 \Rightarrow D=-3$

$\therefore\ I=\int\left[\frac{5(2e^x-5)-3(2e^x)}{(2e^x-5)}\right]dx=5\int dx-3\int\frac{2e^x}{2e^x-5}dx$

$=5x-3\log|2e^x-5|+c$

Hence, $A=5$ and $B=-3$

22. **(b)** : Let $I=\int\frac{x^3-1}{x^3+x}dx=\int\left(1-\frac{x+1}{x^3+x}\right)dx$

$$=\int 1dx-\int\frac{x+1}{x(x^2+1)}dx=x-\int\frac{x+1}{x(x^2+1)}dx \quad \text{...(i)}$$

Now $\frac{x+1}{x(x^2+1)}=\frac{A}{x}+\frac{Bx+C}{x^2+1}$ (By using partial fractions)

$\Rightarrow\ x+1=A(x^2+1)+(Bx+C)x$

$\Rightarrow\ x+1=(A+B)x^2+Cx+A$

Comparing coefficients of x^2, x and constant, we get

$A+B=0,\ C=1,\ A=1 \Rightarrow B=-1$

$\therefore$ From (i), we get $I=x-\int\frac{1}{x}dx-\int\frac{1-x}{x^2+1}dx$

$$=x-\log x-\int\frac{1}{x^2+1}dx+\frac{1}{2}\int\frac{2x}{x^2+1}dx$$

$$=x-\log x-\tan^{-1}x+\frac{1}{2}\log(x^2+1)+c$$

23. **(d)** : Let $I=\int\frac{\cos x+x\sin x}{x^2+x\cos x}dx=\int\frac{\cos x+x\sin x}{x(x+\cos x)}dx$

$f(x)=\frac{\cos x+x\sin x}{x(x+\cos x)}=\frac{A}{x}+\frac{B}{x+\cos x}$

$\cos x+x\sin x=Ax+A\cos x+Bx$

$A+B=\sin x$ & $A=1 \Rightarrow B=-1+\sin x$

$\therefore\ I=\int\frac{1}{x}dx-\int\frac{1-\sin x}{x+\cos x}dx$

$=\log|x|-\log(x+\cos x)+C=\log\left|\frac{x}{x+\cos x}\right|+C$

24. **(c)** : $\int\frac{x+2}{2x^2+6x+5}dx$

Let $x+2=\lambda(4x+6)+\mu$

Comparing coefficients, we get $4\lambda=1$ and $6\lambda+\mu=2 \Rightarrow \lambda=1/4$

$\Rightarrow\ \mu=2-\frac{3}{2}=\frac{4-3}{2}=\frac{1}{2}$

$$\therefore \int \frac{(x+2)dx}{2x^2+6x+5} = \frac{1}{4}\int \frac{4x+6}{2x^2+6x+5}\,dx + \frac{1}{2}\int \frac{dx}{2x^2+6x+5}$$

$$= P\int \frac{4x+6}{2x^2+6x+5}\,dx + \frac{1}{2}\int \frac{dx}{2x^2+6x+5} \Rightarrow P = \frac{1}{4}$$

25. **(d) :** $\int x^3 e^{5x} dx = x^3 . \frac{e^{5x}}{5} - \int\left[3x^2 \cdot \frac{e^{5x}}{5}\right] dx$

$$= x^3 \cdot \frac{e^{5x}}{5} - \left[\frac{3}{5}\cdot x^2 \frac{e^{5x}}{5} - \frac{3}{5}\int\left(2x.\frac{e^{5x}}{5}\right)dx\right]$$

$$= x^3 \cdot \frac{e^{5x}}{5} - \left(\frac{3}{25}x^2 \cdot e^{5x}\right) + \frac{6}{25}\left[\left(\frac{x\cdot e^{5x}}{5}\right) - \int 1.\frac{e^{5x}}{5}dx\right] + c$$

$$= x^3 \cdot \frac{e^{5x}}{5} - \frac{3x^2 e^{5x}}{25} + \frac{6x\cdot e^{5x}}{125} - \frac{6e^{5x}}{625} + c$$

$$= \frac{e^{5x}}{5^4}\left[125x^3 - 75x^2 + 30x - 6\right] + c$$

$\therefore \quad f(x) = 5^3x^3 - 75x^2 + 30x - 6$

26. **(c) :** Let $I = \int e^x \left(\frac{1+\sin x}{1+\cos x}\right) dx$

$$= \int e^x \left(\frac{\cos^2\frac{x}{2} + \sin^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\right) dx$$

$$= \frac{1}{2}\int e^x \left(\frac{\cos\frac{x}{2} + \sin\frac{x}{2}}{\cos\frac{x}{2}}\right)^2 dx = \frac{1}{2}\int e^x\left(1+\tan\frac{x}{2}\right)^2 dx$$

$$= \frac{1}{2}\int e^x\left(1+\tan^2\frac{x}{2} + 2\tan\frac{x}{2}\right)dx = \frac{1}{2}\int e^x\left(\sec^2\frac{x}{2} + 2\tan\frac{x}{2}\right)dx$$

It is of the form $\int e^x(f(x) + f'(x))dx = e^x f(x) + C$

Here, $f(x) = 2\tan\frac{x}{2}$

$$\Rightarrow \quad I = \frac{1}{2}e^x\left(2\tan\frac{x}{2}\right) + C = e^x \tan\left(\frac{x}{2}\right) + C$$

27. **(d) :** Let $I = \int \sec^3 x\, dx$

$$= \int \sec^2 x \cdot \sec x\, dx = \int (1+\tan^2 x)\cdot \sec x\, dx$$

$$= \int \sec x\, dx + \int \tan^2 x \cdot \sec x\, dx$$

$$= \log|\sec x + \tan x| + \int \underset{\text{II}}{(\sec x \tan x)} \cdot \underset{\text{I}}{\tan x}\, dx$$

$$\Rightarrow I = \log|\sec x + \tan x| + \left[\tan x \sec x - \int \sec^2 x \cdot \sec x\, dx\right]$$

$$\Rightarrow 2I = \tan x \sec x + \log|\sec x + \tan x| + C$$

$$\Rightarrow I = \frac{1}{2}[\sec x \tan x + \log|\sec x + \tan x|] + C$$

28. **(a) :** $\int \frac{x\tan^{-1}x}{\sqrt{1+x^2}}dx = \sqrt{1+x^2}\, f(x) + A\log\left(x + \sqrt{1+x^2}\right) + c$

Let $I = \int \frac{x\tan^{-1}x}{\sqrt{1+x^2}}dx$

Using Integration by parts, we get

$$I = \tan^{-1}x\left(\frac{1}{2}\int\frac{2x\,dx}{\sqrt{1+x^2}}\right) - \int\frac{1}{1+x^2}\left(\frac{1}{2}\int\frac{2x\,dx}{\sqrt{1+x^2}}\right)dx + c$$

$$= \frac{1}{2}\tan^{-1}x \times 2\sqrt{1+x^2} - \int\frac{1}{1+x^2}\times\frac{1}{2}\times 2\sqrt{1+x^2}\,dx + c$$

$$= \sqrt{1+x^2}\tan^{-1}x - \int\frac{\sqrt{1+x^2}}{1+x^2}dx + c$$

$$= \sqrt{1+x^2}\tan^{-1}x - \int\frac{1}{\sqrt{1+x^2}}dx + c$$

$$= \sqrt{1+x^2}\tan^{-1}x - \log\left(x + \sqrt{1+x^2}\right) + c$$

By comparing we get, $f(x) = \tan^{-1}x$, $A = -1$.

29. **(a) :** Let $I = \int (\log x)^5\, dx$

Put $\log x = t \Rightarrow x = e^t \Rightarrow dx = e^t\, dt$

$\therefore \quad I = \int (t)^5 \cdot e^t\, dt = t^5 \cdot e^t - \int 5t^4 \cdot e^t dt$

$= t^5 \cdot e^t - 5\,[t^4 \cdot e^t - \int 4t^3 e^t dt]$

$= t^5e^t - [5t^4 e^t] + 5\cdot 4\,[t^3 \cdot e^t - \int(3t^2e^t)dt]$

$= t^5e^t - 5t^4e^t + 20t^3 \cdot e^t - 60\,[t^2e^t - 2\int t\cdot e^t dt]$

$= t^5\cdot e^t - 5t^4\cdot e^t + 20t^3\cdot e^t - 60t^2e^t + 120t\cdot e^t - 120e^t + c$

$= x[(\log x)^5 - 5(\log x)^4 + 20(\log x)^3 - 60(\log x)^2 + 120(\log x) - 120] + c$

$\therefore \quad A + B + C + D + E + F = 1 - 5 + 20 - 60 + 120 - 120 = -44$

30. **(c) :** $\int (3x\cos x - \cos x + \sin x - 2x\sin x)\, dx$

$= f(x)\cos x + g(x)\sin x + C.$

$\Rightarrow \quad 3\int x\cos x\,dx - \sin x - \cos x - 2\int x\sin x\,dx$

$= f(x)\cos x + g(x)\sin x + C.$

$\Rightarrow \quad 3\,[x\cdot\sin x - \int 1\cdot\sin x\,dx] - \sin x - \cos x$

$-2[x\cdot(-\cos x) - \int 1\cdot(-\cos x)\,dx] = f(x)\cos x + g(x)\sin x + C$

$\Rightarrow \quad 3x\sin x + 3\cos x - \sin x - \cos x + 2x\cos x - 2\sin x + C = f(x)\cos x + g(x)\sin x + C$

$\Rightarrow \quad 2(x+1)\cos x + 3(x-1)\sin x + C = f(x)\cos x + g(x)\sin x + C$

$\Rightarrow \quad f(x) = 2(x+1),\ g(x) = 3(x-1)$

31. **(a, b, c) :** Let $I = \frac{1}{\sin(a-b)}\int\frac{\sin(x-b-(x-a))dx}{\cos(x-a)\cos(x-b)}$

$$= \frac{1}{\sin(a-b)}\int(\tan(x-b) - \tan(x-a))dx$$

$$= \frac{1}{\sin(a-b)}[-\log\cos(x-b) + \log\cos(x-a)] + C$$

$$= \frac{1}{\sin(a-b)}\left[\frac{\log.\cos(x-a)}{\log.\cos(x-b)}\right] + C$$

$$= \frac{1}{\sin(a-b)}\left[\frac{\log\sec(x-b)}{\log\sec(x-a)}\right] + C$$

32. **(b, d) :** Dividing by $\cos^2 x$ in both numerator and denominator

$$I = \int\frac{\sec^2 x\,dx}{\tan^2 x(2+\tan^2 x)},$$

Put $\tan x = t$

$\therefore\ I = \int \frac{dt}{t^2(2+t^2)} = \frac{1}{2}\int\left(\frac{1}{t^2} - \frac{1}{2+t^2}\right)dt$

$= -\frac{1}{2t} - \frac{1}{2\sqrt{2}}\tan^{-1}\frac{t}{\sqrt{2}} + c$

$= -\frac{1}{2}\cot x - \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{\tan x}{\sqrt{2}}\right) + c$

33. (a, c) : $I_n = \int \tan^n x\,dx$

$\Rightarrow I_n = \int \tan^{n-2}x\,(\tan^2 x)\,dx$

$\Rightarrow I_n = \int \tan^{n-2}x(\sec^2 x - 1)dx = \frac{1}{n-1}\cdot\tan^{n-1}x - I_{n-2}$

So, we get $a = n - 1$ and $b = 1$

34. (a, b, c) : Let $I = \int x^{\frac{13}{2}}\left(1 + x^{\frac{5}{2}}\right)^{\frac{1}{2}} dx$, Put $t^2 = 1 + x^{5/2}$

$\therefore\ I = \frac{4}{5}\int t^2(t^2-1)^2\,dt = \frac{4}{5}\left[\frac{t^7}{7} - \frac{2}{5}t^5 + \frac{t^3}{3}\right] + c$

$= \frac{4}{35}\left(1+x^{\frac{5}{2}}\right)^{\frac{7}{2}} - \frac{8}{25}\left(1+x^{\frac{5}{2}}\right)^{\frac{5}{2}} + \frac{4}{15}\left(1+x^{\frac{5}{2}}\right)^{\frac{3}{2}} + c$

35. (a, b, d) : $2\sin x + \cos x$
$= a(7\sin x - 5\cos x) + b(7\cos x + 5\sin x)$
$\Rightarrow 7a + 5b = 2,\ -5a + 7b = 1 \Rightarrow a = \frac{9}{74},\ b = \frac{17}{74}$

$\therefore$ The given integral, $I = \int\left[\frac{9}{74} + \frac{17}{74}\left(\frac{7\cos x + 5\sin x}{7\sin x - 5\cos x}\right)\right]dx$

$= \frac{9x}{74} + \frac{17}{74}\ \ln(7\sin x - 5\cos x) + c$

$\therefore\ a + b = \frac{9}{74} + \frac{17}{74} = \frac{13}{37}$.

36. (a, b, d) : Let $I = \int \frac{dx}{x^n(1+x^n)^{\frac{1}{n}}} = \int \frac{dx}{x^{n+1}\left(\frac{1}{x^n}+1\right)^{\frac{1}{n}}}$,

Put $t = \frac{1}{x^n} + 1 = -\frac{1}{n}\int t^{-\frac{1}{n}}dt$

$= \frac{1}{1-n}t^{1-\frac{1}{n}} + c = \frac{1}{1-n}\left(\frac{1}{x^n}+1\right)^{\frac{n-1}{n}} + c$

$\therefore\ ab = \left(\frac{1}{1-n}\right)\left(\frac{n-1}{n}\right) = -\frac{1}{n}$

37. (a, b, c) : Let $I = \int \frac{(1+x^n)^{\frac{1}{n}}}{x^{n+2}}dx = \int \frac{\left(\frac{1}{x^n}+1\right)^{\frac{1}{n}}}{x^{n+1}}dx$, Put $t = \frac{1}{x^n} + 1$

$\therefore\ I = -\frac{1}{n}\int t^{\frac{1}{n}}dt = -\frac{1}{n+1}t^{\frac{n+1}{n}} + c$

$= -\frac{1}{n+1}\left(\frac{1}{x^n}+1\right)^{\frac{n+1}{n}} + c \Rightarrow n = 4,\ a = -\frac{1}{5},\ b = \frac{5}{4}$

$\therefore\ a + b = \frac{21}{20}$.

38. (a, b, c) : Let $I = \int \frac{x^{\frac{3}{2}}dx}{x^4 + x^{\frac{5}{2}}} = \int \frac{dx}{\left(1+\frac{1}{x^{\frac{3}{2}}}\right)x^{\frac{5}{2}}}$, Put $t = 1 + \frac{1}{x^{3/2}}$

$\therefore\ I = -\frac{2}{3}\int\frac{dt}{t} = -\frac{2}{3}\ln\left(1 + x^{-\frac{3}{2}}\right) + c$

$\therefore\ ab = \left(-\frac{2}{3}\right)\left(-\frac{3}{2}\right) = 1.$

39. (d) : (A)-(r), (B)-(s), (C)-(p), (D)-(q)

(A) Let $I = \int\frac{x\,dx}{1+x^4} = \frac{1}{2}\int\frac{dt}{1+t^2}$, Put $t = x^2$

$\therefore\ I = \frac{1}{2}\ \tan^{-1} t + c = \frac{1}{2}\ \tan^{-1} x^2 + c.$

(B) Let $I = \int\cos\sqrt{x}\,dx = 2\int\cos t\cdot t\,dt$, Put $t^2 = x$

$\therefore\ I = 2(t\sin t + \cos t) + c$

$= 2\left(\sqrt{x}\sin\sqrt{x} + \cos\sqrt{x}\right) + c$.

(C) $\int x\sec^2 2x\,dx = \frac{1}{2}\int x\,d(\tan 2x)$

$= \frac{1}{2}\left[x\tan 2x - \int\tan 2x\,dx\right] = \frac{1}{2}\left[x\tan 2x + \frac{1}{2}\ln\cos 2x\right] + c$

$= \frac{x}{2}\tan 2x + \frac{1}{4}\ln\cos 2x + c$

(D) $\int\frac{dx}{(x^2+1)(x^2+4)} = \frac{1}{3}\int\left(\frac{1}{x^2+1} - \frac{1}{x^2+4}\right)dx$

$= \frac{1}{3}\ \tan^{-1}x - \frac{1}{6}\ \tan^{-1}\frac{x}{2} + c$

$A = -\frac{1}{6},\ B = \frac{1}{3},\ A + B = \frac{1}{6}$.

40. (a) : (A)-(s), (B)-(p), (C)-(q), (D)-(r)

(A) Let $I = \int\frac{(x+1)e^x dx}{\cos^2(xe^x)}$, Put $xe^x = t \Rightarrow (x+1)e^x dx = dt$

$\therefore\ I = \int\sec^2 t\,dt = \tan t + c = \tan(xe^x) + c.$

(B) Let $I = \int x^5\cdot e^{x^2}dx$, Put $x^2 = t$

$\therefore I = \frac{1}{2}\int t^2e^t dt = \frac{1}{2}[t^2 - 2t + 2]e^t + c = e^{x^2}\left(\frac{x^4}{2} - x^2 + 1\right) + c.$

(C) Let $I = \int\frac{a^{\sqrt{x}}}{\sqrt{x}}dx$

Put $x = t^2 \Rightarrow dx = 2t\,dt$

$\therefore\ I = 2\int a^t dt = \frac{2a^t}{\ln a} + c = 2a^{\sqrt{x}}\log_a e + c.$

(D) Let $I = \int\frac{\sin x\cos x\,dx}{3\sin^2 x + \cos^2 x} = \int\frac{\sin 2x\,dx}{3(1-\cos 2x) + 1 + \cos 2x}$

$= \int\frac{\sin 2x}{4 - 2\cos 2x}dx = \frac{1}{4}\ln(4 - 2\cos 2x) + c_1$

$= \frac{1}{4}\ln(2 - \cos 2x) + c.$

41. **(b) :** Let $I=\int \dfrac{x-2/x^3}{\sqrt{x^2+1+\frac{2}{x^2}}}dx$, Put $x^2+\dfrac{2}{x^2}+1=t$

$\therefore \quad I=\dfrac{1}{2}\int\dfrac{dt}{\sqrt{t}}=\sqrt{t}+c$

42. **(c) :** Let $I=\int\dfrac{x^2-1}{(x+1)^2\sqrt{x^3+x^2+x}}dx=\int\dfrac{\left(1-\frac{1}{x^2}\right)}{\left(x+\frac{1}{x}+2\right)\sqrt{x+\frac{1}{x}+1}}dx$

Put $x+\dfrac{1}{x}+1=t^2$

$\therefore \quad I=\int\dfrac{2t\,dt}{(t^2+1)t}=2\tan^{-1}t+c$

43. **(b) :** Let $I=\int(f(x)+g(x))dx$

$=\int\left(\dfrac{3}{x^4+3x^2+9}+\dfrac{x^2}{x^4+3x^2+9}\right)dx$

$=\int\dfrac{x^2+3}{x^4+3x^2+9}dx$ =Divide numerator and denominator by x^2,

substitute $x-\dfrac{3}{x}=t$, $dt=\left(1+\dfrac{3}{x^2}\right)dx$

$\therefore \quad I=\int\dfrac{1+\frac{3}{x^2}}{x^2+3+\frac{9}{x^2}}dx=\int\dfrac{dt}{t^2+6+3}=\int\dfrac{dt}{t^2+(3)^2}$

$=\dfrac{1}{3}\tan^{-1}(t/3)+C=\dfrac{1}{3}\tan^{-1}\left(\dfrac{x-\frac{3}{x}}{3}\right)+C$

44. **(c) :** Let $I=\int(g(x)-f(x))dx$

$\int\dfrac{x^2-3}{x^4+3x^2+9}dx$ Divide numerator and denominator by x^2,

substitute $x+\dfrac{3}{x}=u\Rightarrow x^2+\dfrac{9}{x^2}+6=u^2$

$\therefore \quad I=\int\dfrac{\left(1-\frac{3}{x^2}\right)dx}{x^2+3+\frac{9}{x^2}}=\int\dfrac{du}{u^2-6+3}=\int\dfrac{du}{u^2-3}$

$=\dfrac{1}{2\sqrt{3}}\log\left|\dfrac{u-\sqrt{3}}{u+\sqrt{3}}\right|+C=\dfrac{1}{2\sqrt{3}}\log\left(\dfrac{x^2-\sqrt{3}x+3}{x^2+\sqrt{3}x+3}\right)+C$

45. **(a) :** $\int f(x)dx$

$=\dfrac{1}{2}\int\dfrac{6}{x^4+3x^2+9}dx=\dfrac{1}{2}\int\dfrac{(x^2+3)-(x^2-3)}{x^4+3x^2+9}dx$

$=\dfrac{1}{2}\int\dfrac{x^2+3}{x^4+3x^2+9}dx-\dfrac{1}{2}\int\dfrac{x^2-3}{x^4+3x^2+9}dx$

46. **(4) :** Let $I=\int\dfrac{\cos^4 x\,dx}{\sin^3 x(\sin^5 x+\cos^5 x)^{3/5}}=\int\dfrac{\sec^2 x\,dx}{\tan^6 x\left(1+\frac{1}{\tan^5 x}\right)^{3/5}}$

Put $\tan x=t \Rightarrow \sec^2 x\,dx=dt$

$\therefore \quad I=\int\dfrac{dt}{t^6\left(1+\frac{1}{t^5}\right)^{3/5}}$

Again, put $1+\dfrac{1}{t^5}=u\Rightarrow\dfrac{-5}{t^6}dt=du$

$\therefore \quad I=-\dfrac{1}{2}\left(\dfrac{1+\tan^5 x}{\tan^5 x}\right)^{2/5}+K$

$\therefore \quad A+5B=4$

47. **(2) :** Let $I=\int\sin 4x\cdot e^{\tan^2 x}dx$

$I=4\int\tan x\cdot\sec^2 x\cdot\cos^6 x\,(1-\tan^2 x)\,e^{\tan^2 x}dx$

Put $\tan^2 x=t \Rightarrow 2\tan x\sec^2 x\,dx=dt$

$\therefore \quad I=2\int\dfrac{(1-t)\,e^t}{(1+t)^3}dt=-2\int\dfrac{(t+1-2)\,e^t}{(1+t)^3}dt$

$\Rightarrow I=\dfrac{-2e^t}{(1+t)^2}+K \Rightarrow I=-2\cos^4 x\cdot e^{\tan^2 x}+K$

$\therefore \quad a+b=2$

48. **(5) :** $I=\int\sin^{-1}\dfrac{2x+2}{\sqrt{4x^2+8x+13}}dx=\int\tan^{-1}\left(\dfrac{2x+2}{3}\right)dx$

$=\dfrac{3}{2}\int\tan^{-1}t\,dt$, Put $t=\dfrac{2x+2}{3}$

$\therefore \quad I=\dfrac{3}{2}\left[t\tan^{-1}t-\int\dfrac{t}{1+t^2}dt\right]$

$=\dfrac{3}{2}t\tan^{-1}t-\dfrac{3}{4}\ln(1+t^2)+D_1$

$=(x+1)\tan^{-1}\left(\dfrac{2x+2}{3}\right)-\dfrac{3}{4}\ln(4x^2+8x+13)+D$

$\therefore \quad 4(A+B+C)=\left(1+1-\dfrac{3}{4}\right)\times 4=\dfrac{5}{4}\times 4=5$

49. **(1) :** $\int\sin\left(2\tan^{-1}\sqrt{\dfrac{1-x}{1+x}}\right)dx$

$=\int\sqrt{1-x^2}\,dx=\dfrac{x}{2}\sqrt{1-x^2}+\dfrac{1}{2}\sin^{-1}x+c$; $A+B=\dfrac{1}{2}+\dfrac{1}{2}=1.$

50. **(4) :** Since $\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}=\dfrac{\pi}{2}$

$\therefore I=\dfrac{2}{\pi}\int\left(2\sin^{-1}\sqrt{x}-\dfrac{\pi}{2}\right)dx$

$=\int\left(\dfrac{4}{\pi}\sin^{-1}\sqrt{x}-1\right)dx$, Put $x=\sin^2\theta$

$\therefore \quad I=-x+\dfrac{4}{\pi}\int\theta\,d(\sin^2\theta)=-x+\dfrac{4}{\pi}\left[\theta\sin^2\theta-\int\sin^2\theta\,d\theta\right]$

$=-x+\dfrac{4x}{\pi}\sin^{-1}\sqrt{x}-\dfrac{2}{\pi}\int(1-\cos 2\theta)d\theta$

$=-x+\dfrac{4x}{\pi}\sin^{-1}\sqrt{x}-\dfrac{2}{\pi}\sin^{-1}\sqrt{x}+\dfrac{1}{\pi}\sin 2\theta+D$

$=-x+\left(\dfrac{4x}{\pi}-\dfrac{2}{\pi}\right)\sin^{-1}\sqrt{x}+\dfrac{2}{\pi}\sqrt{x-x^2}+D$

$\therefore \quad \pi\times(A+B+C)=\pi\times\left(\dfrac{4}{\pi}-\dfrac{2}{\pi}+\dfrac{2}{\pi}\right)=\pi\times\dfrac{4}{\pi}=4$

JEE MATHEMATICS

DAILY PRACTICE SHEET

Definite Integrals and Area Bounded by Curves

SOLUTIONS

1. **(d) :** Let $y = \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)...\left(1+\frac{n}{n}\right)\right]^{1/n}$

$\Rightarrow \log y = \lim_{n\to\infty}\frac{1}{n}\times\log\left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)...\left(1+\frac{n}{n}\right)\right]$

$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\log\left(1+\frac{r}{n}\right) = \int_0^1 \log(1+x)\,dx$

$= [x\log(1+x)]_0^1 - \int_0^1 \frac{x}{1+x}dx$

$= \log 2 - \int_0^1 \frac{(1+x)-1}{1+x}dx = \log 2 - \int_0^1\left(1-\frac{1}{1+x}\right)dx$

$= \log 2 - [x - \log(1+x)]_0^1$ = log 2 – [(1– log2) – 0]

$= 2\log 2 - \log e = \log\frac{4}{e}$

$\therefore\ y = \frac{4}{e}$

2. **(d) :** $x\int_1^x y(t)dt = x\int_1^x ty(t)dt + \int_1^x ty(t)dt$

$= \int_1^2 e^x \log e^x dx + \int_1^2 \frac{1}{x}\log e^x dx + \int_1^2 e^x dx$

Differentiating w.r.t. x, we get $\int_1^x y(t)\,dt + x[y(x) - y(1)]$

$= \int_1^x ty\,(t)dt + x[xy(x) - y(1)] + xy(x) - y(1)$

$\Rightarrow \int_1^x y\,(t)dt = \int_1^x t\,y(t)dt + x^2 y(x) - y(1)$

Again differentiating w.r.t. x, we get

$y(x) - y(1) = xy(x) - y(1) + 2xy(x) + x^2y'(x)$

$\Rightarrow \frac{y'(x)}{y(x)} = \frac{1-3x}{x^2} \Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{1-3x}{x^2} \Rightarrow \int\frac{1}{y}dy = \int\frac{1}{x^2}dx - \int\frac{3}{x}dx$

$\Rightarrow \ln y = -\frac{1}{x} - 3\ln x + \ln C \Rightarrow \ln\left|\frac{yx^3}{C}\right| = -\frac{1}{x}$

$\Rightarrow \frac{yx^3}{C} = e^{-1/x} \Rightarrow y = \frac{Ce^{-\frac{1}{x}}}{x^3}$

3. **(c) :** $\int_1^2 e^x\left(\log_e x + \frac{1}{x}\right)dx + \int_1^2 e^x dx$

$= \int_1^2 e^x \log_e x\,dx + \int_1^2 \frac{1}{x}\log_e x\,dx + \int_1^2 e^x dx$

$= [(\log_e x)(e^x)]_1^2 - \int_1^2 \frac{1}{x}\cdot e^x dx + \int_1^2 \frac{1}{x}e^x dx + [e^x]_1^2$

$= e^2\log_e 2 - 0 + (e^2 - e) = e^2(1+\log_e 2) - e$

4. **(a) :** We have, $I(m,n) = I = \int_0^1 t^m\,(1+t)^n\,dt$

$\Rightarrow I(m,n) = \left[(1+t)^n\cdot\frac{t^{m+1}}{m+1}\right]_0^1 - \frac{n}{m+1}\int_0^1(1+t)^{n-1}\cdot t^{m+1}dt$

$= \frac{2^n}{m+1} - \frac{n}{m+1}\cdot I(m+1, n-1)$

5. **(b) :** $f(x) = \int_1^x e^{-t^2/2}(1-t^2)\,dt$

$f'(x) = e^{-x^2/2}\,(1 - x^2)$

$f'(x) = 0 \Rightarrow 1 - x^2 = 0 \Rightarrow x = \pm 1$

$f''(x) = e^{-x^2/2}(-2x) + (1-x^2)e^{-x^2/2}\times\left(-\frac{2x}{2}\right)$

$= e^{-x^2/2}\,[-2x - x(1-x^2)]$

$f''(x)\big|_{x=1} = e^{-1/2}[-2 - 1(0)] < 0$

$\therefore$ $x = 1$ is point of maxima.

$f''(x)\big|_{x=-1} > 0$

$\therefore$ $x = -1$ is point of minima.

6. **(b) :** $f(x) = (1-x)^2\sin^2 x + x^2, x \in R$

$g(x) = \int_1^x\left(\frac{2(t-1)}{t+1} - \ln t\right)f(t)\,dt\ \forall\ x \in (1,\infty)$

We have $g'(x) = \left(\frac{2(x-1)}{x+1} - \ln x\right)f(x) = A(x)f(x)$

Let $A(x) = \frac{2(x-1)}{x+1} - \ln x$

Then, $A'(x) = \frac{4}{(x+1)^2} - \frac{1}{x} = \frac{-(x-1)^2}{(x+1)^2 x} < 0$

As $A(1) = 0$ $\therefore$ $A(x) < 0\ \forall\ x > 1$

As $g'(x) = A(x)\,f(x) < 0\ \forall\ x > 1$ [$\because$ $A(x) < 0, f(x) < 0$]

Thus, g is decreasing.

7. **(a) :** $I_n = \int_0^1 x^n \tan^{-1} x\,dx$

Also, $I_{n+2} = \int_0^1 x^{n+2}\cdot\tan^{-1} x\,dx$

Then, $(n+1)\,I_n + (n+3)\,I_{n+2}$

$$= \int_0^1 (n+1)\cdot x^n \tan^{-1} x\,dx + \int_0^1 (n+3)\cdot x^{n+2} \tan^{-1} x\,dx$$

$$= \int_0^1 \{(n+1)x^n + (n+3)x^{n+2}\} \tan^{-1} x\,dx$$

$$= \left|\tan^{-1} x\cdot\{x^{n+1} + x^{n+3}\}\right|_0^1 - \int_0^1 \frac{1}{1+x^2}(x^{n+1}+x^{n+3})\,dx$$

$$= \frac{\pi}{2} - \frac{\left|x^{n+2}\right|_0^1}{n+2} = \frac{\pi}{2} - \frac{1}{n+2}$$

Thus, $(n+1)I_n + (n+3)I_{n+2} = \pi/2 - 1/(n+2)$

So, $b_n = (n+1)$ or some multiple of $(n+1)$.

Thus, $b_1, b_2, b_3, ...$ are in A.P.

8. **(a) :** $F(x) = \frac{1}{x^2}\int_4^x (4t^2 - 2F'(t))\,dt$

$$F'(x) = \frac{-2}{x^3}\int_4^x (4t^2 - 2F'(t))\,dt + \frac{1}{x^2}(4x^2 - 2F'(x))$$

Now $F'(4) = \frac{-2}{x^3}\int_4^4 (4t^2 - 2F'(t))\,dt + \frac{1}{4^2}[4\cdot 4^2 - 2F'(4)]$

$$\Rightarrow F'(4) = 0 + 4 - \frac{F'(4)}{8} \Rightarrow F'(4) + \frac{F'(4)}{8} = 4$$

$$\Rightarrow 9F'(4) = 4\times 8 \Rightarrow F'(4) = \frac{32}{9}$$

9. **(a) :** Let $I = \int_1^2 \frac{dx}{(x^2-2x+4)^{3/2}} = \int_1^2 \frac{dx}{((x-1)^2+3)^{3/2}}$

Put $x - 1 = \sqrt{3}\tan\theta \Rightarrow dx = \sqrt{3}\sec^2\theta d\theta$

When $x = 1$, $\theta = 0$ and when $x = 2$, $\theta = \frac{\pi}{6}$

$$\therefore\quad I = \int_0^{\pi/6} \frac{\sqrt{3}\sec^2\theta d\theta}{(3\tan^2\theta+3)^{3/2}}$$

$$= \int_0^{\pi/6} \frac{\sqrt{3}\sec^2\theta\, d\theta}{3\sqrt{3}(\sec^2\theta)^{3/2}} = \int_0^{\pi/6} \frac{1}{3\sec\theta}\,d\theta$$

$$= \frac{1}{3}\int_0^{\pi/6} \cos\theta\, d\theta = \frac{1}{3}(\sin\theta)_0^{\pi/6} = \frac{1}{3}\left(\frac{1}{2}-0\right) = \frac{1}{6}$$

Now, $\frac{1}{6} = \frac{k}{k+5} \Rightarrow k + 5 = 6k \Rightarrow 5k = 5 \Rightarrow k = 1$

10. **(a) :** Let $I = \int_0^{\pi/4} \frac{\sin x + \cos x}{3 + \sin 2x}dx = \int_0^{\pi/4} \frac{\sin x + \cos x}{4 - (1 - \sin 2x)}dx$

$$= \int_0^{\pi/4} \frac{\sin x + \cos x}{4 - (\sin x - \cos x)^2}dx$$

Put $\sin x - \cos x = t \Rightarrow (\cos x + \sin x)\,dx = dt$

When $x = 0$, $t = -1$ and $x = \frac{\pi}{4}$, $t = 0$

$$\therefore\quad I = \int_{-1}^0 \frac{dt}{4-t^2} = \frac{1}{4}\left[\log\left|\frac{2+t}{2-t}\right|\right]_{-1}^0 = \frac{1}{4}\log(3)$$

11. **(c) :** $\int_{-1/2}^{1/2} \left[\left(\frac{x+1}{x-1}\right)^2 + \left(\frac{x-1}{x+1}\right)^2 - 2\right]^{1/2} dx$

$$= \int_{-1/2}^{1/2} \left[\left(\frac{x+1}{x-1} - \frac{x-1}{x+1}\right)^2\right]^{1/2} dx = \int_{-1/2}^{1/2} \left|\frac{4x}{x^2-1}\right|dx$$

$$= \int_{-1/2}^{0} \left|\frac{4x}{1-x^2}\right|dx + \int_0^{1/2} \left|\frac{4x}{1-x^2}\right|dx = -4\int_{-1/2}^{0} \frac{x}{1-x^2}dx + 4\int_0^{1/2} \frac{x}{1-x^2}dx$$

$$= 2\left|\log(1-x^2)\right|_{-1/2}^0 - 2\left|\log(1-x^2)\right|_0^{1/2}$$

$$= -2\log\left(1-\frac{1}{4}\right) - 2\log\left(1-\frac{1}{4}\right) = -4\log\frac{3}{4} = 4\log\left(\frac{4}{3}\right)$$

12. **(b) :** Let $I = \int_0^{\frac{\pi}{4}} \ln(1+\tan x)dx$ ...(i)

$$I = \int_0^{\frac{\pi}{4}} \ln\left(1+\tan\left(\frac{\pi}{4}-x\right)\right)dx = \int_0^{\frac{\pi}{4}} \ln\left(1+\frac{1-\tan x}{1+\tan x}\right)dx$$

$I = \int_0^{\frac{\pi}{4}} \ln\frac{2}{1+\tan x}dx$ ...(ii)

Adding (i) and (ii), $2I = \int_0^{\frac{\pi}{4}} \left[\ln(1+\tan x) + \ln\left(\frac{2}{1+\tan x}\right)\right]dx$

$$= \int_0^{\frac{\pi}{4}} \ln 2\,dx = \frac{\pi}{4}\ln 2 \Rightarrow I = \frac{\pi}{8}\ln 2.$$

13 **(c) :** Putting $x - \alpha = z^2$, we get $dx = 2z\,dz$

Also, $x = z^2 + \alpha$ and $\beta - x = (\beta - \alpha) - z^2$

$$\therefore\quad \int_\alpha^\beta \frac{dx}{\sqrt{(x-\alpha)(\beta-x)}} = \int_0^{\sqrt{\beta-\alpha}} \frac{2z\,dz}{\sqrt{z^2[(\beta-\alpha)-z^2]}}$$

$$= 2\int_0^{\sqrt{\beta-\alpha}} \frac{dz}{\sqrt{\left(\sqrt{\beta-\alpha}\right)^2 - z^2}} = 2\sin^{-1}\left(\frac{z}{\sqrt{\beta-\alpha}}\right)\Bigg|_0^{\sqrt{\beta-\alpha}}$$

$$= 2[\sin^{-1}(1) - \sin^{-1}(0)] = \pi$$

14. **(a) :** $I = \int_0^{\pi/4} \tan^{n+1} x\cdot dx + \frac{1}{2}\int_0^{\pi/2} \tan^{n-1}(x/2)\cdot dx$

$$= \int_0^{\pi/4} \tan^{n+1} t\cdot dt + \int_0^{\pi/4} \tan^{n-1} t\cdot dt \qquad \left[\text{Put } \frac{x}{2} = t\right]$$

$$= \int_0^{\pi/4} \tan^{n-1} t(\tan^2 t + 1)dt$$

Put $\tan t = z$ $\therefore$ $I = \int_0^1 z^{n-1}\cdot dz = \left[\frac{z^n}{n}\right]_0^1 = \frac{1}{n}$

15. **(a) :** $\int_{\pi/6}^{\pi/3} \frac{(x+\sin x) - x(1+\cos x)}{x(x+\sin x)}dx$

$$= \int_{\pi/6}^{\pi/3} \left(\frac{1}{x} - \frac{1+\cos x}{x+\sin x}\right)dx = [\log x - \log(x+\sin x)]_{\pi/6}^{\pi/3}$$

$$= \left[\log\frac{x}{x+\sin x}\right]_{\pi/6}^{\pi/3} = \log\frac{\pi/3}{\frac{\pi}{3}+\frac{\sqrt{3}}{2}} - \log\frac{\pi/6}{\frac{\pi}{6}+\frac{1}{2}}$$

$$= \log\frac{2\pi}{2\pi+3\sqrt{3}} - \log\frac{\pi}{\pi+3}$$

$$= \log\left(\frac{2\pi}{2\pi+3\sqrt{3}}\cdot\frac{\pi+3}{\pi}\right) = \log\left\{\frac{2(\pi+3)}{2\pi+3\sqrt{3}}\right\}$$

16. (d) : $I_1 = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{\pi\,dx}{2-\cos\left(|x|+\frac{\pi}{3}\right)}$, dropping the odd term

$$= 2\pi\int_0^{\frac{\pi}{3}}\frac{dx}{2-\cos\left(x+\frac{\pi}{3}\right)}, \left[\text{Put } x+\frac{\pi}{3} = t \Rightarrow dx = dt\right]$$

$$= 2\pi\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}}\frac{dt}{2-\cos t}, \left[\text{Put } u = \tan\frac{t}{2}\right]$$

$$= 4\pi\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}}\frac{du}{2(1+u^2)-(1-u^2)}$$

$$= 4\pi\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}}\frac{du}{1+3u^2} = \frac{4\pi}{\sqrt{3}}\tan^{-1}\sqrt{3}u\Big|_{\frac{1}{\sqrt{3}}}^{\sqrt{3}}$$

$$= \frac{4\pi}{\sqrt{3}}\left(\tan^{-1}3-\tan^{-1}1\right) = \frac{4\pi}{\sqrt{3}}\tan^{-1}\left(\frac{1}{2}\right)$$

17. (b) : $\because \quad I = \int_{10}^{19}\frac{\sin x}{1+x^8}dx$

$$\therefore \quad |I| = \left|\int_{10}^{19}\frac{\sin x}{1+x^8}dx\right| \le \int_{10}^{19}\left|\frac{\sin x}{1+x^8}\right|dx$$

$$\Rightarrow \quad |I| \le \int_{10}^{19}\frac{|\sin x|}{1+x^8}dx \le \int_{10}^{19}\frac{dx}{1+x^8} \qquad [\because |\sin x| \le 1]$$

$$\Rightarrow \quad |I| \le \int_{10}^{19}\frac{dx}{x^8} = \left[\frac{x^{-7}}{-7}\right]_{10}^{19} = \frac{1}{7}(10^{-7}-19^{-7}) < \frac{10^{-7}}{7} < 10^{-7}$$

18. (c) : $\int_0^{100(1)} e^{x-[x]}dx = \int_0^{100(1)} e^{\{x\}}dx = 100\int_0^1 e^{\{x\}}dx$

[$\because$ $\{x\}$ is periodic with period 1]

$$= 100\int_0^1 e^x dx = 100(e-1)$$

19. (b) : $\int_0^1 \ln(1+x^2)dx = x\ln(1+x^2)\Big|_0^1 - 2\int_0^1\frac{x^2}{1+x^2}dx$

$$= \ln 2 - 2\int_0^1\left(1-\frac{1}{1+x^2}\right)dx = \ln 2 - 2 + \frac{\pi}{2}$$

$\ln(1+x^2) > 1 \Rightarrow x > \sqrt{e-1}$

$$\therefore \quad I = \int_0^{\sqrt{e-1}} 1\cdot dx + \int_{\sqrt{e-1}}^1 \ln(1+x^2)dx$$

$$= \sqrt{e-1} + x\ln(1+x^2)\Big|_{\sqrt{e-1}}^1 - 2\int_{\sqrt{e-1}}^1\frac{x^2}{1+x^2}dx$$

$$= \ln 2 - 2\int_{\sqrt{e-1}}^1\left(1-\frac{1}{1+x^2}\right)dx$$

$$= \ln 2 - 2 + 2\sqrt{e-1} + \frac{\pi}{2} - 2\tan^{-1}\sqrt{e-1}$$

$$= \ln 2 - 2 + 2\sqrt{e-1} + \frac{\pi}{2} + \tan^{-1}\left(\frac{2\sqrt{e-1}}{e-2}\right).$$

20. (d) : Let $I = \int_0^{\pi/2}\frac{1}{1+\cos^2 x}dx$

$$= \int_0^{\pi/2}\frac{\sec^2 x}{\sec^2 x+1}dx = \int_0^{\pi/2}\frac{\sec^2 x}{\tan^2 x+2}dx$$

Put $\tan x = t \Rightarrow \sec^2 x dx = dt$

When $x = 0$, $t = 0$ and $x = \frac{\pi}{2}$, $t = \infty$

$$\therefore \quad I = \int_0^{\infty}\frac{dt}{t^2+(\sqrt{2})^2} = \frac{1}{\sqrt{2}}\left[\tan^{-1}\left(\frac{t}{\sqrt{2}}\right)\right]_0^{\infty} = \frac{1}{\sqrt{2}}\left[\frac{\pi}{2}-0\right] = \frac{\pi}{2\sqrt{2}}$$

21. (d) : $\int_{-1}^{+1}\left\{\frac{x^{2013}}{e^{|x|}(x^2+\cos x)}+\frac{1}{e^{|x|}}\right\}dx$

$$= 0 + \int_{-1}^{+1}\frac{dx}{e^{|x|}} \quad [\because \text{1st function is odd}]$$

$$= 2\int_0^1\frac{dx}{e^x} \quad [\because e^{|x|} \text{ is even}] \quad = 2[-e^{-x}]_0^1 = 2(-e^{-1}+e^0) = 2(1-e^{-1})$$

22. (c) : Let $I = \int_{\pi/6}^{\pi/3}\frac{dx}{1+\sqrt{\tan x}} = \int_{\pi/6}^{\pi/3}\frac{1}{1+\sqrt{\cot x}}dx$

$$\text{Adding, } 2I = \int_{\pi/6}^{\pi/3}\left(\frac{1}{1+\sqrt{\tan x}}+\frac{\sqrt{\tan x}}{1+\sqrt{\tan x}}\right)dx$$

$$= \int_{\pi/6}^{\pi/3}1\cdot dx = \left(\frac{\pi}{3}-\frac{\pi}{6}\right) = \frac{\pi}{6} \quad \therefore \quad I = \frac{\pi}{12}$$

Hence, Statement-1 is false and Statement-2 is true.

23. (c) : Let $I = \int_0^{\pi/6}\cos^4 3\theta\sin^2 6\theta d\theta$

Put $3\theta = t \Rightarrow d\theta = \frac{dt}{3}$

When $\theta = 0$, $t = 0$ and $\theta = \frac{\pi}{6}$, $t = \frac{\pi}{2}$

$$\therefore \quad I = \frac{1}{3}\int_0^{\pi/2}\cos^4 t\cdot\sin^2 2t dt = \frac{4}{3}\int_0^{\pi/2}\cos^6 t\cdot\sin^2 t dt$$

$$= \frac{4}{3}\left[\frac{1\times5\times3\times1}{8\times6\times4\times2}\right]\times\frac{\pi}{2} = \frac{5}{192}\pi \quad \left[\begin{array}{c}\text{Using reduction formula}\\ m=2, n=6\end{array}\right]$$

24. (a) : Given equation is equation of a curve in polar form. *i.e.*, $r = a\sin 2\theta$

$r = 0$ when $\sin 2\theta = 0 \Rightarrow 2\theta = \pi \Rightarrow \theta = \pi/2$

Similarly, $\theta = 0, \pi/2, \pi, \frac{3\pi}{2}, 2\pi$

Required Area = Area of one loop

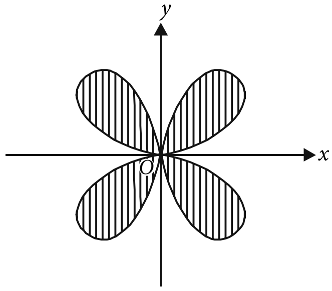
$$= \int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta,\ r = a \sin 2\theta$$

$$= \int_0^{\pi/2} \frac{1}{2}\ a^2 \sin^2 2\theta\ d\theta$$

$$= \frac{a^2}{4} \int_0^{\pi/2} 2\sin^2 2\theta\, d\theta$$

$$= \frac{a^2}{4} \int_0^{\pi/2} (1 - \cos 4\theta)\, d\theta$$

$$= \frac{a^2}{4}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\pi/2} = \frac{a^2}{4}\left[\frac{\pi}{2} - 0\right] = \frac{\pi a^2}{8} \text{ sq. units}$$



Note : Total area = No. of leaf × area of one leaf.

25. **(b) :** $(y - 2)^2 = x - 1$

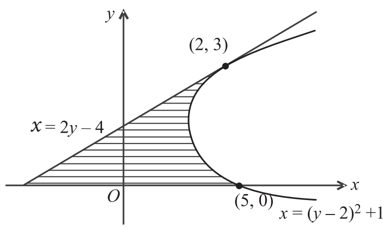
Differentiating w.r.t. x, we have $2(y - 2)y' = 1$

$$\Rightarrow\quad y' = \frac{1}{2(y-2)}$$

At (2, 3), $y' = 1/2$

The equation of the tangent to the parabola at (2, 3) is

$$y - 3 = \frac{1}{2}(x - 2) \Rightarrow x - 2y + 4 = 0$$



The area of the bounded region

$$A = \int_0^3 [(y-2)^2 + 1 - (2y-4)]\, dy$$

$$= \int_0^3 (y^2 - 6y + 9)\, dy = \int_0^3 (y-3)^2\, dy.$$

Put $t = y - 3 \Rightarrow dt = dy$

$$\therefore\quad A = \int_0^3 t^2\, dt = \left[\frac{t^3}{3}\right]_0^3 = \frac{3^3}{3} = 9 \text{ sq. units}$$

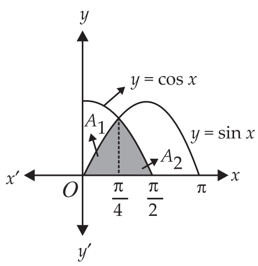
26. **(d) :** Area, $A_1 = \int_0^{\pi/4} \sin x\, dx$

$$= -[\cos x]_0^{\pi/4} = 1 - \frac{1}{\sqrt{2}} = \frac{\sqrt{2}-1}{\sqrt{2}}$$

and $A_2 = \int_{\pi/4}^{\pi/2} \cos x\, dx$

$$= [\sin x]_{\pi/4}^{\pi/2} = \left[1 - \frac{1}{\sqrt{2}}\right] = \frac{\sqrt{2}-1}{\sqrt{2}}$$

$$\therefore\quad A_1 : A_2 = \frac{\sqrt{2}-1}{\sqrt{2}} : \frac{\sqrt{2}-1}{\sqrt{2}} = 1:1$$

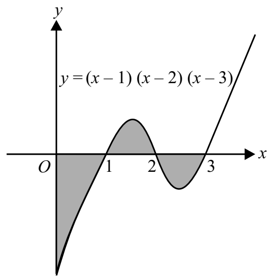


27. **(b) :** $y = (x - 1)(x - 2)(x - 3)$

$= x^3 - 6x^2 + 11x - 6$

∴ Required area

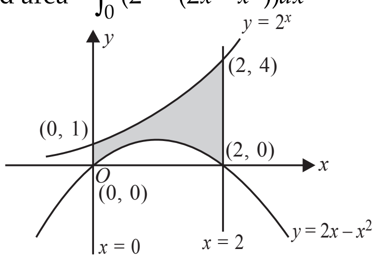
$$= \left|\int_0^1 (x^3 - 6x^2 + 11x - 6)\, dx\right| + \int_1^2 (x^3 - 6x^2 + 11x - 6) dx + \left|\int_2^3 (x^3 - 6x^2 + 11x - 6)\, dx\right|$$



$$= \left|\frac{1}{4} - 2 + \frac{11}{2} - 6\right| + \left[\frac{15}{4} - 14 + \frac{33}{2} - 6\right] + \left|\frac{65}{4} - 38 + \frac{55}{2} - 6\right|$$

$$= \frac{9}{4} + \frac{1}{4} + \frac{1}{4} = \frac{11}{4}$$

28. **(d) :** Required area $= \int_0^2 (2^x - (2x - x^2))dx$



$$= \left[\frac{2^x}{\ln 2} - x^2 + \frac{x^3}{3}\right]_0^2 = \frac{3}{\ln 2} - 4 + \frac{8}{3} = \frac{3}{\ln 2} - \frac{4}{3}.$$

29. **(b) :** The desired area $= \int_0^{\pi/4} \left(\sqrt{\frac{1+\sin x}{\cos x}} - \sqrt{\frac{1-\sin x}{\cos x}}\right) dx$

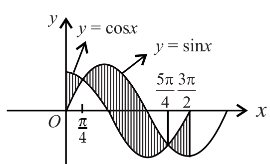
Now $\dfrac{1+\sin x}{\cos x} = \dfrac{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^2}{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}} = \dfrac{\cos\frac{x}{2} + \sin\frac{x}{2}}{\cos\frac{x}{2} - \sin\frac{x}{2}} = \dfrac{1 + \tan\frac{x}{2}}{1 - \tan\frac{x}{2}}$

The expression for area reduces to

$$\int_0^{\pi/4} \left(\sqrt{\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}} - \sqrt{\frac{1-\tan\frac{x}{2}}{1+\tan\frac{x}{2}}}\right) dx = \int_0^{\pi/4} \frac{2\tan\frac{x}{2}}{\sqrt{1-\tan^2\frac{x}{2}}}\, dx$$

Let $\tan\frac{x}{2} = t \Rightarrow$ The expression $= \int_0^{\sqrt{2}-1} \frac{4t}{(1+t^2)\sqrt{1-t^2}}\, dt$

30. **(a) :**



The desired area

$$= \int_0^{\pi/4} (\cos x - \sin x)dx + \int_{\pi/4}^{5\pi/4} (\sin x - \cos x)dx + \int_{5\pi/4}^{3\pi/2} (\cos x - \sin x)dx$$

$= 2[\sin x + \cos x]_0^{\pi/4} + [-\cos x - \sin x]_{\pi/4}^{5\pi/4}$

(As the first and third integrals are equal in magnitude)

$= 2\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 1\right) + \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)$

$= \frac{8}{\sqrt{2}} - 2 = (4\sqrt{2} - 2)$ sq. units

31. **(a, b, c) :** Since $\int_{-a}^{a} f(x)\,dx = \int_0^a (f(x) + f(-x))\,dx$

$I_n = \int_0^{\pi} \frac{\sin nx}{\sin x}\left(\frac{1}{1+\pi^x} + \frac{1}{1+\pi^{-x}}\right)dx \Rightarrow I_n = \int_0^{\pi} \frac{\sin nx}{\sin x}dx$

$I_{n+2} - I_n = \int_0^{\pi} \frac{\sin(n+2)x - \sin nx}{\sin x}\,dx$

$= 2\int_0^{\pi} \cos(n+1)x\,dx = 0 \Rightarrow I_n = I_{n+2}$

$I_0 = I_2 = I_4 = \ldots\ldots\ldots = 0$

$I_1 = I_3 = I_5 = \ldots\ldots\ldots = \pi$

$I_1 + I_3 + \ldots + I_{21} = 10\pi$

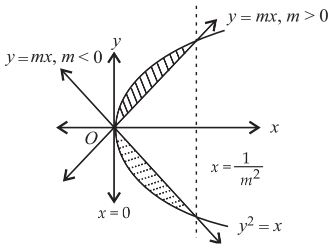
$I_0 + I_2 + \ldots\ldots + I_{20} = 0.$

32. **(a, d) :** $y^2 = x,\ y = mx$

On solving, $m^2x^2 = x$

$\Rightarrow x(m^2x - 1) = 0 \Rightarrow x = 0,\ \frac{1}{m^2}$

$\because$ Area $= \frac{1}{48}$



$\therefore \int_0^{1/m^2} (\sqrt{x} - mx)dx = \frac{1}{48}$

$\Rightarrow \left[\frac{x^{3/2}}{3/2} - \frac{mx^2}{2}\right]_0^{1/m^2} = \frac{1}{48} \Rightarrow \frac{2}{3}\cdot\frac{1}{m^3} - \frac{1}{2m^3} = \frac{1}{48}$

$\Rightarrow \frac{1}{6m^3} = \frac{1}{48} \Rightarrow m^3 = 8 \Rightarrow m = 2$

Similarly, for other chord, $m = -2$. $\therefore m = \pm 2$

33. **(a, b) :** $f(x) = (7\tan^6 x - 3\tan^2 x)(\tan^2 x + 1)$

$= (7\tan^6 x - 3\tan^2 x)\sec^2 x$

Now, $\int_0^{\pi/4} f(x)dx = \int_0^1 (7t^6 - 3t^2)dt = \left|(t^7 - t^3)\right|_0^1 = 0$

Again, $\int_0^{\pi/4} x\,f(x)dx = \int_0^1 \tan^{-1} t(7t^6 - 3t^2)dt$

$= \tan^{-1} t[t^7 - t^3]_0^1 - \int_0^1 (t^7 - t^3)\frac{1}{1+t^2}dt = \int_0^1 t^3(1 - t^2)dt$

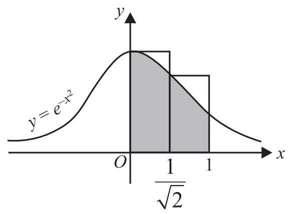
$= \left[\frac{t^4}{4} - \frac{t^6}{6}\right]_0^1 = \frac{1}{4} - \frac{1}{6} = \frac{1}{12}$

34. **(a, b, c) :** $I = \int_{-\pi}^{\pi} \sin mx \cos nx\,dx$

$= \int_{-\pi}^{\pi} \sin m(0 - x)\cos n(0 - x)dx = -\int_{-\pi}^{\pi} \sin mx \cos nx\,dx$

$I = -I \Rightarrow 2I = 0 \Rightarrow I = 0.$

35. **(a, b, d) :** $S = \int_0^1 e^{-x^2}dx$



As $0 \le x^2 \le 1$

$\Rightarrow -1 \le -x^2 \le 0$

$\frac{1}{e} \le e^{-x^2} \le 1$

$S \ge \int_0^1 \frac{1}{e}dx \Rightarrow S \ge \frac{1}{e}$

As $e^{-x^2} \ge e^{-x},\ \forall x \in (0, 1)$

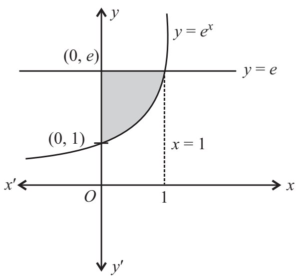
We have $S \ge \int_0^1 e^{-x}dx = [-e^{-x}]_0^1 = 1 - \frac{1}{e}$

Again area of two rectangles

$= 1\left(\frac{1}{\sqrt{2}} - 0\right) + e^{-\frac{1}{2}}\left(1 - \frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{e}}\left(1 - \frac{1}{\sqrt{2}}\right)$

Thus, $S \le \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{e}}\left(1 - \frac{1}{\sqrt{2}}\right)$

36. **(b, c, d) :**



The shaded region represented the area which is equal to

$\int_1^e \ln y\,dy = \int_1^e \ln(e + 1 - y)dy$

or it can be, when considered w.r.t. x as

$e - \int_0^1 e^x dx = e - [e^x]_0^1 = e - [e - 1] = 1 \ne e - 1$

37. **(a, c) :** $I_1 = \int_0^{4\pi} e^t(\sin^6 at + \cos^4 at)dt$

$= \sum_{k=0}^{3} \int_{k\pi}^{(k+1)\pi} e^t(\sin^6 at + \cos^4 at)dt$

$= \sum_{k=0}^{3} \int_0^{\pi} e^{(k\pi + t)}(\sin^6 at + \cos^4 at)dt$

$= \left(\sum_{k=0}^{3} e^{k\pi}\right)\int_0^{\pi} e^t(\sin^6 at + \cos^4 at)dt$

$= (1 + e^{\pi} + e^{2\pi} + e^{3\pi})\int_0^{\pi} e^t(\sin^6 at + \cos^4 at)dt$

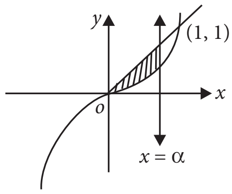
$\therefore$ The given expression $= 1 + e^{\pi} + e^{2\pi} + e^{3\pi} = \frac{e^{4\pi} - 1}{e^{\pi} - 1}$

Also, $a = 2$ and $a = 4$ both cases hold.

38. (a, d) : We have $\int_0^{\alpha}(x-x^3)dx = \frac{1}{2}\int_0^1 (x-x^3)dx$

$\Rightarrow \left[\frac{x^2}{2}-\frac{x^4}{4}\right]_0^{\alpha} = \frac{1}{2}\left[\frac{x^2}{2}-\frac{x^4}{4}\right]_0^1$

$\Rightarrow \frac{\alpha^2}{2}-\frac{\alpha^4}{4} = \frac{1}{8}$



$\therefore \quad 2\alpha^4 - 4\alpha^2 + 1 = 0$

Solving as a quadratic in α^2, we have

$\alpha^2 = \frac{4\pm\sqrt{16-8}}{2\cdot 2} = \frac{4\pm 2\sqrt{2}}{4} = 1\pm\frac{1}{\sqrt{2}}$

As $\alpha < 1$ we have

$\alpha^2 = 1-\frac{1}{\sqrt{2}} = \frac{1}{4}+\frac{3}{4}-\frac{1}{\sqrt{2}} = \frac{1}{4}+\left(\frac{3-2\sqrt{2}}{4}\right) > \frac{1}{4}$

Thus, $\alpha > \frac{1}{2}$

39. (a) : (A)-(s), (B)-(r), (C)-(p), (D)-(s)

(A) $\int_0^2 \sqrt{2x-x^2}\,dx = \int_0^2 \sqrt{1-(x-1)^2}\,dx$,

[Put $x - 1 = \sin\theta \Rightarrow dx = \cos\theta\, d\theta$]

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1-\sin^2\theta}\cos\theta\, d\theta = 2\int_0^{\frac{\pi}{2}} \cos^2\theta\, d\theta$

$= 2\times\frac{\pi}{4} = \frac{\pi}{2}$

(B) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\cos x-\cos^3 x}\,dx = 2\int_0^{\frac{\pi}{2}} \sqrt{\cos x-\cos^3 x}\,dx$

$= 2\int_0^{\frac{\pi}{2}} \sqrt{\cos\cdot x}\,\sin x\,dx = -\frac{4}{3}(\cos x)^{\frac{3}{2}}\Big|_0^{\frac{\pi}{2}} = \frac{4}{3}$

(C) $\int_0^{\frac{\pi^2}{4}} \sin\sqrt{x}\,dx$, [Put $x = t^2$]

$= \int_0^{\frac{\pi}{2}} t\sin t\,dt = -t\cos t+\sin t\Big|_0^{\frac{\pi}{2}} = 2$

(D) $-2\int_0^{\frac{\pi}{2}} \cos 2x \ln\sin x\,dx$

$= -2\left[\frac{\sin 2x}{2}\ln\sin x\Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}}\cos^2 x\,dx\right] = 2\cdot\frac{\pi}{4} = \frac{\pi}{2}$

40. (a) : (A)-(s), (B)-(p), (C)-(q), (D)-(r)

(A) $y^2 = 9x$, $y = 3x \Rightarrow x = 0, 1$

Area $= \int_0^1\left(3\sqrt{x}-3x\right)dx = 3\left(\frac{2}{3}-\frac{1}{2}\right) = \frac{1}{2}$

(B) Area $= \int_0^2 \frac{x}{\sqrt{2x^2+1}}dx = \frac{1}{2}\sqrt{2x^2+1}\Big|_0^2 = \frac{3-1}{2} = 1$

(C) Area $= \int_0^{\frac{\pi}{3}} \sin^2 3x\,dx$, Substitute $3x = t$ we get

Area $= \frac{1}{3}\int_0^{\pi}\sin^2 t\,dt = \frac{1}{3}\cdot\frac{1}{2}\cdot\pi = \frac{\pi}{6}$

(D) The curve is symmetric about x and y-axis.

$\therefore$ Area $= 4\int_0^1\left(1-x^{\frac{2}{3}}\right)^{\frac{3}{2}}dx$, [Put $x = \sin^3\theta$]

$= 4\int_0^{\frac{\pi}{2}}\cos^3\theta\cdot 3\sin^2\theta\cos\theta\,d\theta$

$= 12\int_0^{\frac{\pi}{2}}\cos^4\theta\cdot\sin^2\theta\,d\theta = 12\cdot\frac{3\cdot 1}{6\cdot 4\cdot 2}\frac{\pi}{2} = \frac{3\pi}{8}$

41. (d) : $I = \int_0^{\pi}\sin 2x\sin\left(\frac{\pi}{2}\cos x\right)dx$

$= 2\int_0^{\pi}\cos x\sin\left(\frac{\pi}{2}\cos x\right)\sin x\,dx$

Substitute $\frac{\pi}{2}\cos x = t \Rightarrow -\frac{\pi}{2}\sin x\,dx = dt$

$\Rightarrow I = -\frac{8}{\pi^2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} t\sin t\,dt = -\frac{16}{\pi^2}\int_0^{\frac{\pi}{2}} t\sin t\,dt$

$= -\frac{16}{\pi^2}[-t\cos t+\sin t]_0^{\frac{\pi}{2}} = -\frac{16}{\pi^2}$.

42. (a) : $I = \int_0^{\pi}\frac{\sin 2x\sin\left(\frac{\pi}{2}\cos x\right)}{2x-\pi}dx$

Replacing x by $\pi - x$, we get

$I = \int_0^{\pi}\frac{\sin 2(\pi-x)\sin\left(\frac{\pi}{2}\cos(\pi-x)\right)}{2(\pi-x)-\pi}dx$

$= -\int_0^{\pi}\frac{\sin 2x}{2x-\pi}\sin\left(\frac{\pi}{2}\cos x\right)dx = -I \Rightarrow I = 0.$

43. (c) : $I = \int_0^{\pi}\frac{x\sin 2x\sin\left(\frac{\pi}{2}\cos x\right)}{2x-\pi}dx$

Replacing x by $\pi - x$, we get

$I = -\int_0^{\pi}\frac{(\pi-x)\sin 2x\sin\left(\frac{\pi}{2}\cos x\right)}{(2x-\pi)}dx$

Adding, $2I = \int_0^{\pi} f(x)dx = -\frac{16}{\pi^2} \Rightarrow I = -\frac{8}{\pi^2}$.

44. (a) : The desired area $= \int_a^b f(x)dx$

Integrating by parts, taking $f(x)$ as the first function

$A = [x\cdot f(x)]_a^b - \int_a^b x f'(x)dx = bf(b)-af(a)-\int_a^b xf'(x)dx$ ...(i)

Now, $y^3 - 3y + x = 0$, where $y = f(x)$ (by hyp.)

$\Rightarrow \quad 3y^2y' - 3y' + 1 = 0. \ \therefore \ y' = f'(x)$

$\Rightarrow \quad y'(3y^2 - 3) = -1 \Rightarrow y' = \dfrac{-1}{3[(f(x))^2 - 1]}$

$\therefore$ From (i), $A = bf(b) - af(a) + \displaystyle\int_a^b \frac{x}{3[(f(x))^2 - 1]}dx.$

45. **(d) :** As obtained earlier

$y' = \dfrac{1}{3[1 - \{f(x)\}^2]}$ which is an even function

$\therefore \displaystyle\int_{-1}^{1} g'(x)dx = 2\int_0^1 g'(x)dx = 2g(1)$

46. **(3) :** $\displaystyle\int_0^2 (|x-2| + [x])dx = \int_0^2 |x-2|\,dx + \int_0^2 [x]\cdot dx$

$= \displaystyle\int_0^2 -(x-2)dx + \int_0^1 [x]dx + \int_1^2 [x]dx$

$= \left[2x - \dfrac{x^2}{2}\right]_0^2 + 0 + \displaystyle\int_1^2 1dx$

$= (4 - 2) + (2 - 1) = 2 + 1 = 3$

47. **(3) :** Let $I = \int_0^{\pi} x \sin^4 x \, dx$ ...(i)

$\Rightarrow I = \int_0^{\pi} (\pi - x)\sin^4 x \, dx$ ...(ii)

Adding (i) and (ii), we get

$2I = \pi\int_0^{\pi} \sin^4 x \, dx = 2\pi \int_0^{\pi/2} \sin^4 x \, dx$

$= 2\pi \displaystyle\int_0^{\pi/2} \left(\frac{3}{8} + \frac{\cos 4x}{8} - \frac{\cos 2x}{2}\right)dx$

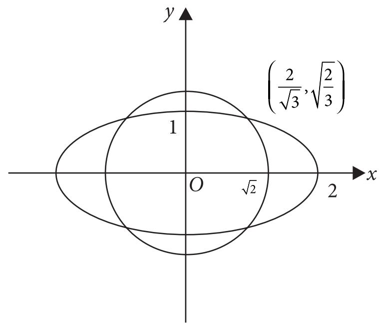
$= 2\pi\left[\dfrac{3}{8}x + \dfrac{\sin 4x}{32} - \dfrac{\sin 2x}{4}\right]_0^{\pi/2}$

$= 2\pi\left[\dfrac{3}{8}\left(\dfrac{\pi}{2}\right)\right] = \dfrac{3\pi^2}{8}$

$\Rightarrow I = \dfrac{3\pi^2}{16}$

48. **(2) :** $x^2 + y^2 = 2, x^2 + 4y^2 = 4$

$\Rightarrow x = \dfrac{2}{\sqrt{3}}$



Common areas $= 4\left[\displaystyle\int_0^{\frac{2}{\sqrt{3}}} \frac{1}{2}\sqrt{4 - x^2}dx + \int_{\frac{2}{\sqrt{3}}}^{\sqrt{2}} \sqrt{2 - x^2}dx\right]$

$= 4\left[\left(\dfrac{x}{4}\sqrt{4 - x^2} + \sin^{-1}\dfrac{x}{2}\right)\Bigg|_0^{\frac{2}{\sqrt{3}}} + \left(\dfrac{x}{2}\sqrt{2 - x^2} + \sin^{-1}\dfrac{x}{\sqrt{2}}\right)_{\frac{2}{\sqrt{3}}}^{\sqrt{2}}\right]$

$= 8\sin^{-1}\dfrac{1}{\sqrt{3}} = 8\tan^{-1}\dfrac{1}{\sqrt{2}} = 4\tan^{-1} 2\sqrt{2}$, then $b/a = 2$

49. **(3) :** $f(\sin x) = \dfrac{1}{\sin^2 x} \Rightarrow f(x) = \dfrac{1}{x^2} \Rightarrow f\left(\dfrac{1}{\sqrt{3}}\right) = 3$

50. **(0) :** $I = \displaystyle\int_{-1}^{2} \frac{x[x^2]\,dx}{2 + [x+1]}$

Splitting the integrand, we get

$= \displaystyle\int_{-1}^{0} \frac{x\cdot 0}{2+0}dx + \int_0^1 \frac{x\cdot 0}{2+1}dx + \int_1^{\sqrt{2}} \frac{x\cdot 1}{2+0}dx + 0$

$= \dfrac{1}{2}\displaystyle\int_1^{\sqrt{2}} x\,dx = \frac{x^2}{4}\Bigg|_1^{\sqrt{2}} = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

$\therefore \quad 4I = 1 \Rightarrow 4I - 1 = 0$

MATHEMATICS

DPS-19

DAILY PRACTICE SHEET

Class XII

Differential Equations

SOLUTIONS

1. **(c) :** $(e^y + 1)\cos x dx + e^y \sin x dy = 0$

$\Rightarrow \dfrac{e^y dy}{e^y + 1} + \dfrac{\cos x}{\sin x} dx = 0$

On integrating both sides, we get

$\log(e^y + 1) + \log(\sin x) = \log c \Rightarrow (e^y + 1)\sin x = c$

2. **(d) :** $y^2\left(\dfrac{d^2y}{dx^2}\right)^2 + x^2y^2 - \sin x = -3x\left(\dfrac{dy}{dx}\right)^{\frac{1}{3}}$

$\left(y^2\left(\dfrac{d^2y}{dx^2}\right)^2 + x^2y^2 - \sin x\right)^3 = -27x^3\left(\dfrac{dy}{dx}\right)$

Here, order = 2 = p

Degree = 6 = q

$\therefore \quad p < q$

3. **(d) :** Differentiating, $x^2 = 4a(y - h)$

$\Rightarrow \quad x = 2ay_1,\ 1 = 2ay_2$

Eliminating a, $xy_2 - y_1 = 0$.

4. **(a) :** On putting $x = \tan A$, $y = \tan B$, we get

$\sec A + \sec B = \lambda(\tan A \sec B - \tan B \sec A)$

$\Rightarrow \cos A + \cos B = \lambda(\sin A - \sin B)$

$\Rightarrow \tan\left(\dfrac{A - B}{2}\right) = \dfrac{1}{\lambda}$

$\tan^{-1} x - \tan^{-1} y = 2\tan^{-1}\dfrac{1}{\lambda}$

On differentiating, $\dfrac{1}{1+x^2} - \dfrac{1}{1+y^2}\dfrac{dy}{dx} = 0$

5. **(a) :** The parametric form of the given equation is $x = t$, $y = t^2$. The equation of any tangent at t is $2xt = y + t^2$, Differentiating, we get $2t = y_1$

Putting this value in the above equation, we get

$$2x\frac{y_1}{2} = y + \left(\frac{y_1}{2}\right)^2 \Rightarrow 4xy_1 = 4y + {y_1}^2$$

The order of this equation is 1.

6. **(a) :** $y = (\cos x + y)^{1/2} \Rightarrow y^2 = \cos x + y$

$\Rightarrow \quad 2y \cdot \dfrac{dy}{dx} = -\sin x + \dfrac{dy}{dx} \Rightarrow (2y - 1)\dfrac{dy}{dx} = -\sin x$

$\Rightarrow \quad (2y - 1)\dfrac{d^2y}{dx^2} + 2\left(\dfrac{dy}{dx}\right)^2 = -\cos x$

7. **(b) :** The given differential equation can be written as

$\dfrac{d^3y}{dx^3} + 7\dfrac{d^2y}{dx^2} + y = \cos x$

[By differentiating both sides w.r.t. x]

Hence, order = 3 and degree = 1.

8. **(b) :** $y^2 = a(b - x^2)$ or $y^2 = ab - ax^2$

Differentiating with respect to x, we get

$$2y\frac{dy}{dx} = -2ax \Rightarrow a = -\frac{y}{x}\frac{dy}{dx}$$

Again, differentiating with respect to x, we have

$0 = -\dfrac{y}{x}\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx}\left[-\dfrac{1}{x}\dfrac{dy}{dx} + \dfrac{y}{x^2}\right]$

$= -xy\dfrac{d^2y}{dx^2} - x\left(\dfrac{dy}{dx}\right)^2 + y\dfrac{dy}{dx}$

$\Rightarrow \quad xy\dfrac{d^2y}{dx^2} + x\left(\dfrac{dy}{dx}\right)^2 - y\dfrac{dy}{dx} = 0$

9. **(d) :** $x = \sin t$, $y = \cos pt$

$\dfrac{dx}{dt} = \cos t; \dfrac{dy}{dt} = -p\sin pt \Rightarrow \dfrac{dy}{dx} = \dfrac{-p\sin pt}{\cos t}$

$\dfrac{d^2y}{dx^2} = \dfrac{-\cos t\, p^2 \cos pt(dt/dx) - p\sin pt \sin t(dt/dx)}{\cos^2 t}$

$\Rightarrow \quad (1 - x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + p^2y = 0$

or $(1 - x^2)y_2 - xy_1 + p^2y = 0$.

10. **(a) :** The circles are $(x - h)^2 + (y - h)^2 = r^2$ ...(i)

On differentiating, we get

$x - h + (y - h)y_1 = 0$ ...(ii)

Again differentiating, we get

$1 + {y_1}^2 + (y - h)y_2 = 0$ ...(iii)

From (ii) $\Rightarrow h = \dfrac{x + yy_1}{1 + y_1}$, $y - h = \dfrac{y - x}{1 + y_1}$

From (iii) $\Rightarrow (1 + y_1^2)(1 + y_1) = (x - y)y_2$

11. **(b) :** $xe^{\frac{-y}{x}} = a + bx \Rightarrow \dfrac{d^2}{dx^2}\left(xe^{-\frac{y}{x}}\right) = 0$

$\Rightarrow \quad \dfrac{d}{dx}\left[e^{-\frac{y}{x}} + xe^{-\frac{y}{x}}\dfrac{(xy_1 - y)(-1)}{x^2}\right] = 0$

$\Rightarrow \quad \dfrac{d}{dx}\left[e^{-\frac{y}{x}}\left(1 - y_1 + \dfrac{y}{x}\right)\right] = 0$

$$\Rightarrow -\frac{(xy_1-y)}{x^2}\left(1-y_1+\frac{y}{x}\right)-y_2+\frac{xy_1-y}{x^2}=0$$

$$\Rightarrow x^3y_2=(xy_1-y)^2$$

12. **(b) :** Consider the general equation of all conics

$ax^2+2hxy+by^2+2gx+2fy+c=0$ ...(i)

As centre of the conic is (0, 0) $\Rightarrow g=0, f=0$

$\therefore$ (i) becomes $ax^2+2hxy+by^2=-c$

or $\left(\frac{a}{-c}\right)x^2+\left(\frac{2h}{-c}\right)xy+\left(\frac{b}{-c}\right)y^2=1$

or $Ax^2+2Hxy+By^2=1$ is the general equation of all conics whose centre lie at origin.

Since, it has three arbitrary constants.

Hence, order of the differential equation is 3.

13. **(b) :** The equation of straight line touching the circle $x^2+y^2=a^2$ is

$x\cos\theta+y\sin\theta=a$ ...(i)

On differentiating w.r.t. x, keeping θ as a constant

$\cos\theta+y'\sin\theta=0$ ...(ii)

From (i) and (ii), we get

$$\cos\theta=\frac{ay'}{xy'-y} \quad\text{and}\quad \sin\theta=-\frac{a}{xy'-y}$$

$\because \cos^2\theta+\sin^2\theta=1 \ \therefore \ \frac{a^2y'^2+a^2}{(xy'-y)^2}=1$

$$\Rightarrow \left(y-x\frac{dy}{dx}\right)^2=a^2\left[1+\left(\frac{dy}{dx}\right)^2\right]$$

14. **(b) :** Since, the parabola is symmetric about a line parallel to x-axis.

So, the parabola may be leftward or rightward.

Let us consider the parabola is rightward.

$\therefore$ The family of parabola (rightward) whose axis of symmetry is parallel to x-axis is represented by the equation.

$(y-h)^2=4a(x-k)$, where h, k and a are arbitrary constants.

Since, there are three arbitrary constants in the equation.

So, order of differential equation = 3.

15. **(a) :** $\sqrt{1+x}-a\sqrt{1+y}=1$ ...(i)

Since the given equation of curve has only one constant, therefore it is differentiated only once.

$$\frac{1}{2\sqrt{1+x}}-\frac{a}{2\sqrt{1+y}}\frac{dy}{dx}=0$$

$$\Rightarrow a\frac{dy}{dx}=\frac{\sqrt{1+y}}{\sqrt{1+x}} \Rightarrow a=\sqrt{\frac{1+y}{1+x}}\cdot\frac{dx}{dy}$$

Put value of a in (i), we get

$$\sqrt{1+x}-\sqrt{\frac{1+y}{1+x}}\cdot\sqrt{1+y}\cdot\frac{dx}{dy}=1 \Rightarrow (1+x)-(1+y)\cdot\frac{dx}{dy}=\sqrt{1+x}$$

Hence degree of differential equation is 1.

16. **(d) :** We have, length of normal = c

$$\Rightarrow y\sqrt{1+\left(\frac{dy}{dx}\right)^2}=c \Rightarrow y^2\left[1+\left(\frac{dy}{dx}\right)^2\right]=c^2$$

Clearly, this is the differential equation of degree 2.

17. **(a) :** Since the given solution contains two constants so, we differentiate the differential equation two times.

Hence, the order of differential equation is 2.

18. **(a) :** Equation of line whose slope = y intercept is $y=cx+c$

$\Rightarrow y=c(x+1)$ ...(i)

Now differentiating w.r.t. x, we get $\frac{dy}{dx}=c$ ...(ii)

$\therefore$ Required diff. equation is $y=\frac{dy}{dx}(x+1)$

(By substituting (ii) in (i))

$$\Rightarrow \frac{dy}{dx}(x+1)-y=0$$

19. **(a) :** Put $x=\sin\alpha$, $y=\sin\beta$, $\sqrt{1-x^2}+\sqrt{1-y^2}=a(x-y)$

$\Rightarrow \cos\alpha+\cos\beta=a(\sin\alpha-\sin\beta)$

$$2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)=a\left(2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)\right)$$

$$\Rightarrow \cot\left(\frac{\alpha-\beta}{2}\right)=a \Rightarrow \frac{\alpha-\beta}{2}=\cot^{-1}a$$

$\Rightarrow \alpha-\beta=2\cot^{-1}a \Rightarrow \sin^{-1}x-\sin^{-1}y=2\cot^{-1}a$

On differentiating w.r.t x, we get

$$\Rightarrow \frac{1}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-y^2}}\frac{dy}{dx}=0$$

$\therefore$ Degree of the differential equation is 1.

20. **(d) :** $y=ae^{-3x}+b$

$\therefore \frac{dy}{dx}=-3ae^{-3x}$ ...(i) and $\frac{d^2y}{dx^2}=9ae^{-3x}$ ...(ii)

Eliminate a from (i) and (ii) by dividing, we get

$$\frac{\frac{dy}{dx}}{\frac{d^2y}{dx^2}}=\frac{-1}{3}\Rightarrow \frac{d^2y}{dx^2}+3\frac{dy}{dx}=0$$

21. **(b) :** Rewrite the differential equation as

$$\frac{dy}{dx}=\frac{1}{8\sqrt{x}\sqrt{9+\sqrt{x}}}\cdot\frac{1}{\sqrt{4+\sqrt{9+\sqrt{x}}}}$$

Let $P=\sqrt{4+\sqrt{9+\sqrt{x}}}$

Then, $\frac{dP}{dx}=\frac{1}{2\sqrt{4+\sqrt{9+\sqrt{x}}}}\cdot\frac{1}{2\sqrt{9+\sqrt{x}}}\cdot\frac{1}{2\sqrt{x}}$

Now, we have $\frac{dy}{dx}=\frac{dP}{dx}$

Then, $y=P(x)+\lambda$

$\Rightarrow y=\sqrt{4+\sqrt{9+\sqrt{x}}}+\lambda$

But $x=0 \Rightarrow y=\sqrt{4+3}+\lambda$

As $y=\sqrt{7} \therefore \lambda=0$

Then, $y=\sqrt{4+\sqrt{9+\sqrt{x}}}$

$\therefore$ At $x=256$, $y=\sqrt{4+\sqrt{9+\sqrt{256}}}=\sqrt{4+\sqrt{9+16}}$

$=\sqrt{4+5}=3$

22. **(a) :** $x^2+y^2=r^2 \Rightarrow x\,dx+y\,dy=r\,dr$

$x\,dy-y\,dx=r\cos\theta\,(dr\sin\theta+r\cos\theta\,d\theta)$

$-r\sin\theta\,(dr\cos\theta-r\sin\theta\,d\theta)=r^2\,d\theta.$

The D.E. reduces to $\frac{rdr}{r^2d\theta}=\sqrt{\frac{1-r^2}{r^2}}, \frac{dr}{\sqrt{1-r^2}}=d\theta$

Integrating, $\sin^{-1}r=\theta+\alpha$

$r = \sin(\theta + \alpha) = \sin\theta\cos\alpha + \cos\theta\sin\alpha = \frac{y}{r}\cos\alpha + \frac{x}{r}\sin\alpha$

$\therefore \quad x^2 + y^2 - x\sin\alpha - y\cos\alpha = 0.$

which represents the family of circles through the origin with radius

$$\sqrt{\frac{\sin^2\alpha}{4} + \frac{\cos^2\alpha}{4}} = \frac{1}{2}.$$

23. (b) : $\frac{d^2y}{dx^2} = e^{-2x} \Rightarrow \frac{dy}{dx} = \frac{e^{-2x}}{-2} + c$

$\Rightarrow \quad y = \frac{e^{-2x}}{4} + cx + d.$

24. (c) : $\frac{dt}{dx} - t\frac{g'(x)}{g(x)} = -\frac{t^2}{g(x)}$

$\Rightarrow \quad -\frac{1}{t^2}\frac{dt}{dx} + \frac{1}{t}\frac{g'(x)}{g(x)} = \frac{1}{g(x)}$

let $\frac{1}{t} = z \therefore -\frac{1}{t^2}\frac{dt}{dx} = \frac{dz}{dx}$ ...(i)

Now by (i), we have

$\frac{dz}{dx} + z\frac{g'(x)}{g(x)} = \frac{1}{g(x)}$

It is of type $\frac{dy}{dx} + Py = Q \;\therefore$ IF $e^{\int\frac{g'(x)}{g(x)}dx} = g(x)$

$\Rightarrow \quad \frac{d}{dx}(zg(x)) = 0$

$\Rightarrow \quad z\,g(x) = x + c \quad \Rightarrow \frac{1}{t}g(x) = x + c \;\Rightarrow\; t = \frac{g(x)}{x+c}$

25. (b) : The given differential equation can be written as

$(3x^2y^4 + 2xy)dx = (x^2 - 2x^3y^3)dy$

$\Rightarrow \quad 3x^2y^4dx + 2x^3y^3dy + 2xy\,dx - x^2dy = 0$

$\Rightarrow \quad y^2(3x^2y^2dx + 2x^3\,ydy) + 2xy\,dx - x^2dy = 0$

$\Rightarrow \quad y^2[y^2d(x)^3 + x^3d(y)^2] + yd(x)^2 - x^2dy = 0$

$\Rightarrow \quad y^2d\,(x)^3 + x^3d\,(y^2) + \frac{yd(x)^2 - x^2dy}{y^2} = 0$

$\Rightarrow \; d(x^3y^2) + d\left(\frac{x^2}{y}\right) = 0$

On integration, we obtain $x^3y^2 + \frac{x^2}{y} = c,$ as the required solution.

26. (a) : From given

$t = 1 + \left((ty)\frac{dy}{dt}\right) + \frac{\left((ty)\frac{dy}{dt}\right)^2}{2!} + ...\infty \;\Rightarrow\; t = e^{ty\left(\frac{dy}{dt}\right)}$

$\Rightarrow \quad \log t = ty\frac{dy}{dt} \Rightarrow ydy = \frac{\log t}{t}dt$

$\Rightarrow \quad \frac{y^2}{2} = \frac{(\log t)^2}{2} + k$

$\Rightarrow \quad y = \pm\sqrt{(\log t)^2 + c}$

27. (a) : $\frac{dy}{dx} + \left(\frac{1}{\sin x} + \cot x + \frac{1}{x}\right)y = \frac{1}{x}$

I.F. $= \exp\int\left(\frac{1}{\sin x} + \cot x + \frac{1}{x}\right)dx = \exp\ln\left(x\tan\frac{x}{2}\sin x\right)$

$= x\tan\frac{x}{2} \times 2\sin\frac{x}{2}\cos\frac{x}{2} = x(1 - \cos x)$

Solution, $yx(1 - \cos x) = \int\frac{1}{x}\cdot x(1-\cos x)dx = x - \sin x + c$

$y\left(\frac{\pi}{2}\right) = 1 - \frac{2}{\pi} \Rightarrow c = 0$

$\therefore \quad y(x) = \frac{x - \sin x}{x(1-\cos x)}$

$$= \frac{x - \left(x - \frac{x^3}{6} + \ldots\right)}{x\left(1 - \left(1 - \frac{x^2}{2} + \ldots\right)\right)} = \frac{x^2}{6}\frac{1}{\frac{x^2}{2}} = \frac{1}{3} \text{ as } x \to 0.$$

28. (a) : The given equation can be written as

$\left(\frac{dx}{x} - \frac{dy}{y}\right) + \frac{(x^2dy - y^2dx)}{(x-y)^2} = 0$

$\Rightarrow \quad \left(\frac{dx}{x} - \frac{dy}{y}\right) + \frac{\left(\frac{dy}{y^2} - \frac{dx}{x^2}\right)}{\left(\frac{1}{y} - \frac{1}{x}\right)^2} = 0$

$\Rightarrow \quad \left(\frac{dx}{x} - \frac{dy}{y}\right) + \frac{\frac{dy}{y^2} - \frac{dx}{x^2}}{\left(\frac{1}{x} - \frac{1}{y}\right)^2} = 0$

Integrating, we get

$\ln|x| - \ln|y| - \frac{1}{\left(\frac{1}{x} - \frac{1}{y}\right)} = c$

or $\ln\left|\frac{x}{y}\right| - \frac{xy}{(y-x)} = c$ or $\ln\left|\frac{x}{y}\right| + \frac{xy}{x-y} = c$

29. (d) : We have, $\frac{dy}{dx} + \frac{y}{2}\sec x = \frac{\tan x}{2y}$

$\Rightarrow \quad 2y\frac{dy}{dx} + y^2\sec x = \tan x$ ...(i)

Put $y^2 = t \Rightarrow 2y\frac{dy}{dx} = \frac{dt}{dx}$

$\therefore$ Equation (i) becomes, $\frac{dt}{dx} + t\sec x = \tan x$

I.F. $= e^{\int\sec x dx} = e^{\ln(\sec x + \tan x)} = \sec x + \tan x$

$\therefore$ Solution is given by

$t(\sec x + \tan x) = \int\tan x(\sec x + \tan x)dx$

$\Rightarrow \quad t(\sec x + \tan x) = \sec x + \tan x - x + c$

$\Rightarrow \quad t = 1 - \frac{x+c}{\sec x + \tan x} \Rightarrow y^2 = 1 - \frac{x+c}{\sec x + \tan x}$

Now, $y(0) = 1 \Rightarrow 1 = 1 - 0 + c \Rightarrow c = 0$

$\therefore$ Particular solution is $y^2 = 1 - \frac{x}{\sec x + \tan x}$

30. **(d) :** We have, $\dfrac{\sqrt{x}dx+\sqrt{y}dy}{\sqrt{x}dx-\sqrt{y}dy}=\sqrt{\dfrac{y^3}{x^3}}$

$$\Rightarrow \frac{d(x^{3/2})+d(y^{3/2})}{d(x^{3/2})-d(y^{3/2})}=\frac{y^{3/2}}{x^{3/2}}$$

$$\Rightarrow \frac{du+dv}{du-dv}=\frac{v}{u}, \text{ where } u=x^{3/2} \text{ and } v=y^{3/2}$$

$\Rightarrow$ $udu + udv = vdu - vdv \Rightarrow udu + vdv = vdu - udv$

$$\Rightarrow \frac{udu+vdv}{u^2+v^2}=\frac{vdu-udv}{u^2+v^2}$$

$$\Rightarrow \frac{d(u^2+v^2)}{u^2+v^2}=-2d\tan^{-1}\left(\frac{v}{u}\right)$$

On integrating, we get

$$\log(u^2+v^2)=-2\tan^{-1}\left(\frac{v}{u}\right)+c$$

$$\Rightarrow \frac{1}{2}\log(x^3+y^3)+\tan^{-1}\left(\frac{y}{x}\right)^{3/2}=\frac{c}{2}$$

31. **(a, b, c, d) :** Equation of the normal at a point $P(x, y)$ is given by

$$Y-y=-\frac{1}{dy/dx}(X-x) \quad \text{...(i)}$$

Let the point Q at the x-axis be $(x_1, 0)$. From (i) we get

$$y\frac{dy}{dx}=x_1-x \quad \text{...(ii)}$$

Now given that $PQ^2 = k^2$

We have $(x - x_1)^2 + y^2 = k^2$

or $x-x_1=\pm\sqrt{k^2-y^2}$

Hence, using (i), we obtain $y\dfrac{dy}{dx}=\pm\sqrt{k^2-y^2}$ ...(iii)

(iii) is the required differential equation for such curves

Now solving (iii), we get $\displaystyle\int\frac{-ydy}{\sqrt{k^2-y^2}}=\int -dx$

or $x^2 + y^2 = k^2$

which is equation of circle with centre at origin passes through $(0, k)$ and $(k, 0)$

32. **(a, d) :** If (x, y) is any point on the curve, the subtangent at $(x, y)=y\dfrac{dx}{dy}$

$\therefore$ $y\dfrac{dx}{dy}=nx$ (given) or $n\dfrac{dy}{y}=\dfrac{dx}{x}$

Integrating, $n\log y = \log x + \log c$

or $\log y^n = \log cx$

or $y^n = cx$ .....(i), which is the required equation of the family of curves.

Putting $x = 2, y = 3$ in (i), we have $3^n=2c$ or $c=\dfrac{3^n}{2}$

Putting this value of c in (i), we get

$$y^n=\frac{3^n}{2}x \text{ or } 2y^n=3^n x \quad \text{(ii)}$$

which is the particular curve passing through the point (2, 3).

Putting $n = 1$ in (ii), we have $2y = 3x$, which is a straight line.

Putting $n = 2$ in (ii), we have $2y^2 = 9x$ which is a parabola.

33. **(a, b, c) :** $p^2 + 2py\cot x - y^2 = 0$

$$p=\frac{-2y\cot x\pm\sqrt{4y^2\cot^2 x+4y^2}}{2}$$

$$\Rightarrow \frac{dy}{dx}=(-\cot x\pm \text{cosec}x)y \Rightarrow \frac{dy}{dx}+y\frac{(\cos x\pm 1)}{\sin x}=0$$

On integrating, we get

$$\log y+2\log\sin\frac{x}{2}=\log c'$$

$\Rightarrow$ $y(1 - \cos x) = 2c' = c$ and $y(1 + \cos x) = c$

$$\Rightarrow y\left(2\sin^2\frac{x}{2}\right)=c \Rightarrow \sin^2\frac{x}{2}=\frac{c}{2y}$$

$$\Rightarrow x=2\sin^{-1}\sqrt{\frac{c}{2y}}$$

34. **(a, b, c) :** Given $\dfrac{d}{dx}\left(y\dfrac{dy}{dx}\right)=x \Rightarrow y\dfrac{dy}{dx}=\dfrac{x^2}{2}+c$

$$\Rightarrow \frac{d}{dx}\left(\frac{y^2}{2}\right)=\frac{x^2}{2}+c$$

$$\Rightarrow y^2=\frac{x^3}{3}+\alpha x+\beta \text{ given } y(0)=1, y(1)=1$$

$$\Rightarrow \beta=1, \alpha=\frac{-1}{3}$$

$$\therefore y^2=f(x)=\frac{x^3-x+3}{3}, f'(x)=\frac{(3x^2-1)}{3}>0 \text{ for } x>1$$

$\therefore$ $f(x)$ is monotonically increasing $\forall\, x \in (1, \infty)$

$$f\left(\frac{1}{\sqrt{3}}\right)f\left(-\frac{1}{\sqrt{3}}\right)>0$$

$f(x) = 0$ has only one real root.

35. **(a, b) :** The given differential equation can be written as

$$\frac{xdx+ydy}{ydx-xdy}=\frac{x\sin^2(x^2+y^2)}{y^3}$$

$$\Rightarrow \frac{2xdx+2ydy}{ydx-xdy}=\frac{2x\sin^2(x^2+y^2)}{y^3}$$

$$\Rightarrow \frac{d(x^2+y^2)}{\sin^2(x^2+y^2)}=\frac{2x}{y^3}(ydx-xdy)$$

$$\Rightarrow \text{cosec}^2(x^2+y^2)\,d(x^2+y^2)=2\left(\frac{x}{y}\right)d\left(\frac{x}{y}\right)$$

On integrating, we get

$$\int\text{cosec}^2(x^2+y^2)\,d(x^2+y^2)=2\int\left(\frac{x}{y}\right)d\left(\frac{x}{y}\right)$$

$$\Rightarrow -\cot(x^2+y^2)=\left(\frac{x}{y}\right)^2+c,$$

which is the required solution.

36. **(a, c) :** $\left(\dfrac{dy}{dx}\right)^2+\dfrac{dy}{dx}(\cos x+\sec x)+1=0$

$$\Rightarrow \frac{dy}{dx}=\frac{-(\cos x+\sec x)\pm(\cos x-\sec x)}{2}$$

$$\Rightarrow \frac{dy}{dx} = -\sec x \text{ and } \frac{dy}{dx} = -\cos x$$

$\Rightarrow$ $y = -\log e\ |\sec x + \tan x| + c$ and $y = -\sin x + c$

37. **(a, c)** : $(1+e^x)\frac{dy}{dx} + ye^x = 1$

$$\Rightarrow \frac{dy}{dx} + \frac{e^x}{1+e^x}y = \frac{1}{1+e^x}$$

This is linear D.E., so $\text{I.F.} = e^{\int \frac{e^x}{1+e^x}dx} = e^{\ln(1+e^x)} = 1+e^x$

The solution is given by $y(1+e^x) = \int \frac{1}{1+e^x}(1+e^x)dx + \lambda$

$\Rightarrow y(1 + e^x) = x + \lambda$

(0, 2) is a point $\Rightarrow$ $2\cdot 2 = \lambda$ $\therefore$ $\lambda = 4$

So, $y(1 + e^x) = x + 4$ ...(i)

(a) $x = -4$ $\Rightarrow$ $y(1 + e^{-4}) = 0$ $\therefore$ $y = 0$, (a) is true

(b) $x = -2 \Rightarrow y = \frac{2}{1+e^{-2}}$. So, (b) is not true.

(c) Differentiating (i) w.r.t. x, we get

$$\text{Also, } \frac{dy}{dx} = \frac{(e^x+1)\cdot 1 - (x+4)e^x}{(e^x+1)^2} = \frac{-e^x(x+3)+1}{(e^x+1)^2}$$

$$\text{Let } f(x) = \frac{-e^x(x+3)+1}{(e^x+1)^2}$$

$$f(0) = \frac{-2}{2^2} < 0,\ f(-1) = \frac{\frac{-2}{e}+1}{\left(\frac{1}{e}+1\right)^2} > 0,\ f(0)\cdot f(-1) < 0$$

Hence, $f(x)$ vanish at least once in $(-1, 0)$. So, (c) is true.

38. **(a, d)** : $\frac{dy}{dx} - y\tan x = 2x\sec x$

It is linear

$\therefore$ $\text{I.F.} = e^{-\int \tan x dx} = e^{-\ln(\sec x)} = \cos x$

The solution reads $y\cdot\cos x = \int 2x\sec x\cos x\, dx = x^2 + c$

We have, $y(0) = 0$ $\Rightarrow$ $c = 0$ $\therefore$ $y = x^2\sec x$

$$y\left(\frac{\pi}{4}\right) = \frac{\pi^2}{16}\cdot\sqrt{2} = \frac{\pi^2}{8\sqrt{2}}$$

$y' = 2x\sec x + x^2\sec x\tan x$

$$y'\left(\frac{\pi}{3}\right) = 2\cdot\frac{\pi}{3}\cdot 2 + \frac{\pi^2}{9}2\cdot\sqrt{3} = \frac{4\pi}{3} + \frac{2\pi^2\sqrt{3}}{9}$$

39. **(c) : (A)-(s), (B)–(r), (C)–(q), (D)–(p)**

The solution of the D.E., $y_2 - (\alpha + \beta)y_1 + \alpha\beta y = 0$ is $y = Ae^{\alpha x} + Be^{\beta x}$ if $\alpha \neq \beta$

(A) $\alpha = 2, \beta = -3, \alpha + \beta = -1, \alpha\beta = -6$, D.E. : $y_2 + y_1 - 6y = 0$

(B) $\alpha = -2, \beta = 3, \alpha + \beta = 1, \alpha\beta = -6$, D.E. : $y_2 - y_1 - 6y = 0$

(C) $\alpha = 2, \beta = 3, \alpha + \beta = 5, \alpha\beta = 6$, D.E.: $y_2 - 5y_1 + 6y = 0$

(D) $\alpha = -2, \beta = -3, \alpha + \beta = -5, \alpha\beta = 6$, D.E.: $y_2 + 5y_1 + 6y = 0$.

40. **(b) : (A)–(p), (B)–(q), (C)–(s), (D)–(r)**

(A) $x^2 + y^2 = 2cy$. Differentiating, $x + yy_1 = cy_1$.
Eliminating c, $2y(x + yy_1) = y_1(x^2 + y^2) \Rightarrow (x^2 - y^2)y_1 = 2xy$

(B) $x^2 + y^2 = 2cx$. Differentiating, $x + yy_1 = c$
Eliminating c, $2x(x + yy_1) = x^2 + y^2 \Rightarrow 2xyy_1 = y^2 - x^2$

(C) $(x - h)^2 + y^2 = 1$. Differentiating, $x - h + yy_1 = 0$.
Eliminating $x - h$, $1 - y^2 = y^2y_1^2 \Rightarrow (1 + y_1^2)y^2 = 1$

(D) $x^2 + (y - k)^2 = 1$. Differentiating, $x + (y - k)y_1 = 0$.
Eliminating $y - k$, $y_1^2(1 - x^2) = x^2$

41. **(a)** : $\frac{dy}{dx} = e^{x-y}(e^x - e^y)$

$$e^y\frac{dy}{dx} = e^{2x} - e^{x+y} \Rightarrow e^y\frac{dy}{dx} + e^xe^y = e^{2x}$$

Let $e^y = u$, so that $e^y\frac{dy}{dx} = \frac{du}{dx}$

We have, $\frac{du}{dx} + e^x\cdot u = e^{2x}$

$\text{I.F.} = e^{\int e^x dx = e^{e^x}}$ $\therefore$ Solution is $u\cdot e^{e^x} = \int e^{e^x}.e^{2x}dx$

$= \int e^t\cdot t\,dt$, where $t = e^x$

$= e^t(t-1) + c$ $= e^{e^x}(e^x - 1) + c$

Then $u = e^x - 1 + ce^{-e^x} \Rightarrow e^y = e^x - 1 + ce^{-e^x}$

42. **(c)** : $\frac{dy}{dx} + \frac{y\ln y}{x} = \frac{y}{x^2}(\ln y)^2$

We have, $\frac{1}{y(\ln y)^2}\frac{dy}{dx} + \frac{1}{x}.\frac{1}{\ln y} = \frac{1}{x^2}$

Let $\frac{-1}{\ln y} = u$, so that $\frac{du}{dx} - \frac{u}{x} = \frac{1}{x^2}$

$\text{I.F.} = e^{-\int\frac{dx}{x}} = e^{-\ln x} = 1/x$

Then the solution is $u\cdot\frac{1}{x} = \int\frac{1}{x^3}dx + c$

which on simplification gives $2x = (1 + 2cx^2)\ln y$

43. **(a)** : $\frac{dx}{dy} = \frac{y^2+e^{2x}}{y^3} \Rightarrow \frac{dx}{dy} = \frac{1}{y} + \frac{e^{2x}}{y^3}$

$$\Rightarrow e^{-2x}\frac{dx}{dy} - e^{-2x}\frac{1}{y} = \frac{1}{y^3}$$

Put $\frac{-e^{-2x}}{2} = u$, so that $e^{-2x}\frac{dx}{dy} = \frac{du}{dy}$

$\Rightarrow \frac{du}{dy} + \frac{2}{y}u = \frac{1}{y^3}$, $\text{I.F.} = e^{\int\frac{2}{y}dy} = e^{2\ln y} = y^2$

Solution is $u\cdot y^2 = \int\frac{1}{y}dy + k$

$\Rightarrow$ $uy^2 = \ln y + k$ $\Rightarrow$ $-\frac{e^{-2x}}{2}y^2 = \ln y + k$

$\Rightarrow$ $-e^{-2x}y^2 = 2\ln y + 2k$ $\therefore$ $e^{-2x}y^2 + 2\ln y = c$

44. **(b)** : $\because$ $\frac{dT}{dt} = -k(T - 290) \Rightarrow \frac{dT}{(T-290)} = -k\,dt$

On integrating, we get

$\Rightarrow$ $\ln(T - 290) = -kt + c$ ...(i)

Initially, $T = 370$ K and $t = 0$, then $\ln(80) = c$ ...(ii)

From Eq. (i) and (ii), $\ln(T - 290) = -kt + \ln 80$

$$\ln\left(\frac{T-290}{80}\right) = -kt \Rightarrow \frac{T-290}{80} = e^{-kt}$$

or $T = 290 + 80e^{-kt}$

45. **(d) :** For $t = 10$ min, $T = 330$
Then, $330 - 290 = 80e^{-10k}$

$$\Rightarrow \quad \frac{1}{2} = e^{-10k} \Rightarrow e^{10k} = 2 \Rightarrow 10k = \ln 2 \text{ or } k = \frac{\ln 2}{10}$$

46. **(3) :** The given equation is $y = 2px + y^2p^3$ ...(i)

Solving for x, $x = \frac{y}{2p} - \frac{1}{2}y^2p^2$

Differentiating w.r.t y, we get

$$\frac{dx}{dy} = \frac{1}{p} = \frac{1}{2p} - \frac{y}{2p^2}\cdot\frac{dp}{dy} - yp^2 - y^2p\cdot\frac{dp}{dy}$$

or $2p = p - y\frac{dp}{dy} - 2yp^4 - 2y^2p^3\frac{dp}{dy}$

or $p(1+2yp^3) + y\frac{dp}{dy}(1+2yp^3) = 0$

or $(1+2yp^3)\left(p + y\frac{dp}{dy}\right) = 0$

Neglecting the first factor which does not involve $\frac{dp}{dy}$, we have

$p + y\frac{dp}{dy} = 0 \Rightarrow \frac{dp}{p} + \frac{dy}{y} = 0$

Integrating, $\log p + \log y = \log c$

or $\log py = \log c \Rightarrow py = c$ ...(ii)

Eliminating p between (i) and (ii)

$$y = 2x\cdot\frac{c}{y} + y^2\cdot\frac{c^3}{y^3} \quad \text{or} \quad y = \frac{2cx}{y} + \frac{c^3}{y}$$

or $y^2 = 2cx + c^3$, which is the required solution.

47. **(1) :** Equation of normal at the point $P(x, y)$ is

$$Y - y = -\frac{dx}{dy}(X - x)\left(\text{let } m = \frac{dy}{dx}\right)$$

Let $m = \frac{dy}{dx} \Rightarrow X + mY - (x + my) = 0$ ...(i)

Distance of perpendicular from the origin to line (i) is

$$\frac{|x+my|}{\sqrt{1+m^2}} = |y|$$

$$\Rightarrow \frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$$

This is homogeneous equation.
Let $y = zx$,

$$\Rightarrow \frac{dy}{dx} = z + x\frac{dz}{dx} \Rightarrow \frac{2z}{1+z^2}dz = -\frac{dx}{x}$$

Integrating $\int \frac{2z}{1+z^2}dz = -\int \frac{dx}{x}$

$\Rightarrow \log(1 + z^2) = -\log x + c \Rightarrow (x^2 + y^2) = x.e^c$
This curve passes through (1, 1)
$\Rightarrow 1 + 1 = 1\cdot e^c \Rightarrow e^c = 2$
The required equation of the curve is
$x^2 + y^2 = 2x$

48. **(0) :** $(2ny + xy\log_e x)dx = x\log_e x dy$

$$\Rightarrow \frac{dy}{y} = \left(\frac{2n}{x\log_e x} + 1\right)dx$$

$\Rightarrow \log(y) = 2n\log|\log x| + x + c$ and $c = 0$

$$\text{Now, } g(x) = (0.020)\lim_{n\to\infty} f(x) = \begin{cases} \to\infty, & \text{if } x < \frac{1}{e} \\ \to 0, & \text{if } \frac{1}{e} < x < e \\ \to\infty, & \text{if } x > e \end{cases}$$

$$\therefore \int_{1/e}^{e} g(x)dx = 0$$

49. **(3) :** Put $e^x = X$ and $e^y = Y$
So that $e^x dx = dX$ and $e^y dy = dY$

$$\Rightarrow \frac{e^y}{e^x}\cdot\frac{dy}{dx} = \frac{dY}{dX} \text{ or } \frac{Y}{X}p = P \text{ (say) or } p = \frac{X}{Y}\cdot P$$

The given equation becomes

$$X^3\left(\frac{X}{Y}P - 1\right) + \frac{X^3}{Y^3}.Y^2P^3 = 0$$

or $XP - Y + P^3 = 0$ or $Y = PX + P^3$
Which is the Clairaut's form.
$\therefore$ The solution is $Y = cX + c^3$ or $e^y = ce^x + c^3$

50. **(2) :** Put $x = \frac{1}{X}$ and $y = \frac{1}{Y}$

$$dx = -\frac{1}{X^2}dX \text{ and } dy = -\frac{1}{Y^2}dY$$

$$\Rightarrow p = \frac{dy}{dx} = \frac{X^2}{Y^2}\frac{dY}{dX} = \frac{X^2}{Y^2}P$$

The given equation becomes

$$\frac{1}{Y^2}\left(\frac{1}{Y} - \frac{1}{X}\cdot\frac{X^2}{Y^2}P\right) = \frac{1}{X^4}\cdot\frac{X^4}{Y^4}P^2$$

$\Rightarrow Y - XP = P^2$ or $Y = PX + P^2$
which is the Clairaut's form
$\therefore$ The solution is $Y = cX + c^2$

or $\frac{1}{y} = \frac{c}{x} + c^2$

JEE MATHEMATICS

Class XII

DPS-20

DAILY PRACTICE SHEET

Vector Algebra

SOLUTIONS

1. **(b) :** $\vec{a} = 2\hat{i} + 2\hat{j} - \hat{k}$ and $\vec{b} = \alpha\hat{i} + \beta\hat{j} + 2\hat{k}$

$\therefore \vec{a} + \vec{b} = (2+\alpha)\hat{i} + (2+\beta)\hat{j} + (-1+2)\hat{k}$

and $\vec{a} - \vec{b} = (2-\alpha)\hat{i} + (2-\beta)\hat{j} + (-1-2)\hat{k}$

Now, $|\vec{a}+\vec{b}| = \sqrt{(2+\alpha)^2 + (2+\beta)^2 + 1}$

$|\vec{a}-\vec{b}| = \sqrt{(2-\alpha)^2 + (2-\beta)^2 + 9}$

Now, $|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}|$

$\Rightarrow$ $(2+\alpha)^2 + (2+\beta)^2 + 1 = (2-\alpha)^2 + (2-\beta)^2 + 9$

$\Rightarrow$ $4 + \alpha^2 + 4\alpha + 4 + \beta^2 + 4\beta + 1 = 4 + \alpha^2 - 4\alpha + 4 + \beta^2 - 4\beta + 9$

$\Rightarrow$ $8\alpha + 8\beta = 17 - 9 = 8 \Rightarrow \alpha + \beta = 1$

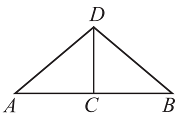
2. **(a) :** $2|\hat{i} + x\hat{j} + 3\hat{k}| = |4\hat{i} + (4x-2)\hat{j} + 2\hat{k}|$

$\Rightarrow$ $4(1 + x^2 + 9) = (16 + (4x-2)^2 + 4)$

$\Rightarrow$ $(3x+2)(x-2) = 0 \Rightarrow x = \frac{-2}{3}, 2$

3. **(b) :** Since C is middle point of AB

$\therefore \quad \overrightarrow{AC} = \frac{\overrightarrow{AB}}{2}$ and $\overrightarrow{AC} = -\overrightarrow{BC}$ ... (i)



In ΔPAC, $\overrightarrow{PA} + \overrightarrow{AC} = \overrightarrow{PC}$ ...(ii)

(using Δ law of addition of vectors)

In ΔPBC, $\overrightarrow{PB} + \overrightarrow{BC} = \overrightarrow{PC}$ ...(iii)

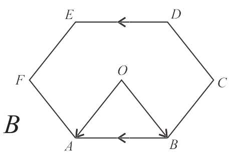
Adding (ii) and (iii), we get

$\overrightarrow{PA} + \overrightarrow{PB} + (\overrightarrow{AC} + \overrightarrow{BC}) = 2\overrightarrow{PC}$

or $\quad \overrightarrow{PA} + \overrightarrow{PB} + (\overrightarrow{AC} - \overrightarrow{AC}) = 2\overrightarrow{PC}$ (using (i))

$\therefore \quad \overrightarrow{PA} + \overrightarrow{PB} = 2\overrightarrow{PC}$

4. **(a) :** In the regular hexagon

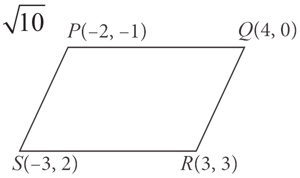


$\overrightarrow{DE}$ is parallel to $\overrightarrow{BA}$.

So $\overrightarrow{DE} = \overrightarrow{BA} =$ P.V. of A – P.V. of B

$= (4\hat{i} + 3\hat{j} - \hat{k}) - (-3\hat{i} + \hat{j} + \hat{k}) = 7\hat{i} + 2\hat{j} - 2\hat{k}$

5. **(a) :** We have, $PS = \sqrt{1^2 + 3^2} = \sqrt{10}$



$SR = \sqrt{6^2 + 1^2} = \sqrt{37}$,

$RQ = \sqrt{1^2 + 3^2} = \sqrt{10}$

and $QP = \sqrt{6^2 + 1^2} = \sqrt{37}$

Then $PQRS$ is a parallelogram. It can be a rectangle, so let's check the product of slopes of PS and SR

$$(m_{PS}) \cdot (m_{SR}) = \frac{3}{-1} \times \frac{1}{6} = -\frac{1}{2} \neq -1$$

Thus PS and SR are not perpendicular. So it is not a rectangle.

Also $m_{PR} \cdot m_{QS} = \frac{4}{5} \times \frac{2}{-7} = -\frac{8}{35} \neq -1$

Thus PR and QS are also not perpendicular. So it is not a rhombus either.

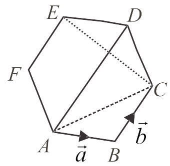
6. **(c) :** $\vec{Q} = \frac{\vec{a} + \vec{b} + \vec{c}}{3}$

$\therefore \quad \overrightarrow{QA} = \vec{a} - \frac{\vec{a} + \vec{b} + \vec{c}}{3} = \frac{2\vec{a} - \vec{b} - \vec{c}}{3}$

Similarly, $\overrightarrow{QB} = \frac{-\vec{a} + 2\vec{b} - \vec{c}}{3}$, $\overrightarrow{QC} = \frac{-\vec{a} - \vec{b} + 2\vec{c}}{3}$

$\therefore \quad \overrightarrow{QA} + \overrightarrow{QB} + \overrightarrow{QC} = \vec{0}$

7. **(c) :** In ΔABC, $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$



$\Rightarrow \quad \overrightarrow{AC} = \vec{a} + \vec{b}$

AD is parallel to BC and

$\overrightarrow{AD} = 2\,\overrightarrow{BC} \quad \Rightarrow \quad \overrightarrow{AD} = 2\vec{b}$

In ΔACD,

$\overrightarrow{AC} + \overrightarrow{CD} = \overrightarrow{AD} \quad \Rightarrow \quad \overrightarrow{CD} = 2\vec{b} - (\vec{a} + \vec{b}) = \vec{b} - \vec{a}$

Now, $\overrightarrow{CE} = \overrightarrow{CD} + \overrightarrow{DE} = \vec{b} - 2\vec{a}$

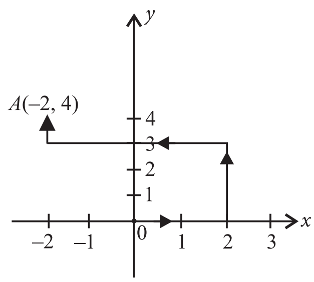
8. **(d) :** $\vec{A} = -2\hat{i} + 4\hat{j}$

Let $\vec{B} = x\hat{i} + y\hat{j}$

Now, $3\vec{A} + 5\vec{B} = 9\hat{i} + 32\hat{j}$

$\Rightarrow \quad -6\hat{i} + 12\hat{j} + 5x\hat{i} + 5y\hat{j}$

$= 9\hat{i} + 32\hat{j}$



Comparing coefficients, we get

$5x - 6 = 9$ and $5y + 12 = 32$

$\Rightarrow \quad x = \frac{15}{5} = 3$

and $y = \frac{20}{5} = 4 \qquad \therefore \quad \vec{B} = (3, 4)$

9. **(c) :** Position vector of centroid $\vec{G} = \frac{\vec{a} + \vec{b} + \vec{c}}{3}$

Position vector of circumcentre $\vec{P} = \frac{\vec{a} + \vec{b} + \vec{c}}{4}$

Let $\vec{r}$ be the orthocentre of the triangle.

Now, we know that, $\vec{G} = \frac{2\vec{P} + \vec{r}}{3} \Rightarrow 3\vec{G} = 2\vec{P} + \vec{r}$

[$\because$ Centroid divides orthocentre and circumcentre in 2:1].

$\Rightarrow \vec{r} = 3\vec{G} - 2\vec{P} = (\vec{a} + \vec{b} + \vec{c}) - 2\left(\frac{\vec{a} + \vec{b} + \vec{c}}{4}\right) = \frac{\vec{a} + \vec{b} + \vec{c}}{2}$

10. **(a) :** $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, $\overrightarrow{OC} = \vec{c}$ (say)

$\therefore \quad$ Position vector of D, E and F are

$\frac{\vec{b}+\vec{c}}{2}, \frac{\vec{c}+\vec{a}}{2}$ and $\frac{\vec{a}+\vec{b}}{2}$ respectively.

Position vector of $\vec{G}$ is $\frac{\vec{a}+\vec{b}+\vec{c}}{3}$

Now, $\overrightarrow{GD} = \overrightarrow{OD} - \overrightarrow{OG} = \frac{\vec{b}+\vec{c}}{2} - \frac{\vec{a}+\vec{b}+\vec{c}}{3} = \frac{\vec{b}+\vec{c}-2\vec{a}}{6}$

In the same way, $\overrightarrow{GE} = \frac{\vec{c}+\vec{a}-2\vec{b}}{6}$

Also, $\overrightarrow{GF} = \frac{\vec{a}+\vec{b}-2\vec{c}}{6}$

$$\overrightarrow{GD} + \overrightarrow{GE} + \overrightarrow{GF} = \frac{\vec{b}+\vec{c}-2\vec{a}+\vec{c}+\vec{a}-2\vec{b}+\vec{a}+\vec{b}-2\vec{c}}{6} = \vec{0}$$

11. (b) : $\overrightarrow{AC} = 4\overrightarrow{AB}$

$\Rightarrow \overrightarrow{AC} - \overrightarrow{AB} = 3\overrightarrow{AB} \Rightarrow \overrightarrow{BC} = 3\overrightarrow{AB} \Rightarrow \overrightarrow{CB} = -3\overrightarrow{AB}$

So, C divides AB externally in the ratio 4 : 3.

Thus, P.V. of $C = \frac{4\vec{b}-3\vec{a}}{4-3} = 4\vec{b}-3\vec{a}$

12. (a) : P.V.of $A, B, C,$ and D are $6\vec{a}-4\vec{b}+4\vec{c}$, $-4\vec{c}, -\vec{a}-2\vec{b}-3\vec{c}$ and $\vec{a}+2\vec{b}-5\vec{c}$ respectively

$\Rightarrow \overrightarrow{AB} = -6\vec{a}+4\vec{b}-8\vec{c}$ and $\overrightarrow{CD} = 2\vec{a}+4\vec{b}-2\vec{c}$

$\therefore \vec{r} = (6\vec{a}-4\vec{b}+4\vec{c}) + \lambda(-6\vec{a}+4\vec{b}-8\vec{c})$

and $\vec{r} = (-\vec{a}-2\vec{b}-3\vec{c}) + \mu(2\vec{a}+4\vec{b}-2\vec{c})$

$\Rightarrow (6 - 6\lambda) = (-1 + 2\mu)$ ...(i) $(-4 + 4\lambda) = (-2 + 4\mu)$ ...(ii)

$(4 - 8\lambda) = (-3 - 2\mu)$ ...(iii)

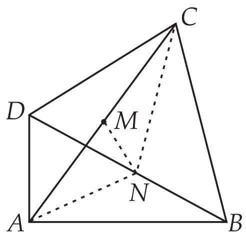
On solving (i), (ii) and (iii), we get $\lambda = 1, \mu = 1/2$

So the point of intersection is :

$(6\vec{a}-4\vec{b}+4\vec{c}) + 1(-6\vec{a}+4\vec{b}-8\vec{c})$

$= 6\vec{a}-4\vec{b}+4\vec{c}-6\vec{a}+4\vec{b}-8\vec{c} = -4\vec{c} = B$

13. (c) : Since, N is mid point of BD



$\therefore \frac{(1)\cdot\overrightarrow{AB} + (1)\cdot\overrightarrow{AD}}{(1+1)} = \overrightarrow{AN}$

$\Rightarrow \overrightarrow{AB} + \overrightarrow{AD} = 2\overrightarrow{AN}$ ...(i)

Also, $\frac{(1)\cdot\overrightarrow{CB} + (1)\cdot\overrightarrow{CD}}{(1+1)} = \overrightarrow{CN}$

$\Rightarrow \overrightarrow{CB} + \overrightarrow{CD} = 2\overrightarrow{CN}$ ...(ii)

Adding (i) and (ii), we get

$\overrightarrow{AB} + \overrightarrow{AD} + \overrightarrow{CB} + \overrightarrow{CD} = 2\overrightarrow{AN} + 2\overrightarrow{CN} = -2\overrightarrow{NA} - 2\overrightarrow{NC}$

$= -2(\overrightarrow{NA} + \overrightarrow{NC}) = -2(2\overrightarrow{NM}) = 4\overrightarrow{MN}$

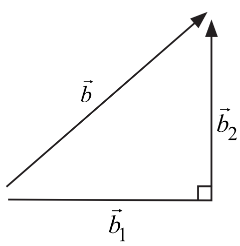
14. (c) : The solution of the equation

$\vec{r} \times \vec{a} = \vec{b}, \vec{a}\cdot\vec{b} = 0$ is $\vec{r} = \lambda\vec{a} + \frac{\vec{a}\times\vec{b}}{\vec{a}\cdot\vec{a}}$

$\vec{r}\cdot\vec{a} = 1 \Rightarrow \lambda = 1 \therefore \vec{r} = \vec{a} + \vec{a}\times\vec{b}$, since $\vec{a}\cdot\vec{a} = 1$.

15. (d) : $\vec{b} = 3\hat{j} + 4\hat{k}, \vec{a} = \hat{i} + \hat{j}$

Given that $\vec{b}_1$ is parallel to $\vec{a}$.



$\therefore \vec{b}_1 = \frac{(\vec{b}\cdot\vec{a})\hat{a}}{|\vec{a}|}$

$= \left\{\frac{(3\hat{j}+4\hat{k})\cdot(\hat{i}+\hat{j})}{\sqrt{2}}\right\}\left(\frac{\hat{i}+\hat{j}}{\sqrt{2}}\right)$

$= \frac{3(\hat{i}+\hat{j})}{\sqrt{2}\times\sqrt{2}} = \frac{3(\hat{i}+\hat{j})}{2}$

Also, $\vec{b}_1 + \vec{b}_2 = \vec{b} \Rightarrow \vec{b}_2 = \vec{b} - \vec{b}_1 = (3\hat{j}+4\hat{k}) - \frac{3}{2}(\hat{i}+\hat{j})$

$\Rightarrow \vec{b}_2 = -\frac{3}{2}\hat{i} + \frac{3}{2}\hat{j} + 4\hat{k}$

Now, $\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{3}{2} & \frac{3}{2} & 0 \\ -\frac{3}{2} & \frac{3}{2} & 4 \end{vmatrix} = \hat{i}(6) - \hat{j}(6) + \hat{k}\left(\frac{9}{4}+\frac{9}{4}\right) = 6\hat{i} - 6\hat{j} + \frac{9}{2}\hat{k}$

16. (a) : $\overrightarrow{OP}\cdot\overrightarrow{OQ} + \overrightarrow{OR}\cdot\overrightarrow{OS} = \overrightarrow{OR}\cdot\overrightarrow{OP} + \overrightarrow{OQ}\cdot\overrightarrow{OS}$ can be written as

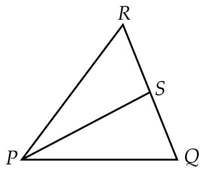
$\overrightarrow{OP}(\overrightarrow{OQ} - \overrightarrow{OR}) = \overrightarrow{OS}(\overrightarrow{OQ} - \overrightarrow{OR})$

$\Rightarrow (\overrightarrow{OQ} - \overrightarrow{OR})(\overrightarrow{OS} - \overrightarrow{OP}) = 0$ which means that $\overrightarrow{RQ}\cdot\overrightarrow{PS} = 0$

So that $\overrightarrow{PS}$ is perpendicular to $\overrightarrow{RQ}$.

Similarly, $\overrightarrow{QS}$ is perpendicular to $\overrightarrow{PR}$ and $\overrightarrow{RS}$ is perpendicular to $\overrightarrow{PQ}$. Hence, S is the orthocentre of ΔPQR.

17. (a) : Let S be the mid-point of QR. Then, PS is bisector of angle P.



Let the position vectors of P, Q and R are $\vec{a}, \vec{b}$ and $\vec{c}$ respectively. Then, the position vector of S is $\frac{(\vec{b}+\vec{c})}{2}$.

$\therefore \overrightarrow{PS} = \frac{\vec{b}+\vec{c}}{2} - \vec{a} \Rightarrow \overrightarrow{PS} = \frac{(\vec{b}-\vec{a})+(\vec{c}-\vec{a})}{2} = \frac{\overrightarrow{PQ}+\overrightarrow{PR}}{2}$

So, Statement-2 is true.

The unit vector along $\overrightarrow{PS}$ is $\frac{\overrightarrow{PS}}{|\overrightarrow{PS}|} = \frac{\overrightarrow{PQ}+\overrightarrow{PR}}{|\overrightarrow{PQ}+\overrightarrow{PR}|}$

$\therefore$ Unit vector bisecting the angle between unit vectors $\vec{a}$ and $\vec{b}$ is $\frac{\vec{a}+\vec{b}}{|\vec{a}+\vec{b}|}$

Now, $|\vec{a}+\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2(\vec{a}\cdot\vec{b})$

$\Rightarrow |\vec{a}+\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}|\cos\theta$

$\Rightarrow |\vec{a}+\vec{b}|^2 = 1 + 1 + 2\cos\theta = 4\cos^2\frac{\theta}{2} \Rightarrow |\vec{a}+\vec{b}| = 2\cos\frac{\theta}{2}$

$\therefore$ Unit vector bisecting the angle between $\vec{a}$ and $\vec{b}$ is $\frac{\vec{a}+\vec{b}}{2\cos\frac{\theta}{2}}$

Hence, both the Statements are true and Statement-2 is a correct explanation of Statement-1.

18. (d) : We have, $\overrightarrow{AB} = -4\hat{i} + 2\hat{j} + (p+1)\hat{k}$

$\overrightarrow{AC} = 2\hat{i} + (q-1)\hat{j} - 3\hat{k}$

ABC is a right angled triangle, right angle at A.

$\therefore \overrightarrow{AB} \perp \overrightarrow{AC} \Rightarrow \overrightarrow{AB}\cdot\overrightarrow{AC} = 0$

$\Rightarrow -8 + 2(q - 1) - 3(p + 1) = 0 \Rightarrow 3p - 2q + 13 = 0$

$\therefore (p, q)$ lies on the line $3x - 2y + 13 = 0$

Now, slope of line $= \frac{3}{2}$

$\therefore$ The point (p, q) lies on a line making acute angle with the positive direction of x-axis.

19. (d) : We have, $\vec{a} + 2\vec{b} = \vec{c}$ ...(i)

Taking cross product on both sides of (i) with $\vec{a}$, we get

$2(\vec{b}\times\vec{a}) = \vec{c}\times\vec{a} \Rightarrow \vec{c}\times\vec{a} = -2(\vec{a}\times\vec{b})$ ...(ii)

Again taking cross product on both sides of (i) with $\vec{c}$, we get

$\vec{a}\times\vec{c} + 2(\vec{b}\times\vec{c}) = \vec{0} \Rightarrow \vec{b}\times\vec{c} = \frac{1}{2}(\vec{c}\times\vec{a}) = -\vec{a}\times\vec{b}$ ...(iii)

[using (ii)]

Now, $\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a} = \lambda(\vec{a}\times\vec{b})$

$\Rightarrow\ \vec{a}\times\vec{b}-\vec{a}\times\vec{b}-2\vec{a}\times\vec{b}=\lambda(\vec{a}\times\vec{b})$ [Using (ii) & (iii)]

$\Rightarrow\ -2(\vec{a}\times\vec{b})=\lambda(\vec{a}\times\vec{b})\ \Rightarrow\ \lambda=-2.$

20. **(b) :** $\vec{r}\cdot\vec{a}=\alpha(\vec{a}\cdot(\vec{b}\times\vec{c}))+\beta(\vec{a}\cdot(\vec{c}\times\vec{a}))+\gamma(\vec{a}\cdot(\vec{a}\times\vec{b}))$

$=\alpha[\vec{a}\ \vec{b}\ \vec{c}]+0+0$

Similarly, $\vec{r}\cdot\vec{b}=\beta[\vec{a}\ \vec{b}\ \vec{c}]$ and $\vec{r}\cdot\vec{c}=\gamma[\vec{a}\ \vec{b}\ \vec{c}]$

$\therefore\ \frac{1}{2}\vec{r}\cdot(\vec{a}+\vec{b}+\vec{c})=\frac{1}{2}(\vec{r}\cdot\vec{a}+\vec{r}\cdot\vec{b}+\vec{r}\cdot\vec{c})$

$=\frac{1}{2}(\alpha+\beta+\gamma)[\vec{a}\ \vec{b}\ \vec{c}]=\frac{1}{2}(\alpha+\beta+\gamma)\times 2=\alpha+\beta+\gamma$

21. **(c) :** Given $\vec{a}\cdot\vec{c}=\frac{1}{2}\Rightarrow$ angle between $\vec{a}$ and $\vec{c}=\frac{\pi}{3}$

Also $\vec{a}\times\vec{b}=\sin\alpha\,\vec{n}_1$ and $\vec{c}\times\vec{d}=\sin\beta\,\vec{n}_2$

where $\vec{n}_1$ and $\vec{n}_2$ are unit vectors.

$(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d})=1$

$\Rightarrow\ \sin\alpha\sin\beta\ \vec{n}_1\cdot\vec{n}_2=1\Rightarrow\sin\alpha\sin\beta\cos\theta=1$

$\therefore\ \alpha=\beta=\pi/2$ and $\theta=0°$.

The possible cases are shown in figures and in any case $\vec{b}$ and $\vec{d}$ are non-parallel vectors.

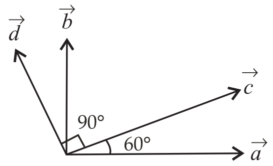


Fig. (i)

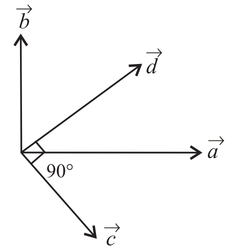


Fig. (ii)

22. **(b) :** We have, $\vec{a}=\frac{1}{y}(\vec{b}+\vec{c}+\vec{d})\Rightarrow\vec{d}=y\vec{a}-\vec{b}-\vec{c}$

Also, $\vec{d}=\frac{1}{x}(\vec{a}+\vec{b}+\vec{c})$

$\Rightarrow\ y\vec{a}-\vec{b}-\vec{c}=\frac{1}{x}\vec{a}+\frac{1}{x}\vec{b}+\frac{1}{x}\vec{c}$

$\Rightarrow\ \vec{a}\left(\frac{1}{x}-y\right)+\vec{b}\left(\frac{1}{x}+1\right)+\vec{c}\left(\frac{1}{x}+1\right)=\vec{0}$

$\Rightarrow\ \frac{1}{x}-y=0,\ \frac{1}{x}+1=0$ $\left[\because\ \vec{a},\ \vec{b}\text{ and }\vec{c}\text{ are non-coplanar}\right]$

$\Rightarrow\ x=-1$ and $y=-1$

$\therefore\ \vec{d}=\frac{1}{x}(\vec{a}+\vec{b}+\vec{c})=-\vec{a}-\vec{b}-\vec{c}$

$\therefore\ \frac{1}{xy}(\vec{a}+\vec{b}+\vec{c}+\vec{d})=\vec{0}$

23. **(a) :** Volume of parallelopiped

$$V=\begin{vmatrix}1 & a & 1\\ 0 & 1 & a\\ a & 0 & 1\end{vmatrix}=1+a^3-a$$

$\therefore\ \frac{dV}{da}=3a^2-1$ and $\frac{d^2V}{da^2}=6a$

$3a^2-1=0\Rightarrow\ a=\pm\frac{1}{\sqrt{3}},\ \frac{d^2V}{da^2}>0$ for $a=\frac{1}{\sqrt{3}}$

Thus V is minimum for $a=\frac{1}{\sqrt{3}}$.

24. **(c) :** Volume of parallelopiped with $\vec{a},\vec{b}$ and $\vec{c}$ as coterminous edges is $[\vec{a}\ \vec{b}\ \vec{c}]=\vec{a}\cdot(\vec{b}\times\vec{c})=40$ (given)

Volume of the parallelopiped having $\vec{b}+\vec{c},\vec{c}+\vec{a}$ and $\vec{a}+\vec{b}$ as coterminous edges $=(\vec{b}+\vec{c})\cdot\{(\vec{c}+\vec{a})\times(\vec{a}+\vec{b})\}$

$=(\vec{b}+\vec{c})\cdot\{\vec{c}\times\vec{a}+\vec{c}\times\vec{b}+\vec{a}\times\vec{b}\}$

$=\vec{b}\cdot(\vec{c}\times\vec{a})+\vec{c}\cdot(\vec{a}\times\vec{b})=[\vec{a}\ \vec{b}\ \vec{c}]+[\vec{a}\ \vec{b}\ \vec{c}]$

$=2[\vec{a}\ \vec{b}\ \vec{c}]=2\times 40=80$ cubic units.

25. **(c) :** The sides of parallelogram $\overline{PS}$ and $\overline{PQ}$ are given by

$\overline{PS}=\frac{\overline{PR}-\overline{SQ}}{2}$ and $\overline{PQ}=\frac{\overline{PR}+\overline{SQ}}{2}$

The volume V is given by

$V=\left|\left[\overline{PQ}\ \overline{PS}\ \overline{PT}\right]\right|=\frac{1}{4}\left|\left[\overline{PR}+\overline{SQ},\ \overline{PR}-\overline{SQ},\ \overline{PT}\right]\right|$

$=\frac{1}{2}\left|\left[\overline{PR},\ \overline{SQ},\ \overline{PT}\right]\right|$

[Recall that $[\vec{a}\ \vec{a}\ \vec{b}]=0$ and $[\vec{a}\ \vec{b}\ \vec{c}]=[\vec{b}\ \vec{c}\ \vec{a}]$, etc.]

$$=\frac{1}{2}\begin{vmatrix}3 & 1 & -2\\ 1 & -3 & -4\\ 1 & 2 & 3\end{vmatrix}=\frac{1}{2}|3(-1)-1(7)-2(5)|=\frac{1}{2}|(-20)|=10$$

26. **(a) :** $\vec{a}=2\hat{i}+\hat{j}-2\hat{k}\ \Rightarrow\ |\vec{a}|=3$ and $\vec{b}=\hat{i}+\hat{j}$

Now, $\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 1 & -2\\ 1 & 1 & 0\end{vmatrix}=2\hat{i}-2\hat{j}+\hat{k}$

$\therefore\ |\vec{a}\times\vec{b}|=3$

We also have,

$|(\vec{a}\times\vec{b})\times\vec{c}|=|\vec{a}\times\vec{b}||\vec{c}||\sin 30°||\hat{n}|$

$=3|\vec{c}|\cdot\frac{1}{2}|\hat{n}|$

$\Rightarrow\ 3=3|\vec{c}|\cdot\frac{1}{2}\ \Rightarrow\ |\vec{c}|=2$

Since, $|\vec{c}-\vec{a}|=3$ ...(i)

On squaring (i), we get $c^2+a^2-2\vec{c}\cdot\vec{a}=9$

$\Rightarrow\ 4+9-2\vec{a}\cdot\vec{c}=9\ \Rightarrow\ \vec{a}\cdot\vec{c}=2$

27. **(d) :** $\vec{a}\times\{\vec{a}\times\{\vec{a}\times(\vec{a}\times\vec{b})\}\}$

$\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}$

So, $\vec{a}\times\{\vec{a}\times\{\vec{a}\times(\vec{a}\times\vec{b})\}\}=\vec{a}\times\{\vec{a}\times\{(\vec{a}\cdot\vec{b})\vec{a}-(\vec{a}\cdot\vec{a})\vec{b}\}\}$

But, $\vec{a}\cdot\vec{b}=0$ (given)

So, $\vec{a}\times\{\vec{a}\times\{-(\vec{a}\cdot\vec{a})\vec{b}\}\}=-|\vec{a}|^2\cdot\{\vec{a}\times(\vec{a}\times\vec{b})\}$

$=-|\vec{a}|^2\{(\vec{a}\cdot\vec{b})\vec{a}-(\vec{a}\cdot\vec{a})\vec{b}\}=-|\vec{a}|^2\{-(\vec{a}\cdot\vec{a})\vec{b}\}=|\vec{a}|^4\vec{b}$

$[\because\ \vec{a}\cdot\vec{b}=0]$

28. **(b) :** Let $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$

Now, $\hat{i}\times(\vec{a}\times\hat{i})=(\hat{i}\cdot\hat{i})\vec{a}-(\hat{i}\cdot\vec{a})\hat{i}=\vec{a}-a_1\hat{i}$

Similarly, $\hat{j}\times(\vec{a}\times\hat{j})=\vec{a}-a_2\hat{j}$

and $\hat{k}\times(\vec{a}\times\hat{k})=\vec{a}-a_3\hat{k}$

$\therefore\ \hat{i}\times(\vec{a}\times\hat{i})+\hat{j}\times(\vec{a}\times\hat{j})+\hat{k}\times(\vec{a}\times\hat{k})$

$=3\vec{a}-(a_1\hat{i}+a_2\hat{j}+a_3\hat{k})=3\vec{a}-\vec{a}=2\vec{a}$

29. **(c) :** Since $(\vec{a}\times\vec{b})\cdot\left((\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a})\right)=9$ (given)

Let $\vec{a}\times\vec{b}=\vec{x}$; $\vec{b}\times\vec{c}=\vec{y}$; $\vec{c}\times\vec{a}=\vec{z}$ $\therefore\ \vec{x}\cdot(\vec{y}\times\vec{z})=9$

$\Rightarrow\ [\vec{x}\ \vec{y}\ \vec{z}]=9$ ...(i)

Required volume of parallelopiped is

$(\vec{x}\times\vec{y})\cdot((\vec{y}\times\vec{z})\times(\vec{z}\times\vec{x}))=(\vec{x}\times\vec{y})\cdot([\vec{y}\ \vec{z}\ \vec{x}]\vec{z}-[\vec{y}\ \vec{z}\ \vec{z}]\vec{x})$

(Scalar triple product of two same vectors is zero)

$=(\vec{x}\times\vec{y})\cdot\vec{z}\ \ ([\vec{x}\ \vec{y}\ \vec{z}])=[\vec{x}\ \vec{y}\ \vec{z}]^2=(9)^2=81$ (using (i))

30. **(c) :** Since, $\vec{a}\times(\vec{b}\times\vec{c})=\frac{1}{2}\vec{b} \Rightarrow (\vec{a}\cdot\vec{c})\,\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}=\frac{1}{2}\vec{b}$

On comparing both sides, we get

$\vec{a}\cdot\vec{c}=\frac{1}{2}$ and $\vec{a}\cdot\vec{b}=0$ Now, $\vec{a}\cdot\vec{c}=\frac{1}{2} \Rightarrow |\vec{a}||\vec{c}|\cos\theta_2=\frac{1}{2}$

$\Rightarrow \cos\theta_2=\frac{1}{2}$ ($\because \vec{a}$ and $\vec{c}$ are unit vectors)

$\Rightarrow \cos\theta_2=\cos\frac{\pi}{3} \Rightarrow \theta_2=\frac{\pi}{3}$ and $\vec{a}\cdot\vec{b}=0 \Rightarrow |\vec{a}||\vec{b}|\cos\theta_1=0$

$\Rightarrow \cos\theta_1=\cos\frac{\pi}{2}$ ($\because \vec{a}$ and $\vec{b}$ are unit vectors)

$\Rightarrow \theta_1=\frac{\pi}{2}$. Hence, $\theta_1=\frac{\pi}{2}$ and $\theta_2=\frac{\pi}{3}$.

31. **(a, d) :** $|\vec{v}|=|\vec{w}|=v, |\vec{z}|=z$

Given $\vec{w}=x\vec{v}+y\vec{z}$

$\vec{w}\cdot\vec{v}=|\vec{w}|\cdot|\vec{v}|\cos 2\beta = v^2\cos 2\beta$

$\vec{w}\cdot\vec{z}=|\vec{w}|\cdot|\vec{z}|\cos\left(\frac{\pi}{2}-\beta\right) = vz\sin\beta$

$\vec{v}\cdot\vec{z}=|\vec{v}||\vec{z}|\cos\left(\frac{\pi}{2}+\beta\right)=-vz\sin\beta$

Now $\vec{w}=x\vec{v}+y\vec{z}$

Talking dot product with $\vec{v}$, we get

$\vec{w}\cdot\vec{v}=x\vec{v}\cdot\vec{v}+y\vec{z}\cdot\vec{v}$

$\Rightarrow v^2\cos 2\beta = v^2x + y(-vz\sin\beta)$

$\Rightarrow (v)x-(z\sin\beta)y = v\cos 2\beta$ ...(i)

Taking dot product with $\vec{w}$, we get

$\vec{w}\cdot\vec{w}=x\vec{v}\cdot\vec{w}+y\vec{z}\cdot\vec{w}$

$\Rightarrow v^2=xv^2\cos 2\beta+y\,vz\sin\beta$

$\Rightarrow (v\cos 2\beta)x+(z\sin\beta)y=v$ ...(ii)

Solving equation (i) and (ii), we have $x=1$ and $y=\frac{2v\sin\beta}{z}$

32. **(b, d) :** Plane P_1 is parallel to $\vec{a}$ and $\vec{b}$. The normal to P_1 is along $\vec{a}\times\vec{b}$. Plane P_2 is parallel to $\vec{c}$ and $\vec{d}$. The normal to P_2 is along $\vec{c}\times\vec{d}$.

$\vec{A}$ is along the line of intersection of planes P_1 and P_2

$\therefore$ $\vec{A}$ is along $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})$

$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\0&2&3\\0&4&-3\end{vmatrix}=-18\hat{i}$ and

$\vec{c}\times\vec{d}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\0&1&-1\\3&3&0\end{vmatrix}=3\hat{i}-3\hat{j}-3\hat{k}=3(\hat{i}-\hat{j}-\hat{k})$

$\therefore$ $\vec{A}$ is along $\hat{i}\times(\hat{i}-\hat{j}-\hat{k})=\hat{j}-\hat{k}$

The angle between $\vec{A}$ and $2\hat{i}+\hat{j}-2\hat{k}$ is θ

$\cos\theta=\frac{\vec{A}}{|\vec{A}|}\cdot\frac{(2\hat{i}+\hat{j}-2\hat{k})}{3}$

$=\pm\frac{(\hat{j}-\hat{k})\cdot(2\hat{i}+\hat{j}-2\hat{k})}{3\sqrt{2}}=\frac{\pm 3}{3\sqrt{2}}=\pm\frac{1}{\sqrt{2}}$

and $\cos\theta=\pm\frac{1}{\sqrt{2}} \Rightarrow \theta=\frac{\pi}{4},\frac{3\pi}{4}$.

33. **(a, c) :** $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{c}=\vec{c}\cdot\vec{a}=\cos\theta$

$\Rightarrow \vec{g}=\frac{\vec{a}+\vec{b}+\vec{c}}{3}, |\vec{g}|=\frac{1}{3}\sqrt{|\vec{a}+\vec{b}+\vec{c}|^2}$

$=\frac{1}{3}\sqrt{1+1+1+6\cos\theta}=\frac{1}{\sqrt{3}}\sqrt{1+2\cos\theta}$ ...(i)

$\frac{\pi}{3}\le\theta\le\frac{\pi}{2}\Rightarrow 0\le\cos\theta\le\frac{1}{2}$

$\Rightarrow 0\le 2\cos\theta\le 1 \Rightarrow 1\le 1+2\cos\theta\le 2$

$\therefore |\vec{g}|\in\left[\frac{1}{\sqrt{3}},\sqrt{\frac{2}{3}}\right]$ [by (i)]

34. **(b, c, d) :** $\vec{a}\cdot\vec{b}=2-2=0$.

The general solution of the vector equation

$\vec{r}\times\vec{a}=\vec{b}, \vec{a}\cdot\vec{b}=0$ is $\vec{r}=\frac{\vec{a}\times\vec{b}}{\vec{a}\cdot\vec{a}}+\alpha\vec{a}$

Now, $\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\2&-1&2\\1&2&0\end{vmatrix}=-4\hat{i}+2\hat{j}+5\hat{k}$ and $\vec{a}\cdot\vec{a}=9$

$\therefore \vec{r}=\frac{-4\hat{i}+2\hat{j}+5\hat{k}}{9}+\alpha(2\hat{i}-\hat{j}+2\hat{k})$

$\Rightarrow 9\vec{r}=-4\hat{i}+2\hat{j}+5\hat{k}+9\alpha(2\hat{i}-\hat{j}+2\hat{k})$

$\alpha=\frac{1}{9}\Rightarrow 9\vec{r}=-2\hat{i}+\hat{j}+7\hat{k}$, $\alpha=\frac{1}{3}\Rightarrow 9\vec{r}=2\hat{i}-\hat{j}+11\hat{k}$

$\alpha=-\frac{1}{9}\Rightarrow 9\vec{r}=-6\hat{i}+3\hat{j}+3\hat{k}$.

35. **(c, d) :** Since $[\vec{a}\,\vec{b}\,\vec{c}]=0$

$\vec{a},\vec{b}$ and $\vec{c}$ are coplanar vectors.

Further since $\vec{d}$ is equally inclined to $\vec{a},\vec{b}$ and $\vec{c}$

$\therefore \vec{d}\cdot\vec{a}=\vec{d}\cdot\vec{b}=\vec{d}\cdot\vec{c}=0 \therefore \vec{d}\cdot\vec{r}=0$

36. **(b, c, d):** $\vec{c}+\vec{a}=(\vec{b}+\vec{c})+(\vec{a}-\vec{b})$

$\vec{a}+\vec{b}=(\vec{b}-\vec{c})+(\vec{c}+\vec{a})$

$\vec{b}+\vec{c}=(\vec{a}+\vec{b})+(\vec{c}-\vec{a})$

So vectors in options (b), (c) and (d) are coplanar.

37. **(a, b, d) :** Volume $=|[2\vec{b}\times\vec{c}\;\; 3\vec{c}\times\vec{a}\;\; 4\vec{a}\times\vec{b}]|=18$

$\Rightarrow 24[\vec{a}\,\vec{b}\,\vec{c}]^2=18 \Rightarrow |[\vec{a}\,\vec{b}\,\vec{c}]|=\frac{\sqrt{3}}{2}$

Now, $[\vec{a}\,\vec{b}\,\vec{c}]=\begin{vmatrix}(1+\sin\theta)&\cos\theta&\sin 2\theta\\\sin\left(\theta+\frac{2\pi}{3}\right)&\cos\left(\theta+\frac{2\pi}{3}\right)&\sin\left(2\theta+\frac{4\pi}{3}\right)\\\sin\left(\theta-\frac{2\pi}{3}\right)&\cos\left(\theta-\frac{2\pi}{3}\right)&\sin\left(2\theta-\frac{4\pi}{3}\right)\end{vmatrix}$

Applying $R_1\to R_1+R_2+R_3$ and expanding

$|[\vec{a}\,\vec{b}\,\vec{c}]|=\sqrt{3}|\cos 3\theta|=\frac{\sqrt{3}}{2}$

$\Rightarrow \cos 3\theta=\pm\frac{1}{2}\Rightarrow 3\theta=\frac{\pi}{3},\frac{2\pi}{3},\frac{4\pi}{3} \Rightarrow \theta=\frac{\pi}{9},\frac{2\pi}{9},\frac{4\pi}{9}$.

38. **(a, c) :** $(\vec{a}-\vec{b})\times[(\vec{b}+\vec{a})\times(2\vec{a}+\vec{b})]=\vec{b}+\vec{a}$

$\Rightarrow \{(\vec{a}-\vec{b})\cdot(2\vec{a}+\vec{b})\}(\vec{b}+\vec{a}) - \{(\vec{a}-\vec{b})\cdot(\vec{b}+\vec{a})\}(2\vec{a}+\vec{b})=\vec{b}+\vec{a}$

$\Rightarrow (2-\vec{a}\cdot\vec{b}-1)(\vec{b}+\vec{a})=\vec{b}+\vec{a} \Rightarrow \vec{b}+\vec{a}=\vec{0}$ or $1-\vec{a}\cdot\vec{b}=1$

$\Rightarrow \vec{b}=-\vec{a}$ or $\vec{a}\cdot\vec{b}=0 \Rightarrow \theta=\pi$ or $\theta=\frac{\pi}{2}$

39. **(a) : (A) - (r), (B) - (p), (C) - (s), (D) - (q)**

(A) If $\vec{a},\vec{b},\vec{c}$ are mutually perpendicular,

$[\vec{a}\times\vec{b}\;\vec{b}\times\vec{c}\;\vec{c}\times\vec{a}]=[\vec{a}\,\vec{b}\,\vec{c}]^2=(|\vec{a}||\vec{b}||\vec{c}|)^2=16$

(B) Given $\vec{a},\vec{b}$ are two unit vectors *i.e.*, $|\vec{a}|=|\vec{b}|=1$ and angle between them is $\pi/3$.

$\therefore \sin\theta=\frac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}\Rightarrow \sin\pi/3=|\vec{a}\times\vec{b}| \Rightarrow \frac{\sqrt{3}}{2}=|\vec{a}\times\vec{b}|$

Now, $[\vec{a}\ \vec{b}+\vec{a}\times\vec{b}\ \vec{b}]=[\vec{a}\ \vec{b}\ \vec{b}]+[\vec{a}\ \vec{a}\times\vec{b}\ \vec{b}]$

$=0+[\vec{a}\ \vec{a}\times\vec{b}\ \vec{b}]=(\vec{a}\times\vec{b})\cdot(\vec{b}\times\vec{a})=-(\vec{a}\times\vec{b})\cdot(\vec{a}\times\vec{b})=-|\vec{a}\times\vec{b}|^2=-\frac{3}{4}$

(C) If $\vec{b}$ and $\vec{c}$ are orthogonal *i.e.*, $\vec{b}\cdot\vec{c}=0$ and also given $\vec{b}\times\vec{c}=\vec{a}$

Now $[\vec{a}+\vec{b}+\vec{c}\ \ \vec{a}+\vec{b}\ \ \vec{b}+\vec{c}]$

$=[\vec{a}\ \ \vec{a}+\vec{b}\ \ \vec{b}+\vec{c}]+[\vec{b}+\vec{c}\ \ \vec{a}+\vec{b}\ \ \vec{b}+\vec{c}]=[\vec{a}\ \vec{b}\ \vec{c}]$

$=\vec{a}\cdot(\vec{b}\times\vec{c})=\vec{a}\cdot\vec{a}=|\vec{a}|^2=1$ [$\because \vec{a}$ is unit vector]

(D) $[\vec{x}\ \vec{y}\ \vec{a}]=0$

Hence, $\vec{x},\vec{y}$ and $\vec{a}$ are coplanar. ...(i)

$[\vec{x}\ \vec{y}\ \vec{b}]=0$

Hence, $\vec{x},\vec{y}$ and $\vec{b}$ are coplanar ...(ii)

Also, $[\vec{a}\ \vec{b}\ \vec{c}]=0$

Hence, $\vec{a},\vec{b}$ and $\vec{c}$ are coplanar. ...(iii)

From (i) (ii) and (iii), we get that $\vec{x},\vec{y}$ and $\vec{c}$ are coplanar. $\therefore [\vec{x}\ \vec{y}\ \vec{c}]=0$

40. (b) : (A) - (r), (B) - (p), (C) - (q), (D) - (s)

(A) $|\vec{a}\times\vec{b}|=|\vec{b}\times\vec{c}|=|\vec{c}\times\vec{a}|=2(\text{area of }\Delta ABC)$

Also directions of $\vec{a}\times\vec{b},\vec{b}\times\vec{c}$ and $\vec{c}\times\vec{a}$ are the same

Hence, $\vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{c}\times\vec{a}$

(B) For regular tetrahendron, all sides are of equal length, hence $|\vec{a}|=|\vec{b}|=|\vec{c}|$. Also, all the faces are equilateral triangle. Therefore, angle between $\vec{a}$ and $\vec{b}$ is 60°, $\vec{b}$ and $\vec{c}$ is 60° and $\vec{a}$ and $\vec{c}$ is 60°

Hence, $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{c}=\vec{c}\cdot\vec{a}$.

(C) Since, $\vec{a}\times\vec{b}=\vec{c}\Rightarrow\vec{a}\perp\vec{c}$ and $\vec{b}\perp\vec{c}$

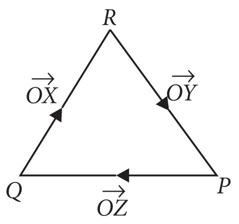
and $\vec{b}\times\vec{c}=\vec{a}\Rightarrow\vec{b}\perp\vec{a}$ and $\vec{c}\perp\vec{a}$

So, $\vec{a},\vec{b},\vec{c}$ are mutually perpendicular.

(D) Since $\vec{a}+\vec{b}+\vec{c}=0$

$\Rightarrow\vec{a}^2+\vec{b}^2+\vec{c}^2+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})=0\ \Rightarrow\ \vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}=-\frac{3}{2}$

41. (c) :



$\overrightarrow{OX}\times\overrightarrow{OY}=\sin R\,\hat{n}$

$\therefore\ |\overrightarrow{OX}\times\overrightarrow{OY}|=\sin R=\sin(\pi-(P+Q))=\sin(P+Q)$

42. (a) : $\cos(P+Q)+\cos(Q+R)+\cos(R+P)$
$=-\cos R-\cos P-\cos Q=-(\cos P+\cos Q+\cos R)$

In a triangle, we know that

$\cos P+\cos Q+\cos R\le\frac{3}{2}$

This is a standard result that gives

$-(\cos P+\cos Q+\cos R)\ge-\frac{3}{2}$

Thus the minimum value of $-(\cos P+\cos Q+\cos R)$ is $-3/2$.

43. (c) : On opening the given relations, we get

$|\vec{a}|^2+\vec{a}\cdot\vec{b}-2|\vec{b}|^2=0$; $2|\vec{a}|^2+(2\lambda+3)\vec{a}\cdot\vec{b}+3\lambda|\vec{b}|^2=0$

On eliminating $|\vec{a}|^2$, we get $(2\lambda+1)\vec{a}\cdot\vec{b}+(3\lambda+4)|\vec{b}|^2=0$

$\Rightarrow\ \vec{a}\cdot\vec{b}=-\frac{3\lambda+4}{2\lambda+1}|\vec{b}|^2$

44. (c) : $|\vec{a}|^2=2|\vec{b}|^2-\vec{a}\cdot\vec{b}=2|\vec{b}|^2+\frac{3\lambda+4}{2\lambda+1}|\vec{b}|^2=\left(\frac{7\lambda+6}{2\lambda+1}\right)|\vec{b}|^2$

$\Rightarrow\cos\theta=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}=\frac{\pm\frac{3\lambda+4}{2\lambda+1}|\vec{b}|^2}{\sqrt{\frac{7\lambda+6}{2\lambda+1}}|\vec{b}||\vec{b}|}=\pm\frac{3\lambda+4}{\sqrt{(7\lambda+6)(2\lambda+1)}}$

45. (d) : We must have $\cos^2\theta\le1\Rightarrow\frac{9\lambda^2+24\lambda+16}{14\lambda^2+19\lambda+6}\le1$

$\lambda\in(-\infty,-1)\cup\left(-\frac{6}{7},-\frac{1}{2}\right)\cup(2,\infty)$

46. (6) : Let $\vec{c}=x\vec{a}+y\vec{b}$, where x, y are scalars

$\Rightarrow\ \vec{c}=x(\hat{i}-\hat{j}+2\hat{k})+y(2\hat{i}-\hat{j}+\hat{k})$

$\Rightarrow\ \vec{c}=\hat{i}(x+2y)+\hat{j}(-x-y)+\hat{k}(2x+y)$

But $\vec{c}\cdot\vec{a}=0$; $6x+5y=0\Rightarrow y=\frac{-6x}{5}$

so, $\vec{c}=\frac{-7x}{5}\hat{i}+\frac{x}{5}\hat{j}+\frac{4x}{5}\hat{k}$

We have $\frac{49x^2+x^2+16x^2}{25}=1\ \Rightarrow x^2=\frac{25}{66}$

$\therefore\ \vec{c}=\pm\frac{5}{\sqrt{66}}\left(\frac{-7}{5}\hat{i}+\frac{1}{5}\hat{j}+\frac{4}{5}\hat{k}\right)$; $p=|\vec{c}\cdot\hat{b}|=\frac{\sqrt{11}}{6}$

so $\frac{\sqrt{11}}{p}=6$

47. (1) : $\vec{a}\cdot\vec{c}=\vec{b}\cdot\vec{c}=\cos\theta$; $\vec{c}=\alpha\vec{a}+\beta\vec{b}+\gamma(\vec{a}\times\vec{b})$

Taking dot product with $\vec{a}$ both sides, $\cos\theta=\alpha$

Taking dot product with $\vec{b}$ both sides, $\cos\theta=\beta$

Taking dot product with $\vec{c}$ both sides, $1=\alpha\cos\theta+\beta\cos\theta+\gamma[\vec{a}\ \vec{b}\ \vec{c}]$

But $[\vec{a}\ \vec{b}\ \vec{c}]^2=\begin{vmatrix}1&0&\cos\theta\\0&1&\cos\theta\\\cos\theta&\cos\theta&1\end{vmatrix}=1-2\cos^2\theta$

So, $1=\cos^2\theta+\cos^2\theta+\gamma\sqrt{1-2\cos^2\theta}\ \Rightarrow\ \gamma=\sqrt{1-2\cos^2\theta}$

So, $\alpha^2+\beta^2+\gamma^2=1$

48. (2) : $(\vec{a}\times\vec{b})\times(\vec{r}\times\vec{c})=[\vec{a}\ \vec{b}\ \vec{c}]\vec{r}-[\vec{a}\ \vec{b}\ \vec{r}]\vec{c}=[\vec{a}\ \vec{r}\ \vec{c}]\vec{b}-[\vec{b}\ \vec{r}\ \vec{c}]\vec{a}$

$\Rightarrow\ [\vec{a}\ \vec{b}\ \vec{c}]\vec{r}=[\vec{a}\ \vec{b}\ \vec{r}]\vec{c}+[\vec{b}\ \vec{c}\ \vec{r}]\vec{a}+[\vec{c}\ \vec{a}\ \vec{r}]\vec{b}$

Now $(\vec{a}\times\vec{b})\times(\vec{r}\times\vec{c})+(\vec{b}\times\vec{c})\times(\vec{r}\times\vec{a})+(\vec{c}\times\vec{a})\times(\vec{r}\times\vec{b})$

$=[\vec{a}\ \vec{b}\ \vec{c}]r-[\vec{a}\ \vec{b}\ \vec{r}]c+[\vec{b}\ \vec{c}\ \vec{a}]\vec{r}-[\vec{b}\ \vec{c}\ \vec{r}]\vec{a}+[\vec{c}\ \vec{a}\ \vec{b}]\vec{r}-[\vec{c}\ \vec{a}\ \vec{r}]\vec{b}$

$=3[\vec{a}\ \vec{b}\ \vec{c}]\vec{r}-[\vec{a}\ \vec{b}\ \vec{c}]\vec{r}=2[\vec{a}\ \vec{b}\ \vec{c}]\vec{r}$ So, $\lambda=2$

49. (2) : The co-ordinates of vertices of projected triangle will be $A'(-1,1,0)$, $B'(1,-1,0)$ $C'(1,1,0)$

Area of triangle $=\frac{1}{2}\begin{vmatrix}-1&1&1\\1&-1&1\\1&1&1\end{vmatrix}=2$ square units.

50. (8) : Volume of the parallelopiped $=\left|[\vec{a}\ \vec{b}\ \vec{c}]\right|$

$\Rightarrow$ (area of the base parallellogram) $h=\left|[\vec{a}\ \vec{b}\ \vec{c}]\right|$

$\Rightarrow\ |\vec{a}\times\vec{b}|h=\left|[\vec{a}\ \vec{b}\ \vec{c}]\right|\ \Rightarrow\ \sqrt{38}h=4\Rightarrow h=\frac{4}{\sqrt{38}}$;

$38h^2=4\times4\Rightarrow19h^2=8$

JEE MATHEMATICS

Class XII

DPS-21

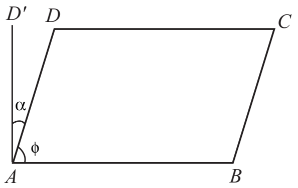
DAILY PRACTICE SHEET

Three Dimensional Geometry

SOLUTIONS

1. **(b) :** We have

$$\cos\phi = \frac{-2+20+22}{\sqrt{2^2+10^2+11^2}\sqrt{1^2+2^2+2^2}} = \frac{40}{15\times 3} = \frac{8}{9}$$



We have $\frac{\sqrt{17}}{9} = \sin\phi$ then, $\cos\alpha = \sin\phi = \frac{\sqrt{17}}{9}$

2. **(d) :** The equation of the line passing through (3, b, 1) and (5, 1, a) is $\frac{x-5}{2} = \frac{y-1}{1-b} = \frac{z-a}{a-1} = \mu$ (say)

The line crosses the yz plane where $x = 0$, *i.e.*

$-5 = 2\mu \;\therefore\; \mu = -\frac{5}{2}$

Again, $y = \mu(1-b)+1 = \frac{17}{2} \;\Rightarrow\; -\frac{5}{2}(1-b)+1 = \frac{17}{2}$

$\Rightarrow\; -\frac{5}{2}(1-b) = \frac{15}{2} \;\Rightarrow\; (1-b) = -3 \;\therefore\; b = 4$

Again, $z = \mu(a-1)+a = -\frac{13}{2}$

$\Rightarrow\; -\frac{5}{2}(a-1)+a = -\frac{13}{2} \;\Rightarrow\; -\frac{3}{2}a+\frac{5}{2} = -\frac{13}{2}$

$\Rightarrow\; -\frac{3}{2}a = -9 \;\Rightarrow a = 6$

3. **(c) :** Let P and Q be point on the lines, then
$P(-2+2s, -6+3s, 34-10s)$, $Q(-6+4t, 7-3t, 7-2t)$
The *d.r's* of PQ are $(-4+4t-2s, 13-3t-3s, -27-2t+10s)$
PQ is perpendicular to the two lines with *d.r's* 2, 3, –10 and 4, –3, –2.
$\therefore\;$ $2(-4+4t-2s)+3(13-3t-3s)-10(-27-2t+10s) = 0$
$4(-4+4t-2s)-3(13-3t-3s)-2(-27-2t+10s) = 0$
$\Rightarrow\;$ $113s-19t = 301$, $29t-19s = 1$
Solving $s = 3$, $t = 2$, $P \equiv (4, 3, 4) = (a, b, c) \Rightarrow a+b+c = 11$.

4. **(a) :** If the lines $\vec{r} = \vec{a}+s\vec{b}$, $\vec{r} = \vec{c}+t\vec{d}$ intersect, then

$[\vec{c}-\vec{a}\;\; \vec{b}\;\; \vec{d}] = 0$ ...(i)

Let the desired line be $\frac{x}{\alpha} = \frac{y}{\beta} = \frac{z}{\gamma}$.

from (i), $\begin{vmatrix} -1 & 3 & -5 \\ 2 & 4 & 3 \\ \alpha & \beta & \gamma \end{vmatrix} = 0$

$\therefore\;$ $29\alpha - 7\beta - 10\gamma = 0$ ...(ii)

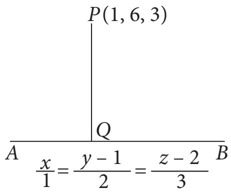
Further $\begin{vmatrix} 4 & -3 & 14 \\ 2 & 3 & 4 \\ \alpha & \beta & \gamma \end{vmatrix} = 0$

$\therefore\;$ $9\alpha - 2\beta - 3\gamma = 0$ ...(iii)

By cross multiplication rule, (ii) and (iii) gives $\frac{\alpha}{1} = \frac{\beta}{-3} = \frac{\gamma}{5}$

$\therefore\;$ The line is $\frac{x}{1} = \frac{y}{-3} = \frac{z}{5}$.

5. **(a) :** Let Q be the foot of $\perp PQ$ lie on line $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$



Let $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3} = k$(say)

$\Rightarrow\;$ $x = k$, $y = 2k+1$, $z = 3k+2$

$\Rightarrow\;$ $Q \equiv (k, 2k+1, 3k+2)$ lie on line AB.

Since $PQ \perp AB$

$\therefore\;$ $(k-1)\cdot 1 + (2k+1-6)\cdot 2 + (3k+2-3)\cdot 3 = 0$

$\Rightarrow\;$ $k-1+2(2k-5)+(3k-1)\cdot 3 = 0$

$\Rightarrow\;$ $k-1+4k-10+9k-3 = 0$

$\Rightarrow\;$ $14k-14 = 0 \Rightarrow k = 1$

$\therefore\;$ Required point $Q \equiv (1, 3, 5)$

6. **(c) :** We have,

$x+y+z+1 = 0$, $2x-y+z+3 = 0$ ... (i)

$\therefore\;$ Point of intersection of above lines are $P(0, 1, -2)$

Given equation of line is $\frac{x-1}{\alpha} = \frac{y+1}{-1} = \frac{z}{1}$ ...(ii)

$\therefore\;$ Point Q (1, –1, 0) lies on above line

$\therefore\;$ $\overrightarrow{PQ} = \hat{i}-2\hat{j}+2\hat{k}$

Also, $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 2 & -1 & 1 \end{vmatrix} = 2\hat{i}+\hat{j}-3\hat{k}$ ...(iii) (from (i))

Now $\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \alpha & -1 & 1 \\ 2 & 1 & -3 \end{vmatrix}$ (from (ii) and (iii))

$= 2\hat{i}+\hat{j}(3\alpha+2)+\hat{k}(\alpha+2)$

Shortest distance between lines is given by

S. D. = $\overrightarrow{PQ}.\hat{n} = \frac{2-2(3\alpha+2)+2(\alpha+2)}{\sqrt{4+(3\alpha+2)^2+(\alpha+2)^2}} = \frac{1}{\sqrt{3}}$

$\Rightarrow\;$ $3(2-4\alpha)^2 = 10\alpha^2+16\alpha+12$

$\Rightarrow\;$ $19\alpha^2-32\alpha = 0 \;\Rightarrow\; \alpha = \frac{32}{19}$

7. **(c) :** $\cos 2\alpha + \cos 2\beta + \cos 2\gamma + \sin^2\alpha + \sin^2\beta + \sin^2\gamma$
$= (\cos^2\alpha - \sin^2\alpha) + (\cos^2\beta - \sin^2\beta) + (\cos^2\gamma - \sin^2\gamma)$
$+ \sin^2\alpha + \sin^2\beta + \sin^2\gamma$
$= \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

8. **(c) :** Given, $l + m + n = 0, \Rightarrow l = -m - n$ and $l^2 + m^2 - n^2 = 0$
$\therefore \quad (-m-n)^2 + m^2 - n^2 = 0$
$\Rightarrow \quad 2m^2 + 2mn = 0 \Rightarrow 2m(m+n) = 0 \Rightarrow m = 0$ or $m + n = 0$
If $m = 0$, then $l = -n$
$\therefore \quad \frac{l_1}{-1} = \frac{m_1}{0} = \frac{n_1}{1}$
and if $m + n = 0 \Rightarrow m = -n$, then $l = 0$
$\therefore \quad \frac{l_2}{0} = \frac{m_2}{-1} = \frac{n_2}{1}$
i.e., $(l_1, m_1, n_1) = (-1, 0, 1)$ and $(l_2, m_2, n_2) = (0, -1, 1)$
$\therefore \quad \cos\theta = \frac{0+0+1}{\sqrt{1+0+1}\sqrt{0+1+1}} = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$

9. **(d) :** $\vec{r} = (-3\hat{i} + 6\hat{j}) + s(-4\hat{i} + 3\hat{j} + 2\hat{k})$ ...(i)
and $\vec{r} = (-2\hat{i} + 7\hat{k}) + t(-4\hat{i} + \hat{j} + \hat{k})$ ...(ii)
Comparing (i) and (ii) with $\vec{r} = \vec{a}_1 + s\vec{b}_1$ and $\vec{r} = \vec{a}_2 + t\vec{b}_2$ respectively.
$\vec{a}_1 = -3\hat{i} + 6\hat{j}$, $\vec{b}_1 = -4\hat{i} + 3\hat{j} + 2\hat{k}$
$\vec{a}_2 = -2\hat{i} + 7\hat{k}$, $\vec{b}_2 = -4\hat{i} + \hat{j} + \hat{k}$
$\vec{a}_2 - \vec{a}_1 = \hat{i} - 6\hat{j} + 7\hat{k}$

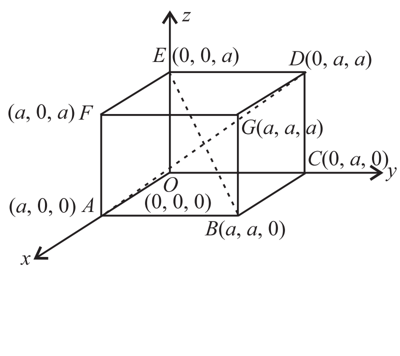
$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -4 & 3 & 2 \\ -4 & 1 & 1 \end{vmatrix} = \hat{i} - 4\hat{j} + 8\hat{k}$$

$|\vec{b}_1 \times \vec{b}_2| = \sqrt{1+16+64} = \sqrt{81} = 9$
Hence, the shortest distance between the given lines is given by

$$d = \left|\frac{(\vec{b}_1 \times \vec{b}_2)\cdot(\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|}\right| = \left|\frac{1+24+56}{9}\right| = \frac{81}{9} = 9 \text{ units.}$$

10. **(d) :** Given equation of line is
$\vec{r} = (-5\hat{i} + 2\hat{j} + 3\hat{k}) + \mu(9\hat{i} - 5\hat{j} + 3\hat{k})$
Let us suppose that (–50, 27, –12) is a point on line then it must satisfy given equation
$\therefore \quad -5 + 9\mu = -50,\ 2 - 5\mu = 27,\ 3 + 3\mu = -12$
$9\mu = -45, -5\mu = 25, 3\mu = -15$
$\mu = -5, \mu = -5, \mu = -5$
$\mu = -5$ comes in all equations.
$\therefore$ (–50, 27, –12) satisfy given equation.

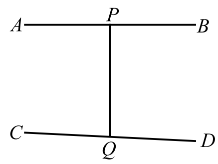
11. **(a) :** Let the diagonals of cube be AD and BE
Direction ratios of AD are $-a, a, a$
Direction ratios of BE are $a, a, -a$
Let θ be the angle between AD and BE



$$\therefore \quad \cos\theta = \left|\frac{(-a)\times a + a\times a + a\times(-a)}{\sqrt{(-a)^2 + a^2 + a^2} \times \sqrt{a^2 + a^2 + (-a)^2}}\right|$$

$$\Rightarrow \quad \cos\theta = \left|\frac{-a^2}{\sqrt{3}a \times \sqrt{3}a}\right| \Rightarrow \cos\theta = \frac{1}{3} \Rightarrow \theta = \cos^{-1}\left(\frac{1}{3}\right)$$

12. **(b) :** Let equation of line AB is
$\frac{x-0}{1} = \frac{y+a}{1} = \frac{z-0}{1} = k(\text{say})$ and equation of line CD is
$\frac{x+a}{2} = \frac{y}{1} = \frac{z}{1} = \lambda(\text{say})$



$\therefore$ Coordinates of P are $(k, k - a, k)$ and coordinates of Q are $(2\lambda - a, \lambda, \lambda)$
Direction ratios of PQ are
$(k - 2\lambda + a, k - a - \lambda, k - \lambda)$
Also, we have given that direction cosines of PQ are proportional to 2, 1, 2.
$\therefore \quad \frac{k - 2\lambda + a}{2} = \frac{k - a - \lambda}{1} = \frac{k - \lambda}{2}$
On solving first and second fraction, we get
$k - 2\lambda + a = 2(k - a - \lambda) \Rightarrow k - 3a = 0 \Rightarrow k = 3a$
On solving second and third fraction, we get
$2(k - a - \lambda) = k - \lambda \Rightarrow 2k - 2a - 2\lambda = k - \lambda$
$\Rightarrow k - \lambda = 2a \Rightarrow 3a - \lambda = 2a \Rightarrow \lambda = a$
So, coordinates of P are $(3a, 2a, 3a)$ and coordinates of Q are (a, a, a).

13. **(b) :** The plane which contains the two given lines $\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z}{3}$ and $\frac{x}{2} = \frac{y-2}{-1} = \frac{z+1}{3}$ which are parallel must pass through (1, –1, 0) and (0, 2, –1) and must be parallel to the line having direction ratios 2, –1, 3.
Any plane passing through (1, –1, 0) is
$a(x - 1) + b(y + 1) + c(z) = 0$ ...(i)
If this plane passes through the point (0, 2, –1)
$\Rightarrow a(-1) + b(3) - c = 0 \Rightarrow -a + 3b - c = 0$ ...(ii)
If plane (i) is parallel to the line having direction ratios 2, –1, 3 then $2a - b + 3c = 0$ ...(iii)
Cross multiplying (ii) and (iii), we get
$\frac{a}{9-1} = \frac{-b}{-3+2} = \frac{c}{1-6} \Rightarrow \frac{a}{8} = \frac{-b}{-1} = \frac{c}{-5} = k$
$a = 8k, b = k, c = -5k$
Putting in (i), we get
$8k(x - 1) + k(y + 1) - 5k(z) = 0$
$\Rightarrow \quad 8x + y - 5z - 7 = 0$

14. **(a) :** The equation of plane in the intercept form is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$
Volume of tetrahedron $OABC$ is $V = \frac{abc}{6} = 64 \Rightarrow abc = 384$
Foot of perpendicular from (0, 0, 0) on this plane is

$$\frac{x}{1/a} = \frac{y}{1/b} = \frac{z}{1/c} = \frac{1}{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}} = k$$

$$\Rightarrow \quad x = \frac{k}{a}, y = \frac{k}{b}, z = \frac{k}{c} \text{ and } \frac{1}{k} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$$

$$\Rightarrow \quad \frac{1}{k} = \frac{x^2 + y^2 + z^2}{k^2} \Rightarrow x^2 + y^2 + z^2 = k$$

$\therefore \quad (x^2 + y^2 + z^2)^3 = k^3 = abc\,xyz = 384\,xyz$ is the required locus.

15. **(a) :** Given equations are $x + 8y + 7z = 0$ ...(i)

$9x + 2y + 3z = 0$ ...(ii)

and $x + y + z = 0$ ...(iii)

Subtracting (iii) from (i), we get

$7y + 6z = 0$ ...(iv)

Multiplying (iii) by 2 and then subtracting from (ii), we get

$7x + z = 0$ ... (v)

Let $x = \lambda$

Then, from (v), $z = -7\lambda$

From (iv), $y = \frac{-6z}{7} = \frac{-6}{7}(-7\lambda) = 6\lambda$

Given that solution of system lies on the plane $x + 2y + z = 6$

$\therefore \quad \lambda + 2(6\lambda) + (-7\lambda) = 6$

$\Rightarrow \quad \lambda + 12\lambda - 7\lambda = 6 \Rightarrow 6\lambda = 6 \Rightarrow \lambda = 1$

$\therefore \quad x = 1, y = 6, z = -7$

So, $2a + b + c = 2(1) + 6 + (-7) = 1$

16. **(b) :** Clearly, $\vec{a}$ is perpendicular to the normals to the two planes determined by the given pairs of vectors.

We have,

$\vec{n}_1 =$ Normal vector to the plane determined by $\hat{i}$ and $\hat{i}+\hat{j}$

$\Rightarrow \quad \vec{n}_1 = \hat{i}\times(\hat{i}+\hat{j}) = \hat{k}$

$\vec{n}_2 =$ Normal vector to the plane determined by $\hat{i}-\hat{j}$ and $\hat{i}+\hat{k}$

$\Rightarrow \quad \vec{n}_2 = (\hat{i}-\hat{j})\times(\hat{i}+\hat{k}) = -\hat{i}-\hat{j}+\hat{k}$

Since $\vec{a}$ is perpendicular to $\vec{n}_1$ and $\vec{n}_2$.

$\therefore \quad \vec{a} = \lambda(\vec{n}_1\times\vec{n}_2) = \lambda\{\hat{k}\times(-\hat{i}-\hat{j}+\hat{k})\} = \lambda(-\hat{j}+\hat{i})$

Let θ be the angle between $\vec{a}$ and $\hat{i}-2\hat{j}+2\hat{k}$. Then,

$$\cos\theta = \frac{\lambda(1+2+0)}{\lambda\sqrt{2}\sqrt{1+4+4}} = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}$$

17. **(c) :** We have $(2\lambda + 1)x + (3 - \lambda)y + z = 4$

Consider, $\frac{x}{1} = \frac{y}{2} = \frac{z-4}{-7} = k$ (say)

$\Rightarrow \quad x = k, y = 2k$ and $z = -7k + 4$

If a plane passes through a line then the point of intersection satisfies both the equations.

Putting $x = k, y = 2k$ and $z = -7k + 4$ in

$(2\lambda + 1)x + (3 - \lambda)y + z = 4$ we get,

$(2\lambda + 1)k + (3 - \lambda)2k + (-7k + 4) = 4$

$\Rightarrow \quad 2\lambda k + k + 6k - 2\lambda k - 7k + 4 = 4$

$\Rightarrow \quad 7k - 7k + 4 = 4 \quad \Rightarrow \quad 4 = 4$

$\Rightarrow \quad x = k, y = 2k$ and $z = -7k + 4$ satisfy the equation of plane also.

$\therefore$ The plane $(2\lambda + 1)x + (3 - \lambda)y + z = 4$ always passes through the line $\frac{x}{1} = \frac{y}{2} = \frac{z-4}{-7}$

18. **(a) :** A vector normal to the plane $x + y - 2z = 3$ is $\vec{n} = \hat{i}+\hat{j}-2\hat{k}$ and $\overrightarrow{PQ} = 4\hat{i}+4\hat{j}-8\hat{k}$

Clearly, $\overrightarrow{PQ} = 4\vec{n}$. Therefore, PQ is parallel to the normal to the plane π. So, Statement-1 is true.

Let R be the mid-point of PQ. Then, coordinates of R are (4, 3, 2). Clearly, it lies on the plane π. So, Q is the image of P.

Hence, Statement-2 is true.

19. **(a) :** Plane is passing through the point $A(1, 2, 3)$ and is at maximum distance from the point $B(-1, 0, 2)$. So, plane is perpendicular to the line AB.

So, direction ratios of normal to the plane are 2, 2, 1.

Hence equation of plane is $2(x - 1) + 2(y - 2) + 1(z - 3) = 0$

or $\quad 2x + 2y + z = 9$

20. **(d) :** We have two equation of planes *i.e.*,

$\vec{r}\cdot(3\hat{i}-\hat{j}+\hat{k}) = 1$ and $\vec{r}\cdot(\hat{i}+4\hat{j}-2\hat{k}) = 2$

The planes have normal vector

$\vec{n}_1 = (3,-1,1)$ and $\vec{n}_2 = (1,4,-2)$

Then, $\vec{n} = \vec{n}_1\times\vec{n}_2$ is parallel to line of intersection (L).

$$\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -1 & 1 \\ 1 & 4 & -2 \end{vmatrix} = \hat{i}(-2) - \hat{j}(-7) + \hat{k}(13)$$

$\therefore \quad \vec{n} = -2\hat{i}+7\hat{j}+13\hat{k}$

Now to find a point on the line of intersection L, we need to solve the two equations :

$3x - y + z = 1$ and $x + 4y - 2z = 2$

We consider the point to be the point on plane $z = 0$.

Put $z = 0$ in systems above, we get

$3x - y = 1$ and $x + 4y = 2$

On solving, we get $x = 6/13$ and $y = 5/13$

$\therefore \quad$ Point of intersection is $\left(\frac{6}{13}, \frac{5}{13}, 0\right)$

Hence, equation of line of intersection to the given planes is

$$\frac{x-6/13}{-2} = \frac{y-5/13}{7} = \frac{z-0}{13}$$

or $$\frac{x-6/13}{2} = \frac{y-5/13}{-7} = \frac{z}{-13}$$

21. **(a) :** Equation of any plane through the intersection of $\vec{r}\cdot\vec{n}_1 = q_1$ and $\vec{r}\cdot\vec{n}_2 = q_2$ is of the form

$\vec{r}\cdot\vec{n}_1 + \lambda(\vec{r}\cdot\vec{n}_2) = q_1 + \lambda q_2$ ...(i) where λ is a parameter

So, $\vec{n}_1 + \lambda\vec{n}_2$ is normal to the plane (i). Now, plane (i) is parallel to the line of intersection of the planes $\vec{r}\cdot\vec{n}_3 = q_3$ and $\vec{r}\cdot\vec{n}_4 = q_4$

$\Rightarrow \quad [\vec{n}_1 + \lambda\vec{n}_2]\cdot[\vec{n}_3\times\vec{n}_4] = 0$

$\Rightarrow \quad [\vec{n}_1\ \vec{n}_3\ \vec{n}_4] + \lambda[\vec{n}_2\vec{n}_3\vec{n}_4] = 0$

$\Rightarrow \quad \lambda = -\frac{[\vec{n}_1\vec{n}_3\vec{n}_4]}{[\vec{n}_2\vec{n}_3\vec{n}_4]}$

On putting this value in equation (i), we have the equation of required plane as

$$\vec{r}\cdot\vec{n}_1 - q_1 = \frac{[\vec{n}_1\ \vec{n}_3\ \vec{n}_4]}{[\vec{n}_2\ \vec{n}_3\ \vec{n}_4]}(\vec{r}\cdot\vec{n}_2 - q_2)$$

$\Rightarrow \quad [\vec{n}_2\ \vec{n}_3\ \vec{n}_4](\vec{r}\cdot\vec{n}_1 - q_1) = [\vec{n}_1\ \vec{n}_3\ \vec{n}_4](\vec{r}\cdot\vec{n}_2 - q_2)$

22. **(c) :** Let the equation of plane passing through the point (1, 2, 2) be

$a(x - 1) + b(y - 2) + c(z - 2) = 0$ ...(i)

Since, it is perpendicular to the planes

$x - y + 2z = 3$ and $2x - 2y + z + 12 = 0$

$\therefore \quad a - b + 2c = 0$ and $2a - 2b + c = 0$ ...(ii)

Solving equations in (ii), we get $c = 0$ and $a = b$

$\therefore \quad$ From (i) equation of plane is $x + y - 3 = 0$

$\therefore \quad$ Distance of point (1, –2, 4) from plane

$x + y - 3 = 0$ is $D = \frac{|1-2-3|}{\sqrt{1+1}} = \frac{4}{\sqrt{2}} = 2\sqrt{2}$

23. **(b) :** Vector perpendicular to plane containing

$\vec{A}(1,-1,2), \vec{B}(2,0,-1), \vec{C}(0,2,1)$ is $\vec{A}\times\vec{B}+\vec{B}\times\vec{C}+\vec{C}\times\vec{A}$

Now, $$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 2 \\ 2 & 0 & -1 \end{vmatrix} = \hat{i}+5\hat{j}+2\hat{k}$$

$$\vec{B}\times\vec{C}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 0 & -1\\ 0 & 2 & 1\end{vmatrix}=2\hat{i}-2\hat{j}+4\hat{k}$$

$$\vec{C}\times\vec{A}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 0 & 2 & 1\\ 1 & -1 & 2\end{vmatrix}=5\hat{i}+\hat{j}+(-2)\hat{k}$$

$\therefore\quad \vec{A}\times\vec{B}+\vec{B}\times\vec{C}+\vec{C}\times\vec{A}=8\hat{i}+4\hat{j}+4\hat{k}$

24. **(b) :** Since straight line $\vec{r}=\vec{a}+\lambda\vec{b}$ ...(i)

meets the plane $\vec{r}\cdot\hat{n}=p$ at point P. ...(ii)

$\therefore\quad (\vec{a}+\lambda\vec{b})\cdot\hat{n}=p$ [from (i) and (ii)]

$\Rightarrow\quad (\vec{a}\cdot\hat{n})+\lambda(\vec{b}\cdot\hat{n})=p$ or $\lambda=\dfrac{p-(\vec{a}\cdot\hat{n})}{\vec{b}\cdot\hat{n}}$

Substituting value of λ in (i), we get position vector of point P, *i.e.*,

$\vec{r}=\vec{a}+\left(\dfrac{p-\vec{a}\cdot\hat{n}}{\vec{b}\cdot\hat{n}}\right)\vec{b}$

25. **(d) :** We have, $\vec{r}=(\hat{i}+2\hat{j}+3\hat{k})+\lambda(\hat{i}+2\hat{j}-\hat{k})$

$\Rightarrow\quad \vec{r}=(1+\lambda)\hat{i}+(2+2\lambda)\hat{j}+(3-\lambda)\hat{k}$

As it cuts the plane $\vec{r}\cdot\hat{k}=0$

$\Rightarrow\quad [(1+\lambda)\hat{i}+(2+2\lambda)\hat{j}+(3-\lambda)\hat{k}]\cdot\hat{k}=0$

$\Rightarrow\quad 3-\lambda=0 \Rightarrow \lambda=3$

So, the point of intersection is $(1+3, 2+2\times3, 3-3)=(4,8,0)$

Distance of $(4, 8, 0)$ from origin $=\sqrt{4^2+8^2}=4\sqrt{5}$

26. **(b) :** Let direction ratios of the line be (a, b, c), then

$2a-b+c=0$ and $a-b-2c=0$ *i.e.*, $\dfrac{a}{3}=\dfrac{b}{5}=\dfrac{c}{-1}$

$\therefore$ Direction ratios of the line are $(3, 5, -1)$

Any point on the line is $(2+\lambda, 2-\lambda, 3-2\lambda)$, it lies on the plane π if $2(2+\lambda)-(2-\lambda)+(3-2\lambda)=4$

$\Rightarrow\quad 4+2\lambda-2+\lambda+3-2\lambda=4 \Rightarrow \lambda=-1$

$\therefore$ The point of intersection of the line and the plane is $(1, 3, 5)$

$\therefore$ Equation of the required line is $\dfrac{x-1}{3}=\dfrac{y-3}{5}=\dfrac{z-5}{-1}$.

27. **(c) :** We must have $\vec{b}\cdot\vec{n}=0$ and $\vec{a}\cdot\vec{n}\neq q$

28. **(b) :** Lines in Statement-1 pass through $(1, 0, -1)$ and $(2, -1, 0)$ respectively.

We have, $\begin{vmatrix}2-1 & -1-0 & 0+1\\ 1 & -1 & 1\\ 1 & 2 & 3\end{vmatrix}=0$

So, the given lines are coplanar.

The equation of the plane containing them is

$$\begin{vmatrix}x-1 & y & z+1\\ 1 & -1 & 1\\ 1 & 2 & 3\end{vmatrix}=0$$

$\Rightarrow\quad -5x+5-2y+3z+3=0$

$\Rightarrow\quad 5x+2y-3z-8=0$

So, Statement-1 is true.

A vector parallel to the given line is $\vec{b}=\hat{i}+2\hat{j}+3\hat{k}$ and vectors normal to given planes are $\vec{n}_1=3\hat{i}+6\hat{j}+9\hat{k}$ and $\vec{n}_2=\hat{i}+\hat{j}-\hat{k}$ respectively.

Clearly, $\vec{n}_1=3\vec{b}$ and $\vec{b}\cdot\vec{n}_2=0$

So, given line is perpendicular to $3x+6y+9z-8=0$ and parallel to $x+y-z=0$

Hence, Statement-2 is true.

29. **(d) :** Any point on the line $\dfrac{x}{2}=\dfrac{y-1}{3}=\dfrac{z-2}{1}$ is $B(2\lambda, 1+3\lambda, 2+\lambda)$

Now, AB is parallel to the plane $x+y+z=2$, where $A\equiv(1, 0, 3)$.

$\therefore\quad (2\lambda-1)+(1+3\lambda)+(\lambda-1)=0 \Rightarrow \lambda=\dfrac{1}{6}$

$\therefore\quad B\equiv\left(\dfrac{1}{3},\dfrac{3}{2},\dfrac{13}{6}\right)$

Direction ratios of AB are $\dfrac{1}{3}-1,\dfrac{3}{2}-0,\dfrac{13}{6}-3\equiv\dfrac{-2}{3},\dfrac{3}{2},\dfrac{-5}{6}$

$\therefore$ Its equation is $\dfrac{x-(1/3)}{-(2/3)}=\dfrac{y-(3/2)}{(3/2)}=\dfrac{z-(13/6)}{-(5/6)}$

or $\dfrac{3x-1}{2}=\dfrac{2y-3}{-3}=\dfrac{6z-13}{5}$

30. **(b) :** Plane through the intersection of given planes is

$(x-y+z+3)+\lambda(x+y+2z+1)=0$ ...(i)

or $(1+\lambda)x+(-1+\lambda)y+(1+2\lambda)z+3+\lambda=0$

If it is parallel to x-axis, then $(1+\lambda)=0 \Rightarrow \lambda=-1$

Putting $\lambda=-1$ in (i), we get

$(x-y+z+3)-1(x+y+2z+1)=0$

$\Rightarrow\quad x-y+z+3-x-y-2z-1=0$

$\Rightarrow\quad -2y-z+2=0 \Rightarrow 2y+z-2=0 \Rightarrow 2y+z=2$

31. **(b, c) :** Direction ratio's of given line are $< 1, -2, 3 >$ and d.r.'s of normal to the given plane are $< 1, 2, 1 >$.

Since $1\times1+(-2)\times2+3\times1=0$ therefore the line is parallel to the plane.

Also, the base point of the line $(1, 2, 1)$ lies in the given plane

Hence, the given line lies in the given plane.

Alternatively, any point on the given line is $(t+1, -2t+2, 3t+1)$.

It lies in the given plane $x+2y+z=6$ if $t+1+2(-2t+2)+3t+1=6$, which is true for all real t.

Hence, every point on the given line lies in the given plane *i.e.*, the line lies in the plane.

32. **(b, d) :** The line l is perpendicular to l_1 and l_2. Hence the direction ratios of l are given by the vector

$$\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 2 & 2\\ 2 & 2 & 1\end{vmatrix}=-2\hat{i}+3\hat{j}-2\hat{k}$$

Let $P\equiv(2r, -3r, 2r)$

As it lies in l_1, we have $\dfrac{2r-3}{1}=\dfrac{-3r+1}{2}=\dfrac{2r-4}{2}$

Thus we have $r=1$ and $P\equiv(2, -3, 2)$

A point on l_2 is $(3+2s, 3+2s, 2+s)$

Then $(3+2s-2)^2+(3+2s+3)^2+(2+s-2)^2=17$

$\Rightarrow\quad 9s^2+28s+20=0 \Rightarrow s=-2, -10/9$

Thus the points are $(-1, -1, 0)$ and $\left(\dfrac{7}{9},\dfrac{7}{9},\dfrac{8}{9}\right)$.

33. **(b, c) :** Let the locus of point D is (x, y, z). Since volume of tetrahedron $ABCD$ is $\dfrac{1}{6}\left|[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}]\right|=1$ cubic units.

$$\Rightarrow \begin{vmatrix} -1 & 1 & -1 \\ 1 & 1 & -1 \\ x-0 & y-1 & z-2 \end{vmatrix} = \pm 6$$

$\Rightarrow -2(y-1)-2(z-2)=\pm 6 \Rightarrow y-1+z-2=\pm 3$
$\Rightarrow y+z=6$ or $y+z=0$

34. **(b, d) :** The plane P_3 is given by
$(x + z - 1) + \lambda y = 0$ *i.e.*, $x + \lambda y + z - 1 = 0$

Distance of the plane from (0, 1, 0) is $\left|\frac{\lambda-1}{\sqrt{2+\lambda^2}}\right| = 1$

$\Rightarrow \quad \lambda^2 - 2\lambda + 1 = \lambda^2 + 2 \Rightarrow 2\lambda = -1 \;\therefore\; \lambda = -\frac{1}{2}$

Then the plane is $2x - y + 2z - 2 = 0$
The distance of (α, β, γ) from P_3 is 2

$\Rightarrow \quad \left|\frac{2\alpha-\beta+2\gamma-2}{\sqrt{9}}\right| = 2 \Rightarrow 2\alpha - \beta + 2\gamma - 2 = \pm 6$

$\Rightarrow \quad 2\alpha - \beta + 2\gamma - 8 = 0$ or $2\alpha - \beta + 2\gamma + 4 = 0$

35. **(a, b) :** By geometrical condition, line L is parallel to the line of intersection of P_1 and P_2.
A vector along L is $(\hat{i}+2\hat{j}-\hat{k})\times(2\hat{i}-\hat{j}+\hat{k}) = \hat{i}-3\hat{j}-5\hat{k}$
Any point on L is $A(\lambda, -3\lambda, -5\lambda)$
The foot of perpendicular from A to plane P_1 is

$$\frac{\alpha-\lambda}{1} = \frac{\beta+3\lambda}{2} = \frac{\gamma+5\lambda}{-1} = -\frac{\lambda-6\lambda+5\lambda+1}{1+4+1} = -\frac{1}{6}$$

$\therefore \quad$ The foot of perpendicular is $\left(\lambda-\frac{1}{6}, -3\lambda-\frac{1}{3}, -5\lambda+\frac{1}{6}\right)$

Only options (a) and (b) match for some $\lambda \in R$.

36. **(b, c) :** The lines $\vec{r} = \vec{a} + s\vec{b}, \vec{r} = \vec{c} + t\,\vec{d}$ intersect if

$\left[(\vec{c}-\vec{a})\;\vec{b}\;\;\vec{d}\right] = 0$

$$\therefore \begin{vmatrix} 2-1 & 3-2 & 1-3 \\ k & 2 & 3 \\ 3 & k & 2 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} 1 & 1 & -2 \\ k & 2 & 3 \\ 3 & k & 2 \end{vmatrix} = 0$$

$\Rightarrow \quad 4 - 3k + 9 - 2k - 2(k^2 - 6) = 0 \Rightarrow 2k^2 + 5k - 25 = 0$

$\Rightarrow \quad k = -5, \frac{5}{2}$

37. **(a, b) :** Let the line be $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$. It intersects $\frac{x-3}{2} = \frac{y-3}{1} = \frac{z}{1}$

$$\therefore \begin{vmatrix} 3 & 3 & 0 \\ l & m & n \\ 2 & 1 & 1 \end{vmatrix} = 0, \text{ or } l - m - n = 0 \quad \text{...(i)}$$

The angle between the lines is 60°.

$$\therefore \quad \frac{2l+m+n}{\sqrt{6\left(l^2+m^2+n^2\right)}} = \cos 60° = \frac{1}{2}$$

Squaring, $5l^2 - m^2 - n^2 + 4mn + 8ln + 8ml = 0$ ...(ii)
Eliminating l from (i) and (ii), $2m^2 + 5mn + 2n^2 = 0$
$\Rightarrow \quad (m + 2n)(2m + n) = 0 \;\Rightarrow\; m = -2n \Rightarrow l = -n$
or $\quad n = -2m \Rightarrow l = -m$
The *d.r.*'s are –1, –2, 1 and –1, 1, –2. The required lines are
$\frac{x}{1} = \frac{y}{2} = \frac{z}{-1}$ and $\frac{x}{1} = \frac{y}{-1} = \frac{z}{2}$.

38. **(b, d) :** $\begin{vmatrix} p & \frac{3}{2} & \frac{1}{2} \\ \frac{3}{2} & 1 & 1 \\ \frac{1}{2} & 1 & q \end{vmatrix} = 0$

$\Rightarrow \quad 4pq - 4p - 9q + 5 = 0$ ...(i)
For perpendicular planes $a + b + c = 0$
i.e. $\quad p + 1 + q = 0$ ...(ii)
Eliminating q from (i) and (ii), we get $4p^2 - p - 14 = 0$
Solving, $p = 2, -\frac{7}{4}$

39. **(a) : (A)-(s), (B)-(r), (C)-(q), (D)-(p)**

(A) $\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}$

$\Rightarrow \quad x = a + \frac{l}{m}(y-b) = Ay + B,$
$z = c + \frac{n}{m}(y-b) = Cy + D$
Only 4 constants.

(B) The line is the z-axis. Distance $= \sqrt{1^2+2^2} = \sqrt{5}$

(C) $\begin{vmatrix} 1-3 & -1-k & 1-0 \\ 2 & 3 & 4 \\ 1 & 2 & 1 \end{vmatrix} = 0 \Rightarrow k = \frac{9}{2}$.

(D) $\frac{x-3}{2} = \frac{y-5}{1} = \frac{z-7}{1} = \frac{-2(6+12)}{6} = -6$

$a = 3 - 12 = -9, b = 5 - 6 = -1, c = 7 - 6 = 1, a + b + c = -9.$

40. **(b) : (A)-(r), (B)-(q), (C)-(p), (D)-(s)**
The coefficient determinant is

$$\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 3abc - (a^3+b^3+c^3)$$

$= -(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$
(A) $a + b + c \neq 0, \Sigma a^2 = \Sigma ab \Rightarrow a = b = c \Rightarrow$ identical planes
(B) $a + b + c = 0, \Sigma a^2 \neq \Sigma ab \Rightarrow x = y = z$, a line.
(C) $a + b + c \neq 0, \Sigma a^2 \neq \Sigma ab \Rightarrow \Delta \neq 0 \Rightarrow x = y = z = 0$ single point
(D) $a + b + c = 0, \Sigma a^2 = \Sigma ab \Rightarrow a = b = c = 0$
$\Rightarrow \quad x, y, z$ are arbitrary.

41. **(d) :** $H(\alpha, \beta, \gamma) \Rightarrow AH \perp BC, BH \perp CA$

$\Rightarrow \quad \frac{\alpha}{1} = \frac{\beta}{2} = \frac{\gamma}{1}$

H lies on the plane $\frac{x}{2} + y + \frac{z}{2} = 1 \Rightarrow \gamma = \frac{1}{3}$

42. **(c) :** $H\left(\frac{1}{3}, \frac{2}{3}, \frac{1}{3}\right), G\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right) \Rightarrow S\left(\frac{5}{6}, \frac{1}{6}, \frac{5}{6}\right)$

43. **(a) :** $A \equiv (2r, r, 1), B \equiv (2r, -r, -1)$ where $r \in R$.
Let $P \equiv (\alpha, \beta, \gamma)$
$\therefore (2r - \alpha)^2 + (r - \beta)^2 + (1 - \gamma)^2$ is minimum

$\Rightarrow \quad r = \frac{2\alpha+\beta}{5}$

$\therefore \quad A \equiv \left(\frac{2}{5}(2\alpha+\beta), \frac{2\alpha+\beta}{5}, 1\right)$

For the second line, $(2r - \alpha)^2 + (r + \beta)^2 + (1 + \gamma)^2$ is minimum

$\Rightarrow \quad r = \frac{2\alpha-\beta}{5}$.

$\therefore \quad B \equiv \left(\frac{2}{5}(2\alpha-\beta), \frac{-1}{5}(2\alpha-\beta), -1\right)$

$OA \perp OB \Rightarrow \frac{4}{25}(4\alpha^2-\beta^2)-\frac{1}{25}(4\alpha^2-\beta^2)-1=0$

$\Rightarrow \quad 3(4\alpha^2 - \beta^2) = 25$

The locus is $3(4x^2 - y^2) = 25$.

44. **(b) :** $AP \perp BP$

$\Rightarrow \left(\frac{2}{5}(2\alpha+\beta)-\alpha\right)\left(\frac{2}{5}(2\alpha-\beta)-\alpha\right)$

$+\left(\frac{2\alpha+\beta}{5}-\beta\right)\left(-\frac{1}{5}(2\alpha-\beta)-\beta\right) + (1-\gamma)(-1-\gamma) = 0$

$\Rightarrow \quad 3(\alpha^2 - 4\beta^2) = 25 \ (\gamma^2 - 1)$.

The locus is $3(x^2 - 4y^2) = 25\ (z^2 - 1)$.

45. **(b) :** $PA^2 = PB^2$

$\Rightarrow \quad \frac{1}{5}(\alpha-2\beta)^2+(1-\gamma)^2 = \frac{1}{5}(\alpha+2\beta)^2+(1+\gamma)^2$

$\Rightarrow \quad 2\alpha\beta + 5\gamma = 0$. Locus is $2xy + 5z = 0$.

46. **(5) :** The $d.r's$ of the first line are given by

$l - m + n = 0, 3l - m + an = 0$

$\Rightarrow \quad l : m : n = 1 - a : 3 - a : 2$

The $d.r's$ of the second line are given by $l+4m-n = 0$, $l + 2m + bn = 0$

$\therefore \quad l : m : n = 4b + 2 : -(b + 1) : -2$

The lines are perpendicular

$(1 - a)(4b + 2) - (3 - a)(b + 1) - 4 = 0$

$\Rightarrow \quad 3ab + a - b + 5 = 0$

47. **(4) :** The lines are $\frac{x}{0}=\frac{y-b}{b}=\frac{z}{-c}$ and $\frac{x-a}{a}=\frac{y}{0}=\frac{z}{c}$

$(b\hat{j}-c\hat{k})\times(a\hat{i}+c\hat{k}) = bc\hat{i}-ca\hat{j}-ab\hat{k}$

Now, $\vec{n} = \frac{bc\hat{i}-ca\hat{j}-ab\hat{k}}{\sqrt{b^2c^2+c^2a^2+a^2b^2}}$

The points on the lines are $(\vec{a}\,\hat{i} - b\hat{j})$

$\Rightarrow L = \frac{2abc}{\sqrt{a^2b^2+b^2c^2+c^2a^2}}$

$\therefore \quad L^2\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right) = 4.$

48. **(6) :** The plane containing the lines $\vec{r} = \vec{a} + \lambda\vec{b}$ and $\vec{r} = \vec{c} + \mu\vec{d}$ is $[\vec{r}-\vec{a} \quad \vec{b} \quad \vec{d}] = 0$

$\therefore \quad \begin{vmatrix} x-1 & y-2 & z-3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = 0 \Rightarrow x - 2y + z = 0 \qquad \text{...(i)}$

$Ax - 2y + z = d$ is parallel to (i)

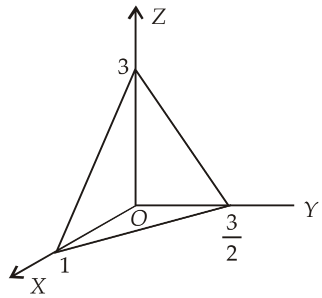
$\therefore \quad A = 1$. Now the planes are $x - 2y + z = 0$, $x - 2y + z = d$

Distance between these parallel planes is

$\frac{|d|}{\sqrt{1^2+2^2+1^2}} = \sqrt{6}$ (given)

$\therefore \ |d| = 6.$

49. **(6) :** The volume is invariant under translation.



If $X = x - \frac{4}{3}, Y = y - \frac{3}{2}, Z = z + 4,$

then the planes become

$3|X| + 2|Y| + |Z| = 3$

The planes $3X + 2Y + Z = \pm 3$,

$-3X + 2Y + Z = \pm 3$,

$3X - 2Y + Z = \pm 3$,

$3X + 2Y - Z = \pm 3$ form an octahedron.

The plane $3X + 2Y + Z = 3$ forms a tetrahedron with coordinate planes in the positive octant of

volume $\frac{1}{6}\cdot 1\cdot\frac{3}{2}\cdot 3 = \frac{3}{4}$.

Total volume $= 8\times\frac{3}{4} = 6.$

50. **(6) :** $P \equiv (2 + s, 1 - 2s, -1 + s)$

$Q \equiv \left(\frac{8}{3}+2t, -3-t, 1+t\right)$

O, P, Q are collinear.

$\therefore \quad \frac{2+s}{\frac{8}{3}+2t} = \frac{1-2s}{-3-t} = \frac{-1+s}{1+t}$

Solving, $s = 3, t = \frac{1}{3}$

$\Rightarrow \quad P \equiv (5, -5, 2), Q \equiv \left(\frac{10}{3}, -\frac{10}{3}, \frac{4}{3}\right)$

$d^2 = (PQ)^2 = \frac{25}{9} + \frac{25}{9} + \frac{4}{9} = 6.$

JEE MATHEMATICS **Class XII**

DPS-22

DAILY PRACTICE SHEET

Probability

SOLUTIONS

1. **(c) :** S = smoker, S' = Non-smoker, D = death by lung cancer
Using conditional probability, we can write
$P(D) = P(S)\,P(D \mid S)$ or $P(S')\,P(D \mid S')$

$$0.006 = \frac{20}{100}\cdot P(D|S) + \frac{80}{100}\cdot P(D|S') = \frac{1}{5}\cdot x + \frac{4}{5}\cdot\frac{x}{10}$$

[Let $P(D \mid S) = x$ and given $P(D \mid S) = 10.\ P(D|S')$]

$$\Rightarrow\ 0.006 = \frac{10x+4x}{50} \Rightarrow x = \frac{0.006\times 50}{14} \Rightarrow x = \frac{3}{140}$$

2. **(c) :** $P(A\cap B) = \frac{7}{10}, P(B) = \frac{17}{20}$

$$\therefore\ P(A \mid B) = \frac{P(A\cap B)}{P(B)} = \frac{7}{10}\times\frac{20}{17} = \frac{14}{17}$$

3. **(d) :** 3 numbers are chosen from {1, 2, 3, ..., 8} without replacement. Let A be the event that the maximum of chosen numbers is 6 and B be the event that the minimum of chosen numbers is 3.

$$P(B \mid A) = \frac{P(A\cap B)}{P(A)} = \frac{\frac{1\cdot 1\cdot 2}{^8C_3}}{\frac{^5C_2}{^8C_3}} = \frac{2}{10} = \frac{1}{5}$$

4. **(b) :** Given , $P(X_1) = \frac{1}{2}, P(X_2) = \frac{1}{4}$ and $P(X_3) = \frac{1}{4}$

$P(X) = P(E_1E_2E_3) + P(\bar{E}_1 E_2E_3) + P(E_1 \bar{E}_2 E_3) + P(E_1E_2\bar{E}_3)$

$$= \frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{4} + \frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{4} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{4} + \frac{1}{2}\cdot\frac{1}{4}\cdot\frac{3}{4}$$

$$= \frac{1}{32} + \frac{1}{32} + \frac{3}{32} + \frac{3}{32} = \frac{8}{32} = \frac{1}{4}$$

P(exact two functioning $|X$)

$$= \frac{\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{4} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{4} + \frac{1}{2}\cdot\frac{1}{4}\cdot\frac{3}{4}}{\frac{1}{4}} = \frac{\frac{1}{32}+\frac{3}{32}+\frac{3}{32}}{\frac{1}{4}} = \frac{7}{32}\cdot\frac{4}{1} = \frac{7}{8}$$

$$P(X \mid X_1) = \frac{\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{4} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{4} + \frac{1}{2}\cdot\frac{1}{4}\cdot\frac{3}{4}}{\frac{1}{2}} = \frac{\frac{1+3+3}{32}}{\frac{1}{2}} = \frac{7}{32}\cdot 2 = \frac{7}{16}$$

5. **(a) :** $P(B|A\cup B^c) = \frac{P(B\cap(A\cup B^c))}{P(A\cup B^c)}$

$$= \frac{P((B\cap A)\cup(B\cap B^c))}{P(A)+P(B^c)-P(A\cap B^c)}$$

$$= \frac{P((A\cap B)\cup\phi)}{(1-0.3)+(1-0.4)-0.5} = \frac{P(A\cap B)}{0.8}$$

$$= \frac{P(A)-P(A\cap B^c)}{0.8} = \frac{(1-0.3)-0.5}{0.8} = \frac{1}{4}$$

6. **(b) :** Let A and B be two events such that
A = getting number 2 at least once
B = getting 7 as the sum of the numbers on two dice
We have,
A = {(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6) (1, 2), (3, 2), (4, 2), (5, 2), (6, 2)} and
B = {(2, 5), (5, 2), (6, 1), (1, 6), (3, 4), (4, 3)}

$$\therefore\ P(A) = \frac{11}{36}, P(B) = \frac{6}{36}, P(A\cap B) = \frac{2}{36}$$

Now, required probability $P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{2/36}{6/36} = \frac{2}{6} = \frac{1}{3}$

7. **(c) :** Let A denotes the event that Indian man is seated adjacent to his wife and B denotes the event that each American man is seated adjacent to his wife.
$P(A \cap B)$ denotes the event that each man is seated adjacent to his wife. Consider each couple as one entity.
Thus, there are 5 entities to be arranged and husbands and wife was interchange their seats in 2! ways.

$$\therefore\ P(A\cap B) = \frac{4!(2!)^5}{9!}$$

Next, consider each American couple as an entity. Thus, there are 6 entities to be arranged including the Indian and his wife.

$$\therefore\ P(B) = \frac{5!(2!)^4}{9!}\qquad \therefore\ P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{2}{5}.$$

8. **(d) :** $P(E_1) = \frac{6}{6\cdot 6} = \frac{1}{6}$

$$P(E_2) = \frac{6}{6\cdot 6} = \frac{1}{6};\ P(E_3) = \frac{3\cdot 3\cdot 2}{6\cdot 6} = \frac{1}{2}$$

$$P(E_1\cap E_2) = \frac{1}{6\cdot 6} = P(E_1)\cdot P(E_2)$$

$$P(E_1\cap E_3) = \frac{1\cdot 3}{6\cdot 6} = P(E_1)\cdot P(E_3)$$

$$P(E_2\cap E_3) = \frac{1\cdot 3}{6\cdot 6} = P(E_2)\cdot P(E_3)$$

As $P(X \cap Y) = P(X)\cdot P(Y)$, the events X and Y are independent.
Also, $P(E_1 \cap E_2 \cap E_3) = 0$ as the event cannot happen.
So, E_1, E_2, E_3 are pairwise independent, but they together are not independent.

9. **(b):** $P(X|Y) = \frac{1}{2} \Rightarrow \frac{P(X\cap Y)}{P(Y)} = \frac{1}{2} \Rightarrow P(Y) = \frac{2}{6} = \frac{1}{3}$

Also, $\frac{P(X\cap Y)}{P(X)} = \frac{1}{3} \Rightarrow P(X) = \frac{3}{6} = \frac{1}{2}$

$P(X \cup Y) = P(X) + P(Y) - P(X \cap Y)$

$$= \frac{1}{2} + \frac{1}{3} - \frac{1}{6} = \frac{1}{2} + \frac{1}{6} = \frac{4}{6} = \frac{2}{3}$$

$$P(X)P(Y) = \frac{1}{2}\cdot\frac{1}{3}. \text{ Also } P(X\cap Y) = \frac{1}{6}$$

As $P(X \cap Y) = P(X)P(Y)$, So X and Y are independent.

10. **(a) :** $P(E \cap F) = P(E) \cdot P(F)$

$P(E \cap F') + P(E' \cap F) = \frac{11}{25}$ and $P(E' \cap F') = \frac{2}{25}$

Let $P(E) = x$, $P(F) = y$, then we have

$x(1-y) + y(1-x) = \frac{11}{25}$ or $x + y - 2xy = \frac{11}{25}$

Also, $(1-x)(1-y) = \frac{2}{25} \Rightarrow 1-(x+y)+xy = \frac{2}{25}$

Solving for $x + y$ and xy gives

$x + y = \frac{7}{5}$ and $xy = \frac{12}{25}$

Thus, x & y are roots of $t^2 - \frac{7}{5}t + \frac{12}{25} = 0$

$\Rightarrow \left(t-\frac{4}{5}\right)\left(t-\frac{3}{5}\right) = 0 \quad \therefore \quad t = \frac{4}{5}, \frac{3}{5}$

Thus, $P(E) = \frac{4}{5}$ and $P(F) = \frac{3}{5}$ or $P(E) = \frac{3}{5}$ and $P(F) = \frac{4}{5}$

11. **(d) :** We have $P(A\cap B) = P(A)P(B) = \frac{4}{10}\cdot\frac{b}{10}$

where we have assumed that $n(B) = b$.

Now, $P(A\cap B) = \frac{n(A\cap B)}{n(s)} = \frac{n(A\cap B)}{10}$

Thus $\frac{n(A\cap B)}{10} = \frac{4}{10}\cdot\frac{b}{10} \Rightarrow n(A\cap B) = \frac{4b}{10} = \frac{2b}{5}$

As $n(A \cap B)$ has to be an integer, we have $b = 5$ or 10.

12. **(b) :** A and B are independent events.

and $P(A^C \cap B) = \frac{2}{15} \Rightarrow P(A^C)\,P(B) = \frac{2}{15}$

$\Rightarrow P(B) - P(A)P(B) = \frac{2}{15}$ ...(i)

and $P(A\cap B^C) = \frac{1}{6} \Rightarrow P(A)P(B^C) = \frac{1}{6}$

$\Rightarrow P(A) - P(A)P(B) = \frac{1}{6}$ ...(ii)

On solving (i) and (ii), we get $P(B) = \frac{1}{6}$ or $\frac{4}{5}$.

13. **(c) :** E, F, G are pairwise independent events.

$\therefore P(E \cap F) = P(E) \cdot P(F)$

$P(F \cap G) = P(F) \cdot P(G)$, $P(G \cap E) = P(G) \cdot P(E)$

$$P\left((E^c\cap F^c)\,|\,G\right) = \frac{P\left((E^c\cap F^c)\cap G\right)}{P(G)}$$

$$= \frac{P(G) - P(G\cap E) - P(G\cap F)}{P(G)}$$

$$= \frac{P(G)\,(1-P(E)-P(F))}{P(G)} = 1 - P(E) - P(F) = P(E^c) - P(F).$$

14. **(b) :** $P(A\cup B) = 1 - P(\overline{A\cup B}) = 1 - \frac{1}{6} = \frac{5}{6}$

We have, $P(\bar{A}) = \frac{1}{4} \Rightarrow P(A) = \frac{3}{4}$

Again, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\Rightarrow \frac{5}{6} = \frac{3}{4} + P(B) - \frac{1}{4} \quad \therefore \quad P(B) = \frac{1}{3}$

Now, $P(A \cap B) = P(A)\,P(B)$ is seem to be true.

Thus, A and B are independent.

As $P(A) \neq P(B)$, A and B are not equally likely.

15. **(a) :** Statement-2 is false.

P(P and Q contradict each other)

$= p(1 - 2p) + 2p(1 - p) = 1/2$

$\Rightarrow 8p^2 - 6p + 1 = 0.$

$\Rightarrow (2p - 1)(4p - 1) = 0 \Rightarrow p = 1/2, 1/4.$

16. **(b) :** Let M be the event that A's blood type matches the guilty party. Let $P(A)$ be probability that A is guilty and $P(B)$ be the probability that B is guilty.

$$P(A/M) = \frac{P(M/A)P(A)}{P(M/A)P(A) + P(M/B)P(B)}$$

$$= \frac{\frac{1}{2}}{\frac{1}{2} + \left(\frac{1}{5}\right)\left(\frac{1}{2}\right)} = \frac{\frac{1}{2}}{\frac{1}{2}+\frac{1}{10}} = \frac{\frac{1}{2}}{\frac{6}{10}} = \frac{1}{2\times 6}\times 10 = \frac{5}{6}$$

17. **(c) :** Let E_1 and E_2 be the events that marble is green and blue respectively in the bag.

Let A be the event of picking up a green marble.

Then, $P(E_1) = P(E_2) = 1/2$, $P(A/E_1) = 1$, $P(A/E_2) = 1/2$

Now, if the marble taken out is green, then probability that remaining marble is also green is $P(E_1/A)$

$$P(E_1/A) = \frac{P(E_1)P(A/E_1)}{P(E_1)P(A/E_1) + P(E_2)P(A/E_2)}$$

$$= \frac{\frac{1}{2}\times 1}{\frac{1}{2}\times 1 + \frac{1}{2}\times\frac{1}{2}} = \frac{\frac{1}{2}}{\frac{2+1}{4}} = \frac{1}{2}\times\frac{4}{3} = \frac{2}{3}$$

18. **(d) :** Let events T_1, T_2, T_3 be the following :

T_1 : the item is manufactured by factory F_1.

T_2 : the item is manufactured by factory F_2.

T_3 : the item is manufactured by factory F_3.

Clearly, T_1, T_2, T_3 are mutually exclusive and exhaustive events.

$P(T_1) = 30\% = 0.3$, $P(T_2) = 20\% = 0.2$, $P(T_3) = 50\% = 0.5$

Let E be the event that item is defective

Now, $P(E|T_1) = 2\% = 0.02$, $P(E|T_2) = 3\% = 0.03$

$P(E|T_3) = 4\% = 0.04$. Hence, by Bayes' theorem, we have

$$P(T_1\,|\,E) = \frac{P(T_1)P(E\,|\,T_1)}{P(T_1)P(E\,|\,T_1) + P(T_2)P(E\,|\,T_2) + P(T_3)P(E\,|\,T_3)}$$

$$= \frac{0.3\times 0.02}{0.3\times 0.02 + 0.2\times 0.03 + 0.5\times 0.04}$$

$$= \frac{0.006}{0.006 + 0.006 + 0.020} = \frac{0.006}{0.032} = \frac{6}{32} = \frac{3}{16}$$

19. **(c) :** Let the following events :

E = He knows the correct answer

F = He randomly ticks

G = He answered correctly

$\therefore$ Required probability = $P(E\,|\,G)$

$$= \frac{P(G|E)\cdot P(E)}{P(G|E)\cdot P(E) + P(G|F)\cdot P(F)} = \frac{1\times p}{1\times p + \frac{1}{5}\times(1-p)} = \frac{5p}{4p+1}$$

20. **(d) :** Let E_1 and E_2 be the events that unbiased and biased coins are selected respectively and H be the event that head has occurred.

$$\therefore \quad P(E_1) = \frac{1}{2},\; P(E_2) = \frac{1}{2},\; P(H|E_1) = \frac{1}{2},\; P(H|E_2) = \frac{3}{4}$$

$$\therefore \quad P(E_1|H) = \frac{P(H|E_1)\cdot P(E_1)}{P(H|E_1)\cdot P(E_1) + P(H|E_2)\cdot P(E_2)}$$

$$= \frac{\frac{1}{2}\cdot\frac{1}{2}}{\frac{1}{2}\cdot\frac{1}{2}+\frac{3}{4}\cdot\frac{1}{2}} = \frac{2}{5}$$

21. **(b) :** Let E_1 = event that item is manufactured by machine M_1.
E_2 = event that item is manufactured by machine M_2.
and A = event that selected item is defective.

$\therefore\ P(E_1) = \frac{2}{3},\ P(E_2) = \frac{1}{3}$

$\therefore\ P(A|E_1) = \frac{4}{100},\ P(A|E_2) = \frac{3}{100}$

$\therefore$ By theorem of total probability,

$P(A) = P(E_1)\,P(A \mid E_1) + P(E_2)\,P(A \mid E_2)$

$= \frac{2}{3}\times\frac{4}{100}+\frac{1}{3}\times\frac{3}{100} = \frac{11}{300}$

22. **(c) :** Let E_1 = event of choosing bag (i) & E_2 the event of choosing the bag (ii) & 'A' be the event of drawing red ball

$\therefore\ P(E_1) = P(E_2) = \frac{1}{2}$

$\therefore\ P(A/E_1) = \frac{3}{7},\ \ P(A/E_2) = \frac{5}{11}$

$$\therefore\ P(E_2/A) = \frac{P(E_2)P(A/E_2)}{P(A)} = \frac{\frac{1}{2}\times\frac{5}{11}}{\frac{1}{2}\times\frac{3}{7}+\frac{1}{2}\times\frac{5}{11}} = \frac{35}{68}$$

23. **(b) :** Let X denote the sum of the numbers obtained when two fair dice are rolled. So, X may have values 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 and 12.

$p\,(x=2) = \frac{1}{36}$; $p\,(x=3) = \frac{2}{36}$; $p\,(x=4) = \frac{3}{36}$; $p(x=5) = \frac{4}{36}$;

$p\,(x=6) = \frac{5}{36}$; $p(x=7) = \frac{6}{36}$; $p(x=8) = \frac{5}{36}$;

$p\,(x=9) = \frac{4}{36}$; $p(x=10) = \frac{3}{36}$; $p(x=11) = \frac{2}{36}$; $p\,(x=12) = \frac{1}{36}$

x	2	3	4	5	6	7	8	9	10	11	12
$p(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Mean $(\overline{X}) = \Sigma xp(x)$

$$= \frac{\begin{bmatrix}(2\times1)+(3\times2)+(4\times3)+(5\times4)+(6\times5)+\\(7\times6)+(8\times5)+(9\times4)+(10\times3)+\\(11\times2)+(12\times1)\end{bmatrix}}{36} = \frac{252}{36} = 7$$

Variance = $\Sigma x^2p(x) - (\Sigma xp(x))^2$

$$= \frac{\begin{bmatrix}(2^2\times1)+(3^2\times2)+(4^2\times3)+(5^2\times4)\\+(6^2\times5)+(7^2\times6)+(8^2\times5)+(9^2\times4)\\+(10^2\times3)+(11^2\times2)+(12^2\times1)\end{bmatrix}}{36} - 7^2$$

$$= \frac{1974}{36} - 49 = \frac{1974-1764}{36} = \frac{210}{36} = \frac{35}{6}$$

$\therefore$ Variance $= \frac{35}{6}$. Hence, S.D. $= \sqrt{\frac{35}{6}}$

24. **(b) :** Standard deviation $= \sqrt{\sum p_i x_i^2 - (\Sigma p_i x_i)^2}$

$\sum p_i x_i^2 = (0.1)\times(1)^2 + (0.2)\times(2)^2 + (0.3)\times(3)^2 + (0.4)\times(4)^2 = 10$

$\sum p_i x_i = 1\times0.1 + 2\times0.2 + 3\times0.3 + 4\times0.4 = 3$

$\therefore$ Standard deviation $= \sqrt{10-3^2} = \sqrt{1} = 1$

25. **(a) :** Since, the man is one step away from starting point means that either

(i) man has taken 6 steps forward and 5 steps backward

or

(ii) man has taken 5 steps forward and 6 steps backward.

Taking, movement 1 step forward as success and 1 step backward as failure

$\therefore\ p$ = probability of success = 0.4

q = probability of failure = 0.6

$\therefore$ Required probability

$= P(X = 6 \text{ or } X = 5) = P(X = 6) + P(X = 5)$

$= {}^{11}C_6\, p^6q^5 + {}^{11}C_5\, p^5q^6 = {}^{11}C_6\,[p^6q^5 + p^5q^6]$

$= {}^{11}C_6\,[(0.4)^6\,(0.6)^5 + (0.4)^5\,(0.6)^6]$

$= {}^{11}C_6\,(0.4)^5\,(0.6)^5\,[0.4 + 0.6] = {}^{11}C_6\,(0.24)^5$

26. **(a) :** Probability of at least one failure = 1 – P(no failure) = $1 - p^5$

Now $1 - p^5 \geq \frac{31}{32} \Rightarrow p^5 \leq \frac{1}{32}$ thus $p \leq \frac{1}{2}$ $\therefore\ p \in [0, 1/2]$

27. **(d) :** $E(Y) = \sum_{i=1}^{n} y_i p_i$

$\Rightarrow\ E(Y) = \sum_{i=1}^{n} 10x_i p_i\ \ (\because Y = 10X)\ \ \Rightarrow\ E(Y) = 10\sum_{i=1}^{n} x_i p_i$

$\Rightarrow E(Y) = 10 \times E(X) = 10 \times 3 \Rightarrow E(Y) = 30$

Now, $V(X) = \sum_{i=1}^{n} x_i^2 p_i - [E(X)]^2$

$\Rightarrow\ 2 = \sum_{i=1}^{n} x_i^2 p_i - (3)^2\ \Rightarrow\ \sum_{i=1}^{n} x_i^2 p_i = 11$

$\therefore\ V(Y) = \sum_{i=1}^{n} y_i^2 p_i - [E(Y)]^2$

$\Rightarrow\ V(Y) = \sum_{i=1}^{n} (10x_i)^2 p_i - (30)^2 = 10^2 \sum_{i=1}^{n} x_i^2 p_i - 900$

$\Rightarrow V(Y) = (100 \times 11) - 900 \Rightarrow V(Y) = 200$

28. **(d) :** Probability of at least one success

$= 1 - P(\text{No success}) = 1 - {}^nC_0\, q^n$ where $q = 1 - p = 3/4$

we want $1 - \left(\frac{3}{4}\right)^n \geq \frac{9}{10}\ \Rightarrow\ \frac{1}{10} \geq \left(\frac{3}{4}\right)^n\ \Rightarrow\ \left(\frac{3}{4}\right)^n \leq \frac{1}{10}$

Taking logarithm on base 10 we have,

$n \log_{10}(3/4) \leq \log_{10}10^{-1}$

$\Rightarrow n(\log_{10}3 - \log_{10}4) \leq -1$

$\Rightarrow n(\log_{10}4 - \log_{10}3) \geq 1$

$\Rightarrow n \geq \frac{1}{\log_{10}4 - \log_{10}3}$

29. **(b) :** Possibility of getting a score 9 are (5, 4), (4, 5), (6, 3), (3, 6)

Probability of getting score 9 in a single throw = $p = \frac{4}{36} = \frac{1}{9}$

$\therefore$ Required probability

= probability of getting score 9 exactly twice

$= {}^3C_2\left(\frac{1}{9}\right)^2 \times \left(\frac{8}{9}\right) = \frac{8}{243}.$

30. (b) :

x_i	0	1	2	3	4	5	6
f_i	0.17	0.29	0.27	0.16	0.07	0.03	0.01
$f_i x_i$	0	0.29	0.54	0.48	0.28	0.15	0.06

$$\text{Mean} = \sum_{i=1}^{7} f_i x_i = 0 + 0.29 + 0.54 + 0.48 + 0.28 + 0.15 + 0.06$$
= 1.80

31. (a, c, d) : $P(E_2 / E_1) = \dfrac{P(E_1 \cap E_2)}{P(E_1)}$

$\Rightarrow \dfrac{1}{2} = \dfrac{P(E_1 \cap E_2)}{1/4} \Rightarrow P(E_1 \cap E_2) = \dfrac{1}{8}$

Also, $P(E_1 \cap E_2) = P(E_2) \cdot P(E_1 / E_2) = P(E_2) \cdot \dfrac{1}{4}$

$\Rightarrow P(E_2) = \dfrac{1}{2}$

Since $P(E_1 \cap E_2) = \dfrac{1}{8} = P(E_1) \cdot P(E_2)$

$\Rightarrow$ Events E_1 and E_2 are independent.

Also $P(E_1 \cup E_2) = \dfrac{1}{2} + \dfrac{1}{4} - \dfrac{1}{8} = \dfrac{5}{8}$

$\Rightarrow$ E_1 and E_2 are non exhaustive.

32. (a, b, d) :

Urn	Red marbles	White marbles	Blue marbles
A	5	3	8
B	3	5	0

$$P(E_1) = P(R) = \left(\frac{2}{3}\right)\left(\frac{5}{16}\right) + \left(\frac{1}{3}\right)\left(\frac{3}{8}\right) = \frac{10}{48} + \frac{6}{48} = \frac{1}{3}$$

$$P(E_2) = P(W) = \left(\frac{2}{3}\right)\left(\frac{3}{16}\right) + \left(\frac{1}{3}\right)\left(\frac{5}{8}\right) = \frac{6}{48} + \frac{10}{48} = \frac{1}{3}$$

$$P(E_3) = P(B) = \left(\frac{2}{3}\right)\left(\frac{8}{16}\right) = \frac{1}{3}$$

Now, let E be the event that urn A is chosen

$$P(E | R) = \frac{P(E \cap R)}{P(R)} = \frac{\left(\frac{2}{3}\right)\left(\frac{5}{16}\right)}{\frac{1}{3}} = \left(\frac{10}{48}\right)(3) = \frac{5}{8}$$

$$P(E | W) = \frac{P(E \cap W)}{P(W)} = \frac{\left(\frac{2}{3}\right)\left(\frac{3}{16}\right)}{\frac{1}{3}} = \left(\frac{6}{48}\right)(3) = \frac{3}{8}$$

$$P(\text{face five} | W) = \left(\frac{3}{8}\right)\left(\frac{1}{4}\right) = \frac{3}{32}$$

33. (a, c) : The probability of exactly one of A and B to occur

$= P(A\bar{B}) + P(\bar{A}B)$

$= P(A)P(\bar{B}) + P(\bar{A})P(B) = P(A)\{1 - P(B)\} + \{1 - P(A)\}P(B)$

$= P(A) + P(B) - 2P(A)P(B)$

$= P(A) + P(B) - 2P(A \cap B)$

$= 1 - P(\bar{A}) + 1 - P(\bar{B}) - 2\{1 - P(\bar{A})\}\{1 - P(\bar{B})\}$

$= P(\bar{A}) + P(\bar{B}) - 2P(\bar{A}) \cdot P(\bar{B})$

34. (a, b, c, d) : $P(E_1) = P$(choosing 5 and two tickets from 6 to 15)

$$= \frac{{}^{10}C_2}{{}^{15}C_3} = \frac{10 \times 9}{2} \times \frac{3 \times 2}{15 \times 14 \times 13} = \frac{9}{91}$$

$P(E_2) = P$(choosing 10 and two tickets from 1 to 9)

$$= \frac{{}^{9}C_2}{{}^{15}C_3} = \frac{9 \times 8}{2} \times \frac{3 \times 2}{15 \times 14 \times 13} = \frac{36}{455}$$

$P(E_1 \cap E_2) = P$(choosing 5 and 10 and one ticket from 6 to 9)

$$= \frac{{}^{4}C_1}{{}^{15}C_3} = 4 \times \frac{3 \times 2}{15 \times 14 \times 13} = \frac{4}{455}$$

$P(E_1/E_2) = \dfrac{1}{9}$

35. (a, b, c, d) : Probability that atleast one blue ball is drawn
= 1 – Probability that all the balls drawn are red.

$$= 1 - \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{3}{5} = 1 - \frac{1}{10} = \frac{9}{10} = 0.9$$

Probability that exactly one blue ball is drawn

$$= \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{2}{5} + \frac{2}{3} \cdot \frac{1}{4} \cdot \frac{2}{5} + \frac{1}{3} \cdot \frac{2}{4} \cdot \frac{2}{5} = 0.2$$

Probability that all the drawn balls are red given that all the drawn balls are of the same colour

$$= \frac{\frac{1}{10}}{\frac{1}{10} + \frac{4}{10}} = \frac{1}{5} = 0.2$$

Probability that atleast one red ball is drawn

$$= 1 - \left(\frac{2}{3} \cdot \frac{3}{4} \cdot \frac{4}{5}\right) = 0.6$$

36. (a, b, c) : Given, $a = np$, $b = npq$
Since $0 < q < 1 \Rightarrow 0 < npq < np$

$\Rightarrow 0 < b < a \Rightarrow \dfrac{a}{b} > 1$

Also $\dfrac{a^2}{a - b} = \dfrac{n^2 p^2}{np - npq} = \dfrac{np}{1 - q} = n$, which is an integer.

37. (a, b, c, d) : A, B are exhaustive events
$\Rightarrow A \cup B = S \Rightarrow P(A \cup B) = 1$
Now, $2P(B) = P(A) + P(A \cup B) = K + 1$

$\Rightarrow P(B) = \dfrac{K + 1}{2}$

$\Rightarrow -P(A \cap B) = P(A \cup B) - P(A) - P(B)$

$$= 1 - K - \frac{(K + 1)}{2} = \frac{2 - 2K - K - 1}{2}$$

$\Rightarrow P(A \cap B) = \dfrac{3K - 1}{2}$

Also, $P(A' \cup B') = \dfrac{3(1 - k)}{2}$

38. (a, b, d) : $P(A) = P(A/B) = 1/4$
$\Rightarrow$ A and B are independent,
hence, $P(B/A) = P(B) = 1/2$
Now A', B'; A', B will also be independent

$$\therefore P\left(\frac{A'}{B}\right) = P(A') = 1 - P(A) = \frac{3}{4}$$

$$P\left(\frac{B'}{A'}\right) = P(B') = 1 - P(B) = \frac{1}{2}$$

$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 5/8$
$P(A \cap B) = 1/8$

39. **(b) : (A) – (q), (B) – (p), (C) – (q), (D) – (r)**

(A) Total no. of ways of putting the balls in boxes

$$= \frac{6!}{2!2!2!} + \frac{6!3!}{1!2!4!} + \frac{6!3!}{1!2!3!} = 540$$

No. of favourable ways $= \frac{6!}{2!2!2!} = 90$

Required probability = $\frac{1}{6}$

(B) Probability $= \frac{3^6 - {}^3C_1 \cdot 2^6 + 3}{3^6} = \frac{20}{27}$

(C) No. of ways A can throw greater than 9 is 6

Required probability = $\frac{1}{6}$

(D) $1 - P(B) + P(A \cap B) = 0.8$

$\Rightarrow P(B)(0.7) = 0.2 \Rightarrow P(B) = \frac{2}{7}$

40. **(a) : (A) – (r) , (B) – (p) , (C) – (s) (D) – (q)**

(A) P (no pair) = $\frac{20}{20} \cdot \frac{18}{19} \cdot \frac{16}{18} \cdot \frac{14}{17} = \frac{224}{323}$

(B) P (atleast one pair) = $1 - \frac{224}{323} = \frac{99}{323}$

(C) P (exactly two pairs) = $\frac{{}^{10}C_2}{{}^{20}C_4} = \frac{3}{323}$

(D) P (exactly one pair) = $1 - \left[\frac{224}{323} + \frac{3}{323}\right] = \frac{96}{323}$

41. **(a) :** The event associated with $X = 5$ is
T T T T T H
$\Rightarrow P(X = 5) = (1 - p)^5 . p$

42. **(d) :** $P(3 \le X \le 6) = (1 - p)^3 p + (1 - p)^4 p + (1 - p)^5 p + (1 - p)^6 p$
$= (1 - p)^3 - (1 - p)^7$

43. **(c) :** $P(A/T) = 0.8$; $P(A/S) = 0.6$; $P(A/F) = 0.4$
Now, $P(A) = P(A \cap T) + P(A \cap S) + P(A \cap F)$
$= P(T) \times P(A/T) + P(S) \times P(A/S) + P(F) \times P(A/F)$
$= (0.1)(0.8) + (0.2)(0.6) + (0.7)(0.4)$
$= 0.08 + 0.12 + 0.28 = 0.48$

44. **(b) :** $P(F/A) = \frac{P(F \cap A)}{P(A)} = \frac{0.28}{0.48} = \frac{7}{12}$

45. **(d) :** $P(F/\bar{A}) = \frac{P(F \cap \bar{A})}{P(\bar{A})} = \frac{P(F) - P(F \cap A)}{0.52}$

$$= \frac{(0.7) - 0.28}{0.52} = \frac{0.42}{0.52} = \frac{21}{26}.$$

46. **(1) :** We may include 2 women or all the 3 women. When 2 women are include the number of ways $= {}^3C_2 \cdot {}^7C_3$ and when all the three women are included the number of ways $= {}^3C_3 \cdot {}^7C_2$

Hence the required probability

$$= \frac{{}^3C_2 \cdot {}^7C_3 + {}^3C_3 \cdot {}^7C_2}{{}^{10}C_5} = \frac{1}{2}$$

47. **(1) :** $n(S) = {}^{100}C_3$
Since A.M. of 3 numbers = 25
Then their sum is 75
$x_1 + x_2 + x_3 = 75, x_1 \ge 1, x_2 \ge 1, x_3 \ge 1$
No. of +ve integral solutions = ${}^{74}C_2 = 2701$

$\therefore$ Required probability = $\frac{2701}{{}^{100}C_3}$

48. **(5) :** $\frac{\frac{1}{9} \times \frac{K}{9}}{\frac{1}{9} \times \frac{1}{9} + ... + \frac{1}{9} \times \frac{9}{9}} = \frac{1}{9}$

$$\Rightarrow \frac{K}{\left(\frac{9 \times 10}{2}\right)} = \frac{1}{9} \Rightarrow \frac{K}{45} = \frac{1}{9} \Rightarrow K = 5$$

49. **(3) :** Number of ways in which $m - 1$ remaining cars can take their places (excluding the car of a man)
$= {}^{n-1}C_{m-1}$
No. of ways in which remaining $(m - 1)$ cars can take places keeping the two places on two sides of his car vacant $= {}^{n-3}C_{m-1}$

Probability = $\frac{{}^{n-3}C_{m-1}}{{}^{n-1}C_{m-1}} = \frac{(n-m)(n-m-1)}{(n-1)(n-2)}$

$\Rightarrow A = 1, B = 2$
$\Rightarrow A + B = 3$

50. **(7) :** $P(G_k) \propto k^2 \Rightarrow P(G_k) = \lambda k^2$

$\sum_{k=0}^{n} P(G_k) = 1$ (as these are mutually exclusive and exhaustive events)

$$\Rightarrow \lambda \sum_{k=0}^{n} k^2 = 1 \Rightarrow \lambda = \frac{6}{n(n+1)(2n+1)}$$

$$P(A) = \sum_{k=0}^{n} P(G_k) P(A/G_k)$$

$$= \sum_{k=0}^{n} \lambda k^2 \cdot \frac{k}{n} = \frac{\lambda}{n} \cdot \frac{n^2(n+1)^2}{4} = \frac{3(n+1)}{2(2n+1)}$$

Taking $n = 15$, we get

$$P(A) = \frac{24}{31} = \frac{p}{q}$$

Then, $q - p = 31 - 24 = 7$.

JEE MATHEMATICS

DPS-23

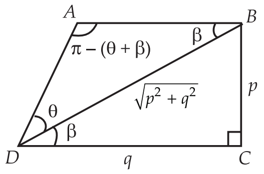
DAILY PRACTICE SHEET

Model Test Paper

SOLUTIONS

1. **(d) :** Using sine rule in triangle ABD, we get

$$\frac{AB}{\sin\theta}=\frac{BD}{\sin(\theta+\beta)} \quad \Rightarrow \quad AB=\frac{\sqrt{p^2+q^2}\sin\theta}{\sin(\theta+\beta)}$$



As $\tan\beta=\frac{p}{q}$, we have $\sin(\theta+\beta)=\sin\theta\cos\beta+\cos\theta\sin\beta$

$$=\sin\theta\cdot\frac{q}{\sqrt{p^2+q^2}}+\cos\theta\cdot\frac{p}{\sqrt{p^2+q^2}}=\frac{p\cos\theta+q\sin\theta}{\sqrt{p^2+q^2}}$$

$$\therefore \quad AB=\frac{(p^2+q^2)\sin\theta}{p\cos\theta+q\sin\theta}$$

2. **(d) :** Range of $|\sin 2x|+|\cos 2x|$ is $[1,\sqrt{2}]$

$[|\sin 2x|+|\cos 2x|]=1$

$\therefore\ f(x)=0 \ \Rightarrow f(x)$ is a constant function

$\therefore\ f(x)$ is periodic with no fundamental period.

Clearly $f(x)$ is into function and f is not invertible.

3. **(a) :** $\vec{p},\vec{q}$ are parallel if $\vec{p}\times\vec{q}=0$ and

$\vec{p}\times\vec{q}=(3ax^3+2(x-1)^2b)\hat{k}$

$\therefore \quad |\vec{p}\times\vec{q}|=3ax^3+2(x-1)^2b=f(x)$, (say)

$f(0)=2b, f(1)=3a, f(0)f(1)=6ab<0$ (given)

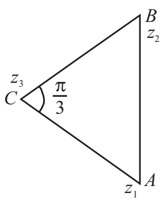
By intermediate value theorem, there exists x in $(0, 1)$ such that $f(x)=0$, $\vec{p}\times\vec{q}=0$.

4. **(d) :** $z_1+z_2\omega+z_3\omega^2=0$

$z_1+z_2\omega-z_3(1+\omega)=0$

$z_1-z_3=-\omega(z_2-z_3)$

$$-\omega=\frac{1}{2}-\frac{i\sqrt{3}}{2}=\text{cis}\left(-\frac{\pi}{3}\right)$$



AC is obtained by rotating CB through $\frac{\pi}{3}$ clockwise about C.

$\therefore \quad z_1, z_2, z_3$ form an equilateral triangle.

5. **(b) :** We have,

p	q	$\sim p$	$\sim q$	$\sim p\vee\sim q$	$\sim q\vee p$	$(p\wedge q)$	$\sim(p\wedge q)$	$p\wedge\sim q$	$q\vee p$
T	T	F	F	F	T	T	F	F	T
T	F	F	T	T	T	F	T	T	T
F	T	T	F	T	F	F	T	F	T
F	F	T	T	T	T	F	T	F	F

So, $\sim(p\wedge q)\equiv\sim p\vee\sim q$

6. **(c) :** $\frac{dy}{y\sqrt{y^2-1}}=\frac{dx}{x\sqrt{x^2-1}}$

Integrating, $\sec^{-1}y=\sec^{-1}x+c$

$$y(2)=\frac{2}{\sqrt{3}} \Rightarrow \sec^{-1}\frac{2}{\sqrt{3}}=\sec^{-1}2+c$$

$$\therefore \quad c=\cos^{-1}\frac{\sqrt{3}}{2}-\cos^{-1}\frac{1}{2}=\frac{\pi}{6}-\frac{\pi}{3}=\frac{-\pi}{6}$$

$$\therefore \quad \sec^{-1}y=\sec^{-1}x-\frac{\pi}{6} \qquad \text{...(i)}$$

$$\Rightarrow \quad y=\sec\left(\sec^{-1}x-\frac{\pi}{6}\right)$$

From (i) $\Rightarrow \cos^{-1}\frac{1}{y}=\cos^{-1}\frac{1}{x}-\cos^{-1}\frac{\sqrt{3}}{2}$

$$\Rightarrow \quad \frac{1}{y}=\cos\left(\cos^{-1}\frac{1}{x}-\cos^{-1}\frac{\sqrt{3}}{2}\right)=\frac{\sqrt{3}}{2x}+\sqrt{1-\frac{1}{x^2}}\times\frac{1}{2}$$

or $\quad \frac{1}{y}=\frac{\sqrt{3}}{2x}+\frac{1}{2}\sqrt{1-\frac{1}{x^2}}$.

7. **(d) :** We have, $f(2-x)-1=1-f(x)$

$\Rightarrow \quad g(2-x)=-g(x)$

Replacing x by $x+1$, we get, $g(1-x)=-g(1+x)$

So, $g(x)$ is symmetrical about the point $(1, 0)$.

8. **(b) :** We have,

$$A_1=\frac{3a+b}{4}, A_2=\frac{a+b}{2}, A_3=\frac{a+3b}{4}$$

$G_1=(a^3b)^{1/4}$, $G_2=(ab)^{1/2}$, $G_3=(ab^3)^{1/4}$ and

$$H_1=\frac{4ab}{(a+3b)}, H_2=\frac{2ab}{(a+b)}, H_3=\frac{4ab}{(3a+b)}$$

$\Rightarrow \quad A_2H_2=ab=G_2{}^2$

$G_2{}^2=A_1H_3=A_2H_2=A_3H_1=ab$

9. **(a) :** Let E_1 be the event that one of first n urns is chosen, E_2 be the event that $(n+1)^{\text{th}}$ urn is chosen.

A be the event that two balls drawn are black.

$$\therefore \quad P(E_1)=\frac{n}{n+1};\ P(E_2)=\frac{1}{n+1}$$

$$P\left(\frac{A}{E_1}\right)=\frac{{}^6C_2}{{}^{10}C_2}=\frac{1}{3};\ P\left(\frac{A}{E_2}\right)=\frac{{}^5C_2}{{}^{10}C_2}=\frac{2}{9}$$

Also, $$P\left(\frac{E_2}{A}\right)=\frac{P(E_2)\cdot P\left(\frac{A}{E_2}\right)}{P(E_1)P\left(\frac{A}{E_1}\right)+P(E_2)\cdot P\left(\frac{A}{E_2}\right)}$$

$$\Rightarrow \quad \frac{1}{16}=\frac{\frac{1}{n+1}\cdot\frac{2}{9}}{\frac{n}{n+1}\cdot\frac{1}{3}+\frac{1}{n+1}\cdot\frac{2}{9}} \Rightarrow n=10$$

10. **(a) :** $99^{100} - 1 = (1 - 100)^{100} - 1$

$$= -100 \cdot 100 + \binom{100}{2} \cdot 100^2 - \ldots$$

It is divisible by $100^2 = 10^4$

The number of zeroes at the end is 4.

11. **(b) :** The number of ways of 3 letters going to correct destinations is $\binom{7}{3} = 35$

The number of ways of 4 letters going to wrong destinations is

$$4!\left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right) = 9$$

$\therefore$ The number of desired ways is $35 \times 9 = 315$.

12. **(c) :** Number of words starting with

CC is 4! words, CH is 4! words

CI is 4! words, CN is 4! words

The next word is COCHIN.

There are 4(4!) = 96 words before COCHIN.

13. **(b) :** $\cos^2 x + \cos^2 2x + \cos^2 3x = 1$

$\Rightarrow \cos^2 2x + \cos^2 3x = \sin^2 x$

$\Rightarrow 1 + \cos 4x + 1 + \cos 6x = 2\sin^2 x$

$\Rightarrow \cos 4x + \cos 6x + 2\cos^2 x = 0$

$\Rightarrow 2\cos 5x \cos x + 2\cos^2 x = 0$

$\Rightarrow \cos x(\cos 5x + \cos x) = 0$

$\Rightarrow \cos x \cdot \cos 3x \cdot \cos 2x = 0$

$\Rightarrow \cos x = 0 \Rightarrow x = (2n+1)\frac{\pi}{2}$,

$\cos 2x = 0$

$\Rightarrow x = (2n+1)\frac{\pi}{4}$

$\cos 3x = 0$

$\Rightarrow x = (2n+1)\frac{\pi}{6}$.

14. **(c) :** Let A be the event that the man selected the winning horse.

$\therefore$ Exhaustive cases = ${}^5C_2$

Favourable cases = $1 \times {}^4C_1$

$$\therefore \ P(A) = \frac{1 \times {}^4C_1}{{}^5C_2} = \frac{2}{5}$$

15. **(d) :** $\cot^{-1}\left(r^2 + \frac{3}{4}\right) = \tan^{-1}\frac{1}{r^2 + \frac{3}{4}}$

$$= \tan^{-1}\frac{1}{1 + r^2 - \frac{1}{4}} = \tan^{-1}\frac{\left(r + \frac{1}{2}\right) - \left(r - \frac{1}{2}\right)}{1 + \left(r^2 - \frac{1}{4}\right)}$$

$$= \tan^{-1}\left(r + \frac{1}{2}\right) - \tan^{-1}\left(r - \frac{1}{2}\right)$$

$$\therefore \sum_{r=1}^{n} \cot^{-1}\left(r^2 + \frac{3}{4}\right) = \sum_{r=1}^{n}\left(\tan^{-1}\left(r + \frac{1}{2}\right) - \tan^{-1}\left(r - \frac{1}{2}\right)\right)$$

$$= \tan^{-1}\left(n + \frac{1}{2}\right) - \tan^{-1}\frac{1}{2}$$

$$= \tan^{-1}\frac{n + \frac{1}{2} - \frac{1}{2}}{1 + \left(n + \frac{1}{2}\right)\frac{1}{2}} = \tan^{-1}\frac{4n}{2n+5}.$$

16. **(c) :** The slope of the line $3x - y = 7$ *i.e.*, $y = 3x - 7$ is $\tan\theta = 3$.

or $\frac{P}{B} = \frac{3}{1} \Rightarrow H = \sqrt{9+1} = \sqrt{10}$

$\therefore \ \sin\theta = \frac{3}{\sqrt{10}}, \ \cos\theta = \frac{1}{\sqrt{10}}$

The equation of line passing through (1, 2) and parallel to $y = 3x - 7$ is

$$\frac{x-1}{\cos\theta} = \frac{y-2}{\sin\theta} \quad \ldots\text{(i)}$$

Let r be the required distance.

$\therefore$ $(1 + r\cos\theta, 2 + r\sin\theta)$ lies on $x + y + 5 = 0$

$\Rightarrow 1 + r\cos\theta + 2 + r\sin\theta + 5 = 0$

$\Rightarrow 1 + r\frac{1}{\sqrt{10}} + 2 + r\frac{3}{\sqrt{10}} + 5 = 0 \Rightarrow r = -2\sqrt{10}$

But distance can't be negative

$\therefore$ Required distance $= \sqrt{40}$

17. **(c) :** $xy = 4 \Rightarrow y + x\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{y}{x}$

$\therefore$ Any point on $xy = 4$ may be taken as $P\left(2t, \frac{2}{t}\right)$

$$\therefore \left[\frac{dy}{dx}\right]_{\text{at } P} = -\frac{2/t}{2t} = -\frac{1}{t^2}$$

If $ax + by + c = 0$ be tangent to $xy = 4$ at P, then

$$-\frac{1}{t^2} = -\frac{a}{b} \Rightarrow t^2 = \frac{b}{a} \qquad (\because t^2 > 0)$$

$\therefore \frac{b}{a} > 0 \Rightarrow$ Either $a < 0, b < 0$ or $a > 0, b > 0$

18. **(b) :** Equation of normal at $P(x, y)$ is

$$Y - y = -\frac{dx}{dy}(X - x) \Rightarrow G \equiv \left(x + y\cdot\frac{dy}{dx}, 0\right)$$

$$\left|x + y\frac{dy}{dx}\right| = |2x| \Rightarrow y\frac{dy}{dx} = x \text{ or } y\frac{dy}{dx} = -3x$$

$\Rightarrow ydy = xdx$ or $ydy = -3xdx$

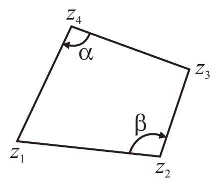
After integrating, we get

$$\frac{y^2}{2} = \frac{x^2}{2} + c \text{ or } \frac{y^2}{2} = -\frac{3x^2}{2} + c$$

$\Rightarrow x^2 - y^2 = -2c$ or $3x^2 + y^2 = 2c$

$\Rightarrow x^2 - y^2 = c_1$ or $3x^2 + y^2 = c_2$

19. **(a) :** $\arg\left(\frac{z_1 - z_4}{z_3 - z_4}\right) = \alpha$



and $\arg\left(\frac{z_3 - z_2}{z_1 - z_2}\right) = \beta$

$$\arg\frac{(z_1 - z_4)(z_2 - z_3)}{(z_1 - z_2)(z_3 - z_4)} =$$

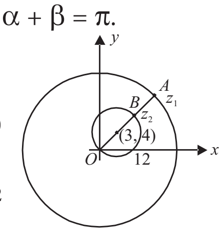
$$\arg\left(\frac{z_1 - z_4}{z_3 - z_4}\right) + \arg\left(\frac{z_2 - z_3}{z_1 - z_2}\right) = \alpha + \arg\frac{(z_3 - z_2)(-1)}{(z_1 - z_2)}$$

$= \alpha + \beta + \arg(-1) = \pi - \pi = 0$, since $\alpha + \beta = \pi$.

20. **(b) :** $|z_1| = 12$ is circle with centre (0, 0) and radius 12.

$|z_2 - 3 - 4i| = 5$ is circle with centre (3, 4) and radius 5.

The minimum distance between z_1 and z_2 is $AB = 12 - 10 = 2$.



21. **(a) :** Let $A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$.

We have, $A^2 = \begin{bmatrix} \alpha^2 + \beta\gamma & \beta(\alpha+\delta) \\ \gamma(\alpha+\delta) & \delta^2 + \beta\gamma \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow \quad \alpha^2 + \beta\gamma = 1 = \delta^2 + \beta\gamma$

and $\gamma(\alpha + \delta) = \beta(\alpha + \delta) = 0$

As $A \neq I$, $A \neq -I$, we have $\alpha = -\delta$

$$\det A = \begin{vmatrix} \sqrt{1-\beta\gamma} & \beta \\ \gamma & -\sqrt{1-\beta\gamma} \end{vmatrix} = -1 + \beta\gamma - \beta\gamma = -1$$

$\therefore$ Statement-1 is true.

$Tr(A) = \alpha + \delta = 0 \qquad \{\because \alpha = -\delta\}$

Statement-2 is false, because $Tr(A) = 0$

22. **(c) :** $B, s_1, s_2, s_3, s_4, s_5, D$

For a particular class the total number of different tickets from first intermediate station is 5.

Number of different tickets from second intermediate station is 4.

So, the total no. of different tickets is 5 + 4 + 3 + 2 +1 = 15

For the remaining classes the total no. of different tickets = 2 ×15 = 30

$\therefore$ Total number of different tickets for all three classes = 3 × 15 = 45

Required number of ways = ${}^{45}C_{10}$

23 **(a) :** $a + b + c + d + e + f + g = 265$

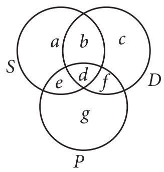
$a + b + d + e = 200$

$b + c + d + f = 110$

$d + e + f + g = 55$

$b + d = 60$, $d + e = 30$, $d = 10$

On solving,



a	b	c	d	e	f	g
120	50	40	10	20	10	15

$\therefore$ Number of persons who like only dancing and painting = f = 10

24. **(d) :** Required number of ways

$$= 3^5 - \binom{3}{1}2^5 + \binom{3}{2}1^5 = 243 - 96 + 3 = 150.$$

25. **(d) :** For $(a, b), (c, d) \in N \times N$

$(a, b)\ R\ (c, d) \Rightarrow ad\ (b + c) = bc\ (a + d)$

Reflexive : Since, $ab\ (b + a) = ba\ (a + b)$, $\forall\ a,b \in N$

$\therefore \quad (a, b)\ R\ (a, b)$, So, R is reflexive.

Symmetric : For $(a, b), (c, d) \in N \times N$,

Let $(a, b)\ R\ (c, d)$

$\therefore \quad ad\ (b + c) = bc\ (a + d) \Rightarrow bc\ (a + d) = ad\ (b + c)$

$\Rightarrow \quad cb\ (d + a) = da\ (c + b) \Rightarrow (c, d)\ R\ (a, b)$

So, R is symmetric.

Transitive : For $(a, b), (c, d), (e, f) \in N \times N$

Let $(a, b)\ R\ (c, d)$, $(c, d)\ R\ (e, f)$

$\therefore \quad ad\ (b + c) = bc\ (a + d)$, $cf\ (d + e) = de\ (c + f)$

$\Rightarrow \quad adb + adc = bca + bcd \qquad$ ...(i)

and $cfd + cfe = dec + def \qquad$ ...(ii)

On multiplying Eq. (i) by ef and Eq. (ii) by ab and then adding, we get

$adbef + adcef + cfdab + cfeab = bcaef + bcdef + decab + defab$

$\Rightarrow \quad adcf\ (b + e) = bcde\ (a + f) \Rightarrow af\ (b + e) = be\ (a + f)$

$\Rightarrow \quad (a, b)\ R\ (e, f)$

So, R is transitive. Hence, R is an equivalence relation.

26. **(b) :** $f(x) = \log_a x$, $a > 0$

The domain of the logarithmic function is the set of all positive real numbers *i.e.*, $(0, \infty)$ and the range is the set R of all real numbers.

27. **(c) :** $\begin{vmatrix} b^2 - ab & b - c & bc - ac \\ ab - a^2 & a - b & b^2 - ab \\ bc - ac & c - a & ab - a^2 \end{vmatrix}$

$$= \begin{vmatrix} b(b-a) & b-c & c(b-a) \\ a(b-a) & a-b & b(b-a) \\ c(b-a) & c-a & a(b-a) \end{vmatrix} = (b-a)^2 \begin{vmatrix} b & b-c & c \\ a & a-b & b \\ c & c-a & a \end{vmatrix} = 0$$

28. **(b) :** We have, $4y^2 + 12x - 12y + 39 = 0$

$\Rightarrow \quad 4(y^2 - 3y) + 12x + 39 = 0$

$\Rightarrow \quad 4\left[\left(y - \frac{3}{2}\right)^2 - \frac{9}{4}\right] + 12x + 39 = 0$

$\Rightarrow \quad 4\left(y - \frac{3}{2}\right)^2 + 12x + 30 = 0$

$\Rightarrow \quad \left(y - \frac{3}{2}\right)^2 = -3\left(x + \frac{5}{2}\right)$

which represents a leftward parabola with vertex $\left(-\frac{5}{3}, \frac{3}{2}\right)$

$a = \frac{3}{4}$ and line of axis is parallel to x-axis.

$\therefore$ Equation of directrix, $x = -\frac{5}{2} + \frac{3}{4} = -\frac{7}{4}$

29. **(d) : Statement-1 :** The sum of the series

1 + (1 + 2 + 4) + (4 + 6 + 9) + ... + (361 + 380 + 400) is 8000

Here, $T_1 = 1$, $T_2 = 7 = 8 - 1$,

$T_3 = 19 = 27 - 8 \Rightarrow T_n = n^3 - (n - 1)^3$

Statement-2 : $\sum_{k=1}^{n} (k^3 - (k-1)^3) = n^3$

$\therefore$ Statement-2 is a correct explanation of Statement-1.

30. **(d) :** $AB = B$

So, A should be unit matrix

Now $BA = A$

So, B should be unit matrix

Square of unit matrix is also unit matrix

So, $A^2 = A$ and $B^2 = B$

So, both A and B are idempotent.

31. **(b, d) :** $\sec x - 1 = (\sqrt{2} - 1)\tan x = \tan\frac{\pi}{8}\tan x$

$$\Rightarrow \quad \frac{1}{\cos x} - 1 = \frac{\sin\frac{\pi}{8}\sin x}{\cos\frac{\pi}{8}\cos x}$$

$$\Rightarrow \quad \cos\frac{\pi}{8} - \cos\frac{\pi}{8}\cos x = \sin\frac{\pi}{8}\sin x$$

$$\Rightarrow \quad \cos\frac{\pi}{8}\cos x + \sin\frac{\pi}{8}\sin x = \cos\frac{\pi}{8}$$

$\Rightarrow \cos\left(x-\frac{\pi}{8}\right)=\cos\frac{\pi}{8}$ or $x-\frac{\pi}{8}=2n\pi\pm\frac{\pi}{8}$

or $x=2n\pi, 2n\pi+\frac{\pi}{4}$ and $y=0, \sqrt{2}-1$

$(x, y)=(2n\pi, 0), \left(2n\pi+\frac{\pi}{4}, \sqrt{2}-1\right)$.

32. (a, c, d) : As A and B are skew hermitian

$\therefore \quad A^\theta=-A, B^\theta=-B$

$(A+B)^\theta=A^\theta+B^\theta=-A-B=-(A+B)$ ((a) is true)

$(AB)^\theta=B^\theta A^\theta=(-B)(-A)=BA$

$=AB$ ((b) is false and (d) is true)

It is well-known as diagonal elements in hermitian matrix must be real.

33. (a, b) : $x^2+y^2+2gx+2fy+c=0$

(There are three arbitrary constants g, f and c)

$\therefore \quad 2x+2y\cdot\frac{dy}{dx}+2g+2f\cdot\frac{dy}{dx}=0$ ...(i)

$\therefore \quad 2+2y_1^2+2yy_2+2fy_2=0$ ...(ii)

$\therefore \quad 4y_1y_2+2y_1y_2+2yy_3+2fy_3=0$ ...(iii)

From (i), (ii) and (iii), $y_3(1+y_1^2)-3y_1y_2^2=0$

34. (b, d) : Let $t=\log_{10}x$

$T_9=\binom{10}{8}\frac{10}{x^{5t}}\cdot x^{\left(8+\frac{4}{t}\right)}=450$

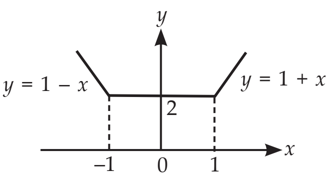
$\therefore x^{8+\frac{4}{t}-5t}=1$. Taking logarithm with base 10 on both sides, we get

$\left(8+\frac{4}{t}-5t\right)=0$

$\Rightarrow \quad 5t^2-8t-4=0 \Rightarrow t=2, \frac{-2}{5}$

$\therefore \quad x=10^t=10^2, 10^{-2/5}$

35. (a, c) :



$f(x)$ is continuous for all x and differentiable for all x except $x=\pm 1$

36. (a, b, d) : $C_0-C_1+C_2-C_3+...+(-1)^{m-1}C_{m-1}$

$={}^nC_0-{}^nC_1+{}^nC_2-{}^nC_3+...+(-1)^{m-1}\cdot{}^nC_{m-1}$

$={}^nC_0-({}^{n-1}C_0+{}^{n-1}C_1)+({}^{n-1}C_1+{}^{n-1}C_2)$

$-({}^{n-1}C_2+{}^{n-1}C_3)+....+(-1)^{m-1}({}^{n-1}C_{m-2}+{}^{n-1}C_{m-1})$

$=({}^nC_0-{}^{n-1}C_0)+(-1)^{m-1}\cdot{}^{n-1}C_{m-1}$

$=0+(-1)^{m-1}\cdot{}^{n-1}C_{m-1}$ (option (b))

$=(-1)^{m-1}\cdot\frac{(n-1)(n-2)...(n-m+1)}{(m-1)!}$ (option (a))

Also from option (b) we get,

$C_0-C_1+C_2-C_3+...+(-1)^{m-1}\cdot C_{m-1}$

$=(-1)^{m-1}\cdot{}^{n-1}C_{m-1}$

$=(-1)^{m-1}\cdot{}^{n-1}C_{n-m}$ (option (d)) $[\because {}^nC_r={}^nC_{n-r}]$

37. (a, c, d) : Given, $A^T=A$ and $B^T=-B$

(a) $(ABA^T)^T=AB^TA^T=-(ABA^T)$

So, ABA^T is skew-symmetric matrix.

(b) $AB^T+BA^T=-AB+BA$

$\Rightarrow (BA-AB)^T=(BA)^T-(AB)^T$

$=A^TB^T-B^TA^T$

$=-AB+BA$

So, AB^T+BA^T is symmetric matrix.

(c) $((A+B)(A-B))^T$

$=(A-B)^T(A+B)^T$

$=(A^T-B^T)(A^T+B^T)$

$=(A+B)(A-B)$

So, $(A+B)(A-B)$ is symmetric.

(d) $((A+I)(B-I))^T=(B^T-I)(A^T+I)$

$=-(B+I)(A+I)$

So, $(A+I)(B-I)$ is neither symmetric nor skew symmetric.

38. (b, c) : The circles $x^2+y^2-2x-15=0$ and $x^2+y^2-1=0$ has the radical axis $-2x-14=0$. $\therefore \; x+7=0$

Hence any circle orthogonal to them has its centre as $(-7, -\beta)$

Let its equation be

$S: x^2+y^2+14x+2\beta y+\lambda=0$

Orthogonality with 2nd circle gives

$\lambda-1=0. \quad \therefore \; \lambda=1$

Again, (0, 1) lies on $S \Rightarrow 1+2\beta+1=0. \; \therefore \; \beta=-1$

Equation is $x^2+y^2+14x-2y+1=0$

Radius $=\sqrt{7^2+1^2-1}=7$

39. (a) : (A)-(q), (B)-(r), (C)-(s), (D)-(s)

(A) $x=x^2 \Rightarrow x=0, 1$ at which $f(x)$ is continuous

(B) $f(x)$ is not differentiable at $x=0, -1, 1$

(C) $|x+1|+|x-1|=2|x| \Rightarrow x\in(-\infty,-1]\cup[1,\infty)$.

(D) $[x-1]+[x+1]=0 \Rightarrow [x]-1+[x]+1=0$

$\Rightarrow [x]=0 \Rightarrow x\in[0, 1)$

40. (b) : (A)-(q), (B)-(p), (C)-(q), (D)-(r)

(A) Total no. of ways of putting the balls in boxes

$=\frac{6!}{2!2!2!}+\frac{6!3!}{1!2!4!}+\frac{6!3!}{1!2!3!}=540$

No. of favourable ways $=\frac{6!}{2!2!2!}=90$

Required probability $=\frac{90}{540}=\frac{1}{6}$

(B) Probability $=\frac{3^6-{}^3C_1\cdot 2^6+3}{3^6}=\frac{20}{27}$

(C) No. of ways A can throw greater than 9 is 6

Required probability $=\frac{1}{6}$

(D) $1-P(B)+P(A\cap B)=0.8 \Rightarrow P(B)(.7)=.2 \Rightarrow P(B)=\frac{2}{7}$

41. (d) **42. (b)**

(41-42) : $\frac{S_a(1)}{S_b(1)}=\frac{a_1}{b_1}=\frac{14}{10}=\frac{7}{5}$

Also, $\frac{S_a(2)}{S_b(2)}=\frac{19}{12}=\frac{a_1+a_2}{b_1+b_2} \Rightarrow \frac{7k+4}{5k+b_2}=\frac{19}{12}$

$\Rightarrow \quad 84k+48=95k+19b_2$

$\Rightarrow \quad \frac{48-11k}{19}=b_2$ ...(i)

Now $\frac{S_a(3)}{S_b(3)} = \frac{2a_1 + 2d_a}{2b_1 + 2d_b} = \frac{a_1 + d_a}{b_1 + d_b} = \frac{a_2}{b_2}$

$\therefore \ \frac{a_2}{b_2} = \frac{24}{14}$

$\Rightarrow \ b_2 = \frac{7}{12} \times 4 = \frac{7}{3}$ (As, $a_2 = 4$) ...(ii)

Also, $\frac{7}{3} = \frac{48 - 11k}{19}$ [Using (ii) in (i)]

$\Rightarrow \ 133 = 144 - 33k \Rightarrow \ k = \frac{1}{3}$

So, $a_1 = \frac{7}{3}; a_2 = 4 \Rightarrow d_a = \frac{5}{3}$

Now, $b_1 = \frac{5}{3}; b_2 = \frac{7}{3} \Rightarrow d_b = \frac{2}{3}$

$\therefore \ b_4 = \frac{5}{3} + 3 \cdot \left(\frac{2}{3}\right) = \frac{5}{3} + 2 = \frac{11}{3}$

43. (b) : The vector perpendicular to L_1 and L_2 is

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{vmatrix} = -\hat{i} - 7\hat{j} + 5\hat{k}.$$

$\hat{n}$ = unit vector is $\frac{-\hat{i} - 7\hat{j} + 5\hat{k}}{5\sqrt{3}}$.

44. (d) : The points on L_1 and L_2 are $A(-1, -2, -1)$ and $C(2, -2, 3)$

The shortest distance is $\overrightarrow{AC} \cdot \hat{n}$

$= (3\hat{i} + 4\hat{k}) \cdot \frac{(-\hat{i} - 7\hat{j} + 5\hat{k})}{5\sqrt{3}} = \frac{17}{5\sqrt{3}}$.

45. (c) : The *d.r's* of the normal are –1, –7, 5

$\therefore$ The plane is

$-(x + 1) - 7(y + 2) + 5(z + 1) = 0 \Rightarrow x + 7y - 5z + 10 = 0$

The distance of (1, 1, 1) from plane is $\frac{13}{\sqrt{75}} = \frac{13}{5\sqrt{3}}$.

46. (4) : $n(M \cup P \cup C) = 50$, $n(M) = 37$, $n(P) = 24$, $n(C) = 43$, $n(M \cap P) \le 19$, $n(M \cap C) \le 29$, $n(P \cap C) \le 20$.

We have,

$n(M \cup P \cup C) = n(M) + n(P) + n(C) - n(M \cap P) - n(M \cap C) - n(P \cap C) + n(M \cap P \cap C)$

$\Rightarrow \ 50 = 37 + 24 + 43 - n(M \cap P) - n(M \cap C) - n(P \cap C) + n(M \cap P \cap C)$

$\Rightarrow \ n(M \cap P \cap C) = n(M \cap P) + n(M \cap C) + n(P \cap C) - 54$

$\Rightarrow \ n(M \cap P \cap C) \le 19 + 29 + 20 - 54 = 14$

$\Rightarrow \ n(M \cap P \cap C) \le 14$.

47. (4) : $t^2 - 9t + 14 = 0 \Rightarrow (t - 2)(t - 7) = 0$

$\Rightarrow \ [x^2 - 2] = 2$ or $[x^2 - 2] = 7$

$\Rightarrow \ 2 \le x^2 - 2 < 3$ or $7 \le x^2 - 2 < 8$

$\Rightarrow \ 4 \le x^2 < 5$ or $9 \le x^2 < 10$

$\Rightarrow 2 \le |x| < \sqrt{5}$ or $3 \le |x| < \sqrt{10}$

$\Rightarrow \ x \in (-\sqrt{5}, -2] \cup [2, \sqrt{5})$

or $x \in (-\sqrt{10}, -3] \cup [3, \sqrt{10})$

$\Rightarrow$ The integral values of x are –2, 2, –3, 3

48. (4) : Let $y = (1 + a)(1 + b)$

$= 1 + b + a + ab = 2 + a + b = 2 + a + \frac{1}{a}$ $(\because ab = 1)$

Differentiating both sides with respect to a, we get

$\frac{dy}{da} = 0 + 1 - \frac{1}{a^2} = 1 - \frac{1}{a^2}$

Put $\frac{dy}{da} = 0 \Rightarrow a = \pm 1$

But $a > 0$, so $a = 1$

Further, $\frac{d^2y}{da^2} = \frac{2}{a^3} > 0$ for $a = 1$

So, y is least at $a = 1$

At $a = 1$, $b = 1$

Now, the least value of $y = (1 + a)(1 + b)$ is $= (1 + 1)(1 + 1) = 4$

49. (1) : $f = g^{-1}$ *i.e.* $g^{-1}(x) = f(x)$

where $g(x) = y = 2\tan^{-1}(e^x) - \frac{\pi}{2}$

$\Rightarrow \tan^{-1}(e^x) = \frac{\pi}{4} + \frac{y}{2}$

$\Rightarrow x = \log \tan\left(\frac{\pi}{4} + \frac{y}{2}\right)$

$\Rightarrow g^{-1}(y) = \log \tan\left(\frac{\pi}{4} + \frac{y}{2}\right)$

$\Rightarrow g^{-1}(x) = f(x) = \log \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$

$\Rightarrow f'(x) = \frac{1}{\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)} \cdot \sec^2\left(\frac{\pi}{4} + \frac{x}{2}\right) \times \frac{1}{2}$

$\therefore \ f'(0) = \frac{1}{\tan\frac{\pi}{4}} \times \sec^2\frac{\pi}{4} \times \frac{1}{2} = 1$

50. (1) : $y = x^2 = \frac{4 \cdot 1}{4} x^2$

Equation of tangent to parabola is $y = mx + \frac{1}{4m}$ ... (i)

On differentiating, we get $\frac{dy}{dx} = m$ ... (ii)

From (i) and (ii), we get

$y = \frac{dy}{dx} x + \frac{1}{4(dy/dx)} \Rightarrow 4y\left(\frac{dy}{dx}\right) = 4x\left(\frac{dy}{dx}\right)^2 + 1$

This is differential equation of tangent.

Clearly order is 1, degree is 2.

JEE MATHEMATICS

DPS-24

DAILY PRACTICE SHEET

Model Test Paper

SOLUTIONS

1. **(d) :** $f(x) = e^{x^2} + e^{-x^2}$

$f'(x) = 2x(e^{x^2} - e^{-x^2}) \geq 0 \quad \forall\ x \in [0, 1]$

Thus f is increasing and so the maximum value of f is

$f(1) = e + \dfrac{1}{e}$

Again $g(x) = xe^{x^2} + e^{-x^2}$,

So, $g'(x) = (1 + 2x)\ e^{x^2} - 2xe^{-x^2} \geq 0\ \forall\ x \in [0, 1]$

Thus g is increasing and so $g(1) = e + \dfrac{1}{e}$ is the maximum value.

Also $h(x) = x^2e^{x^2} + e^{-x^2}$

$h'(x) = 2x[e^{x^2} + x^2e^{x^2} - e^{-x^2}] \geq 0\ \forall\ x \in [0, 1]$

Thus h is increasing and so $h(1) = e + \dfrac{1}{e}$ is the maximum value.

Hence $a = b = c$.

2. **(b) :** $g(x) = \sin(\sin^{-1} \sqrt{\{x\}}\ ) + \cos\ (\sin^{-1} \sqrt{\{x\}}) - 1$

$= \sqrt{\{x\}} + \cos(\cos^{-1} \sqrt{1-\{x\}}) - 1\ = \sqrt{\{x\}} + \sqrt{1-\{x\}} - 1$

If $x \in I$ then $\{x\} = 0 \Rightarrow g(x) = 0 \quad \Rightarrow g(-x) = g(x) \Rightarrow g$ is even

If $x \notin I$ then $\{-x\} = 1 - \{x\}$

$\Rightarrow \quad g(-x) = \sqrt{1-\{x\}} + \sqrt{\{x\}} - 1 = g(x) \quad \Rightarrow$'$g$' is even function

$g(x) = \begin{cases} 0, & x \in I \\ g(-x), & x \notin I \end{cases} \quad \Rightarrow$ 'g' is periodic function

3. **(c) :** Here,

$\dfrac{\cos 2A}{a^2} - \dfrac{\cos 2B}{b^2} = \dfrac{1-2\sin^2 A}{a^2} - \dfrac{1-2\sin^2 B}{b^2}$

$= \dfrac{1}{a^2} - \dfrac{1}{b^2} - \dfrac{2\sin^2 A}{a^2} + \dfrac{2\sin^2 B}{b^2}$

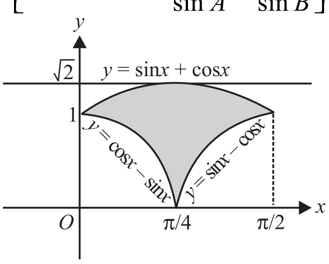
$= \dfrac{1}{a^2} - \dfrac{1}{b^2} - 2\left(\dfrac{\sin^2 A}{a^2} - \dfrac{\sin^2 B}{b^2}\right)$

$= \dfrac{1}{a^2} - \dfrac{1}{b^2} \qquad \left[\because \text{In } \Delta ABC,\ \dfrac{a}{\sin A} = \dfrac{b}{\sin B}\right]$

4. **(b) :** The curves are

$y = \sin x + \cos x,\ x \in [0, \pi/2]$

$y = \begin{cases} \cos x - \sin x, x \in [0, \pi/4) \\ \sin x - \cos x, x \in (\pi/4, \pi/2] \end{cases}$



$\therefore$ Required area $= \displaystyle\int_0^{\pi/4} \{(\sin x + \cos x) - (\cos x - \sin x)\}\ dx$

$\displaystyle + \int_{\pi/4}^{\pi/2} \{(\sin x + \cos x) - (\sin x - \cos x)\} dx$

$\displaystyle = 2\int_0^{\pi/4} \sin x dx + 2\int_{\pi/4}^{\pi/2} \cos x dx = 2(-\cos x)\Big|_0^{\pi/4} + 2(\sin x)\Big|_{\pi/4}^{\pi/2}$

$= 2\left(1 - \dfrac{1}{\sqrt{2}}\right) + 2\left(1 - \dfrac{1}{\sqrt{2}}\right) = 4\left(1 - \dfrac{1}{\sqrt{2}}\right)$

$= 2\sqrt{2}(\sqrt{2} - 1)$ sq. units

5. **(d) :** We have, $\dfrac{dy}{dx} + \dfrac{y}{2}\sec x = \dfrac{\tan x}{2y}$

$\Rightarrow\ 2y\dfrac{dy}{dx} + y^2 \sec x = \tan x$ …(i)

Put $y^2 = t \Rightarrow 2y\dfrac{dy}{dx} = \dfrac{dt}{dx}$

$\therefore$ Equation (i) becomes, $\dfrac{dt}{dx} + t\sec x = \tan x$

I.F. $= e^{\int \sec x dx} = e^{\ln(\sec x + \tan x)} = \sec x + \tan x$

$\therefore$ Solution is given by

$t(\sec x + \tan x) = \int \tan x(\sec x + \tan x) dx$

$\Rightarrow t(\sec x + \tan x) = \sec x + \tan x - x + c$

$\Rightarrow\ t = 1 - \dfrac{x + c}{\sec x + \tan x} \ \Rightarrow\ y^2 = 1 - \dfrac{x + c}{\sec x + \tan x}$

Now, $y(0) = 1 \Rightarrow 1 = 1 - 0 + c \Rightarrow c = 0$

$\therefore$ Particular solution is $y^2 = 1 - \dfrac{x}{\sec x + \tan x}$

6. **(a) :** $\sim (p \vee q) \vee (\sim p \wedge q)$

$\equiv (\sim p \wedge \sim q) \vee (\sim p \wedge q) \equiv \sim p \wedge (\sim q \vee q) \equiv \sim p \wedge t \equiv \sim p$

7. **(d) :** $|\vec{b}|^2 = |\vec{c}|^2 = 9x^2 + y^2 + 4z^2$

$\Rightarrow\ \angle(\vec{a}, \vec{b}) = \angle(\vec{a}, \vec{c}) \Rightarrow \vec{a}\cdot\vec{b} = \vec{a}\cdot\vec{c}$

$\Rightarrow xy - 2yz + 3zx = 2zx + 3xy - yz$

$\Rightarrow 2xy + yz - zx = 0$ ....(i)

$\Rightarrow\ \vec{a}\cdot\vec{d} = 0 \Rightarrow x - y + 2z = 0$ .....(ii)

Eliminating y from (i), (ii), we get

$(2x + z)\ (x + 2z) - zx = 0$

$\Rightarrow (x + z)^2 = 0 \Rightarrow z = -x,\ y = -x$ [by (ii)]

$|\vec{a}| = 2\sqrt{3} \Rightarrow 12 = x^2 + y^2 + z^2 = 3x^2$

$\therefore\ x = \pm 2$

$\Rightarrow (x, y, z) = (2, -2, -2), (-2, 2, 2)$

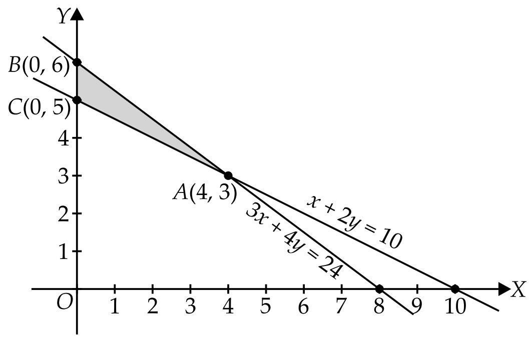
$\vec{a}\cdot\vec{b} = xy - 2yz + 3zx = -4 - 8 - 12 = -24$.

8. **(c) :** The corners of the feasible region are $A(4, 3), B(0, 6), C(0, 5)$.

$z = 2x + 5y$

$\Rightarrow z(A) = 23, z(B) = 30, z(C) = 25$

$\therefore$ Minimum $z = 23$ at A.



9. **(d) :** As a_{912}, a_{951} and a_{480} are divisible by 3, none of them is prime. For a_{91}, we have

$$a_{91} = \frac{1}{9}\underbrace{(99...9)}_{91 \text{ times}} = \frac{1}{9}(10^{91} - 1)$$

$$= \frac{1}{9}[(10^7)^{13} - 1] = \left[\frac{(10^7)^{13} - 1}{10^7 - 1}\right]\left[\frac{10^7 - 1}{10 - 1}\right]$$

$= [(10^7)^{12} + (10^7)^{11} + ... + 10^7 + 1] \times [10^6 + 10^5 + ... + 10 + 1]$

$\Rightarrow a_{91}$ is not prime.

10. **(b) :** Coefficient of $(r-1)^{\text{th}}$, r^{th} and $(r+1)^{\text{th}}$ term in expansion of $(1+x)^n$ are ${}^nC_{r-2}$, ${}^nC_{r-1}$, ${}^nC_r$ respectively.

$\Rightarrow {}^nC_{r-2} : {}^nC_{r-1} : {}^nC_r = 1 : 3 : 5$

$$\Rightarrow \frac{{}^nC_{r-2}}{{}^nC_{r-1}} = \frac{1}{3} \text{ and } \frac{{}^nC_{r-1}}{{}^nC_r} = \frac{3}{5}$$

$$\Rightarrow \frac{r-1}{n-r+2} = \frac{1}{3} \text{ and } \frac{r}{n-r+1} = \frac{3}{5}$$

$\Rightarrow 3r - 3 = n - r + 2$ and $5r = 3n - 3r + 3$

$\Rightarrow 4r - n = 5$....(i) and $8r - 3n = 3$....(ii)

Solving (i) and (ii), we get $r = 3$ and $n = 7$

11. **(d) :** Let $f(x) = \dfrac{(1+x)^{3/5}}{1+x^{3/5}}$

When $f'(x) = 0 \Rightarrow x = 1$

Also, $f(0) = 1$, and $f(1) = \dfrac{2^{0.6}}{2} = 2^{-0.4}$

$\therefore f(x) \in (2^{-0.4}, 1)$

12. **(c) :** $S_1 = \sum_{j=1}^{10} j(j-1)\, {}^{10}C_j = \sum j(j-1) \cdot \dfrac{10(10-1)}{j(j-1)} \cdot {}^8C_{j-2}$

$$= 9 \times 10 \sum_{j=2}^{10} {}^8C_{j-2} = 90 \times 2^8$$

$$S_2 = \sum_{j=1}^{10} j \cdot {}^{10}C_j = 10 \sum_{j=1}^{10} {}^9C_{j-1} = 10 \times 2^9$$

$$S_3 = \sum_{j=1}^{10} j^2 \cdot {}^{10}C_j = \sum_{j=1}^{10} (j(j-1) + j) \cdot {}^{10}C_j$$

$$= \sum_{j=1}^{10} j(j-1)\, {}^{10}C_j + \sum_{j=1}^{10} j \cdot {}^{10}C_j$$

$= 90 \cdot 2^8 + 10 \cdot 2^9 = (45 + 10)2^9 = 55 \cdot 2^9$.

Thus Statement-1 is true and Statement-2 is false.

13. **(c) :** Number of words starting with
CC is 4! words, CH is 4! words
CI is 4! words, CN is 4! words
The next word is COCHIN.
There are 4(4!) = 96 words before COCHIN.

14. **(c) :** Summing the infinite G.P.' s,

$$x = \frac{1}{1-\cos^2\theta} = \frac{1}{\sin^2\theta}, \; y = \frac{1}{1-\sin^2\theta} = \frac{1}{\cos^2\theta}$$

$$\frac{1}{x} + \frac{1}{y} = \sin^2\theta + \cos^2\theta = 1 \Rightarrow x + y = xy \quad ...(i)$$

$$z = \frac{1}{1-\cos^2\theta\sin^2\theta} \Rightarrow \frac{1}{z} = 1 - \cos^2\theta\sin^2\theta$$

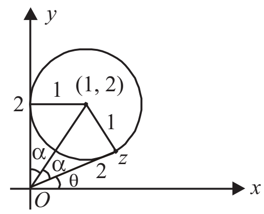
$$\Rightarrow \frac{1}{z} = 1 - \left(\frac{1}{x}\right)\left(\frac{1}{y}\right) = 1 - \frac{1}{xy}$$

$\Rightarrow z = \dfrac{xy}{xy-1} \Rightarrow xyz = z + xy \Rightarrow xyz = z + x + y$ (by (i))

15. **(c) :** $\tan\theta = \tan(\alpha + \theta - \alpha)$

$$= \frac{\tan(\alpha+\theta) - \tan\alpha}{1 + \tan(\alpha+\theta)\tan\alpha}$$

$$= \frac{\frac{2}{1} - \frac{1}{2}}{1 + \frac{2}{1} \cdot \frac{1}{2}} = \frac{3}{4}$$



$$\theta = \tan^{-1}\frac{3}{4}$$

$$\Rightarrow \text{least } \arg z = \tan^{-1}\frac{3}{4}$$

16. **(d) :** Let $\Delta = \begin{vmatrix} \sin\alpha & \cos\alpha & \sin(\alpha+\gamma) \\ \sin\beta & \cos\beta & \sin(\beta+\gamma) \\ \sin\delta & \cos\delta & \sin(\gamma+\delta) \end{vmatrix}$

Applying $C_3 \to C_3 - \cos\gamma\, C_1 - \sin\gamma\, C_2$, we get

$$\Delta = \begin{vmatrix} \sin\alpha & \cos\alpha & 0 \\ \sin\beta & \cos\beta & 0 \\ \sin\delta & \cos\delta & 0 \end{vmatrix} = 0$$

17. **(a) :** $\bar{x} = \dfrac{1 + 2 + 2 + ... + n + n}{1 + 2 + 3 + ... + n}$

$$= \frac{1^2 + 2^2 + 3^2 + ... + n^2}{n(n+1)/2} = \frac{n(n+1)(2n+1)}{6n(n+1)/2} = \frac{2n+1}{3}$$

$$\sigma^2 = \frac{1 \cdot 1^2 + 2 \cdot 2^2 + ... + n \cdot n^2}{n(n+1)/2} - \left(\frac{2n+1}{3}\right)^2$$

$$= \frac{1^3 + 2^3 + 3^3 + ... + n^3}{n(n+1)/2} - \left(\frac{2n+1}{3}\right)^2$$

$$= \frac{n(n+1)}{2} - \frac{(2n+1)^2}{9} = \frac{n^2 + n - 2}{18} = \frac{(n-1)(n+2)}{18}$$

18. **(b) :** Point of intersection of lines $x + y = 4$ and $x - y = 2$ is (3, 1)

Line through (3, 1) and making angle $\tan^{-1}\left(\dfrac{3}{4}\right)$ with x-axis is

$$y - 1 = \frac{3}{4}(x-3) \Rightarrow y = \frac{3x-5}{4} \quad ...(i)$$

On solving (i) with $y^2 = 4(x-3)$, we get

$9x^2 - 30x + 25 = 64(x-3) \Rightarrow 9x^2 - 94x + 217 = 0$

According to question, its roots are x_1, x_2

$$\therefore x_1 + x_2 = \frac{94}{9}; \; x_1x_2 = \frac{217}{9}$$

$$\therefore |x_1 - x_2| = \sqrt{(x_1+x_2)^2 - 4x_1x_2} = \frac{32}{9}$$

19. **(b) :** Let $\lim_{x\to\infty} f(x) = l$. Then,

$\lim_{x\to\infty} f(x+1) = \lim_{x\to\infty} f(x+2) = l.$

Now, $f(x) = \frac{1}{3}\left\{f(x+1) + \frac{5}{f(x+2)}\right\}$

$\Rightarrow \lim_{x\to\infty} f(x) = \frac{1}{3}\left\{\lim_{x\to\infty} f(x+1) + \frac{5}{\lim_{x\to\infty} f(x+2)}\right\}$

$\Rightarrow l = \frac{1}{3}\left(l + \frac{5}{l}\right) \Rightarrow 3l^2 = l^2 + 5 \Rightarrow 2l^2 = 5 \Rightarrow l = \sqrt{\frac{5}{2}}$

20. **(d) :** The given equation can be rewritten as

$\frac{(x-\sqrt{2})^2}{2/\pi} + \frac{y^2}{8/\pi} = 1$, which represents an ellipse

Here, $a = \sqrt{\frac{2}{\pi}}$ and $b = \sqrt{\frac{8}{\pi}}$

Area enclosed in an ellipse = πab

$= \pi\sqrt{\frac{2}{\pi}}\sqrt{\frac{8}{\pi}} = \sqrt{16} = 4$ sq. units

21. **(a)** : The 3 girls have to sit together in back seats *a, b, c, d* or *p, q, r, s*.

The number of ways they can be seated is 4 (3!) = 4!

The 9 boys can sit in the remaining 11 seats in the two vans in $^{11}P_9$ ways.

The desired number is $4!\cdot\frac{11!}{2!} = 12!$.

22. **(d) : Case (i) :** 4 correct + 1 incorrect.

For correct answer, he/she has to tick the correct option only, while for incorrect answer, he/she can give wrongly in 3 ways for 1st three questions each and in 1 way each for last 2 questions.

$\therefore$ Probability $= \frac{3+3+3+1+1}{4\times4\times4\times2\times2} = \frac{11}{256}$

Case (ii) : All 5 correct.

Probability $= \frac{1}{4\times4\times4\times2\times2} = \frac{1}{256}$

Required probability $= \frac{11}{256} + \frac{1}{256} = \frac{12}{256} = \frac{3}{64}$

23. **(a) :** $\sin^{-1}\sqrt{\frac{2-\sqrt{3}}{4}} = \sin^{-1}\sqrt{\frac{4-2\sqrt{3}}{8}}$

$= \sin^{-1}\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right) = 15°$

$\cos^{-1}\frac{\sqrt{12}}{4} = \cos^{-1}\frac{\sqrt{3}}{2} = 30°$

$\sec^{-1}\sqrt{2} = \cos^{-1}\frac{1}{\sqrt{2}} = 45°$

The given expression becomes $\sin^{-1}\cot(15° + 30° + 45°)$

$= \sin^{-1}(\cot 90°) = \sin^{-1} 0 = 0$.

24. **(c) :** Since $\vec{c}$ is a unit vector coplanar with $\vec{a}$ and $\vec{b}$,

let $\vec{c} = x\vec{a} + y\vec{b} = x(\hat{i}+\hat{j}-\hat{k}) + y(\hat{i}-\hat{j}+\hat{k})$

$= (x+y)\hat{i} + (x-y)\hat{j} - (x-y)\hat{k}$ ...(i)

where $(x+y)^2 + (x-y)^2 + (x-y)^2 = 1$ ...(ii)

Since $\vec{c} \perp \vec{a}$, $x+y+x-y+x-y = 0$

$\Rightarrow 3x = y$ ...(iii)

Using (iii) in (ii), we get $x = \frac{1}{2\sqrt{6}}$ (taking+ ve sign)

$\therefore$ From (i), $\vec{c} = \frac{1}{\sqrt{6}}(2\hat{i} - \hat{j} + \hat{k})$

Let $\vec{d} = d_1\hat{i} + d_2\hat{j} + d_3\hat{k}$

Since $\vec{d} \perp \vec{a}$ and $\vec{d} \perp \vec{c}$

$\therefore d_1 + d_2 - d_3 = 0$ and $2d_1 - d_2 + d_3 = 0$

$\Rightarrow d_1 = 0$ and $d_2 = d_3$.

Also, $|\vec{d}| = 1 \Rightarrow d_1^2 + d_2^2 + d_3^2 = 1$

$\Rightarrow d_2 = d_3 = \frac{1}{\sqrt{2}} \quad \therefore \vec{d} = \frac{1}{\sqrt{2}}(\hat{j} + \hat{k})$.

25. **(a) :** Given, $n(A) = 3$, $n(B) = 4$

$\therefore n(A \times A \times B) = 3 \times 3 \times 4 = 36$

26. **(c) :** Direction cosines of the line,

$L = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 1 \\ 1 & 3 & 2 \end{vmatrix} = 3\hat{i} - 3\hat{j} + 3\hat{k}$. Then, $\cos\alpha = \frac{3}{\sqrt{9+9+9}} = \frac{1}{\sqrt{3}}$.

27. **(d) :** $\because aRb$ iff $\sin^2 a + \cos^2 b = 1$

$\because \sin^2 a + \cos^2 a = 1$ for all $a \in \mathbf{R}$

$\therefore aRa \Rightarrow R$ is reflexive.

$\because \sin^2 a + \cos^2 b = 1 \Rightarrow 1 - \cos^2 a + 1 - \sin^2 b = 1$

$\Rightarrow \sin^2 b + \cos^2 a = 1 \Rightarrow bRa$

$\therefore R$ is symmetric. Further, let aRb and bRc

i.e. $\sin^2 a + \cos^2 b = 1$ ...(i) and $\sin^2 b + \cos^2 c = 1$ ...(ii)

Adding (i) & (ii), $\sin^2 a + 1 + \cos^2 c = 2$

$\Rightarrow \sin^2 a + \cos^2 c = 1 \Rightarrow aRc$

$\therefore R$ is also transitive. $\therefore R$ is an equivalence relation.

28. **(b) :** $\because y\frac{dy}{dx} + by^2 = a\cos x$

$\therefore 2y\frac{dy}{dx}\cdot e^{2bx} + y^2\cdot e^{2bx}\cdot 2b = 2a\cos x\cdot e^{2bx}$

[On multiplying by $2e^{2bx}$]

$\Rightarrow \frac{d}{dx}(y^2\cdot e^{2bx}) = 2a\cos x\cdot e^{2bx}$

On integrating, $y^2\cdot e^{2bx} = 2a\int\cos x\cdot e^{2bx}dx$

$= 2a\cdot\frac{e^{2bx}(2b\cos x + \sin x)}{(2b)^2 + 1^2} + c'$

$\Rightarrow (4b^2 + 1)y^2 = 2a(2b\cos x + \sin x) + c'(4b^2 + 1)e^{-2bx}$

$= 2a(2b\cos x + \sin x) + ce^{-2bx}$

29. **(c) :** Number of A.P.'s with common difference 1 = 17

Number of A.P.'s with common difference 2 = 14

Number of A.P.'s with common difference 3 = 11

Number of A.P.'s with common difference 4 = 8

Number of A.P.'s with common difference 5 = 5

Number of A.P.'s with common difference 6 = 2

Number of favourable outcomes = 57

The total number of outcomes = $^{20}C_4$

$\therefore$ Required probability $= \frac{57}{^{20}C_4} = \frac{57\times24}{20\times19\times18\times17} = \frac{1}{85}$

Now Statement-2 is false and Statement-1 is true.

30. **(d) :** Let the points A and B be t_1 and t_2.

The tangents at A and B meet at

$P(t_1t_2, t_1 + t_2) = (-2, -1) \quad \therefore t_1 = 1, t_2 = -2$

The normal at t is $xt + y = t^3 + 2t$.

The normals are $x + y = 3$, $-2x + y = -12$

Solving, $C \equiv (5, -2)$, But $A \equiv (1, 2)$ and $B \equiv (4, -4)$

$\therefore$ Area of $\Delta ABC = \frac{1}{2}\begin{vmatrix} 1 & 2 & 1 \\ 4 & -4 & 1 \\ 5 & -2 & 1 \end{vmatrix} = 6$.

31. **(a, d) :** We have $\alpha x^2 - x + \alpha = 0$

$\therefore\ D = 1 - 4\alpha^2$

For distinct real roots, $1 - 4\alpha^2 > 0$

i.e., $\alpha \in \left(-\frac{1}{2}, \frac{1}{2}\right)$

Now, $|x_1 - x_2| < 1$ *i.e.* $(x_1 - x_2)^2 < 1$

i.e., $\frac{b^2-4ac}{a^2} < 1 \Rightarrow 1 - 4\alpha^2 < \alpha^2$

$\Rightarrow\ \alpha \in \left(-\infty, -\frac{1}{\sqrt{5}}\right) \cup \left(\frac{1}{\sqrt{5}}, \infty\right)$

Combining the two bounds, we have

$\alpha \in \left(-\frac{1}{2}, -\frac{1}{\sqrt{5}}\right) \cup \left(\frac{1}{\sqrt{5}}, \frac{1}{2}\right)$

$\therefore\ S = \left(-\frac{1}{2}, -\frac{1}{\sqrt{5}}\right) \cup \left(\frac{1}{\sqrt{5}}, \frac{1}{2}\right)$

32. **(a,c)** : The equation is $\{(x-3)^2 + y^2\} + \lambda y = 0$

i.e., $x^2 + y^2 - 6x + \lambda y + 9 = 0$

Radius $= \sqrt{7+9} = \sqrt{16} = 4$

Now, $9 + \frac{\lambda^2}{4} - 9 = 16 \Rightarrow \lambda^2 = 64 \quad \therefore\ \lambda = \pm 8$

33. **(b, d) :** $I = x(2\sqrt{1+e^x}) - 2\int \sqrt{1+e^x}\,dx$

$= 2x\sqrt{1+e^x} - 2\int \frac{2t^2}{t^2-1}dt \qquad (t = \sqrt{1+e^x})$

$= 2x\sqrt{1+e^x} - 4\left(t + \frac{1}{2}\ln\frac{t-1}{t+1}\right) + C$

$= 2x\sqrt{1+e^x} - 4(\sqrt{1+e^x}) - 2\ln\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1} + C$

34. **(a, b, c, d) :** As $\vec{a} + \vec{b} + \vec{c} = 0$

$\Rightarrow\ a^2 = b^2 + c^2 + 2\vec{b}\cdot\vec{c}$

$\Rightarrow\ 144 = 48 + c^2 + 48 \Rightarrow\ c^2 = 48 \Rightarrow c = 4\sqrt{3}$

Again, $c^2 = a^2 + b^2 + 2\vec{a}\cdot\vec{b}$

$\Rightarrow\ \vec{a}\cdot\vec{b} = \frac{48-144-48}{2} = -72$

$|\vec{a}\times\vec{b} + \vec{c}\times\vec{a}| = |\vec{a}\times\vec{b} + \vec{a}\times\vec{b}| = 2|\vec{a}\times\vec{b}|$

$= 2\sqrt{a^2b^2 - (\vec{a}\cdot\vec{b})^2} = 2\sqrt{12^2\cdot 48 - (-72)^2}$

$= 2\cdot 12\sqrt{48-36} = 2\cdot 12\cdot 2\sqrt{3} = 48\sqrt{3}$

35. **(a, d)** : $ydx + dy = -e^x y^2 dy$

$\Rightarrow\ \frac{-(e^{-x}ydx + e^{-x}dy)}{y^2} = dy \Rightarrow d\left(\frac{e^{-x}}{y}\right) = dy$

$\Rightarrow e^{-x} = y^2 + cy \quad f(0) = 1, c = 0$

$\Rightarrow e^{-x} = y^2 \Rightarrow y = e^{-x/2}$

$\Rightarrow\ A = \int_0^1 (e^x - e^{-x/2})dx = e + \frac{2}{\sqrt{e}} - 3$

36. **(a, b) :** $f(\cos 4\theta) = \frac{2}{2-\sec^2\theta}$

Let $\cos 4\theta = t$

$\Rightarrow 2\cos^2 2\theta - 1 = t \Rightarrow \cos^2 2\theta = \frac{t+1}{2}$

For $t = \frac{1}{3}$, we have $\cos^2 2\theta = \frac{2}{3}$

$\cos 2\theta = \sqrt{\frac{2}{3}}$ or $\cos 2\theta = -\sqrt{\frac{2}{3}}$

$f(\cos 4\theta) = \frac{2}{2-\frac{1}{\cos^2\theta}} = \frac{2\cos^2\theta}{2\cos^2\theta - 1} = \frac{1+\cos 2\theta}{\cos 2\theta} = 1 + \frac{1}{\cos 2\theta}$

Hence, $f\left(\frac{1}{3}\right) = 1 + \sqrt{\frac{3}{2}}$ or $1 - \sqrt{\frac{3}{2}}$

37. **(a, c) :** $z = \sin\theta - i\cos\theta = \cos\left(\frac{\pi}{2} - \theta\right) - i\sin\left(\frac{\pi}{2} - \theta\right)$

$\therefore\ z^n = \cos\left(\frac{n\pi}{2} - n\theta\right) - i\sin\left(\frac{n\pi}{2} - n\theta\right)$

$z^{-n} = \cos\left(\frac{n\pi}{2} - n\theta\right) + i\sin\left(\frac{n\pi}{2} - n\theta\right)$

$\Rightarrow\ z^n + z^{-n} = 2\cos\left(\frac{n\pi}{2} - n\theta\right)$

and $z^n - z^{-n} = -2i\sin\left(\frac{n\pi}{2} - n\theta\right) = 2i\sin\left(n\theta - \frac{n\pi}{2}\right)$

38. **(a, b) :** $f(x) = (7\tan^6 x - 3\tan^2 x)(\tan^2 x + 1) = (7\tan^6 x - 3\tan^2 x)\sec^2 x$

Now, $\int_0^{\pi/4} f(x)dx = \int_0^1 (7t^6 - 3t^2)dt = \left(t^7 - t^3\right)\Big|_0^1 = 0$

Again, $\int_0^{\pi/4} x\,f(x)dx = \int_0^1 \tan^{-1}t(7t^6 - 3t^2)dt$

$= \tan^{-1}t[t^7 - t^3]_0^1 - \int_0^1 (t^7 - t^3)\frac{1}{1+t^2}dt$

$= \int_0^1 t^3(1-t^2)dt = \frac{t^4}{4} - \frac{t^6}{6}\Big|_0^1 = \frac{1}{4} - \frac{1}{6} = \frac{1}{12}$

39. **(b) : (A) — (p), (B) — (q), (C) — (r), (D) — (s)**

$A = \begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}$ is orthogonal

$\therefore\ AA^T = I$

or $\begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}\begin{bmatrix} a & c & b \\ b & a & c \\ c & b & a \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

or $\begin{bmatrix} p & q & q \\ q & p & q \\ q & q & p \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\Rightarrow p = a^2 + b^2 + c^2 = 1$ and $q = ab + bc + ca = 0$

$\Rightarrow (a+b+c)^2 = 1$

$\Rightarrow a + b + c = \pm 1$

$a^3 + b^3 + c^3 - 3abc$

$= (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca) = \pm 1$

$\Rightarrow a^3 + b^3 + c^3 = \pm 1 + 3 = 4$ or 2

40. **(a) : (A)-(s), (B)-(r), (C)-(q), (D)-(p)**

(A) $\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}$

$\Rightarrow\ x = a + \frac{l}{m}(y-b) = Ay + B,$

$z = c + \frac{n}{m}(y-b) = Cy + D$

Only 4 constants.

(B) The line is the z-axis. Distance = $\sqrt{1^2+2^2}=\sqrt{5}$

(C) $\begin{vmatrix} 1-3 & -1-k & 1-0 \\ 2 & 3 & 4 \\ 1 & 2 & 1 \end{vmatrix} = 0 \Rightarrow k = \frac{9}{2}$.

(D) $\frac{x-3}{2}=\frac{y-5}{1}=\frac{z-7}{1}=\frac{-2(6+12)}{6}=-6$

$a = 3-12 = -9, b = 5-6 = -1, c = 7-6 = 1, a+b+c = -9.$

41. **(d) :** Given $a + ar^{n+1} = 66$ ... (i)

$ar \cdot ar^n = 128$ ... (ii)

$S_{n+2} = 126$ ... (iii)

From (i) $a(1+r^{n+1}) = 66$

$\Rightarrow a\left(1+\frac{128}{a^2}\right)=66$ (by (ii))

$\Rightarrow a\left(\frac{a^2+128}{a^2}\right)=66$

$\Rightarrow a^2 - 66a + 128 = 0$

$\Rightarrow (a-2)(a-64) = 0 \quad \therefore \quad a = 2, 64$

When $a = 2$, then from (ii) $r^{n+1} = 32 = 2^5$

Now from (iii), we get

$\frac{2(r^{n+2}-1)}{r-1}=126$

$\Rightarrow r^{n+1}\cdot r - 1 = 63(r-1)$

$\Rightarrow 32r - 1 = 63r - 63$

$\Rightarrow 31r = 62 \Rightarrow r = 2$

$\therefore$ $r^{n+1} = 2^5$ gives $n = 4$

$\therefore$ number of terms = $n + 2 = 6$

when $a = 64$, then r becomes fraction and then G.P. becomes decreasing.

42. **(b) :** The G.P. is decreasing when $a = 64$, $r=\frac{1}{2}$

When $a = 64$, then from (ii)

$r^{n+1}=\frac{128}{64\times 64}=\frac{1}{32}=\left(\frac{1}{2}\right)^5$

$r=\left(\frac{1}{2}\right)^{\frac{5}{n+1}}=\left(\frac{1}{2}\right)^{\frac{5}{4+1}}=\frac{1}{2}$

$\therefore$ Sum $=\frac{64\left(1-\frac{1}{2^6}\right)}{1-\frac{1}{2}}=64\times 2\times\frac{63}{64}=126$

43. **(d) :** The smallest term is 2 and the greatest term is $2\cdot 2^5 = 64$

$\therefore$ The required difference is 62.

44. (b) **45. (c)**

(44 – 45): $f(x)=\sin x+\sin x\int_{-\pi/2}^{\pi/2} f(t)dt+\cos x\int_{-\pi/2}^{\pi/2} t\ f(t)\,dt$

$f(x) = A \sin x + B \cos x$

$\therefore \quad A=1+\int_{-\pi/2}^{\pi/2} f(t)dt$

$=1+\int_{-\pi/2}^{\pi/2}(A\sin t+B\cos t)dt=1+2B\int_{0}^{\pi/2}\cos t\,dt$

$\Rightarrow A = 1 + 2B$ .....(i)

$B=\int_{-\pi/2}^{\pi/2} t.\,f(t)dt=\int_{-\pi/2}^{\pi/2} t(A\sin t+B\cos t)dt$

$=2A\int_0^{\pi/2} t\sin t\,dt$

$\Rightarrow B = 2A$ ......(ii)

Solving (i) & (ii), we get $A = -1/3, B = -2/3$

$\therefore \quad f(x)=-\frac{1}{3}(\sin x+2\cos x)$

Range of $f(x)$ is $\left[\frac{-\sqrt{5}}{3},\frac{\sqrt{5}}{3}\right]$

$\int_0^{\pi/2} f(x)dx=\frac{-1}{3}\int_0^{\pi/2}(\sin x+2\cos x)dx=-1$

46. **(3) :** $3^{100} = 9^{50} = (10-1)^{50} = (1-10)^{50}$

$=1-50\cdot 10+\frac{50\cdot 49}{2}\cdot 10^2-\frac{50\cdot 49\cdot 48}{6}\cdot 10^3$ + multiple of 10^4.

$= 1 - 500 + 122500$ + multiple of 10^4

$= 12001$ + multiple of 10^4.

47. **(1) :** $(a+c)^2 - b^2 < 0$

$\Rightarrow (a-b+c)(a+b+c) < 0$

$\Rightarrow f(-1)\,f(1) < 0$

$\Rightarrow f(x) = 0$ has exactly one solution in $(-1, 1)$

48. **(6) :** $n(M \cap P' \cap C')$

$= n(M) - [n(M\cap P) + n(M\cap C) - n(M\cap C\cap P)]$

$= 100 - 30 - 28 + 18 = 60$

49. **(2) :** RHL = $\lim_{x\to -1^+}\frac{\pi-\cos^{-1}(x)}{\sqrt{x+1}}\cdot\frac{1}{\sqrt{\pi}+\sqrt{\cos^{-1}x}}$

$=\lim_{x\to -1^+}\frac{\cos^{-1}(-x)}{\sqrt{x+1}}\cdot\frac{1}{\sqrt{\pi}+\sqrt{\cos^{-1}x}}$

put $\cos^{-1}(-x) = t$

$=\sqrt{2}\cdot\frac{1}{\sqrt{\pi}+\sqrt{\pi}}=\frac{1}{\sqrt{2\pi}}$

$\therefore \lambda = 2$

50. **(4) :** $\int_0^{1-m}(x-x^2-mx)dx=\pm\frac{9}{2}$

$\Rightarrow \frac{(1-m)^3}{6}=\pm\frac{9}{2}\Rightarrow m=4,-2$

JEE MATHEMATICS

DPS-25

DAILY PRACTICE SHEET

Model Test Paper

SOLUTIONS

1. **(b)** : Let $\tan^{-1} x = \theta, \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$3\theta + \cos^{-1}(\cos 3\theta) = 0$

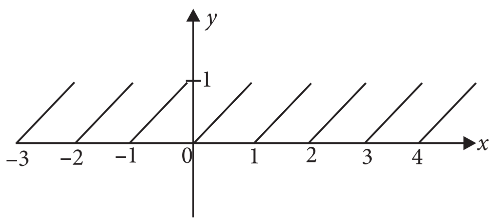
$\cos^{-1}(\cos 3\theta) = -3\theta \Rightarrow -\pi \le 3\theta \le 0$

$\Rightarrow -\frac{\pi}{3} \le \theta \le 0$

$\Rightarrow x \in \left[-\sqrt{3}, 0\right]$

So, number of integral solutions is 2.

2. **(b)** : The graph of $y = x - [x]$ is as shown below.



When x is an integer, $x - [x] = 0$

[$\because$ $x \to [x]$ as x tends to an integer.]

Hence, $f(x) = 0$ when x is an integer

As $x \to 1, \frac{x}{1+x} \to \frac{1}{2}$

Hence, the range of $f(x)$ is $\left[0, \frac{1}{2}\right)$.

3. **(b)** : Equation of family of lines through the intersection of the given lines be

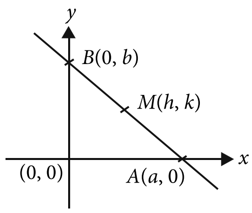
$(x + 2y - 1) + \lambda(2x - y - 1) = 0$

$\Rightarrow x(1 + 2\lambda) + y(2 - \lambda) = 1 + \lambda$

$\Rightarrow \frac{x}{\frac{1+\lambda}{1+2\lambda}} + \frac{y}{\frac{1+\lambda}{2-\lambda}} = 1$

$\Rightarrow \frac{x}{a} + \frac{y}{b} = 1$

where $a = \frac{1+\lambda}{1+2\lambda}, b = \frac{1+\lambda}{2-\lambda}$



Now, $M(h, k)$ be mid point of AB

$\therefore$ $a = 2h$ and $b = 2k$

$\therefore$ $2h = \frac{1+\lambda}{1+2\lambda}$ and $2k = \frac{1+\lambda}{2-\lambda}$ ...(i)

By eliminating λ from (i), we get

$10hk = h + 3k$

Replacing $h \to x$ and $k \to y$, we have $x + 3y = 10xy$ be the required equation of locus.

4. **(b)** : Number of numbers formed $= \frac{6 \times 6!}{2! \times 3!} = 360$

5. **(c)** : Here $f(x) = \log_4(\log_5(\log_3(18x - x^2 - 77)))$

The given function is defined, when

$\log_5(\log_3(18x - x^2 - 77)) > 0$ [$\because$ $\log x$ is defined $\forall\ x > 0$]

$\Rightarrow \log_3(18x - x^2 - 77) > 5^0$

$\Rightarrow 18x - x^2 - 77 > 3^1 \Rightarrow x^2 - 18x + 80 < 0$

$\Rightarrow (x - 8)(x - 10) < 0 \Rightarrow 8 < x < 10$

$\therefore x \in (8, 10)$

6. **(c)** : Equation of tangent is $y = mx + \frac{a}{m}$

$\Rightarrow m^2x - my + a = 0 \Rightarrow m_1 + m_2 = \frac{y}{x}, m_1m_2 = \frac{a}{x}$

$\tan 45^\circ = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right| \Rightarrow \left(\frac{y}{x}\right)^2 - 4\left(\frac{a}{x}\right) = \left(1 + \frac{a}{x}\right)^2$

$\Rightarrow x^2 - y^2 + 6ax + a^2 = 0$

7. **(a)** : Centres and radii of the given circles are $C_1 = (1, 3)$, $r_1 = r$ and $C_2 = (4, -1)$, $r_2 = 3$ respectively, since circles intersect in two distinct points, then

$|r_1 - r_2| < C_1C_2 < r_1 + r_2$

$\Rightarrow |r - 3| < 5 < r + 3$ ...(i)

From last two relations, $r > 2$

From first two relations, $|r - 3| < 5$

$\Rightarrow -5 < r - 3 < 5 \Rightarrow -2 < r < 8$ ...(ii)

From equations (i) and (ii), we get $2 < r < 8$

8. **(c)** : $x^2 + 6xy + 9y^2 + 4x + 12y - 5 = 0$ ....(i)

Consider the homogeneous part equal to zero to find the lines

$\therefore$ $x^2 + 6xy + 9y^2 = 0 \Rightarrow (x + 3y)^2 = 0$

$\therefore$ Required line be $x + 3y + c_1 = 0$, $x + 3y + c_2 = 0$

$\therefore$ $(x + 3y + c_1)(x + 3y + c_2)$

$= x^2 + 6xy + 9y^2 + 4x + 12y - 5 = 0$

$\Rightarrow (x + 3y)(x + 3y) + c_2(x + 3y) + c_1(x + 3y) + c_1c_2$

$= x^2 + 6xy + 9y^2 + 4x + 12y - 5$

$\Rightarrow x(c_2 + c_1) + y(3c_1 + 3c_2) + c_1c_2 = 4x + 12y - 5$

$\Rightarrow c_1 + c_2 = 4, c_1c_2 = -5$

$\Rightarrow$ $c_1 = -1, c_2 = 5$ or $c_1 = 5, c_2 = -1$

$\therefore$ Required lines are $x + 3y - 1 = 0$ & $x + 3y + 5 = 0$

$\therefore$ Lines $x + 3y - 1 = 0$ & $x + 3y + 5 = 0$ are parallel

$\therefore$ Distance between them $= \frac{|c_1 - c_2|}{\sqrt{a^2 + b^2}} = \frac{6}{\sqrt{10}}$

Hence, Statement-1 is true, Statement-2 is true, Statement-2 is a correct explanation for Statement-1.

9. **(d)** : $\log_6(abc) = 6 \Rightarrow (abc) = 6^6$

Let $a = b/r$ and $c = br$

$\Rightarrow b = 36$ and $a = \frac{36}{r} \Rightarrow r = 2, 3, 4, 6, 9, 12, 18$

Also $b - a = 36\left(1 - \frac{1}{r}\right)$ is a perfect cube.

$\therefore r = 4$

$\Rightarrow a + b + c = 36 + 9 + 144 = 189$

10. **(d) :** Let A and B denote the set of employees who like bananas and apples respectively. Further, let the total number of employees be 100.

$\therefore\ n(A) = 70, n(B) = 64, n(U) = 100$ and $n(A \cap B) = x$

Now $n(A \cup B) \le 100$

$\therefore\ n(A) + n(B) - n(A \cap B) \le 100$

$\Rightarrow 70 + 64 - x \le 100$

$\Rightarrow x \ge 34$ ...(i)

Again $A \cap B \subseteq B$. $\therefore$ $n(A \cap B) \le n(B)$

$\Rightarrow x \le 64$ ...(ii)

From (i) and (ii), we get

$34 \le x \le 64$

11. **(c) :** $L = \lim\limits_{x\to 0}(1^{\text{cosec}^2 x} + 2^{\text{cosec}^2 x} + ... + n^{\text{cosec}^2 x})^{\sin^2 x}$

$$= \lim_{x\to 0}\left(\left(\frac{1}{n}\right)^{\text{cosec}^2 x} + \left(\frac{2}{n}\right)^{\text{cosec}^2 x} + ... + \left(\frac{n-1}{n}\right)^{\text{cosec}^2 x} + 1\right)^{\sin^2 x} \cdot n$$

$= (0 + 0 + 0 + ... + 1)^0 \cdot n = n$

12. **(c) :** *Reflexive property :* For all $(a, b) \in N \times N$, $(a, b)\ R\ (a, b)$ since $a + b = b + a$. Hence, $(a, b)\ R\ (a, b)\ \forall\ (a, b) \in N \times N$

$\therefore$ R is reflexive.

Symmetric property :

If $(a, b)\ R\ (c, d)$ then $a + d = b + c$.

Now, when $a + d = b + c$ then $c + b = d + a$.

Hence, $(c, d)\ R\ (a, b)$.

Hence, if $(a, b)\ R\ (c, d)$ then $(c, d)\ R\ (a, b)\ \forall$ all $(a, b), (c, d) \in N \times N$.

$\therefore$ R is symmetric.

Transitive property:

Let $(a, b)\ R\ (c, d)$ and $(c, d)\ R\ (e, f)$.

Then $a + d = b + c$ and $c + f = d + e$

$\therefore\ a + d + c + f = b + c + d + e$

i.e. $a + f = b + e$. Hence $(a, b)\ R\ (e, f)$

Hence if $(a, b)\ R\ (c, d)$ and $(c, d)\ R\ (e, f)$ then

$(a, b)\ R\ (e, f)\ \forall\ (a, b), (c, d), (e, f) \in N \times N$

$\therefore$ R is transitive.

Since R is reflexive, symmetric and transitive, hence R is an equivalence relation on $N \times N$.

13. **(b) :** $\dfrac{3 + \dfrac{\cos 80° \cos 20°}{\sin 80° \sin 20°}}{\dfrac{\cos 80°}{\sin 80°} + \dfrac{\cos 20°}{\sin 20°}}$

$$= \frac{2\sin 80° \sin 20° + \cos 80° \cos 20° + \sin 80° \sin 20°}{\sin 20° \cos 80° + \cos 20° \sin 80°}$$

$$= \frac{\cos 60° - \cos 100° + \cos 60°}{\sin 100°} = \frac{1 - \cos 100°}{\sin 100°} = \tan 50°$$

14. **(c) :** Let us define the events first.

E_1 = the examinee guesses the answer

E_2 = the examinee copies the answer

E_3 = the examinee knows the answer

and A = the examinee has answered the question correctly.

$\therefore\ P(E_1) = \frac{1}{3},\ P(E_2) = \frac{1}{6}$ and $P(E_3) = 1 - \frac{1}{3} - \frac{1}{6} = \frac{1}{2}$

[$\because$ E_1, E_2, E_3 are mutually exclusive and exhaustive events, therefore, $P(E_1) + P(E_2) + P(E_3) = 1$]

Now, $P(A\,|\,E_1) = \frac{1}{4},\ P(A\,|\,E_2) = \frac{1}{8},\ P(A\,|\,E_3) = 1$

Now,

$$P(E_3\,|\,A) = \frac{P(E_3)\,P(A\,|\,E_3)}{P(E_1)P(A\,|\,E_1) + P(E_2)P(A\,|\,E_2) + P(E_3)P(A\,|\,E_3)}$$

$$= \frac{\frac{1}{2}\cdot 1}{\frac{1}{3}\cdot\frac{1}{4} + \frac{1}{6}\cdot\frac{1}{8} + \frac{1}{2}\cdot 1} = \frac{24}{29}$$

15. **(b) :** Let E, W, N, S stand for one unit movement along +ve, –ve x-direction, +ve, –ve y-direction respectively. The sequence of 6 steps are

ENNNSS with $\dfrac{6!}{3!2!} = 60$ ways

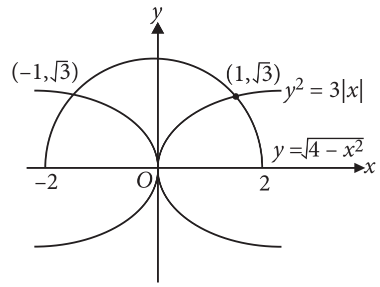
EEWNNS with $\dfrac{6!}{2!2!} = 180$ ways

EEEWWN with $\dfrac{6!}{3!2!} = 60$ ways

The desired number of ways = 60 + 180 + 60 = 300

16. **(c) :** Required area $= 2\displaystyle\int_0^1\left(\sqrt{4 - x^2} - \sqrt{3x}\right)dx$

$$= 2\left(\frac{x}{2}\sqrt{4 - x^2} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right) - \frac{\sqrt{3}\cdot 2x^{3/2}}{3}\right)_0^1 = \frac{2\pi - \sqrt{3}}{3}$$



17. **(a) :** $p(x^2 - x) + x + 5 = 0 \Rightarrow px^2 - (p - 1)x + 5 = 0$

$$\therefore\quad \alpha + \beta = \frac{p-1}{p} \text{ and } \alpha\beta = \frac{5}{p}$$

Now, $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha} = \dfrac{4}{5} \Rightarrow \dfrac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta} = \dfrac{4}{5}$

$$\Rightarrow \frac{(p-1)^2 - 10p}{5p} = \frac{4}{5} \Rightarrow p^2 - 16p + 1 = 0$$

$\therefore\ p_1 + p_2 = 16,\ p_1p_2 = 1$

Now, $\dfrac{p_1}{p_2} + \dfrac{p_2}{p_1} = \dfrac{(p_1 + p_2)^2 - 2p_1p_2}{p_1p_2}$

$\therefore$ Required value is 254.

18. **(b) :** $\tan^{-1}\left(\dfrac{\dfrac{a}{b+c} + \dfrac{b}{c+a}}{1 - \dfrac{a}{b+c}\cdot\dfrac{b}{c+a}}\right)$ as $\dfrac{ab}{(b+c)(c+a)} < 1$

But in right angled ΔABC $c^2 = a^2 + b^2$ $\therefore$ $\tan^{-1}(1) = \dfrac{\pi}{4}$

19. **(c) :** $B = \displaystyle\int_1^{\text{cosec}\theta}\frac{1}{t(1+t^2)}\,dt$

Let $\frac{1}{t}=u \Rightarrow B=\int_{1}^{\sin\theta}\frac{-u\,du}{1+u^2} \Rightarrow A+B=0 \ \Rightarrow A=-B$

$$\therefore \begin{vmatrix} A & A^2 & -A \\ e^0 & A^2 & -1 \\ 1 & 2A^2 & -1 \end{vmatrix}=0$$

20. **(b) :** We have to find the number of integral solutions if $x_1+x_2+x_3+x_4+x_5=6$ and that equals ${}^{5+6-1}C_{5-1}={}^{10}C_4$

Thus, Statement-1 is false.

Number of different ways of arranging 6A's and 4B's in a row $=\frac{\lfloor 10}{\lfloor 6\times\lfloor 4}={}^{10}C_4=$ Number of different ways the child can buy the six ice-creams.

$\therefore$ Statement-2 is true.

21. **(b) :** Let X = {selected coin is fair}

H = {head turns up}

Thus $P(H)=P(X)\,P(H/X)+P(X')\,P(H/X')$

$$=\frac{n}{n+1}\times\frac{1}{2}+\frac{1}{n+1}\times 1=\frac{n+2}{2(n+1)}$$

$$P(X/H)=\frac{P(X)P(H/X)}{P(H)}=\frac{\frac{n}{2(n+1)}}{\frac{n+2}{2(n+1)}}=\frac{n}{n+2}$$

22. **(b) :** Let $y=f(x)=(5-(x-8)^5)^{1/3}$, then

$y^3=5-(x-8)^5 \ \Rightarrow \ (x-8)^5=5-y^3$

$\Rightarrow \ x=8+(5-y^3)^{1/5}$

Let, $z=g(x)=8+(5-x^3)^{1/5}$

Now, $f(g(x))=(5-[(5-x^3)^{1/5}]^5)^{1/3}=(5-5+x^3)^{1/3}=x$

Similarly, we can show that $g(f(x))=x$

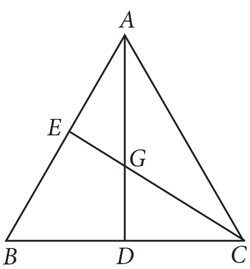
Hence, $g(x)=8+(5-x^3)^{1/5}$ is the inverse of $f(x)$.

23. **(a) :** As f is a positive increasing function, we have $f(x)<f(2x)<f(3x)$

Dividing by $f(x)$ leads to $1<\frac{f(2x)}{f(x)}<\frac{f(3x)}{f(x)}$

As $\lim_{x\to\infty}\frac{f(3x)}{f(x)}=1$, we have by Squeez theorem or Sandwich theorem, $\lim_{x\to\infty}\frac{f(2x)}{f(x)}=1.$

24. **(b) :** $AG=\frac{2}{3}AD=\frac{10}{3}$



$$\frac{GC}{\sin\frac{\pi}{8}}=\frac{AG}{\sin\frac{\pi}{4}} \ \Rightarrow \ GC=\frac{10}{3}\times\frac{\sin\frac{\pi}{8}}{\sin\frac{\pi}{4}}$$

$\therefore$ Area of ΔABC = 3 Area of ΔAGC

$$3\left(\frac{1}{2}\times\frac{10}{3}\times\left(\frac{10}{3}\times\frac{\sin\frac{\pi}{8}}{\sin\frac{\pi}{4}}\right)\right)\times\sin\left(\frac{\pi}{2}+\frac{\pi}{8}\right)=\frac{25}{3}$$

25. **(b) :** Let $I=\int_0^{\pi}x\sin^4 x\,dx$ ...(i)

$\Rightarrow I=\int_0^{\pi}(\pi-x)\sin^4 x\,dx$ ...(ii)

Adding (i) and (ii), we get

$2I=\pi\int_0^{\pi}\sin^4 x\,dx=2\pi\int_0^{\pi/2}\sin^4 x\,dx$

$$=2\pi\int_0^{\pi/2}\left(\frac{3}{8}+\frac{\cos 4x}{8}-\frac{\cos 2x}{2}\right)dx=2\pi\left[\frac{3}{8}x+\frac{\sin 4x}{32}-\frac{\sin 2x}{4}\right]_0^{\pi/2}$$

$$=2\pi\left[\frac{3}{8}\left(\frac{\pi}{2}\right)\right]=\frac{3\pi^2}{8}\Rightarrow I=\frac{3\pi^2}{16}$$

26. **(c) :** Let $f(x)=x^7+14x^5+16x^3+30x-560$

$\therefore f'(x)=7x^6+70x^4+48x^2+30\Rightarrow f'(x)>0\ \forall\ x\in R$

i.e., $f(x)$ is an strictly increasing function.

So, it can have at the most one solution. It can be shown that it has exactly one solution.

27. **(c) :** The perpendicular distance of the origin (0, 0, 0) from the plane $x+y+z=p$ is

$$\left|\frac{-p}{\sqrt{1+1+1}}\right|=\frac{|p|}{\sqrt{3}}$$

If the coordinates of P are (l, m, n), then we must have

$$\left|\frac{l+m+n-p}{\sqrt{3}}\right|=\frac{|p|}{\sqrt{3}} \ \Rightarrow \ |l+m+n-p|=|p|$$

which is satisfied by (c)

28. **(b) :** $f(x)=\begin{cases}\frac{[x^2]-1}{x^2-1}, & \text{for } x^2\neq 1\\ 0, & \text{for } x^2=1\end{cases}$

$$=\begin{cases}\frac{-1}{x^2-1}, & \text{for } 0<x^2<1\\ 0, & \text{for } x^2=1\\ 0, & \text{for } 1<x^2<2\end{cases}$$

$\therefore$ R.H.L at $x=1$ is 0

Also L.H.L. at $x=1$ is $-\infty$

29. **(a) :** Selected year may be a non leap year with a probability 3/4.

Selected year may leap year with a probability 1/4.

Let E be an event that randomly selected year contains 53 Mondays

$$P(E)=\frac{3}{4}\times\frac{1}{7}+\frac{1}{4}\times\frac{2}{7}=\frac{5}{28}\ \Rightarrow P\left(\frac{\text{leap year}}{E}\right)=\frac{\frac{2}{28}}{\frac{5}{28}}=\frac{2}{5}$$

30. **(b) :** The letters other than vowels are : PRMTTN

Number of permutations with no two vowels together is $\frac{6!}{2!}\times{}^7C_5\times 5!$

Further among these permutations, the number of cases in which T's are together is $5!\times{}^6C_5\times 5!$

So, the required number

$$=\frac{6!}{2!}\,{}^7C_5\times 5!-5!\times{}^6C_5\times 5!=57\times(5!)^2$$

31. **(c, d) :** $(1+x)^{11}={}^{11}C_0+{}^{11}C_1x+{}^{11}C_2x^2+\ldots+{}^{11}C_{11}x^{11}$ ... (i)

$(1+x)^9={}^9C_0+{}^9C_1x+{}^9C_2x^2+\ldots+{}^9C_9x^9$ ... (ii)

Multiply (i) & (ii) and compare coefficient of x^{11} on both sides and put $x=1$

${}^{20}C_{11}={}^{11}C_{11}{}^9C_0+{}^{11}C_{10}{}^9C_1+\ldots+{}^{11}C_2{}^9C_9$

$\therefore\ {}^{20}C_9-1={}^{11}C_{10}{}^9C_1+\ldots+{}^{11}C_2{}^9C_9$

32. (b, c) : As (0, 0) and $(x, g(x))$ are two vertices of an equilateral triangle, therefore length of the side of Δ is

$$= \sqrt{(x-0)^2 + (g(x)-0)^2} = \sqrt{x^2 + (g(x))^2}$$

$\therefore$ The area of equilateral $\Delta = \frac{\sqrt{3}}{4}(x^2 + (g(x))^2)$

Also area $= \frac{\sqrt{3}}{4}$ (Given)

$\therefore$ We get, $\frac{\sqrt{3}}{4}(x^2 + (g(x))^2) = \frac{\sqrt{3}}{4}$

$\Rightarrow (g(x))^2 = 1 - x^2 \Rightarrow g(x) = \pm\sqrt{1-x^2}$

$\therefore$ (b), (c) are the correct answers as (a) is not a function ($\because$ image of x is not unique).

33. (b, d) : Let the 1st and $(2n-1)$th terms be p and q.

Then middle term of the A.P. $= \frac{p+q}{2} = a$

Middle term of the G.P. $= \sqrt{pq} = b$

Middle term of the H.P. $= \frac{2pq}{p+q} = c$

Now $ac = pq = b^2 \Rightarrow ac - b^2 = 0$

Also, A.M. ≥ G.M. ≥ H.M. $\Rightarrow a \geq b \geq c$

Here $a = b = c$ is true if all the terms are equal.

34. (a, b, c, d) : $I_n + I_{n+2} = \int_0^{\pi/4} \tan^n x \cdot \sec^2 x\, dx$

$$= \int_0^1 t^n dt = \frac{1}{n+1}$$

$$I_{12} + I_{10} + I_{10} + I_8 = \frac{1}{11} + \frac{1}{9} = \frac{20}{99}$$

$$I_8 + I_{10} - (I_{10} + I_{12}) = \frac{1}{9} - \frac{1}{11} = \frac{2}{99}$$

35. (b, c) : The given equation is $x^{\frac{3}{4}(\log_2 x)^2 + \log_2 x - \frac{5}{4}} = \sqrt{2}$

Put $\log_2 x = t \Rightarrow x = 2^t$

$\therefore x^{\frac{3}{4}t^2 + t - \frac{5}{4}} = \sqrt{2}$

Taking logarithm to the base x, $\frac{3}{4}t^2 + t - \frac{5}{4} = \frac{1}{2t}$

$\Rightarrow 3t^3 + 4t^2 - 5t - 2 = 0$

$\Rightarrow (t-1)(3t^2 + 7t + 2) = 0 \Rightarrow (t-1)(t+2)(3t+1) = 0$

$\therefore t = \log_2 x = 1, -2, -\frac{1}{3}$

$\therefore x = 2, 2^{-2}, 2^{-1/3}$ or $x = 2, \frac{1}{4}, \frac{1}{2^{1/3}}$

36. (b, c, d) : $P(A) = 1/4$, $P(B \mid A) = 5/7$,

$P(B \mid \bar{A}) = 6/7$, $P(B) = P(A)P(B \mid A) + P(\bar{A})P(B \mid \bar{A})$

$= (1/4 \times 5/7) + (3/4 \times 6/7) = 23/28$

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{4} \times \frac{5}{7}}{\frac{1}{4} \times \frac{5}{7} + \frac{3}{4} \times \frac{6}{7}} = \frac{5}{23}$$

$$P(A \mid \bar{B}) = \frac{P(A \cap \bar{B})}{P(\bar{B})} = \frac{\frac{1}{4} \times \frac{2}{7}}{1 - \frac{23}{28}} = \frac{2}{5}$$

37. (a, b, d) : $A^2 - 4A - 5I_3$

$$= \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - 4\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} + \begin{bmatrix} -4 & -8 & -8 \\ -8 & -4 & -8 \\ -8 & -8 & -4 \end{bmatrix} + \begin{bmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

$\therefore A^2 - 4A - 5I_3 = O \Rightarrow A^{-1}A^2 - 4A^{-1}A - 5A^{-1}I_3 = O$

$\Rightarrow (A^{-1}A)A - 4I_3 - 5A^{-1} = O \Rightarrow IA - 4I_3 - 5A^{-1} = O$

$\therefore A^{-1} = \frac{1}{5}(A - 4I_3)$ Also, $|A^2| = \begin{vmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{vmatrix}$

$= 9(81 - 64) - 8(72 - 64) + 8(64 - 72)$

$= (9 \times 17) - (8 \times 8) + 8 \times (-8)$

$= 25 \neq 0$

$\therefore A^2$ is invertible.

and $A^3 = A \cdot A^2 = A(4A + 5I_3) = 4A^2 + 5A$

$$= \begin{bmatrix} 36 & 32 & 32 \\ 32 & 36 & 32 \\ 32 & 32 & 36 \end{bmatrix} + \begin{bmatrix} 5 & 10 & 10 \\ 10 & 5 & 10 \\ 10 & 10 & 5 \end{bmatrix} = \begin{bmatrix} 41 & 42 & 42 \\ 42 & 41 & 42 \\ 42 & 42 & 41 \end{bmatrix}$$

$\therefore |A^3| \neq 0 \therefore A^3$ is invertible.

38. (a, d) : If (x, y) is any point on the curve, the subtangent at $(x, y) = y\frac{dx}{dy}$

$\therefore y\frac{dx}{dy} = nx$ (given) or $n\frac{dy}{y} = \frac{dx}{x}$

Integrating, $n\log y = \log x + \log c$ or $\log y^n = \log cx$

or $y^n = cx$ .....(i), which is the required equation of the family of curves.

Putting $x = 2$, $y = 3$ in (i), we have $3^n = 2c$ or $c = \frac{3^n}{2}$

Putting this value of c in (i), we get

$y^n = \frac{3^n}{2}x \Rightarrow 2y^n = 3^n x$ ...(ii)

which is the particular curve passing through the point (2, 3).

Putting $n = 1$ in (ii), we have $2y = 3x$, which is a straight line.

Putting $n = 2$ in (ii), we have $2y^2 = 9x$

which is a parabola.

39. (b) : (A)-(s), (B)-(r), (C)-(q), (D)-(q)

(A) $|1 - i|^n = 2^n \Rightarrow n/2 = 0 \Rightarrow n = 0$

(B) $x^3 + 2x^2 + 2x + 1 = 0 \Rightarrow x = -1, \omega, \omega^2$

But $x = \omega, \omega^2$ will only satisfy $x^{2000} + x^{2002} + 1 = 0$

(C) $x + 2xy = 0$ and $x^2 - y^2 + y = 0$

$\Rightarrow i, \frac{\sqrt{3}}{2} - \frac{i}{2}, -\frac{\sqrt{3}}{2} - \frac{i}{2}$

(D) $x^2 - y^2 + \sqrt{x^2 + y^2} = 0$ and $2xy = 0 \Rightarrow z = 0, i, -i$

40. (a) : (A)-(r), (B)-(s), (C)-(q), (D)-(p)

(A) By L' Hospital Rule

$$\lim_{x \to 3} \frac{(x^3 + 27)\frac{1}{x-2} + \log(x-2).3x^2}{2x} = \frac{54}{6} = 9$$

(B) Let $L = \lim_{x\to 0}\left(\frac{e^x-1}{x}\right)^{\left(\frac{x}{x+1-e^x}\right)} = \lim_{x\to 0}\left(\frac{e^x-1}{x}\right)^{\left(\frac{1}{1-\frac{e^x-1}{x}}\right)}$

Put $\frac{e^x-1}{x} = t$, As $x \to 0, t \to 1$

$\therefore \quad L = \lim_{t\to 1} t^{\frac{1}{1-t}} \quad [1^\infty \text{ form}]$

Taking logarithm, we get

$\log L = \lim_{t\to 1}\frac{1}{1-t}\log t \quad \left(\frac{0}{0} \text{ form}\right)$

$= \lim_{t\to 1}\frac{\frac{1}{t}}{-1} = -1$ (by L' Hospital rule)

$\Rightarrow L = e^{-1}$

(C) We know that $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ....$

Let $L = \lim_{x\to 0}\frac{ax + x\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-....\right) - b\left(x - \frac{x^3}{3!}+....\right)}{x^3}$

$= \lim_{x\to 0}(a-b+1)\times\frac{1}{x^2} + \left(\frac{b}{3!}-\frac{1}{2!}\right) +$ terms containing x

As $L = 1$, we must have $a - b + 1 = 0$ and $\frac{b}{3!} - \frac{1}{2!} = 1$

$\Rightarrow b = 9$ and $a = 8$

(D) Let $L = \lim_{x\to 0}\frac{-f(x)+3f(2x)-3f(3x)+f(4x)}{x^3}\left(\frac{0}{0}\text{ form}\right)$

$= \lim_{x\to 0}\frac{-f'(x)+6f'(2x)-9f'(3x)+4f'(4x)}{3x^2}\left(\frac{0}{0}\text{ form}\right)$

$= \lim_{x\to 0}\frac{-f''(x)+12f''(2x)-27f''(3x)+16f''(4x)}{6x}\left(\frac{0}{0}\text{ form}\right)$

$= \lim_{x\to 0}\frac{-f'''(x)+24f'''(2x)-81f'''(3x)+64f'''(4x)}{6}$

$= \frac{6.f'''(0)}{6} = f'''(0)$

But $L = 12 \Rightarrow f'''(0) = 12$

41. (d) **42. (a)**

(41-42) : We have $f(x) = 2^x + 2^{|x|}$

For $x \ge 0$, $f(x) = 2\cdot 2^x$

And for $x < 0$; $f(x) = 2^x + 2^{-x}$ and in this case

$f'(x) = 2^x\ln 2 - 2^{-x}\ln 2 = \ln 2\left(\frac{2^{2x}-1}{2^x}\right)$

$= \frac{\ln 2}{2^x}.(2^x+1)(2^x-1) < 0$

So, $f(x)$ is decreasing in $x \in (-\infty, 0)$ and increasing in $x \in (0, \infty)$.

So, $f(x)$ is many–one.

Also, Range of $f(x)$ is $[2, \infty)$

$\therefore$ f is many–one into.

Since $\ln \pi < 2$

So, the equation $2^x + 2^{|x|} = \ln \pi$ has no solution.

43. (c) : Let $I = \int\frac{dx}{(1+x^2)(\sqrt{1-x^2})} = \int\frac{dx}{x^3\left(1+\frac{1}{x^2}\right)\sqrt{\frac{1}{x^2}-1}}$

Put $\frac{1}{x^2} - 1 = t^2 \Rightarrow -\frac{2}{x^3}dx = 2t\,dt$

$\therefore I = -\int\frac{tdt}{(t^2+2)t} = -\frac{1}{\sqrt{2}}\tan^{-1}\frac{t}{\sqrt{2}} + c = -\frac{1}{\sqrt{2}}\tan^{-1}\frac{\sqrt{1-x^2}}{\sqrt{2}x} + c$

$= -\frac{1}{\sqrt{2}}\sin^{-1}\sqrt{\frac{1-x^2}{1+x^2}} + c \therefore A = -\frac{1}{\sqrt{2}}$

44. (b) : $f(x) = \frac{1-x^2}{1+x^2}$

45. (a) : $\int f(x)dx = \int\frac{1-x^2}{1+x^2}dx = \int\left(\frac{2}{1+x^2}-1\right)dx = 2\tan^{-1}x - x + c$

46. (3) : From the given equation, we have

$\left(\frac{n}{2}-\left[\frac{n}{2}\right]\right) + \left(\frac{n}{3}-\left[\frac{n}{3}\right]\right) + \left(\frac{n}{5}-\left[\frac{n}{5}\right]\right) = 0$

$\Rightarrow \left\{\frac{n}{2}\right\} + \left\{\frac{n}{3}\right\} + \left\{\frac{n}{5}\right\} = 0$; where $\{.\}$ is a fractional part

But each of the fraction part function is positive and their sum is zero. Hence, each of the fraction part function is zero. Consequently, each of $\frac{n}{2}, \frac{n}{3}, \frac{n}{5}$ is an integer. The L.C.M. of 2, 3, 5 is 30.

Therefore, we can take $n = 30k$, where k is an integer.

Hence, the number of solutions such that $1 \le n \le 100$ is 3 (n = 30, 60 and 90).

47. (2) : Put $x = \frac{1}{X}$ and $y = \frac{1}{Y}$

$dx = -\frac{1}{X^2}dX$ and $dy = -\frac{1}{Y^2}dY$

$\Rightarrow \quad p = \frac{dy}{dx} = \frac{X^2}{Y^2}\frac{dY}{dX} = \frac{X^2}{Y^2}P$

The given equation becomes

$\frac{1}{Y^2}\left(\frac{1}{Y} - \frac{1}{X}\cdot\frac{X^2}{Y^2}P\right) = \frac{1}{X^4}\cdot\frac{X^4}{Y^4}P^2$

$\Rightarrow Y - XP = P^2$ or $Y = PX + P^2$ which is the Clairaut's form

$\therefore$ The solution is $Y = cX + c^2$

or $\frac{1}{y} = \frac{c}{x} + c^2$

48. (6) : Given expression is

$\frac{(a+b)}{c} + \frac{(b+c)}{a} + \frac{(c+a)}{b} = \frac{a}{c} + \frac{b}{c} + \frac{b}{a} + \frac{c}{a} + \frac{c}{b} + \frac{a}{b}$

Using A.M. $\ge$ G.M., we get

$$\frac{\frac{a}{c}+\frac{b}{c}+\frac{b}{a}+\frac{c}{a}+\frac{c}{b}+\frac{a}{b}}{6} \geq \sqrt[6]{\frac{a}{c}\frac{b}{c}\frac{b}{a}\frac{c}{a}\frac{c}{b}\frac{a}{b}}$$

$$\Rightarrow \quad \frac{a+b}{c}+\frac{b+c}{a}+\frac{c+a}{b} \geq 6$$

49. **(3) :** $x^3 + y^3$ is divisible by $3 \Rightarrow x + y$ is divisible by $3 \Rightarrow$ x, y are multiples of 3 or one leaves remainder 1 and the other 2 when divided by 3.

3, 6, 9, ... , 30 are multiples of 3; 1, 4, 7, ..., 28 leave remainder 1; 2, 5, 8, ..., 29 leave remainder 2.

$$\therefore \text{ Required Probability } = \frac{\binom{10}{1}\binom{10}{1}+\binom{10}{2}}{\binom{30}{2}} = \frac{145}{15\times 29} = \frac{1}{3}$$

50. **(1) :** Projection of $\vec{V}$ on $\vec{r}$ is $\frac{\vec{V}\cdot\vec{r}}{|\vec{r}|} = \frac{4}{\sqrt{6}}$

$$\Rightarrow \frac{(\vec{p}\cdot\vec{r})+\lambda(\vec{q}\cdot\vec{r})}{|\vec{r}|} = \frac{4}{\sqrt{6}} \Rightarrow \frac{-1+\lambda(5)}{\sqrt{6}} = \frac{4}{\sqrt{6}} \quad \Rightarrow \lambda = 1$$