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CHAPTERWISE QUESTION BANK

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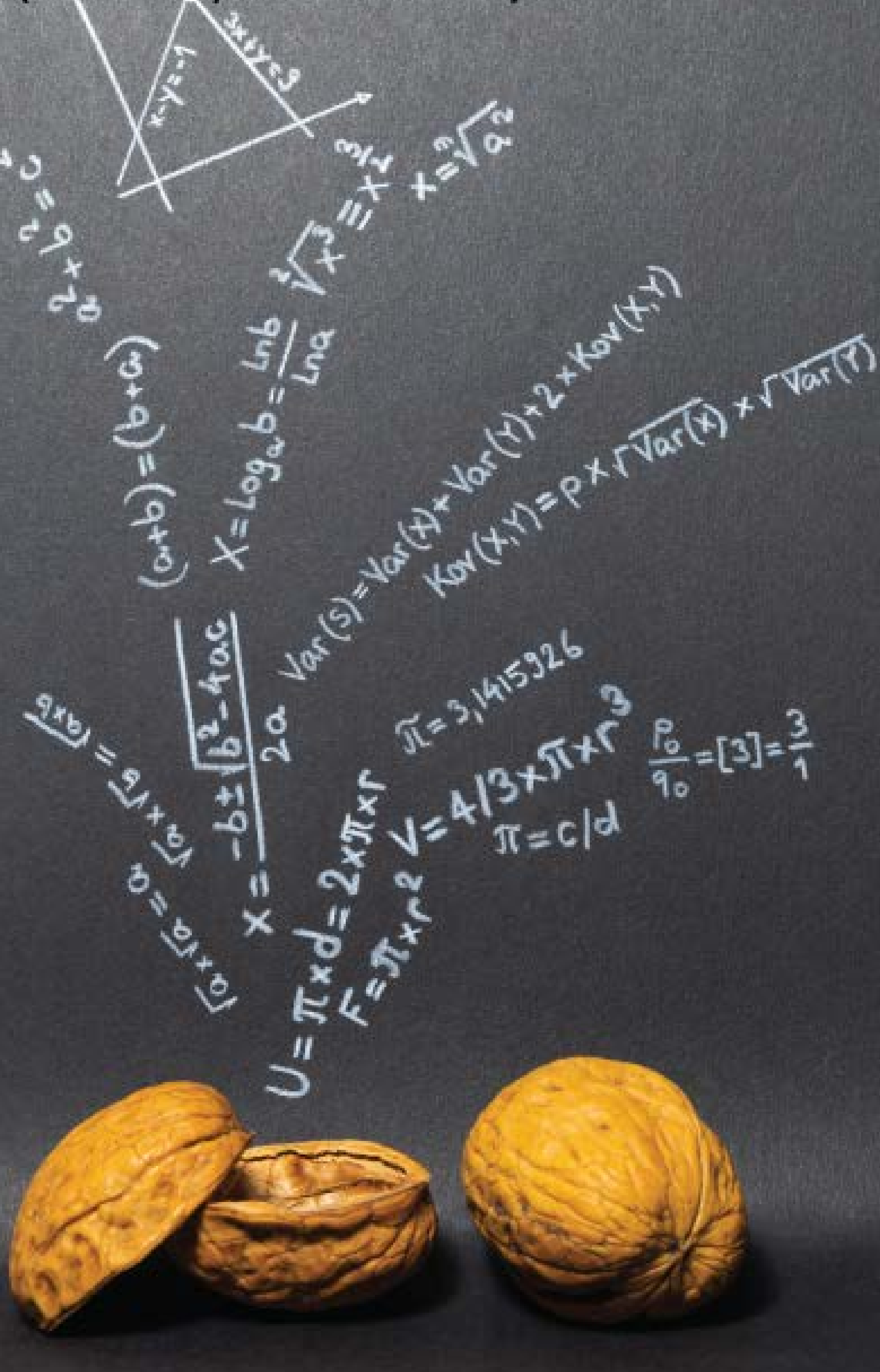

CBSE 2025

MATHEMATICS

Including Case Based Questions

CLASS 12

Chapter-wise Question Bank

Based on Previous 20 Years 63 Papers

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CONTENTS

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CHAPTER 1

RELATION AND FUNCTION

OBJECTIVE QUESTIONS

1. A function $f: R \to R$ defined as $f(x) = x^2 - 4x + 5$ is
   (a) injective but not surjective.
   (b) surjective but not injective.
   (c) both injective and surjective.
   (d) neither injective nor surjective.

   Sol : OD 2024

   We have $f: R \to R$ where $f(x) = x^2 - 4x + 5$

   One-One function :

   Let $x_1, x_2 \in R$, such that

   $$f(x_1) = f(x_2)$$

   $$x_1^2 - 4x_1 + 5 = x_2^2 - 4x_2 + 5$$

   $$x_1^2 - x_2^2 - 4x_1 + 4x_2 = 0$$

   $$(x_1 - x_2)(x_1 + x_2) - 4(x_1 - x_2) = 0$$

   $$(x_1 - x_2)(x_1 + x_2 - 4) = 0$$

   Thus $x_1 + x_2 - 4 = 0$ and $x_1 = x_2$.

   Both are possible for real numbers.

   Hence $f(x)$ is not one-one.

   Onto function :

   Now $y = x^2 - 4x + 5$

   $$= (x-2)^2 + 5 - 4$$

   $$= (x-2)^2 + 1$$

   As $(x-2)^2 \geq 0$

   $$y - 1 \geq 0$$

   $$y \geq 1$$

   Range $(f) \in [1, \infty]$ and co-domain $\in R$

   Since Range $\neq$ Co-domain $f(x)$ is not onto.

2. Select the correct option out of the four given options Let R be a relation in the set N given by $R = \{(a,b): a = b - 2, b > 6\}$

   Then,
   (a) $(8,7) \in R$
   (b) $(6,8) \in R$
   (c) $(3,8) \in R$
   (d) $(8,7) \in R$

   Sol : OD 2023

   We have $R = \{(a,b): a = b - 2, b > 6\}$

   Here $b > 6$. so we substituting use $b = 7$ and 8.

   When $b = 7$, then $a = 7 - 2 = 5$

   When $b = 8$, then $a = 8 - 2 = 6$

   So, $(5,7) \in R$

   and $(6,8) \in R$

   Thus (b) is correct option.

3. Let $A = \{3,5\}$. Then, number of reflexive relations on A is
   (a) 2
   (b) 4
   (c) 0
   (d) 8

   Sol : OD 2023

   Here, $n(A) = 2$

   The number of reflexive relations are $2^{n^2 - n}$, where n is the number of elements in the set.

   Thus number of reflexive relation are

   $$= 2^{2^2 - 2} = 2^{4-2} = 2^2 = 4$$

   Thus (b) is correct option.

4. The relation R defined in the set $A = \{1,2,3,4,5,6,7\}$ by $R = \{(a,b)$: both a and b are either odd or even$\}$. Then, R is
   (a) symmetric
   (b) transitive
   (c) an equivalence relation
   (d) reflexive

   Sol : Delhi 2017

   Given, any element a in A, both a and a must be either odd or even, so that $(a,a) \in R$. Further, $(a,b) \in R \Rightarrow$ both a and b must be either odd or even $\Rightarrow (b,a) \in R$.

Similarly, $(a,b) \in R$ and $(b,c) \in R \Rightarrow$ all elements a,b,c must be either even or odd simultaneously $\Rightarrow$ $(a,c) \in R$ Hence, R is an equivalence relation.

Thus (c) is correct option.

5. Let R be the relation defined in the set $A = \{1,2,3,4,5,6,7\}$ by $R = \{(a,b)$: both a and b are either odd or even$\}$.

Now, consider the following statements

I. All the elements of the subset $\{1,3,5,7\}$ are related to each other.

II All the elements of the subset $\{2,4,6\}$ are related to each other.

III Some elements of the subset $\{1,3,5,7\}$ are related to some elements of the subset $\{2,4,6\}$.

Choose the correct option

(a) I and III are true

(b) I and II are true

(c) II and III are true

(d) All the true

Sol : Foreign 2010

All the elements of the subset $\{1,3,5,7\}$ are related to each other, as all the elements of this subset are odd.

Similarly, all the elements of the subset $\{2,4,6\}$ are related to each other, as all of them are even. Also, no element of the subset $\{1,3,5,7\}$ can be related to any element ot $\{2,4,6\}$ as elements of $\{1,3,5,7\}$ are odd, while elements of $\{2,4,6\}$ are even.

Thus (b) is correct option.

6. Let $f : I \to I$ be defined by $f(x) = x + i$ where i is a fixed integer, then f is

(a) one-one but not onto

(b) onto but not one-one

(c) non-invertible

(d) both one-one and onto

Sol : SQP 2020

Let
$$\begin{aligned} f(x_1) &= f(y_1) \\ x_1 + i &= x_2 + i \\ x_1 &= x_2 \end{aligned}$$

and for any integer y, we have

$$\begin{aligned} y &= x + i \\ x &= y - i \end{aligned}$$

ie,
$$f(y - i) = y$$

Hence, f is both one-one and onto.

Thus (d) is correct option.

7. Let C be the set of complex numbers. The mapping $f \quad C \to R$ given by $f(z) = |z|$, $\forall z \in C$, is

(a) one-one and onto

(b) one-one but not onto

(c) not one-one but onto

(d) neither one-one nor onto

Sol : Delhi 2015, OD 2011

Here,
$$\begin{aligned} f(z) &= |z| \quad \forall z \in C \\ f(1) &= |1| = 1 \\ f(-1) &= |-1| = 1 \\ f(1) &= f(-1) \end{aligned}$$

But
$$1 \neq -1$$

Therefore, it is not one-one.

Now, let $f(z) = |z|$. Here, there is not pre-image of negative numbers. Hence, it is not onto.

Thus (d) is correct option.

8. Let R be the relation in the set Z of all integers defined by $R = \{(x,y): x - y \text{ is an integer}\}$.Then R is

(a) reflexive

(b) symmetric

(c) transitive

(d) an equivalence relation

Sol : Comp 2017

Here, $R = \{(x,y) : x - y$ is an integer$\}$ is a relation in the set of integers.

Reflexive :

Putting $y = x, x - x = 0$ which is an integer for all $x \in Z$. So, R is reflexive in Z.

Symmetric :

Let $(x,y) \in R$, then $(x - y)$ is an integer λ (say) and also $y - x = -\lambda$. $(\lambda \in Z \Rightarrow -\lambda \in Z)$

$y - x$ is an integer $\Rightarrow (y,x) \in R$, So, R is symmetric.

Transitive :

Let $(x,y) \in R$, and $(y,z) \in R$,, so $x - y$ = integer and $y - z$ = integers, then $x - z$ is also an integer.

$(x,z) \in R$, So, R is transitive.

Thus (d) is correct option.

9. The function $f : R \to R$ defined by $f(x) = x^2 + x$ is

(a) one-one (b) onto

(c) many-one (d) None of the above

Sol : Foreign 2011

The given function $f : R \to R$ defined by

$$f(x) = x^2 + x$$

Now, for $x = 0$ and -1 we have

$$f(0) = 0$$

and $$f(-1) = 0$$

Hence, $$f(0) = f(-1)$$

but $$0 \neq -1$$

Thus f is not one-one. Function f is not many-one.

Thus (c) is correct option.

10. If R is the relation defined in the set $\{1,2,3,4,5,6\}$ as $R = \{(a,b) : b = a+1\}$, then R is

(a) reflexive (b) symmetric

(c) transitive (d) None of these

Sol : OD 2010, Comp 2007

Let $$A = \{1,2,3,4,5,6\}$$

Reflexive :

A relation R is defined on set A is

$$R = \{(a,b): b = a+1\}.$$

Therefore, $$R = \{(1,2),(2,3),(3,4),(4,5),(5,6)\}$$

Now, $6 \in A$ but $(6,6) \notin R$.

Therefore, R is not reflexive.

Symmetric :

It can be observed that $(1,2) \in R$ but $(2,1) \notin R$. Therefore, R is not symmetric.

Transitive :

Now, $(1,2),(2,3) \in R$ but $(1,3) \notin R$. Therefore, R is not transitive.

Hence, R is neither reflexive nor symmetric nor transitive.

Thus (d) is correct option.

11. Let $A = \{1,2,3\}$ and $R = \{(1,2),(2,3)\}$ be a relation in A. Then, the minimum number of ordered pairs may be added, so that R becomes an equivalence relation, is

(a) 7 (b) 5

(c) 1 (d) 4

Sol : OD 2018

The given relation is $R = \{(1,2),(2,3)\}$ in the set $A = \{1,2,3\}$.

Now, R is reflexive, if $(1,1),(2,2),(3,3) \in R$.

R is symmetric, if $(2,1),(3,2) \in R$.

R is transitive, if $(1,3)$ and $(3,1) \in R$.

Thus, the minimum number of ordered pairs which are to be added, so that R becomes an equivalence relation, is 7.

Thus (a) is correct option.

12. A function $f : x \to y$ is said to be one-one , if for every $x_1, x_2 \in X$,

(a) $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

(b) $f(x_1) = f(x_2) \Rightarrow x_1 \neq x_2$

(c) $f(x_1) \neq f(x_2) \Rightarrow x_1 = x_2$

(d) None of these

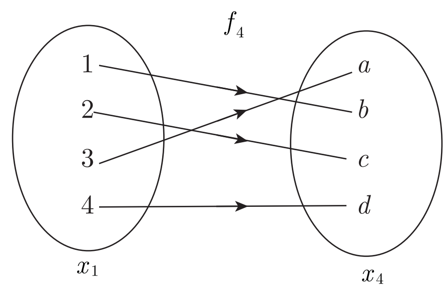
Sol : Foreign 2014, Delhi 2012

A function $f : x \to y$ is defined to be one-one (or injective), if the images of distinct elements of x under f are distinct, i.e., for every $x_1, x_2 \in x, f(x_1) = f(x_2)$ implies $x_1 = x_2$.

Otherwise, f is called many-one.

Thus (a) is correct option.

13. The function f_4 defined by



(a) one-one only (b) onto only

(c) bijective (d) many-one

Sol : OD 2016

Since, distinct elements of x_1 have distinct images in x_4 and every element in x_4 has a unique pre image in x_1, the function f_4 is both one-one and onto. Thus f_4 is bijective.

Thus (c) is correct option.

14. The function $f : N \to N$ given by $f(x) = 2x$ is

(a) surjective (b) bijective

(c) injective (d) many-one

Sol : Delhi 2010

The function f is one-one, as

$$f(x_1) = f(x_2)$$

$$2x_1 = 2x_2$$

$$x_1 = x_2.$$

Further, f is not onto, as for $1 \in N$, there does not exist any x in N such that $f(x) = 2x = 1$.

Thus (c) is correct option.

15. The function $f : X \to Y$ defined by $f(x) = \sin x$ is one-one but not onto, if X and Y respectively equal to

(a) R and R

(b) $[0,\pi]$and $[0,1]$

(c) $[0,\frac{\pi}{2}]$ and $[-1,1]$

(d) $[-\frac{\pi}{2},\frac{\pi}{2}]$ and $[-1,1]$

Sol : Foreign 2013, Delhi 2010

Since, $f : x \to y$

and $f(x) = \sin x$

Now, take option (c).

Domain $= \left[0, \dfrac{\pi}{2}\right],$

Range $= [-1, 1]$

For every value of x, we get unique value of y. But the value of every y in $[-1,\ 0]$ does not have image on X.

Thus (a) is correct option.

16. Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 4\}$, then $f = \{(1,1),(1,2),(2,1),(3,4)\}$ is a

(a) one-one function from A to B

(b) bijection from A to B

(c) surjection from A to B

(d) none of the above

Sol : OD 2014

Here, f is not a function from A to B as $f(1)$ is not unique.

Thus (d) is correct option.

17. The function f defined by $f(x) = (1-x)^{1/3}$ is

(a) one-one and onto

(b) many-one and onto

(c) one-one and into

(d) many-one and into

Sol : Delhi 2017

We have $f(x) = (1-x)^{1/3}$

$$f'(x) = -\frac{1}{3(1-x)^{2/3}} < 0$$

Thus, function is decreasing from $-\infty$ to ∞. Also, f is continuous everywhere. Hence, f is one-one and onto.

Thus (a) is correct option.

18. If $f : [0, \frac{\pi}{2}] \to [0, \infty]$ be a function defined by $y = \sin(\frac{x}{2})$, then f is

(a) injective (b) surjective

(c) bijective (d) none of these

Sol : Comp 2018

Since, $0 \le x \le \dfrac{\pi}{2}$

$$0 \le \frac{x}{2} \le \frac{\pi}{4}$$

$$0 \le \sin\frac{x}{2} \le \frac{1}{\sqrt{2}}$$

$$\left(0, \frac{1}{\sqrt{2}}\right) \subset [0, \infty)$$

Hence, function is injective.

Thus (a) is correct option.

19. Let the function $f \quad R \to R$ be defined by $f(x) = \cos x$, $\forall x \in R$. The function f is

(a) one-one and onto

(b) one-one but not onto

(c) not one-one but onto

(d) neither one-one nor onto

Sol : SQP 2013

Here, $f(x) = \cos x \forall x \in R$

Let $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right] \in f(x)$

$$f\left(-\frac{\pi}{2}\right) = \cos\left(-\frac{\pi}{2}\right) = \cos\frac{\pi}{2} = 0$$

$$\cos\left(\frac{\pi}{2}\right) = \cos\frac{\pi}{2} = 0$$

Thus $f\left(-\dfrac{\pi}{2}\right) = f\left(\dfrac{\pi}{2}\right) = 0$

But $-\dfrac{\pi}{2} \ne \dfrac{\pi}{2}$

Therefore, the given function is not one-one. Also it it is not onto function as no pre-image of all real number belongs to the range of $\cos x$ i.e., $[-1,1]$.

Thus (d) is correct option.

20. Consider the non-empty set consisting of children in a family and a relation R defined as $a R b$, if a is brother of b. Then R is

(a) symmetric but not transitive

(b) transitive but not symmetric

(c) neither symmetric nor transitive

(d) both symmetric and transitive

Sol : Delhi 2017, Foreign 2013

Reflexive :

Here, $a R b \Rightarrow a$ is brother of b.

$a R a \Rightarrow a$ is a brother of a which is not true.

So, R is not reflexive.

Symmetric :

$a\,R\,b \Rightarrow a$ is a brother of b.

$b\,R\,a \Rightarrow$ which is not true because b may be sister of a. Thus $a\,R\,b \neq b\,R\,a$. So, R is not symmetric.

Transitive :

Now, $a\,R\,b$, $b\,R\,c \Rightarrow a\,R\,c$

a is the brother of b and b is the brother of c.

a is also the brother of c.

So, R is transitive.

Hence, correct answer is (b).

21. The maximum number of equivalence relations on the set $A = \{1,2,3\}$ are

(a) 1 (b) 2

(c) 3 (d) 5

Sol : Comp 2018

Here $A = \{1,2,3\}$

The number of equivalence relations are as follows:

$$R_1 = \{(1,1),(1,2),(2,1),(2,3),(1,3)\}$$

$$R_2 = \{(2,2),(1,3),(3,1),(2,3),(1,2)\}$$

$$R_3 = \{(3,3),(1,2),(2,3),(1,3),(3,2)\}$$

Hence, correct answer is (d)

22. Let us define a relation R in R as $a\,R\,b$ if $a \geq b$. Then R is

(a) an equivalence relation

(b) reflexive, transitive but not symmetric

(c) symmetric, transitive but not reflexive

(d) neither transitive nor reflexive but symmetric.

Sol : OD 2013, Comp 2007

Reflexive :

Here, aRb if $a \geq b$

$a\,R\,b \Rightarrow a \geq a$ which is true, so it is reflexive.

Symmetric :

Let $a\,R\,a \Rightarrow a \geq b$, but $b \ngeq a$, so $b\,R\,a$

R is not symmetric.

Transitive :

Now, $a \geq b$,

$b \geq c$

$a \geq c$ which is true.

So, R is transitive.

Hence, correct answer is (b).

23. Let $A = \{1,2,3\}$ and consider the relation $R = \{(1,1),(2,2),(3,3),(1,2),(2,3),(1,3)\}$, then R is

(a) reflexive but not symmetric

(b) reflexive but not transitive

(c) symmetric and transitive

(d) neither symmetric nor transitive.

Sol : OD 2012

We have

$$R = \{(1,1),(2,2),(3,3),(1,2),(2,3),(1,3)\}$$

Here, $(1,1),(2,2),(3,3)$ are element of so R is reflexive

Here (1, 2), (2, 3) and (1, 3) are element of R but (2, 1), (3, 2) and (3, 1) are not element of R. Thus R is not symmetric

Here (1, 2) and (2, 3) are element of R and (1, 3) is also element of R. Thus R is transitive.

Hence, the correct answer is (a).

24. Let $f \quad R \to R$ be defined by $f(x) = \frac{1}{x}$, $\forall\, x \in R$ then f is

(a) one-one (b) onto

(c) bijective (d) f is not defined

Sol : Delhi 2010

We have $f(x) = \dfrac{1}{x}$

At $x = 0$ $f(x) = \dfrac{1}{0} = \infty$

So, $f(x)$ is not defined.

Thus (d) is correct option.

25. Which of the following functions from Z to Z are bijective?

(a) $f(x) = x^3$ (b) $f(x) = x + 2$

(c) $f(x) = 2x + 1$ (c) $f(x) = x^2 + 1$

Sol : SQP 2017

We have $f \quad Z \to Z$

Let $x_1,\ x_2 \in f(x)$

$$f(x_1) = x_1 + 2,$$

$$f(x_2) = x_2 + 2$$

$$f(x_1) = f(x_2)$$

$$x_1 + 2 = x_2 + 2$$

$$x_1 = x_2$$

So, $f(x)$ is one-one function.

Now, let $y = x + 2$

$$x = y - 2 \in Z \forall\, y \in Z$$

So, $f(x)$ is onto function.

Thus $f(x)$ is bijective function.

Hence, the correct answer is (b).

26. A relation R in set $A=\{1,2,3\}$ is defined as $R=\{(1,1),(1,2),(2,2),(3,3)\}$.Which of the following ordered pair in R shall be removed to make it an equivalence relation in A?

(a) (1,1) (b) (1,2)

(c) (2,2) (d) (3,3)

Sol : SQP 2022-I

We have, $S=\{1,2,3\}$

and $R=\{(1,1),(1,2),(2,2),(3,3)\}$

Here we have (1, 2), so we need to have (2, 1) to make it symmetric. But if we remove (1, 2), this relation will be symmetric, reflexive and transitive i.e. equivalence relation.

So, if we remove $(1,2)$ then R becomes equivalence relation.

27. Let the relation R in the set $A=\{x\in Z:0\le x\le 12\}$, given by $R=\{(a,b):|a-b|$ is a multiple of 4 $\}$.Then [1], the equivalence class containing 1 is.

(a) $\{1,5,9\}$ (b) $\{0,1,2,5\}$

(c) ϕ (d) A

Sol : SQP 2022-I

$$A=\{0,1,2,3,.....12\}$$

$$R=\{(a,b):|a-b|\text{ is multiple of 4}\}$$

The equivalence class containing 1 is {1,5,9}.

Thus (a) is correct option.

28. The function $f:R\to R$ defined as $f(x)=x^3$ is

(a) One-one but not onto

(b) not one-one but onto

(c) neither one-one nor onto

(d) both one-one and onto

Sol : SQP 2022-I

We have, $f(x)=x^3$

One-one function :

Let $x_1,x_2\in R$, $f(x_1)=f(x_2)$

$$x_1^3=x_2^3$$

$$x_1^3-x_2^3=0$$

$$(x_1-x_2)(x_1^2+x_1x_2+x_2^2)=0$$

$$x_1=x_1$$

and $x_1^2+x_1x_2+x_2^2\ne 0$

Thus $f(x)$ is one-one function.

Onto Function :

Range of $f(x)=(-\infty,\infty)$

Range = Co-domain

Thus $f(x)$ is also onto function.

Thus (d) is correct option.

29. Let $A=\{1,2,3\}$, $B=\{4,5,6,7\}$ and let $f=\{(1,4),(2,5),(3,6)\}$ be a function from A to B. Based on the given information f is best defined as

(a) surjective function

(b) injective function

(c) bijective function

(d) None of the above

Sol : SQP 2022-I

We have, $A=\{1,2,3\}$,

$B=\{4,5,6,7\}$

Function $f=\{(1,4),(2,5),(3,6)\}$

Clearly, f is injective function.

Now, range of $f=\{4,5,6\}$

Co-domain $=\{4,5,6,7\}$

Here, Range $\ne$ Co-domain

Thus f is not surjective function.

Thus (b) is correct option.

30. If $A=\{1,2,3\}$, $B=\{6,7,8\}$ and $f:A\to B$ is a function such that $f(x)=x+5$ then what type of a function is f?

(a) into

(b) one-one onto

(c) many-one onto

(d) Constant function

Sol : OD 2010

We have $f:A\to B$

where $f(x)=x+5$

$A=\{1,2,3\}$, $B=\{6,7,8\}$

Hence, $f(x)=\{(1,6),(2,7),(3,8)\}$

Since, each element of A has unique image in f, so $f(x)$ is one-one.

Also, co-domain = Range $=\{6,7,8\}$, thus $f(x)$ is onto.

Hence, $f(x)$ is one-one onto.

Thus (b) is correct option.

31. If R is a relation on the set N, defined by $\{(x,y): 2x - y = 10\}$, then R is
(a) reflexive
(b) symmetric
(c) transitive
(d) None of the above

Sol : Delhi 2007

Reflexive :
As $20 \in N$ but $(20, 20) \notin R$. So, it is not reflexive.
Symmetric :
As $(20, 30) \in R$ but $(30, 20) \notin R$. So, it is not symmetric.
Transitive :
As $(20, 30) \in R$, $(30, 50) \in R$ but $(20, 50) \notin R$.
So, it is not transitive.
Thus (d) is correct option.

32. The function $f(x) = x^2 + bx + c$, where b and c are real constants, describes
(a) one-one mapping
(b) onto mapping
(c) not one-one but onto mapping
(d) neither one-one nor onto mapping

Sol : Foreign 2009

We have $f(x) = x^2 + bx + c$
It is a quadratic equation in x.
So, we will get a parabola either downward or upward. Hence, it is many-one mapping and not onto mapping. Hence, it is neither one-one nor onto mapping.
Thus (d) is correct option.

33. What type of a relation is "Less than" in the set of real numbers?
(a) only symmetric
(b) only transitive
(c) only reflexive
(d) equivalence relation

Sol : Comp 2010, Foreign 2007

Let A be the set of all real numbers, and R be the relation 'less than' i.e., $<$ on A, then R is
Reflexive :
The relation R is not reflective, since a is not less than for any natural number a.
Symmetric :
The relation R is not symmetric, since if $(a,b) \in R$, then a is less than b but b is not less than a, i.e., $(b,a) \notin R$.
Transitive :
The relation R is transitive, since $a < b$ and $b < c \Rightarrow a < c$ i.e., $(a,b) \in R$, $(b,c) \in R$ $\Rightarrow (a,c) \in R$.
Thus (b) is correct option.

34. What type of relation is $R = \{(1,3),(4,2), (2,4), (2,3),(3,1)\}$ on the set $A = \{1,2,3,4\}$?
(a) Reflexive
(b) Transitive
(c) Symmetric
(d) None of these

Sol : Delhi 2015, OD 2011

Reflexive :
Since, $(a,a) \notin R$. Hence R on set A is not reflexive.
Symmetric :
Since $(2,3) \in R$ but $(3,2) \notin R$. Hence R on set A is not symmetric.
Transitive :
Here, $(1,3) \in R$ and $(3,1) \in R$ but $(1,1) \notin R$. Hence, R on set A is not transitive relation.
Thus (d) is correct option.

35. If R be a relation on A such that
$R = \{ (2, 2), (3, 3), (2, 3), (3, 2), (3, 1), (2, 1)\}$ on the set $A = (1, 2, 3)$, then R is
(a) reflexive (b) symmetric
(c) equivalence (d) transitive

Sol : Delhi 2017

We have,
$$R = \{(2,2),(3,3),(2,3),(3,2),(3,1),(2,1)\}$$
Reflexive :
Since $(1, 1) \notin R$, therefore R is not reflexive
Symmetric :
Since $(3, 1) \in R$, but $(1, 3) \notin R$, therefore R is not symmetric.
Transitive :
Since, for every element $(a, b) \in R$ and $(b, c) \in R$, $(a,c) \in R$, therefore R is transitive.
Thus (d) is correct option.

36. If A and B are two equivalence relations defined on set C, then
(a) $A \cap B$ is an equivalence relation
(b) $A \cap B$ is not an equivalence relation
(c) $A \cap B$ is an equivalence relation
(d) $A \cap B$ is not an equivalence relation

Sol : Comp 2012

If A and B are equivalence relations, then $A \cap B$ is also an equivalence relation.

Thus (a) is correct option.

37. If $f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \forall x_1, \; x_2 \in A$, then the function $f: A \to B$ is

(a) one-one (b) constant

(c) onto (d) many one

Sol : Delhi 2015

If $f(x_1) = f(x_2)$ then

$$x_1 = x_2 \;\forall\; x_1 \; x_2 \in A$$

Since there exist a unique value of $f(x)$ in set B Therefore $f: A \to B$ is one-one.

Thus (a) is correct option.

38. $f: A \to B$ will be an onto function if

(a) $f(A) \subset B$ (b) $f(A) = B$

(c) $f(A) \supset B$ (d) $f(A) \neq B$

Sol : Comp 2018

If Range = Co-domain of function f then function will be onto.

$$f(A) = \text{Range}$$

and $$B = \text{Set of co-domain}$$

$$f(A) = B$$

Thus (b) is correct option.

39. If $A = \{1, 3, 5, 7\}$ and $B = \{1, 2, 3, 4, 5, 6, 7, 8\}$, then the number of one-one function from A into B is

(a) 1340 (b) 1860

(c) 1430 (d) 1680

Sol : Foreign 2016, OD 2014

We have $$A = \{1, 3, 5, 7\}$$

and $$B = \{1, 2, 3, 4, 5, 6, 7, 8\}$$

Here, $$n(A) = 4$$

and $$n(B) = 8$$

Number of one-one function from A into B

$$^8P_4 = 8 \cdot 7 \cdot 6 \cdot 5 = 1680$$

40. A mapping $f: n \to N$, where N is the set of natural numbers is define as

$$f(n) = \begin{cases} n^2, & \text{for } n \text{ odd} \\ 2n+1, & \text{for } n \text{ even} \end{cases}$$

for $n \in N$. Then, f is

(a) Surjective but not injective

(b) Injective but not surjective

(c) Bijective

(d) neither injective nor surjective

Sol : Comp 2007

We have $$f(n) = \begin{cases} n^2, & \text{if } n \text{ is odd.} \\ 2n+1, & \text{if } n \text{ is even} \end{cases}$$

$$f(1) = 1^2 = 1$$

$$f(2) = 2(2) + 1 = 5$$

$$f(3) = 3^2 = 9$$

$$f(4) = 2(4) + 1 = 9$$

$$f(3) = f(4)$$

So, f is not injective.

Also, f is not surjective as some element of N (co-domain) is not the image of any element of N.

Thus (d) is correct option.

41. **Assertion (A) :** The relation $f: \{1, 2, 3, 4\} \to \{x, y, z, p\}$ defined by $f = \{(1, x), (2, x), (3, z)\}$ is a bijective function.

Reason (R) : The function $f: \{1, 2, 3\} \to \{x, y, z, p\}$ such that $f = \{(1, x), (2, x), (3, z)\}$ is one-one.

(a) Both (A) and (R) are true and (R) is the correct explanation of (A).

(b) Both (A) and (R) are true but (R) is not the correct explanation of (A).

(c) (A) is true but (R) is false.

(d) (A) is false but (R) is true

Sol : SQP 2023

We have $$f: \{1, 2, 3, 4\} \to \{x, y, z, p\}$$

and $$f = \{(1, x), (2, y), (3, z)\}$$

Since, 4 has no image under f, so relation f is not a function.

Assertion is false.

Now, $$f: \{1, 2, 3\} \to \{x, y, z, p\}$$

and $$f = \{(1, x), (2, y), (3, z)\}$$

Since, every element $\{1, 2, 3\}$ has different image in $\{x, y, z, p\}$ under f, so the given relation f is one-one.

Reason is true.

Thus (d) is correct option.

42. **Assertion (A) :** Let $F: N \to Y$ be a function defined as $f(x) = 9x + 3$, where $Y = \{y : y = 9x + 3, x \in N\}$ then f is one-one.

Reason (R) : For $x_1,\ x_2 \in N$, $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

(a) Both (A) and (R) are true and (R) is the correct explanation of (A).

(b) Both (A) and (R) are true but (R) is not the correct explanation of (A).

(c) (A) is true but (R) is false.

(d) (A) is false but (R) is true

Sol : Comp 2013

We have $\quad f(x) = 9x+3$

Let, $\quad f(x_1) = f(x_2)$

$$9x_1 + 3 = 9x_2 + 3$$

$$9x_1 = 9x_2$$

$$x_1 = x_2$$

For any function to be one-one, if $f(x_1) = f(x_2)$

$$x_1 = x_2$$

Hence, $f(x)$ is one-one function.

Thus both (A) and (R) are true and (R) is the correct explanation of (A).

Thus (a) is correct option.

43. **Assertion (A) :** The function $f: R \rightarrow R$ given by $f(x) = x^3$ is injective.

Reason (R) : The function $f: X \rightarrow Y$ is injective, if $f(x) = f(y) \Rightarrow x = y$ for all $x, y \in X$.

(a) Both (A) and (R) are true and (R) is the correct explanation of (A).

(b) Both (A) and (R) are true but (R) is not the correct explanation of (A).

(c) (A) is true but (R) is false.

(d) (A) is false but (R) is true

Sol : Delhi 2014, OD 2010

Here, $f: R \rightarrow R$ is given as

$$f(x) = x^3$$

Suppose $\quad f(x) = f(y) \quad$ where $x, y \in R$

$$x^3 = y^3 \qquad ...(i)$$

Suppose $x \neq y$, their cubes will also be not equal.

$$x^3 \neq y^3$$

However, this will be a contradiction to Eq.(i).

Therefore $x = y$ and hence f is injective.

Thus both (A) and (R) are true and (R) is the correct explanation of (A).

Thus (a) is correct option.

44. **Assertion (A) :** Let A and B be sets. Then, the function $f: A \times B \rightarrow B \rightarrow A$ such that $f(a, b) = (b, a)$ is bijective.

Reason (R) : A function f is said to be bijective, if it is both one-one and onto.

(a) Both (A) and (R) are true and (R) is the correct explanation of (A).

(b) Both (A) and (R) are true but (R) is not the correct explanation of (A).

(c) (A) is true but (R) is false.

(d) (A) is false but (R) is true

Sol : OD 2018

Here, $f: A \times B \rightarrow B \times A$ is defined as

$$f(a, b) = (b, a)$$

Let $(a_1, b_1), (a_2, b_2) \in A \times B$ such that

$$f(a_1, b_1) = f(a_2, b_2)$$

$$(b_1, a_1) = (b_2, a_2)$$

$$b_1 = b_2$$

and $\quad a_1 = a_2$

$$(a_1, b_1) = (a_2, b_2)$$

Therefore, f is one-one.

Now, let $(b, a) \in B \times A$ be any element.

Then, from definition of f there exists $(a, b) \in A \times B$ such that $f(a, b) = (b, a)$.

Therefore, f is onto. Hence, f is bijective.

Thus both (A) and (R) are true and (R) is the correct explanation of (A).

Thus (a) is correct option.

45. **Assertion (A) :** Let L be a set of lines in a plane. R is a relation on L defined as $R = \{L_1, L_2\}$, L_1 is parallel to L_2. Then R is an equivalence relation.

Reason (R) : R is not transitive relation.

(a) Both (A) and (R) are true and (R) is the correct explanation of (A).

(b) Both (A) and (R) are true but (R) is not the correct explanation of (A).

(c) (A) is true but (R) is false.

(d) (A) is false but (R) is true

Sol : Delhi 2011, OD 2008

Reflexive :

Reflexive Relation on L is said to be reflexive if every element of L is related to itself. Since every line is parallel to itself. Therefore, relation R is reflexive relation.

Symmetric :

A relation R on a set L is said to be symmetric relation iff $(a,b) \in R \Rightarrow (b,a) \in R$ for all $a,b \in L$. If $L_1 \| L_2 \Rightarrow L_2 \| L_2$. Thus, given relation is also symmetric.

Transitive :

A relation R on a set L is said to be transitive relation iff $(a,b) \in R$, $(b,c) \in R \Rightarrow (a,c) \in R$ for all a, b, $c \in L$. If $L_1 \| L_2$ and $L_2 \| L_3 \Rightarrow L_1 \| L_3$.Therefore given relation is transitive relation.

Relation which is reflexive, symmetric and transitive is called equivalence relation.

Thus (A) is true but (R) is false.

Thus (c) is correct option.

46. Assertion (A) : The relation R in $A = \{1,2,3,4,5,6\}$ defined as $R = \{(x,y): y \text{ is divisible by } x\}$ is not an equivalence relation.

Reason (R) : The relation R will be an equivalence relation, if it is reflexive, symmetric and transitive.

(a) Both (A) and (R) are correct and (R) is the correct explanation of (A).

(b) Both (A) and (R) are correct but (R) is not the correct explanation of (A).

(c) (A) is correct but (R) is false.

(d) (A) is correct but (R) is true

Sol : Foreign 2014

Here, $R = \{(x,y): y$ is divisible by $x\}$ is a relation in the set $A = \{1,2,3,4,5,6\}$.

Reflexive :

We know that x is divisible by x, which is true for all $x \in A$. $(x,x) \in R$ for all $x \in A$. So, R is reflexive.

Symmetric :

We observe that 6 is divisible by 2 . This means that $(2,6) \in R$ but $(6,2) \notin R$ So, R is not symmetric.

Transitive :

Let $(x,y) \in R$ and $(y,z) \in R$

Here y is divisible by x and z is divisible by y.

thus z is divisible by x. It means $(x,y) \in R$. For example, 2 is divisible by l and 4 is divisible by 2.

So, 4 is divisible by 1. So, R is transitive

Thus assertion is correct and reason is correct explanation.

Thus (a) is correct option.

47. **Assertion:** Let $A = \{-1,1,2,3\}$ and $B = \{1,4,9\}$ where $f: A \to B$ given by $f(x) = x^2$, then f is a many-one function.

Reason: If $x_1 \neq x^2 \Rightarrow f(x_1) \neq f(x_2)$, for every $x_1, x_2 \in$ domain then f is one-one or else many

(a) Assertion is true, Reason is true; Reason is a correct explanation for Assertion.

(b) Assertion is true, Reason is true; Reason is not a correct explanation for Assertion.

(c) Assertion is true; Reason is false.

(d) Assertion is false; Reason is true.

Sol : Comp 2014

Here, $f(-1) = 1,\ f(1) = 1,$

$$f(2) = 4,\ f(3) = 9$$

Two elements 1 and -1 have the same image $1 \in B$.

So, f is a many-one function .

Assertion and reason are true and reason is the correct explanation of assertion.

Thus (a) is correct option.

48. **Assertion:** The relation R in a set $A = \{1,2,3,4\}$ defined by $R = \{(x,y): 3x - y = 0)\}$ have the domain $= \{1,2,3,4\}$ and range $= \{3,6,9,12\}$

Reason: Domain and range of the relation (R) is respectively the set of all first & second entries of the distinct ordered pair of the relation.

(a) Assertion is true, Reason is true; Reason is a correct explanation for Assertion.

(b) Assertion is true, Reason is true; Reason is not a correct explanation for Assertion.

(c) Assertion is true; Reason is false.

(d) Assertion is false; Reason is true.

Sol : Delhi 2010

$$R = \{(x,y): y = 3x, x \in A\}$$

$$R = \{(1,3),(2,6),(3,9),(4,12)\}$$

Domain of the relation is $\{1,2,3,4\}$ and range of the relation is $\{3,6,9,12\}$

Thus (a) is correct option.

VERY SHORT ANSWER QUESTIONS

49. How many equivalence relations on the set $\{1,2,3\}$ containing (1,2) and (2,1) are there in all? Justify your answer.

Sol : SQP 2017

Equivalence relations on the set $\{1,2,3\}$ containing $(1,2)$ and $(2,1)$ could be the following $\{(1,1),(2,2),(3,3),(1,2),(2,1)\}$ and $\{(1,1),(2,2),(3,3),(1,2),(1,3),(2,1),(2,3),(3,1),(3,2)\}$

50. How many reflexive relations are possible in a set A whose $n(A) = 3$?

Sol : SQP 2021

If a set has n elements, then number of reflexive relation in the set is $2^{n(n-1)}$.

Required number of reflexive relations that are possible in the set

$$\begin{aligned} A &= 2^{3(3-1)} \\ &= 2^{3\times 2} \\ &= 2^6 \\ &= 64 \end{aligned}$$

51. Check whether the function $f:R\to R$ defined as $f(x)=x^3$ is one-one or not.

Sol : OD 2018

We have, $f(x) = x^3$

Let $x_1, x_2 \in R$ such that

$$f(x_1) = f(x_2)$$

$$x_1^3 = x_2^3 \Rightarrow x_1 = x_2$$

Thus $f(x)$ is one-one function.

52. A relation R in $S=\{a,b,c\}$ is defined as $R=\{(a,a),(a,b),(b,b),(c,c)\}$. Which elements(s) of relation R be removed to make R an equivalence relation?

Sol : Comp 2012, Delhi 2010

We have, $S = \{a,b,c\}$

and $R = \{(a,a),(a,b),(b,b),(c,c)\}$

Here we have (a,b), so we need to have (b,a) to make it symmetric. But if we remove (a,b), this relation will be symmetric, reflexive and transitive i.e. equivalence relation.

So, if we remove (a,b) then R becomes equivalence relation.

53. An equivalence relation R in A divides it into equivalence classes A_1, A_2, A_3. What is the value of $A_1\cup A_2\cup A_3$ and $A_1\cap A_2\cap A_3$?

Sol : Delhi 2007

Equivalence relation R defined on a set A partitions the set A into pairwise disjoint subsets and collection of all equivalence classes form set A.

$$A_1\cup A_2\cup A_3 = A$$

and $A_1\cap A_2\cap A_3 = \phi$

54. A relation R in the set of real number R defined as $R=\{(a,b):\sqrt{a}=b\}$ is a function or not. Justify.

Sol : Foreign 2018

Since, $\sqrt{a}$ is not defined for $a\in(-\infty,0)$ thus R is not a function.

55. A function $f:A\to B$ defined as $f(x)=2x$ is both one-one and onto. If $A=\{1,2,3,4\}$, then find the set B.

Sol : OD 2023

We have $f:A\to B$.

Such that $f(x)=2x$ is both one-one and onto

and $A = \{1,2,3,4\}$

$$\begin{aligned} f(1) &= 2\times 1 = 2 \\ f(2) &= 2\times 2 = 4 \\ f(3) &= 2\times 3 = 6 \end{aligned}$$

and $f(4) = 2\times 4 = 8$

Since, $f(x)$ is onto.

Co-domain of f = Range of f

$$B = \{2,4,6,8\}$$

56. A relation in a set A is called .... relation, if each element of A is related to itself.

Sol : OD 2020

If R be any relation on set A and for all $a\in A$ $(a,a)\in R$ then R is known as reflexive relation.

57. Check if the relation R is the set R of real number defined as $R=\{(a,b):a<b\}$ is (i) symmetric, (ii) transitive.

Sol : OD 2020

Given, $A=$ set of real number and $R=\{(a,b):a<b\}$

Symmetric :

Let $(a,b)\in R$, then $a<b$ or $a=b$

If $a=b$, then $b=a$, but if we consider $a<b$, then $b\nless a$. Thus $(b,a)\notin R$ e.g. $4<5$ but $5\nless 4$

So, R is not symmetric.

Transitive :

Let (a,b), $(b,c)\in R$, then

$$(a,b)\in R,\ a<b \qquad \text{(i)}$$

$$(b,c)\in R,\ b<c \qquad \text{(ii)}$$

From Eqs. (i) and (ii), we get $a<c$, thus $(a,c)\in R$. So, R is transitive.

58. If $R=\{(a,a^3)$: a is a prime number less than 5$\}$ be a relation. Find the range of R.

Sol : Foreign 2014, OD 2011

Given, $R=\{(a,a^3)$: a is a prime number less than 5$\}$

Since 2 and 3 are the prime numbers less than 5, a can take value 2 and 3.

Thus, $R = \{(2, 2^3), (3, 3^3)\}$

$= \{(2, 8), (3, 27)\}$

Therefore, the range of R is $\{8, 27\}$.

59. Let R is the equivalence relation in the set $A = \{0, 1, 2, 3, 4, 5\}$ given by $R = \{(a, b): 2$ divides $(a - b)\}$. Write the equivalence class $[0]$.

Sol : Delhi 2014 C

We have $R = \{(a, b): 2 \text{ divides } (a - b)\}$

and $A = \{0, 1, 2, 3, 4, 5\}$

Clearly, $[0] = \{b \in A : (0, b) \in R\}$

$= \{b \in A : 2 \text{ divides } (0 - b)\}$

$= \{b \in A : 2 \text{ divides } (-b)\}$

$= \{0, 2, 4\}$.

60. If $A = \{1, 2, 3\}, B = \{4, 5, 6, 7\}$ and $f = \{(1, 4), (2, 5), (3, 6)\}$ is a function from A to B. State whether f is one-one or not.

Sol : Comp 2011

We have $A = \{1, 2, 3\}$

$B = \{4, 5, 6, 7\}$

Now $f : A \to B$ is defined as

$f = \{(1, 4), (2, 5), (3, 6)\}$

i.e. $f(1) = 4,\ f(2) = 5\ f(3) = 6$

It may be seen that the images of distinct elements of A under f are distinct. So, f is one-one.

61. If $f : R \to R$ is defined by $f(x) = 3x + 2$, then define $f[f(x)]$.

Sol : Foreign 2011, Delhi 2010f

We have $f(x) = 3x + 2$

$f[f(x)] = f(3x + 2) = 3$

$= 3(3x + 2) + 2$

$= 9x + 6 + 2$

$= 9x + 8$

62. Write fog, if $f : R \to R$ and $g : R \to R$ are given by $f(x) = |x|$ and $g(x) = |5x - 2|$.

Sol : Foreign 2011

We have $f(x) = |x| g(x) = |5x - 2|$

$fog(x) = f[g(x)]$

$= f\{|5x - 2|\}$

$= ||5x - 2||$

$= |5x - 2| \qquad ||x|| = |x|$

63. State the reason for the relation R in the set $\{1, 2, 3\}$ given by $R = \{(1, 2), (2, 1)\}$ not to be transitive.

Sol : Delhi 2011

For a relation to be transitive,
$(x, y) \in R$ and $(y, z) \in R$ $(x, z) \in R$.
Here, $(1, 2) \in R$ and $(2, 1) \in R$ but $(1, 1) \notin R$.
Thus R is not transitive.

64. What is the range of the function

$$f(x) = \frac{|x - 1|}{x - 1},\ x \neq 1?$$

Sol : Delhi 2010

We have $f(x) = \dfrac{|x - 1|}{x - 1}, x \neq 1$

The above function can be written as

$$f(x) = \begin{cases} \dfrac{x - 1}{x - 1}, & \text{if } x > 1 \\ -\dfrac{(x - 1)}{x - 1}, & \text{if } x < 1 \end{cases}$$

$$f(x) = \begin{cases} 1, & \text{if } x > 1 \\ -1, & \text{if } x < 1 \end{cases}$$

Hence, the range of $f(x)$ is $\{-1, 1\}$.

SHORT ANSWER QUESTIONS

65. A relation R is defined on a set or real number R as $R = \{(x, y) : x \cdot y \text{ is an irrational number}\}$.
Check whether R is reflexive, symmetric and transitive.

Sol : CBSE 2023

Relation R is defined on a set of real number R. such that

$$R = \{(x, y) : x \cdot y \text{ is an irrational number}\}$$

Reflexive :
$\sqrt{2}$ is a real number but $\sqrt{2} \cdot \sqrt{2} = 2$ is not an irrational number

$$(\sqrt{2}, \sqrt{2}) \notin R$$

Thus R is not reflexive.

Symmetric :
Consider $\sqrt{3}$ and $\sqrt{5}$ are two real numbers

Clearly, $\sqrt{3}\cdot\sqrt{5} = \sqrt{15}$ is an irrational number

Thus R is symmetric.

Transitive :

Consider three real number 2, $\sqrt{5}$ and 3.

Clearly, $2\cdot\sqrt{5} = 2\sqrt{5}$ is an irrational number

$\sqrt{5}\cdot 3 = \sqrt{15}$ is an irrational number

But $2\cdot 3$ is not and irrational number.

Thus R is not transitive.

66. Let $f:N\to R$ be a function defined as $f(x)=4x^2+12x+15$ Show that $f:N\to S$, where S is the range of f, is one - one and onto function.

Sol : Foreign 2015

Function $f:N\to N$ is defined as

$$f(x) = 4x^2+12x+15$$

One-One function :

Let $x_1,\ x_2\in N$, such that

$$f(x_1) = f(x_2)$$

Then, $4x_1^2+12x_1+15 = 4x_2^2+12x_2+15$

$$4x_1^2+12x_1 = 4x_2^2+12x_2$$

$$x_1^2+3x_1 = x_2^2+3x_2$$

$$(x_1^2-x_2^2)+3(x_1-x_2) = 0$$

$$(x_1-x_2)(x_1+x_2+3) = 0$$

Since $x_1, x_2\in N$, then

$$(x_1+x_2+3) \neq 0$$

Thus $x_1-x_2 = 0$

$$x_1 = x_2$$

Therefore, f is one-one function.

Onto function :

Obviously, $f:N\to S$ is an onto function, because S is the range of f.

67. Thus, $f:N\to S$ is one-one and onto function. Show that the relation S in the set R of real number defined as $S=\{(a,b):a,b\in R \text{ and } a\leq b^3\}$ is neither reflexive nor symmetric nor transitive.

Sol : Delhi 2010, OD 2007

Here, the result is disproved by using some specific examples.

We have $S = \{(a,b)\}: a,b\in R$ and $a\leq b^3\}$.

Reflexive :

As $\frac{1}{2}\leq\left(\frac{1}{2}\right)^3$, where $\frac{1}{2}\in R$, is not true.

$\left(\frac{1}{2},\frac{1}{2}\right)\notin S$ Thus, S is not reflexive.

Symmetric :

As $-2\leq(3)^3$, where $-2,3\in R$, is true but $3\leq(-2)^3$ is not true,

i.e. $(-2,3)\in S$ but $(3,-2)\notin S$.

Therefore, S is not symmetric.

Transitive :

As $3\leq\left(\frac{3}{2}\right)^3$ and $\frac{3}{2}\leq\left(\frac{4}{3}\right)^3$, where

$3,\frac{3}{2},\frac{4}{3}\in R$, are true but $3\leq\left(\frac{4}{3}\right)^3$ is not true.

i.e. $\left(3,\frac{3}{2}\right)\in S$ and $\left(\frac{3}{2},\frac{4}{3}\right)\in S$ but $\left(3,\frac{4}{3}\right)\notin S$.

Therefore, S is not transitive.

Hence, S is neither reflexive nor symmetric nor transitive.

68. Prove that the function f is surjective,where $f:N\to N$ such that

$$f(n)=\begin{cases}\frac{n+1}{2}, \text{if } n \text{ is odd}\\ \frac{n}{2}, \text{if } n \text{ is even}\end{cases}$$

Is the function injective? Justify your answer.

Sol : SQP 2023

Consider a natural number n in co-domain N.

Case I : When n is odd

Then $n = 2r+1$ for some $r\in N$.

There exists

$4r+1\in N$ such that

$$f(4r+1) = \frac{4r+1+1}{2}$$

$$= 2r+1$$

Therefore, f is onto.

Case II : When n is even

Then $n = 2r$ for some $r\in N$.

There exists $4r\in N$ such that

$$f(4r) = \frac{4r}{2} = 2r$$

Hence, f is surjective.

Now, it can be observed that

$$f(1) = \frac{1+1}{2} = \frac{2}{2} = 1$$

and $f(2) = \frac{2}{2} = 1$

Here, $f(1) = f(2)$

but $1 \neq 2$

Hence, f is not injective.

69. Prove that the function, $f: N \rightarrow N$ is defined by $f(x) = x^2 + x + 1$ is one-one but not onto.

Sol : Delhi 2019

We have $f: N \rightarrow N$

where $f(x) = x^2 + x + 1$

One-One function :

Let $x,\ y \in N$ such that

$$f(x) = f(y)$$

$$x^2 + x + 1 = y^2 + y + 1$$

$$(x - y)(x + y + 1) = 0 \qquad [x + y + 1 \neq 0]$$

$$x = y$$

Therefore $f: N \rightarrow N$ is one-one.

Onto function :

Function f is not onto because $x^2 + x + 1 \geq 3$, $\forall x \in N$ and so, 1, 2 does not have their pre images.

70. Consider $f: R^+ \rightarrow [4, \infty)$ given by $f(x) = x^2 + 4$, Show that f is one-one and onto.

Sol : Comp 2013, Foreign 2011

Function $f: R^+ \rightarrow [4, \infty)$ is given by

$$f(x) = x^2 + 4$$

One-One function :

Let $x,\ y \in R^+$, such that

$$f(x) = f(y)$$

$$x^2 + 4 = y^2 + 4$$

$$x^2 = y^2$$

$$x = y$$

Here we take only positive sign as $x, y \in R^+$

Therefore, f is a one-one function.

Onto Function :

For $y \in [4, \infty)$, such that

$$y = x^2 + 4$$

$$x^2 = y - 4 \geq 0 \qquad [y \geq 4]$$

$$x = \sqrt{y - 4} \geq 0$$

[we take only positive sign, as $x \in R^+$]

Therefore, for any $y \in R^+$ (co-domain), there exists

$x = \sqrt{y-4} \in R^+$ (co-domain), such that

$$f(x) = f(\sqrt{y - 4})$$

$$= (\sqrt{y-4})^2 + 4$$

$$= y - 4 + 4$$

$$= y$$

Therefore, f is onto function.

71. Let $F: N \rightarrow Y$ be a function defined as $f(x) = 7x + 5$, where $Y = \{y : y = 9x + 3, x \in N\}$ then show that f is one-one.

Sol : Delhi 2015

We have $f(x) = 7x + 5$

Let, $f(x_1) = f(x_2)$

$$7x_1 + 5 = 7x_2 + 5$$

$$7x_1 = 7x_2$$

$$x_1 = x_2$$

For any function to be one-one, if $f(x_1) = f(x_2)$

$$x_1 = x_2$$

Hence, $f(x)$ is one-one function.

LONG ANSWER QUESTIONS

72. Check whether the relation R in the set Z of integers defined as $R = \{(a, b) : a + b \text{ is divisible by } 2\}$ is reflexive, symmetric or transitive. Write the equivalence class containing 0 i.e.[0].

Sol : OD 2018

We have $R = \{(a, b) : a + b \text{ is divisible by } 2\}$

Reflexive :

For any $a = \in Z$, we have

$a + a = 2a$, which is divisible by 2.

$$(a, a) \in R$$

Thus, $(a, a) \in R$ for all $a \in Z$

So, R is reflexive.

Symmetric :

Let $(a, b) \in R$.

Then, $(a, b) \in R$

$(a + b)$ is divisible by 2 and $(b + a)$ is also divisible by 2.

Thus $(b, a) \in R$

So R is symmetric.

Transitive :

Let $(a,b) \in R$

and $(b,c) \in R.$

Then, $(a,b) \in R$

and $(b,c) \in R.$

$(a+b)$ is divisible by 2 and $(b+c)$ is divisible by 2.

$$a+b = 2\lambda \quad \text{...(i)}$$

and $$b+c = 2\mu \quad \text{...(ii)}$$

Adding Eqs. (i) and (ii),we get

$$a+2b+c = 2(\lambda+\mu)$$

$$a+c = 2(\lambda+\mu-b)$$

$$a+c = 2k,$$

where $$k = k+\mu-b$$

Thus $a+c$ is divisible by 2.

$$(a,c) \in R$$

So, R is transitive.

Since, R is reflexive, symmetric and transitive, so R is equivalence relation.

Let $b \in Z$

such that $(0,b) \in R$

$0+b$ is divisible by 2.

$$b = \{..., -6, -4, -2, 0, 2, 4, 6.....\}$$

73. Show that a function $f:R \to R$ defined as $f(x) = \frac{5x-3}{4}$ is both one-one and onto.

Sol : OD 2023

We have $f:R \to R$ defined as

$$f(x) = \frac{5x-3}{4}$$

One-One function :

Let $x_1, x_2 \in R$

such that $f(x_1) = f(x_2)$

$$\frac{5x-3}{4} = \frac{5x_2-3}{4}$$

$$5x_1 - 3 = 5x_2 - 3$$

$$5x_1 = 5x_2$$

$$x_1 = x_2$$

Thus f is one-one function.

Onto Function :

Let $y \in R$ (co-domain) be any arbitrary element.

Then, $$y = f(x)$$

$$y = \frac{5x-3}{4}$$

$$4y = 5x-3$$

$$5x = 4y+3$$

$$x = \frac{4y+3}{5}$$

Thus, for each $y \in R$,there exists

$$x = \frac{4y+3}{5} \in R$$

such that $$f(x) = f\left(\frac{4y+3}{5}\right)$$

$$= \frac{5\left(\frac{4y+3}{5}\right)-3}{4}$$

$$= \frac{4y+3-3}{4} = y$$

Thus f is onto.

74. Let $f:R-\left\{-\frac{4}{3}\right\} \to R$ be a function defined as $f(x) = \frac{4x}{3x+4}$. Show that f is a one-one function. Also, check whether f is an onto function or not.

Sol : OD 2023

We have $$f(x) = \frac{4x}{3x+4}$$

and $$f:R-\left\{-\frac{4}{3}\right\} \to R$$

One-One function :

Let $$f(x_1) = f(x_2),$$

for some $$x_1, x_2 \in R-\left\{-\frac{4}{3}\right\}$$

Now $$\frac{4x_1}{3x_1+4} = \frac{4x_2}{3x_2+4}$$

$$(x_1)(3x_2+4) = (x2)(3x_1+4)$$

$$3x_1x_2 + 4x_1 = 3x_1x_2 + 4x_2$$

$$4x_1 = 4x_2$$

$$x_1 = x_2$$

Thus $f(x)$ is one-one function.

Onto Function :

Let $$y = \frac{4x}{3x+4}$$

$$3xy+4y = 4x$$

$$4x-3xy = 4y$$

$$x(4-3y) = 4y$$

$$x = \frac{4y}{4-3y}$$

Clearly, x will not define, if

$$4 - 3y = 0$$

$$y = \frac{4}{3}$$

Range of $f(x) = R - \left\{-\frac{4}{3}\right\}$

Range of $f(x) \neq$ Co-domain of $f(x)$

Therefore $f(x)$ is not an onto function.

75. A function $f:[-4,4] \to [0,4]$ is given by $f(x) = \sqrt{16 - x^2}$. Show that f is an onto function but not a one-one function. Further, find all possible values of a for which $f(a) = \sqrt{7}$.

Sol : OD 2023

We have $f:[-4,4] \to [0,4]$

defined by $f(x) = \sqrt{16 - x^2}$

One-One function :

Let $x_1, x_2 \in [-4,4]$

such that $f(x_1) = f(x_2)$

$$\sqrt{16 - x_1^2} = \sqrt{16 - x_2^2}$$

$$16 - x_1^2 = 16 - x_2^2$$

$$x_1^2 = x_2^2$$

$$(x_1^2 - x_2^2) = 0$$

$$(x_1 - x_2)(x_1 + x_2) = 0$$

$$x_1 - x_2 = 0$$

$$x_1 + x_2 = 0$$

$$x_1 = x_2$$

$$x_1 = -x_2$$

Thus $f(x)$ is not one-one function.

Onto Function :

Let $y \in [0,4]$ (co-domain) be any arbitrary element.

Then, $y = f(x)$

$$y = \sqrt{16 - x^2}$$

$$y^2 = 16 - x^2$$

$$x^2 = 16 - y^2$$

$$x = \sqrt{16 - y^2}$$

Thus, for each $y \in [0,4]$, there exists

$$x = \sqrt{16 - y^2} \in [-4,4]$$

Such that $f(x) = y$

Thus $f(x)$ is onto.

Also, given $f(a) = \sqrt{7}$

$$\sqrt{16 - a^2} = \sqrt{7}$$

$$16 - a^2 = 7$$

$$a^2 = 9$$

$$a = \pm 3$$

76. If N denotes the set of all natural numbers and R be the relation on $N \times N$ defined by $(a,b)R(c,d)$, if $ad(b+c) = bc(a+d)$. Show that R is an equivalence relation.

Sol : Delhi 2023, Delhi 2015

We have, a relation R on $N \times N$ defined by

$$(a,b)\,R\,(c,d), \text{ if } ad(b+c) = bc(a+d).$$

Reflexive :

Let $(a,b) \in N \times N$ be any arbitrary element.

Since, $(a,b) \in N \times N$ was arbitrary, therefore R is reflexive.

Symmetric :

Let $(a,b),(c,d) \in N \times N$ such that $(a,b)\,R\,(c,d)$, i.e.

$$ad(b+c) = bc(a+d)$$

$$ad(b+c) = bc(a+d)$$

$$da(c+b) = cd(d+a)$$

Natural numbers are commutative under usual addition and multiplication

$$cb(d+a) = da(c+b)$$

Thus $(c,d)\,R\,(a,b)$ and hence R is symmetric.

Transitive :

Let (a,b), (c,d) and $(e,f) \in N \times N$ such that $(a,b)\,R\,(c,d)$ and $(c,d)\ R\,(e,f)$.

Now from $(a,b)\,R\,(c,d)$ so we have

$$ad(b+c) = bc(a+d)$$

$$\frac{b+c}{bc} = \frac{a+d}{ad}$$

$$\frac{1}{b} + \frac{1}{c} = \frac{1}{a} + \frac{1}{d} \quad ...(1)$$

From $(c,d)\,R\,(e,f)$ we have

$$cf(d+e) = de(c+f)$$

$$\frac{d+e}{de} = \frac{c+f}{cf}$$

$$\frac{1}{d} + \frac{1}{e} = \frac{1}{c} + \frac{1}{f} \quad ...(2)$$

Adding Eqs. (1) and (2), we get

$$\left(\frac{1}{b}+\frac{1}{c}\right)+\left(\frac{1}{d}+\frac{1}{e}\right) = \left(\frac{1}{a}+\frac{1}{d}\right)+\left(\frac{1}{c}+\frac{1}{f}\right)$$

$$\frac{1}{b}+\frac{1}{e} = \frac{1}{a}+\frac{1}{f}$$

$$\frac{e+b}{be} = \frac{f+a}{af}$$

$$af(e+b) = be(f+a)$$

$$af(b+e) = be(a+f)$$

Thus $(a,b)R(e,f)$ and hence R is transitive.

Since R is reflexive, symmetric and transitive, hence R is an equivalence relation.

77. Given, a non-empty set X, define the relation R in $P(X)$ as follow
For $A,B \in P(X),(A,B) \in R$ iff $A \subset B$. Prove that R is reflexive, transitive and not symmetric.

Sol : Delhi 2023

Given, a relation R in the set $P(X)$, where X is a non-empty set as $(A,B) \in R$ iff $A \subset B$.

Let $A \in P(X)$

Then, $A \subset A$

$$(A,A) \in R$$

Hence, R is reflexive.

Now, let $\phi, A \in P(X)$

Such that $\phi \subset A$

$$(\phi, A) \in R$$

But $A \not\subset \phi$

$$(A,\phi) \notin R$$

Hence, R is not symmetric.

Let $A,B,C \in P(X)$

Such that

$$(A,B),(B,C) \in R$$

$$A \subset B$$

and $B \subset C$

$$A \subset C \Rightarrow (A,C) \in R$$

Hence, R is transitive.

78. Let N be the set of all natural numbers and R be a relation on $N \times N$ defined by $(a,b)R(c,d)$ $\Leftrightarrow ad = bc$ for all $(a,b),(c,d) \in N \times N$.Show that R is an equivalence relation on $N \times N$. Also, find the equivalence class of $(2,6)$ i.e. $[(2,6)]$.

Sol : Foreign 2018

Relation R in $N \times N$ is defined as

$$(a,b)R(c,d) \text{ if } ad = bc$$

Reflexive :

For any $(a,b) \in N \times N$

$$ab = ba$$

Since multiplication is commutative on N.

Thus $(a,b)\ R(a,b)$

Thus, R is reflexive.

Symmetric :

Let $(a,b),(c,d)$ be arbitrary elements of $N \times N$

$$(a,b)\ R(c,d)$$

$$ad = bc$$

$$bc = ad$$

$$cb = da$$

Since $a,b,c,d \in N$ and multiplication is commutative on N.

Thus $(c,d)\ R(a,b) R$ is symmetric.

Transitive :

Let $(a,b),(c,d),(e,f)$ be arbitrary elements of $N \times N$ such that

$$(a,b)\ R(c,d)$$

and $(c,d)\ R(e,f)$

Then, $ad = bc$

and $cf = de$

$$(ad)(cf) = (bc)(de)$$

$$af = be$$

$$(a,b)\ R(e,f)$$

Thus R is transitive.

Hence, R is an equivalence relation.

Now $[(2,6)] = \{(x,y) \in N \times N : (x,y)\,R(2,6)\}$

$$6x = 2y$$

$$3x = y$$

$$[(2,6)] = \{(x,y) \in N \times N : (x,y)\,R(2,6)\}$$

$$= \{(x,3x) : x \in N\}$$

$$= \{(1,3),(2,6),(3,9)...\}$$

79. Show that the function $f : R \rightarrow \{x \in R : -1 < x < 1\}$ defined by $f(x) = \frac{x}{1+|x|}$, $x \in R$ is one-one and onto function.

Sol : Comp 2016, Foreign 2011

We have, $$f(x) = \begin{cases} \dfrac{x}{1+x}, & x > 0 \\ \dfrac{x}{1-x}, & x < 0 \end{cases}$$

Case I : When $x \geq 0$, we have

$$f(x) = \frac{x}{1+x}$$

One-One function :

Let $x_1, x_2 \in R$ such that

$$\begin{aligned} f(x_1) &= f(x_2) \\ \frac{x_1}{1+x_1} &= \frac{x_2}{1+x_2} \\ x_1 + x_1x_2 &= x_2 + x_1x_2 \\ x_1 &= x_2 \end{aligned}$$

Thus $f(x)$ is one-one.

Onto Function :

Let $f(x) = y$, such that $y \in (-1,1)$.

$$\begin{aligned} y &= \frac{x}{1+x} \\ y + xy &= x \\ x - xy &= y \\ x(1-y) &= y \\ x &= \frac{y}{1-y} \end{aligned}$$

Here x is defined at $\forall y \in (-1,1)$.
Thus $f(x)$ is onto.
Case II : When $x < 0$,we have

$$f(x) = \frac{x}{1-x}$$

One-One function :

Let $x_1, x_2 \in R$ such that

$$\begin{aligned} f(x_1) &= f(x_2) \\ \frac{x_1}{1-x_1} &= \frac{x_2}{1-x_2} \\ x_1 - x_2x_1 &= x_2 - x_2x_1 \\ x_1 &= x_2 \end{aligned}$$

Thus $f(x)$ is one-one.

Onto Function :

Let $\quad y = f(x)$

such that $\quad y \in (-1,1)$

$$\begin{aligned} y &= \frac{x}{1-x} \\ y - yx &= x \\ x &= \frac{y}{1+y} \end{aligned}$$

Here x is defined at $\forall y \in (-1,1)$.
Thus $f(x)$ is onto.
Hence, $f(x)$ is one-one and onto function.

80. Show that the function $f: R \to R$ defined by $f(x) = \frac{x}{x^2+1}$, is neither one-one nor onto.

Sol : Foreign 2023; Delhi 2020, 2018

We have $f: R \to R$, defined by

$$f(x) = \frac{x}{x^2+1}, \forall x \in R$$

One-One function :

Let $x_1, x_2 \in R$ such that

$$\begin{aligned} f(x_1) &= f(x_2) \\ \frac{x_1}{x_1^2+1} &= \frac{x_2}{x_2^2+1} \\ x_1x_2^2 + x_1 &= x_2x_1^2 + x_2 \\ x_1x_2^2 - x_2x_1^2 + x_1 - x_2 &= 0 \\ x_1x_2(x_2 - x_1) - 1(x_2 - x_1) &= 0 \\ (x_2 - x_1) - 1(x_1x_2 - 1) &= 0 \\ x_2 &= x_1 \text{ or } x_1x_2 = 1 \end{aligned}$$

Thus $\quad x_1 = x_2$

or $\quad x_1 = \dfrac{1}{x_2}$

Here f is not one-one, as if we take $x_1 = 3$ and $x_2 = \frac{1}{3}$,

$$f(3) = \frac{3}{10} = f\left(\frac{1}{3}\right),$$

but $\quad 3 = \neq \dfrac{1}{3}$

Onto Function :

Now, let $k \in R$ be any arbitrary element and let

$$\begin{aligned} f(x) &= k \\ \frac{x}{x^2+1} &= k \\ kx^2 + k &= x \\ kx^2 - x + k &= 0 \\ x &= \frac{1 \pm \sqrt{1-4k^2}}{2k} \notin R, \end{aligned}$$

If $\quad 1 - 4k^2 < 0$

or $(1-2k)(1+2k) < 0$ *i.e.* $k > \frac{1}{2}$ or $k < -\frac{1}{2}$

Thus f is not onto.

Hence, f is neither one-one nor onto.

81. If $A = R - \{3\}$ and $B = R - \{1\}$. Consider the function $f : A \to B$ defined by $f(x) = \frac{x-2}{x-3}$, for all $x \in A$. Then, show that f is bijective.

Sol : Delhi 2014, 2012

We have a function $f : A \to B$, defined by

$$f(x) = \frac{x-2}{x-3}$$

where $A = R - \{3\}$ and $B = R - \{1\}$,

One-One function :

Let $x_1,\ x_2 \in A$ such that

$$f(x_1) = f(x_2)$$

Then, $\frac{x_1 - 2}{x_1 - 3} = \frac{x_2 - 2}{x_2 - 3}$

$$(x_1 - 2)(x_2 - 3) = (x_2 - 2)(x_1 - 3)$$

$$x_1x_2 - 3x_1 - 2x_2 + 6 = x_1x_2 - 3x_2 - 2x_1 + 6$$

$$-3x_1 - 2x_2 = -3x_2 - 2x_1$$

$$-3(x_1 - x_2) + 2(x_1 - x_2) = 0$$

$$-(x_1 - x_2) = 0$$

$$x_1 - x_2 = 0$$

$$x_1 = x_2$$

Thus for $f(x_1) = f(x_2)$, $x_1 = x_2$, $\forall\, x_1, x_2 \in A$

Therefore $f(x)$ is a one-one function.

Onto Function :

Let $y \in B$ be any arbitrary element.

Then, $f(x) = y$

$$\frac{x-2}{x-3} = y$$

$$x - 2 = xy - 3y$$

$$x - xy = 2 - 3y$$

$$x(1-y) = 2 - 3y$$

$$x = \frac{2-3y}{1-y}$$

or $x = \frac{3y-2}{y-1}$ ...(i)

Clearly x is a real number for all $y \neq 1$.

Now let $\frac{3y-2}{y-1} = 3$

$$3y - 2 = 3y - 3$$

$2 = 3$ which is absurd

Thus $\frac{3y-2}{y-1} \neq 3$

Thus, for each $y \in B$, there exists

$$x = \frac{3y-2}{y-1} \in A$$

Such that $f(x) = f\left(\frac{3y-2}{y-1}\right)$

$$= \frac{\left(\frac{3y-2}{y-1}\right) - 2}{\frac{3y-2}{y-1} - 3}$$

$$= \frac{3y - 2 - 2y + 2}{3y - 2 - 3y + 3} = y$$

Hence, $f(x)$ is an onto function.

Since function $f(x)$ is one-one and onto, therefore, $f(x)$ is a bijective function.

82. If $A = R - \{2\}, B = R - \{1\}$ and $f; A \to B$ is a function define by $f(x) = \frac{x-1}{x-2}$, then show that f is bijective.

Sol : Delhi 2013

We have a function $f : A \to B$ defined by

$$f(x) = \frac{x-1}{x-2}$$

where $A = R - \{3\}$ and $B = R - \{1\}$

One-One function :

Let $x_1,\ x_2 \in A$ such that

$$f(x_1) = f(x_2)$$

Then, $\frac{x_1 - 1}{x_1 - 2} = \frac{x_2 - 1}{x_2 - 2}$

$$(x_1 - 1)(x_2 - 2) = (x_2 - 1)(x_1 - 2)$$

$$x_1x_2 - 2x_1 - x_2 + 2 = x_1x_2 - 2x_2 - x_1 + 2$$

$$-2x_1 - x_2 = -2x_2 - x_1$$

$$-x_1 = -x_2$$

$$-(x_1 - x_2) = 0$$

$$x_1 - x_2 = 0$$

$$x_1 = x_2$$

Thus for $f(x_1) = f(x_2)$, $x_1 = x_2$, $\forall\, x_1, x_2 \in A$

Therefore $f(x)$ is a one-one function.

Onto Function :

Let $y \in B$ be any arbitrary element.

Then, $f(x) = y$

$$\frac{x-1}{x-2} = y$$

$$x - 1 = xy - 2y$$

$$x - xy = 1 - 2y$$

$$x(1-y) = 1 - 2y$$

$$x = \frac{1-2y}{1-y}$$

or $$x = \frac{2y-1}{y-1} \qquad ...(i)$$

Clearly x is a real number for all $y \neq 1$.

Now let $$\frac{2y-1}{y-1} = 2$$
$$2y-1 = 2y-2$$
$$-1 = -2 \qquad \text{which is absurd}$$

Thus $$\frac{2y-1}{y-1} \neq 2$$

Thus, for each $y \in B$, there exists

$$x = \frac{2y-1}{y-1} \in A$$

Such that $$f(x) = f\left(\frac{2y-1}{y-1}\right)$$
$$= \frac{\left(\frac{2y-1}{y-1}\right)-1}{\frac{2y-1}{y-1}-2}$$
$$= \frac{2y-1-y+1}{2y-1-2y+2} = y$$

Hence, $f(x)$ is an onto function.

Since function $f(x)$ is one-one and onto, therefore, $f(x)$ is a bijective function.

83. Show that the function f in $A = R - \{\frac{2}{3}\}$ defined as $f(x) = \frac{4x+3}{6x-4}$, is one-one and onto.

Sol : Delhi 2013

We have $$f(x) = \frac{4x+3}{6x-4}$$

where, $$x \in A = R - \left\{\frac{2}{3}\right\}$$

One-One function :

Let $x_1, x_2 \in A$ such

$$f(x_1) = f(x_2)$$

Then, $$\frac{4x_1+3}{6x_1-4} = \frac{4x_2+3}{6x_2-4}$$
$$(4x_1+3)(6x_2-4) = (4x_2+3)(6x_1-4)$$
$$24x_1x_2 - 16x_1 + 18x_2 - 12 = 24x_1x_2 - 16x_2 + 18x_1 - 12$$
$$-34x_1 = -34x_2$$
$$x_1 = x_2$$

So, f is one-one function.

Onto Function :

Let y be an arbitrary element of A (co-domain).

Then $$f(x) = y$$

$$\frac{4x+3}{6x-4} = y$$
$$4x+3 = 6xy - 4y$$
$$4x - 6xy = -4y - 3$$
$$x(4-6y) = -(4y+3)$$
$$x = \frac{-(4y+3)}{4-6y}$$
$$x = \frac{4y+3}{6y-4}$$

Clearly, $x = \frac{4y+3}{6y-4}$ is a real number for all

Now let $y = \frac{2}{3}$, then we have

$$\frac{4y+3}{6y-4} = \frac{2}{3}$$
$$12y+9 = 12y-8$$
$$9 = -8 \qquad \text{which is absurd.}$$

Thus $y \neq \frac{2}{3}$.

Thus, for each $y \in A$ (co-domain) there exist $x = \frac{4y+3}{6y-4} \in A$ such that

$$f(x) = f\left(\frac{4y+3}{6y-4}\right)$$
$$= \frac{4\left(\frac{4y+3}{6y-4}\right)+3}{6\left(\frac{4y+3}{6y-4}\right)-4}$$
$$= \frac{16y+12+18y-12}{24y+18-24+16}$$
$$= \frac{34y}{34}$$
$$= y$$

Hence, f is onto function.

84. Show that the function $f: R \to R$ defined by $f(x) = \frac{x}{x^2+1}, \forall x \in R$ is neither one-one nor onto.

Sol : OD 2018

We have, a function $f: R \to R$ defined as

$$f(x) = \frac{x}{x^2+1} \forall x \in R$$

One-One function :

Let $x_1,\ x_2 \in R$ such that

$$f(x_1) = f(x_2)$$
$$\frac{x_1}{x_1^2+1} = \frac{x_2}{x_2^2+1}$$
$$x_1(x_2^2+1) = x_2(x_1^2+1) \qquad [1]$$
$$x_1x_2^2 + x_1 = x_2x_1^2 + x_2$$
$$x_1x_2(x_2 - x_1) = (x_2 - x_1)$$

$$(x_2 - x_1)(x_1 - x_2 - 1) = 0$$

$$x_2 = x_1 \text{ or } x_1 x_2 = 1$$

$$x_1 = x_2 \text{ or } x_1 = \frac{1}{x_2}$$

If we take $x_1 = 2$ and $x_2 = \frac{1}{2}$, we get

$$f(2) = \frac{2}{4+1} = \frac{2}{5}$$

and $$f\left(\frac{1}{2}\right) = \frac{\frac{1}{2}}{\frac{1}{4}+1} = \frac{2}{5}$$

Here $f(2) = f\left(\frac{1}{2}\right)$ but $2 \neq \frac{1}{2}$

Thus f is not one-one.

Onto function :

Let $y \in R$ (co-domain) be any arbitrary element.

Consider, $$y = f(x)$$

$$y = \frac{x}{x^2+1}$$

$$x^2 y + y = x$$

$$x^2 y - x + y = 0$$

$$x = \frac{1 \pm \sqrt{1-4y^2}}{2y}$$

which does not exist for $1 - 4y^2 < 0$, i.e. for $y > \frac{1}{2}$ and $y < -\frac{1}{2}$.

In particular for $y = 1 \in R$ (co-domain), there does not exist any $x \in R$ (domain) such that

$$f(x) = y$$

Therefore f is not onto.

Hence, f is neither one-one nor onto.

85. Show that the function $f : R_+ \to [-5, \infty)$ given by $f(x) = 9x^2 + 6x - 5$ is one - one and onto function. where R_+ is the set of all non-negative real numbers.

Sol : Delhi 2017, Foreign 2010

Function $f : R_+ \to [-5, \infty)$ is given as

$$f(x) = 9x^2 + 6x - 5$$

One-One function :

Let $x_1, x_2 \in R_+$ such that

$$f(x_1) = f(x_2)$$

Then, $$9x_1^2 + 6x_1 - 5 = 9x_2^2 + 6x_2 - 5$$

$$9(x_1^2 - x_2^2) + 6(x_1 - x_2) = 0$$

$$3(x_1 + x_2)(x_1 - x_2) + 2(x_1 - x_2) = 0$$

$$(x_1 - x_2)(3x_1 + 3x_2 + 2) = 0$$

Since $x_1, x_2 \in R_+$, then

$$(3x_1 + 3x_2 + 2) \neq 0$$

Thus $$x_1 - x_2 = 0$$

$$x_1 = x_2, \forall x_1, x_2 \in R_+$$

Therefore, $f(x)$ is one-one function.

Onto function :

Let y be any arbitrary element of $[-5, \infty)$.

Then, $$y = f(x)$$

$$y = 9x^2 + 6x - 5$$

$$y = (3x+1)^2 - 1 - 5$$

$$y = (3x+1)^2 - 6$$

$$y + 6 = (3x+1)^2$$

$$(3x+1)^2 = y + 6$$

$$3x + 1 = \sqrt{y+6}$$

As $y \geq -5$, $\quad y + 6 \geq 0$

$$x = \frac{\sqrt{y+6} - 1}{3}$$

Therefore f is onto and range of f is $[-5, \infty)$.

86. Consider $f : R - \{-\frac{4}{3}\} \to R - \{\frac{4}{3}\}$ given by

$$f(x) = \frac{4x+3}{3x+4}$$

Show that f is bijective.

Sol : Comp 2017

Function $f : R - \{-\frac{4}{3}\} \to R - \{\frac{4}{3}\}$ is defined as

$$f(x) = \frac{4x+3}{3x+4}$$

One-One function :

Let, x_1, $x_2 \in R - \{-\frac{4}{3}\}$ such that

$$f(x_1) = f(x_2)$$

$$\frac{4x_1+3}{3x_1+4} = \frac{4x_2+3}{3x_2+4}$$

$$(4x_1+3)(3x_2+4) = (3x_1+4)(4x_2+3)$$

$$12x_1x_2 + 16x_1 + 9x_2 + 12 = 12x_1x_2 + 9x_1 + 16x_2 + 12$$

$$7x_1 = 7x_2$$

Thus for $f(x_1) = f(x_2)$,

$$x_1 = x_2,$$

Therefore f is one-one function.

Onto function :

Let, $y \in R - \{\frac{4}{3}\}$, then $y \neq \frac{4}{3}$

The function f is onto if there exist $x \in R - \{-\frac{4}{3}\}$, such that $f(x) = y$.

Now,
$$f(x) = y$$
$$\frac{4x+3}{3x+4} = y$$
$$4x+3 = y(3x+4)$$
$$4x+3 = 3xy+4y$$
$$4x-3xy = 4y-3$$
$$x(4-3y) = 4y-3$$
$$x = \frac{4y-3}{4-3y} \in R - \left\{\frac{-4}{3}\right\} \qquad \left(y \neq \frac{4}{3}\right)$$

Thus, for any $y \in R - \{\frac{4}{3}\}$, there exist
$$\frac{4y-3}{4-3y} \in R - \left\{-\frac{4}{3}\right\}$$

Thus f is onto function.

Therefore, $f(x)$ is a bijective function.

87. Let $f: N \to N$ be a function defined as $f(x) = 9x^2 + 6x - 5$ Show that $f: N \to S$, where S is the range of f, is invertible. Find the inverse of f and hence find $f^{-1}(43)$ and $f^{-1}(163)$

Sol : Delhi 2016

Function $f: N \to N$ is defined as
$$f(x) = 9x^2 + 6x - 5$$

One-One function :

Let $x_1,\ x_2 \in N$, such that
$$f(x_1) = f(x_2)$$

Then,
$$9x_1^2 + 6x_1 - 5 = 9x_2^2 + 6x_2 - 5$$
$$9x_1^2 + 6x_1 = 9x_2^2 + 6x_2$$
$$9(x_1^2 - x_2^2) + 6(x_1 - x_2) = 0$$
$$3(x_1 - x_2)(x_1 + x_2) + 2(x_1 - x_2) = 0$$
$$(x_1 - x_2)(3x_1 + 3x_2 + 2) = 0$$

Since $x_1, x_2 \in N$, then
$$(3x_1 + 3x_2 + 2) \neq 0$$

Thus
$$x_1 - x_2 = 0$$
$$x_1 = x_2$$

Therefore, f is one-one function.

Onto function :

Obviously, $f: N \to S$ is an onto function, because S is the range of f.

Thus, $f: N \to S$ is one-one and onto function.

88. Consider $f: R^+ \to [-9, \infty)$ given by $f(x) = 5x^2 + 6x - 9$ Prove that f is one - one and onto function

Sol : Comp 2015, OD 2010

Function $f: R^+ \to [-9, \infty)$ is given as
$$f(x) = 5x^2 + 6x - 9$$

One-One function :

Let $x_1,\ x_2 \in R_+$ such that
$$f(x_1) = f(x_2)$$

Then,
$$5x_1^2 + 6x_1 - 9 = 5x_2^2 + 6x_2 - 9$$
$$5x_1^2 + 6x_1 = 5x_2^2 + 6x_2$$
$$5(x_1^2 - x_2^2) + 6(x_1 - x_2) = 0$$
$$(x_1 - x_2)(5x_1 + 5x_2 + 6) = 0$$

Since $x_1, x_2 \in R_+$, then
$$(5x_1 + 5x_2 + 6) \neq 0$$

Thus
$$x_1 - x_2 = 0$$
$$x_1 = x_2, \forall x_1, x_2 \in R_+$$

Therefore, $f(x)$ is one-one function.

Onto function :

Let y be any arbitrary element of $[-9, \infty)$.

Then,
$$y = f(x)$$
$$y = 5x^2 + 6x - 9$$
$$0 = 5x^2 + 6x - 9 - y$$
$$5x^2 + 6x - (9 + y) = 0$$
$$x = \frac{-6 \pm \sqrt{36 + 4 \times 5(9 + y)}}{10}$$
$$= \frac{-6 \pm \sqrt{216 + 20y}}{10}$$
$$= \frac{-3 \pm \sqrt{54 + 5y}}{5}$$
$$= \pm\frac{\sqrt{54 + 5y} - 3}{5}$$
$$= \frac{\sqrt{54 + 5y} - 3}{5} \qquad x \in R_+$$

Obviously $x \in R_+$ for all $y \in [-9,\ \infty)$.

Therefore f is onto function. Hence, f is one - one and onto function.

If $f: W \to W$ is defined as $f(x) = x - 1$, if x is odd and $f(x) = x + 1$, if x is even. Show that f is bijective.

Sol : Foreign 2014, AI 2011

Function $f: W \to W$ is defined as
$$f(x) = \begin{cases} x - 1, \text{if } x \text{ is odd} \\ x + 1, \text{if } x \text{ is even} \end{cases}$$

One-One function :

Let $x_1,\ x_2 \in W$ be any two numbers such that

$$f(x_1) = f(x_2)$$

Case I : When x_1 and x_2 are odd.

In this case $f(x_1) = f(x_2)$

$$x_1 - 1 = x_2 - 1$$

$$x_1 = x_2$$

Case II : When x_1 and x_2 are even.

In this case $f(x_1) = f(x_2)$

$$x_1 + 1 = x_2 + 1$$

$$x_1 = x_2$$

Thus, in both cases,

$$f(x_1) = f(x_2)$$

$$x_1 = x_2$$

Case III : When x_1 is odd and x_2 is even.

In this case $x_1 \neq x_2$

Also, $f(x_1)$ is even and $f(x_2)$ is odd.

So, $f(x_1) \neq f(x_2)$

Thus, $x_1 \neq x_2$

$$f(x_1) \neq f(x_2)$$

Case IV : When x_1 is even and x_2 is odd.

In this case $x_1 \neq x_2$

Also, $f(x_1)$ is odd and $f(x_2)$ is even.

So, $f(x_1) \neq f(x_2)$

Thus, $x_1 \neq x_2$

$$f(x_1) \neq f(x_2)$$

Hence, from case I, II, III and IV we can see that, $f(x)$ is a one-one function.

Onto function :

Clearly, any odd number $2y+1$ in the co-domain W, is the image of $2y$ in the domain W.

Also, any even number $2y$ in the co-domain W, is the image of $2y+1$ in the domain W.

Thus, every element in W (domain). So, f is onto.

Therefore, f is bijective.

89. If R is a relation defined on the set of natural numbers N as follows:

$R = \{(x,y): x \in N, y \in N$ and $2x + y = 24\}$, then find the domain and range of the relation R. Also, find whether R is an equivalence relation or not.

Sol : Comp 2014

We have $R = \{(x,y): x \in N, y \in N$ and $2x + y = 24\}$

Now $2x + y = 24$

$$y = 24 - 2x$$

If $x = 1$, then $y = 22$

If $x = 2$ then $y = 20$

If $x = 3$, then $y = 18$

If $x = 4$ then $y = 16$

If $x = 5$, then $y = 14$

If $x = 6$, then $y = 12$

If $x = 7$, then $y = 10$

If $x = 8$, then $y = 8$

If $x = 9$, then $y = 6$

If $x = 10$, then $y = 4$

If $x = 11$, then $y = 2$

If $x = 12$, then $y = 0 \notin N$

If $x = 13$, then $y = -2 \notin N$

If $x > 11$, $y \notin N$

So, domain of R is $\{1, 2, 3, ..., 11\}$

Range of R is $\{2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22\}$ and

$$R = \{(1,22), (2,20), (3,18), (4,16), (5,14), (6,12)$$
$$(7,10), (8,8), (9,6), (10,4), (11,\ 2)\}$$

Reflexive :

Since, for $1 \in$ domain of R, $(1,1) \notin R$, thus R is not reflexive.

Symmetric :

Since $(1,\ 22) \in R$ but $(22,\ 1) \notin R$, thus R is not symmetric.

Thus, R is neither reflexive nor symmetric. So, R is not an equivalence relation. We do not need to check transitivity.

90. If $A = \{1, 2, 3, ..., 9\}$ and R is the relation in $A \times A$ defined by $(a,b)\,R\,(c,d)$, if $a + d = b + c$ for (a,b), (c,d) in $A \times A$. Prove that R is an equivalence relation. Also, obtain the equivalence classes $[(2,5)]$.

Sol : Delhi 2014

Relation R in $A \times A$ is defined as

$$(a,b)\,R\,(c,d),\ \text{if}\ a + d = b + c.$$

where $A = \{1, 2, 3, ..., 9\}$,

Reflexive :

Let (a,b) be any arbitrary element of $A\times A$ i.e. $(a,b)\in A\times A$, where $a,\ b\in A$.

Now $\quad a+b = b+a \qquad$ [addition is commutative]

Hence $(a,b)\,R\,(a,b)$, and R is reflexive.

Symmetric :

Let $(a,b),(c,d)\in A\times A$, such that $(a,b)\,R\,(c,d)$. Then,

$$a+d = b+c$$

$$b+c = a+d$$

$$c+b = d+a \qquad \text{[addition is commutative]}$$

Thus $(c,d)\,R\,(a,b)$ and hence R is symmetric.

Transitive :

Let $(a,b),\ (c,d),\ (e,f)\in A\times A$ such that $(a,b)\,R\,(c,d)$ and $(c,d)\,R\,(e,f)$

Then, $\qquad a+d = b+c$

and $\qquad c+f = d+e$

Adding the above equations, we get

$$a+d+c+f = b+c+d+e$$

$$a+f = b+e\ \ (a,b)\,R\,(e,f)$$

Thus R is transitive.

Since R is reflexive, symmetric and transitive, hence, R is an equivalence relation.

Now, for $[(2,5)]$, we will find $(c,d)\in A\times A$ such that $2+d=5+c$ or $d-c=3$

Clearly, $\qquad (2,5)\,R\,(1,4)\ as\ 4-1 = 3$

$$(2,5)\,R\,(2,5)\text{ as }5-2 = 3$$

$$(2,5)\,R\,(3,6)\text{ as }6-3 = 3$$

$$(2,5)\,R\,(4,7)\text{ as }7-4 = 3$$

$$(2,5)\,R\,(5,8)\text{ as }8-5 = 3$$

and $\qquad (2,5)\,R\,(6,9)\text{ as }9-6 = 3$

Hence, equivalence class $[(2,\ 5)]$

$$=\{(1,4),(2,5),(3,6),(4,7),(5,8),(6,9)\}.$$

91. Show that $f:N\to N$, given by

$$f(x)=\begin{cases}x+1 & \text{if } x \text{ is odd}\\ x-1 & \text{if } x \text{ is even}\end{cases}$$

is bijective (both one-one and onto).

Sol : Comp 2019, Delhi 2017

Function $f:N\to N$ is defined as

$$f(x) = \begin{cases}x+1, \text{if } x \text{ is odd}\\ x-1, \text{if } x \text{ is even}\end{cases}$$

One-One function :

Let $x_1,\ x_2\in N$ be any two numbers such that

$$f(x_1) = f(x_2)$$

Case I : When x_1 and x_2 are odd.

In this case $\qquad f(x_1) = f(x_2)$

$$x_1+1 = x_2+1$$

$$x_1 = x_2$$

Case II : When x_1 and x_2 are even.

In this case $\qquad f(x_1) = f(x_2)$

$$x_1-1 = x_2-1$$

$$x_1 = x_2$$

Thus, in both cases,

$$f(x_1) = f(x_2)$$

$$x_1 = x_2$$

Case III : When x_1 is odd and x_2 is even.

In this case $\qquad x_1 \neq x_2$

Also, $f(x_1)$ is even and $f(x_2)$ is odd.

So, $\qquad f(x_1) \neq f(x_2)$

Thus, $\qquad x_1 \neq x_2$

$$f(x_1) \neq f(x_2)$$

Case IV : When x_1 is even and x_2 is odd.

In this case $\qquad x_1 \neq x_2$

Also, $f(x_1)$ is odd and $f(x_2)$ is even.

So, $\qquad f(x_1) \neq f(x_2)$

Thus, $\qquad x_1 \neq x_2$

$$f(x_1) \neq f(x_2)$$

Hence, from case I, II, III and IV we can see that, $f(x)$ is a one-one function.

Onto Function :

Let $y\in N$ (co-domain) be any arbitrary number.

If y is odd, then there exists an even number $y+1\in N$ (domain) such that.

$$f(y+1) = (y+1)+1$$

$$= y$$

If y is even, then there exists an odd number $y-1\in N$ (domain) such that.

$$f(y-1) = (y-1)+1$$

$$= y$$

Thus, every element in N (co-domain) has a pre-image in N (domain)

Therefore, $f(x)$ is onto function.

Hence, the function $f(x)$ is bijective.

92. If $f:R \to R$ is the function defined by $f(x) = 4x^3 + 7$, then show that f is a bijective.

Sol : Comp 2011

We have function $f:R \to R$ defined as

$$f(x) = 4x^3 + 7$$

One-One function :

Let $x_1, x_2 \in R$ such that

$$f(x_1) = f(x_2)$$

$$4x_1^3 + 7 = 4x_2^3 + 7$$

$$4x_1^3 = 4x_2^3$$

$$x_1^3 - x_2^3 = 0$$

Using $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$ we obtain

$$(x_1 - x_2)(x_1^2 + x_1 x_2 + x_2^2) = 0$$

$$x_1 - x_2 = 0$$

$$x_1 = x_2$$

Thus for $f(x_1) = f(x_2)$,

$$x_1 = x_2, \ \forall x_1, x_2 \in R$$

Therefore, $f(x)$ is a one-one function.

Onto Function :

Let $y \in R$ (co-domain)be any arbitrary number.

Then, $$f(x) = y$$

$$4x^3 + 7 = y$$

$$4x^3 = y - 7$$

$$x^3 = \left(\frac{y-7}{4}\right)$$

Thus $$x = \left(\frac{y-7}{4}\right)^{\frac{1}{3}} \in R \text{ such that}$$

$$f(x) = f\left[\left(\frac{y-7}{4}\right)^{\frac{1}{3}}\right]$$

$$= 4\left[\left(\frac{y-7}{4}\right)^{\frac{1}{3}}\right]^3 + 7$$

$$= 4\left(\frac{y-7}{4}\right) + 7$$

$$= y - 7 + 7 = y$$

Thus $f(x)$ is an onto function.

Since, $f(x)$ is both one-one and onto, so it is a bijective.

93. If Z is the set of all integers and R is the relation on Z defined as $R = \{(a,b) : a, b \in Z \text{ and } a - b \text{ is divisible by } 5\}$. Prove that R is an equivalence relation.

Sol : Delhi 2010

We have

$$R = \{a,b\} : a, b \in Z \text{ and } a - b \text{ is divisible by } 5\}.$$

Reflexive :

As for any $x \in Z$ we have $x - x = 0$, which is divisible by 5.

Thus $(x - x)$ is divisible by 5.

Thus $(x,x) \in R$, $\forall x \in Z$ and R is reflexive.

Symmetric :

Let $(x,y) \in R$, where x, $y \in Z$

By definition of R, $(x-y)$ is divisible by 5.

$$x - y = 5A \text{ for some } A \in Z.$$

$$y - x = 5(-A)$$

Thus $(y-x)$ is divisible by 5 where $(y,x) \in R$

Therefore, R is symmetric.

Transitive :

Let $(x,y) \in R$, where x, $y \in Z$

Since $(x-y)$ is divisible by 5, we have

$$x - y = 5A \text{ for some } A \in Z$$

Again, let $(y,z) \in R$, where x, $y \in Z$

Since $(x-y)$ is divisible by 5,

$$y - z = 5B \text{ for some } B \in Z.$$

Now,$(x-y) + (y-z) = 5A + 5B$

$$x - z = 5(A+B)$$

Thus $(x-z)$ is divisible by 5 for some $(A+B) \in Z$.

Thus $(x,z) \in R$

Therefore, R is transitive.

Since R is reflexive, symmetric and transitive, it is an equivalence relation.

94. Show that the relation S defined on set $N \times N$ by $(a,b) S (c,d) \Rightarrow a + d = b + c$ is an equivalence relation.

Sol : Comp 2010, Delhi 2007

Relation S in $N \times N$ is defined as

$$(a,b) S (c,d), \text{ if } a + d = b + c$$

Reflexive :

Let (a,b) be any arbitrary element of $N \times N$ i.e. $(a,b) \in N \times N$, where a, $b \in N$.

Now $a + b = b + a$ [addition is commutative]

Hence $(a,b) S (a,b)$, and S is reflexive.

Symmetric :

Let $(a,b),(c,d) \in N \times N$, such that $(a,b)S(c,d)$. Then,

$$a+d = b+c$$

$$b+c = a+d$$

$$c+b = d+a \qquad \text{[addition is commutative]}$$

Thus $(c,d)S(a,b)$ and hence S is symmetric.

Transitive :

Let (a,b), (c,d), $(e,f) \in A \times A$ such that $(a,b)S(c,d)$ and $(c,d)R(e,f)$

Then, $\quad a+d = b+c$

and $\quad c+f = d+e$

Adding the above equations, we get

$$a+d+c+f = b+c+d+e$$

$$a+f = b+e \ (a,b)S(e,f)$$

Thus S is transitive.

Since S is reflexive, symmetric and transitive, hence, S is an equivalence relation.

95. If $f:X \to Y$ is a function, define a relation R on X given by $R = \{(a,b): f(a) = f(b)\}$. Show that R is an equivalence relation on X.

Sol : Comp 2010

The given function is $f: X \to Y$ and relation on X is $R = \{(a,b): f(a) = f(b)\}$

Reflexive :

Since, for every $x \in X$, we have

$$f(x) = f(x) \ (x,x) \in R, \forall x \in X$$

Therefore, R is reflexive.

Symmetric :

Let $(x,y) \in R$, then we have

$$f(x) = f(y)$$

$$f(y) = f(x) \ (x,y) \in R$$

Thus, $(x,y) \in R \ (y,x) \in R, \forall x, y \in X$

Therefore, R is symmetric.

Transitive :

Let $x, y, z \in X$ such that

$$(x,y) \in R \text{ and } (y,z) \in R$$

Then, $\quad f(x) = f(y) \qquad ...(i)$

and $\quad f(y) = f(z) \qquad ...(ii)$

From Eqs. (i) and (ii), we get

$$f(x) = f(z) \ (x,z) \in R$$

Thus, $(x,y) \in R$ and $(y,z) \in R$

Thus $(x,z) \in R$, $\forall x, y, z \in X$, hence R is transitive.

Since, R is reflexive, symmetric and transitive, so it is an equivalence relation.

96. Show that a function $f:R \to R$ given by $f(x) = ax+b, a, b \in R \, a \neq 0$ is a bijective.

Sol : Comp 2008

We have function $f:R \to R$ defined as

$$f(x) = ax+b;\ b \in R,\ a \neq 0.$$

One-One function :

Let $x_1,\ x_2 \in R$ such that

$$f(x_1) = f(x_2)$$

Then, $\quad ax_1+b = ax_2+b$

$$ax_1 = ax_2$$

$$x_1 = x_2 \qquad [a \neq 0]$$

Thus for $f(x_1) = f(x_2)$ we get

$$x_1 = x_2,\ \forall x_1, x_2 \in R$$

Therefore, $f(x)$ is a one-one function.

Onto function :

Let $y \in R$ (co-domain) be any arbitrary element.

Then, $\quad f(x) = y$

$$ax+b = y$$

$$x = \frac{y-b}{a} \qquad [y \in R\ ]$$

Clearly, x is a real number.

Thus, for each $y \in R$ (co-domain), there exists

$$x = \frac{y-b}{a} \in R \text{ (domain) such that}$$

$$\begin{aligned} f(x) &= f\left(\frac{y-b}{a}\right) \\ &= a\left(\frac{y-b}{a}\right)+b \\ &= y-b+b = y \end{aligned}$$

Therefore, $f(x)$ is an onto function.

As $f(x)$ is both one-one and onto, so it is a bijective function.

97. Let $A = \{x \in Z: 0 \leq x \leq 12\}$. Show that $R = \{(a,b): a, b \in A,\ |a-b| \text{ is divisible by } 4\}$ is an equivalence relation. Find the set of all elements related to 1. Also , write the equivalence class.

Sol : OD 2018, Delhi 2015

We have

$R = \{(a,b): |a-b| \text{ is divisible by 4 and } a,\ b \in A\}$

and $A = \{x : x \in Z \text{ and } 0 \leq x \leq 12\}$.

Now, A can be written as

$$A = \{0, 1, 2, 3, ..., 12\}$$

Reflexive :

For any $x \in A$, we get $|x - x| = 0$, which is divisible by 4. Thus $(x, x) \in R$, $\forall x \in A$

Therefore, R is reflexive.

Symmetric :

Using definition of given relation, for any $(x, y) \in R$, we get $|x - y|$ is divisible by 4.

$$|x - y| = 4\lambda, \text{ for some } \lambda \in Z$$

$$|y - x| = 4\lambda, \text{ for some } \lambda \in Z, (y, x) \in R$$

Thus, $(x, y) \in R$ $(y, x) \in R$, $\forall x, y \in A$

Therefore, R is symmetric.

Transitive :

Using definition of given relation, for any $(x, y) \in R$ and $(y, z) \in R$, we get $|x - y|$ is divisible by 4 and $|y - z|$ is divisible by 4.

$$|x - y| = 4\lambda \qquad \text{For some } \lambda \in Z.$$

and $$|y - z| = 4\mu \qquad \text{For some } \mu \in Z.$$

Now, $$x - z = (x - y) + (y - z)$$

$$= \pm 4\lambda \pm 4\mu$$

$$= \pm 4(\lambda + \mu)$$

Thus $|x - z|$ is divisible by 4 and $(x, z) \in R$.

So we get $(x, y) \in R$, $(y, z) \in R$ and $(x, z) \in R$, $\forall x, y, z \in A$.

Therefore, R is transitive.

Since, R is reflexive, symmetric and transitive, so it is an equivalence relation.

Set of all elements related to $\{1\}$ is $\{1, 5, 9\}$.

Thus set of all elements related to $\{2\}$ is

$$= \{a \in A : |2 - a| \text{ is divisible by } 4\}$$

$$= \{2, 6, 10\}$$

98. Show that the relation R on the set Z of all integers defined by $(x, y) \in R \Leftrightarrow (x - y)$ is divisible by 3 is an equivalence relation.

Sol : Comp 2018

We have

$$R = \{a, b\} : a, b \in Z \text{ and } a - b \text{ is divisible by } 3\}.$$

Reflexive :

As for any $x \in Z$ we have $x - x = 0$, which is divisible by 3.

Thus $(x - x)$ is divisible by 3.

Thus $(x, x) \in R$, $\forall x \in Z$ and R is reflexive.

Symmetric :

Let $(x, y) \in R$, where x, $y \in Z$

By definition of R, $(x - y)$ is divisible by 3.

$$x - y = 3A \text{ for some } A \in Z.$$

$$y - x = 3(-A)$$

Thus $(y - x)$ is divisible by 3 where $(y, x) \in R$

Therefore, R is symmetric.

Transitive :

Let $(x, y) \in R$, where x, $y \in Z$

Since $(x - y)$ is divisible by 3, we have

$$x - y = 3A \text{ for some } A \in Z$$

Again, let $(y, z) \in R$, where x, $y \in z$

Since $(x - y)$ is divisible by 3,

$$y - z = 3B \text{ for some } B \in Z.$$

Now, $(x - y) + (y - z) = 3A + 3B$

$$x - z = 3(A + B)$$

Thus $(x - z)$ is divisible by 3 for some $(A + B) \in Z$.

Thus $(x, z) \in R$

Therefore, R is transitive.

Since R is reflexive, symmetric and transitive, it is an equivalence relation.

99. Show that the relation R in the set $A = \{1, 2, 3, 4, 5\}$ given by $R = \{(a, b) : |a - b| \text{ is divisible by } 2\}$ is an equivalence relation. Write all the equivalence classes of R.

Sol : Comp 2015, Delhi 2008

Relation

$$R = \{(a, b) : |a - b| \text{ is divisible by 2 and } a, b \in A\}$$

and A $A = \{1, 2, 3, 4, 5\}$

Reflexive :

For any $x \in A$, we get $|x - x| = 0$, which is divisible by 4. Thus $(x, x) \in S$, $\forall x \in A$

Therefore, S is reflexive.

Symmetric :

Using definition of given relation, for any $(x, y) \in R$, we get $|x - y|$ is divisible by 2.

$$|x - y| = 2\lambda, \text{ for some } \lambda \in Z$$

$$|y - x| = 2\lambda, \text{ for some } \lambda \in Z\,(y, x) \in R$$

Thus, $(x, y) \in R$ $(y, x) \in R$, $\forall x, y \in A$

Therefore, R is symmetric.

Transitive :

Using definition of given relation, for any $(x, y) \in R$ and $(y, z) \in R$, we get $|x-y|$ is divisible by 2 and $|y-z|$ is divisible by 2.

$$|x-y| = 2\lambda \qquad \text{For some } \lambda \in Z.$$

and
$$|y-z| = 2\mu \qquad \text{For some } \mu \in Z.$$

Now,
$$\begin{aligned} x-z &= (x-y)+(y-z) \\ &= \pm 2\lambda \pm 2\mu \\ &= \pm 2(\lambda+\mu) \end{aligned}$$

Thus $|x-z|$ is divisible by 2 and $(x, z) \in R$.

So we get $(x, y) \in R$, $(y, z) \in R$ and $(x, z) \in R$, $\forall x, y, z \in A$.

Therefore, R is transitive.

Since, R is reflexive, symmetric and transitive, so it is an equivalence relation.

Clearly
$$\begin{aligned} [1] &= \{1, 3, 5\} \\ [2] &= \{2, 4\} \\ [3] &= \{1, 3, 5\} \\ [4] &= \{2, 4\} \end{aligned}$$

and
$$[5] = \{1, 3, 5\}$$

Thus
$$[1] = [3] = [5] = \{1, 3, 5\}$$
$$[2] = [3] = \{2, 4\}$$

CASE BASED QUESTIONS

100. Students of a school are taken to a railway museum to learn about railways heritage and its history.

An exhibit in the museum depicted many rail lines on the track near the railway station. Let L be the set of all rail lines on the railway track and R be the relation on L defined by

$R = \{(l_1, l_2) : l_1 \text{ is parallel to } l_2\}$

On the basis of the above information, answer the following questions :

(i) Find whether the relation R is symmetric or not.

(ii) Find whether the relation R is transitive or not.

(iii) If one of the rail lines on the railway track is represented by the equation $y = 3x+2$, then find the set of rail lines in R related to it.

(iv) Let S be the relation defined by $S = \{(l_1, l_2) : l_1$ is perpendicular to $l_2\}$ check whether the relation S is symmetric and transitive.

Sol : OD 2024

We have
$$R = \{(l_1, l_2) : l_1 \text{ is parallel to } l_2\}$$

(i) If l_1 is parallel to l_2, then l_2 is parallel to l_1. Thus if $(l_1, l_2) \in R$ then $(l_2, l_1) \in R$. Thus R is symmetric.

(ii) If l_1 is parallel to l_2 and l_2 is parallel to l_3, then l_1 is parallel to l_3. So, if $(l_1, l_2) \in R$ and $(l_2, l_3) \in R$ then $(l_1, l_3) \in R$. Thus R is transitive.

(iii)
$$R = \{(l_1, l_2) : l_1 \text{ is parallel to } l_2\}$$

Set of all lines related to $y = 3x+2$ is set of all lines that are parallel to $y = 3x+2$. Let equation of line parallel to $y = 3x+2$ be $y = mx+c$ where m is slope of line.

Since $y = 3x+2$ and $y = mx+c$ are parallel, slope of both the lines will be equal. Thus $m = 3$. Hence required line is $y = 3x+c$ where $c \in R$.

(iv)
$$S = \{(l_1, l_2) : l_1 \text{ is perpendicular to } l_2\}$$

If l_1 is perpendicular to l_2, then l_2 is perpendicular to l_1. So $(l_1, l_2) \in S$ then $(l_2, l_1) \in S$. Thus S is symmetric.

If l_1 is perpendicular to l_2 and l_2 is perpendicular to l_3, then l_1 is not perpendicular to l_3. It is parallel to l_3. So, if $(l_1, l_2) \in S$, and $(l_2, l_3) \in S$ then $(l_1, l_3) \notin S$. Thus S is not transitive.

101. A function $f : [-4, 4] \to [0, 4]$ is given by $f(x) = \sqrt{16-x^2}$. Show that f is an onto function but not a one-one function. Further, find all possible values of a for which $f(a) = \sqrt{7}$.

Sol : OD 2023

Let $y \in [\,0, 4\,]$ such that

$$\begin{aligned} f(x) &= y \\ 0 &\le \sqrt{16-x^2} \le 4 \\ 0 &\le 16-x^2 \le 16 \\ 0 &\le x^2 \le 16 \\ -4 &\le x \le 4 \end{aligned}$$

which is the domain of the function i.e. the function f is onto.

Let $\quad f(x) = 1 \in [\,0, 4\,]$

$$\sqrt{16 - x^2} = 1$$

$$x^2 = 15$$

$$x = \pm\sqrt{15}$$

As an element in the co-domain has two pre-images in its domain, hence the function f is not one-one function.

Given, $\quad f(a) = \sqrt{7}$

$$\sqrt{16 - a^2} = \sqrt{7}$$

$$16 - a^2 = 7$$

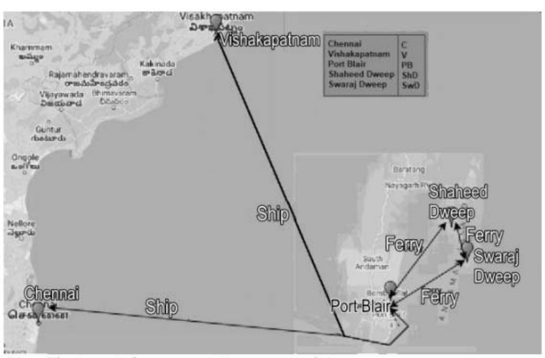
$$a^2 = 9$$

$$a = \pm 3$$

102. Port Blair, the capital city of Andaman and Nicobar Islands is directly connected to Chennai and Vishakapatnam via ship route. The ships sailfrom Chennai/Vishakapatnam to Port Blair and vice versa.

Swaraj Dweep and Shaheed Dweep are two popular tourist tourist islands in Andaman Islands. One has to take a ferry from Port Blair to reach these islands. There are ferries that sail frequently between the three islands - Port Blair (PB), Swaraj Dweep (SwD) and Shaheed Dweep (ShD).

Shown below is a schematic representation of the ship routes and ferry routes.



(Note: The image is for representation purpose only.)

X is the set of all 5 places and Y is the set of 3 places in Andaman Islands. That is ,

$$X = \{C, V, PB, SwD\}$$

and $\quad Y = \{PB, SwD, ShD\}$

A relation R defined on the set X which is given by, $R\,R = \{(x_1, x_2)\}$: there is a direct ship or direct ferry from x_1 to $x_2\}$.

A function $f : Y \to X$ is defined by,

$$f(PB) = V,$$

$$f(SwD) = PB,$$

$$f(ShD) = SwD.$$

Answer the questions based on the given information.

(i) List all the elements of R.

(ii) Is the relation R symmetric? Give a valid reason.

(iii) Is the relation R transitive? Give a valid reason.

(iv) Check whether the function f is one-one and onto. Give valid reasons.

Sol :

(i) List all the elements of R

List all the elements of R is given below

$R = \{(C, PB), (PB, C), (V, PB), (PB, V), (PB, SwD),$
$(SwD, PB), (PB, ShD), (ShD, PB), (SwD, ShD),$
$(ShD, SwD)\}$

(ii) Is the relation R symmetric?

Yes. For example, for every $(x_1, x_2) \in R, (x_2, x_1) \in R$ as every direct ship/direct ferry runs in both the directions.

(iii) Is the relation R transitive? No. For example, $(C, PB) \in R$ as there is a direct ship from Chennai to Port Blair. $(PB, SwD) \in R$as there is a direct ferry from Port Blair to Swaraj Dweep. But $(C, SwD) \notin R$ as there is no direct ship/ferry from Chennai to Swaraj Dweep.

(iv) Whether the function f is one-one and onto.

Function f is one-one. For example, no two elements of set Y are mapped to a common element in set X.

Function f is not onto. For example, $C \in X$(co - domain of f) but it has no pre-image in Y.

103. Sherlin and Danju are playing Ludo at home during Covid-19. While rolling the dice, Sherlin's sister Raji observed and noted the possible outcomes of the throw every time belongs to set $\{1, 2, 3, 4, 5, 6\}$. Let A be the set of players while B be the set of all possible outcomes.

We have $A = \{S, D\}$ and $B = \{1, 2, 3, 4, 5, 6\}$

(i) Let $R: B \longrightarrow B$ be defined by
$R = \{(x,y): y \text{ is divisible by } x\}$
Show that above relation is reflexive and transitive but not symmetric.

(ii) Raji wants to know the number of functions from A to B. How many number of functions are possible?

(iii) Let R be a relation on B defined by $R = \{(1,2), (2,2), (1,3), (3,4), (3,1), (4,3), (5,5)\}$. Show that R is not reflexive, not transitive and not symmetric.

(iv) Raji wants to know the number of relations possible from A to B. How many numbers of relations are possible?

(v) Let $R: B \longrightarrow B$ be a relation defined by $R = \{(1,1),(1,2), (2,2), (3,3), (4,4), (5,5),(6,6)\}$. Show that R is an equivalence relation .

Sol :

(i) Since every number x is divisible by itself x. Thus R is reflexive.

Since $(1,2) \in R$ but $(2,1) \notin R$. Thus R is not symmetric.

Since $(1,2),(2,4) \in R$ and $(1,4) \in R$, thus R is transitive.

(ii) Since $n(A) = 2 \text{ and } n(B) = 6$, total possible function 6^2

(iii) $R = \{(1,2),(2,2),(1,3),(3,4),(3,1),(4,3),(5,5)\}$

Since $(3,3) \notin R$, R is not reflexive .

$(1,2) \in R$ but $(2,1) \notin R$. Thus R is not symmetric.

Since for all element of B there does not exist, $(a,b)(b,c) \in R$ and $(a,c) \in R$. Thus R is not transitive.

(iv) Since $n(A) = 2 \text{ and } n(B) = 6$, total possible relations are from A to B are $2^{2\times 6} = 2^{12}$

(v) Here $(1,2) \in R$ but $(2,1) \notin R$. Thus R is not symmetric.

For all values of a,b and c, $(a,b)(b,c) \in R$ and $(a,c) \in R$. Thus R is transitive.

For example $(1,1),(1,2) \in R$ and $(1,2) \in R$.

104. An organization conducted bike race under 2 different categories - boys and girls. Totally there were 250 participants. Among all of them finally three from Category 1 and two from Category 2 were selected for the final race. Ravi forms two set B and G with these participants for his college project. Let $B = \{b_1, b_2, b_3\}$ $G = \{g_1, g_2\}$ where B represents the set of boys selected and G the set of girls who were selected for the final race.

Ravi decides to explore these sets for various types of relations and functions

(i). Ravi wishes to form all the relations possible from B to G. How many such relations are possible?

(ii). Let $R: B \to B$ be defined by $R = \{ (x,y): x$ and y are students of same sex$\}$, Show that this relation R is equivalence relation.

(iii) Ravi wants to know among those relations, how many functions can be formed from B to G?

(iv) Let $R: B \to G$ be defined by $R = \{(b_1, g_1),(b_2, g_2),(b_3, g_1)\}$, then R is __________

(v) Ravi wants to find the number of injective functions from B to G. How many numbers of injective functions are possible?

Sol :

(i) Since $n(B) = 3 \text{ and } n(G) = 2$, total possible relations are from B to G are $2^{3\times 2} = 2^6$

(ii) We have $R: B \to B$ be defined by
$R = \{(x,y): x$ and y are students of same sex $\}$

Since $(x,x) \in R$, R is reflexive.

Since $(x,y) \in R$ and $(y,x) \in R$, Thus R is symmetric.

Since $a,b,c \in B$, $(a,b)(b,c) \in R$, $(a,c) \in R$. Thus R is transitive.

Therefore R is equivalence relation.

(iii) Since $n(B) = 3 \text{ and } n(G) = 2$, number of function from B to G are 2^3.

(v) If A and B are finite sets having m and n elements respectively then the number of injective function from A to B is

$$\begin{cases} {}^nP_m, & n \geqslant m \\ 0 & n < m \end{cases}$$

Here $n(B) = 3 \text{ and } n(G) = 2$, total number of injective relation from B to G is 0.

105. Students of Grade 9 planned to plant saplings along straight lines, parallel to each other to one side of the playground ensuring that they had enough play area. Let us assume that they planted one of the rows of the saplings along the line $y = x - 4$.Let L be the set of all lines which are parallel on the ground and R be a relation on L.

(i). Let relation R be defined by $R = \{(L_1, L_2) : L_1 \parallel L_2 \text{ where } L_1, L_2 \in L\}$ then Show that R is an equivalence relation.

(ii). Let relation R be defined by $R = \{(L_1, L_2) : L_1 \perp L_2 \text{ where } L_1, L_2 \in L\}$ then show that R is symmetric but neither reflexive nor transitive relation.

(iii) Show that the function $f : R \to R$ defined by $f(x) = x - 4$ is bijective.

(iv) Let $f : R \to R$ defined by $f(x) = x - 4$, then find the range of $f(x)$.

(v) Let $R = \{(L_1, L_2) : L_1 \parallel L_2 \text{ and } L_1 : y = x - 4\}$ then show that $2x - 2y + 5$ can be taken as L_2.

Sol :

(i) If L_1 is always parallel to itself L_2. Thus (l_2, l_1) $(L_1, L_1) \in R$. Thus R is reflexive.

If L_1 is parallel to L_2, then L_2 is parallel to L_1. Thus if $(L_1, L_2) \in R$ then $(L_2, L_1) \in R$. Thus R is symmetric.

If L_1 is parallel to L_2 and L_2 is parallel to L_3, then L_1 is parallel to L_3. So, if $(L_1, L_2) \in R$ and $(L_2, L_3) \in R$ then $(L_1, L_3) \in R$. Thus R is an equivalence relation.

(ii) If L_1 is always parallel to itself L_2 but not perpendicular to itself. $(L_1, L_1) \notin R$. Thus R is not reflexive.

If L_1 is perpendicular to L_2, then L_2 is perpendicular to L_1. So $(L_1, L_2) \in R$ then $(L_2, L_1) \in R$. Thus R is symmetric.

If L_1 is perpendicular to L_2 and L_2 is perpendicular to L_3, then L_1 is not perpendicular to L_3. It is parallel to L_3. So, if $(L_1, L_2) \in R$, and $(L_2, L_3) \in R$ then $(L_1, L_3) \notin R$. Thus R is not transitive.

(iii) A liner function defined from R to R is always one-one and onto. Thus $f(x)$ is bijective.

(iv) We have $f(x) = x - 4$. For all real value of x we can get a real number $f(x)$. Thus range of $f(x)$ is $\boldsymbol{R}$.

(v) Since, $L_1 \parallel L_2$ then slope of both the lines should be same.

Slope of L_1 is 1, and slope of $2x - 2y + 5$ is also 1. Thus $2x - 2y + 5$ can be taken as L_2.

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CHAPTER 2

INVERSE TRIGONOMETRIC FUNCTIONS

OBJECTIVE QUESTIONS

1. $\left[\sin^{-1}\frac{\pi}{3}+\sin^{-1}\left(\frac{1}{2}\right)\right]$ is equal to

(a) 1 (b) $\frac{1}{2}$

(c) $\frac{1}{3}$ (d) $\frac{1}{4}$

Sol : OD 2023

We have

$$\sin\left[\frac{x}{2}+\sin^{-1}\left(\frac{1}{2}\right)\right] = \sin\left[\frac{\pi}{3}+\frac{\pi}{6}\right]$$
$$= \sin\frac{3\pi}{6}$$
$$= \sin\frac{\pi}{2}$$
$$= 1$$

Thus (a) is correct option.

2. If $\tan^{-1}x = y$, then

(a) $-1 < y < 1$ (b) $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

(c) $-\frac{\pi}{2} < y < \frac{\pi}{2}$ (d) $y \in \left\{-\frac{\pi}{2},\frac{\pi}{2}\right\}$

Sol : OD 2022 Term i

If $\tan^{-1}x = \alpha$, then the principle range of α is $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$.

Here, $\tan^{-1}x = y$

So, the range of y will be $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$, where $x \in R$.

Thus (c) is correct option.

3. $\sin\left[\frac{\pi}{3}-\sin^{-1}\left(-\frac{1}{2}\right)\right]$ is equal to

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$

(c) -1 (d) 1

Sol : SQP 2022 Term i

Let x be the value of given trigonometric expression.

Thus $$x = \sin\left[\frac{\pi}{3}-\sin^{-1}\left(-\frac{1}{2}\right)\right]$$

Let $$\sin^{-1}\left(-\frac{1}{2}\right) = \theta$$
$$-\frac{1}{2} = \sin\theta$$
$$\sin\left(-\frac{\pi}{6}\right) = \sin\theta$$
$$\theta = -\frac{\pi}{6}$$

Now $$x = \sin\left[\frac{\pi}{3}-\sin^{-1}\left(-\frac{1}{2}\right)\right]$$
$$= \sin\left[\frac{\pi}{3}-\left(-\frac{\pi}{6}\right)\right]$$
$$= \sin\frac{\pi}{2}$$
$$= 1$$

Thus (d) is correct option.

4. $\sin(\tan^{-1}x)$, where $|x| < 1$ is equal to

(a) $\frac{x}{\sqrt{1-x^2}}$ (b) $\frac{1}{\sqrt{1-x^2}}$

(c) $\frac{1}{\sqrt{1+x^2}}$ (d) $\frac{x}{\sqrt{1+x^2}}$

Sol : SQP 2022 Term i

We have, $\sin(\tan^{-1}x)$, where $|x| < 1$.

Let $$\tan^{-1}x = \theta$$
$$\tan\theta = x$$

Now $$\sin\theta = \frac{x}{\sqrt{1+x^2}}$$
$$\theta = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$$
$$\sin(\tan^{-1}x) = \sin\left[\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)\right]$$
$$= \frac{x}{\sqrt{1+x^2}}$$

Thus (d) is correct option.

5. If $\cos^{-1}x > \sin^{-1}x$, then

(a) $\frac{1}{\sqrt{2}} < x \leq 1$ (b) $0 \leq x < \frac{1}{\sqrt{2}}$

(c) $-1 \leq x < \frac{1}{\sqrt{2}}$ (d) $x > 0$

Sol : Comp 2018

From $\sin^{-1}x+\cos^{-1}x = \frac{\pi}{2}$ we have

$$\cos^{-1}x = \tfrac{\pi}{2} - \sin^{-1}x$$

Since $\cos^{-1}x > \sin^{-1}x$, we get

$$\tfrac{\pi}{2} - \sin^{-1}x > \sin^{-1}x$$

$$\tfrac{\pi}{2} > 2\sin^{-1}x$$

$$\sin^{-1}x < \tfrac{\pi}{4}$$

$$x < \tfrac{1}{\sqrt{2}}$$

$$-1 \leq x < \tfrac{1}{\sqrt{2}}$$

Thus (c) is correct option.

6. The value of $\sin(2\sin^{-1}0.8)$ is

(a) 0.48 (b) $\sin 1.2^\circ$

(c) $\sin 1.6^\circ$ (d) 0.96

Sol : Delhi 2010, Foreign 2008

Let $\sin^{-1}0.8 = \theta$

$$\sin\theta = 0.8$$

then $\cos\theta = 0.6$

Now $\sin(2\sin^{-1}0.8) = \sin(2\theta)$

$$= 2\sin\theta\cos\theta$$

$$= 2 \times 0.8 \times 0.6$$

$$= 0.96$$

Thus (d) is correct option.

7. The principal value of $\sin^{-1}\left[\sin\dfrac{2\pi}{3}\right]$ is

(a) $\dfrac{-2\pi}{3}$ (b) $\dfrac{2\pi}{3}$

(c) $\dfrac{4\pi}{3}$ (d) None of these

Sol : OD 2015

$$\sin^{-1}\left[\sin\frac{2\pi}{3}\right] = \sin^{-1}\left[\sin\left(\pi - \frac{\pi}{3}\right)\right]$$

$$= \sin^{-1}\left(\sin\frac{\pi}{3}\right)$$

$$= \frac{\pi}{3}$$

Thus (d) is correct option.

8. If x takes negative permissible value, then $\sin^{-1}x$ is equal to

(a) $-\cos^{-1}\sqrt{1-x^2}$ (b) $\cos^{-1}\sqrt{x^2-1}$

(c) $\pi - \cos^{-1}\sqrt{1-x^2}$ (d) $\cos^{-1}\sqrt{1-x^2}$

Sol : SQP 2018

Let $x = -y,\ y > 0$

$$\sin^{-1}x = \sin^{-1}(-y)$$

$$= -\sin^{-1}y$$

$$= -\cos^{-1}\sqrt{1-y^2}$$

$$= -\cos^{-1}\sqrt{1-x^2}$$

Thus (a) is correct option.

9. If $-\frac{\pi}{2} < \sin^{-1}x < \frac{\pi}{2}$, then $\tan(\sin^{-1}x)$ is equal to

(a) $\dfrac{x}{1-x^2}$ (b) $\dfrac{x}{1+x^2}$

(c) $\dfrac{x}{\sqrt{1-x^2}}$ (d) $\dfrac{1}{\sqrt{1-x^2}}$

Sol : Foreign 2014

$$\tan(\sin^{-1}x) = \tan\left(\tan^{-1}\frac{x}{\sqrt{1-x^2}}\right)$$

$$= \frac{x}{\sqrt{1-x^2}}$$

Thus (c) is correct option.

10. If $\sin^{-1}x + \sin^{-1}y = \frac{\pi}{2}$, then $\cos^{-1}x + \cos^{-1}y$ is equal to

(a) $\dfrac{\pi}{2}$ (b) $\dfrac{\pi}{4}$

(c) π (d) $\dfrac{3\pi}{4}$

Sol : Comp 2012

We have $\sin^{-1}x + \sin^{-1}y = \dfrac{\pi}{2}$

$$\frac{\pi}{2} - \cos^{-1}x + \frac{\pi}{2} - \cos^{-1}y = \frac{\pi}{2}$$

$$\cos^{-1}x + \cos^{-1}y = \frac{\pi}{2}$$

Thus (a) is correct option.

11. If $\sin^{-1}x - \cos^{-1}x = \frac{\pi}{6}$, then x is equal to

(a) $\dfrac{1}{2}$ (b) $\dfrac{\sqrt{3}}{2}$

(c) $\dfrac{-1}{2}$ (d) None of these

Sol : Delhi 2009, OD 2007

We have $\sin^{-1}x - \cos^{-1}x = \dfrac{\pi}{6}$

$$\left(\frac{\pi}{2} - \cos^{-1}x\right) - \cos^{-1}x = \frac{\pi}{6}$$

$$\frac{\pi}{2} - 2\cos^{-1}x = \frac{\pi}{6}$$

$$\frac{\pi}{2} - \frac{\pi}{6} = 2\cos^{-1}x$$

$$\frac{2\pi}{6} = 2\cos^{-1}x$$

$$\cos^{-1}x = \frac{\pi}{6}$$

$$x = \cos\frac{\pi}{6}$$

$$= \frac{\sqrt{3}}{2}$$

Thus (b) is correct option.

12. The value of $\tan\left\{\cos^{-1}\left(-\frac{2}{7}\right)-\frac{\pi}{2}\right\}$ is

(a) $\frac{2}{3\sqrt{5}}$ (b) $\frac{2}{3}$

(c) $\frac{1}{\sqrt{5}}$ (d) $\frac{4}{\sqrt{5}}$

Sol : SQP 2019

$$\begin{aligned}\tan\left\{\cos^{-1}\left(-\frac{2}{7}\right)-\frac{\pi}{2}\right\} &= \tan\left\{\pi-\cos^{-1}\frac{2}{7}-\frac{\pi}{2}\right\}\\ &= \tan\left\{\frac{\pi}{2}-\cos^{-1}\frac{2}{7}\right\}\\ &= \tan\left\{\sin^{-1}\frac{2}{7}\right\}\\ &= \tan\left\{\tan^{-1}\frac{2}{3\sqrt{5}}\right\}\\ &= \frac{2}{3\sqrt{5}}\end{aligned}$$

Thus (a) is correct option.

13. The domain of $\sin^{-1}x$ is

(a) $(-\pi,\pi)$ (b) $[-1,1]$

(c) $(0,2\pi)$ (d) $(-\infty,\infty)$

Sol : OD 2017

Since $-1 \le \sin x \le 1$

Hence domain of $\sin^{-1}x$ is $[-1,1]$.

Thus (b) is correct option.

14. $\cos^{-1}\frac{1}{2}+2\sin^{-1}\frac{1}{2}=$

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$

(c) $\frac{2\pi}{3}$ (d) $\frac{5\pi}{6}$

Sol : Delhi 2015

$$\begin{aligned}\cos^{-1}\frac{1}{2}+2\sin^{-1}\frac{1}{2} &= \cos^{-1}\frac{1}{2}+\sin^{-1}\frac{1}{2}+\sin^{-1}\frac{1}{2}\\ &= \frac{\pi}{2}+\frac{\pi}{6}\\ &= \frac{2\pi}{3}\end{aligned}$$

Thus (c) is correct option.

15. $\sin\left[\sin^{-1}\frac{1}{2}+\cos^{-1}\frac{1}{2}\right]=$

(a) 1 (b) 2

(c) $\frac{1}{2}$ (d) $\frac{1}{4}$

Sol : Foreign 2008

$$\begin{aligned}\sin\left[\sin^{-1}\frac{1}{2}+\cos^{-1}\frac{1}{2}\right] &= \sin\frac{\pi}{2}\\ &= 1\end{aligned}$$

Thus (a) is correct option.

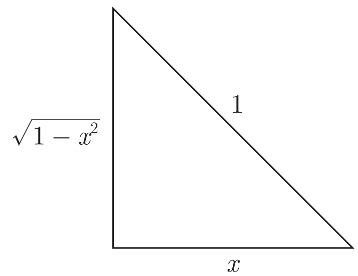
16. $\tan(\cos^{-1}x)=$

(a) $\sqrt{1+x^2}$ (b) $\sqrt{1-x^2}$

(c) $\frac{\sqrt{1-x^2}}{x}$ (d) $\frac{x}{\sqrt{1-x^2}}$

Sol : Comp 2012, OD 2010

$$\begin{aligned}\tan(\cos^{-1}x) &= \tan\left[\tan^{-1}\frac{\sqrt{1-x^2}}{x}\right]\\ &= \frac{\sqrt{1-x^2}}{x}\end{aligned}$$



Thus (c) is correct option.

17. If $4\sin^{-1}x+\cos^{-1}x=\pi$ then $x=$

(a) $\frac{1}{5}$ (b) $\frac{1}{4}$

(c) $\frac{1}{3}$ (d) $\frac{1}{2}$

Sol : OD 2015

$$\begin{aligned}4\sin^{-1}x+\cos^{-1}x &= \pi\\ 3\sin^{-1}x+\sin^{-1}x+\cos^{-1}x &= \pi\\ 3\sin^{-1}x+\frac{\pi}{2} &= \pi\\ 3\sin^{-1}x &= \frac{\pi}{2}\\ x &= \frac{1}{2}\end{aligned}$$

Thus (d) is correct option.

18. $2\cos^{-1}\frac{1}{2}+\sin^{-1}\frac{1}{2}=$

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$

(c) $\frac{2\pi}{3}$ (d) $\frac{5\pi}{6}$

Sol : Delhi 2007

$$\begin{aligned}2\cos^{-1}\frac{1}{2}+\sin^{-1}\frac{1}{2} &= \cos^{-1}\frac{1}{2}+\cos^{-1}\frac{1}{2}+\sin^{-1}\frac{1}{2}\\ &= \frac{\pi}{3}+\frac{\pi}{2}\\ &= \frac{5\pi}{3}\end{aligned}$$

Thus (c) is correct option.

19. If $\sin\left(\sin^{-1}\frac{1}{5}+\cos^{-1}x\right)=1$, then $x=$

(a) 1 (b) 0

(c) $\frac{4}{5}$ (d) $\frac{1}{5}$

Sol : Foreign 2018, Delhi 2009

$$\begin{aligned}\sin\left(\sin^{-1}\frac{1}{5}+\cos^{-1}x\right) &= 1\\ \sin^{-1}\frac{1}{5}+\cos^{-1}x &= \sin^{-1}1\\ \sin^{-1}\frac{1}{5}+\cos^{-1}x &= \frac{\pi}{2}\\ \sin^{-1}\frac{1}{5} &= \frac{\pi}{2}-\cos^{-1}x\\ \sin^{-1}\frac{1}{5} &= \sin^{-1}x\\ x &= \frac{1}{5}\end{aligned}$$

Thus (d) is correct option.

20. The principal value of $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ is

(a) $-\frac{2\pi}{3}$ (b) $-\frac{\pi}{3}$

(c) $\frac{4\pi}{3}$ (d) $\frac{5\pi}{3}$

Sol : Comp 2015

$$\begin{aligned}\sin^{-1}\left(\frac{-\sqrt{3}}{2}\right) &= -\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\\ &= -\sin^{-1}\left(\sin\frac{\pi}{3}\right)\\ &= -\frac{\pi}{3}\end{aligned}$$

Thus (b) is correct option.

21. The value of $\cos^{-1}\left(\cos\frac{5\pi}{3}\right)+\sin^{-1}\left(\sin\frac{5\pi}{3}\right)$ is

(a) 0 (b) $\frac{\pi}{2}$

(c) $\frac{2\pi}{3}$ (d) $\frac{10\pi}{3}$

Sol : SQP 2016

$$\begin{aligned}x &= \cos^{-1}\left(\cos\frac{5\pi}{3}\right)+\sin^{-1}\left(\sin\frac{5\pi}{3}\right)\\ &= \cos^{-1}\left\{\cos\left(2\pi-\frac{\pi}{3}\right)\right\}+\sin^{-1}\left\{\sin\left(2\pi-\frac{\pi}{3}\right)\right\}\\ &= \frac{\pi}{3}-\frac{\pi}{3}\\ &= 0\end{aligned}$$

Thus (a) is correct option.

22. The value of $\cos^{-1}\left(\cos\frac{5\pi}{3}\right)+\sin^{-1}\left(\cos\frac{5\pi}{3}\right)$ is

(a) $\frac{\pi}{2}$ (b) $\frac{5\pi}{3}$

(c) $\frac{10\pi}{3}$ (d) 0

Sol : Delhi 2009

$$\cos^{-1}\left(\cos\frac{5\pi}{3}\right)+\sin^{-1}\left(\cos\frac{5\pi}{3}\right) = \frac{\pi}{2}$$

Thus (a) is correct option.

23. If $\sin^{-1}\frac{x}{5}+\operatorname{cosec}^{-1}\frac{5}{4}=\frac{\pi}{2}$, then x is

(a) 1 (b) 3

(c) 5 (d) 9

Sol : OD 2012, Comp 2007

$$\begin{aligned}\sin^{-1}\frac{x}{5}+\operatorname{cosec}^{-1}\frac{5}{4} &= \frac{\pi}{2}\\ \sin^{-1}\frac{x}{5}+\cos^{-1}\frac{3}{5} &= \frac{\pi}{2}\\ \sin^{-1}\frac{x}{5} &= \frac{\pi}{2}-\cos^{-1}\frac{3}{5}\\ \sin^{-1}\frac{x}{5} &= \sin^{-1}\frac{3}{5}\\ x &= 3\end{aligned}$$

Thus (b) is correct option.

24. The value of $\sin^{-1}\frac{\sqrt{3}}{2}-\sin^{-1}\frac{1}{2}$ is

(a) 45° (b) 90°

(c) 15° (d) 30°

Sol : Comp 2011

$$\begin{aligned}\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)-\sin^{-1}\left(\frac{1}{2}\right) &= 60^\circ-30^\circ\\ &= 30^\circ\end{aligned}$$

Thus (d) is correct option.

25. If $\sin^{-1}x+\sin^{-1}y=\frac{2\pi}{3}$, then $\cos^{-1}x+\cos^{-1}y=$

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$

(c) $\frac{\pi}{4}$ (d) $\frac{2\pi}{3}$

Sol : Delhi 2013

$$\begin{aligned}\sin^{-1}x+\sin^{-1}y &= \frac{2\pi}{3}\\ &= \pi-\frac{\pi}{3}\\ \sin^{-1}x+\sin^{-1}y &= \frac{\pi}{2}+\frac{\pi}{2}-\frac{\pi}{3}\\ \frac{\pi}{3} &= \frac{\pi}{2}-\sin^{-1}x+\frac{\pi}{2}-\sin^{-1}y\\ \frac{\pi}{3} &= \cos^{-1}x+\cos^{-1}y\\ \cos^{-1}x+\cos^{-1}y &= \frac{\pi}{3}\end{aligned}$$

Thus (b) is correct option.

26. If $|x|\le 1$, which of the following four is different from the other three?

Ch 7 : Integrals

PDF	PDF	PDF	DOC	PPT
Text Book	Worksheet	Solutions	Worksheet	Presentation

Ch 8 : Application of Integrals

PDF	PDF	PDF	DOC	PPT
Text Book	Worksheet	Solutions	Worksheet	Presentation

Ch 9 : Differential Equations

PDF	PDF	PDF	DOC	PPT
Text Book	Worksheet	Solutions	Worksheet	Presentation

Ch 10 : Vector Algebra

PDF	PDF	PDF	DOC	PPT
Text Book	Worksheet	Solutions	Worksheet	Presentation

Ch 11 : Three Dimensional Geometry

PDF	PDF	PDF	DOC	PPT
Text Book	Worksheet	Solutions	Worksheet	Presentation

Ch 12 : Linear Programming

PDF	PDF	PDF	DOC	PPT
Text Book	Worksheet	Solutions	Worksheet	Presentation

Ch 13 : Probability

PDF	PDF	PDF	DOC	PPT
Text Book	Worksheet	Solutions	Worksheet	Presentation

Class 10th

Maths | Science | Social Science | English LL

Info Tech. | Comp. App. | AI | English Comm.

हिंदी अ | हिंदी ब | संस्कृत

Class 9th

Maths | Science | Social Science | English LL

Info Tech. | Comp. App. | AI | English Comm.

हिंदी अ | हिंदी ब | संस्कृत

Class 8th

Maths | Science | Social Science | English

हिंदी | संस्कृत

Class 7th

Maths | Science | Social Science | English

हिंदी | संस्कृत

Class 6th

Maths | Science | Social Science | English

हिंदी | संस्कृत

Class 5th

Maths | English | हिंदी | EVS

Class 4th

Maths | English | हिंदी | EVS

Class 3th

Maths | English | हिंदी | EVS

Class 2th

Maths | English | हिंदी | EVS

Class 1th

Maths | English | हिंदी | EVS

Prep/KG 3rd Year

Maths | English | हिंदी | EVS

Prep/KG 2nd Year

Maths | English | हिंदी | EVS

Prep/KG 1st Year

Maths | English | हिंदी | EVS

(a) $\sin(\cos^{-1}x)$ (b) $\cos(\sin^{-1}x)$

(c) $\sqrt{1-x^2}$ (d) $\frac{\sqrt{1-x^2}}{x}$

Sol : Foreign 2014

$$\sin(\cos^{-1}x) = \cos(\sin^{-1}x) = \sqrt{1-x^2}$$

Thus option (d) is different from the other three.

Thus (d) is correct option.

27. $\cos^{-1}[\cos(-\frac{\pi}{3})]$ is equal to

(a) $-\frac{\pi}{3}$ (b) $\frac{\pi}{3}$

(c) $\frac{2\pi}{3}$ (d) $\frac{5\pi}{3}$

Sol : Comp 2012, Delhi 2011

$$\cos^{-1}\left[\cos\left(-\frac{\pi}{3}\right)\right] = \cos^{-1}\left(\cos\frac{\pi}{3}\right) = \frac{\pi}{3}$$

Thus (b) is correct option.

28. The value of $\sin[2\cos^{-1}(-\frac{3}{5})]$ is

(a) $\frac{24}{25}$ (b) $-\frac{24}{25}$

(c) $\frac{7}{25}$ (d) none of these

Sol : OD 2008

$$\begin{aligned} x &= \sin\left[2\cos^{-1}\left(-\frac{3}{5}\right)\right] \\ &= 2\sin\left[\cos^{-1}\left(-\frac{3}{5}\right)\right]\cos\left[\cos^{-1}\left(-\frac{3}{5}\right)\right] \\ &= 2\sqrt{1-\left(-\frac{3}{5}\right)^2}\left(-\frac{3}{5}\right) \\ &= -\frac{24}{25} \end{aligned}$$

Thus (b) is correct option.

29. If $\cos^{-1}x+\cos^{-1}y = 2\pi$, then the value of $\sin^{-1}x+\sin^{-1}y$ is

(a) 0 (b) π

(c) $-\pi$ (d) $\frac{\pi}{3}$

Sol : SQP 2020

$$\begin{aligned} \sin^{-1}x+\sin^{-1}y &= \left(\frac{\pi}{2}-\cos^{-1}x\right)+\left(\frac{\pi}{2}-\cos^{-1}y\right) \\ &= \pi-(\cos^{-1}x+\cos^{-1}y) \\ &= \pi-2\pi \\ &= -\pi \end{aligned}$$

Thus (c) is correct option.

30. The number of solutions of the equation $\sin^{-1}x-\cos^{-1}x = \sin^{-1}(\frac{1}{2})$ is

(a) 1 (b) 2

(c) 3 (d) infinite

Sol : Comp 2013

We have
$$\begin{aligned} \sin^{-1}x-\cos^{-1}x &= \sin^{-1}\left(\frac{1}{2}\right) \\ \sin^{-1}x-\cos^{-1}x &= \frac{\pi}{6} \\ \sin^{-1}x-\left(\frac{\pi}{2}-\sin^{-1}x\right) &= \frac{\pi}{6} \\ 2\sin^{-1}x &= \frac{2\pi}{3} \\ x &= \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} \end{aligned}$$

Thus (a) is correct option.

31. The value of $\sec^2(\tan^{-1}2)+\mathrm{cosec}^2(\cot^{-1}3)$ is

(a) 15 (b) 5

(c) 13 (d) 14

Sol : OD 2008

$$\begin{aligned} x &= \sec^2(\tan^{-1}2)+\mathrm{cosec}^2(\cot^{-1}3) \\ &= \{1+\tan^2(\tan^{-1}2)\}+\{1+\cot^2(\cot^{-1}3)\} \\ &= 1+\{\tan(\tan^{-1}2)\}^2+1+\{\cot(\cot^{-1}3)\}^2 \\ &= 1+2^2+1+3^2 \\ &= 15 \end{aligned}$$

Thus (a) is correct option.

32. $\cos^{-1}(\cos\frac{8\pi}{5}) =$

(a) $\frac{8\pi}{5}$ (b) $\frac{12\pi}{5}$

(c) $\frac{2\pi}{5}$ (d) $\frac{4\pi}{5}$

Sol : Delhi 2011

$$\begin{aligned} \cos^{-1}\left(\cos\frac{8\pi}{5}\right) &= \cos^{-1}\left[\cos\left(2\pi-\frac{2\pi}{5}\right)\right] \\ &= \cos^{-1}\left[\cos\left(\frac{2\pi}{5}\right)\right] \\ &= \frac{2\pi}{5} \end{aligned}$$

Thus (c) is correct option.

33. $\cos^{-1}(2x-1) =$

(a) $2\cos^{-1}x$ (b) $\cos^{-1}\sqrt{x}$

(c) $2\cos^{-1}\sqrt{x}$ (d) None of these

Sol : Foreign 2019, OD 2013

Let, $x = \cos^2 A$

or $A = \cos^{-1}\sqrt{x}$ ...(1)

Then given equation becomes

$$\cos^{-1}(2x-1) = = \cos^{-1}(2\cos^2 A - 1)$$
$$= \cos^{-1}(\cos 2A)$$
$$= 2A$$

From Eq.(1),

$$\cos^{-1}(2x-1) = = 2\cos^{-1}\sqrt{x}$$

Thus (c) is correct option.

34. $\tan^{-1}(1) + \cos^{-1}(\frac{-1}{2}) + \sin^{-1}(\frac{-1}{2}) =$

(a) $\frac{\pi}{4}$ (b) $\frac{3\pi}{4}$

(c) $\frac{-\pi}{4}$ (d) $\frac{\pi}{2}$

Sol : Delhi 2015

Using $\cos^{-1}x + \sin^{-1}x = \frac{\pi}{2}$ we have

$$\tan^{-1}(1) + \cos^{-1}\left(\frac{-1}{2}\right) + \sin^{-1}\left(\frac{-1}{2}\right) = \frac{\pi}{4} + \frac{\pi}{2}$$
$$= \frac{3\pi}{4}$$

Thus (b) is correct option.

35. The principal value of $\cos^{-1}(-\frac{1}{2})$ is

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{6}$

(c) $\frac{2\pi}{3}$ (d) $\frac{3\pi}{4}$

Sol : Foreign 2014, OD 2009

Let, $\cos^{-1}\left(\frac{-1}{2}\right) = \theta$

Then $-\frac{1}{2} = \cos\theta$

Since principal value of $\cos^{-1}\theta$ is $[0,\pi]$

$$\theta = \frac{2\pi}{3}$$

Thus (c) is correct option.

36. The principal value of $\sin^{-1}\frac{\sqrt{3}}{2}$ is

(a) $\frac{2\pi}{3}$ (b) $\frac{\pi}{6}$

(c) $\frac{\pi}{4}$ (d) $\frac{\pi}{3}$

Sol : Comp 2017

Let, $\sin^{-1}\frac{\sqrt{3}}{2} = \theta$

$$\frac{\sqrt{3}}{2} = \sin\theta$$

$$\sin\theta = \sin\frac{\pi}{3}$$

$$\theta = \frac{\pi}{3}$$

Thus (d) is correct option.

37. The value of $\sin(2\tan^{-1}x)$, $|x| \le 1$ is

(a) $\frac{1}{x}$ (b) x

(c) $\frac{1}{x^2}$ (d) $\frac{2x}{1+x^2}$

Sol : SQP 2008

$$\sin(2\tan^{-1}x) = \sin\left(\sin^{-1}\frac{2x}{1+x^2}\right)$$
$$= \frac{2x}{1+x^2}, \qquad -1 \le x \le 1$$

Thus (d) is correct option.

38. If $\sin^{-1}(1-x) - 2\sin^{-1}x = \frac{\pi}{2}$, then $x =$

(a) $(0, -\frac{1}{2})$ (b) $(\frac{1}{2}, 0)$

(c) (0) (d) $(-1, 0)$

Sol : OD 2007

$$\sin^{-1}(1-x) - 2\sin^{-1}x = \frac{\pi}{2}$$
$$-2\sin^{-1}x = \frac{\pi}{2} - \sin^{-1}(1-x)$$
$$-2\sin^{-1}x = \cos^{-1}(1-x)$$
$$\cos(-2\sin^{-1}x) = 1-x$$
$$1 - 2\sin^2(\sin^{-1}x) = 1-x$$
$$1 - 2x^2 = 1-x$$
$$x = 0, \frac{1}{2}$$

But $\frac{1}{2}$ does not satisfy the equation. Hence $x = 0$

Alternative :

Since, $\sin^{-1}(1-x)$ is defined only, when

$$-1 \le 1-x \le 1$$
$$0 \le x \le 2 \qquad ...(1)$$

and $\sin^{-1}x$ is defined only, when

$$-1 \le x \le 1 \qquad ...(2)$$

From Eqs. (1) and (1), we get

$$0 \le x \le 1$$

At $x = 0$, we have

$$\text{LHS} = \sin^{-1}(1) - 2\sin^{-1}(0) = \frac{\pi}{2} = \text{RHS}$$

Hence, 0 is a solution of the given equation.

Again, at $x = 1$ we have,

$$\text{LHS} = \sin^{-1}(0) - 2\sin^{-1}(1)$$
$$= -\pi \neq \text{RHS}$$

Hence, 1 is not a solution of a given equation.

Thus (c) is correct option.

39. **Assertion (A) :** Domain of $y = \cos^{-1}(x)$ is $[-1,1]$.

Reason (R) : The range of the principal value branch of $y = \cos^{-1}(x)$ is $[0,\pi] - \left\{\frac{\pi}{2}\right\}$.

(a) Both Assertion (A) and Reason (R) are true and the Reason (R) is the correct explanation of the Assertion (A).

(b) Both Assertion (A) and Reason (R) are true and Reason (R) is not the correct explanation of the Assertion (A).

(c) Assertion (A) is true, but Reason (R) is false.

(d) Assertion (A) is false, but Reason (R) is true. 19. Assertion (A) : The vectors

Sol : OD 2024

We have $y = \cos^{-1}x$

Domain of y is equivalent to the range of value of x for which y exists

Let
$$y = \theta$$
$$\cos^{-1}x = \theta$$
$$x = \cos\theta$$

as we know range of $\cos\theta$ is $[-1,1]$ therefore range of x is $[-1,1]$. Hence, domain of y is $[-1,1]$.

Also, when $x = 0$, then

$$y = \cos^{-1}(0) = \frac{\pi}{2}$$

Hence, $\frac{\pi}{2}$ is included in the principal value branch of y.

Assertion (A) is correct but Reason (R) is false.

Thus (c) is correct option.

40. **Assertion (A) :** Maximum value of $(\cos^{-1}x)^2$ is π^2.

Reason (R) : Range of the principal value branch of $\cos^{-1}x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

(a) Both (A) and (R) are true and (R) is the correct explanation of (A).

(b) Both (A) and (R) are true, but (R) is not the correct explanation of (A)

(c) (A) is true, but (R) is false.

(d) (A) is false, but (R) is true.

Sol : OD 2023

The maximum value of $(\cos^{-1}x)^2$ is π^2. Assertion is true.

Range of the principal value branch of $\cos^{-1}x$ is $[0,\pi]$. Reason is false.

Thus (c) is correct option.

41. **Assertion (A) :** All trigonometric functions have their inverses over their respective domains.

Reason (R) : The inverse of $\tan^{-1}$ exists for some $x \in R$.

(a) Both (A) and (R) are true and (R) is the correct explanation of (A).

(b) Both (A) and (R) are true but (R) is not the correct explanation of (A).

(c) (A) is true but (R) is false.

(d) (A) is false but (R) is true

Sol : OD 2023

All trigonometric functions have inverse over their restricted domains So, assertion is incorrect.

Now $\tan^{-1} : R \to \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

i.e. inverse of $\tan x$ exists for some $x \in R$.

So, reason is correct.

Thus (d) is correct option.

42. **Assertion (A) :** Range of $[\sin^{-1}x + 2\cos^{-1}x]$ is $[0,\pi]$

Reason (R) : Principal value branch of $\sin^{-1}x$ has range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(a) Both (A) and (R) are true and (R) is the correct explanation of (A).

(b) Both (A) and (R) are true but (R) is not the correct explanation of (A).

(c) (A) is true but (R) is false.

(d) (A) is false but (R) is true

Sol : OD 2023

$$\sin^{-1}x + 2\cos^{-1}x = \sin^{-1}x + \cos^{-1}x + \cos^{-1}x$$
$$= \frac{\pi}{2} + \cos^{-1}x$$

Since $0 \leq \cos^{-1}x \leq \pi$, thus

$$0 + \frac{\pi}{2} \leq \left(\frac{\pi}{2} + \cos^{-1}x\right) \leq \pi + \frac{\pi}{2}$$
$$\frac{\pi}{2} \leq \left(\frac{\pi}{2} + \cos^{-1}x\right) \leq \frac{3\pi}{2}$$

So, assertion is incorrect.

Principle value branch of $\sin^{-1}x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. So, reason is correct.

Thus (d) is correct option.

43. **Assertion (A) :** The range of the function $f(x) = 2\sin^{-1}x + \frac{3\pi}{2}$, where $x \in [-1, 1]$ is $[\frac{\pi}{2}, \frac{5\pi}{2}]$

Reason (R) : The range of the principal value branch of $\sin^{-1}(x)$ is $[0, \pi]$

(a) Both (A) and (R) are true and (R) is the correct explanation of (A).

(b) Both (A) and (R) are true but (R) is not the correct explanation of (A).

(c) (A) is true but (R) is false.

(d) (A) is false but (R) is true

Sol : OD 2023

We have
$$-\frac{\pi}{2} \le \sin^{-1}x \le \frac{\pi}{2}$$
$$-\pi \le 2\sin^{-1}x \le \pi$$
$$-\pi + \frac{3\pi}{2} \le 2\sin^{-1}x + \frac{3\pi}{2} \le \pi + \frac{3\pi}{2}$$
$$\frac{\pi}{2} \le 2\sin^{-1}x + \frac{3\pi}{2} \le \frac{5\pi}{2}$$
$$\frac{\pi}{2} \le f(x) \le \frac{5\pi}{2}$$

The range of $f(x)$ is $\left[\frac{\pi}{2}, \frac{5\pi}{2}\right]$

So, assertion is correct.

Principle value branch of $\sin^{-1}x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. So, reason is correct. So, reason is incorrect.

Thus (c) is correct option.

44. **Assertion (A) :** The domain of the function $\sec^{-1}2x$ is $(-\infty, -\frac{1}{2}] \cup [\frac{1}{2}, \infty)$

Reason (R) : $\sec^{-1}(-2) = -\frac{\pi}{4}$

(a) Both (A) and (R) are true and (R) is the correct explanation of (A).

(b) Both (A) and (R) are true but (R) is not the correct explanation of (A).

(c) (A) is true but (R) is false.

(d) (A) is false but (R) is true

Sol : OD Sample Paper 2023

$\sec^{-1}x$ is defined, if $x \le -1$ or $x \ge 1$.

Hence, $\sec^{-1}2x$ will be defined, if $x \le -\frac{1}{2}$ or $x \ge \frac{1}{2}$.

So, assertion is correct.

Now,
$$\sec^{-1}(-2) = \pi - \sec^{-1}(2)$$
$$= \pi - \frac{\pi}{3}$$
$$= \frac{2\pi}{3}$$
$$\ne \frac{-\pi}{4}$$

Hence, reason is incorrect.

Thus (c) is correct option.

45. **Assertion (A) :** $\cos^{-1}\frac{3}{7} = \pi - \cos^{-1}\frac{3}{7}$

Reason (R) : $0 \le \cos^{-1}x \le \pi$, where $-1 \le x \le 1$

(a) Both (A) and (R) are true and (R) is the correct explanation of (A).

(b) Both (A) and (R) are true but (R) is not the correct explanation of (A).

(c) (A) is true but (R) is false.

(d) (A) is false but (R) is true

Sol : Comp 2012

Assertion : $\cos^{-1}\frac{3}{7} \ne \pi - \cos^{-1}\left(\frac{3}{7}\right)$

Since $\cos^{-1}(-x) = \pi - \cos^{-1}x$ thus
$$\pi - \cos^{-1}\left(\frac{3}{7}\right) = \cos^{-1}\left(\frac{-3}{7}\right)$$

and
$$\frac{3}{7} \ne \frac{-3}{7}$$

Hence (A) is false.

Reason is true because $0 \le \cos^{-1}x \le \pi$ is principle range and $-1 \le x \le 1$ is domain of $\cos^{-1}x$.

Hence (A) is false but (R) is true.

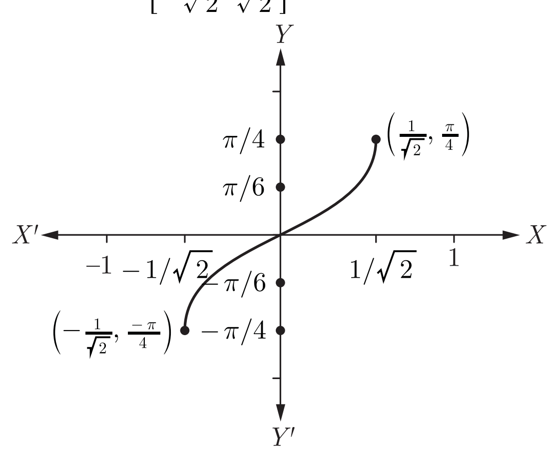
Thus (d) is correct option.

VERY SHORT ANSWER QUESTIONS

46. Draw the graph of $f(x) = \sin^{-1}x, x \in \left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$. Also, write range of $f(x)$.

Sol : OD 2023

$$f(x) = \sin^{-1}x,$$
$$x \in \left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$$



47. Evaluate $3\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)+2\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)+\cos^{-1}(0)$

Sol : OD 2023

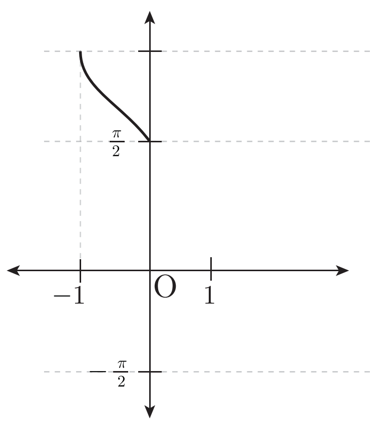
Let x be the value of given trigonometric expression.

$$\begin{aligned} x &= 3\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)+2\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)+\cos^{-1}(0) \\ &= 3\times\frac{\pi}{4}+2\times\frac{\pi}{6}+\frac{\pi}{2} \\ &= \frac{3\pi}{4}+\frac{\pi}{3}+\frac{\pi}{2} \\ &= \frac{9\pi+4\pi+6\pi}{12} \\ &= \frac{19\pi}{12} \end{aligned}$$

48. Draw the graph of $\cos^{-1}x$, where $x\in[-1,0]$. Also write its range.

Sol : OD 2023

Graph of $\cos^{-1}x$ is shown below.



Range of the $\cos^{-1}x$ is $[0,\pi]$.

But since $x\in[-1,0]$, hence range is $\left[\frac{\pi}{2},\pi\right]$

49. Write the domain and range (principle value branch) of the following function $f(x)=\tan^{-1}x$

Sol : OD 2023

Domain and range (principal value branch) of $\tan^{-1}x$ are R and $\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$ respectively.

50. Evaluate $\cos^{-1}\left[\cos\left(-\frac{7x}{3}\right)\right]$

Sol : OD 2023

Let x be the value of given trigonometric expression.

$$\begin{aligned} x &= \cos^{-1}\left[\cos\left(-\frac{7\pi}{3}\right)\right] \\ &= \cos^{-1}\left[\cos\frac{7\pi}{3}\right] \\ &= \cos^{-1}\left[\cos\left(2\pi+\frac{\pi}{3}\right)\right] \\ &= \cos^{-1}\left(\cos\frac{\pi}{3}\right) \\ &= \frac{\pi}{3}\in[0,\pi] \end{aligned}$$

51. Find the value of $\sin^{-1}\left[\cos\frac{33\pi}{5}\right]$

Sol : SQP 2023

$$\begin{aligned} \sin^{-1}\left(\cos\frac{33\pi}{5}\right) &= \sin^{-1}\left[\cos\left(6\pi+\frac{3\pi}{5}\right)\right] \\ &= \sin^{-1}\left(\cos\frac{3\pi}{5}\right) \\ &= \sin^{-1}\left[\cos\left(\frac{\pi}{2}+\frac{\pi}{10}\right)\right] \\ &= \sin^{-1}\left(-\sin\frac{\pi}{10}\right) \\ &= -\sin^{-1}\left(\sin\frac{\pi}{10}\right) \\ &= -\frac{\pi}{10} \end{aligned}$$

52. Find the value of $\sin^{-1}\left[\sin\frac{13\pi}{7}\right]$

Sol : Delhi 2019, Comp 2016

Let x be the value of given trigonometric expression.

$$\begin{aligned} x &= \sin^{-1}\left[\sin\left(\frac{13\pi}{7}\right)\right] \\ &= \sin^{-1}\left[\sin\left(2\pi-\frac{\pi}{7}\right)\right] \\ &= \sin^{-1}\left[\sin\left(\frac{-\pi}{7}\right)\right] \\ &= \frac{-\pi}{7} \qquad \left[\frac{-\pi}{7}\in\left[\frac{-\pi}{2},\frac{\pi}{2}\right]\right] \end{aligned}$$

53. Find the value of $\sin^{-1}\left[\sin\left(-\frac{17\pi}{8}\right)\right]$.

Sol : OD 2020

$$\begin{aligned} x &= \sin^{-1}\left[\sin\left(-\frac{17\pi}{8}\right)\right] \\ &= \sin^{-1}\left(-\sin\frac{17\pi}{8}\right) \\ &= \sin^{-1}\left[-\sin\left(2\pi+\frac{\pi}{8}\right)\right] \\ &= \sin^{-1}\left(-\sin\frac{\pi}{8}\right) \\ &= \sin^{-1}\left[\sin\left(\frac{-\pi}{8}\right)\right] \\ &= -\frac{\pi}{8} \end{aligned}$$

54. Write the value of $\tan^{-1}(\sqrt{3})-\cot^{-1}(-\sqrt{3})$.

Sol : Delhi 2019, OD 2013

Let x be the value of given trigonometric expression.

$$\begin{aligned} x &= \tan^{-1}(\sqrt{3})-\cot^{-1}(-\sqrt{3}) \\ &= \tan^{-1}\sqrt{3}-\{\pi-\cot^{-1}\sqrt{3}\} \\ &= \tan^{-1}\sqrt{3}-\pi+\cot^{-1}\sqrt{3} \end{aligned}$$

$$= (\tan^{-1}\sqrt{3} + \cot^{-1}\sqrt{3}) - \pi$$

$$= \frac{\pi}{2} - \pi = -\frac{\pi}{2}$$

which is the required principal value.

55. Find the principal value of $\tan^{-1}\sqrt{3} - \sec^{-1}(-\sqrt{2})$.

Sol : Comp 2018, OD 2012

Let x be principal the value of given trigonometric expression.

$$\begin{aligned} x &= \tan^{-1}\sqrt{3} - \sec^{-1}(-2) \\ &= \tan^{-1}\left(\tan\frac{\pi}{3}\right) - \sec^{-1}\left(\sec\frac{2\pi}{3}\right) \\ &= \frac{\pi}{3} - \frac{2\pi}{3} = -\frac{\pi}{3} \end{aligned}$$

which is the required principal value.

56. Write the value of $\cos^{-1}\left(-\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right)$.

Sol : Foreign 2014

Let x be the value of given trigonometric expression.

$$\begin{aligned} x &= \cos^{-1}\left(\frac{-1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right) \\ &= \left[\pi - \cos^{-1}\left(\frac{1}{2}\right)\right] + 2\sin^{-1}\left(\frac{1}{2}\right) \\ &= \left[\pi - \cos^{-1}\left(\cos\frac{\pi}{3}\right)\right] + 2\sin^{-1}\left(\sin\frac{\pi}{6}\right) \\ &= \left[\pi - \frac{\pi}{3}\right] + 2\times\frac{\pi}{6} \\ &= \frac{2\pi}{3} + \frac{\pi}{3} \\ &= \frac{2\pi+\pi}{3} = \pi \end{aligned}$$

Thus required value is π.

57. Write the principal value of $\cos^{-1}[\cos(680^\circ)]$.

Sol : Comp 2014

Here angle 680° doesn't lie in the principal value branch $[0, 180^\circ]$. So we reduce the angle such that, it lies in principal value branch.

We write 680° as, $2\times 360^\circ - 40^\circ$

$$\begin{aligned} \text{Now,} \cos^{-1}[\cos(680^\circ)] &= \cos^{-1}[\cos(2\times 360^\circ - 40^\circ)] \\ &= \cos^{-1}(\cos 40^\circ) \qquad \cos(4\pi - \theta) = \cos\theta \\ &= 40^\circ \end{aligned}$$

Since, $40^\circ \in [0, 180^\circ]$ we have

$$\cos^{-1}[\cos(680^\circ)] = 40^\circ$$

which is the required principal value.

58. Write the principal value of $\tan^{-1}\left[\sin\left(\frac{-\pi}{2}\right)\right]$.

Sol : Comp 2014

$$\begin{aligned} \tan^{-1}\left[\sin\left(-\frac{\pi}{2}\right)\right] &= \tan^{-1}\left[-\sin\left(\frac{\pi}{2}\right)\right] \\ &= \tan^{-1}(-1) \\ &= \tan^{-1}\left(-\tan\frac{\pi}{4}\right) \\ &= \tan^{-1}\left[\tan\left(\frac{-\pi}{4}\right)\right] = \frac{-\pi}{4} \end{aligned}$$

which is the required principal value.

59. Find the value of the following. $\cot\left(\frac{\pi}{2} - 2\cot^{-1}\sqrt{3}\right)$

Sol : Comp 2014

Let x be the value of given trigonometric expression.

$$\begin{aligned} x &= \cot\left(\frac{\pi}{2} - 2\cot^{-1}\sqrt{3}\right) \\ &= \tan(2\cot^{-1}\sqrt{3}) \qquad \cot\left(\tfrac{\pi}{2} - \theta\right) = \tan\theta \\ &= \tan\left(2\cot^{-1}\left(\cot\frac{\pi}{6}\right)\right) \\ &= \tan\left(2\times\frac{\pi}{6}\right) \\ &= \tan\left(\frac{\pi}{3}\right) = \sqrt{3} \end{aligned}$$

Thus required value is $\sqrt{3}$.

60. Write the principal value of $\left[\cos^{-1}\frac{\sqrt{3}}{2} + \cos^{-1}\left(-\frac{1}{2}\right)\right]$

Sol : Comp 2013

Let x be principal the value of given trigonometric expression.

$$\begin{aligned} x &= \cos^{-1}\frac{\sqrt{3}}{2} + \cos^{-1}\left(-\frac{1}{2}\right) \\ &= \cos^{-1}\frac{\sqrt{3}}{2} + \left[\pi - \cos^{-1}\left(\frac{1}{2}\right)\right] \\ &= \cos^{-1}\left(\cos\frac{\pi}{6}\right) + \left[\pi - \cos^{-1}\left(\cos\frac{\pi}{3}\right)\right] \\ &= \frac{\pi}{6} + \pi - \frac{\pi}{3} \\ &= \frac{5\pi}{6} \end{aligned}$$

which is the required principal value.

61. Write the principal value of $\tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right)$.

Sol : Delhi 2013

Let x be principal the value of given trigonometric expression.

$$\begin{aligned} x &= \tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right) \\ &= \tan^{-1}\left(\tan\frac{\pi}{4}\right) + \cos^{-1}\left(-\cos\frac{\pi}{3}\right) \\ &= \frac{\pi}{4} + \cos^{-1}\left[\cos\left(\pi - \frac{\pi}{3}\right)\right] \\ &= \frac{\pi}{4} + \cos^{-1}\left(\cos\frac{2\pi}{3}\right) \\ &= \frac{\pi}{4} + \frac{2\pi}{3} \\ &= \frac{11\pi}{12} \end{aligned}$$

which is the required principal value.

Alternate Method

We have,

$$\begin{aligned}\tan^{-1}(1)+\cos^{-1}\left(-\frac{1}{2}\right) &= \tan^{-1}(1)+\pi-\cos^{-1}\left(\frac{1}{2}\right)\\ &= \tan^{-1}\left(\tan\frac{\pi}{4}\right)+\pi-\cos^{-1}\left(\cos\frac{\pi}{3}\right)\\ &= \frac{\pi}{4}+\pi-\frac{\pi}{3}\\ &= \frac{3\pi+12\pi-4\pi}{12}=\frac{11\pi}{12}\end{aligned}$$

which is the required principal value.

62. Write the value of $\tan^{-1}\left[2\sin\left(2\cos^{-1}\frac{\sqrt{3}}{2}\right)\right]$.

Sol : OD 2013

Let x be the value of given trigonometric expression.

$$\begin{aligned}x &= \tan^{-1}\left[2\sin\left(2\cos^{-1}\frac{\sqrt{3}}{2}\right)\right]\\ &= \tan^{-1}\left[2\sin\left\{2\cos^{-1}\left(\cos\frac{\pi}{6}\right)\right\}\right]\\ &= \tan^{-1}\left[2\sin\left\{2\times\frac{\pi}{6}\right\}\right]\\ &= \tan^{-1}\left(2\sin\frac{\pi}{3}\right)\\ &= \tan^{-1}\left(2\cdot\frac{\sqrt{3}}{2}\right)\\ &= \tan^{-1}(\sqrt{3})\\ &= \tan^{-1}\left(\tan\frac{\pi}{3}\right)=\frac{\pi}{3}\end{aligned}$$

63. Write the value of $\cos^{-1}\left(\frac{1}{2}\right)-2\sin^{-1}\left(-\frac{1}{2}\right)$.

Sol : Delhi 2012, OD 2009

Let x be the value of given trigonometric expression.

$$\begin{aligned}x &= \cos^{-1}\left(\frac{1}{2}\right)-2\sin^{-1}\left(\frac{-1}{2}\right)\\ &= \cos^{-1}\left(\frac{1}{2}\right)+2\sin^{-1}\left(\frac{1}{2}\right)\\ &= \frac{\pi}{3}+2\sin^{-1}\left(\sin\frac{\pi}{6}\right)\\ &= \frac{\pi}{3}+2\times\frac{\pi}{6}\\ &= \frac{\pi}{3}+\frac{\pi}{3}=\frac{2\pi}{3}\end{aligned}$$

64. Using the principal values, write the value of $\cos^{-1}\frac{1}{2}+2\sin^{-1}\frac{1}{2}$.

Sol : Comp 2012

Let x be the value of given trigonometric expression.

$$\begin{aligned}x &= \cos^{-1}\left(\frac{1}{2}\right)+2\sin^{-1}\left(\frac{1}{2}\right)\\ &= \cos^{-1}\left(\cos\frac{\pi}{3}\right)+2\sin^{-1}\left(\sin\frac{\pi}{6}\right)\\ &= \frac{\pi}{3}+2\times\frac{\pi}{6}\\ &= \frac{\pi}{3}+\frac{\pi}{3}=\frac{2\pi}{3}\end{aligned}$$

65. Write the value of $\sin\left[\frac{\pi}{3}-\sin^{-1}\left(-\frac{1}{2}\right)\right]$.

Sol : Delhi 2011

Let x be the value of given trigonometric expression.

$$\begin{aligned}x &= \sin\left[\frac{\pi}{3}-\sin^{-1}\left(-\frac{1}{2}\right)\right]\\ &= \sin\left[\frac{\pi}{3}+\sin^{-1}\left(\frac{1}{2}\right)\right]\\ &= \sin\left[\frac{\pi}{3}+\sin^{-1}\left(\sin\frac{\pi}{6}\right)\right]\\ &= \sin\left[\frac{\pi}{3}+\frac{\pi}{6}\right]\\ &= \sin\frac{\pi}{2}=1\end{aligned}$$

66. Write the value of $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$.

Sol : Delhi 2011

Here angle $\frac{3\pi}{4}$ doesn't lie in the principal value branch $[-\frac{\pi}{2},\frac{\pi}{2}]$. So we reduce the angle such that, it lies in principal value branch.

We write as $\frac{3\pi}{4}=\pi-\frac{\pi}{4}$

Let x be the value of given trigonometric expression.

$$\begin{aligned}x &= \tan^{-1}\left(\tan\frac{3\pi}{4}\right)\\ &= \tan^{-1}\left[\tan\left(\pi-\frac{\pi}{4}\right)\right]\\ &= \tan^{-1}\left(-\tan\frac{\pi}{4}\right)\\ &= \tan^{-1}\left[\tan\left(-\frac{\pi}{4}\right)\right]\\ &= -\frac{\pi}{4}\end{aligned}$$

Thus required value is $-\frac{\pi}{4}$.

67. Write the value of $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$.

Sol : Delhi 2011, Foreign 2009, OD 2009

Here angle $\frac{7\pi}{6}$ doesn't lie in the principal value branch $[0,\pi]$. So we reduce the angle such that, it lies in principal value branch.

We write as $\frac{7\pi}{6}=2\pi-\frac{5\pi}{6}$

Let x be the value of given trigonometric expression.

Now,
$$\begin{aligned}x &= \cos^{-1}\left(\cos\frac{7\pi}{6}\right)\\ &= \cos^{-1}\left[\cos\left(2\pi-\frac{5\pi}{6}\right)\right]\\ &= \cos^{-1}\left(\cos\frac{5\pi}{6}\right)\\ &= \frac{5\pi}{6}\end{aligned}$$

Thus required value is $\frac{5\pi}{6}$.

68. What is the principal value of $\cos^{-1}\left(\cos\frac{2\pi}{3}\right)+\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$?

Sol : OD 2011, Comp 2009, Delhi 2008

Principal value branch of $\cos^{-1}x$ is $[0,\pi]$ and for $\sin^{-1}x$ is $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.

Since, $\frac{2\pi}{3}\notin\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, so we write $\frac{2\pi}{3}=\left(\pi-\frac{\pi}{3}\right)$

Let x be the value of given trigonometric expression.

$$\begin{aligned} x &= \cos^{-1}\left(\cos\frac{2\pi}{3}\right)+\sin^{-1}\left(\sin\frac{2\pi}{3}\right)\\ &=\frac{2\pi}{3}+\sin^{-1}\left[\sin\left(\pi-\frac{\pi}{3}\right)\right]\\ &=\frac{2\pi}{3}+\sin^{-1}\left(\sin\frac{\pi}{3}\right)\\ &=\frac{2\pi}{3}+\frac{\pi}{3}\\ &=\frac{3\pi}{3}=\pi \end{aligned}$$

which is the required principal value.

69. What is the principal value of $\tan^{-1}(-1)$?

Sol : Foreign 2011, Comp 2008

$$\begin{aligned} \tan^{-1}(-1) &= \tan^{-1}\left(-\tan\frac{\pi}{4}\right)\\ &=\tan^{-1}\left[\tan\left(-\frac{\pi}{4}\right)\right]\\ &=-\frac{\pi}{4} \end{aligned}$$

which is the required principal value.

Alternative :

$$\begin{aligned} \tan^{-1}(-1) &= -\tan^{-1}(1)\\ &=-\tan^{-1}\left(\tan\frac{\pi}{4}\right)\\ &=\frac{-\pi}{4} \end{aligned}$$

which is the required principal value.

70. Using the principal values, write the value of $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$.

Sol : Comp 2011, Delhi 2010

$$\begin{aligned} \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) &= \sin^{-1}\left(-\sin\frac{\pi}{3}\right)\\ &=\sin^{-1}\left[\sin\left(-\frac{\pi}{3}\right)\right]\\ &=-\frac{\pi}{3} \end{aligned}$$

which is the required value.

71. Write the principal value of $\sin^{-1}\left(-\frac{1}{2}\right)$.

Sol : Comp 2011

$$\begin{aligned} \sin^{-1}\left(-\frac{1}{2}\right) &= \sin^{-1}\left(-\sin\frac{\pi}{6}\right)\\ &=\sin^{-1}\left[\sin\left(-\frac{\pi}{6}\right)\right]\\ &=-\frac{\pi}{6} \end{aligned}$$

which is the required value.

72. What is the principal value of $\sec^{-1}(-2)$?

Sol : OD 2010

$$\begin{aligned} \sec^{-1}(-2) &= \pi-\sec^{-1}(2)\\ &=\pi-\sec^{-1}(\sec\frac{\pi}{3})\\ &=\pi-\frac{\pi}{3}\\ &=\frac{2\pi}{3} \end{aligned}$$

which is the required principal value.

73. What is the domain of the function $\sin^{-1}x$?

Sol : Foreign 2010

The domain of the function $\sin^{-1}x$ is $[-1,1]$.

74. Using the principal values, find the value of $\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$.

Sol : Comp 2010

Here angle $\frac{13\pi}{6}$ doesn't lie in the principal value branch $[0,\pi]$. So we reduce the angle such that, it lies in principal value branch.

We write as $\frac{13\pi}{6}=2\pi+\frac{\pi}{6}$

Let x be principal the value of given trigonometric expression.

Now,
$$\begin{aligned} x &= \cos^{-1}\left(\cos\frac{13\pi}{6}\right)\\ &=\cos^{-1}\left[\cos\left(2\pi+\frac{\pi}{6}\right)\right]\\ &=\cos^{-1}\left(\cos\frac{\pi}{6}\right)\\ &=\frac{\pi}{6} \end{aligned}$$

which is the required value.

75. If $\tan^{-1}(\sqrt{3})+\cos^{-1}x=\frac{\pi}{2}$, then find the value of x.

Sol : Comp 2010

We have
$$\begin{aligned} \tan^{-1}\sqrt{3}+\cot^{-1}x &= \frac{\pi}{2}\\ \tan^{-1}\sqrt{3} &= \frac{\pi}{2}-\cot^{-1}x\\ \tan^{-1}\sqrt{3} &= \tan^{-1}x\\ x &= \sqrt{3} \end{aligned}$$

which is the required value.

SHORT ANSWER QUESTIONS

76. If $a=\sin^{-1}\frac{\sqrt{2}}{2}+\cos^{-1}\left(-\frac{1}{2}\right)$ and $b=\tan^{-1}\sqrt{3}-\cot^{-1}\left(-\frac{1}{\sqrt{3}}\right)$ then find the value of $a+b$

Sol : OD 2024

We have
$$\begin{aligned} a &= \sin^{-1}\left(\frac{\sqrt{2}}{2}\right)+\cos^{-1}\left(-\frac{1}{2}\right)\\ &= \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)+\cos^{-1}\left(-\frac{1}{2}\right)\\ &= \sin^{-1}\left(\sin\frac{\pi}{4}\right)+\cos^{-1}\left(\cos\frac{2\pi}{3}\right)\\ &= \frac{\pi}{4}+\frac{2\pi}{3}\\ a &= \frac{11\pi}{12}\end{aligned}$$

and
$$\begin{aligned} b &= \tan^{-1}\sqrt{3}-\cot^{-1}\left(-\frac{1}{\sqrt{3}}\right)\\ &= \frac{\pi}{3}-\left(\pi-\cot^{-1}\left(\frac{1}{\sqrt{3}}\right)\right)\\ &= \frac{\pi}{3}-\left(\pi-\frac{\pi}{3}\right)\\ &= \frac{\pi}{3}-\frac{2\pi}{3}\\ b &= -\frac{\pi}{3}\\ a+b &= \frac{11\pi}{12}+\left(\frac{-\pi}{3}\right)\\ &= \frac{11\pi}{12}-\frac{4\pi}{12}=\frac{7\pi}{12}\end{aligned}$$

77. Show that $\sin^{-1}(2x\sqrt{1-x^2})=2\cos^{-1}x, \frac{1}{\sqrt{2}}\le x\le 1$.

Sol : Delhi 2023

We have $\sin^{-1}(2x\sqrt{1-x^2}) = 2\cos^{-1}x,$

Substituting $x=\cos\theta\Rightarrow\cos^{-1}x=\theta$ we get
$$\begin{aligned}\text{LHS} &= \sin^{-1}(2x\sqrt{1-x^2})\\ &= \sin^{-1}(2\cos\theta\sqrt{1-\cos^2\theta})\\ &= \sin^{-1}(2\cos\theta\sin\theta)\\ &= \sin^{-1}(\sin 2\theta)\\ &= 2\theta\\ &= 2\cos^{-1}\\ x &= \text{RHS}\end{aligned}$$

78. Find the value of $\tan^{-1}\left[2\cos\left(2\sin^{-1}\frac{1}{2}\right)\right]+\tan^{-1}1$

Sol : OD 2023

Let x be the value of given trigonometric expression.
$$\begin{aligned} x &= \tan^{-1}\left[2\cos\left(2\sin^{-1}\tfrac{1}{2}\right)\right]+\tan^{-1}1\\ &= \tan^{-1}\left[2\cos\left(2\sin^{-1}\tfrac{1}{2}\right)\right]+\tan^{-1}1\\ &= \tan^{-1}\left[2\cos\left(2\sin^{-1}(\sin\tfrac{\pi}{6})\right)\right]+\tan^{-1}\left(\tan\tfrac{\pi}{4}\right)\\ &= \tan^{-1}\left[2\cos\left(2\times\tfrac{\pi}{6}\right)\right]+\tfrac{\pi}{4}\\ &= \tan^{-1}\left[2\cos\tfrac{\pi}{3}\right]+\tfrac{\pi}{4}\\ &= \tan^{-1}\left[2\times\tfrac{1}{2}\right]+\tfrac{\pi}{4}\\ &= \tan^{-1}\left[2\times\tfrac{1}{2}\right]+\tfrac{\pi}{4}\\ &= \tan^{-1}(1)+\tfrac{\pi}{4}\\ &= \tfrac{\pi}{4}+\tfrac{\pi}{4}\\ &= \tfrac{\pi}{2}\end{aligned}$$

79. Find the domain of $y=\sin^{-1}(x^2-4)$

Sol : SQP 2023

We have,
$$\begin{aligned} y &= \sin^{-1}(x^2-4)\\ -1 &\le x^2-4\le 1\\ -1+4 &\le x^2\le 1+4\\ 3 &\le x^2\le 5\\ \sqrt{3} &\le |x|\le\sqrt{5}\\ x &\in\left[-\sqrt{5},-\sqrt{3}\right]\cup\left[\sqrt{3},\sqrt{5}\right]\end{aligned}$$

The domain of y is
$$\left[-\sqrt{5},-\sqrt{3}\right]\cup\left[\sqrt{3},\sqrt{5}\right]$$

80. If $\cos\left(\cos^{-1}x+\sin^{-1}\frac{1}{2}\right)=0$, then find x.

Sol : Delhi 2014, OD 2011

$$\begin{aligned}\cos\left(\cos^{-1}x+\sin^{-1}\frac{1}{2}\right) &= 0\\ \cos\left(\cos^{-1}x+\frac{\pi}{6}\right) &= 0\\ \cos^{-1}x+\frac{\pi}{6} &= \cos 0\\ \cos^{-1}x+\frac{\pi}{6} &= \frac{\pi}{2},\frac{3\pi}{2}\end{aligned}$$

Taking $\frac{\pi}{2}$ we have, $\cos^{-1}x=\frac{\pi}{2}-\frac{\pi}{6}=\frac{\pi}{3}$

Taking $\frac{3\pi}{2}$ we have, $\cos^{-1}x=\frac{3\pi}{2}-\frac{\pi}{6}=\frac{4\pi}{3}$

But $\cos^{-1}x$ cannot be greater than π, thus we have
$$\begin{aligned}\cos^{-1}x &= \frac{\pi}{3}\\ x &= \cos\frac{\pi}{3}\\ x &= \frac{1}{2}\end{aligned}$$

81. Evaluate $\sin^{-1}\left(\sin\frac{3\pi}{4}\right)+\cos\left(\cos^{-1}\pi\right)+\tan^{-1}1$.

Sol : Foreing 2023

Let x be the value of given trigonometric expression.

$$\begin{aligned} x &= \sin^{-1}\left(\sin\tfrac{3\pi}{4}\right)+\cos\left(\cos^{-1}\pi\right)+\tan^{-1}1 \\ &= \sin^{-1}\left(\sin\left(\pi-\frac{\pi}{4}\right)\right)+\pi+\frac{\pi}{4} \\ &= \sin^{-1}\left(\sin\frac{\pi}{4}\right)+\pi+\frac{\pi}{4} \\ &= \frac{\pi}{4}+\pi+\frac{\pi}{4} = \frac{3\pi}{2} \end{aligned}$$

82. Solve for $x:\sin^{-1}(1-x)-2\sin^{-1}(x)=\frac{\pi}{2}$.

Sol : OD 2020

We have
$$\begin{aligned} \sin^{-1}(1-x)-2\sin^{-1}x &= \frac{\pi}{2} \\ -2\sin^{-1}x &= \frac{\pi}{2}-\sin^{-1}(1-x) \\ -2\sin^{-1}x &= \cos^{-1}(1-x) \\ \cos(-2\sin^{-1}x) &= (1-x) \\ \cos[2\sin^{-1}(-x)] &= 1-x \\ 1-2\sin^2\{\sin^{-1}(-x)\} &= 1-x \qquad [\cos 2\theta = 1-2\sin^2\theta] \\ 1-2\{\sin(\sin^{-1}(-x))\}^2 &= 1-x \\ 1-2(-x)^2 &= 1-x \\ 2x^2-x &= 0 \\ x(2x-1) &= 0 \\ x &= 0,\ \frac{1}{2} \end{aligned}$$

But, $x=\frac{1}{2}$ does not satisfy the given equation because

$$\sin^{-1}\left(1-\frac{1}{2}\right)-2\sin^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{6}-2\times\frac{\pi}{6}\neq\frac{\pi}{2}$$

83. If $\sin\left(\sin^{-1}\frac{1}{5}+\cos^{-1}x\right)=1$, then find the value of x.

Sol : Delhi 2014

We have
$$\begin{aligned} \sin\left(\sin^{-1}\frac{1}{5}+\cos^{-1}x\right) &= 1 \\ \sin^{-1}\frac{1}{5}+\cos^{-1}x &= \sin^{-1}(1) \\ \sin^{-1}\frac{1}{5}+\cos^{-1}x &= \sin^{-1}\left(\sin\frac{\pi}{2}\right) \\ \sin^{-1}\frac{1}{5}+\cos^{-1}x &= \frac{\pi}{2} \\ \sin^{-1}\frac{1}{5} &= \frac{\pi}{2}-\cos^{-1}x \\ \sin^{-1}\frac{1}{5} &= \sin^{-1}x \end{aligned}$$

Thus $x=\frac{1}{5}$

84. Prove that

$$3\sin^{-1}x = \sin^{-1}(3x-4x^3),\ x\in\left[-\frac{1}{2},\frac{1}{2}\right].$$

Sol : OD 2018

Let $x=\sin\theta$ then $\theta=\sin^{-1}x$. Now we consider RHS.

$$\begin{aligned} \text{RHS} &= \sin^{-1}(3x-4x^3) \qquad \text{...(i)} \\ &= \sin^{-1}(3\sin\theta-4\sin^3\theta) \\ &= \sin^{-1}(\sin 3\theta) \qquad \sin 3A = 3\sin A-4\sin^3 A \\ &= 3\theta \\ &= 3\sin^{-1}x \\ &= \text{LHS} \end{aligned}$$

Hence proved.

85. Prove that $3\cos^{-1}x=\cos^{-1}(4x^3-3x),\ x\in\left[\frac{1}{2},1\right]$.

Sol : Comp 2018

Let $x=\cos\theta\Rightarrow\theta=\cos^{-1}x$. Now we consider RHS.

$$\begin{aligned} \text{RHS} &= \cos^{-1}(4x^3-3x) \qquad \text{...(i)} \\ &= \cos^{-1}(4\cos^3\theta-3\cos\theta) \\ &= \cos^{-1}(\cos 3\theta) \qquad \cos 3A = 4\cos^3 A-3\cos A \\ &= 3\theta \\ &= 3\cos^{-1}x = \text{LHS} \end{aligned}$$

Hence proved.

86. Solve for x,

$$2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x).$$

Sol : Delhi 2016, Foreign 2015, Comp 2014, OD 2009

We have $2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x)$

Using $2\tan^{-1}x=\tan^{-1}\left(\frac{2x}{1-x^2}\right); -1<x<1$ we have

$$\begin{aligned} \tan^{-1}\left(\frac{2\cos x}{1-\cos^2 x}\right) &= \tan^{-1}\left(\frac{2}{\sin x}\right) \\ \frac{2\cos x}{\sin^2 x} &= \frac{2}{\sin x} \\ \sin x\cos x-\sin^2 x &= 0 \\ \sin x(\cos x-\sin x) &= 0 \end{aligned}$$

If $\sin x=0$ then,

$$\sin x = \sin 0 \Rightarrow x=0$$

If $\cos x=\sin x$ then

$$\cot x = 1 = \cot\tfrac{\pi}{4} \Rightarrow x=\tfrac{\pi}{4}$$

But here at $x=0$, the given equation does not exist.

Here, $x=\frac{\pi}{4}$ in the only solution.

87. Prove that $\cos^{-1}(x)+\cos^{-1}\left\{\frac{x}{2}+\frac{\sqrt{3-3x^2}}{2}\right\}=\frac{\pi}{3}$.

Sol : Comp 2014

We have $\cos^{-1}(x)+\cos^{-1}\left\{\frac{x}{2}+\frac{\sqrt{3-3x^2}}{2}\right\}=\frac{\pi}{3}$

Substituting $\cos^{-1}x=\alpha \Rightarrow x=\cos\alpha$ we have

$$\begin{aligned}\text{LHS} &= \alpha+\cos^{-1}\left[\frac{\cos\alpha}{2}+\frac{\sqrt{3}}{2}\sqrt{1-\cos^2\alpha}\right]\\ &= \alpha+\cos^{-1}\left[\cos\frac{\pi}{3}\cos\alpha+\sin\frac{\pi}{3}\sin\alpha\right]\end{aligned}$$

Now using $\cos A\cos B+\sin A\sin B=\cos(A-B)$ we have

$$\begin{aligned}\text{LHS} &= \alpha+\cos^{-1}\left[\cos\left(\frac{\pi}{3}-\alpha\right)\right]\\ &= \alpha+\frac{\pi}{3}-\alpha \text{ RHS}\\ &= \frac{\pi}{3}=\text{RHS}\end{aligned}$$

Hence proved.

88. Solve the following equation for x.

$$\cos(\tan^{-1}x)=\sin\left(\cot^{-1}\frac{3}{4}\right)$$

Sol : Delhi 2017, Foreign 2014; OD 2013

We know that

$$\tan^{-1}x=\cos^{-1}\frac{1}{\sqrt{1+x^2}}$$

and $$\cot^{-1}y=\sin^{-1}\frac{1}{\sqrt{1+y^2}}$$

Now $$\begin{aligned}\cos(\tan^{-1}x) &= \sin\left(\cot^{-1}\frac{3}{4}\right)\\ \cos\left(\cos^{-1}\frac{1}{\sqrt{1+x^2}}\right) &= \sin\left(\sin^{-1}\frac{4}{5}\right)\\ \frac{1}{\sqrt{1+x^2}} &= \frac{4}{5}\\ 16(x^2+1) &= 25\\ 16x^2 &= 9\\ x &= \pm\frac{3}{4}\end{aligned}$$

But $x=\frac{-3}{4}$ does not satisfy the given equation.

Hence, the required solution is $x=\frac{3}{4}$.

89. If $\sin[\cot^{-1}(x+1)]=\cos(\tan^{-1}x)$, then find x.

Sol : Delhi 2015

We know that $$\tan^{-1}z=\cos^{-1}\frac{1}{\sqrt{1+z^2}}$$

and $$\cot^{-1}y=\sin^{-1}\frac{1}{\sqrt{1+y^2}}$$

Now $$\begin{aligned}\sin[\cot^{-1}(x+1)] &= \cos(\tan^{-1}x)\\ \sin\left(\sin^{-1}\frac{1}{\sqrt{1+(x+1)^2}}\right) &= \cos\left(\cos^{-1}\frac{1}{\sqrt{1+x^2}}\right)\\ \frac{1}{\sqrt{1+(1+x)^2}} & \quad \frac{1}{\sqrt{1+x^2}}\\ 1+(1+x)^2 &= 1+x^2\\ (1+x)^2 &= x^2\\ x^2+2x+1 &= x^2\\ 2x+1 &= 0\\ x &= -\frac{1}{2}\end{aligned}$$

Hence, the required solution is $x=-\frac{1}{2}$.

90. Prove that $\sin^{-1}\frac{8}{17}+\sin^{-1}\frac{3}{5}=\cos^{-1}\frac{36}{85}$.

Sol : Comp 2014, Delhi 2012, Foreign 2010

We have $\sin^{-1}\left(\frac{8}{17}\right)+\sin^{-1}\left(\frac{3}{5}\right)=\cos^{-1}\left(\frac{36}{85}\right)$.

Let $\sin^{-1}\left(\frac{8}{17}\right)=x$ and $\sin^{-1}\left(\frac{3}{5}\right)=y$;

Then, $\sin x=\frac{8}{17}$ and $\sin y=\frac{3}{5}$

$\cos x=\frac{15}{17}$ and $\cos y=\frac{4}{5}$

Now $$\begin{aligned}\cos(x+y) &= \cos x\cos y-\sin x\sin y\\ \cos(x+y) &= \left(\frac{15}{17}\times\frac{4}{5}\right)-\left(\frac{8}{17}\times\frac{3}{5}\right)\\ \cos(x+y) &= \frac{60}{85}-\frac{24}{85}=\frac{36}{85}\\ x+y &= \cos^{-1}\left(\frac{36}{85}\right)\end{aligned}$$

$$\sin^{-1}\frac{8}{17}+\sin^{-1}\frac{3}{5}=\cos^{-1}\frac{36}{85}$$

Hence proved.

91. Find the value of the following if $|x|<1$, $y>0$ and $xy<1$:

$$\tan\frac{1}{2}\left[\sin^{-1}\left(\frac{2x}{1+x^2}\right)+\cos^{-1}\left(\frac{1-y^2}{1+y^2}\right)\right],$$

Sol : Delhi 2013

Let $x=\tan\theta$ and $y=\tan\phi$, then

$$\begin{aligned}&\tan\frac{1}{2}\left[\sin^{-1}\left(\frac{2x}{1+x^2}\right)+\cos^{-1}\left(\frac{1-y^2}{1+y^2}\right)\right]\\ &= \tan\frac{1}{2}\left[\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)+\cos^{-1}\left(\frac{1-\tan^2\phi}{1+\tan^2\phi}\right)\right]\\ &= \tan\frac{1}{2}[\sin^{-1}(\sin 2\theta)+\cos^{-1}(\cos 2\phi)]\\ &= \tan\frac{1}{2}[2\theta+2\phi]=\tan(\theta+\phi)\\ &= \tan(\tan^{-1}x+\tan^{-1}y)\\ &= \tan\left[\tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]\\ &= \frac{x+y}{1-xy}\end{aligned}$$

92. Prove that $\tan^{-1}\left[\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1+x}}\right]=\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x$
$-\frac{1}{\sqrt{2}}\le x\le 1$.

Sol : OD 2014, Comp 2014, Delhi 2011

We have $\tan^{-1}\left[\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right] = \dfrac{\pi}{4}-\dfrac{1}{2}\cos^{-1}x$

Substituting $x=\cos 2\theta \Rightarrow \theta = \dfrac{1}{2}\cos^{-1}x$ we have

$$\begin{aligned}\text{LHS} &= \tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)\\ &= \tan^{-1}\left(\frac{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}\right)\\ &= \tan^{-1}\left(\frac{\sqrt{2}\cos\theta-\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta+\sqrt{2}\sin\theta}\right)\\ &= \tan^{-1}\left(\frac{1-\tan\theta}{1+\tan\theta}\right)\\ &= \tan^{-1}\left[\tan\left(\frac{\pi}{4}-\theta\right)\right] \qquad \tfrac{1-\tan\theta}{1+\tan\theta}=\tan\left(\tfrac{\pi}{4}-\theta\right)\\ &= \frac{\pi}{4}-\theta\\ &= \frac{\pi}{4}-\frac{1}{2}\cos^{-1}x\\ &= \text{RHS}\end{aligned}$$

Hence proved.

93. Solve that $\sin^{-1}\dfrac{8}{17}+\sin^{-1}\dfrac{3}{5}=\tan^{-1}\dfrac{77}{36}$.

Sol : Comp 2013

We have $\sin^{-1}\left(\dfrac{8}{17}\right)+\sin^{-1}\left(\dfrac{3}{5}\right) = \tan^{-1}\dfrac{77}{36}$.

Let $\sin^{-1}\left(\dfrac{8}{17}\right) = x$ and $\sin^{-1}\left(\dfrac{3}{5}\right)= y$;

Then, $\sin x = \dfrac{8}{17}$ and $\sin y = \dfrac{3}{5}$

$\cos x = \dfrac{15}{17}$ and $\cos y = \dfrac{4}{5}$

$\tan x = \dfrac{8}{15}$ and $\tan y = \dfrac{3}{4}$

Now,
$$\begin{aligned}\text{LHS} &= x+y = \tan^{-1}\frac{8}{15}+\tan^{-1}\frac{3}{4}\\ &= \tan^{-1}\left(\frac{\frac{8}{15}+\frac{3}{4}}{1-\frac{8}{15}\times\frac{3}{4}}\right)\\ &= \tan^{-1}\left(\frac{32+45}{60-24}\right)\\ &= \tan^{-1}\left(\frac{77}{36}\right) = \text{RHS}\end{aligned}$$

Hence proved.

94. Prove that

$\cos^{-1}\left(\dfrac{4}{5}\right)+\cos^{-1}\left(\dfrac{12}{13}\right)=\cos^{-1}\left(\dfrac{33}{65}\right)$.

Sol : OD 2012, Delhi 2012, Foreign 2005

We have $\cos^{-1}\left(\dfrac{4}{5}\right)+\cos^{-1}\left(\dfrac{12}{13}\right) = \cos^{-1}\left(\dfrac{33}{65}\right)$

Let $\cos^{-1}\dfrac{4}{5} = x$ and $\cos^{-1}\left(\dfrac{12}{13}\right)= y$

$\cos x = \dfrac{4}{5}$ and $\cos y = \dfrac{12}{13}$ ...(i)

Now $\sin x = \dfrac{3}{5}$ and $\sin y = \dfrac{5}{13}$

Now
$$\begin{aligned}\cos(x+y) &= \cos x\cos y-\sin x\sin y\\ &= \left(\frac{4}{5}\times\frac{12}{13}\right)-\left(\frac{3}{5}\times\frac{5}{13}\right)\\ &= \frac{48}{65}-\frac{15}{65}=\frac{33}{65}\end{aligned}$$

or $\cos(x+y) = \dfrac{33}{65}$

$x+y = \cos^{-1}\dfrac{33}{65}$

$\cos^{-1}\dfrac{4}{5}+\cos^{-1}\dfrac{12}{13} = \cos^{-1}\dfrac{33}{65}$ Hence proved.

95. Prove that

$\tan^{-1}\left(\dfrac{\cos x}{1+\sin x}\right) = \dfrac{\pi}{4}-\dfrac{x}{2},\ x\in\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)$.

Sol : Delhi 2012

We have $\tan^{-1}\left(\dfrac{\cos x}{1+\sin x}\right) = \dfrac{\pi}{4}-\dfrac{x}{2},\ x\in\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)$.

$$\begin{aligned}\text{LHS} &= \tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)\\ &= \tan^{-1}\left(\frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}\right)\\ &= \tan^{-1}\left[\frac{(\cos\frac{x}{2}-\sin\frac{x}{2})(\cos\frac{x}{2}+\sin\frac{x}{2})}{(\cos\frac{x}{2}+\sin\frac{x}{2})^2}\right]\\ &= \tan^{-1}\left(\frac{\cos\frac{x}{2}-\sin\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}}\right)\\ &= \tan^{-1}\left(\frac{\frac{\cos\frac{x}{2}}{\cos\frac{x}{2}}-\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}}{\frac{\cos\frac{x}{2}}{\cos\frac{x}{2}}+\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}}\right)\\ &= \tan^{-1}\left(\frac{1-\tan\frac{x}{2}}{1+\tan\frac{x}{2}}\right)\\ &= \tan^{-1}\left[\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right]\\ &= \text{RHS}\end{aligned}$$

96. Prove that $\sin^{-1}\left(\dfrac{63}{65}\right) = \sin^{-1}\left(\dfrac{5}{13}\right)+\cos^{-1}\left(\dfrac{3}{5}\right)$.

Sol : Foreign 2012

We have $\sin^{-1}\left(\dfrac{63}{65}\right) = \sin^{-1}\left(\dfrac{5}{13}\right)+\cos^{-1}\left(\dfrac{3}{5}\right)$

Let $\sin^{-1}\dfrac{5}{13} = x$ and $\cos^{-1}\dfrac{3}{5}= y$,

$\sin x = \dfrac{5}{13}$ and $\cos y = \dfrac{3}{5}$

$\cos x = \dfrac{12}{13}$ and $\sin y = \dfrac{4}{5}$

Now,
$$\begin{aligned}\sin(x+y) &= \sin x\cos y+\cos x\sin y\\ &= \frac{5}{13}\cdot\frac{3}{5}+\frac{12}{13}\cdot\frac{4}{5}\\ &= \frac{15}{65}+\frac{48}{65}=\frac{63}{65}\end{aligned}$$

$\sin(x+y) = \dfrac{63}{65}$

$x+y = \sin^{-1}\left(\dfrac{63}{65}\right)$

$$\sin^{-1}\left(\frac{5}{13}\right)+\cos^{-1}\left(\frac{3}{5}\right)=\sin^{-1}\left(\frac{63}{65}\right)$$ Hence proved.

97. Solve for x,

$$2\tan^{-1}(\sin x) = \tan^{-1}(2\sec x),\ x \neq \frac{\pi}{2}.$$

Sol : SQP 2012

We have $2\tan^{-1}(\sin x)=\tan^{-1}(2\sec x),\ x \neq \frac{\pi}{2}$

Using $2\tan^{-1}x=\tan^{-1}\left(\frac{2x}{1-x^2}\right); -1 < x < 1$ we have

$$\tan^{-1}\left(\frac{2\sin x}{1-\sin^2 x}\right) = \tan^{-1}\left(\frac{2}{\cos x}\right)$$

$$\frac{2\sin x}{\cos^2 x} = \frac{2}{\cos x}$$

$$\sin x\cos x-\cos^2 x = 0$$

$$\cos x(\sin x-\cos x) = 0$$

If $\cos x = 0$ then,

$$\cos x = \cos\tfrac{\pi}{2} \Rightarrow x=\tfrac{\pi}{2}$$

If $\sin x=\cos x$ then

$$\tan x = 1=\tan\tfrac{\pi}{4} \Rightarrow x=\tfrac{\pi}{4}$$

But here at $x=0$, the given equation does not exist.

Here, $x=\frac{\pi}{4}$ in the only solution.

98. Solve for x, $\sin^{-1}(1-x)-2\sin^{-1}x=\frac{\pi}{2}$.

Sol : Comp 2013, OD 2010

We have $\sin^{-1}(1-x)-2\sin^{-1}x = \frac{\pi}{2}$

$$\sin^{-1}(1-x) = \frac{\pi}{2}+2\sin^{-1}x$$

$$1-x = \sin\left(\frac{\pi}{2}+2\sin^{-1}x\right)$$

$$1-x = \cos(2\sin^{-1}x) \qquad \sin\left(\tfrac{\pi}{2}+\theta\right)=\cos\theta$$

Substituting $\sin^{-1}x=\theta$, $\Rightarrow x=\sin\theta$ we have

$$1-x = \cos(2\theta)$$

$$1-x = 1-2\sin^2\theta \qquad \cos 2A=1-2\sin^2 A$$

$$1-x = 1-2x^2$$

$$2x^2-x = 0$$

$$x(2x-1) = 0$$

Thus $x=0$ or $x=\frac{1}{2}$

For $x=\frac{1}{2}$, LHS $=\sin^{-1}\left(\frac{1}{2}\right)-2\sin^{-1}\left(\frac{1}{2}\right)$

$$=\frac{\pi}{6}-\frac{2\pi}{6}=\frac{-\pi}{6}\neq\frac{\pi}{2}$$

Thus $x=\frac{1}{2}$ is not a solution of given equation.

Hence, $x=0$ is the only solution.

99. Solve for x, $\cos(2\sin^{-1}x)=\frac{1}{9}$; $x>0$.

Sol : Comp 2011

We have $\cos(2\sin^{-1}x) = \frac{1}{9},\ x>0$

Substituting $\sin^{-1}x = y$ we have $x = \sin y$ given equation becomes

$$\cos 2y = = \frac{1}{9}$$

$$1-2\sin^2 y = \frac{1}{9}$$

$$2\sin^2 y = 1-\frac{1}{9}=\frac{8}{9}$$

$$\sin^2 y = \frac{4}{9}$$

$$x^2 = \frac{4}{9} \Rightarrow x = \pm\frac{2}{3}$$

But it given that, $x>0$ thus $x = \frac{2}{3}$

100. Prove that $\cos^{-1}\left(\frac{12}{13}\right)+\sin^{-1}\left(\frac{3}{5}\right) = \sin^{-1}\left(\frac{56}{65}\right)$.

Sol : Delhi 2010

We have $\cos^{-1}\left(\frac{12}{13}\right)+\sin^{-1}\left(\frac{3}{5}\right) = \sin^{-1}\left(\frac{56}{65}\right)$

Let $\cos^{-1}\left(\frac{12}{13}\right)= x$ and $\sin^{-1}\left(\frac{3}{5}\right)= y$,

$$\cos x = \frac{12}{13} \text{ and } \sin y = \frac{3}{5}$$

$$\sin x = \frac{5}{13} \text{ and } \cos y=\frac{4}{5}$$

Now, $\sin(x+y) = \sin x\cos y+\cos x\sin y$

$$= \frac{5}{13}\cdot\frac{4}{5}+\frac{12}{13}\cdot\frac{3}{5}$$

$$= \frac{20}{65}+\frac{36}{65}=\frac{56}{65}$$

or $\sin(x+y) = \frac{56}{65}$

$$x+y = \sin^{-1}\left(\frac{56}{65}\right)$$

$$\cos^{-1}\left(\frac{12}{13}\right)+\sin^{-1}\left(\frac{3}{5}\right) = \sin^{-1}\left(\frac{56}{65}\right)$$ Hence Proved

LONG ANSWER QUESTIONS

101. Express $\tan^{-1}\left(\frac{\cos x}{1-\sin x}\right), \frac{-3\pi}{2}<x<\frac{\pi}{2}$ in the simplest form.

Sol : SQP 2023

Let x be the value of given trigonometric expression in simplest form.

$$x = \tan^{-1}\left(\frac{\cos x}{1-\sin x}\right)$$

$$= \tan^{-1}\left[\frac{\sin\left(\frac{\pi}{2}-x\right)}{1-\cos\left(\frac{\pi}{2}-x\right)}\right]$$

$$= \tan^{-1}\left[\frac{2\sin\left(\frac{\pi}{4}-\frac{x}{2}\right)\cos\left(\frac{\pi}{4}-\frac{x}{2}\right)}{2\sin^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}\right]$$

$$= \tan^{-1}\cot\left(\frac{\pi}{4}-\frac{x}{2}\right)$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{2}-\left(\frac{\pi}{4}-\frac{x}{2}\right)\right)\right]$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right]$$

Now, $-\frac{3\pi}{2} < x < \frac{\pi}{2}$

$$-\frac{3\pi}{2} < \frac{x}{2} < \frac{\pi}{4}$$

$$-\frac{\pi}{2} < \frac{x}{2}+\frac{\pi}{4} < \frac{\pi}{2}$$

$$x = \tan^{-1}\left(\frac{\cos x}{1-\sin x}\right)$$

$$= \tan^{-1}\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$$

$$= \frac{\pi}{4}+\frac{x}{2}$$

102. Prove that if $\frac{1}{2} \leq x \leq 1$, then

$$\cos^{-1}x + \cos^{-1}\left(\frac{x}{2}+\frac{\sqrt{3-3x^2}}{2}\right) = \frac{\pi}{3}$$

Sol : SQP 2018

Let $\cos^{-1}x = \theta$

Then, for all $x \in \left[\frac{1}{2}, 1\right]$, $\theta \in \left[0, \frac{\pi}{3}\right]$

Thus $x = \cos\theta$

We have, $\cos^{-1}x + \cos^{-1}\left(\frac{x}{2}+\frac{\sqrt{3-3x^2}}{2}\right) = \frac{\pi}{3}$

$$\text{LHS} = \cos^{-1}x + \cos^{-1}\left(\frac{x}{2}+\frac{\sqrt{3-3x^2}}{2}\right)$$

$$= \theta + \cos^{-1}\left[\frac{\cos\theta}{2}+\frac{\sqrt{3-3\cos^2\theta}}{2}\right]$$

$$= \theta + \cos^{-1}\left[\frac{\cos\theta}{2}+\frac{\sqrt{3}\sqrt{1-\cos^2\theta}}{2}\right]$$

$$= \theta + \cos^{-1}\left[\frac{\cos\theta}{2}+\frac{\sqrt{3}\sqrt{\sin^2\theta}}{2}\right]$$

$$= \theta + \cos^{-1}\left[\frac{\cos\theta}{2}+\frac{\sqrt{3}\sin\theta}{2}\right]$$

$$= \theta + \cos^{-1}\left[\cos\theta\cdot\frac{1}{2}+\frac{\sqrt{3}}{2}\cdot\sin\theta\right]$$

$$= \theta + \cos^{-1}\left[\cos\theta\cdot\cos\frac{\pi}{3}+\sin\frac{\pi}{3}\sin\theta\right]$$

$$= \theta + \cos^{-1}\left[\cos\left(\theta-\frac{\pi}{3}\right)\right]$$

$$= \theta + \cos^{-1}\left[\cos\left(\frac{\pi}{3}-\theta\right)\right]$$

$$= \theta + \frac{\pi}{3} - \theta$$

$$= \frac{\pi}{3}$$

$$= \text{RHS}$$

Hence Proved

103. Find simplest form of $\tan^{-1}\left(\frac{\sqrt{1+\cos x}+\sqrt{1-\cos x}}{\sqrt{1+\cos x}-\sqrt{1-\cos x}}\right)$, $\pi < x < \frac{3\pi}{2}$.

Sol :

Let x be the simplest form of given trigonometric expression.

$$x = \tan^{-1}\left(\frac{\sqrt{1+\cos x}+\sqrt{1-\cos x}}{\sqrt{1+\cos x}-\sqrt{1-\cos x}}\right) \quad \pi < x < \tfrac{3\pi}{2}$$

$$= \tan^{-1}\left(\frac{\sqrt{2}\left|\cos\frac{x}{2}\right|+\sqrt{2}\left|\sin\frac{x}{2}\right|}{\sqrt{2}\left|\cos\frac{x}{2}\right|-\sqrt{2}\left|\sin\frac{x}{2}\right|}\right) \quad \pi < x < \tfrac{3\pi}{2}$$

$$= \tan^{-1}\left(\frac{-\cos\frac{x}{2}+\sin\frac{x}{2}}{-\cos\frac{x}{2}-\sin\frac{x}{2}}\right) \quad \left[\begin{array}{l}\cos\frac{x}{2}<0\ x\in\left(\pi,\frac{3\pi}{2}\right)\\ \sin\frac{x}{2}>0\ x\in\left(\pi,\frac{3\pi}{2}\right)\end{array}\right]$$

$$= \tan^{-1}\left(\frac{\cos\frac{x}{2}-\sin\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}}\right)$$

$$= \tan^{-1}\left(\frac{1-\tan\frac{x}{2}}{1+\tan\frac{x}{2}}\right)$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right]$$

$$= \frac{\pi}{4}-\frac{x}{2}$$

104. Find the value of $\sin\left(\cos^{-1}\frac{4}{5}+\tan^{-1}\frac{2}{3}\right)$.

Sol : OD 2019, Comp 2017

Let $x = \cos^{-1}\frac{4}{5}$ then $x = \tan^{-1}\frac{3}{4}$

Now,

$$\sin\left(\cos^{-1}\frac{4}{5}+\tan^{-1}\frac{2}{3}\right) = \sin\left(\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{2}{3}\right)$$

$$= \sin\left(\tan^{-1}\left(\frac{\frac{3}{4}\times\frac{2}{3}}{1-\frac{3}{4}\times\frac{2}{3}}\right)\right)$$

$$= \sin\left(\tan^{-1}\frac{17}{6}\right)$$

Now, let $\tan^{-1}\left(\frac{17}{6}\right) = y$ then $\tan y = \frac{17}{6}$

$$\sin y = \frac{17}{\sqrt{6^2+17^2}} = \frac{17}{\sqrt{325}} = \frac{17}{5\sqrt{13}}$$

$$y = \sin^{-1}\left(\frac{17}{5\sqrt{13}}\right)$$

$$= \tan^{-1}\left(\frac{17}{6}\right)$$

Hence, $\sin\left(\cos^{-1}\frac{4}{5}+\tan^{-1}\frac{2}{3}\right) = \sin\left(\sin^{-1}\frac{17}{5\sqrt{13}}\right)$

$$= \frac{17}{5\sqrt{13}}$$

which is the required value.

105. Prove that

$$\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right) = \frac{x}{2},$$

where$0 < x < \frac{\pi}{2}$, or $x \in \left(0, \frac{\pi}{4}\right)$.

Sol : Foreign 2016, Delhi 2014, Delhi 2011, OD 2009

$$1+\sin x = \sin^2\frac{x}{2}+\cos^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}$$

$$= \left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2$$

$$\sqrt{1+\sin x} = \cos\frac{x}{2}+\sin\frac{x}{2} \quad (1)$$

Similarity we have

$$\sqrt{1-\sin x} = \cos\frac{x}{2}-\sin\frac{x}{2} \quad (2)$$

Adding (1) and (2) we have

$$\sqrt{1+\sin x}+\sqrt{1-\sin x} = 2\cos\frac{x}{2}$$

Subtracting (2) from (1) we have

$$\sqrt{1+\sin x}-\sqrt{1-\sin x} = 2\sin\frac{x}{2}$$

Now LHS of given equation becomes

$$\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right) = \cot^{-1}\left(\frac{2\cos\frac{x}{2}}{2\sin\frac{x}{2}}\right)$$

$$= \cot^{-1}\left(\cot\frac{x}{2}\right)$$

$$= \frac{x}{2}$$

Since LHS = RHS Hence proved.

Alternate Method

$$\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}$$

$$= \frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}} \times \frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}+\sqrt{1-\sin x}}$$

$$= \frac{\left(\sqrt{1+\sin x}+\sqrt{1-\sin x}\right)^2}{\left(\sqrt{1+\sin x}\right)^2-\left(\sqrt{1-\sin x}\right)^2}$$

$$= \frac{1+\sin x+1-\sin x+2\sqrt{1-\sin^2 x}}{1+\sin x-1+\sin x}$$

$$= \frac{2+2\cos x}{2\sin x}$$

$$= \cot\frac{x}{2}$$

Now $\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right) = \cot^{-1}\left(\cot\frac{x}{2}\right)$

$$= \frac{x}{2}$$

Since LHS = RHS Hence proved.

106. If $(\tan^{-1}x)^2+(\cot^{-1}x)^2 = \frac{5\pi^2}{8}$, then find x.

Sol : Delhi 2015

We have $(\tan^{-1}x)^2+(\cot^{-1}x)^2 = \frac{5\pi^2}{8}$

$$(\tan^{-1}x)^2+\left(\frac{\pi}{2}-\tan^{-1}\right)^2 = \frac{5\pi^2}{8}$$

$$(\tan^{-1}x)^2+\left(\frac{\pi}{2}\right)^2-2\times\frac{\pi}{2}\times\tan^{-1}x+(\tan^{-1}x)^2 = \frac{5\pi^2}{8}$$

$$2(\tan^{-1}x)^2+\frac{\pi^2}{4}-\pi\tan^{-1}x = \frac{5\pi^2}{8}$$

$$2(\tan^{-1}x)^2-\pi\tan^{-1}x = \frac{5\pi^2}{8}-\frac{\pi^2}{4}$$

$$2(\tan^{-1}x)^2-\pi\tan^{-1}x = \frac{3\pi^2}{8}$$

Let $\tan^{-1}x = \theta$, where $\theta \in \left(\frac{-\pi}{2},\frac{\pi}{2}\right)$

Then we have $2\theta^2-\pi\theta = \frac{3\pi^2}{8}$

$$16\theta^2-8\pi\theta-3\pi^2 = 0$$

$$16\theta^2-12\pi\theta+4\pi\theta-3\pi^2 = 0$$

$$4\theta(4\theta-3\pi)+\pi(4\theta-3\pi) = 0$$

$$(4\theta+\pi)(4\theta-3\pi) = 0$$

Thus $\theta = -\frac{\pi}{4}$ or $\theta = \frac{3\pi}{4}$

But $\theta \in \left(\frac{-\pi}{2},\frac{\pi}{2}\right)$, so $\theta \neq \frac{3\pi}{4}$

Now, $\theta = \frac{-\pi}{4}$

$$\tan^{-1}x = \frac{-\pi}{4}$$

$$x = \tan\left(-\frac{\pi}{4}\right) = -\tan\frac{\pi}{4}$$

Thus $x = -1$

107. If $\tan^{-1}\left(\frac{1}{1+1\cdot 2}\right)+\tan^{-1}\left(\frac{1}{1+2\cdot 3}\right)+...+\tan^{-1}\left(\frac{1}{1+n(n+1)}\right) = \tan^{-1}\theta$, then find the value fo θ.

Sol : Foreign 2015, OD 2012

Here we convert each inverse trigonometric function in the form of $\tan^{-1}\left(\frac{x-1}{1+xy}\right)$ and then use the formula

$$\tan^{-1}\left(\frac{x-y}{1+xy}\right) = \tan^{-1}x-\tan^{-1}y;\ xy > -1.$$

Here n th therm is

$$\tan^{-1}\left(\frac{1}{1+n(n+1)}\right) = \tan^{-1}\left(\frac{(n+1)-n}{1+n(n+1)}\right)$$

$$= \tan^{-1}(n-1)-\tan^{-1}(n)$$

Thus given equation reduce to

$$\tan^{-1}\left(\frac{1}{1+1\cdot 2}\right)+\tan^{-1}\left(\frac{1}{1+2\cdot 3}\right)+...+\tan^{-1}\left(\frac{1}{1+n(n+1)}\right) = \tan^{-1}\theta$$

$$\tan^{-1}(2)-\tan^{-1}(1)+\tan^{-1}(3)-\tan^{-1}(2)+...+\tan^{-1}(n+1)-\tan^{-1}(n) = \tan^{-1}\theta$$

Thus $\tan^{-1}(n+1)-\tan^{-1}(1) = \tan^{-1}\theta$

$$\tan^{-1}\left(\frac{n+1-1}{1+(n+1)\cdot 1}\right) = \tan^{-1}\theta$$

$$\tan^{-1}\left(\frac{n}{1+n+1}\right) = \tan^{-1}\theta$$

$$\theta = \frac{n}{2+n}$$

108. Show that $\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right) = \frac{4-\sqrt{7}}{3}$.

Sol : OD 2013

We have $\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right) = \frac{4-\sqrt{7}}{3}$

$$\text{LHS} = \tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right) \quad ...(1)$$

Let $\frac{1}{2}\sin^{-1}\left(\frac{3}{4}\right) = \theta$ ...(2)

Then, $\sin^{-1}\left(\frac{3}{4}\right) = 2\theta \Rightarrow \sin 2\theta = \frac{3}{4}$

Now, $\sin 2\theta = \frac{3}{4}$

$$\frac{2\tan\theta}{1+\tan^2\theta} = \frac{3}{4} \qquad \sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}$$

$$8\tan\theta = 3 + 3\tan^2\theta$$

$$3\tan^2\theta - 8\tan\theta + 3 = 0$$

Now, by quadratic formula we have

$$\tan\theta = \frac{-(-8)\pm\sqrt{(-8)^2 - 4\times 3\times 3}}{2\times 3}$$

$$= \frac{8\pm\sqrt{64-36}}{6}$$

$$= \frac{8\pm\sqrt{28}}{6}$$

Thus $\tan\theta = \frac{4\pm\sqrt{7}}{3}$

As, $-\frac{\pi}{2} \le 2\theta \le \frac{\pi}{2}$

, $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$

$$-1 \le \tan\theta \le 1$$

$$\tan\theta = \frac{4-\sqrt{7}}{3} \qquad \frac{4+\sqrt{7}}{3} > 1$$

$$\theta = \tan^{-1}\left(\frac{4-\sqrt{7}}{3}\right)$$

$$\frac{1}{2}\sin^{-1}\left(\frac{3}{4}\right) = \tan^{-1}\left(\frac{4-\sqrt{7}}{3}\right)$$

$$\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right) = \left(\frac{4-\sqrt{7}}{3}\right)$$

$$\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right) = \frac{4-\sqrt{7}}{3} = \text{RHS}$$

Hence proved.

109. Prove the following.

$$\cos\left(\cos^{-1}\frac{3}{5} + \cot^{-1}\frac{3}{2}\right) = \frac{6}{5\sqrt{13}}$$

Sol : OD 2012

We have $\cos\left(\sin^{-1}\frac{3}{5} + \cot^{-1}\frac{3}{2}\right) = \frac{6}{5\sqrt{13}}$

Let $\sin^{-1}\frac{3}{5} = x$ then

$$\sin x = \frac{3}{5} \text{ and } \cos x = \frac{4}{5}$$

Let $\cot^{-1}\left(\frac{3}{2}\right) = y$ then

$$\cot y = \frac{3}{2},\ \sin y = \frac{2}{\sqrt{13}} \text{ and } \cos y = \frac{3}{\sqrt{13}}$$

Now,

$$\cos(x+y) = \cos x\cos y - \sin x\sin y$$

$$= \frac{4}{5}\times\frac{3}{\sqrt{13}} - \frac{3}{5}\times\frac{2}{\sqrt{13}}$$

$$= \frac{12}{5\sqrt{13}} - \frac{6}{5\sqrt{13}} \text{ RHS}$$

$$= \frac{6}{5\sqrt{13}} = \text{RHS}$$

Hence proved.

110. Prove that

$$\frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right) = \frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right).$$

Sol : Foreign 2011, Comp 2010

We have $\frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right) = \frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$.

Let $\sin^{-1}\left(\frac{1}{3}\right) = x$ and $\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) = y$ then

$$\sin x = \frac{1}{3} \text{ and } \sin y = \frac{2\sqrt{2}}{3}$$

$$\cos x = \frac{2\sqrt{2}}{3} \text{ and } \cos y = \frac{1}{3}$$

Now, $\sin(x+y) = \sin x\cos y + \cos x\sin y$

$$= \frac{1}{3}\times\frac{1}{3} + \frac{2\sqrt{2}}{3}\times\frac{2\sqrt{2}}{3}$$

$$= \frac{1}{9} + \frac{8}{9} = \frac{9}{9} = 1$$

$$\sin(x+y) = 1$$

$$x + y = \sin^{-1}(1)$$

$$= \sin^{-1}\left(\sin\frac{\pi}{2}\right) = \frac{\pi}{2}$$

Thus $x + y = \frac{\pi}{2}$

$$\sin^{-1}\left(\frac{1}{3}\right) + \sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) = \frac{\pi}{2}$$

$$\frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right) + \frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) = \frac{9\pi}{8}$$

$$\frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right) = \frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) \text{ Hence proved.}$$

Alternative :

$$\text{LHS} = \frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right)$$

$$= \frac{9}{4}\left[\frac{\pi}{2} - \sin^{-1}\left(\frac{1}{3}\right)\right]$$

$$= \frac{9}{4}\left[\cos^{-1}\left(\frac{1}{3}\right)\right]$$

$$= \frac{9}{4}\sin^{-1}\left(\sqrt{1-\frac{1}{9}}\right)$$

$$= \frac{9}{4}\sin^{-1}\left(\sqrt{\frac{8}{9}}\right)$$
$$= \frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$$
$$= \text{RHS}$$

Hence proved.

111. Solve for x,

$$\tan^{-1}\left(\frac{2x}{1-x^2}\right) + \cot^{-1}\left(\frac{1-x^2}{2x}\right) = \frac{\pi}{3},\ -1 < x < 1.$$

Sol : Comp 2011

$$\tan^{-1}\left(\frac{2x}{1-x^2}\right) + \cot^{-1}\left(\frac{1-x^2}{2x}\right) = \frac{\pi}{3}$$
$$\tan^{-1}\left(\frac{2x}{1-x^2}\right) + \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \frac{\pi}{3}$$
$$2\tan^{-1}\left(\frac{2x}{1-x^2}\right) = \frac{\pi}{3}$$
$$\tan^{-1}\left(\frac{2x}{1-x^2}\right) = \frac{\pi}{6}$$
$$\frac{2x}{1-x^2} = \tan\frac{\pi}{6}$$
$$\frac{2x}{1-x^2} = \frac{1}{\sqrt{3}}$$
$$2\sqrt{3}\,x = 1-x^2$$
$$x^2 + 2\sqrt{3}\,x - 1 = 0$$

By quadratic formula we have

$$x = \frac{-2\sqrt{3} \pm \sqrt{12+4}}{2}$$
$$= \frac{-2\sqrt{3} \pm 4}{2}$$
$$= \frac{4-2\sqrt{3}}{2},\ \frac{-4-2\sqrt{3}}{2}$$

Thus $x = 2-\sqrt{3}$ or $-(2+\sqrt{3})$

But it is given that $-1 < x < 1$, so $x = -(2+\sqrt{3})$ is rejected, hence $x = 2-\sqrt{3}$.

112. Prove that

$$\tan^{-1}\sqrt{x} = \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right),\ x \in (0,1).$$

Sol : Delhi 2010, OD 2008

We have $\tan^{-1}\sqrt{x} = \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right),\ x \in (0,1).$

$$\text{RHS} = \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right)$$
$$= \frac{1}{2}\cos^{-1}\left[\frac{1-(\sqrt{x})^2}{1+(\sqrt{x})^2}\right]$$

Substituting $\sqrt{x} = \tan\theta$, we have

$$\text{RHS} = \frac{1}{2}\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$$
$$= \frac{1}{2}\cos^{-1}(\cos 2\theta) \qquad \frac{1-\tan^2 A}{1+\tan^2 A} = \cos 2A$$
$$= \frac{1}{2}(2\theta) = \theta$$
$$= \tan^{-1}\sqrt{x}$$
$$= \text{LHS}$$

Hence proved.

113. Prove that $\cos[\tan^{-1}\{\sin(\cot^{-1}x)\}] = \sqrt{\frac{1+x^2}{2+x^2}}$.

Sol : OD 2010

We have $\cos[\tan^{-1}\{\sin(\cot^{-1}x)\}] = \sqrt{\frac{1+x^2}{2+x^2}}$

$$\text{LHS} = \cos[\tan^{-1}\{\sin(\cot^{-1}x)\}]$$

Substituting $\cot^{-1}x = \theta \Rightarrow x = \cot\theta$ we have

$$\text{LHS} = \cos[\tan^{-1}(\sin\theta)]$$
$$= \cos\left[\tan^{-1}\left(\frac{1}{\operatorname{cosec}\theta}\right)\right]$$
$$= \cos\left[\tan^{-1}\left(\frac{1}{\sqrt{1+\cot^2\theta}}\right)\right]$$
$$= \cos\left[\tan^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)\right]$$
$$= \cos\phi$$

where, $\tan^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right) = \phi$ or $\tan\phi = \frac{1}{\sqrt{1+x^2}}$

Now, $\text{LHS} = \cos\phi$

$$= \frac{1}{\sec\phi}$$
$$= \frac{1}{\sqrt{1+\tan^2\phi}}$$
$$= \frac{1}{\sqrt{1+\frac{1}{1+x^2}}}$$
$$= \frac{1}{\sqrt{\frac{1+x^2+1}{1+x^2}}}$$
$$= \sqrt{\frac{1+x^2}{2+x^2}} = \text{RHS}$$

Hence proved.

114. Solve for x, $\cos^{-1}x + \sin^{-1}\left(\frac{x}{2}\right) = \frac{\pi}{6}$.

Sol : Comp 2010

We have $\cos^{-1}x + \sin^{-1}\left(\frac{x}{2}\right) = \frac{\pi}{6}$

$$\cos^{-1}x = \frac{\pi}{6} - \sin^{-1}\frac{x}{2}$$
$$x = \cos\left(\frac{\pi}{6} - \sin^{-1}\frac{x}{2}\right)$$

Using $\cos(x-y) = \cos x\cos y + \sin x\sin y$ we have

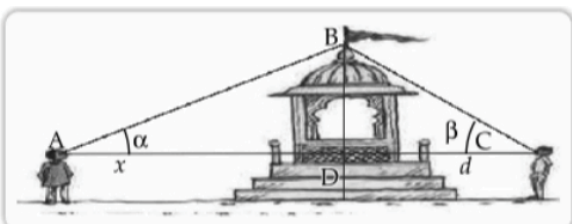
$$x = \cos\frac{\pi}{6}\cos\left(\sin^{-1}\frac{x}{2}\right) + \sin\frac{\pi}{6}\sin\left(\sin^{-1}\frac{x}{2}\right)$$
$$x = \frac{\sqrt{3}}{2}\cos\left(\cos^{-1}\sqrt{1-\frac{x^2}{4}}\right) + \frac{x}{4}$$
$$x = \frac{\sqrt{3}}{2}\left(\sqrt{1-\frac{x^2}{4}}\right) + \frac{x}{4}$$

$$x - \frac{x}{4} = \frac{\sqrt{3}}{2}\left(\sqrt{1-\frac{x^2}{4}}\right)$$
$$\frac{3x}{4} = \frac{\sqrt{3}}{2}\left(\sqrt{1-\frac{x^2}{4}}\right)$$
$$\frac{9x^2}{16} = \frac{3}{4}\left(1-\frac{x^2}{4}\right)$$
$$\frac{3}{4}x^2 = 1-\frac{x^2}{4}$$
$$\frac{3}{4}x^2 + \frac{x^2}{4} = 1$$
$$\frac{4x^2}{4} = 1$$
$$x^2 = 1 \Rightarrow x = \pm 1$$

But $x = -1$, does not satisfy the given equation. Hence, $x = 1$ satisfy the given equation.

CASE BASED QUESTIONS

115. Two mean on either side of a temple of 5 meters high observe its top at the angles of elevation α and β respectively. (as shown in the figure above).



The distance between the two men is 25 metres and the distance between the first person A and the temple is 12 meters.

(i) Find the value of α.

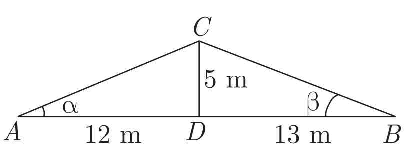
(ii) Find the value of $\cos^{-1}\frac{5}{13}$

(iii) Find the value of β.

(ii) Find the value of $\cot^{-1}\frac{13}{5}$

Sol :

We make the figure as shown below.



(i) In ΔADC

$$AC^2 = AD^2 + CD^2$$
$$= 12^2 + 5^2$$
$$= 13^2$$
$$AC = 13 \text{ m}$$

Now $\sin\alpha = \frac{CD}{AC} = \frac{5}{13}$

i.e. $\alpha = \sin^{-1}\frac{5}{13}$

(ii) $\cos^{-1}\frac{5}{13}$

$$\cos^{-1}a + \sin^{-1}a = \frac{\pi}{2}$$
$$\cos^{-1}\tfrac{5}{13} + \sin^{-1}\tfrac{5}{13} = \frac{\pi}{2}$$
$$\cos^{-1}\tfrac{5}{13} = \tfrac{\pi}{2} - \sin^{-1}\tfrac{5}{13}$$

(iii) In ΔBDC

$$\tan\beta = \frac{CD}{DB} = \frac{13}{5}$$

i.e. $\beta = \tan^{-1}\frac{13}{5}$

(iv) $\cot^{-1}\frac{13}{5}$

$$\tan^{-1}a + \cot^{-1}a = \frac{\pi}{2}$$
$$\tan^{-1}\tfrac{13}{5} + \cot^{-1}\tfrac{13}{5} = \frac{\pi}{2}$$
$$\cot^{-1}\tfrac{13}{5} = \tfrac{\pi}{2} - \tan^{-1}\tfrac{13}{5}$$

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CHAPTER 3

MATRICES

OBJECTIVE QUESTIONS

1. The product of matrix P and Q is equal to a diagonal matrix. If the order of matrix Q is 3×2, then order of matrix P is

(a) 2×2 (b) 3×3

(c) 2×3 (d) 3×2

Sol : OD 2024

Let P is $m \times n$ matrix. Given PQ is diagonal matrix Since diagonal matrix is always square matrix, PQ is square matrix. Thus PQ is defined

Given Q is 3×2 matrix. For PQ has to be square matrix m has to be 2 and n has to be 3

i.e. $m = 2$

$n = 3$

2. Find the matrix A^2, where $A = [a_{ij}]$ is a 2×2 matrix whose elements are given by $a_{ij} =$ maximum $(i, j) -$ minimum

(a) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

Sol : OD 2024

We have $A = [a_{ij}]$ is 2×2 matrix

Now $a_{ij} =$ maximum $(i, j) -$ minimum (i, j)

$$a_{11} = \max(1,1) - \min(1,1)$$
$$= 1 - 1 = 0$$
$$a_{12} = \max(1,2) - \min(1,2)$$
$$= 2 - 1 = 1$$
$$a_{21} = \max(2,1) - \min(2,1)$$
$$= 2 - 1 = 1$$
$$a_{22} = \max(2,2) - \min(2,2)$$
$$= 2 - 2 = 0$$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
$$A^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus (c) is correct option.

3. If A is a 2×3 matrix such that AB and AB' both are defined, then order of matrix B is

(a) 2×2 (b) 2×1

(c) 3×2 (d) 3×3

Sol : OD 2023

Let order of matrix B is $m \times n$.

AB and AB' both are defined.

$$[A]_{2 \times 3}[B]_{m \times n} = [AB]_{2 \times n}$$

and $m = 3$

Similarly, $[A]_{2 \times 3}[B']_{n \times m} = [AB']_{2 \times m}$

and $n = 3$

Therefore, the order of matrix B is 3×3.

Thus (d) is correct option.

4. If $\begin{bmatrix} 2 & 0 \\ 5 & 4 \end{bmatrix} = P + Q$ where P is a symmetric and Q is a skew symmetric matrix, then Q is equal to :

(a) $\begin{bmatrix} 2 & \frac{5}{2} \\ \frac{5}{2} & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & -\frac{5}{2} \\ \frac{5}{2} & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & \frac{5}{2} \\ -\frac{5}{2} & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & -\frac{5}{2} \\ \frac{5}{2} & 4 \end{bmatrix}$

Sol : OD 2023

Let $A = \begin{bmatrix} 2 & 0 \\ 5 & 4 \end{bmatrix}$

Any matrix can be written as the sum of a symmetric matrix and a skew-symmetric matrix

$$A = P + Q$$
$$= \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$$

Here $$Q = \frac{1}{2}(A - A^T)$$

$$= \frac{1}{2}\left(\begin{bmatrix} 2 & 0 \\ 5 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 5 \\ 0 & 4 \end{bmatrix}\right)$$

$$= \frac{1}{2}\left(\begin{bmatrix} 0 & -5 \\ 5 & 0 \end{bmatrix}\right)$$

$$Q = \begin{bmatrix} 0 & -\frac{5}{2} \\ \frac{5}{2} & 0 \end{bmatrix}$$

Thus (b) is correct option.

5. If A is 3×4 matrix and B is a matrix such that $A'B$ and BA' are both defined, then B is of the type

(a) 4×4 (b) 3×4

(c) 4×3 (d) 3×3

Sol : Delhi 2010, OD 2008

Since A is 3×4 matrix. So, A' is 4×3 matrix. Since, $A'B$ is defined, therefore number of column in A' must be equal to the number of row in B. Thus number of row in B is 3.

Since, BA' is defined, therefore number of column in B must be equal to the number of row in A'. Thus number of column in B is 4.

Hence, B is the type 3×4.

Thus option (b) is correct.

6. The symmetric part of the matrix $A = \begin{bmatrix} 1 & 2 & 4 \\ 6 & 8 & 2 \\ 2 & -2 & 7 \end{bmatrix}$ is equal to

(a) $\begin{bmatrix} 0 & -2 & -1 \\ -2 & 0 & -2 \\ -1 & -2 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 4 & 3 \\ 4 & 8 & 0 \\ 3 & 0 & 7 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & -2 & 1 \\ 2 & 0 & 2 \\ -1 & 2 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 4 & 3 \\ 4 & 8 & 0 \\ 3 & 0 & 7 \end{bmatrix}$

Sol : Comp 2007

We have $$A = \begin{bmatrix} 1 & 2 & 4 \\ 6 & 8 & 2 \\ 2 & -2 & 7 \end{bmatrix}$$

Symmetric part of A,

$$\frac{1}{2}[A + A'] = \frac{1}{2}\left\{\begin{bmatrix} 1 & 2 & 4 \\ 6 & 8 & 2 \\ 2 & -2 & 7 \end{bmatrix} + \begin{bmatrix} 1 & 6 & 2 \\ 2 & 8 & -2 \\ 4 & 2 & 7 \end{bmatrix}\right\}$$

$$= \frac{1}{2}\begin{bmatrix} 2 & 8 & 6 \\ 8 & 16 & 0 \\ 6 & 0 & 14 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 3 \\ 4 & 8 & 0 \\ 3 & 0 & 7 \end{bmatrix}$$

Thus option (b) is correct.

7. If $AB = A$ and $BA = B$, then B^2 is equal to

(a) B (b) A

(c) $-B$ (d) B^3

Sol : Foreign 2018, OD 2011

Given, $AB = A$ and $BA = B$

Now, $$BA = B$$

$$BAB = BB$$

$$B(A) = B^2$$

$$B = B^2$$

Thus option (a) is correct.

8. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, then

(a) $A^2 + 7A - 5I = O$

(b) $A^2 - 7A + 5I = O$

(c) $A^2 + 5A - 7I = O$

(d) $A^2 - 5A + 7I = O$

Sol : Delhi 2017

We have, $$A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$$

and $$5A = \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix}$$

Now, $$A^2 - 5A + 7I = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - \begin{bmatrix} 15 & 5 \\ 5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

Thus option (d) is correct.

9. Choose the correct option for the values of x, y and z from the following equations $\begin{bmatrix} 4 & x-z \\ 2+y & xz \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ -1 & 10 \end{bmatrix}$.

(a) $x = -5,\ y = 3,\ z = 2$

(b) $x = 5,\ y = -3,\ z = 2$

(c) $x = 5,\ y = 3,\ z = -2$

(d) $x = 5,\ y = -3,\ z = -2$

Sol : Comp 2010

We have $$\begin{bmatrix} 4 & x-z \\ 2+y & xz \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ -1 & 10 \end{bmatrix}$$

Comparing the corresponding elements, we get

$$x - z = 3 \qquad \text{...(1)}$$

$$2+y = -1$$
$$y = -1-2 = -3 \quad ..(2)$$

and $\quad xz = 10 \quad ...(3)$

Now, $\quad (x+z)^2 = (x-z)^2 + 4xz$
$$= (3)^2 + 4\times 10$$
$$= 9+40$$
$$(x+z)^2 = 49$$
$$x+z = 7 \quad ...(4)$$

Adding Eqs. (1) and (4), we get
$$2x = 10 \Rightarrow x = 5$$

From Eq. (1), we have
$$5-z = 3 \Rightarrow -z = 3-5$$
$$z = 2$$

Hence, $x=5$, $y=-3$ and $z=2$.
Thus option (b) is correct.

10. Choose the correct option for $\begin{bmatrix} -1 & 3 & 0 \\ 2 & 1 & 4 \end{bmatrix} \times \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ -2 & -1 \end{bmatrix}$.

(a) $\begin{bmatrix} 1 & 3 \\ 6 & -4 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & -3 \\ 6 & 4 \end{bmatrix}$

(c) $\begin{bmatrix} -1 & 3 \\ -6 & 4 \end{bmatrix}$ (d) $\begin{bmatrix} -1 & 3 \\ -6 & -4 \end{bmatrix}$

Sol : SQP 2020

$$\begin{bmatrix} -1 & 3 & 0 \\ 2 & 1 & 4 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & 2 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} -1+0+0 & -3+6+0 \\ 2+0-8 & 6+2-4 \end{bmatrix}$$
$$= \begin{bmatrix} -1 & 3 \\ -6 & 4 \end{bmatrix}$$

Thus option (c) is correct.

11. If $A = \begin{bmatrix} 3 & x-1 \\ 2x+3 & x+2 \end{bmatrix}$ is a symmetric matrix, then the value of x is

(a) 4 (b) 3

(c) -4 (d) -3

Sol : OD 2017, Delhi 2013

Since $A = \begin{bmatrix} 3 & x-1 \\ 2x+3 & x+2 \end{bmatrix}$ is a symmetric matrix, i.e. $A^T = A$,

$$\begin{bmatrix} 3 & 2x+3 \\ x-1 & x+2 \end{bmatrix} = \begin{bmatrix} 3 & x-1 \\ 2x+3 & x+2 \end{bmatrix}$$

Equating the corresponding elements, we get
$$2x+3 = x-1$$
$$x = -4$$

Thus option (c) is correct.

12. If $A = [a_{ij}]_{2\times 2}$, where $a_{ij} = i+j$, then A is equal to

(a) $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$

Sol : Foreign 2014

We have $\quad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$
$$= \begin{bmatrix} 1+1 & 1+2 \\ 2+1 & 2+2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$$

Thus option (d) is correct.

13. If $A = \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}$ and A^2 is the identity matrix, then x is equal to

(a) -1 (b) 0

(c) 1 (d) 2

Sol : Delhi 2016, OD 2012

We have $\quad A = \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}$

Now, $\quad A^2 = \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}$
$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} x^2+1 & x \\ x & 1 \end{bmatrix}$$

Equating the corresponding elements, we get $x=0$
Thus option (b) is correct.

14. If A and B are two symmetric matrices of same order then, the matrix $AB-BA$ is equal to

(a) a symmetric matrix

(b) a skew-symmetric matrix

(c) a null matrix

(d) the identity matrix

Sol : Comp 2009

We have $\quad A = A',\ B = B'$

Now, $(AB-BA)' = (AB)' - (BA)'$
$$= B'A' - A'B'$$
$$= BA - AB$$
$$= -(AB-BA)$$

Hence, $AB-BA$ is a skew-symmetric matrix.
Thus option (b) is correct.

15. If $A = \begin{bmatrix} 1 & -2 \\ 4 & 5 \end{bmatrix}$ and $f(A) = A^2 - 3A + 7$, then $f(A) + \begin{bmatrix} 3 & 6 \\ -12 & -9 \end{bmatrix}$ is equal to

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

Sol : SQP 2015

We have $$A^2 = \begin{bmatrix} 1 & -2 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} -7 & -12 \\ 24 & 17 \end{bmatrix}$$

Now, $$f(A) = A^2 - 3A + 7$$

$$= \begin{bmatrix} -7 & -12 \\ 24 & 17 \end{bmatrix} - 3\begin{bmatrix} 1 & -2 \\ 4 & 5 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & -6 \\ 12 & 9 \end{bmatrix}$$

$$f(A) + \begin{bmatrix} 3 & 6 \\ -12 & -9 \end{bmatrix} = \begin{bmatrix} -3 & -6 \\ 12 & 9 \end{bmatrix} + \begin{bmatrix} 3 & 6 \\ -12 & -9 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Thus option (b) is correct.

16. If A is square matrix, A' is its transpose, then $\frac{1}{2}(A - A')$ is

(a) a symmetric matrix

(b) a skew-symmetric matrix

(c) a unit matrix

(d) an elementary matrix

Sol : OD 2013

$$\frac{1}{2}(A - A')' = \frac{1}{2}(A' - A)$$

$$= -\frac{1}{2}(A - A')$$

Hence, it is a skew-symmetric matrix.

Thus option (b) is correct.

17. If $A(\alpha) = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, then the matrix $A^2(\alpha)$ is equal to

(a) $A(2\alpha)$ (b) $A(\alpha)$

(c) $A(3\alpha)$ (d) $A(4\alpha)$

Sol : Delhi 2017, Foreign 2012

$$A^2(\alpha) = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\alpha - \sin^2\alpha & 2\cos\alpha\sin\alpha \\ -2\sin\alpha\cos\alpha & \cos^2\alpha - \sin^2\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos 2\alpha & \sin 2\alpha \\ -\sin 2\alpha & \cos 2\alpha \end{bmatrix} = A(2\alpha)$$

Thus option (a) is correct.

18. Two matrix are said to be equal if

(a) Their order is same

(b) Their corresponding elements are identical

(c) Their order is not same

(d) Their order is same and corresponding elements are identical

Sol : Comp 2007

Two matrix are said to be equal if their order is same and corresponding elements are identical.

Thus option (d) is correct.

19. A matrix in which number of column is more than number of rows is called

(a) Horizontal Matrix

(b) Triangular Matrix

(c) Vertical matrix

(d) Scalar Matrix

Sol : Delhi 2010

A matrix in which number of column is more than number of rows is called horizontal matrix.

Thus option (a) is correct.

20. If all elements in a diagonal matrix in the principal diagonal are equal, the matrix called as

(a) Vector matrix

(b) Scalar matrix

(c) Singular matrix

(d) Single Element matrix

Sol : SQP 2018

If all elements in a diagonal matrix in the principal diagonal are equal, the matrix called as scalar matrix.

Thus option (b) is correct.

21. True for matrix multiplication is

(a) Commutative law

(b) Associative law

(c) Both of laws

(d) None of these

Sol : Delhi 2013, Comp 2008

Multiplication of Matrices is not always commutative but associative law is obeyed.

Thus option (b) is correct.

22. Choose the correct option

(a) Every unit matrix is a scalar matrix

(b) Every scalar matrix is a unit matrix

(c) Every Diagonal matrix is a unit matrix

(d) A square matrix in which all elements are 1 is a unit matrix

Sol : OD 2018

Every Unit matrix is a scalar matrix
Thus option (a) is correct.

23. A matrix in which number of rows is more than number of column is called

(a) Vertical Matrix

(b) Horizontal Matrix

(c) Identity Matrix

(d) Scalar Matrix

Sol : Foreign 2014

A matrix in which number of rows is more than number of column is called vertical matrix
Thus option (a) is correct.

24. Two matrices are said to be comparable if they have

(a) Same number of rows

(b) Same number of columns

(c) Same number of rows and columns

(d) None of these

Sol : OD 2012, Comp 2009

Two matrices are said to be comparable if they have same number of rows and columns.
Thus option (c) is correct.

25. A Identity matrix is that square matrix

(a) Whose all elements are zero

(b) Elements present on diagonal are zero

(c) All elements are zero and elements of diagonal are unity

(d) All elements are non-zero

Sol : Delhi 2010

A Identity matrix is that square matrix all elements are zero and element of diagonal are unity.
Thus option (c) is correct.

26. $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & -1 \end{bmatrix} =$

(a) $[-1]$

(b) $\begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 1 & -1 \\ -2 & -1 & 1 \\ 4 & 2 & -2 \end{bmatrix}$

(d) Not defined

Sol : SQP 2017

$$\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & -1 \\ -2 & -1 & 1 \\ 4 & 2 & -2 \end{bmatrix}$$

Thus option (c) is correct.

27. If a matrix A is such that $4A^3 + 2A^2 + 7A + I = 0$, then $A^{-1} =$

(a) $(6A^2 + 2A - 5I)$

(b) $(6A^2 + 2A + 5I)$

(c) $-(4A^2 + 2A + 7I)$

(d) $-(4A^2 - 2A + 7I)$

Sol : Delhi 2016

$$\begin{aligned} 4A^3 + 2A^2 + 7A + I &= 0 \\ A^{-1}(4A^3 + 2A^2 + 7A + I) &= 0 \\ 4A^2 + 2A + 7I + A^{-1}I &= 0 \\ A^{-1} &= -(4A^2 + 2A + 7I) \end{aligned}$$

Thus option (c) is correct.

28. If $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, then $A^{100} =$

(a) $2^{100}A$

(b) $2^{99}A$

(c) $100A$

(d) $99A$

Sol : Comp 2010, OD 2008

$$\begin{aligned} A^2 &= \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} = 2A \\ A^3 &= 2^2 A, \\ A^n &= 2^{n-1}, \\ A^{100} &= 2^{99}A \end{aligned}$$

Thus option (b) is correct.

29. If A and B are 2×2 matrices, then which of the following is true ?

(a) $(A + B)(A - B) = A^2 - B^2$

(b) $(A - B)^2 = A^2 + B^2 - 2AB$

(c) $(A - B)(A + B) = A^2 + AB - BA - B^2$

(d) $(A + B)^2 = A^2 + B^2 + 2AB$

Sol : OD 2012

$$\begin{aligned} (A + B)(A - B) &= A^2 - AB + BA - B^2 \\ &\neq A^2 - B^2 \qquad [AB \neq BA] \end{aligned}$$

$$\begin{aligned}(A-B)^2 &= A^2-AB-BA+B^2\\ &\neq A^2+B^2-2AB \qquad [AB\neq BA]\\ (A-B)(A+B) &= A^2+AB-BA-B^2\\ (A+B)^2 &= A^2+AB+BA+B^2\\ &\neq A^2+B^2+2AB \qquad [AB\neq BA]\end{aligned}$$

Thus option (c) is correct.

30. If $A=\begin{bmatrix}1&2\\3&0\end{bmatrix}$ and $B=\begin{bmatrix}3&4\\1&6\end{bmatrix}$ then transpose matrix of AB is

(a) $\begin{bmatrix}5&16\\9&12\end{bmatrix}$ (b) $\begin{bmatrix}5&9\\16&12\end{bmatrix}$

(c) $\begin{bmatrix}5&12\\9&16\end{bmatrix}$ (d) None of these

Sol : Delhi 2011

$$AB=\begin{bmatrix}1&2\\3&0\end{bmatrix}\begin{bmatrix}3&4\\1&6\end{bmatrix}=\begin{bmatrix}5&16\\9&12\end{bmatrix}$$

$$(AB)'=\begin{bmatrix}5&9\\16&12\end{bmatrix}$$

Thus option (b) is correct.

31. If $A^2-A+I=0$, then $A^{-1}=$

(a) A (b) $I+A$

(c) $I-A$ (d) $A-I$

Sol : OD 2017, Foreign 2013

Multiplying both sides by A^{-1}, we get

$$\begin{aligned}A^{-1}(A^2)-A^{-1}(A)+A^{-1}(I) &= A^{-1}(0)\\ A-I+A^{-1} &= 0\\ A^{-1} &= I-A\end{aligned}$$

32. If $A=\begin{bmatrix}1&2&2\\2&1&2\\2&2&1\end{bmatrix}$, then $A^2-4A=$

(a) $2I_3$ (b) $3I_3$

(c) $4I_3$ (d) $5I_3$

Sol : OD 2014

$$A^2=\begin{bmatrix}1&2&2\\2&1&2\\2&2&1\end{bmatrix}\begin{bmatrix}1&2&2\\2&1&2\\2&2&1\end{bmatrix}=\begin{bmatrix}9&8&8\\8&9&8\\8&8&9\end{bmatrix}$$

$$A^2-4A=\begin{bmatrix}9&8&8\\8&9&8\\8&8&9\end{bmatrix}-\begin{bmatrix}4&8&8\\8&4&8\\8&8&4\end{bmatrix}$$

$$=\begin{bmatrix}5&0&0\\0&5&7\\0&0&0\end{bmatrix}=5I_3$$

Thus option (d) is correct.

33. If $A=\begin{bmatrix}x&1\\1&0\end{bmatrix}$ and A^2 is the identity matrix, then x is

(a) -1 (b) 0

(c) 1 (d) 2

Sol : SQP 2017

$$A^2=\begin{bmatrix}x&1\\1&0\end{bmatrix}\begin{bmatrix}x&1\\0&1\end{bmatrix}$$

$$\begin{bmatrix}x^2+1&x\\x&1\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

$$x^2+1=1 \text{ or } x=0$$

Thus option (b) is correct.

34. If $A=\begin{bmatrix}i&0\\0&i\end{bmatrix}$, then $A^2=$

(a) $\begin{bmatrix}1&0\\0&-1\end{bmatrix}$ (b) $\begin{bmatrix}-1&0\\0&-1\end{bmatrix}$

(c) $\begin{bmatrix}1&0\\0&1\end{bmatrix}$ (d) $\begin{bmatrix}-1&0\\0&1\end{bmatrix}$

Sol : OD 2015, Delhi 2011

$$A=\begin{bmatrix}i&0\\0&i\end{bmatrix}$$

$$A\times A=\begin{bmatrix}i&0\\0&i\end{bmatrix}\times\begin{bmatrix}i&0\\0&i\end{bmatrix}$$

$$A^2=\begin{bmatrix}i^2+0&0+0\\0+0&0+i^2\end{bmatrix}=\begin{bmatrix}-1&0\\0&-1\end{bmatrix}$$

Thus option (b) is correct.

35. If A^T, B^T are transpose matrices of the square matrices A, B respectively, then $(AB)^T=$

(a) A^TB^T (b) AB^T

(c) BA^T (d) B^TA^T

Sol : OD 2013

$$(AB)^T=B^TA^T$$

Thus option (d) is correct.

36. If $A=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$, then $A^2+2A=$

(a) A (b) $2A$

(c) $3A$ (d) $4A$

Sol : Foreign 2018, OD 2014

$$A^2=A\times A$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$2A = 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$A^2 + 2A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 3A$$

Thus option (c) is correct.

37. The matrix $\begin{bmatrix} 0 & k \\ -7 & 0 \end{bmatrix}$ is a skew-symmetric matrix if k is

(a) -7 (b) 7

(c) 7 (d) -7

Sol : Delhi 2017

Given, $\begin{bmatrix} 0 & k \\ -7 & 0 \end{bmatrix}$ is skew-symmetric matrix.

Let $A = \begin{bmatrix} 0 & k \\ -7 & 0 \end{bmatrix}$

Since, A is skew-symmetric matrix, we have

$$A = -A'$$

$$\begin{bmatrix} 0 & k \\ -7 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 7 \\ -k & 0 \end{bmatrix}$$

$$-k = -7$$

$$k = 7$$

Thus option (c) is correct.

38. **Assertion:** If the matrix $\begin{bmatrix} 0 & a & 3 \\ 2 & b & -1 \\ c & 1 & 0 \end{bmatrix}$ is a skew symmetric matrix, then value of $(a-b-c)$ is 1

Reason: A square matrix $A = [a_{ij}]$ is said to be skew-symmetric if $A' = -A$

(a) Assertion is true, Reason is true; Reason is a correct explanation for Assertion.

(b) Assertion is true, Reason is true; Reason is not a correct explanation for Assertion.

(c) Assertion is true; Reason is false.

(d) Assertion is false; Reason is true.

Sol : OD 2011

Let $A = \begin{bmatrix} 0 & a & 3 \\ 2 & b & -1 \\ c & 1 & 0 \end{bmatrix}$

Since, A is a skew symmetric matrix , So $A' = -A$

$$\begin{bmatrix} 0 & 2 & c \\ a & b & 1 \\ 3 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -a & -3 \\ -2 & -b & 1 \\ -c & -1 & 0 \end{bmatrix}$$

$$2 = -a,\ b = -b,\ 3 = -c$$

$$a = -2,\ 2b = 0,\ c = -3$$

Therefore, $a = -2$, $b = 0$,and $c = -3$

Now,

$$a - b - c = -2 - 0 + 3 = 1$$

Hence, Assertion is true, reason is true and reason is a correct explanation for assertion.

Thus (a) is correct option.

39. **Assertion:** Scalar matrix $A = [a_{ij}] = \begin{cases} k: & i = j \\ 0: & i \neq j \end{cases}$ where k is a scalar, in an identity matrix when $k = 1$

Reason: Every identity matrix is not a scalar matrix.

(a) Assertion is true, Reason is true; Reason is a correct explanation for Assertion.

(b) Assertion is true, Reason is true; Reason is not a correct explanation for Assertion.

(c) Assertion is true; Reason is false.

(d) Assertion is false; Reason is true.

Sol : Foreign 2014, Delhi 2010

A scalar matrix $A = [a_{ij}]$

$$\begin{cases} k, & i = j \\ 0 & i \neq j \end{cases}$$

Is an identity matrix when $k = 1$. But every identity matrix is clearly a scalar matrix.

Thus (c) is correct option.

40. **Assertion:** If $A = \begin{bmatrix} \sqrt{3} & 1 \\ 2 & 2 \end{bmatrix}$ and $b = \begin{bmatrix} x & y & z \\ a & b & c \end{bmatrix}$, then orders of $(A+B)$ is 2×3

Reason: If $[aij]$ and $[bij]$ are two matrics of the same order, then order of $A + B$ is the same as the order of A or B

(a) Assertion is true, Reason is true; Reason is a correct explanation for Assertion.

(b) Assertion is true, Reason is true; Reason is not a correct explanation for Assertion.

(c) Assertion is true; Reason is false.

(d) Assertion is false; Reason is true.

Sol : OD 2018

$A + B$ is defined only if A and B have same order. Here, A is of order 2×2 and B is order of 2×3. As

A and B are not of the same order, so $(A+B)$ is not defined, therefore it is not possible to find the sum of A and B.

Hence, Assertion is false and reason is true.

Thus (d) is correct option.

41. **Assertion :** The matrix $A=\frac{1}{3}\begin{bmatrix}1&-2&2\\-2&1&2\\-2&-2&-1\end{bmatrix}$ is an orthogonal matrix.

Reason : If A and B are orthogonal, then AB is also orthogonal.

(a) Assertion is true, reason is true, reason is a correct explanation for assertion.

(b) Assertion is true,reason is true, reason is not a correct explanation for assertion.

(c) Assertion is true, reason is false.

(d) Assertion is false, reason is true.

Sol : **Comp 2016**

As we know that, if A and B are orthogonal matrix, then AB is also orthogonal matrix. So given Reason is true.

We have $A=\frac{1}{3}\begin{bmatrix}1&-2&2\\-2&1&2\\-2&-2&-1\end{bmatrix}$

Now, $A^1=\frac{1}{3}\begin{bmatrix}1&-2&-2\\-2&1&-2\\2&2&-1\end{bmatrix}$

So,
$$\begin{aligned}AA'&=\frac{1}{3}\begin{bmatrix}1&-2&2\\-2&1&2\\-2&-2&-1\end{bmatrix}\cdot\frac{1}{3}\begin{bmatrix}1&-2&-2\\-2&1&-2\\2&2&-1\end{bmatrix}\\&=\frac{1}{9}\begin{bmatrix}1+4+4&-2-2+4&-2+4-2\\-2-2+4&4+1+4&4-2-2\\-2+4-2&4-2-2&4+4+1\end{bmatrix}\\&=\frac{1}{9}\begin{bmatrix}9&0&0\\0&9&0\\0&0&9\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=\mathrm{I}\end{aligned}$$

So, A is orthogonal matrix.

Hence, both Assertion and reason are true but reason is not a correct explanation for assertion.

Thus (b) is correct option.

42. For any square matrix A with real number entries consider the following statements.

Assertion: $A+A'$ is a symmetric matrix.

Reason: $A-A'$ is a skew-symmetric matrix.

(a) Assertion is true, Reason is true; Reason is a correct explanation for Assertion.

(b) Assertion is true, Reason is true; Reason is not a correct explanation for Assertion.

(c) Assertion is true; Reason is false.

(d) Assertion is false; Reason is true.

Sol : **Foreign 2013**

Let $B=A+A'$ then
$$\begin{aligned}B'&=(A+A')'=A'+(A')'\\&=A'+A=A+A'=B\end{aligned}$$

Therefore,
$$B=A+A' \text{ is a symmetric matrix.}$$

Now, let $C=A-A'$
$$\begin{aligned}C'&=(A-A')'=A'-(A')'\\&=A'-A=-(A-A')=-C\end{aligned}$$

Therefore,$C=A-A'$ is a skew-symmetric matrix.

Here assertion is true, Reason is true; Reason is not a correct explanation for Assertion.

Thus option (b) is correct.

43. **Assertion:** If $A=\frac{1}{3}\begin{bmatrix}1&-2&2\\-2&1&2\\-2&-2&-1\end{bmatrix}$, then $(A^T)A=I$

Reason: For any square matrix A, $(A^T)^T=A$

(a) Assertion is true, Reason is true; Reason is a correct explanation for Assertion.

(b) Assertion is true, Reason is true; Reason is not a correct explanation for Assertion.

(c) Assertion is true; Reason is false.

(d) Assertion is false; Reason is true.

Sol : **Comp 2007**

$$\begin{aligned}AA^T&=\frac{1}{3}\begin{bmatrix}1&-2&2\\-2&1&2\\-2&-2&-1\end{bmatrix}\cdot\frac{1}{3}\begin{bmatrix}1&-2&-2\\-2&1&-2\\2&2&-1\end{bmatrix}\\&=\frac{1}{9}\begin{bmatrix}9&0&0\\0&9&0\\0&0&9\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=I\end{aligned}$$

Assertion is true, Reason is true; Reason is not a correct explanation for Assertion.

Thus (b) is correct option.

44. **Assertion:** $f(\theta)=\begin{vmatrix}\cos(\theta+\alpha)&\cos(\theta+\beta)&\cos(\theta+\gamma)\\\sin(\theta+\alpha)&\sin(\theta+\beta)&\sin(\theta+\gamma)\\\sin(\beta-\gamma)&\sin(\gamma-\alpha)&\sin(\alpha-\beta)\end{vmatrix}$ is independent of θ

Reason: If $f(\theta)=c$ then $f(\theta)$ is independent of θ

(a) Assertion is true, Reason is true; Reason is a correct explanation for Assertion.

(b) Assertion is true, Reason is true; Reason is not a correct explanation for Assertion.

(c) Assertion is true; Reason is false.

(d) Assertion is false; Reason is true.

Sol : Delhi 2011

(a) Assertion is true, Reason is true; Reason is a correct explanation for Assertion.

Let,

$$f(\theta) = \begin{vmatrix} \cos(\theta+\alpha) & \cos(\theta+\beta) & \cos(\theta+\gamma) \\ \sin(\theta+\alpha) & \sin(\theta+\beta) & \sin(\theta+\gamma) \\ \sin(\beta-\gamma) & \sin(\gamma-\alpha) & \sin(\alpha-\beta) \end{vmatrix}$$

$$f'(\theta) = \begin{vmatrix} -\sin(\theta+\alpha) & -\sin(\theta+\beta) & -\sin(\theta+\gamma) \\ \sin(\theta+\alpha) & \sin(\theta+\beta) & \sin(\theta+\gamma) \\ \sin(\beta-\gamma) & \sin(\gamma-\alpha) & \sin(\alpha-\beta) \end{vmatrix}$$

$$+\begin{vmatrix} \cos(\theta+\alpha) & \cos(\theta+\beta) & \cos(\theta+\gamma) \\ \cos(\theta+\alpha) & \cos(\theta+\beta) & \cos(\theta+\gamma) \\ \sin(\beta-\gamma) & \sin(\gamma-\alpha) & \sin(\alpha-\beta) \end{vmatrix}$$

$$+\begin{vmatrix} \cos(\theta+\alpha) & \cos(\theta+\beta) & \cos(\theta+\gamma) \\ \sin(\theta+a) & \sin(\theta+\beta) & \sin(\theta+\gamma) \\ 0 & 0 & 0 \end{vmatrix}$$

$$= 0+0+0$$

$$= 0$$

Thus $f'(\theta) = 0$ and $f(\theta) = c$

Assertion is true, Reason is true; Reason is a correct explanation for Assertion.

Thus option (a) is correct.

VERY SHORT ANSWER QUESTIONS

45. If $3A - B = \begin{bmatrix} 5 & 0 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$, then find the value of matrix A.

Sol : Delhi 2019

We have $\quad 3A - B = \begin{bmatrix} 5 & 0 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$

$$3A - \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 1 & 1 \end{bmatrix}$$

$$3A = \begin{bmatrix} 5 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 5+4 & 0+3 \\ 1+2 & 1+5 \end{bmatrix}$$

$$3A = \begin{bmatrix} 9 & 3 \\ 3 & 6 \end{bmatrix}$$

$$A = \frac{1}{3}\begin{bmatrix} 9 & 3 \\ 3 & 6 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$$

46. Find the value of $x - y$, if

$$2\begin{bmatrix} 1 & 3 \\ 0 & x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$$

Sol : Delhi 2019

We have $\quad 2\begin{bmatrix} 1 & 3 \\ 0 & x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$

$$\begin{bmatrix} 2 & 6 \\ 0 & 2x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 2+y & 6+0 \\ 0+1 & 2x+2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$$

Here, both matrices equal, so we equate the corresponding elements,

$$2+y = 5 \text{ and } 2x+2 = 8$$

$$y = 3 \text{ and } 2x = 6 \Rightarrow x = 3$$

Therefore, $x - y = 3 - 3 = 0$

47. If A is a square matrix such that $A^2 = I$, then find the simplified value of $(A-I)^3 + (A+I)^3 - 7A$.

Sol : Delhi 2016, OD 2009

We have $\quad A^2 = I$

$$(A-I)^3 + (A+I)^3 - 7A$$

$$= (A^3 - 3A^2I + 3AI^2 - I^3)$$

$$+ (A^3 + 3A^2I + 3AI^2 + I^3) - 7A$$

$$= A^3 - 3A^2 + 3A - I + A^3 + 3A^2 + 3A + I - 7A$$

$$= 2A^3 + 6A - 7A$$

$$= 2A^2A + 6A - 7A$$

$$= 2IA - A \qquad A^2 = I$$

$$= 2A - A$$

$$= A$$

48. Write the number of all possible matrices of order 2×2 with each entry 1, 2 or 3.

Sol : OD 2016

A matrix of order 2×2 has 4 entries. Since, each entry has 3 choices, namely 1, 2 or 3, therefore number of required matrices

$$3^4 = 3 \times 3 \times 3 \times 3 = 81.$$

49. If $\begin{bmatrix} 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} -1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = A$, then write the order of matrix A.

Sol : Foreign 2016

We have, $$A = \begin{bmatrix} 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} -1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} -2-1 & 1+3 & -2+3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 4 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$= [-3-1]$$

$$= [-4]_{1\times 1}$$

Order of matrix A is 1×1.

50. Write the elements a_{23} of a 3×3 matrix $A = [a_{ij}]$, whose elements a_{ij} are given by $a_{ij} = \frac{|i-j|}{2}$.

Sol : Delhi 2015, OD 2012

We have $$A = |a_{ij}|_{3\times 3}$$

where, $$a_{ij} = \frac{|i-j|}{2}$$

Substituting $i = 2$ and $j = 3$ we have

$$a_{23} = \frac{|2-3|}{2} = \frac{|-1|}{2} = \frac{1}{2}$$

51. If $\begin{bmatrix} 2x & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix}\begin{bmatrix} x \\ 3 \end{bmatrix} = O$, find x.

Sol : Comp 2015

We have $$\begin{bmatrix} 2x & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix}\begin{bmatrix} x \\ 3 \end{bmatrix} = O$$

$$\begin{bmatrix} 2x-9 & 4x \end{bmatrix}\begin{bmatrix} x \\ 3 \end{bmatrix} = O$$

$$[2x^2 - 9x + 12x] = [0]$$

$$2x^2 + 3x = 0$$

$$x(2x+3) = 0$$

Thus $x = 0$ or $x = -\frac{3}{2}$

52. If $2\begin{bmatrix} 3 & 4 \\ 5 & x \end{bmatrix} + \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 10 & 5 \end{bmatrix}$, then find $(x-y)$.

Sol : Delhi 2014, Comp 2010

We have $$2\begin{bmatrix} 3 & 4 \\ 5 & x \end{bmatrix} + \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 10 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 8 \\ 10 & 2x \end{bmatrix} + \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 10 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 7 & 8+y \\ 10 & 2x+1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 10 & 5 \end{bmatrix}$$

Equating the corresponding elements, we get

$$8 + y = 0 \text{ and } 2x + 1 = 5$$

$$y = -8 \text{ and } x = \frac{5-1}{2} = 2$$

$$x - y = 2 - (-8) = 10$$

53. Solve the following matrix equation

$$\begin{bmatrix} x & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix} = O$$

Sol : Delhi 2014

We have $$\begin{bmatrix} x & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix} = O$$

Using matrix multiplication, we get

$$\begin{bmatrix} x-2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

Equating the corresponding elements, we get

$$x - 2 = 0$$

$$x = 2$$

54. If A is a square matrix such that $A^2 = A$, then write the value of $7A - (I + A)^3$, where I is an identity matrix.

Sol : OD 2014

We have $$A^2 = A$$

$$7A - (I+A)^3 = 7A - [I^3 + A^3 + 3IA(I+A)]$$

$$= 7A - [I + A^2 \cdot A + 3A(I+A)]$$

$$= 7A - (I + A \cdot A + 3AI + 3A^2)$$

$$= 7A - (I + A + 3A + 3A)$$

$$= 7A - (I + 7A) = -I$$

55. If $\begin{bmatrix} x-y & z \\ 2x-y & w \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 0 & 5 \end{bmatrix}$, then find the value of $x + y$.

Sol : OD 2014

We have $$\begin{bmatrix} x-y & z \\ 2x-y & w \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 0 & 5 \end{bmatrix}$$

Equating the corresponding elements, we get

$$x - y = -1 \quad \text{...(i)}$$

and $$2x - y = 0 \quad \text{...(ii)}$$

Solving the Eq. (i) and (ii), we get

$$x = 1 \text{ and } y = 2$$

$$x + y = 1 + 2 = 3$$

56. If $\begin{bmatrix} a+4 & 3b \\ 8 & -6 \end{bmatrix} = \begin{bmatrix} 2a+2 & b+2 \\ 8 & a-8b \end{bmatrix}$, then write the value of $a - 2b$.

Sol : Foreign 2014, OD 2011

We have $\begin{bmatrix} a+4 & 3b \\ 8 & -6 \end{bmatrix} = \begin{bmatrix} 2a+2 & b+2 \\ 8 & a-8b \end{bmatrix}$

Equating the corresponding elements, we get

$$a + 4 = 2a + 2 \quad ...(i)$$

$$3b = b + 2 \quad ...(ii)$$

and $$-6 = a - 8b \quad ...(iii)$$

Solving the eqs. (i), (ii) and (iii), we get

$$a = 2 \text{ and } b = 1$$

Now, $$a - 2b = 2 - 2(1) = 2 - 2 = 0$$

57. If $\begin{bmatrix} x \cdot y & 4 \\ z+6 & x+y \end{bmatrix} = \begin{bmatrix} 8 & w \\ 0 & 6 \end{bmatrix}$, then write the value of $(x + y + z)$.

Sol : Delhi 2014C

We have $\begin{bmatrix} x \cdot y & 4 \\ z+6 & x+y \end{bmatrix} = \begin{bmatrix} 8 & w \\ 0 & 6 \end{bmatrix}$

Equating the corresponding elements, we get

$$x \cdot y = 8 \quad ...(i)$$

$$z + 6 = 0 \Rightarrow z = -6 \quad ...(ii)$$

and $$x + y = 6 \quad ...(iii)$$

Adding eqs. (ii) and (iii), we get

$$x + y + z = 6 + (-6) = 0$$

58. The elements a_{ij} of a 3×3 matrix are given by $a_{ij} = \frac{1}{2}|-3i + j|$. Write the value of element a_{32}.

Sol : OD 2014C

We have $A = |a_{ij}|_{3 \times 3}$

where, $a_{ij} = \frac{1}{2}|-3i + j|$

Substituting $i = 3$ and $j = 2$ we have

$$a_{32} = \frac{|-3 \times 3 + 2|}{2} = \frac{|-7|}{2} = \frac{7}{2}$$

59. If $\begin{bmatrix} 2x & 4 \end{bmatrix} \begin{bmatrix} x \\ -8 \end{bmatrix} = O$, then find the positive value of x.

Sol : Comp 2014, Foreign 2013

We have $$\begin{bmatrix} 2x & 4 \end{bmatrix} \begin{bmatrix} x \\ -8 \end{bmatrix} = O$$

$$\begin{bmatrix} 2x^2 & -32 \end{bmatrix} = [0]$$

Equating the corresponding elements, we get

$$2x^2 - 32 = 0$$

$$x^2 = 16 \Rightarrow x = \pm 4$$

Thus positive value of x is 4.

60. If $2\begin{bmatrix} 1 & 3 \\ 0 & x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$, then find the value of $(x + y)$.

Sol : Delhi 2013C, OD 2012

We have $$2\begin{bmatrix} 1 & 3 \\ 0 & x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 6 \\ 0 & 2x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 2+y & 6 \\ 1 & 2x+2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$$

Equating the corresponding elements, we get

$$2 + y = 5 \text{ and } 2x + 2 = 8$$

$$y = 3 \text{ and } x = 3$$

Thus $$x + y = 3 + 3 = 6$$

61. Find the value of a, if

$$\begin{bmatrix} a-b & 2a+c \\ 2a-b & 3c+d \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}.$$

Sol : Delhi 2013

We have $\begin{bmatrix} a-b & 2a+c \\ 2a-b & 3c+d \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}$

Equating the corresponding elements, we get

$$a - b = -1 \quad ...(i)$$

and $$2a - b = 0 \quad ...(ii)$$

Subtracting Eqs. (i) from (ii), we get

$$a = 1$$

62. If $\begin{bmatrix} 9 & -1 & 4 \\ -2 & 1 & 3 \end{bmatrix} = A + \begin{bmatrix} 1 & 2 & -1 \\ 0 & 4 & 9 \end{bmatrix}$, then find the matrix A.

Sol : Delhi 2013

We have $$\begin{bmatrix} 9 & -1 & 4 \\ -2 & 1 & 3 \end{bmatrix} = A + \begin{bmatrix} 1 & 2 & -1 \\ 0 & 4 & 9 \end{bmatrix}$$

$$A = \begin{bmatrix} 9 & -1 & 4 \\ -2 & 1 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 2 & -1 \\ 0 & 4 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 9-1 & -1-2 & 4+1 \\ -2-0 & 1-4 & 3-9 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & -3 & 5 \\ -2 & -3 & -6 \end{bmatrix}$$

NOTE : Two matrices can be subtracted only when their orders are same.

63. If matrix $A=\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$ and $A^2=kA$, then write the value of k.

Sol : OD 2013

We have $A = \begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$ ...(i)

and $A^2 = kA$ ...(ii)

Now, $A^2 = A\cdot A$

$$= \begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\cdot\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

$$= \begin{bmatrix}1+1 & -1-1\\ -1-1 & 1+1\end{bmatrix}$$

$$= \begin{bmatrix}2 & -2\\ -2 & 2\end{bmatrix}$$

$$= 2\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

$A^2 = 2A$ [from Eq. (i)]

Comparing with Eq. (ii), we get $k=2$

64. If matrix $A=\begin{bmatrix}2 & -2\\ -2 & 2\end{bmatrix}$ and $A^2=pA$, then write the value of p.

Sol : OD 2013, Comp 2008

We have $A = \begin{bmatrix}2 & -2\\ -2 & 2\end{bmatrix}$ (1)

and $A^2 = pA$ (2)

Now, $A^2 = A\cdot A$

$$= \begin{bmatrix}2 & -2\\ -2 & 2\end{bmatrix}\cdot\begin{bmatrix}2 & -2\\ -2 & 2\end{bmatrix}$$

$$= \begin{bmatrix}2+2 & -2-2\\ -2-2 & 2+2\end{bmatrix}$$

$$= \begin{bmatrix}4 & -4\\ -4 & 4\end{bmatrix}$$

$$= 4\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

$A^2 = 4A$ from (1)

Comparing with Eq. (2), we get $p=2$.

65. If matrix $A=\begin{bmatrix}3 & -3\\ -3 & 3\end{bmatrix}$ and $A^2=\lambda A$, then write the value of λ.

Sol : OD 2013

We have $A = \begin{bmatrix}3 & -3\\ -3 & 3\end{bmatrix}$ (1)

and $A^2 = \lambda A$ (2)

Now, $A^2 = A\cdot A$

$$= \begin{bmatrix}3 & -3\\ -3 & 3\end{bmatrix}\cdot\begin{bmatrix}3 & -3\\ -3 & 3\end{bmatrix}$$

$$= \begin{bmatrix}9+9 & -9-9\\ -9-9 & 9+9\end{bmatrix}$$

$$= \begin{bmatrix}18 & -18\\ -18 & 18\end{bmatrix}$$

$$= 6\begin{bmatrix}3 & -3\\ -3 & 3\end{bmatrix}$$

$A^2 = 6A$ from (1)

Comparing with Eq. (2), we get $\lambda=6$.

66. Simplify $\cos\theta\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}+\sin\theta\begin{bmatrix}\sin\theta & -\cos\theta\\ \cos\theta & \sin\theta\end{bmatrix}$.

Sol : Delhi 2012

Let a be the simplified value of given expression.

$$a = \cos\theta\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}+\sin\theta\begin{bmatrix}\sin\theta & -\cos\theta\\ \cos\theta & \sin\theta\end{bmatrix}$$

$$= \begin{bmatrix}\cos^2\theta & \sin\theta\cos\theta\\ -\sin\theta\cos\theta & \cos^2\theta\end{bmatrix}+\begin{bmatrix}\sin^2\theta & -\sin\theta\cos\theta\\ \sin\theta\cos\theta & \sin^2\theta\end{bmatrix}$$

$$= \begin{bmatrix}\cos^2\theta+\sin^2\theta & \sin\theta\cos\theta-\sin\theta\cos\theta\\ -\sin\theta\cos\theta+\sin\theta\cos\theta & \cos^2\theta+\sin^2\theta\end{bmatrix}$$

$$= \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$= I =$ unit matrix.

67. If $\begin{bmatrix}2 & 3\\ 5 & 7\end{bmatrix}\begin{bmatrix}1 & -3\\ -2 & 4\end{bmatrix}=\begin{bmatrix}-4 & 6\\ -9 & x\end{bmatrix}$, write the value of x.

Sol : Delhi 2012, OD 2010

We have $\begin{bmatrix}2 & 3\\ 5 & 7\end{bmatrix}\begin{bmatrix}1 & -3\\ -2 & 4\end{bmatrix} = \begin{bmatrix}-4 & 6\\ -9 & x\end{bmatrix}$

$$\begin{bmatrix}2-6 & -6+12\\ 5-14 & -15+28\end{bmatrix} = \begin{bmatrix}-4 & 6\\ -9 & x\end{bmatrix}$$

$$\begin{bmatrix}-4 & 6\\ -9 & 13\end{bmatrix} = \begin{bmatrix}-4 & 6\\ -9 & x\end{bmatrix}$$

Equating the corresponding elements, we get

$$x = 13$$

68. If $x\begin{bmatrix}2\\ 3\end{bmatrix}+y\begin{bmatrix}-1\\ 1\end{bmatrix}=\begin{bmatrix}10\\ 5\end{bmatrix}$, then write the value of x.

Sol : Foreign 2012

We have $x\begin{bmatrix}2\\ 3\end{bmatrix}+y\begin{bmatrix}-1\\ 1\end{bmatrix} = \begin{bmatrix}10\\ 5\end{bmatrix}$

$$\begin{bmatrix} 2x \\ 3x \end{bmatrix} + \begin{bmatrix} -y \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 2x-y \\ 3x+y \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$$

Equating the corresponding elements, we get

$$2x - y = 10 \qquad ...(i)$$

and $$3x + y = 5 \qquad ...(ii)$$

Adding Eqs. (i) and (ii), we get

$$5x = 15$$

$$x = 3$$

69. If $3A - B = \begin{bmatrix} 5 & 0 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$, then find the matrix A.

Sol : Comp 2012

We have $B = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$ and $3A - B = \begin{bmatrix} 5 & 0 \\ 1 & 1 \end{bmatrix}$

$$3A = \begin{bmatrix} 5 & 0 \\ 1 & 1 \end{bmatrix} + B = \begin{bmatrix} 5 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 5+4 & 0+3 \\ 1+2 & 1+5 \end{bmatrix} = \begin{bmatrix} 9 & 3 \\ 3 & 6 \end{bmatrix}$$

$$3A = 3\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$$

70. Write the value of $x - y + z$ from following equation.

$$\begin{bmatrix} x+y+z \\ x+z \\ y+z \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ 7 \end{bmatrix}$$

Sol : Foreign 2011, OD 2007

We have $$\begin{bmatrix} x+y+z \\ x+z \\ y+z \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ 7 \end{bmatrix}$$

Equating the corresponding elements, we get

$$x + y + z = 9 \qquad ...(i)$$

$$x + z = 5 \qquad ...(ii)$$

and $$y + z = 7 \qquad ...(iii)$$

Substituting the value of $x + z$ from Eq. (ii) in Eq. (i), we get $y + 5 = 9 \Rightarrow y = 4$

Substituting $y = 4$ in Eq. (iii), we get $z = 3$

Again, substituting $z = 3$ in Eq. (ii), we get $x = 2$

$$x - y + z = 2 - 4 + 3 = 1$$

71. Write the order of product matrix $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\begin{bmatrix} 2 & 3 & 4 \end{bmatrix}$.

Sol : Foreign 2011

If a matrix A has order $m \times n$ and other matrix B has order $n \times z$, then the matrix AB has order $m \times z$.

Let $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix}$

Here, order of matrix $A = 3 \times 1$

and order of matrix $B = 1 \times 3$

Order of product matrix $AB = 3 \times 3$.

72. If a matrix has 5 elements, then write all possible orders it can have.

Sol : OD 2011

If a matrix has order $m \times n$, then total number of elements in that matrix is mn.

Given, a matrix has 5 elements. So, possible order of this matrix are 5×1 and 1×5.

73. For a 2×2 matrix, $A = [a_{ij}]$ whose elements are given by $a_{ij} = i/j$, write the value of a_{12}.

Sol : Delhi 2011

We have $A = |a_{ij}|_{2 \times 2}$

where, $a_{ij} = \dfrac{i}{j}$

Substituting $i = 1$ and $j = 2$ we have

$$a_{23} = \frac{1}{2}$$

74. If $\begin{bmatrix} x & x-y \\ 2x+y & 7 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 8 & 7 \end{bmatrix}$, then find the value of y.

Sol : Delhi 2011C

We have $\begin{bmatrix} x & x-y \\ 2x+y & 7 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 8 & 7 \end{bmatrix}$

Equating the corresponding elements, we get

$x = 3$ and $x - y = 1$

$$y = x - 1 = 3 - 1 = 2$$

75. From the following matrix equation, find the value of x.

$$\begin{bmatrix} x+y & 4 \\ -5 & 3y \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ -5 & 6 \end{bmatrix}$$

Sol : Foreign 2010, Comp 2008

We have $\begin{bmatrix} x+y & 4 \\ -5 & 3y \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ -5 & 6 \end{bmatrix}$

Equating the corresponding elements, we get

$$x + y = 3 \quad ...(i)$$

and $$3y = 6 \quad ...(ii)$$

From Eq. (ii), we get

$$y = 2$$

Substituting $y = 2$ in Eq. (i), we get

$$x + 2 = 3 \Rightarrow x = 1$$

76. Find x from the matrix equation

$$\begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} x \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}.$$

Sol : Foreign 2010

We have $$\begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} x \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} x + 6 \\ 4x + 10 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

Equating the corresponding elements, we get

$$x + 6 = 5$$

$$x = -1$$

77. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, then find $A + A'$.

Sol : Comp 2010, Delhi 2009

We have $$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Interchange the elements of rows and columns we have

$$A' = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

Now, $$A + A' = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 5 \\ 5 & 8 \end{bmatrix}$$

78. If $\begin{bmatrix} 3 & 4 \\ 2 & x \end{bmatrix}\begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} 19 \\ 15 \end{bmatrix}$, then find the value of x.

Sol : Foreign 2010

We have $$\begin{bmatrix} 3 & 4 \\ 2 & x \end{bmatrix}\begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} 19 \\ 15 \end{bmatrix}$$

$$\begin{bmatrix} 3x + 4 \\ 2x + x \end{bmatrix} = \begin{bmatrix} 19 \\ 15 \end{bmatrix}$$

Equating the corresponding elements, we get

$$3x + 4 = 19$$

$$3x = 15$$

$$x = 5$$

79. If $A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$, then for what value of α, A is an identity matrix?

Sol : Delhi 2010

We have $$A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

For A to be an identity matrix, we must have

$$\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Equating the element a_{11} of both matrices, we get

$$\cos\alpha = 1$$

$$\cos\alpha = \cos 0^\circ$$

$$\alpha = 0$$

Hence, for $\alpha = 0$, A is an identity matrix.

80. If $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 7 & 11 \\ k & 23 \end{bmatrix}$, then write the value of k.

Sol : Delhi 2010

We have $$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 7 & 11 \\ k & 23 \end{bmatrix}$$

$$\begin{bmatrix} 3+2 & 1+10 \\ 9+8 & 3+20 \end{bmatrix} = \begin{bmatrix} 7 & 11 \\ k & 23 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 6 \\ -9 & 13 \end{bmatrix} = \begin{bmatrix} -4 & 6 \\ -9 & x \end{bmatrix}$$

Equating the corresponding elements, we get

$$k = 9 + 8 = 17$$

81. If A is a matrix of order 3×4 and B is a matrix of order 4×3, then find order of matrix (AB).

Sol : Comp 2010

If a matrix A has order $m \times n$ and other matrix B has order $n \times z$, then the matrix AB has order $m \times z$.

Here, order of matrix $A = 3 \times 4$

and order of matrix $B = 4 \times 3$

Order of product matrix $AB = 3 \times 3$.

82. If $\begin{bmatrix} x + y & 1 \\ 2y & 5 \end{bmatrix} = \begin{bmatrix} 7 & 1 \\ 4 & 5 \end{bmatrix}$, then find the value of x.

Sol : Delhi 2010, OD 2008

We have $$\begin{bmatrix} x + y & 1 \\ 2y & 5 \end{bmatrix} = \begin{bmatrix} 7 & 1 \\ 4 & 5 \end{bmatrix}$$

Equating the corresponding elements, we get

$$x+y=7 \quad ...(i)$$

and $$2y=4 \quad ...(ii)$$

From Eq. (ii), we get $y=2$

Substituting $y=2$ in Eq. (i), we get

$$x+2=7 \Rightarrow x=5$$

83. If $\begin{bmatrix} 2x+y & 3y \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 4 \end{bmatrix}$, then find the value of x.

Sol : Comp 2010

We have $$\begin{bmatrix} 2x+y & 3y \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 4 \end{bmatrix}$$

Equating the corresponding elements, we get

$$2x+y=6 \quad ...(i)$$

and $$3y=0 \quad ...(ii)$$

From Eq. (ii), we get

$$y=0$$

Substituting $y=0$ in Eq. (i), we get

$$2x=6 \Rightarrow x=3$$

84. If $\begin{bmatrix} 3y-x & -2x \\ 3 & 7 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ 3 & 7 \end{bmatrix}$, then find the value of y.

Sol : Comp 2010, Delhi 2007

We have $$\begin{bmatrix} 3y-x & -2x \\ 3 & 7 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ 3 & 7 \end{bmatrix}$$

Equating the corresponding elements, we get

$$3y-x=5 \quad ...(i)$$

and $$-2x=-2 \quad ...(ii)$$

From Eq. (ii), we get

$$x=1$$

Substituting $x=1$ in Eq. (i), we get

$$3y-1=5$$

$$3y=5+1=6$$

$$y=2$$

85. Write 2×2 matrix which is both symmetric and skew-symmetric.

Sol : Comp 2014

A null matrix of order 2×2, i.e. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is both symmetric and skew-symmetric.

SHORT ANSWER QUESTIONS

86. If $A+B=\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $A-2B=\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$, then A is

Sol : OD 2020

We have, $$A+B=\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad (1)$$

$$A-2B=\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \quad (2)$$

Multiplying eq (1) be 2 we have

$$2A+2B=2\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad (3)$$

Adding eq (2) and (3) we have

$$\begin{aligned} 3A &= \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \\ A &= \frac{1}{3}\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} \end{aligned}$$

87. For what value of x, is the matrix $A=\begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ x & -3 & 0 \end{bmatrix}$ a skew-symmetric matrix?

Sol : OD 2013

We have, $$A=\begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ x & -3 & 0 \end{bmatrix}$$

If A is a skew-symmetric matrix, then

$$A=-A^T$$

$$\begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ x & -3 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & -1 & x \\ 1 & 0 & -3 \\ -2 & 3 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ x & -3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -x \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}$$

Equating the corresponding element, we get $x=2$

88. If $A^T=\begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 0 & 1 \end{bmatrix}$ and $B=\begin{bmatrix} -1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$, then find A^T-B^T.

Sol : OD 2012

We have, $A^T = \begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$

Transpose of B, $B^T = \begin{bmatrix} -1 & 1 \\ 2 & 2 \\ 1 & 3 \end{bmatrix}$

Now,
$$A^T - B^T = \begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 & 1 \\ 2 & 2 \\ 1 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} 3+1 & 4-1 \\ -1-2 & 2-2 \\ 0-1 & 1-3 \end{bmatrix}$$
$$= \begin{bmatrix} 4 & 3 \\ -3 & 0 \\ -1 & -2 \end{bmatrix}$$

89. Let $A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 2 \\ 7 & 4 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix}$, find a matrix D such that $CD - AB = O$.

Sol : Delhi 2017, SQP 2015

We have $A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 2 \\ 7 & 4 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix}$

Let matrix $D = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$

According to the questions, $CD - AB = 0$

$$\begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix}\begin{bmatrix} x & y \\ z & w \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 2 \\ 7 & 4 \end{bmatrix} = 0$$
$$\begin{bmatrix} 2x+5z & 2y+5w \\ 3x+8z & 3y+8w \end{bmatrix} - \begin{bmatrix} 10-7 & 4-4 \\ 15+28 & 6+16 \end{bmatrix} = 0$$
$$\begin{bmatrix} 2x+5z & 2y+5w \\ 3x+8z & 3y+8w \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 43 & 22 \end{bmatrix}$$

Equating corresponding elements both sides, we get

$$2x+5z = 3,\ 3x+8z = 43$$

and $$2y+5w = 0,\ 3y+8w = 22$$

After solving, we get

$$x = -191,\ y = -110,\ z = 77 \text{ and } w = 44$$
$$D = \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} -191 & -110 \\ 77 & 44 \end{bmatrix}$$

90. Show that all the diagonal elements of a skew-symmetric matrix are zero.

Sol : Delhi 2017

Let $A = [a_{ij}]$ be a skew-symmetric matrix.

Then, $a_{ji} = -a_{ij}$ for all i, j

Now substituting $i = j$, we get

$$a_{ii} = -a_{ii} \text{ for all values of } i$$
$$2a_{ii} = 0$$
$$a_{ii} = 0 \text{ for all values of } i$$
$$a_{11} = a_{22} = a_{33} = \dots = a_{nn} = 0$$

Hence, all the diagonal elements of a skew-symmetric matrix are zero. Hence proved

91. If $\begin{bmatrix} 2x+y & 3y \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 6 & 4 \end{bmatrix}'$, then find the value of x.

Sol : OD 2010

We have
$$\begin{bmatrix} 2x+y & 3y \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 6 & 4 \end{bmatrix}'$$
$$\begin{bmatrix} 2x+y & 3y \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 6 \\ 0 & 4 \end{bmatrix}$$

If two matrices are equal, then their corresponding elements are equal.

$$2x+y = 6 \quad \text{...(i)}$$

and $$3y = 6 \quad \text{...(ii)}$$

From Eq. (ii), we get

$$y = 2$$

Substituting $y = 2$ in Eq. (i), we get

$$2x+2 = 6$$
$$x+1 = 3$$
$$x = 2$$

92. If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & x \\ -2 & 2 & -1 \end{bmatrix}$ is a metric satisfying $AA' = 9I$, find x.

Sol : Comp 2018, Delhi 2015

We have, $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & x \\ -2 & 2 & -1 \end{bmatrix}$

Also, $AA' = 9I$

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & x \\ -2 & 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & x & -1 \end{bmatrix} = 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 1+4+4 & 2+2+2x & -2+4-2 \\ 2+2+2x & 4+1+x^2 & -4+2-x \\ -2+4-2 & -4+2-x & 4+4+1 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$
$$\begin{bmatrix} 9 & 4+2x & 0 \\ 4+2x & 5+x^2 & -2-x \\ 0 & -2-x & 9 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

Equating the corresponding elements, we get

$$4 + 2x = 0 \text{ and } 5 + x^2 = 9$$
$$x = -2 \text{ and } x^2 = 4$$
$$x = -2 \text{ and } x = \pm 2$$

The value of x is -2.

93. If the matrix $A = \begin{bmatrix} 0 & a & -3 \\ 2 & 0 & -1 \\ b & 1 & 0 \end{bmatrix}$ is skew-symmetric, find the values of a and b.

Sol : OD 2018

Since $A = \begin{bmatrix} 0 & a & -3 \\ 2 & 0 & -1 \\ b & 1 & 0 \end{bmatrix}$ is a skew-symmetric matrix,

$$A^T = -A$$

$$\begin{bmatrix} 0 & a & -3 \\ 2 & 0 & -1 \\ b & 1 & 0 \end{bmatrix}^T = -\begin{bmatrix} 0 & a & -3 \\ 2 & 0 & -1 \\ b & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2 & b \\ a & 0 & 1 \\ -3 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -a & 3 \\ -2 & 0 & 1 \\ -b & -1 & 0 \end{bmatrix}$$

Equating the corresponding elements, we get

$$a = -2 \text{ and } b = 3$$

94. Matrix $A = \begin{bmatrix} 0 & 2b & -2 \\ 3 & 1 & 3 \\ 3a & 3 & -1 \end{bmatrix}$ is given to be symmetric, find find the values of a and b.

Sol : Delhi 2016, OD 2012

Since $A = \begin{bmatrix} 0 & 2b & -2 \\ 3 & 1 & 3 \\ 3a & 3 & -1 \end{bmatrix}$ is a symmetric matrix.

$$A^T = A$$

$$\begin{bmatrix} 0 & 2b & -2 \\ 3 & 1 & 3 \\ 3a & 3 & -1 \end{bmatrix}^T = \begin{bmatrix} 0 & 2b & -2 \\ 3 & 1 & 3 \\ 3a & 3 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 3 & 3a \\ 2b & 1 & 3 \\ -2 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 2b & -2 \\ 3 & 1 & 3 \\ 3a & 3 & -1 \end{bmatrix}$$

Equating the corresponding elements, we get

$$3 = 2b \text{ and } 3a = -2$$
$$b = \frac{3}{2} \text{ and } a = -\frac{2}{3}$$

95. Find the value of $y - x$ from following equation.

$$2\begin{bmatrix} x & 5 \\ 7 & y-3 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$$

Sol : OD 2012

We have $2\begin{bmatrix} x & 5 \\ 7 & y-3 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$

$$\begin{bmatrix} 2x & 10 \\ 14 & 2y-6 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$$

$$\begin{bmatrix} 2x+3 & 10-4 \\ 14+1 & 2y-6+2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$$

$$\begin{bmatrix} 2x+3 & 6 \\ 15 & 2y-4 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$$

Equating the corresponding elements, we get

$$2x + 3 = 7 \Rightarrow x = \frac{7-3}{2} = 2$$

and $2y - 4 = 14 \Rightarrow y = \frac{14-4}{2} = 5$

Now $y - x = 5 - 2 = 3$

96. If $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, then find α satisfying $0 < \alpha < \frac{\pi}{2}$ when $A + A^T = \sqrt{2}\, I_2$; where A^T is transpose of A.

Sol : OD 2016, Comp 2014

We have $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$

Also, given $A + A^T = \sqrt{2}\, I_2$

$$\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} + \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}^T = \sqrt{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} + \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$$

$$\begin{bmatrix} \cos\alpha + \cos\alpha & \sin\alpha - \sin\alpha \\ -\sin\alpha + \sin\alpha & \cos\alpha + \cos\alpha \end{bmatrix} = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$$

$$\begin{bmatrix} 2\cos\alpha & 0 \\ 0 & 2\cos\alpha \end{bmatrix} = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$$

Equating the corresponding elements, we get

$$2\cos\alpha = \sqrt{2}$$
$$\cos\alpha = \frac{\sqrt{2}}{2}$$
$$\cos\alpha = \frac{1}{\sqrt{2}}$$

Thus $\alpha = \frac{\pi}{4}$, which is satisfying $0 < \alpha < \frac{\pi}{2}$.

97. If $A = \begin{bmatrix} 3 & 5 \\ 7 & 9 \end{bmatrix}$ is written as $A = P + Q$, where P is a symmetric matrix and Q is skew-symmetric matrix, then write the matrix P.

Sol : Foreign 2016

We have, $A = \begin{bmatrix} 3 & 5 \\ 7 & 9 \end{bmatrix}$ and $A = P + Q$, where P is

Ch 7 : Integrals

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Ch 8 : Application of Integrals

PDF	PDF	PDF	DOC	PPT
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Ch 9 : Differential Equations

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Class 10th

Maths
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Class 9th

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Comp. App.
AI
English Comm.
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हिंदी ब
संस्कृत

Class 8th

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संस्कृत

Class 7th

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Science
Social Science
English
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Class 6th

Maths
Science
Social Science
English
हिंदी
संस्कृत

Class 5th

Maths | English | हिंदी | EVS

Class 4th

Maths | English | हिंदी | EVS

Class 3th

Maths | English | हिंदी | EVS

Class 2th

Maths | English | हिंदी | EVS

Class 1th

Maths | English | हिंदी | EVS

Prep/KG 3rd Year

Maths | English | हिंदी | EVS

Prep/KG 2nd Year

Maths | English | हिंदी | EVS

Prep/KG 1st Year

Maths | English | हिंदी | EVS

symmetric matrix and Q is skew-symmetric matrix.

We have, $A = \begin{bmatrix} 3 & 5 \\ 7 & 9 \end{bmatrix}$

Now
$$P = \frac{1}{2}(A + A^T)$$
$$= \frac{1}{2}\left\{\begin{bmatrix} 3 & 5 \\ 7 & 9 \end{bmatrix} + \begin{bmatrix} 3 & 5 \\ 7 & 9 \end{bmatrix}^T\right\}$$
$$= \frac{1}{2}\left\{\begin{bmatrix} 3 & 5 \\ 7 & 9 \end{bmatrix} + \begin{bmatrix} 3 & 7 \\ 5 & 9 \end{bmatrix}\right\}$$
$$= \frac{1}{2}\begin{bmatrix} 6 & 12 \\ 12 & 18 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 6 & 9 \end{bmatrix}$$

98. If $A = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$, show that $(A - 2I)(A - 3I) = O$.

Sol : OD 2019

We have $A = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$

$$\text{LHS} = (A - 2I)(A - 3I)$$
$$= \left\{\begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right\}\left\{\begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\right\}$$
$$= \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$$
$$= \begin{bmatrix} 2-2 & 4-4 \\ -1+1 & -2+2 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O = \text{RHS}$$

Hence proved.

99. Find a matrix A such that $2A - 3B + 5C = O$, where $B = \begin{bmatrix} -2 & 2 & 0 \\ 3 & 1 & 4 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 0 & -2 \\ 7 & 1 & 6 \end{bmatrix}$.

Sol : Delhi 2019, Comp 2017

We have $2A - 3B + 5C = O$

$$2A = 3B - 5C$$
$$2A = 3\begin{bmatrix} -2 & 2 & 0 \\ 3 & 1 & 4 \end{bmatrix} - 5\begin{bmatrix} 2 & 0 & -2 \\ 7 & 1 & 6 \end{bmatrix}$$
$$= \begin{bmatrix} -6 & 6 & 0 \\ 9 & 3 & 12 \end{bmatrix} - \begin{bmatrix} 10 & 0 & -10 \\ 35 & 5 & 30 \end{bmatrix}$$
$$= \begin{bmatrix} -6-10 & 6-0 & 0-(-10) \\ 9-35 & 3-5 & 12-30 \end{bmatrix}$$
$$2A = \begin{bmatrix} -16 & 6 & 10 \\ -26 & -2 & -18 \end{bmatrix}$$
$$A = \frac{1}{2}\begin{bmatrix} -16 & 6 & 10 \\ -26 & -2 & -18 \end{bmatrix}$$
$$= \begin{bmatrix} -8 & 3 & 5 \\ -13 & -1 & -9 \end{bmatrix}$$

LONG ANSWER QUESTIONS

100. If $f(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$, then prove that

$$f(\alpha)\cdot f(-\beta) = f(\alpha - \beta).$$

Sol : Delhi 2020

We have $f(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$f(-\beta) = \begin{bmatrix} \cos(-\beta) & -\sin(-\beta) & 0 \\ \sin(-\beta) & \cos(-\beta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} \cos\beta & \sin\beta & 0 \\ -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$f(\alpha)\cdot f(-\beta) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}\cdot\begin{bmatrix} \cos\beta & \sin\beta & 0 \\ -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} \cos\alpha\cos\beta + \sin\alpha\sin\beta & \cos\alpha\sin\beta - \sin\alpha\cos\beta & 0 \\ \sin\alpha\cos\beta - \cos(\alpha-\beta) & \sin\alpha\sin\beta + \cos\alpha\cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} \cos(\alpha-\beta) & \sin(\beta-\alpha) & 0 \\ \sin(\alpha-\beta) & \cos(\alpha-\beta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} \cos(\alpha-\beta) & -\sin(\alpha-\beta) & 0 \\ \sin(\alpha-\beta) & \cos(\alpha-\beta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= f(\alpha-\beta)$$

(Proved)

101. If $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$, $B = \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}$ and $(A+B)^2 = A^2 + B^2$, then find the values of a and b.

Sol : Foreign 2015

We have $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}$

Now, $A + B = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}$

$$= \begin{bmatrix} 1+a & -1+1 \\ 2+b & -1-1 \end{bmatrix} = \begin{bmatrix} 1+a & 0 \\ 2+b & -2 \end{bmatrix}$$
$$(A+B)^2 = \begin{bmatrix} 1+a & 0 \\ 2+b & -2 \end{bmatrix}\cdot\begin{bmatrix} 1+a & 0 \\ 2+b & -2 \end{bmatrix}$$
$$= \begin{bmatrix} 1+a^2+2a & 0 \\ 2+2a+b+ab-4-2b & 4 \end{bmatrix}$$

$$= \begin{bmatrix} a^2+2a+1 & 0 \\ 2a-b+ab-2 & 4 \end{bmatrix}$$

$$A^2+B^2 = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix} \cdot \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} a^2+b & a-1 \\ ab-b & b+1 \end{bmatrix}$$

$$= \begin{bmatrix} a^2+b-1 & a-1 \\ ab-b & b \end{bmatrix}$$

Now, $(A+B)^2 = A^2+B^2$

$$\begin{bmatrix} a^2+2a+1 & 0 \\ 2a-b+ab-2 & 4 \end{bmatrix} = \begin{bmatrix} a^2+b-1 & a-1 \\ ab-b & b \end{bmatrix}$$

Equating the corresponding elements, we get

$$a^2+2a+1 = a^2+b-1$$

$$2a-b = -2 \qquad \text{...(i)}$$

$$a-1 = 0 \Rightarrow a = 1 \qquad \text{...(ii)}$$

$$2a-b+ab-2 = ab-b$$

$$2a-2 = 0 \Rightarrow a = 1 \qquad \text{...(iii)}$$

and $\qquad b = 4 \qquad$ ...(iv)

Since, $a=1$ and $b=4$ also satisfy Eq. (i), therefore $a=1$ and $b=4$.

102. If $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$, the find $A^2-5A+4I$ and hence find a matrix X such that $A^2-5A+4I+X=O$.

Sol : Delhi 2015

We have $\qquad A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4+0+1 & 0+0-1 & 2+0+0 \\ 4+2+3 & 0+1-3 & 2+3+0 \\ 2-2+0 & 0-1+0 & 1-3+0 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix}$$

Now, consider $A^2-5A+4I$

$$A^2-5A+4I = \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix} - 5\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} + 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix} - \begin{bmatrix} 10 & 0 & 5 \\ 10 & 5 & 15 \\ 5 & -5 & 0 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & -1 & -3 \\ -1 & -7 & -10 \\ -5 & 4 & -2 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -1 & -3 \\ -1 & -3 & -10 \\ -5 & 4 & 2 \end{bmatrix}$$

Now $\qquad A^2-5A+4I+X = O$

$$A^2-5A+4I+X-X = O-X$$

$$A^2-5A+4I+O = -X$$

$$X = -(A^2-5A+4I)$$

$$= -\begin{bmatrix} -1 & -1 & -3 \\ -1 & -3 & -10 \\ -5 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & 10 \\ 5 & -4 & -2 \end{bmatrix}$$

103. Find matrix A such that

$$\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix} A = \begin{bmatrix} -1 & -8 \\ 1 & -2 \\ 9 & 22 \end{bmatrix}$$

Sol : OD 2017

Let the order of A is $m \times n$ where $m=2$, $n=2$

Let $\qquad A = \begin{bmatrix} x & y \\ s & t \end{bmatrix} \qquad$ ...(i)

Now $\qquad \begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix} A = \begin{bmatrix} -1 & -8 \\ 1 & -2 \\ 9 & 22 \end{bmatrix}$

$$\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} x & y \\ s & t \end{bmatrix} = \begin{bmatrix} -1 & -8 \\ 1 & -2 \\ 9 & 22 \end{bmatrix}$$

$$\begin{bmatrix} 2x-s & 2y-t \\ x & y \\ -3x+4s & -3y+4t \end{bmatrix} = \begin{bmatrix} -1 & -8 \\ 1 & -2 \\ 9 & 22 \end{bmatrix}$$

Equating corresponding elements both sides, we get

$$2x-s = -1,\ x=1,\ y=-2 \text{ and } 2y-t=-8$$

At $\qquad x = 1,\ 2x-s=-1 \Rightarrow 2\times 1-s=-1$

$$-s = -1-2 \Rightarrow s=3 \text{ and at}$$

$$y = -2,\ 2y-t=-8,$$

$$2\times(-2)-t = -8$$

$$-4-t = -8$$

$$t = 4$$

Substituting $x=1$, $y=-2$, $s=3$ and $t=4$ in Eq. (i), we get

$$A = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$$

104. If $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$ and $A^3 - 6A^2 + 7A + kI_3 = O$, find the values of k.

Sol : OD 2015, Comp 2010

We have $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$

Now, $A^2 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 1+0+4 & 0+0+0 & 2+0+6 \\ 0+0+2 & 0+4+0 & 0+2+3 \\ 2+0+6 & 0+0+0 & 4+0+9 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix}$$

and $A^3 = A \cdot A^2$

$$= \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}\begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix}$$

$$= \begin{bmatrix} 5+0+16 & 0+0+0 & 8+0+26 \\ 0+4+8 & 0+8+0 & 0+10+13 \\ 10+0+24 & 0+0+0 & 16+0+39 \end{bmatrix}$$

$$= \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix}$$

Also, given, $A^3 - 6A^2 + 7A + kI_3 = O$

$$\begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix} - 6\begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} + k\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix} - \begin{bmatrix} 30 & 0 & 48 \\ 12 & 24 & 30 \\ 48 & 0 & 78 \end{bmatrix} + \begin{bmatrix} 7 & 0 & 14 \\ 0 & 14 & 7 \\ 14 & 0 & 21 \end{bmatrix} + \begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 21-30+7+k & 0-0+0+0 & 34-48+14+0 \\ 12-12+0+0 & 8-24+14+k & 23-30+7+0 \\ 34-48+14+0 & 0-0+0+0 & 55-78+21+k \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -2+k & 0 & 0 \\ 0 & -2+k & 0 \\ 0 & 0 & -2+k \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Equating the corresponding elements, we get

$$-2+k = 0$$

$$k = 2$$

105. If $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$, then find the values of $(A^2 - 5A)$.

Sol : Delhi 2019

We have $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$

$A^2 = A \times A$

$$= \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} \times \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4+0+1 & 0+0-1 & 2+0+0 \\ 4+2+3 & 0+1-3 & 2+3+0 \\ 2-2+0 & 0-1+0 & 1-3-0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix}$$

Now, $A^2 - 5A = \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix} - 5\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix} - \begin{bmatrix} 10 & 0 & 5 \\ 10 & 5 & 15 \\ 5 & -5 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 5-15 & -1-0 & 2-5 \\ 9-10 & -2-5 & 5-15 \\ 0-5 & -1+5 & -2-0 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & -1 & -3 \\ -1 & -7 & -10 \\ -5 & 4 & -2 \end{bmatrix}$$

106. If $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$, then find values of $A^2 - 3A + 2I$.

Sol : OD 2010

We have $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4+0+1 & 0+0-1 & 2+0+0 \\ 4+2+3 & 0+1-3 & 2+3+0 \\ 2-2+0 & 0-1+0 & 1-3+0 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix}$$

$$A^2 - 3A + 2I = \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix} - 3\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix} - \begin{bmatrix} 6 & 0 & 3 \\ 6 & 3 & 9 \\ 3 & -3 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -1 & -1 \\ 3 & -5 & -4 \\ -3 & 2 & -2 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & -1 \\ 3 & -3 & -5 \\ -3 & 2 & 0 \end{bmatrix}$$

107. Express the matrix $A = \begin{bmatrix} 2 & 4 & -6 \\ 7 & 3 & 5 \\ 1 & -2 & 4 \end{bmatrix}$ as the sum of a symmetric and skew-symmetric matrix.

Sol : Comp 2015, Delhi 2012

Any square matrix A can be expressed as the sum of a symmetric matrix and skew-symmetric matrix, i.e.

$$A = \frac{A + A'}{2} + \frac{A - A'}{2},$$

where $\frac{A+A'}{2}$ and $\frac{A-A'}{2}$ are symmetric and skew-symmetric matrices respectively.

We have $A = \begin{bmatrix} 2 & 4 & -6 \\ 7 & 3 & 5 \\ 1 & -2 & 4 \end{bmatrix}$

$$A' = \begin{bmatrix} 2 & 7 & 1 \\ 4 & 3 & -2 \\ -6 & 5 & 4 \end{bmatrix}$$

Now, let $P = \frac{1}{2}(A + A')$

$$= \frac{1}{2}\left\{\begin{bmatrix} 2 & 4 & -6 \\ 7 & 3 & 5 \\ 1 & -2 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 7 & 1 \\ 4 & 3 & -2 \\ -6 & 5 & 4 \end{bmatrix}\right\}$$

$$= \frac{1}{2}\begin{bmatrix} 4 & 11 & -5 \\ 11 & 6 & 3 \\ -5 & 3 & 8 \end{bmatrix} = \begin{bmatrix} 2 & \frac{11}{2} & -\frac{5}{2} \\ \frac{11}{2} & 3 & \frac{3}{2} \\ -\frac{5}{2} & \frac{3}{2} & 4 \end{bmatrix}$$

and $Q = \frac{1}{2}(A - A')$

$$= \frac{1}{2}\left\{\begin{bmatrix} 2 & 4 & -6 \\ 7 & 3 & 5 \\ 1 & -2 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 7 & 1 \\ 4 & 3 & -2 \\ -6 & 5 & 4 \end{bmatrix}\right\}$$

$$= \frac{1}{2}\begin{bmatrix} 0 & -3 & -7 \\ 3 & 0 & 7 \\ 7 & -7 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{3}{2} & -\frac{7}{2} \\ \frac{3}{2} & 0 & \frac{7}{2} \\ \frac{7}{2} & -\frac{7}{2} & 0 \end{bmatrix}$$

Clearly, $P' = \begin{bmatrix} 2 & \frac{11}{2} & -\frac{5}{2} \\ \frac{11}{2} & 3 & \frac{3}{2} \\ -\frac{5}{2} & \frac{3}{2} & 4 \end{bmatrix} = P$

and $Q' = \begin{bmatrix} 0 & \frac{3}{2} & \frac{7}{2} \\ -\frac{3}{2} & 0 & -\frac{7}{2} \\ -\frac{7}{2} & \frac{7}{2} & 0 \end{bmatrix} = -Q$

So, P is a symmetric matrix and Q is a skew-symmetric matrix.

Now, $P + Q = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$

$$= \begin{bmatrix} 2 & \frac{11}{2} & -\frac{5}{2} \\ \frac{11}{2} & 3 & \frac{3}{2} \\ -\frac{5}{2} & \frac{3}{2} & 4 \end{bmatrix} + \begin{bmatrix} 0 & -\frac{3}{2} & -\frac{7}{2} \\ \frac{3}{2} & 0 & \frac{7}{2} \\ \frac{7}{2} & -\frac{7}{2} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 4 & -6 \\ 7 & 3 & 5 \\ 1 & -2 & 4 \end{bmatrix} = A$$

Thus, matrix A is expressed as the sum of symmetric matrix and skew-symmetric matrix.

108. For the following matrices A and B, verify that $[AB]' = B'A'$;

$$A = \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}, B = \begin{bmatrix} -1 & 2 & 1 \end{bmatrix}.$$

Sol : OD 2010

We have $A = \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 & 1 \end{bmatrix}$

Here, $AB = \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}_{3 \times 1} \begin{bmatrix} -1 & 2 & 1 \end{bmatrix}_{1 \times 3}$

$$AB = \begin{bmatrix} -1 & 2 & 1 \\ 4 & -8 & -4 \\ -3 & 6 & 3 \end{bmatrix}$$

$$(AB)' = \begin{bmatrix} -1 & 4 & -3 \\ 2 & -8 & 6 \\ 1 & -4 & 3 \end{bmatrix} \quad \text{...(i)}$$

Now, $B' = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$

and $A' = \begin{bmatrix} 1 & -4 & 3 \end{bmatrix}$

$$B'A' = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & -4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 4 & -3 \\ 2 & -8 & 6 \\ 1 & -4 & 3 \end{bmatrix} \quad \text{(ii)}$$

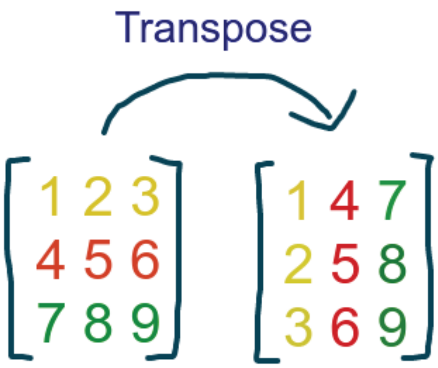
From Eqs. (i) and (ii), we get

$$(AB)' = B'A'$$

LHS = RHS

CASE BASED QUESTIONS

109. In linear algebra, the transpose of a matrix is an operator which flips a matrix over its diagonal; that is, it switches the row and column indices of the matrix A by producing another matrix, often denoted by A^T. The transpose of a matrix was introduced in 1858 by the British mathematician Arthur Cayley.



If $A = [a_{ij}]$ be a $m \times n$ matrix, then the matrix obtained by interchanging the rows and columns of A is called the transpose of A. A square matrix $A = [a_{ij}]$ is said to be symmetric, $A^T = A$ for all possible values of i and j. A square matrix A $[a_{ij}]$ is said to be skew-symmetric, if $A^T = -A$ for all possible values of i and j. Based on the above, information, answer the following questions._

(i) Find the transpose of $[1, -2, -5]$.

(ii) Find the transpose of matrix (ABC).

(iii) Evaluate $(A+B)^T - A$, where $A = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

(iv) Evaluate $(AB)^T$, where $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$

Sol :

(i) $$[1 -2 -5]^T = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$$

(ii) $$(ABC)^T = C^T B^T A^T$$

(iii) $$(A+B) = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix}$$

Now, $$(A+B)^T - A = \begin{bmatrix} 1 & 5 \\ 3 & 3 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 1 & 4 \end{bmatrix}$$

(iv) $$AB = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 1 & 4 \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} 4 & 1 \\ 6 & 4 \end{bmatrix}$$

110. Publishing is the activity of making information, literature, music, software and other content available to the public for sale or for free. Traditionally, the term refers to the creation and distribution of printed works, such as books, newspapers, and magazines.

NODIA Press is a such publishing house having two branch at Jaipur. In each branch there are three offices. In each office, there are 2 peons, 5 clerks and 3 typists. In one office of a branch, 5 salesmen are also working. In each office of other branch 2 head-clerks are also working. Using matrix notations find :

(i) the total number of posts of each kind in all the offices taken together in each branch.

(ii) the total number of posts of each kind in all the offices taken together from both branches.

Sol :

Consider the three row matrices,

$$A_1 = [2\ \ 5\ \ 3\ \ 5\ \ 0]$$

$$A_2 = [2\ \ 5\ \ 3\ \ 0\ \ 0]$$

$$A_3 = [2\ \ 5\ \ 3\ \ 0\ \ 0]$$

The matrices A_1, A_2, A_3 represent the three offices of one branch (say A), where the elements appearing in the row represents the number of peons, clerks, typists, salesman and head-clerks taken in that order working in the three offices.

Now, consider the three row matrices,

$$B_1 = [2\ \ 5\ \ 3\ \ 0\ \ 2]$$

$$B_2 = [2\ \ 5\ \ 3\ \ 0\ \ 2]$$

$$B_3 = [2\ \ 5\ \ 3\ \ 0\ \ 2]$$

The matrices B_1, B_2, B_3 represents the three offices of other branch (say B), where the elements appearing in the row represents the number of peons, clerks, typists, salesman and head-clerks taken in that order working in the three offices.

The total number of posts of each kind in all the offices of branch A are the elements of the matrix:

$$\begin{aligned} A &= A_1 + A_2 + A_3 \\ &= [2\ \ 5\ \ 3\ \ 5\ \ 0] + [2\ \ 5\ \ 3\ \ 0\ \ 0] + [2\ \ 5\ \ 3\ \ 0\ \ 0] \\ &= [2+2+2\ \ 5+5+5\ \ 3+3+3\ \ 5+0+0\ \ 0+0+0] \\ &= [6\ \ 15\ \ 9\ \ 5\ \ 0] \end{aligned}$$

(i) The total number of posts of each kind in all the offices of branch B are the element of the matrix:

$$\begin{aligned} B &= B_1 + B_2 + B_3 \\ &= [2\ \ 5\ \ 3\ \ 0\ \ 2] + [2\ \ 5\ \ 3\ \ 0\ \ 2] + [2\ \ 5\ \ 3\ \ 0\ \ 2] \\ &= [6\ \ 15\ \ 9\ \ 0\ \ 6] \end{aligned}$$

(ii) The total number of posts of each kind in all the offices taken together from both branches A and B are the elements of the matrix:

$$\begin{aligned} A + B &= (A_1 + A_2 + A_3) + (B_1 + B_2 + B_3) \\ &= [12\ \ 30\ \ 18\ \ 5\ \ 6] \end{aligned}$$

111. Fertilizer, natural or artificial substance containing the chemical elements that improve growth and productiveness of plants. Fertilizers enhance the natural fertility of the soil or replace chemical elements taken from the soil by previous crops.

The following matrix gives the proportionate mix of constituents used for three fertilisers:

$$\text{Fertilisers}\ \begin{array}{c} \\ \text{I} \\ \text{II} \\ \text{III} \end{array}\!\!\begin{array}{c} \text{Constituents} \\ \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0.5 & 0 & 0.5 & 0 \\ 0.2 & 0.3 & 0 & 0.5 \\ 0.2 & 0.2 & 0.1 & 0.5 \end{bmatrix} \end{array}$$

(i) If sales are 1000 tins (of one kilogram) per week, 20% being fertiliser I, 30% being fertiliser II and 50% being fertiliser III, how much of each constituent is used.

(ii) If the cost of each constituents is ` 5, ` 6, ` 7.5 and ` 10 per 100 grams, respectively, how much does a one kilogram tin of each fertiliser cost

(iii) What is the total cost per week?

Sol :

The sales of fertilisers I, II and III per week can be expressed as the following matrix:

$$\begin{aligned} P &= 1000\begin{array}{c} \begin{array}{ccc} \text{I} & \text{II} & \text{III} \end{array} \\ \begin{bmatrix} 0.2 & 0.3 & 0.5 \end{bmatrix} \end{array} \\ &= \begin{array}{c} \begin{array}{ccc} \text{I} & \text{II} & \text{III} \end{array} \\ [200\ \ 300\ \ 500] \end{array} \end{aligned}$$

Let the matrix denote the proportionate mix of constituents used for three fertilisers I, II and III, then

$$Q = \text{Fertilisers}\ \begin{array}{c} \\ \text{I} \\ \text{II} \\ \text{III} \end{array}\!\!\begin{array}{c} \text{Constituents} \\ \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0.5 & 0 & 0.5 & 0 \\ 0.2 & 0.3 & 0 & 0.5 \\ 0.2 & 0.2 & 0.1 & 0.5 \end{bmatrix} \end{array}$$

The requirement of each constituents given by the matrix product PQ is:

$$\begin{aligned} PQ &= \begin{array}{c} \begin{array}{ccc} \text{I} & \text{II} & \text{III} \end{array} \\ [200\ \ 300\ \ 500] \end{array} \\ &= \begin{array}{c} \\ \text{I} \\ \text{II} \\ \text{III} \end{array}\!\!\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0.50 & 0 & 0.5 & 0 \\ 0.2 & 0.3 & 0 & 0.5 \\ 0.2 & 0.2 & 0.1 & 0.5 \end{bmatrix} \end{array} \\ &= \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \left[100+600+100\ \ \ 0+90+100\ \ \ 100+0+50\ \ \ 0+150+250\right] \end{array} \\ &= \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ [260\ \ 190\ \ 150\ \ 400] \end{array} \end{aligned}$$

Thus, the requirements of constituents are $A - 260$, $B - 190$, $.C - 150$, and $D - 400$ of each constituents is Rs 5, Rs 6, Rs 7.5 e and Rs 10 per 100 grams, i.e., Rs 50, Rs 60, Rs 75 and Rs 100 per 1000 grams (one kilogram) of each constituent, respectively.

Let R denote the cost of constituents A, B, C and Dper kilogram, then

$$R = \begin{array}{c} A \\ B \\ C \\ D \end{array}\!\!\begin{bmatrix} 50 \\ 60 \\ 75 \\ 100 \end{bmatrix}$$

The cost of one kilogram tin of each fertiliser I, II and III is given by the matrix product QR is:

$$QR = \begin{array}{c} \\ \text{I} \\ \text{II} \\ \text{III} \end{array}\!\!\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0.5 & 0 & 0.5 & 0 \\ 0.2 & 0.3 & 0 & 0.5 \\ 0.2 & 0.2 & 0.1 & 0.5 \end{bmatrix} \end{array} \begin{array}{c} A \\ B \\ C \\ D \end{array}\!\!\begin{bmatrix} 50 \\ 60 \\ 75 \\ 100 \end{bmatrix}$$

$$= \begin{array}{c} \text{I} \\ \text{II} \\ \text{III} \end{array} \begin{bmatrix} 25+0+37.5+0 \\ 10+18+0+50 \\ 10+12+7.5+50 \end{bmatrix}$$

$$= \begin{array}{c} \text{I} \\ \text{II} \\ \text{III} \end{array} \begin{bmatrix} 62.5 \\ 78 \\ 79.5 \end{bmatrix}$$

Thus, the costs per kilogram tin of fertiliser are:

$$\text{I} - \text{`}\,62.5, \text{II} - \text{`}\,78, \text{III} - \text{`}\,79.5.$$

The total cost of fertiliser of 1000 tins (of one kilogram) are needed per week may be calculated by

Either: $P(QR)$

$$= \begin{array}{ccc} \text{I} & \text{II} & \text{III} \\ \end{array} \begin{bmatrix} 200 & 300 & 500 \end{bmatrix} \begin{array}{c} \text{I} \\ \text{II} \\ \text{III} \end{array} \begin{bmatrix} 62.5 \\ 78 \\ 79.5 \end{bmatrix}$$

$$= \begin{bmatrix} 12500 + 23400 + 39750 \end{bmatrix}$$

$$= \begin{bmatrix} 75650 \end{bmatrix}$$

Hence, the total cost per week is ` 75650.

112. Rice is a nutritional staple food which provides instant energy as its most important component is carbohydrate (starch). On the other hand, rice is poor in nitrogenous substances with average composition of these substances being only 8 per cent and fat content or lipids only negligible, i.e., 1 per cent and due to this reason it is considered as a complete food for eating. Rice flour is rich in starch and is used for making various food materials.

Two farmers Ramkishan and Gurcharan Singh cultivate only three varieties of rice namely Basmati, Permal and Naura. The sale (in `) of these varieties of rice by both the farmers in the month of September and October are given by the following matrices A and B.

September Sales (in `)

$$A = \begin{array}{ccc} \text{Basmati} & \text{Permal} & \text{Naura} \end{array}$$
$$A = \begin{bmatrix} 10000 & 20000 & 30000 \\ 50000 & 30000 & 10000 \end{bmatrix} \begin{array}{l} \text{Ramkishan} \\ \text{Gurcharan Singh} \end{array}$$

October Sales (in `)

$$\begin{array}{ccc} \text{Basmati} & \text{Permal} & \text{Naura} \end{array}$$
$$B = \begin{bmatrix} 5000 & 10000 & 6000 \\ 20000 & 10000 & 10000 \end{bmatrix} \begin{array}{l} \text{Ramkishan} \\ \text{Gurcharan Singh} \end{array}$$

(i) Find the combined sales in September and October for each farmer in each variety.

(ii) Find the decrease in sales from September to October.

(iii) If both farmers receive 2% profit on gross sales, compute the profit for each farmer and for each variety sold in October.

Sol :

The sale (in `) of these varieties of rice by both the farmers in the month of September and October are given by the following matrices A and B.

September Sales (in `)

$$\begin{array}{ccc} \text{Basmati} & \text{Permal} & \text{Naura} \end{array}$$
$$A = \begin{bmatrix} 10000 & 20000 & 30000 \\ 50000 & 30000 & 10000 \end{bmatrix} \begin{array}{l} \text{Ramkishan} \\ \text{Gurcharan Singh} \end{array}$$

October Sales (in `)

$$\begin{array}{ccc} \text{Basmati} & \text{Permal} & \text{Naura} \end{array}$$
$$B = \begin{bmatrix} 5000 & 10000 & 6000 \\ 20000 & 10000 & 10000 \end{bmatrix} \begin{array}{l} \text{Ramkishan} \\ \text{Gurcharan Singh} \end{array}$$

(i) Combined sales in September and October for each farmer in each variety is given by

$$\begin{array}{ccc} \text{Basmati} & \text{Permal} & \text{Naura} \end{array}$$
$$A + B = \begin{bmatrix} 10000+5000 & 20000+10000 & 30000+6000 \\ 50000+20000 & 30000+10000 & 10000+10000 \end{bmatrix}$$

$$\begin{array}{ccc} \text{Basmati} & \text{Permal} & \text{Naura} \end{array}$$
$$= \begin{bmatrix} 15000 & 30000 & 36000 \\ 70000 & 40000 & 20000 \end{bmatrix} \begin{array}{l} \text{Ramkishan} \\ \text{Gurcharan Singh} \end{array}$$

(ii) Change in sales from September to October is given by

$$\begin{array}{ccc} \text{Basmati} & \text{Permal} & \text{Naura} \end{array}$$
$$A - B = \begin{bmatrix} 10000-5000 & 20000-10000 & 30000-6000 \\ 50000-20000 & 30000-10000 & 10000-10000 \end{bmatrix}$$

$$\begin{array}{ccc} \text{Basmati} & \text{Permal} & \text{Naura} \end{array}$$
$$= \begin{bmatrix} 5000 & 10000 & 24000 \\ 30000 & 20000 & 0 \end{bmatrix} \begin{array}{l} \text{Ramkishan} \\ \text{Gurcharan Singh} \end{array}$$

(iii) $\quad 2\% \text{ of } B = \frac{2}{100} \times B = 0.02 \times B$

$$\begin{array}{ccc} \text{Basmati} & \text{Permal} & \text{Naura} \end{array}$$
$$= 0.02 \begin{bmatrix} 5000 & 10000 & 6000 \\ 20000 & 10000 & 10000 \end{bmatrix} \begin{array}{l} \text{Ramkishan} \\ \text{Gurcharan Singh} \end{array}$$

$$\begin{array}{ccc} \text{Basmati} & \text{Permal} & \text{Naura} \end{array}$$
$$= \begin{bmatrix} 100 & 200 & 120 \\ 400 & 200 & 200 \end{bmatrix} \begin{array}{l} \text{Ramkishan} \\ \text{Gurcharan Singh} \end{array}$$

Thus, in October Ramkishan receives ` 100, ` 200 and ` 120 as profit in the sale of each variety of rice, respectively, and Gurcharan Singh receives profit of ` 400, ` 200 and ` 200 in the sale of each variety of rice, respectively.

113. Sun Pharmaceutical Industries Limited is an Indian multinational pharmaceutical company headquartered in Mumbai, Maharashtra, that manufactures and sells pharmaceutical formulations and active pharmaceutical ingredients in more than 100 countries across the globe.

Sun Pharmaceutical produces three final chemical products P_1, P_2 and P_3 requiring mixup of three raw material chemicals M_1, M_2 and M_3. The per unit requirement of each product for each material (in litres) is as follows:

$$A = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} M_1 & M_2 & M_3 \end{array} \\ \begin{bmatrix} 2 & 3 & 1 \\ 4 & 2 & 5 \\ 2 & 4 & 2 \end{bmatrix} \end{array}$$

(i) Find the total requirement of each material if the firm produces 100 litres of each product,

(ii) Find the per unit cost of production of each product if the per unit of materials M_1, M_2 and M_3 are ₹5, ₹10 and ₹5 respectively, and

(iii) Find the total cost of production if the firm produces 200 litres of each product.

Sol :

When the firm produces 100 litres of each product, then the production matrix B is given by:

$$B = \begin{array}{c} \begin{array}{ccc} P_1 & P_2 & P_3 \end{array} \\ \begin{bmatrix} 100 & 100 & 100 \end{bmatrix} \end{array}$$

The total requirement of each material is:

$$B = \begin{array}{c} \begin{array}{ccc} P_1 & P_2 & P_3 \end{array} \\ \begin{bmatrix} 100 & 100 & 100 \end{bmatrix} \end{array} \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} M_1 & M_2 & M_3 \end{array} \\ \begin{bmatrix} 2 & 3 & 1 \\ 4 & 2 & 5 \\ 2 & 4 & 2 \end{bmatrix} \end{array}$$

$$= \begin{array}{c} \begin{array}{ccc} M_1 & M_2 & M_3 \end{array} \\ \begin{bmatrix} 200+400+200 & 300+200+400 & 100+500+200 \end{bmatrix} \end{array}$$

$$= \begin{array}{c} \begin{array}{ccc} M_1 & M_2 & M_3 \end{array} \\ \begin{bmatrix} 800 & 900 & 800 \end{bmatrix} \end{array}$$

When the cost per unit of materials M_1, M_2 and M_3 are ₹5, ₹10 and ₹5 respectively, then the cost materials matrix C is given by:

$$C = \begin{array}{c} M_1 \\ M_2 \\ M_3 \end{array}\!\!\begin{bmatrix} 5 \\ 10 \\ 5 \end{bmatrix}$$

The cost of production of each product is given by the matrix

$$AC = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} M_1 & M_2 & M_3 \end{array} \\ \begin{bmatrix} 2 & 3 & 1 \\ 4 & 2 & 5 \\ 2 & 4 & 2 \end{bmatrix} \end{array} \begin{array}{c} \\ M_1 \\ M_2 \\ M_3 \end{array}\!\!\begin{array}{c} \text{Cost (per unit)} \\ \begin{bmatrix} 5 \\ 10 \\ 5 \end{bmatrix} \end{array}$$

$$= \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \end{array}\!\!\begin{array}{c} \text{Cost of (Production)} \\ \begin{bmatrix} 10+30+5 \\ 20+20+25 \\ 10+40+10 \end{bmatrix} \end{array}$$

$$= \begin{array}{c} P_1 \\ P_2 \\ P_3 \end{array}\!\!\begin{bmatrix} 45 \\ 65 \\ 60 \end{bmatrix}$$

When the firm produces 200 litres of each product, then the matrix D is given by:

$$D = \begin{array}{c} \begin{array}{ccc} P_1 & P_2 & P_3 \end{array} \\ \begin{bmatrix} 200 & 200 & 200 \end{bmatrix} \end{array}$$

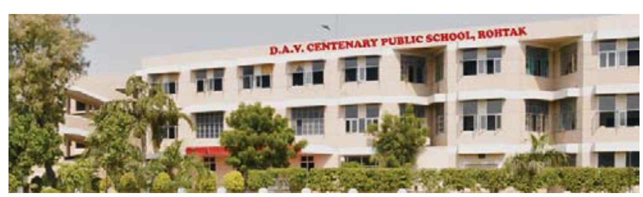
The total cost of production is given by:

$$D(AC) = \begin{array}{c} \begin{array}{ccc} P_1 & P_2 & P_3 \end{array} \\ \begin{bmatrix} 200 & 200 & 200 \end{bmatrix} \end{array} \begin{array}{c} P_1 \\ P_2 \\ P_3 \end{array}\!\!\begin{bmatrix} 45 \\ 65 \\ 60 \end{bmatrix}$$

$$= \begin{bmatrix} 200 \times 45 + 200 \times 65 + 200 \times 60 \end{bmatrix}$$

$$= \begin{bmatrix} 34000 \end{bmatrix}$$

114. The D.A.V. College Managing Committee, familiarly known as DAVCMC, is a non-governmental educational organisation in India and overseas with over 900 schools. 75 colleges and a university. It is based on the ideals of Maharishi Dayanand Saraswati. Full Form of DAV is Dayanand Anglo Vedic.



In a certain city there are 50 colleges and 400 schools. Each school and college has 18 peons, 5 clerks and 1 cashier. Each college in addition has 1 section officer and one librarian. The monthly salary of each of them is as follows:

Peon-Rs 3000, Clerk- Rs 5000, Cashier- Rs 6000, Section Officer-Rs 7000 and Librarian-Rs 9000

Using matrix notation, find

(a) total number of posts of each kind in schools and colleges taken together.

(b) the total monthly salary bill of all the schools and colleges taken together.

Sol :

The number of colleges and schools can be represented by the matrix A as:

$$A = \begin{array}{cc} \text{College} & \text{School} \end{array} \\ \quad = \begin{bmatrix} 50 & 400 \end{bmatrix}$$

The number of posts of each kind in each college and school can be represented by the matrix B as:

$$B = \begin{bmatrix} 18 & 5 & 1 & 1 & 1 \\ 18 & 5 & 1 & 0 & 0 \end{bmatrix}$$

The monthly salary of each of them can be represented by the matrix C as

$$C = \begin{array}{l} \\ \text{Peon} \\ \text{Clerk} \\ \text{Cashier} \\ \text{Section Officer} \\ \text{Librarian} \end{array} \begin{array}{c} \text{Salary} \\ \begin{bmatrix} 3000 \\ 5000 \\ 6000 \\ 7000 \\ 9000 \end{bmatrix} \end{array}$$

The total number of posts of each kind in colleges and schools taken together can be obtained by the matrix product AB as:

$$AB = [50 \;\; 400] \begin{bmatrix} 18 & 5 & 1 & 1 & 1 \\ 18 & 5 & 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 8100 & 2250 & 450 & 50 & 50 \end{bmatrix}$$

Thus there are 8100 peons, 2250 clerks, 450 cashiers, 50 section officers and 50 librarians in all colleges and schools taken together.

The total monthly salary bill of all the colleges and schools taken together is given by the product $(AB)\,C$ as

$$(AB)\,C = [8100 \;\; 2250 \;\; 450 \;\; 50 \;\; 50] \begin{bmatrix} 3000 \\ 5000 \\ 6000 \\ 7000 \\ 9000 \end{bmatrix}$$

$$= [39050000]$$

Hence total monthly salary bill of all the colleges and schools taken together .

(a) There are 8100 peons, 2250 clerks, 450 cashiers, 50 section officers and 50 librarians in all colleges and schools taken together.

(b) Total monthly salary bill of all the colleges and schools taken together Rs 3,905,000.

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CHAPTER 4

DETERMINANTS

OBJECTIVE QUESTIONS

1. If A is a square matrix of order 2 and $|A| = -2$, then value of $|5A'|$ is

(a) -50 (b) -10

(c) 10 (d) 50

Sol : OD 2024

If A is $n \times n$ matrix then $|kA| = k^n|A|$ where k is scalar.

Thus $|5A'| = 5^2|A'|$

$|5A'| = 5^2|A|$ $[|A'| = |A|]$

$|5A'| = 25|A|$

Since $|A| = -2$, we get

$$|5A'| = 25(-2) = -50$$

2. If inverse of matrix $\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ is the matrix $\begin{bmatrix} 1 & 3 & 3 \\ 1 & \lambda & 3 \\ 1 & 3 & 4 \end{bmatrix}$, then value of λ is

(a) –4 (b) 1

(c) 3 (d) 4

Sol : OD 2024

We have $A = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} 1 & 3 & 3 \\ 1 & \lambda & 3 \\ 1 & 3 & 4 \end{bmatrix}$

Now $A = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

$$|A| = 7(1) + 3(-1) - 3(+1)$$

$$= 7 - 3 - 3 = 1$$

We know $AA^{-1} = I$

$$|AA^{-1}| = |I|$$

$$|A| \cdot |A^{-1}| = 1 \qquad |AB| = |A||B|$$

$$1 \cdot |A^{-1}| = 1$$

$$|A^{-1}| = 1$$

$$|A^{-1}| = \begin{vmatrix} 1 & 3 & 3 \\ 1 & \lambda & 3 \\ 1 & 3 & 4 \end{vmatrix}$$

$$= 1(4\lambda - 9) - 3(4 - 3) + 3(3 - \lambda)$$

$$= 4\lambda - 9 - 3(1) + 3(3 - \lambda)$$

$$= 4\lambda - 9 - 3 + 9 - 3\lambda$$

$$|A^{-1}| = \lambda - 3$$

Equating $|A^{-1}|$ to 1 we have

$$(\lambda - 3) = 1$$

$$\lambda - 3 = 1$$

$$\lambda = 4$$

3. If A is a square matrix of order 3 such that the value of $|\text{adj}\, A| = 8$, then the value of $|A^T|$ is

(a) $\sqrt{2}$ (b) $-\sqrt{2}$

(c) 8 (d) $2\sqrt{2}$

Sol : OD 2024

If A is $n \times n$ matrix then

$$|\text{adj}\, A| = |A|^{n-1} \qquad ...(1)$$

Since A is 3×3 matrix, $n = 3$ and given $|\text{adj}\, A| = 8$, Therefore

$$8 = |A|^{3-1}$$

$$8 = |A|^2$$

$$|A| = 2\sqrt{2}$$

Now $|A^T| = |A|$

$$= 2\sqrt{2}$$

Thus (d) is correct option.

Sol : Delhi 2008

$$1\times-\omega-\omega\times\omega=-\omega-\omega^2$$
$$=-(-1)=1$$

Thus option (a) is correct.

10. $\begin{vmatrix} 1 & \log_y x \\ \log_x y & 1 \end{vmatrix}=$

(a) 1 (b) -1

(c) 0 (d) $\log_x y\cdot\log_x y$

Sol : OD 2009

$$1\times1-\log_x y\times\log_y x=1-1=0$$

Thus option (c) is correct.

11. If $\begin{vmatrix} 5 & -1 \\ 3 & n \end{vmatrix}=13$ then $n=$

(a) 1 (b) 2

(c) 3 (d) 4

Sol : Comp 2018

$$(5)(n)-(-1)(3)=13$$
$$5n+3=13$$
$$n=2$$

Thus option (b) is correct.

12. $\begin{vmatrix} \sin 20^\circ & \cos 20^\circ \\ \sin 70^\circ & \cos 70^\circ \end{vmatrix}=$

(a) $-\sin 50^\circ$ (b) $\sin 50^\circ$

(c) $\sin 70^\circ$ (d) $\sin 20^\circ$

Sol : Delhi 2017, OD 2009

$$\begin{vmatrix} \sin 20^\circ & \cos 20^\circ \\ \sin 70^\circ & \cos 70^\circ \end{vmatrix}=\sin 20^\circ\cos 70^\circ-\sin 70^\circ\cos 20^\circ$$
$$=\sin(20^\circ-70^\circ)$$
$$=-\sin 50^\circ$$

Thus option (a) is correct.

13. Value of determinant $\begin{vmatrix} 2 & 8 & 4 \\ -5 & 6 & -10 \\ 1 & 7 & 2 \end{vmatrix}$ is

(a) 488 (b) 328

(c) 0 (d) -440

Sol : Delhi 2010

$$2\begin{vmatrix} 2 & 8 & 2 \\ -5 & 6 & -5 \\ 1 & 7 & 1 \end{vmatrix}=0$$

Thus option (c) is correct.

14. If $\begin{vmatrix} -a^2 & ab & ab \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix}=\lambda a^2b^2c^2$, then $\lambda=$

(a) 1 (b) 2

(c) 3 (d) 4

Sol : SQP 2020

Let $$A=\begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix}$$

Taking a,b and c common from R_1, R_2 and R_3

$$A=abc\begin{vmatrix} -a & b & c \\ a & -b & c \\ a & b & -c \end{vmatrix}$$

Taking a,b and c common from C_1, C_2 and C_3

$$A=(abc)(abc)\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$
$$=a^2b^2c^2[-1(1-1)-1(-1-1)+1(1+1)]$$
$$=4a^2b^2c^2$$
$$\lambda=4$$

Thus option (d) is correct.

15. If $a\neq b\neq c$, then value of x which satisfies the equation $\begin{vmatrix} 0 & x-a & x-b \\ x+a & 0 & x-c \\ x+b & x+c & 0 \end{vmatrix}=0$, is

(a) $x=a$ (b) $x=b$

(c) $x=c$ (d) $x=0$

Sol : Comp 2017, OD 2014

Substituting $x=0$ we have,

$$\begin{vmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{vmatrix}=abc-abc=0$$
$$x=0$$

Thus option (d) is correct.

16. The values of x in the following determinants are

$$\begin{vmatrix} a+x & a-x & a-x \\ a-x & a+x & a-x \\ a-x & a-x & a+x \end{vmatrix}=0$$

(a) $x=0,\ x=4a$ (b) $x=0,\ x=a$

(c) $x=0,\ x=2a$ (d) $x=0,\ x=3a$

Sol : Comp 2015

From given option putting the value of every option $x=0, 3a$

Thus option (d) is correct.

Sol : Foreign 2018, Delhi 2014

We have $\text{adj}\,A = \begin{bmatrix} 4y & -x \\ -x^2 & 1 \end{bmatrix}$

$$\text{adj}(A) + B = \begin{bmatrix} 4y & -x \\ -x^2 & 1 \end{bmatrix} + \begin{bmatrix} -3 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4y-3 & -x+1 \\ -x^2+1 & 1+0 \end{bmatrix}$$

$$4y - 3 = 1 \Rightarrow y = 1$$

and $-x+1 = 0 \Rightarrow x = 1$

Thus option (a) is correct.

23. If $A = \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}$, then A^{-1} is equal to

(a) $-\frac{1}{13}\begin{bmatrix} 1 & -3 \\ -5 & 2 \end{bmatrix}$ (b) $\frac{1}{13}\begin{bmatrix} 1 & -3 \\ -5 & 2 \end{bmatrix}$

(c) $-\begin{bmatrix} 1 & -3 \\ -5 & 2 \end{bmatrix}$ (d) None of these

Sol : OD 2007

We have $A = \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}$

$$|A| = 2 \times 1 - 3 \times 5 = -13$$

$$\text{adj}\,A = \begin{bmatrix} 1 & -3 \\ -5 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{adj}\,A$$

$$= -\frac{1}{13}\begin{bmatrix} 1 & -3 \\ -5 & 2 \end{bmatrix}$$

Thus option (a) is correct.

24. Inverse of the matrix $\begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}$ is

(a) $\begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}$ (b) $\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$

(c) $\begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}$ (d) $\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix}$

Sol : Delhi 2010

Here, cofactors are

$$C_{11} = \cos 2\theta,$$

$$C_{12} = -\sin 2\theta$$

$$C_{21} = \sin 2\theta$$

and $C_{22} = \cos 2\theta$

$$|A| = |\cos^2 2\theta + \sin^2 2\theta| = 1$$

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}^T$$

$$= \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix}$$

Thus option (d) is correct.

25. The value of k such that the lines $2x - 3y + k = 0$, $3x - 4y - 13 = 0$ and $8x - 11y - 33 = 0$ are concurrent, is

(a) 20 (b) -7

(c) 7 (d) -20

Sol : SQP 2014

Since given lines are concurrent, we have

$$\begin{bmatrix} 2 & -3 & k \\ 3 & -4 & -13 \\ 8 & -11 & -33 \end{bmatrix} = 0$$

$$2(132 - 143) + 3(-99 + 104) + k(-33 + 32) = 0$$

$$-22 + 15 - k = 0$$

$$k = -7$$

Thus option (b) is correct.

26. The existence of the unique solution of the system of equations $x + y + z = \beta$; $5x - y + \alpha z = 10$ and $2x + 3y = 6$ depends on

(a) α only (b) β only

(c) Both α and β (d) Neither β nor α

Sol : OD 2018, SQP 2016

We have $x + y + z = \beta$

$$5x - y + az = 10$$

and $2x + 3y - z = 6$

For unique solution, $\begin{vmatrix} 1 & 1 & 1 \\ 5 & -1 & \alpha \\ 2 & 3 & -1 \end{vmatrix} \neq 0$

$$1(1 - 3\alpha) - 1(-5 - 2\alpha) + 1(15 + 2) \neq 0$$

$$1 - 3\alpha + 5 + 2\alpha + 17 = 0$$

$$-\alpha + 23 \neq 0,$$

$$\alpha \neq 23$$

Hence, for the existence of the unique solution, system of equations depends on α only.

Thus option (a) is correct.

27. If $\begin{vmatrix} 6i & -3i & 1 \\ 4 & 3i & -1 \\ 20 & 3 & i \end{vmatrix} = x + iy$, then $(x, y) =$

(a) (2, 3) (b) (0, 3)

(c) (3, 1) (d) (0, 0)

Sol : OD 2018

We have $\begin{vmatrix} x & 5 \\ 5 & x \end{vmatrix} = 0$

Solving determinant we have

$$x^2 - 25 = 0$$

$$x = \pm\sqrt{25}$$

$$= \pm 5$$

Thus option (a) is correct.

34. $\begin{vmatrix} 10 & 2 \\ 35 & 7 \end{vmatrix} =$

(a) 4 (b) 0

(c) 3 (d) 6

Sol : Foreign 2013, OD 2011

We have $\Delta = \begin{vmatrix} 10 & 2 \\ 35 & 7 \end{vmatrix}$

$$= 7 \times 10 - 2 \times 35$$

$$= 70 - 70$$

$$= 0$$

Thus option (b) is correct.

35. The matrix $\begin{bmatrix} 3 & 5 \\ 2 & k \end{bmatrix}$ has no inverse if the value of k is

(a) 0 (b) 5

(c) $\frac{10}{3}$ (d) $\frac{4}{9}$

Sol : Delhi 2006

We have $A = \begin{bmatrix} 3 & 5 \\ 2 & k \end{bmatrix}$

$$|A| = 3k - 10$$

$$A^{-1} = \frac{\text{adj}\,A}{|A|}$$

A^{-1} will not exist if $|A| = 0$ therefore we have

$$3k - 10 = 0$$

$$k = \frac{10}{3}$$

Hence, inverse will not exist if $k = \frac{10}{3}$.
Thus option (c) is correct.

36. $\begin{vmatrix} 2 & 3 & 5 \\ 0 & 4 & 7 \\ 0 & 0 & 5 \end{vmatrix} =$

(a) 40 (b) 0

(c) 3 (d) 25

Sol : Delhi 2008

We have $\Delta = \begin{vmatrix} 2 & 3 & 5 \\ 0 & 4 & 7 \\ 0 & 0 & 5 \end{vmatrix}$

Solving along C_1 we get

$$\Delta = 2(4 \times 5 - 7 \times 0)$$

$$= 2(20 - 0)$$

$$= 40$$

Thus option (a) is correct.

37. $\begin{vmatrix} 3 & 4 & 5 \\ 0 & 2 & 3 \\ 0 & 0 & 7 \end{vmatrix} =$

(a) 40 (b) 50

(c) 42 (d) 15

Sol : Foreign 2014, OD 2008

$$\Delta = \begin{vmatrix} 3 & 4 & 5 \\ 0 & 2 & 3 \\ 0 & 0 & 7 \end{vmatrix}$$

Solving along C_1 we get

$$\Delta = 3\begin{vmatrix} 2 & 3 \\ 0 & 7 \end{vmatrix}$$

$$= 3 \times 14$$

$$= 42$$

Thus option (c) is correct.

38. The value of the determinant $\begin{vmatrix} 3 & 1 & 7 \\ 5 & 0 & 2 \\ 2 & 5 & 3 \end{vmatrix} =$

(a) 124 (b) 125

(c) 134 (d) 144

Sol : SQP 2019

Let, $\Delta = \begin{vmatrix} 3 & 1 & 7 \\ 5 & 0 & 2 \\ 2 & 5 & 3 \end{vmatrix}$

$$= 3\begin{vmatrix} 0 & 2 \\ 5 & 3 \end{vmatrix} - 1\begin{vmatrix} 5 & 2 \\ 2 & 3 \end{vmatrix} + 7\begin{vmatrix} 5 & 0 \\ 2 & 5 \end{vmatrix}$$

$$= 3(0 - 10) - 1(15 - 4) + 7(25 - 0)$$

$$= -30 - 11 + 175$$

$$= 134$$

Thus option (c) is correct.

39. If $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, then adj A is

Sol : SQP 2019

Let $X = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$$|X| = 1 \neq 0$$

Hence X is invertible

$$X^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

Given $XA = I$

$$A = X^{-1}I$$

Hence $A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

Thus option (b) is correct.

46. The matrix $\begin{bmatrix} 2 & 10 & 3 \\ -2 & -4 & 6 \\ -1 & -2 & b \end{bmatrix}$ is a singular matrix, if b is

(a) -5 (b) 5

(c) 3 (d) For any value of b

Sol : Comp 2013, OD 2011

Matrix is singular if $|A| = 0$

$$\begin{vmatrix} 5 & 10 & 3 \\ -2 & -4 & 6 \\ -1 & -2 & b \end{vmatrix} = 0$$

$$0b + 0 = 0 \text{ any value of } b$$

Thus option (d) is correct.

47. If $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ then $|A + A^T| =$

(a) 4 (b) 8

(c) 16 (d) 64

Sol : OD 2007

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$A + A^T = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$|A + A^T| = 4(16 - 0) = 64$$

Thus option (d) is correct.

48. Let A be a non-singular matrix of the order 2×2 then $|A^{-1}| =$

(a) $|A|$ (b) $\frac{1}{|A|}$

(c) 0 (d) 1

Sol : Foreign 2014

$$|A^{-1}| = |A|^{-1}$$

$$= \frac{1}{|A|}$$

Note that only for determinant,

$$|A^{-1}| = |A|^{-1} \cdot A^{-1} \neq (A)^{-1}$$

Thus option (b) is correct.

49. If $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$; then adj $A =$

(a) $\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & -2 \\ -2 & -1 \end{bmatrix}$ (d) $\begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$

Sol : Delhi 2010

We have $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

$$\text{adj}\,A = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}^T$$

$$= \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$$

Thus option (a) is correct.

50. If $A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$, then $|2A| =$

(a) $2|A|$ (b) $4|A|$

(c) $8|A|$ (d) None of these

Sol : Comp 2010, Delhi 2007

We have $A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$

$$|A| = 2 - 8$$

$$= -6$$

Then, $2A = 2\begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 8 & 4 \end{bmatrix}$

$$|2A| = 8 - 32 = -24 = 4 \times -6$$

$$= 4|A|$$

Thus option (b) is correct.

VERY SHORT ANSWER QUESTIONS

55. Find $|AB|$, if $A = \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 5 \\ 0 & 0 \end{bmatrix}$.

Sol : OD 2019

We have $A = \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 5 \\ 0 & 0 \end{bmatrix}$

$$\text{Now,} \quad AB = \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0+0 & 0+0 \\ 0+0 & 0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$|AB| = 0$$

56. If $A = \begin{bmatrix} 5 & 6 & -3 \\ -4 & 3 & 2 \\ -4 & -7 & 3 \end{bmatrix}$, then write the cofactor of the element a_{21} of its 2nd row.

Sol : Foreign 2015, OD 2013

$$\text{We have} \quad A = \begin{bmatrix} 5 & 6 & -2 \\ -4 & 3 & 3 \\ -4 & -7 & 3 \end{bmatrix}$$

$$\text{Cofactor of } a_{21} \quad A_{21} = (-1)\begin{vmatrix} 6 & -3 \\ -7 & 3 \end{vmatrix}$$

$$= -(18-21) = 3$$

57. If $A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix}$, write the value of $|AB|$.

Sol : Comp 2015

$$|A| = \begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} = -1-6 = -7$$

$$\text{and} \quad |B| = \begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix} = 1+3 = 4$$

$$|AB| = |A|\cdot|B| = (-7)(4) = -28$$

58. If $\begin{vmatrix} 3x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & -2 \\ 7 & 3 \end{vmatrix}$, then write the value of x.

Sol : Delhi 2014

$$\text{We have} \quad \begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & -2 \\ 7 & 3 \end{vmatrix}$$

$$2x^2 - 40 = 18 - (-14)$$

$$2x^2 - 40 = 32$$

$$2x^2 = 72$$

$$x^2 = 36$$

$$x = \pm 6$$

59. If $\begin{vmatrix} 3x & 7 \\ -2 & 4 \end{vmatrix} = \begin{vmatrix} 8 & 7 \\ 6 & 4 \end{vmatrix}$, then find the value of x.

Sol : OD 2014

$$\text{We have} \quad \begin{vmatrix} 3x & 7 \\ -2 & 4 \end{vmatrix} = \begin{vmatrix} 8 & 7 \\ 6 & 4 \end{vmatrix}$$

$$12x - (-14) = 32 - 42$$

$$12x + 14 = -10$$

$$12x = -10 - 14 = -24$$

$$x = -\frac{24}{12} = -2$$

60. Write the value of the determinant $\begin{vmatrix} p & p+1 \\ p-1 & p \end{vmatrix}$.

Sol : Comp Delhi 2014

$$\text{We have} \quad \Delta = \begin{vmatrix} p & p+1 \\ p-1 & p \end{vmatrix}$$

Expanding, we get

$$\Delta = p^2 - (p-1)(p+1)$$

$$= p^2 - (p^2 - 1^2)$$

$$= p^2 - p^2 + 1$$

$$= 1$$

61. If $\begin{vmatrix} 2x & x+3 \\ 2(x+1) & x+1 \end{vmatrix} = \begin{vmatrix} 1 & 5 \\ 3 & 3 \end{vmatrix}$, then find the value of x.

Sol : Comp 2013

$$\text{We have} \quad \begin{vmatrix} 2x & x+3 \\ 2(x+1) & x+1 \end{vmatrix} = \begin{vmatrix} 1 & 5 \\ 3 & 3 \end{vmatrix}$$

Expanding we have

$$2x(x+1) - (x+3)(2x+2) = 3 - 15$$

$$2x^2 + 2x - (2x^2 + 8x + 6) = -12$$

$$-6x - 6 = -12$$

$$-6x = -6$$

$$x = 1$$

62. If $\begin{vmatrix} x+1 & x-1 \\ x-3 & x+2 \end{vmatrix} = \begin{vmatrix} 4 & -1 \\ 1 & 3 \end{vmatrix}$, then write the value of x.

Sol : Delhi 2013

$$\text{We have} \quad \begin{vmatrix} x+1 & x-1 \\ x-3 & x+2 \end{vmatrix} = \begin{vmatrix} 4 & -1 \\ 1 & 3 \end{vmatrix}$$

Expanding we have

$$(x+1)(x+2) - (x-1)(x-3) = 12 + 1$$

$$x^2 + 3x + 2 - (x^2 - 4x + 3) = 13$$

$$2x^2 - 15 = 17$$
$$2x^2 = 32$$
$$x^2 = 16 \Rightarrow x = \pm 4$$

Hence, for $x = 4$, given pair of determinants is equal.

72. If A is a square matrix satisfying $A'A = I$, write the value of $|A|$.

Sol : Delhi 2019, Foreign 2014

We have, $A'A = I$

$$|A'A| = |I|$$
$$|A'||A| = 1 \qquad |AB| = |A||B|$$
$$|A|^2 = 1 \qquad |A'| = |A|$$
$$|A| = \pm 1$$

73. If A and B are square matrices of the same order 3, such that $|A| = 2$ and $AB = 2I$. Write the values of $|B|$.

Sol : Delhi 2019

We know that,

(i) $|kA| = k^n|A|$, if A is square matrix of nth order

(ii) $|AB| = |A| \times |B|$

Here, we have $AB = 2I$ and $n = 3$

$$|AB| = |2I|$$
$$= 2^3|I| = 8 \cdot 1 = 8$$
$$|A||B| = 8$$
$$2 \cdot |B| = 8 \Rightarrow |B| = 4$$

74. Let A be the square matrix of order 3×3. Write the value of $|2A|$, where $|A| = 4$.

Sol : OD 2012

For a square matrix A of order n,

$$|kA| = k^n \cdot |A|$$

Here, $|2A| = 2^3 \cdot |A|$ order of A is 3×3

$$= 2^3 \times 4 \qquad |A| = 4$$
$$= 8 \times 4 = 32$$

75. If the determinant of matrix A of order 3×3 is of value 4, then write the values of $|3A|$.

Sol : Comp 2012

For a square matrix A of order n,

$$|kA| = k^n \cdot |A|$$

Here, $|3A| = 3^3 \cdot |A|$ order of A is 3×3

$$= 3^3 \times 4 \qquad |A| = 4$$
$$= 27 \times 4 = 108$$

76. If A is a square matrix of order 3 and $|3A| = k|A|$, then write the value of k.

Sol : Delhi 2010

If A is a square matrix of order n, then

$$|pA| = p^n|A|.$$

Here, the matrix A is of order 3×3.

$$|3A| = (3)^3|A| = 27|A|$$

Comparing with given equation, we get

$$k = 27$$

77. If for any 2×2 square matrix A, $A(\text{adj}\,A) = \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$, then write the value of $|A|$.

Sol : OD 2017, Delhi 2010

We have $A(\text{adj}\,A) = \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$

$$|A(\text{adj}\,A)| = \begin{vmatrix} 8 & 0 \\ 0 & 8 \end{vmatrix}$$
$$|A||\text{adj}\,A| = 64 - 0$$
$$|A||A|^{2-1} = 64 \qquad |\text{adj}\,A| = |A|^{n-1}$$
$$|A|^2 = 64$$
$$|A| = \pm 8$$

78. Find $|\text{adj}\,A|$, if $A = \begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}$.

Sol : Comp 2014

We have $A = \begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}$

$$|A| = \begin{vmatrix} 5 & 2 \\ 7 & 3 \end{vmatrix}$$
$$= 15 - 14 = 1$$

If A is a non-singular matrix of order n, then

$$|\text{adj}\,A| = |A|^{n-1},$$
$$= |A|^{2-1}$$
$$|\text{adj}\,A| = (1)^{2-1} = 1$$

79. If A is a square matrix of order 3 such that $|\text{adj}\,A| = 64$, then find $|A|$.

Sol : Comp 2013

For a square matrix of order n,

$$|\text{adj}\,A| = |A|^{n-1}$$

86. Find the maximum value of $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+\sin\theta & 1 \\ 1 & 1 & 1+\cos\theta \end{vmatrix}$

Sol : Delhi 2016

We have $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+\sin\theta & 1 \\ 1 & 1 & 1+\cos\theta \end{vmatrix}$

Expanding along first row i.e. R_1, we get

$$\Delta = 1\begin{vmatrix} 1+\sin\theta & 1 \\ 1 & 1+\cos\theta \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & 1+\cos\theta \end{vmatrix} + 1\begin{vmatrix} 1 & 1+\sin\theta \\ 1 & 1 \end{vmatrix}$$

$$= [(1+\sin\theta)(1+\cos\theta)-1] - [1+\cos\theta-1] + [1-1-\sin\theta]$$

$$= 1+\cos\theta+\sin\theta+\sin\theta\cos\theta-1-\cos\theta-\sin\theta$$

$$= \sin\theta\cos\theta$$

$$= \left|\frac{1}{2}\right|(2\sin\theta\cos\theta) = \frac{1}{2}\sin 2\theta$$

Since maximum value of $\sin 2\theta$ is 1.

$$\Delta_{\max} = \frac{1}{2}\times 1 = \frac{1}{2}.$$

87. If $\begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix} = 8$, write the value of x.

Sol : Foreign 2015, SQP 2012

We have, $\begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix} = 8$

Expanding along R_1, we get

$$x(-x^2-1) - \sin\theta(-x\sin\theta-\cos\theta) + \cos\theta(-\sin\theta+x\cos\theta) = 8$$

$$-x^3-x+x\sin^2\theta+\sin\theta\cos\theta-\sin\theta\cos\theta + x\cos^2\theta = 8$$

$$-x^3-x+x(\sin^2\theta+\cos^2\theta) = 8$$

$$-x^3-x+x = 8$$

$$-x^3 = 8$$

$$x^3+8 = 0$$

$$x^3+2^3 = 0$$

$$(x+2)(x^2-2x+4) = 0 \Rightarrow x = -2$$

Here $x^2-2x+4=0$, gives imaginary values.

88. In the interval $\frac{\pi}{2} < x < \pi$, find the value of x for which the matrix $\begin{bmatrix} 2\sin x & 3 \\ 1 & 2\sin x \end{bmatrix}$ is singular.

Sol : Comp 2015

We have $A = \begin{bmatrix} 2\sin x & 3 \\ 1 & 2\sin x \end{bmatrix}$

Since A is a singular matrix, we have

$$|A| = 0$$

$$\begin{vmatrix} 2\sin x & 3 \\ 1 & 2\sin x \end{vmatrix} = 0$$

$$4\sin^2 x - 3 = 0$$

$$\sin^2 x = \frac{3}{4}$$

Taking positive square root because $\frac{\pi}{2} < x < \pi$,

$$\sin x = \frac{\sqrt{3}}{2}$$

$$x = \frac{2\pi}{3}$$

89. For what value of x, $A = \begin{bmatrix} 2(x+1) & 2x \\ x & x-2 \end{bmatrix}$ is a singular matrix?

Sol : Comp 2011

We have $A = \begin{bmatrix} 2(x+1) & 2x \\ x & x-2 \end{bmatrix}$

Matrix A is said to be singular, if $|A| = 0$.

Thus $\begin{vmatrix} 2x+2 & 2x \\ x & x-2 \end{vmatrix} = 0$

$$(2x+2)(x-2)-2x^2 = 0$$

$$2x^2-2x-4-2x^2 = 0$$

$$-2x-4 = 0$$

$$x = -2$$

90. For what value of x, then matrix $\begin{bmatrix} 2x+4 & 4 \\ x+5 & 3 \end{bmatrix}$ is a singular matrix?

Sol : Comp 2011

We have $A = \begin{bmatrix} 2x+4 & 4 \\ x+5 & 3 \end{bmatrix}$

If matrix A is singular, then

$$|A| = 0$$

$$\begin{vmatrix} 2x+4 & 4 \\ x+5 & 3 \end{vmatrix} = 0$$

$$(2x+4)\times 3 - (x+5)\times 4 = 0$$

$$6x+12-4x-20 = 0$$

$$2x-8 = 0$$

$$x = 4$$

$$A^{-1} = \frac{1}{|A|}\text{adj}\, A$$
$$= \frac{-1}{19}\begin{bmatrix} -2 & -3 \\ -5 & 2 \end{bmatrix}$$
$$= \frac{1}{19}\begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$$
$$= \frac{1}{19}A$$

96. Compute A^{-1} and show that $2A^{-1} = 9I - A$.if $A = \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$

Sol : OD 2018, Delhi 2011

We have, $A = \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$

Here, $|A| = \begin{vmatrix} 2 & -3 \\ -4 & 7 \end{vmatrix}$
$$= 14 - 12 = 2$$

Since $|A| \neq 0$, therefore A^{-1} exists.

Now, $\text{adj}\, A = \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$

$$A^{-1} = \frac{1}{|A|}\text{adj}(A)$$
$$= \frac{1}{2}\begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix} \quad \text{...(i)}$$

Now $2A^{-1} = 9I - A$

$$9I - A = 9\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$$
$$= \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix} = \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$$
$$= 2A^{-1}$$

Hence proved.

97. If $A = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$ be such that $A^{-1} = kA$, then find the value of k.

Sol : Comp 2018

We have, $A = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$

$$|A| = \begin{vmatrix} 2 & 3 \\ 5 & -2 \end{vmatrix}$$
$$= -4 - 15 = -19$$

Since $|A| \neq 0$, matrix A is non-singular and A^{-1} exists.

Now $\text{adj}\, A = \begin{bmatrix} -2 & -3 \\ -5 & 2 \end{bmatrix}$

$$A^{-1} = \frac{1}{|A|}\text{adj}\, A$$
$$= \frac{-1}{19}\begin{bmatrix} -2 & -3 \\ -5 & 2 \end{bmatrix}$$
$$= \frac{1}{19}\begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$$
$$A^{-1} = \frac{1}{19}A$$

Comparing with $A^{-1} = kA$ (given), we get

$$k = \frac{1}{19}$$

LONG ANSWER QUESTIONS

98. If $A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 0 & -3 \\ 1 & 2 & 0 \end{bmatrix}$, then find A^{-1} and hence solve the following system of equations :

$$x + 2y - 3z = 2$$
$$x - 3z = 2$$
$$x + 2y = 3$$

Sol : OD 2024

We have, $A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 0 & -3 \\ 1 & 2 & 0 \end{bmatrix}$

Here, $|A| = 1(0+6) - 2(0+3) - 3(4-0)$
$$= 1(6) - 2(3) - 3(4)$$
$$= 6 - 6 - 12 = -12$$

Since $|A| \neq 0$, matrix A is non-singular and A^{-1} exists.

Now, cofactors of elements of $|A|$ are

$$A_{11} = (-1)^2\begin{vmatrix} 0 & -3 \\ 2 & 0 \end{vmatrix} = 1(0+6) = 6$$
$$A_{12} = (-1)^3\begin{vmatrix} 2 & -3 \\ 1 & 0 \end{vmatrix} = -1(0+3) = 3$$
$$A_{13} = (-1)^4\begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix} = 1(4-0) = 4$$
$$A_{21} = (-1)^3\begin{vmatrix} 2 & -3 \\ 2 & 0 \end{vmatrix} = -1(0+6) = -6$$
$$A_{22} = (-1)^4\begin{vmatrix} 1 & -3 \\ 1 & 0 \end{vmatrix} = 1(0+3) = 3$$
$$A_{23} = (-1)^5\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = -1(2-2) = 0$$

$$= \frac{1}{67}\begin{bmatrix} 201 \\ -134 \\ 67 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

Comparing corresponding elements, we get

$$x = 3\ ,\ y = -2 \text{ and } z = 1$$

100. If $A = \begin{bmatrix} -3 & -2 & -4 \\ 2 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 0 \\ -2 & -1 & -2 \\ 0 & -1 & 1 \end{bmatrix}$ then find AB and use it to solve the following system of equations :

$$x - 2y = 3$$

$$2x - y - z = 2$$

$$-2y + z = 3$$

Sol : OD 2023

We have
$$x - 2y = 3$$

$$2x - y - z = 2$$

$$-2y + z = 3$$

In matrix form, it can be written as

$$PX = C \qquad ...(1)$$

where, $P = \begin{bmatrix} 1 & -2 & 0 \\ 2 & -1 & -1 \\ 0 & -2 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $C = \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}$

Its solution can be given as

$$X = P^{-1}C \qquad (2)$$

We have $A = \begin{bmatrix} -3 & -2 & -4 \\ 2 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 0 \\ -2 & -1 & -2 \\ 0 & -1 & 1 \end{bmatrix}$

Now,
$$AB = \begin{bmatrix} -3 & -2 & -4 \\ 2 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ -2 & -1 & -2 \\ 0 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -3+4+0 & -6+2+4 & 0+4-4 \\ 2-2+0 & 4-1-2 & 0-2+2 \\ 2-2+0 & 4-1-3 & 0-2+3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$AB = I$$

$$(AB)B^{-1} = I \cdot B^{-1}$$

$$ABB^{-1} = B^{-1}$$

$$A = B^{-1}$$

Thus
$$B^{-1} = \begin{bmatrix} -3 & -2 & -4 \\ 2 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix}$$

Now
$$P = B^T$$

Here
$$P^{-1} = (B^T)^{-1} = (B^{-1})^T$$

$$= \begin{bmatrix} -3 & 2 & 2 \\ -2 & 1 & 1 \\ -4 & 2 & 3 \end{bmatrix}$$

Now substituting C and P^{-1} in (2) we have

$$X = \begin{bmatrix} -3 & 2 & 2 \\ -2 & 1 & 1 \\ -4 & 2 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -9+4+6 \\ -6+2+3 \\ -12+4+9 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Comparing corresponding elements, we get

$$x = 1,\ y = -1 \text{ and } z = 1$$

101. If $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 4 \\ -2 & 2 & 1 \end{bmatrix}$, then find $(A')^{-1}$.

Sol : Delhi 2015, Foreign 2010

We have,
$$A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 4 \\ -2 & 2 & 1 \end{bmatrix}$$

Now,
$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -1 & 4 \\ -2 & 2 & 1 \end{vmatrix}$$

Expanding along R_1 we have

$$|A| = 1(-1-8) + 2(0+8) + 3(0-2)$$

$$= -9 + 16 - 6$$

$$= 1$$

Since $|A| \neq 0$, matrix A is non-singular and A^{-1} exists.

Cofactors of an element of $|A|$ are given by

$$A_{11} = (-1)^{1+1}\begin{vmatrix} -1 & 4 \\ 2 & 1 \end{vmatrix} = (-1-8) = -9$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} 0 & 4 \\ -2 & 1 \end{vmatrix} = -(0+8) = -8$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} 0 & -1 \\ -2 & 2 \end{vmatrix} = (0-2) = -2$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix} -2 & -1 \\ 2 & 2 \end{vmatrix} = -(-2-6) = 8$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 3 \\ -2 & 1 \end{vmatrix} = (1+6) = 7$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 3 \\ -2 & 1 \end{vmatrix} = -(2-4) = 2$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} -2 & 3 \\ -1 & 4 \end{vmatrix} = (-8+3) = -5$$

$$A^2 = \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix} \quad ...(i)$$

Now $$4A - 3I = 4\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 8 & -4 \\ -4 & 8 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix} \quad ...(ii)$$

From eqs. (i) and (ii), we get

$$A^2 = 4A - 3I \quad \text{Hence proved...(iii)}$$

Here, $$|A| = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4 - 1 = 3 \neq 0$$

Since $|A| \neq 0$, matrix A is non-singular and A^{-1} exists.

Now, pre-multiplying both sides of Eq. (iii) by A^{-1}, we get

$$A^{-1} \cdot A^2 = A^{-1} \cdot (4A - 3I)$$
$$(A^{-1} \cdot A) \cdot A = 4A^{-1} \cdot A - 3A^{-1} \cdot I$$
$$IA = 4I - 3A^{-1}$$
$$A = 4I - 3A^{-1}$$
$$3A^{-1} = 4I - A$$
$$3A^{-1} = 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$
$$A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

104. If $A = \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$, verify that

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Sol : Comp 2015, Delhi 2010

We have, $$A = \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$$
$$AB = \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} 2-3 & -4+9 \\ 1+4 & -2-12 \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 5 & -14 \end{bmatrix}$$

Now, $$|A| = \begin{vmatrix} 2 & 3 \\ 1 & -4 \end{vmatrix} = -8 - 3 = -11 \neq 0$$

$$|B| = \begin{vmatrix} 1 & -2 \\ -1 & 3 \end{vmatrix} = 3 - 2 = 1 \neq 0$$

and $$|AB| = \begin{vmatrix} -1 & 5 \\ 5 & -14 \end{vmatrix} = 14 - 25 = -11 \neq 0$$

Thus, A, B and AB are non-singular matrices, so their inverse exists.

Now, $$\operatorname{adj} A = \begin{bmatrix} -4 & -3 \\ -1 & 2 \end{bmatrix}$$
$$\operatorname{adj} B = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$$

and $$\operatorname{adj}(AB) = \begin{bmatrix} -14 & -5 \\ -5 & -1 \end{bmatrix}$$
$$(AB)^{-1} = \frac{1}{|AB|}\operatorname{adj}(AB)$$
$$= \frac{1}{-11}\begin{bmatrix} -14 & -5 \\ -5 & -1 \end{bmatrix}$$
$$(AB)^{-1} = \frac{1}{11}\begin{bmatrix} 14 & 5 \\ 5 & 1 \end{bmatrix} \quad ...(i)$$
$$A^{-1} = \frac{1}{|A|}\operatorname{adj}(A)$$
$$= \frac{1}{-11}\begin{bmatrix} -4 & -3 \\ -1 & 2 \end{bmatrix}$$
$$A^{-1} = \frac{1}{11}\begin{bmatrix} 4 & 3 \\ 1 & -2 \end{bmatrix}$$

and $$B^{-1} = \frac{1}{|B|}\operatorname{adj}(B)$$
$$= \frac{1}{1}\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$$

Now, $$B^{-1}A^{-1} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\frac{1}{11}\begin{bmatrix} 4 & 3 \\ 1 & -2 \end{bmatrix}$$
$$= \frac{1}{11}\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 1 & -2 \end{bmatrix}$$
$$= \frac{1}{11}\begin{bmatrix} 12+2 & 9-4 \\ 4+1 & 3-2 \end{bmatrix}$$
$$= \frac{1}{11}\begin{bmatrix} 14 & 5 \\ 5 & 1 \end{bmatrix}$$
$$= (AB)^{-1} \quad \text{[from Eq. (i)]}$$

Hence, $(AB)^{-1} = B^{-1}A^{-1}$.

105. Show that for the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$, $A^3 - 6A^2 + 5A + 11I = O$. Hence, find A^{-1}.

$$= \begin{bmatrix} -1 & 8 & -3 \\ 1 & -19 & 14 \\ 2 & 6 & -5 \end{bmatrix}^T$$

$$= \begin{bmatrix} -1 & 1 & 2 \\ 8 & -19 & 6 \\ -3 & 14 & -5 \end{bmatrix}$$

and $\quad A^{-1} = \dfrac{1}{|A|}\text{adj}\,A$

$$= \frac{1}{11}\begin{bmatrix} -1 & 1 & 2 \\ 8 & -19 & 6 \\ -3 & 14 & -5 \end{bmatrix}$$

We have $\quad x + 3y + 4z = 8$

$$2x + y + 2z = 5$$

and $\quad 5x + y + z = 7$

In matrix form, it can be written as

$$AX = B \qquad ...(i)$$

where, $\quad A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 1 & 2 \\ 5 & 1 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 8 \\ 5 \\ 7 \end{bmatrix}$

System has a unique solution given by

$$X = A^{-1}B.$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{11}\begin{bmatrix} -1 & 1 & 2 \\ 8 & -19 & 6 \\ -3 & 14 & -5 \end{bmatrix}\begin{bmatrix} 8 \\ 5 \\ 7 \end{bmatrix}$$

$$= \frac{1}{11}\begin{bmatrix} -8+5+14 \\ 64-95+42 \\ -24+70-35 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 11 \\ 11 \\ 11 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Comparing corresponding elements, we get $x = 1$, $y = 1$ and $z = 1$.

107. If $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 3 & 1 & 1 \end{bmatrix}$, find A^{-1}.

Hence, solve the system of equations $x + y + z = 6$, $x + 2z = 7$, $3x + y + z = 12$.

Sol : Delhi 2019

We have, $\quad A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 3 & 1 & 1 \end{bmatrix}$

Here, $\quad |A| = 1(0-2) - 1(1-6) + 1(1-0)$

$$= 1(-2) - 1(-5) + 1(1)$$

$$= -2 + 5 + 1 = 4$$

Since $|A| \neq 0$, matrix A is non-singular and A^{-1} exists.

Now, cofactors of elements of $|A|$ are

$$A_{11} = (-1)^2\begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} = 1(0-2) = -2$$

$$A_{12} = (-1)^3\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = -1(1-6) = 5$$

$$A_{13} = (-1)^4\begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix} = 1(1-0) = 1$$

$$A_{21} = (-1)^3\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = -1(1-1) = 0$$

$$A_{22} = (-1)^4\begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} = 1(1-3) = -2$$

$$A_{23} = (-1)^5\begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} = -1(1-3) = 2$$

$$A_{31} = (-1)^4\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 1(2-0) = 2$$

$$A_{32} = (-1)^5\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = -1(2-1) = -1$$

$$A_{33} = (-1)^6\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 1(0-1) = -1$$

$$\text{adj}\,A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$$

$$= \begin{bmatrix} -2 & 5 & 1 \\ 0 & -2 & 2 \\ 2 & 1 & -1 \end{bmatrix}^T$$

$$= \begin{bmatrix} -2 & 0 & 2 \\ 5 & -2 & 1 \\ 1 & 2 & -1 \end{bmatrix}$$

and $\quad A^{-1} = \dfrac{1}{|A|}\text{adj}\,A$

$$= \frac{1}{4}\begin{bmatrix} -2 & 0 & 2 \\ 5 & -2 & 1 \\ 1 & 2 & -1 \end{bmatrix}$$

We have $\quad x + y + z = 6$

$$x + 2z = 7$$

and $\quad 3x + y + z = 12$

In matrix form, it can be written as

$$AX = B \qquad ...(i)$$

where, $\quad A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 3 & 1 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 6 \\ 7 \\ 12 \end{bmatrix}$

System has a unique solution given by

$$X = A^{-1}B.$$

Now, $$|A| = \begin{vmatrix} 5 & 0 & 4 \\ 2 & 3 & 2 \\ 1 & 2 & 1 \end{vmatrix}$$
$$= 5(3-4) - 0 + 4(4-3)$$
$$= -5 + 4 = -1 \neq 0$$

Since $|A| \neq 0$, matrix A is non-singular and A^{-1} exists.

Now, cofactors of elements of $|A|$ arc

$$A_{11} = (-1)^2 \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} = (3-4) = -1$$
$$A_{12} = (-1)^3 \begin{vmatrix} 2 & 2 \\ 1 & 1 \end{vmatrix} = -(2-2) = 0$$
$$A_{13} = (-1)^4 \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = (4-3) = 1$$
$$A_{21} = (-1)^3 \begin{vmatrix} 0 & 4 \\ 2 & 1 \end{vmatrix} = -(0-8) = 8$$
$$A_{22} = (-1)^4 \begin{vmatrix} 5 & 4 \\ 1 & 1 \end{vmatrix} = (5-4) = 1$$
$$A_{23} = (-1)^5 \begin{vmatrix} 5 & 0 \\ 1 & 2 \end{vmatrix} = -(10-0) = -10$$
$$A_{31} = (-1)^4 \begin{vmatrix} 0 & 4 \\ 3 & 2 \end{vmatrix} = (0-12) = -12$$
$$A_{32} = (-1)^5 \begin{vmatrix} 5 & 4 \\ 2 & 2 \end{vmatrix} = -(10-8) = -2$$
$$A_{33} = (-1)^6 \begin{vmatrix} 5 & 0 \\ 2 & 3 \end{vmatrix} = (15-0) = 15$$

$$\text{adj}\, A = \begin{bmatrix} -1 & 8 & -12 \\ 0 & 1 & -2 \\ 1 & -10 & 15 \end{bmatrix}$$

and $$A^{-1} = \frac{1}{|A|}\text{adj}(A)$$
$$= \frac{1}{-1}\begin{bmatrix} -1 & 8 & -12 \\ 0 & 1 & -2 \\ 1 & -10 & 15 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & -8 & 12 \\ 0 & -1 & 2 \\ -1 & 10 & -15 \end{bmatrix}$$

Now $$(AB)^{-1} = B^{-1}A^{-1}$$
$$= \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & -8 & 12 \\ 0 & -1 & 2 \\ -1 & 10 & -15 \end{bmatrix}$$
$$= \begin{bmatrix} 1+0-3 & -8-3+30 & 12+6-45 \\ 1+0-3 & -8-4+30 & 12+8-45 \\ 1+0-4 & -8-3+40 & 12+6-60 \end{bmatrix}$$
$$= \begin{bmatrix} -2 & 19 & -27 \\ -2 & 18 & -25 \\ -3 & 29 & -42 \end{bmatrix}$$

110. If $A = \begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix}$, find A^{-1}.

Using A^{-1} solve the system of equations

$$\frac{2}{x} + \frac{3}{y} + \frac{10}{z} = 2,\ \frac{4}{x} - \frac{6}{y} + \frac{5}{z} = 5 \text{ and } \frac{6}{x} + \frac{9}{y} - \frac{20}{z} = -4$$

Sol : Delhi 2017, OD 2015

We have, $$A = \begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix}$$

Here, $$|A| = 2(120-45) - 3(-80-30) + 10(36+36)$$
$$= 150 + 330 + 720$$
$$= 1200$$

Since $|A| \neq 0$, matrix A is non-singular and A^{-1} exists.

Now, cofactors of elements of $|A|$ are

$$A_{11} = (-1)^2 \begin{vmatrix} -6 & 5 \\ 9 & -20 \end{vmatrix} = 1(120-45) = 75$$
$$A_{12} = (-1)^3 \begin{vmatrix} 4 & 5 \\ 6 & -20 \end{vmatrix} = -1(-80-30) = 110$$
$$A_{13} = (-1)^4 \begin{vmatrix} 4 & -6 \\ 6 & 9 \end{vmatrix} = 1(36+36) = 72$$
$$A_{21} = (-1)^3 \begin{vmatrix} 3 & 10 \\ 9 & -20 \end{vmatrix} = -1(-60-90) = 150$$
$$A_{22} = (-1)^4 \begin{vmatrix} 2 & 10 \\ 6 & -20 \end{vmatrix} = 1(-40-60) = -100$$
$$A_{23} = (-1)^5 \begin{vmatrix} 2 & 3 \\ 6 & 9 \end{vmatrix} = -1(18-18) = 0$$
$$A_{31} = (-1)^4 \begin{vmatrix} 3 & 10 \\ -6 & 5 \end{vmatrix} = 1(15+60) = 75$$
$$A_{32} = (-1)^5 \begin{vmatrix} 2 & 10 \\ 4 & 5 \end{vmatrix} = -1(10-40) = 30$$
$$A_{33} = (-1)^6 \begin{vmatrix} 2 & 3 \\ 4 & -6 \end{vmatrix} = 1(-12-12) = -24$$

$$\text{adj}\, A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$$
$$= \begin{bmatrix} 75 & 110 & 72 \\ 150 & -100 & 0 \\ 75 & 30 & -24 \end{bmatrix}^T$$
$$= \begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$$

and $$A^{-1} = \frac{1}{|A|}\text{adj}(A)$$

Sol : Delhi 2017, Foreign 2011

We have $x - y + 2z = 1$

$2y - 3z = 1$

and $3x - 2y + 4z = 2$

In matrix form, it can be written as

$$AX = B \quad ...(1)$$

where, $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

Its solution can be given as

$$X = A^{-1}B \quad (2)$$

Now let $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}$ and $C = \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$

$$\text{Now, } AC = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}\begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -2-9+12 & 0-2+2 & 1+3-4 \\ 0+18-18 & 0+4-3 & 0-6+6 \\ -6-18+24 & 0-4-4 & 3+6-8 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$AC = I$$

$$A^{-1}(AC) = A^{-1} \cdot I$$

$$A^{-1}AC = A^{-1}$$

$$C = A^{-1}$$

$$A^{-1} = C$$

$$= \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$$

Now substituting B and A^{-1} in (2) we have

$$X = \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2+0+2 \\ 9+2-6 \\ 6+1-4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 5 \\ 3 \end{bmatrix}$$

Comparing corresponding elements, we get

$$x = 0,\ y = 5 \text{ and } z = 3$$

113. If $A = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$, find $adj\ A$ and verify that $A(adj\ A) = (adj\ A)A = |A|I_3$.

Sol : Foreign 2016

We have, $A = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Cofactors of elements of $|A|$ are given by

$$A_{11} = \cos\alpha;\ A_{12} = -\sin\alpha;\ A_{13} = 0;$$

$$A_{21} = \sin\alpha;\ A_{22} = \cos\alpha;\ A_{23} = 0$$

$$A_{31} = 0;\ A_{32} = 0 \text{ and } A_{33} = 1$$

$$\text{adj}\,A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$$

$$= \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}^T$$

$$= \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A(\text{adj}\,A) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\alpha + \sin^2\alpha & 0 & 0 \\ 0 & \sin^2\alpha + \cos^2\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad ...(i)$$

$$(\text{adj}\,A) \cdot (A) = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\alpha + \sin^2\alpha & 0 & 0 \\ 0 & \sin^2\alpha + \cos^2\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad ...(ii)$$

and $|A| = \begin{vmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{vmatrix}$

Expanding along R_3 we have

$$|A| = 1 \cdot (\cos^2\alpha + \sin^2\alpha) = 1$$

$$3x + 4y - 5z = -5$$

and $$2x - y + 3z = 12$$

In matrix form, it can be written as

$$AX = B \quad \text{...(i)}$$

where, $A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 4 & -5 \\ 2 & -1 & 3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 7 \\ -5 \\ 12 \end{bmatrix}$

Here,
$$\begin{aligned} |A| &= 1(12-5) + 1(9+10) + 2(-3-8) \\ &= 1(7) + 1(19) + 2(-11) \\ &= 7 + 19 - 22 = 4 \end{aligned}$$

Since $|A| \neq 0$, matrix A is non-singular and A^{-1} exists.

Now, cofactors of elements of $|A|$ are

$$A_{11} = (-1)^2 \begin{vmatrix} 4 & -5 \\ -1 & 3 \end{vmatrix} = 1(12-5) = 7$$

$$A_{12} = (-1)^3 \begin{vmatrix} 3 & -5 \\ 2 & 3 \end{vmatrix} = -1(9+10) = -19$$

$$A_{13} = (-1)^4 \begin{vmatrix} 3 & 4 \\ 2 & -1 \end{vmatrix} = 1(-3-8) = -11$$

$$A_{21} = (-1)^3 \begin{vmatrix} -1 & 2 \\ -1 & 3 \end{vmatrix} = -1(-3+2) = 1$$

$$A_{22} = (-1)^4 \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = 1(3-4) = -1$$

$$A_{23} = (-1)^5 \begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} = -1(-1+2) = -1$$

$$A_{31} = (-1)^4 \begin{vmatrix} -1 & 2 \\ 4 & -5 \end{vmatrix} = 1(5-8) = -3$$

$$A_{32} = (-1)^5 \begin{vmatrix} 1 & 2 \\ 3 & -5 \end{vmatrix} = -1(-5-6) = 11$$

$$A_{33} = (-1)^6 \begin{vmatrix} 1 & -1 \\ 3 & 4 \end{vmatrix} = 1(4+3) = 7$$

$$\begin{aligned} \text{adj}\, A &= \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T \\ &= \begin{bmatrix} 7 & -19 & -11 \\ 1 & -1 & -1 \\ -3 & 11 & 7 \end{bmatrix}^T \\ &= \begin{bmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{bmatrix} \end{aligned}$$

and
$$\begin{aligned} A^{-1} &= \frac{1}{|A|} \text{adj}(A) \\ &= \frac{1}{4} \begin{bmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{bmatrix} \end{aligned}$$

Since A^{-1} exists, system has a unique solution given by

$$X = A^{-1}B.$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} == \frac{1}{4} \begin{bmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{bmatrix} \begin{bmatrix} 7 \\ -15 \\ 12 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 49 - 5 - 36 \\ -133 + 5 + 132 \\ -77 + 5 + 84 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 8 \\ 4 \\ 12 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

Comparing corresponding elements, we get $x = 2$, $y = 1$ and $z = 3$

116. Using matrices, solve the following system of linear equations.

$$x + y - z = 3$$

$$2x + 3y + z = 10$$

and $$3x - y - 7z = 1$$

Sol : OD 2012, Comp 2009

We have
$$x + y - z = 3$$

$$2x + 3y + z = 10$$

and $$3x - y - 7z = 1$$

In matrix form, it can be written as

$$AX = B \quad \text{...(i)}$$

where, $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & 1 \\ 3 & -1 & -7 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 3 \\ 10 \\ 1 \end{bmatrix}$

Here,
$$\begin{aligned} |A| &= 1(-21+1) - 1(-14-3) - 1(-2-9) \\ &= 1(-20) - 1(-17) - 1(-11) \\ &= -20 + 17 + 11 = 8 \end{aligned}$$

Since $|A| \neq 0$, matrix A is non-singular and A^{-1} exists.

Now, cofactors of elements of $|A|$ are

$$A_{11} = (-1)^2 \begin{vmatrix} 3 & 1 \\ -1 & -7 \end{vmatrix} = 1(-21+1) = -20$$

$$A_{12} = (-1)^3 \begin{vmatrix} 2 & 1 \\ 3 & -7 \end{vmatrix} = -1(-14-3) = 17$$

$$A_{13} = (-1)^4 \begin{vmatrix} 2 & 3 \\ 3 & -1 \end{vmatrix} = 1(-2-9) = -11$$

$$A_{21} = (-1)^3 \begin{vmatrix} 1 & -1 \\ -1 & -7 \end{vmatrix} = -1(-7-1) = 8$$

Since A^{-1} exists, system has a unique solution given by

$$X = A^{-1}B.$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{62}\begin{bmatrix} -3 & 26 & 19 \\ 9 & -16 & 5 \\ 5 & -2 & -11 \end{bmatrix}\begin{bmatrix} 4 \\ -3 \\ 8 \end{bmatrix}$$

$$= \frac{1}{62}\begin{bmatrix} -12-78+152 \\ 36+48+40 \\ 20+6-88 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 62 \\ 124 \\ -62 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Comparing corresponding elements, we get $x = 1$, $y = 2$ and $z = -1$

118. Using matrices, solve the following system of equations.

$$2x + 3y + 3z = 5,$$

$$x - 2y + z = -4$$

and $$3x - y - 2z = 3$$

Sol : OD 2012, Delhi 2008

We have $$2x + 3y + 3z = 5,$$

$$x - 2y + z = -4$$

and $$3x - y - 2z = 3$$

In matrix form, it can be written as

$$AX = B \quad \text{...(i)}$$

where, $A = \begin{bmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$

Here, $$|A| = 2(4+1) - 3(-2-3) + 3(-1+6)$$

$$= 2(5) - 3(-5) + 3(5)$$

$$= 10 + 15 + 15 = 40$$

Since $|A| \neq 0$, matrix A is non-singular and A^{-1} exists.

Now, cofactors of elements of $|A|$ are

$$A_{11} = (-1)^2\begin{vmatrix} -2 & 1 \\ -1 & -2 \end{vmatrix} = 1(4+1) = 5$$

$$A_{12} = (-1)^3\begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix} = -1(-2-3) = 5$$

$$A_{13} = (-1)^4\begin{vmatrix} 1 & -2 \\ 3 & -1 \end{vmatrix} = 1(-1+6) = 5$$

$$A_{21} = (-1)^3\begin{vmatrix} 3 & 3 \\ -1 & -2 \end{vmatrix} = -1(-6+3) = 3$$

$$A_{22} = (-1)^4\begin{vmatrix} 2 & 3 \\ 3 & -2 \end{vmatrix} = 1(-4-9) = -13$$

$$A_{23} = (-1)^5\begin{vmatrix} 2 & 3 \\ 3 & -1 \end{vmatrix} = -1(-2-9) = 11$$

$$A_{31} = (-1)^4\begin{vmatrix} 3 & 3 \\ -2 & 1 \end{vmatrix} = 1(3+6) = 9$$

$$A_{32} = (-1)^5\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = -1(2-3) = 1$$

$$A_{33} = (-1)^6\begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} = 1(-4-3) = -7$$

$$\text{adj}\,A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$$

$$= \begin{bmatrix} 5 & 5 & 5 \\ 3 & -13 & 11 \\ 9 & 1 & -7 \end{bmatrix}^T$$

$$= \begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}$$

and $$A^{-1} = \frac{1}{|A|}\text{adj}(A)$$

$$= \frac{1}{40}\begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}$$

Since A^{-1} exists, system has a unique solution given by

$$X = A^{-1}B.$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{40}\begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}\begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$$

$$= \frac{1}{40}\begin{bmatrix} 25-12+27 \\ 25+52+3 \\ 25-44-21 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{40}\begin{bmatrix} 40 \\ 80 \\ -40 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Comparing corresponding elements, we get $x = 1$, $y = 2$ and $z = -1$.

119. If $A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 1 \\ 1 & -3 & 1 \end{bmatrix}$, then find A^{-1} and hence solve the system of equations

$$x + 2y + z = 4$$

$$-x + y + z = 0$$

and $$x - 3y + z = 4.$$

$$A_{23} = (-1)^5\begin{vmatrix}1 & 2\\3 & -3\end{vmatrix} = -1(-3-6) = 9$$

$$A_{31} = (-1)^4\begin{vmatrix}2 & -3\\3 & 2\end{vmatrix} = 1(4+9) = 13$$

$$A_{32} = (-1)^5\begin{vmatrix}1 & -3\\2 & 2\end{vmatrix} = -1(2+6) = -8$$

$$A_{33} = (-1)^6\begin{vmatrix}1 & 2\\2 & 3\end{vmatrix} = 1(3-4) = -1$$

$$\text{adj}\,A = \begin{bmatrix}A_{11} & A_{12} & A_{13}\\A_{21} & A_{22} & A_{23}\\A_{31} & A_{32} & A_{33}\end{bmatrix}^T = \begin{bmatrix}-6 & 14 & -15\\17 & 5 & 9\\13 & -8 & -1\end{bmatrix}^T = \begin{bmatrix}-6 & 17 & 13\\14 & 5 & -8\\-15 & 9 & -1\end{bmatrix}$$

and $$A^{-1} = \frac{1}{|A|}\text{adj}(A) = \frac{1}{67}\begin{bmatrix}-6 & 17 & 13\\14 & 5 & -8\\-15 & 9 & -1\end{bmatrix}$$

We have $$x+2y-3z = -4$$

$$2x+3y+2z = 2$$

and $$3x-3y-4z = 11$$

In matrix form, it can be written as

$$AX = B \quad \text{...(i)}$$

where, $A = \begin{bmatrix}1 & 2 & -3\\2 & 3 & 2\\3 & -3 & -4\end{bmatrix}$, $X = \begin{bmatrix}x\\y\\z\end{bmatrix}$ and $B = \begin{bmatrix}-4\\2\\11\end{bmatrix}$

System has a unique solution given by

$$X = A^{-1}B.$$

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = \frac{1}{67}\begin{bmatrix}-6 & 17 & 13\\14 & 5 & -8\\-15 & 9 & -1\end{bmatrix}\begin{bmatrix}-4\\2\\11\end{bmatrix} = \frac{1}{67}\begin{bmatrix}24+34+143\\-56+10-88\\60+18-11\end{bmatrix}$$

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = \frac{1}{67}\begin{bmatrix}201\\-134\\67\end{bmatrix} = \begin{bmatrix}3\\-2\\1\end{bmatrix}$$

Comparing corresponding elements, we get $x = 3$, $y = -2$ and $z = 1$.

121. Using matrix method, solve the following system of equations

$$\frac{2}{x}+\frac{3}{y}+\frac{10}{z} = 4$$

$$\frac{4}{x}-\frac{6}{y}+\frac{5}{z} = 1$$

and $$\frac{6}{x}+\frac{9}{y}-\frac{20}{z} = 2,\ x,y,z \neq 0.$$

Sol : Delhi 2011, OD 2007

We have $$\frac{2}{x}+\frac{3}{y}+\frac{10}{z} = 4,$$

$$\frac{4}{x}-\frac{6}{y}+\frac{5}{z} = 1,$$

and $$\frac{6}{x}+\frac{9}{y}-\frac{20}{z} = 2;\ x,\ y,\ z \neq 0$$

Let $\frac{1}{x} = u$, $\frac{1}{y} = v$ and $\frac{1}{z} = w$, then system of equations can be written as

$$2u+3v+10w = 4$$

and $$4u-6v+5w = 1 \quad \text{...(i)}$$

$$6u+9v-20w = 2$$

In matrix form, it can be written as

$$AX = B$$

where $A = \begin{bmatrix}2 & 3 & 10\\4 & -6 & 5\\6 & 9 & -20\end{bmatrix}$, $X = \begin{bmatrix}u\\v\\w\end{bmatrix}$ and $B = \begin{bmatrix}4\\1\\2\end{bmatrix}$

Here,

$$|A| = 2(120-45)-3(-80-30)+10(36+36)$$

$$= 2(75)-3(-110)+10(72)$$

$$= 150+330+720 = 1200$$

$$|A| = 1200$$

Since, $|A| \neq 0$, so A is non-singular and its inverse exists.

Now, cofactors of elements of $|A|$ arc

$$A_{11} = (-1)^2\begin{vmatrix}-6 & 5\\9 & -20\end{vmatrix} = 1(120-45) = 75$$

$$A_{12} = (-1)^3\begin{vmatrix}4 & 5\\6 & -20\end{vmatrix} = -1(-80-30) = 110$$

$$A_{13} = (-1)^4\begin{vmatrix}4 & -6\\6 & 9\end{vmatrix} = 1(36+36) = 72$$

$$A_{21} = (-1)^3\begin{vmatrix}3 & 10\\9 & -20\end{vmatrix} = -(-60-90) = 150$$

$$A_{22} = (-1)^4\begin{vmatrix}2 & 10\\6 & -20\end{vmatrix} = 1(-40-60) = -100$$

$$= \begin{bmatrix} 0 & -5 & 5 \\ 15 & 0 & -10 \\ -10 & 10 & 5 \end{bmatrix}$$

and $\quad A^{-1} = \frac{1}{|A|}\text{adj}(A)$

$$= \frac{1}{25}\begin{bmatrix} 0 & -5 & 5 \\ 15 & 0 & -10 \\ -10 & 10 & 5 \end{bmatrix}$$

Since A^{-1} exists, system has a unique solution given by

$$X = A^{-1}B.$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{25}\begin{bmatrix} 0 & -5 & 5 \\ 15 & 0 & -10 \\ -10 & 10 & 5 \end{bmatrix}\begin{bmatrix} 60 \\ 45 \\ 70 \end{bmatrix}$$

$$= \frac{1}{25}\begin{bmatrix} 0-225+350 \\ 900+0-700 \\ -600+450+350 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{25}\begin{bmatrix} 125 \\ 200 \\ 200 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 4 \end{bmatrix}$$

Comparing corresponding elements, we get $x = 5$, $y = 8$ and $z = 4$.

123. Using matrices, solve the following system of equations.

$$x + 2y + z = 7$$

$$x + 3z = 11$$

and $\quad 2x - 3y = 1$

Sol : OD 2011, Comp 2008

We have $\quad x + 2y + z = 7$

$$x + 3z = 11$$

and $\quad 2x - 3y = 1$

In matrix form, it can be written as

$$AX = B \qquad \text{...(i)}$$

where, $\quad A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 7 \\ 11 \\ 1 \end{bmatrix}$ and $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Here, $\quad |A| = 1(0+9) - 2(0-6) + 1(-3-0)$

$$= (9) - 2(-6) + 1(-3)$$

$$= 9 + 12 - 3 = 18$$

Since $|A| \neq 0$, matrix A is non-singular and A^{-1} exists.

Now, cofactors of elements of $|A|$ are

$$A_{11} = (-1)^2\begin{vmatrix} 0 & 3 \\ -3 & 0 \end{vmatrix} = 1(0+9) = 9$$

$$A_{12} = (-1)^3\begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} = -1(0-6) = 6$$

$$A_{13} = (-1)^4\begin{vmatrix} 1 & 0 \\ 2 & -3 \end{vmatrix} = 1(-3-0) = -3$$

$$A_{21} = (-1)^3\begin{vmatrix} 2 & 1 \\ -3 & 0 \end{vmatrix} = -1(0+3) = -3$$

$$A_{22} = (-1)^4\begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} = 1(0-2) = -2$$

$$A_{23} = (-1)^5\begin{vmatrix} 1 & 2 \\ 2 & -3 \end{vmatrix} = -1(-3-4) = 7$$

$$A_{31} = (-1)^4\begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix} = 1(6-0) = 6$$

$$A_{32} = (-1)^5\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = -1(3-1) = -2$$

$$A_{33} = (-1)^6\begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} = 1(0-2) = -2$$

$$\text{adj}\,A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$$

$$= \begin{bmatrix} 9 & 6 & -3 \\ -3 & -2 & 7 \\ 6 & -2 & -2 \end{bmatrix}^T$$

$$= \begin{bmatrix} 9 & -3 & 6 \\ 6 & -2 & -2 \\ -3 & 7 & -2 \end{bmatrix}$$

and $\quad A^{-1} = \frac{1}{|A|}\text{adj}(A)$

$$= \frac{1}{18}\begin{bmatrix} 9 & -3 & 6 \\ 6 & -2 & -2 \\ -3 & 7 & -2 \end{bmatrix}$$

Since A^{-1} exists, system has a unique solution given by

$$X = A^{-1}B.$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 9 & -3 & 6 \\ 6 & -2 & -2 \\ -3 & 7 & -2 \end{bmatrix}\begin{bmatrix} 7 \\ 11 \\ 1 \end{bmatrix}$$

$$= \frac{1}{18}\begin{bmatrix} 63-33+6 \\ 42-22-2 \\ -21+77-2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 36 \\ 18 \\ 54 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

Comparing corresponding elements, we get $x = 2$, $y = 1$ and $z = 3$.

$$A_{11} = (-1)^2\begin{vmatrix}-1 & 1\\ 0 & -3\end{vmatrix} = 1(3-0) = 3$$

$$A_{12} = (-1)^3\begin{vmatrix}0 & 1\\ 2 & -3\end{vmatrix} = -1(0-2) = 2$$

$$A_{13} = (-1)^4\begin{vmatrix}0 & -1\\ 2 & 0\end{vmatrix} = 1(0+2) = 2$$

$$A_{21} = (-1)^3\begin{vmatrix}-2 & 1\\ 0 & -3\end{vmatrix} = -1(6-0) = -6$$

$$A_{22} = (-1)^4\begin{vmatrix}1 & 1\\ 2 & -3\end{vmatrix} = 1(-3-2) = -5$$

$$A_{23} = (-1)^5\begin{vmatrix}1 & -2\\ 2 & 0\end{vmatrix} = -1(0+4) = -4$$

$$A_{31} = (-1)^4\begin{vmatrix}-2 & 1\\ -1 & 1\end{vmatrix} = 1(-2+1) = -1$$

$$A_{32} = (-1)^5\begin{vmatrix}1 & 1\\ 0 & 1\end{vmatrix} = -1(1-0) = -1$$

$$A_{33} = (-1)^6\begin{vmatrix}1 & -2\\ 0 & -1\end{vmatrix} = 1(-1-0) = -1$$

$$\text{adj}\,A = \begin{bmatrix}A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33}\end{bmatrix}^T = \begin{bmatrix}3 & 2 & 2\\ -6 & -5 & -4\\ -1 & -1 & -1\end{bmatrix}^T = \begin{bmatrix}3 & -6 & -1\\ 2 & -5 & -1\\ 2 & -4 & -1\end{bmatrix}$$

and $$A^{-1} = \frac{1}{|A|}\text{adj}(A) = \frac{1}{1}\begin{bmatrix}3 & -6 & -1\\ 2 & -5 & -1\\ 2 & -4 & -1\end{bmatrix}$$

Now we have
$$x - 2y + z = 0$$
$$-y + z = -2$$
and
$$2x - 3z = 10$$

In matrix form, it can be written as

$$AX = B \qquad \text{...(i)}$$

where, $A = \begin{bmatrix}1 & -2 & 1\\ 0 & -1 & 1\\ 2 & 0 & -3\end{bmatrix}$, $B = \begin{bmatrix}0\\ -2\\ 10\end{bmatrix}$ and $X = \begin{bmatrix}x\\ y\\ z\end{bmatrix}$

Since A^{-1} exists, system has a unique solution given by

$$X = A^{-1}B.$$

$$\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}3 & -6 & -1\\ 2 & -5 & -1\\ 2 & -4 & -1\end{bmatrix}\begin{bmatrix}0\\ -2\\ 10\end{bmatrix} = \begin{bmatrix}0+12-10\\ 0+0-10\\ -0+8-16\end{bmatrix}$$

$$\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}2\\ 0\\ -2\end{bmatrix}$$

Comparing corresponding elements, we get $x = 2$, $y = 0$ and $z = -2$.

126. If $A = \begin{bmatrix}1 & -2 & 0\\ 2 & 1 & 3\\ 0 & -2 & 1\end{bmatrix}$ and $B = \begin{bmatrix}7 & 2 & -6\\ -2 & 1 & -3\\ -4 & 2 & 5\end{bmatrix}$, then find AB hence solve system of equations

$$x - 2y = 10$$
$$2x + y + 3z = 8$$
and
$$-2y + z = 7.$$

Sol : Delhi 2011, OD 2010

We have
$$x - 2y = 10$$
$$2x + y + 3z = 8$$
and
$$-2y + z = 7.$$

In matrix form, it can be written as

$$AX = C \qquad \text{...(1)}$$

where, $A = \begin{bmatrix}1 & -2 & 0\\ 2 & 1 & 3\\ 0 & -2 & 1\end{bmatrix}$, $X = \begin{bmatrix}x\\ y\\ z\end{bmatrix}$ and $C = \begin{bmatrix}10\\ 8\\ 7\end{bmatrix}$

Its solution can be given as

$$X = A^{-1}C \qquad (2)$$

Now let $A = \begin{bmatrix}1 & -2 & 0\\ 2 & 1 & 3\\ 0 & -2 & 1\end{bmatrix}$ and $B = \begin{bmatrix}7 & 2 & -6\\ -2 & 1 & -3\\ -4 & 2 & 5\end{bmatrix}$

Now,
$$AB = \begin{bmatrix}1 & -2 & 0\\ 2 & 1 & 3\\ 0 & -2 & 1\end{bmatrix}\begin{bmatrix}7 & 2 & -6\\ -2 & 1 & -3\\ -4 & 2 & 5\end{bmatrix}$$
$$= \begin{bmatrix}7+4+0 & 2-2+0 & -6+6+0\\ 14-2-12 & 4+1+6 & -12-3+15\\ 0+4-4 & 0-2+2 & 0+6+5\end{bmatrix}$$
$$= \begin{bmatrix}11 & 0 & 0\\ 0 & 11 & 0\\ 0 & 0 & 11\end{bmatrix} = 11I$$

$$AB = 11I$$

$$A^{-1}(AB) = A^{-1}\cdot 11I$$

Comparing corresponding elements, we get $x=3$, $y=2$ and $z=-1$.

128. If $A=\begin{bmatrix}8&-4&1\\10&0&6\\8&1&6\end{bmatrix}$, then find A^{-1} and hence solve the following system of equations.

$$8x-4y+z=5$$

$$10x+6z=4$$

and $$8x+y+6z=\tfrac{5}{2}$$

Sol : Comp 2010

We have, $$A=\begin{bmatrix}8&-4&1\\10&0&6\\8&1&6\end{bmatrix}$$

Here, $$|A|=8(0-6)+4(60-48)+1(10-0)$$
$$=8(-6)+4(12)+1(10)$$
$$=-48+48+10=10$$

Since $|A|\neq 0$, matrix A is non-singular and A^{-1} exists.

Now, cofactors of elements of $|A|$ are

$$A_{11}=(-1)^2\begin{vmatrix}0&6\\1&6\end{vmatrix}=1(0-6)=-6$$

$$A_{12}=(-1)^3\begin{vmatrix}10&6\\8&6\end{vmatrix}=-1(60-48)=-12$$

$$A_{13}=(-1)^4\begin{vmatrix}10&0\\8&1\end{vmatrix}=1(10-0)=10$$

$$A_{21}=(-1)^3\begin{vmatrix}-4&1\\1&6\end{vmatrix}=-1(-24-1)=25$$

$$A_{22}=(-1)^4\begin{vmatrix}8&1\\8&6\end{vmatrix}=1(48-8)=40$$

$$A_{23}=(-1)^5\begin{vmatrix}8&-4\\8&1\end{vmatrix}=-1(8+32)=-40$$

$$A_{31}=(-1)^4\begin{vmatrix}-4&1\\0&6\end{vmatrix}=1(-24-0)=-24$$

$$A_{32}=(-1)^5\begin{vmatrix}8&1\\10&6\end{vmatrix}=-1(48-10)=-38$$

$$A_{33}=(-1)^6\begin{vmatrix}8&-4\\10&0\end{vmatrix}=1(0+40)=40$$

$$\operatorname{adj}A=\begin{bmatrix}A_{11}&A_{12}&A_{13}\\A_{21}&A_{22}&A_{23}\\A_{31}&A_{32}&A_{33}\end{bmatrix}^T$$

$$=\begin{bmatrix}-6&-12&10\\25&40&-40\\-24&-38&40\end{bmatrix}^T$$

$$=\begin{bmatrix}-6&25&-24\\-12&40&-38\\10&-40&40\end{bmatrix}$$

and $$A^{-1}=\frac{1}{|A|}\operatorname{adj}(A)$$

$$=\frac{1}{10}\begin{bmatrix}-6&25&-24\\-12&40&-38\\10&-40&40\end{bmatrix}$$

Now we have $$8x-4y+z=5$$

$$10x+6z=4$$

and $$8x+y+6z=\tfrac{5}{2}$$

In matrix form, it can be written as

$$AX=B \qquad \text{...(i)}$$

where, $A=\begin{bmatrix}8&-4&1\\10&0&6\\8&1&6\end{bmatrix}$, $B=\begin{bmatrix}5\\4\\\frac{5}{2}\end{bmatrix}$ and $X=\begin{bmatrix}x\\y\\z\end{bmatrix}$

Since A^{-1} exists, system has a unique solution given by

$$X=A^{-1}B.$$

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\frac{1}{10}\begin{bmatrix}-6&25&-24\\-12&40&-38\\10&-40&40\end{bmatrix}\begin{bmatrix}5\\4\\\frac{5}{2}\end{bmatrix}$$

$$=\frac{1}{10}\begin{bmatrix}-30+100-60\\-60+160-95\\50-160+100\end{bmatrix}$$

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\frac{1}{10}\begin{bmatrix}10\\5\\-10\end{bmatrix}=\begin{bmatrix}1\\\frac{1}{2}\\-1\end{bmatrix}$$

Comparing corresponding elements, we get $x=1$, $y=\frac{1}{2}$ and $z=-1$.

129. If $A=\begin{bmatrix}1&-1&0\\2&3&4\\0&1&2\end{bmatrix}$ and $B=\begin{bmatrix}2&2&-4\\-4&2&-4\\2&-1&5\end{bmatrix}$, then find AB. Use this to solve the system of equations

$$x-y=3$$

$$2x+3y+4z=17$$

and $$y+2z=7.$$

Sol : OD 2010, Delhi 2007

We have $$x-y=3$$

$$2x+3y+4z=17$$

and $$y+2z=7.$$

In matrix form, it can be written as

$$AX=C \qquad \text{...(1)}$$

and $\qquad 7x+3y-3z = 7.$

In matrix form, it can be written as

$$AX = B \qquad \text{...(i)}$$

where, $A = \begin{bmatrix} 3 & 2 & 1 \\ 4 & -1 & 2 \\ 7 & 3 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 6 \\ 5 \\ 7 \end{bmatrix}$ and $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Since A^{-1} exists, system has a unique solution given by

$$X = A^{-1}B.$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{62}\begin{bmatrix} -3 & 9 & 5 \\ 26 & -16 & -2 \\ 19 & 5 & -11 \end{bmatrix}\begin{bmatrix} 6 \\ 5 \\ 7 \end{bmatrix}$$

$$= \frac{1}{62}\begin{bmatrix} -18+45+35 \\ 156-80-14 \\ 114+25-77 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{62}\begin{bmatrix} 62 \\ 62 \\ 62 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Comparing corresponding elements, we get $x=1$, $y=1$ and $z=1$.

CASE BASED QUESTIONS

131. Pastry is a dough of flour, water and shortening that may be savoury or sweetened. Sweetened pastries are often described as bakers' confectionery. The word "pastries" suggests many kinds of baked products made from ingredients such as flour, sugar, milk, butter, shortening, baking powder, and eggs.

The Sunrise Bakery Pvt Ltd produces three basic pastry mixes A, B and C. In the past the mix of ingredients has shown in the following matrix:

$$\begin{array}{cc} & \begin{array}{ccc} \text{Flour} & \text{Fat} & \text{Sugar} \end{array} \\ \text{Type} \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 5 & 1 & 1 \\ 6.5 & 2.5 & 0.5 \\ 4.5 & 3 & 2 \end{bmatrix} \end{array}$$

(All quantities in kg)

Due to changes in the consumer's tastes it has been decided to change the mixes using the following amendment matrix:

$$\begin{array}{cc} & \begin{array}{ccc} \text{Flour} & \text{Fat} & \text{Sugar} \end{array} \\ \text{Type} \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ -0.5 & 0.5 & 0.5 \\ 0.5 & 0 & 0 \end{bmatrix} \end{array}$$

Using matrix algebra you are required to calculate:

(i) the matrix for the new mix:

(ii) the production requirement to meet an order for 50 units of type A, 30 units of type B and 20 units of type C of the new mix;

(iii) the amount of each type that must be made to totally use up 370 kg of flour, 170 kg of fat and 80 kg of sugar that are at present in the stores.

Sol :

(i) Matrix for the new mix:

The matrix for the new mix is obtained by adding the amendment matrix to the original mix matrix, i.e

$$\begin{bmatrix} 6 & 1 & 1 \\ 6.5 & 2.5 & 0.5 \\ 4.5 & 3 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ -0.5 & 0.5 & 0.5 \\ 0.5 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 2 & 1 \\ 6 & 3 & 1 \\ 5 & 3 & 2 \end{bmatrix}$$

(ii) Production requirement

The production requirement to meet an order for 50 units of type A, 30 units of type B and 20 units of type C of the new mix is given by the matrix product

$$\begin{bmatrix} 50 & 30 & 20 \end{bmatrix}\begin{bmatrix} 5 & 2 & 1 \\ 6 & 3 & 1 \\ 5 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 530 & 250 & 120 \end{bmatrix}$$

Hence, the production requirement to meet a given order of the new mix is 530 kg of flour, 250 kg of fat, 120 kg of sugar.

(iii) Amount of Each Type

Let x_1, x_2 and x_3 be the amount of mixes of type A, B and C respectively. Then, we have the following set of simultaneous equations:

$$5x_1 + 6x_2 + 5x_3 = 370$$

$$2x_1 + 3x_2 + 3x_3 = 170$$

$$x_1 + x_2 + 2x_3 = 80$$

The above system of equations can be written in matrix form as:

$$\begin{bmatrix} 5 & 6 & 5 \\ 2 & 3 & 3 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 370 \\ 170 \\ 80 \end{bmatrix}$$

or $\qquad AX = B$

where $A = \begin{bmatrix} 5 & 6 & 5 \\ 2 & 3 & 3 \\ 1 & 1 & 2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 370 \\ 170 \\ 80 \end{bmatrix}$

$$= \frac{1}{-26}\begin{bmatrix} -48000 & -900000 & +112000 \\ -1120000 & +50000 & -16000 \\ 128000 & -20000 & -160000 \end{bmatrix}$$

$$= \frac{-1}{26}\begin{bmatrix} -26000 \\ -78000 \\ -52000 \end{bmatrix} = \begin{bmatrix} 1000 \\ 3000 \\ 2000 \end{bmatrix}$$

Thus $x = 1000$, $y = 3000$ and $z = 2000$. Hence, the number of strips produced are: Paingo-1000, X -prene-3000 and Relaxo-2000.

133. A manufacturing company has two service departments, S_1, S_2 and four production departments P_1, P_2, P_3 and P_4.

Overhead is allocated to the production departments for inclusion in the stock valuation. The analysis of benefits received by each department during the last quarter and the overhead expense incurred by each department were:

Service Department	Percentages to be allocated to departments					
	S_1	S_2	P_1	P_2	P_3	P_4
S_1	0	20	30	25	15	10
S_2	30	0	10	35	20	5
Direct overhead expense ` '000	20	40	25	30	20	10

You are required to find out following using matrix method.

(i) Express the total overhead of the service departments in the form of simultaneous equations.

(ii) Express these equations in a matrix form and solve for total overhead of service departments using matrix inverse method.

(iii) Determine the total overhead to be allocated from each of S_1 and S_2 to the production department.

Sol :

Let S_1 be total overhead of service department and S_2 be total overhead of service department.

Then, $S_1 = 20000 + 0.3S_2$

and $S_2 = 40000 + 0.2S_1$

The total overhead of the service departments can be expressed in the form of simultaneous equations as:

$$S_1 - 0.3S_2 = 20000$$

$$-0.2S_1 + S_2 = 40000$$

The above simultaneous equations can be expressed in the matrix form as:

$$\begin{bmatrix} 1 & -0.3 \\ -0.2 & 1 \end{bmatrix}\begin{bmatrix} S_1 \\ S_2 \end{bmatrix} = \begin{bmatrix} 20000 \\ 40000 \end{bmatrix},$$

or $AS = B$

Here $|A| = 1 - 0.06 = 0.94$

Since $|A| \neq 0$ thus A^{-1} exists, so that the unique solution of $AX = B$ is $X = A^{-1}B$.

Here, $A^{-1} = \frac{\text{adj}.A}{|A|}$

$$= \frac{1}{0.94}\begin{bmatrix} 1 & 0.3 \\ 0.2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} S_1 \\ S_2 \end{bmatrix} = S = A^{-1}B$$

$$= \frac{1}{0.94}\begin{bmatrix} 1 & 0.3 \\ 0.2 & 1 \end{bmatrix}\begin{bmatrix} 20000 \\ 40000 \end{bmatrix} = \begin{bmatrix} 34043 \\ 46809 \end{bmatrix}$$

S_1 = ` 34043 and $S_2$ = ` 46809

The allocation of overhead from S_1 and S_2 to the production department becomes:

$$[P_1 \; P_2 \; P_3 \; P_4] = [34043][0.3 \; 0.25 \; 0.15 \; 0.1]$$

$$= [10213 \; 8511 \; 5106 \; 3404]$$

and $$[P_1 \; P_2 \; P_3 \; P_4] = [46809][0.1 \; 0.35 \; 0.2 \; 0.05]$$

$$= [4681 \; 16383 \; 9362 \; 2340]$$

The final allocation becomes:

Department	Total	P_1	P_2	P_3	P_4
S_1	27234	10213	8511	5106	3404
S_2	32766	4681	16383	9362	2340
Total	60000	14894	24894	14468	5744

134. Cross holding, also referred to as cross shareholding, describes a situation where one publicly-traded company holds a significant number of shares of another publicly-traded company. The shares owned bought I and III, 60 customers only products II and II and 80 customers only products only III regardless of the market segmentation groups.

Based on the market segmentation analysis, for product I, the percentage for the income groups are given as (40%, 20% and 40%), for product II (30%, 20% and 50%), for product III (10%, 50% and 40%).

(i) Taking the suitable variable form the system of equation that represent given problem.

(ii) Using matrix method, find out the number of persons in the lower income, middle income and higher income class in the region referred.

Sol :

The number of persons purchasing product I, II and III can be put in the tabular form as:

I	$(200+240+80+120)$	$=640$
II	$(200+60+80+220)$	$=560$
III	$(200+60+240+80)$	$=580$

Let the number of persons in the lower income group, middle income group and higher income group be x, y and z, respectively. Then the given information can be put in the form of simultaneous linear equations as:

$$0.4x+0.2y+0.4z = 640$$

$$2x+y+2z = 3200 \quad \text{(i)}$$

$$0.3x+0.2y+0.5z = 560$$

$$3x+2y+5z = 5600 \quad \text{(ii)}$$

$$0.1x+0.5y+0.4z = 580$$

$$x+5y+4z = 5800 \quad \text{(iii)}$$

The above system of equations can be written in matrix form as:

$$\begin{bmatrix} 2 & 1 & 2 \\ 3 & 2 & 5 \\ 1 & 5 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3200 \\ 5600 \\ 5800 \end{bmatrix}$$

or $\qquad AX = B$

Here $\qquad |A| = 2(8-25)-1(12-5)+2(15-2)$

$$= -34-7+26 = -15$$

Since $|A| \neq 0$ thus A^{-1} exists, so that the unique solution of $AX = B$ is $X = A^{-1}B$.

Here, $\qquad A^{-1} = \dfrac{\text{adj}.A}{|A|}$

$$= \frac{1}{-15}\begin{bmatrix} -17 & 6 & 1 \\ -7 & 6 & -4 \\ 13 & -9 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = X = A^{-1}B$$

$$= \frac{1}{-15}\begin{bmatrix} -17 & 6 & 1 \\ -7 & 6 & -4 \\ 13 & -9 & 1 \end{bmatrix}\begin{bmatrix} 3200 \\ 5600 \\ 5800 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{-1}{15}\begin{bmatrix} -54400 & +33600 & +5800 \\ -22400 & +33600 & -23200 \\ 41600 & -50400 & 5800 \end{bmatrix}$$

$$= \frac{-1}{15}\begin{bmatrix} -15000 \\ -12000 \\ -3000 \end{bmatrix} = \begin{bmatrix} 1000 \\ 800 \\ 200 \end{bmatrix}$$

$$x = 1000,\ y = 800 \text{ and } z = 200$$

Hence, the number of persons in the lower income, middle income and higher income class in the region referred are 1000, 800 and 200 respectively.

136. A car carrier trailer, also known as a car-carrying trailer, car hauler, or auto transport trailer, is a type of trailer or semi-trailer designed to efficiently transport passenger vehicles via truck. Commercial-size car carrying trailers are commonly used to ship new cars from the manufacturer to auto dealerships. Modern car carrier trailers can be open or enclosed. Most commercial trailers have built-in ramps for loading and off-loading cars, as well as power hydraulics to raise and lower ramps for stand-alone accessibility.

A transport company uses three types of trucks T_1, T_2 and T_3 to transport three types of vehicles V_1, V_2 and V_3. The capacity of each truck in terms of three types of vehicles is given below:

	V_1	V_2	V_3
T_1	1	3	2
T_2	2	2	3
T_3	3	2	2

Using matrix method find:

CHAPTER 5

CONTINUITY AND DIFFERENTIABILITY

OBJECTIVE QUESTIONS

1. Derivative of $e^{\sin^2 x}$ with respect to $\cos x$ is

(a) $\sin x e^{\sin^2 x}$ (b) $\cos x e^{\sin^2 x}$

(c) $-2\cos x e^{\sin^2 x}$ (d) $-2\sin^2 x \cos x e^{\sin^2 x}$

Sol : OD 2024

Let $$P = e^{\sin^2} x \quad ...(1)$$

$$Q = \cos x \quad ...(2)$$

Differentiating equation (1) w.r.t. x we have

$$\begin{aligned} \frac{dP}{dx} &= \frac{d}{dx}(e^{\sin^2 x}) \\ &= e^{\sin^2 x}\frac{d}{dx}(\sin^2 x) \\ &= e^{\sin^2 x}(\sin 2x) \\ &= e^{\sin^2 x}(\sin 2x) \quad ...(3) \end{aligned}$$

Differentiating equation (2) w.r.t. x we get

$$\begin{aligned} \frac{dQ}{dx} &= \frac{d}{dx}(\cos x) \\ &= -\sin x \quad ...(4) \end{aligned}$$

Dividing equation (3) by (4) we have

$$\begin{aligned} \frac{\frac{dP}{dx}}{\frac{dQ}{dx}} &= \frac{e^{\sin^2 x}(\sin 2x)}{-\sin x} \\ \frac{dP}{dQ} &= \frac{e^{\sin^2 x}(2\sin x \cos x)}{-\sin x} \\ &= -2e^{\sin^2 x}\cos x \end{aligned}$$

Thus (c) is correct option.

2. If $\sin(xy) = 1$, then $\frac{dy}{dx}$ is equal to

(a) $\frac{x}{y}$ (b) $-\frac{x}{y}$

(c) $\frac{y}{x}$ (d) $-\frac{y}{x}$

Sol : OD 2024

We have $$\sin(xy) = 1$$

Differentiating both sides w.r.t x we have

$$\begin{aligned} \cos(xy)\frac{d}{dx}(xy) &= 0 \\ \cos xy\left(x\frac{dy}{dx} + y\right) &= 0 \\ x\frac{dy}{dx} + y &= 0 \\ x\frac{dy}{dx} &= -y \\ \frac{dy}{dx} &= \frac{-y}{x} \end{aligned}$$

Thus (d) is correct option.

3. The function $f(x) = x|x|$ is :

(a) continuous and differentiable at $x = 0$

(b) continuous but not differentiable at $x = 0$

(c) differentiable but not continuous at $x = 0$

(d) neither differentiable nor continuous at $x = 0$

Sol : OD 2023

We have $$f(x) = \begin{cases} x^2, x \geq 0 \\ -x^2, \quad x < 0 \end{cases}$$

At $x = 0$ we have

$$\lim_{x \to 0^-} -x^2 = 0 \text{ and } \lim_{x \to 0^+} -x^2 = 0$$

As $$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$$

Thus $f(x)$ is continuous at $x = 0$ and

$$f'(x) = \begin{cases} 2x, & x > 0 \\ -2x, & x < 0 \end{cases}$$

At $x = 0$, $$\frac{d}{dx}f(x)\,|_{0^-} = \frac{d}{dx}f(x)\,|_{0^+} = 0$$

Thus $f(x)$ is differentiable at $x = 0$.

Thus (a) is correct option.

4. If $\tan\left(\frac{x+y}{x-y}\right) = k$, then $\frac{dy}{dx}$ is equal to :

(a) $-\frac{y}{x}$ (b) $\frac{y}{x}$

(c) $\sec^2\left(\frac{y}{x}\right)$ (d) $-\sec^2\left(\frac{y}{x}\right)$

Sol : OD 2023

We have $$\tan\left(\frac{x+y}{x-y}\right) = k$$

$$\left(\frac{x+y}{x-y}\right) = \tan^{-1} k$$

Differentiating w.r.t x we have

Ch 7 : Integrals

PDF Text Book	PDF Worksheet	PDF Solutions	DOC Worksheet	PPT Presentation

Ch 8 : Application of Integrals

PDF Text Book	PDF Worksheet	PDF Solutions	DOC Worksheet	PPT Presentation

Ch 9 : Differential Equations

PDF Text Book	PDF Worksheet	PDF Solutions	DOC Worksheet	PPT Presentation

Ch 10 : Vector Algebra

PDF Text Book	PDF Worksheet	PDF Solutions	DOC Worksheet	PPT Presentation

Ch 11 : Three Dimensional Geometry

PDF Text Book	PDF Worksheet	PDF Solutions	DOC Worksheet	PPT Presentation

Ch 12 : Linear Programming

PDF Text Book	PDF Worksheet	PDF Solutions	DOC Worksheet	PPT Presentation

Ch 13 : Probability

PDF Text Book	PDF Worksheet	PDF Solutions	DOC Worksheet	PPT Presentation

Class 10th

Maths	Science	Social Science	English LL
Info Tech.	Comp. App.	AI	English Comm.
हिंदी अ	हिंदी ब	संस्कृत	

Class 9th

Maths	Science	Social Science	English LL
Info Tech.	Comp. App.	AI	English Comm.
हिंदी अ	हिंदी ब	संस्कृत	

Class 8th

Maths	Science	Social Science	English
हिंदी	संस्कृत		

Class 7th

Maths	Science	Social Science	English
हिंदी	संस्कृत		

Class 6th

Maths	Science	Social Science	English
हिंदी	संस्कृत		

Class 5th

Maths	English	हिंदी	EVS

Class 4th

Maths	English	हिंदी	EVS

Class 3th

Maths	English	हिंदी	EVS

Class 2th

Maths	English	हिंदी	EVS

Class 1th

Maths	English	हिंदी	EVS

Prep/KG 3rd Year

Maths	English	हिंदी	EVS

Prep/KG 2nd Year

Maths	English	हिंदी	EVS

Prep/KG 1st Year

Maths	English	हिंदी	EVS

10. The function $f(x) = |x|$ at $x = 0$ is
(a) continuous but not differentiable
(b) discontinuous and differentiable
(c) discontinuous and not differentiable
(d) continuous and differentiable

Sol : Comp 2017, Delhi 2015

We have $f(x) = |x| = \begin{cases} x & x \geq 0 \\ -x & x \leq 0 \end{cases}$

LHS Limit $\lim_{x \to 0-h} f(x) = \lim_{h \to 0} |0-h| = 0$

RHS Limit $\lim_{x \to 0+h} f(x) = \lim_{h \to 0} |0+h| = 0,$

Hence function is continuous at $x = 0$.

Now $Rf'(0) = \lim_{h \to 0} \frac{|0+h| - |0|}{h} = 1$

And $Lf'(0) = -1$ (Not differentiable)

Thus (a) is correct option.

11. If $f(x) = \begin{cases} \frac{3\sin \pi x}{5x}, & x \neq 0 \\ 2k, & x = 0 \end{cases}$ is continuous at $x = 0$, then the value of k is
(a) $\frac{\pi}{10}$
(b) $\frac{3\pi}{10}$
(c) $\frac{3\pi}{2}$
(d) $\frac{3\pi}{5}$

Sol : OD 2016

We have $f(x) = \begin{cases} \frac{3\sin \pi x}{5x}, & x \neq 0 \\ 2k, & x = 0 \end{cases}$

Now,
$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{3 \sin \pi x}{5x}$$
$$= \lim_{x \to 0} \frac{3\pi \sin \pi x}{5 \pi x}$$
$$= \frac{3\pi}{5} \lim_{x \to 0} \left(\sin \frac{\pi x}{\pi x} \right)$$
$$= \frac{3}{5}\pi \times 1 = \frac{3}{5}\pi$$

Also, $f(0) = 2k$

Since, $f(x)$ is continuous at $x = 0$, we have
$$f(0) = \lim_{x \to 0} f(x)$$
$$2k = \frac{3}{5}\pi$$
$$k = \frac{3\pi}{10}$$

Thus (b) is correct option.

12. If $f(x) = \begin{cases} ax+3, & x \leq 2 \\ a^2 x - 1, & x > 2 \end{cases}$, then the values of a for which f is continuous for all x are
(a) 1 and -2
(b) 1 and 2
(c) -1 and 2
(d) -1 and -2

Sol : Foreign 2015

We have $f(x) = \begin{cases} ax+3, & x \leq 2 \\ a^2 x - 1, & x > 2 \end{cases}$
$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (ax+3) = 2a+3$$
$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (a^2 x - 1) = 2a^2 - 1$$

Since, $f(x)$ is continuous for all values of x.
$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x)$$
$$2a+3 = 2a^2 - 1$$
$$2a^2 - 2a - 4 = 0$$
$$a^2 - a - 2 = 0$$
$$a^2 - 2a + a - 2 = 0$$
$$a(a-2) + 1(a-2) = 0$$
$$(a+1)(a-2) = 0$$
$$a = -1, 2$$

Thus (c) is correct option.

13. If $f(x) = \begin{cases} x+\lambda, & x < 3 \\ 4, & x = 3 \\ 3x-5, & x > 3 \end{cases}$ is continuous at $x = 3$, then λ is equal to
(a) 3
(b) 2
(c) 1
(d) 4

Sol : Foreign 2010, OD 2007

Since function is continuous at $x = 3$, both left limit and right limit must be equal.

Hence $x + \lambda = 3x - 5 = 4$
$$\lambda = 1$$

Thus (c) is correct option.

14. If function $f(x) = \begin{cases} x-1, & x < 2 \\ 2x-3, & x \geq 2 \end{cases}$ is a continuous function then
(a) for $x = 2$ only
(b) for all real values of x such that $x \neq 2$
(c) for all real values of x
(d) for all integral values of x only

Sol : OD 2013

LHS $\lim_{x \to 2-h} f(x) = \lim_{h \to 0} (2 - h - 1)$
$$= 1$$

We have $y = \log_{10} x$

$$= \frac{\log_e x}{\log_e 10} \qquad \left(\log_a b = \frac{\log_e b}{\log_e a}\right)$$

$$\frac{dy}{dx} = \frac{1}{\log_e 10} \cdot \frac{1}{x}$$

$$= \frac{\log_{10} e}{x} \qquad \left(\log_a b = \frac{1}{\log_b a}\right)$$

Thus (a) is correct option.

21. Differential co-efficient of $\operatorname{cosec}^{-1} x$ is

(a) $-\frac{1}{x\sqrt{x^2-1}}$ (b) $\frac{1}{x\sqrt{x^2-1}}$

(c) $\frac{x}{\sqrt{x^2-1}}$ (d) None of the above

Sol : Comp 2007

We have $y = \operatorname{cosec}^{-1} x$

or $\operatorname{cosec} y = x$

Now differentiating wrt x we have

$$-\operatorname{cosec} y \cot y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = -\frac{1}{\operatorname{cosec} y \cot y}$$

$$= -\frac{1}{\operatorname{cosec} y\sqrt{\operatorname{cosec}^2 - 1}}$$

$$= -\frac{1}{x\sqrt{x^2-1}}$$

Thus (a) is correct option.

22. Differential coefficient of $x^{\log_e x}$ wrt x is

(a) $x^{\log_e x - 1}\log_e x$ (b) $2x^{\log_e(x-1)}\log_e x$

(c) $x^{\log_e x - 1}$ (d) $\frac{2}{x}\log_e x$

Sol : OD 2007

We have $y = x^{\log_e x}$

$$\log y = \log_e x^{\log_e x}$$

$$= \log_e x \log_e x$$

$$= (\log_e x)^2$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = 2\log_e x \cdot \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{2y}{x}\log_e x$$

$$= \frac{2x^{\log_e x}}{x}\log_e x$$

$$= 2x^{\log_e x - 1}\log_e x$$

Thus (b) is correct option.

23. If $f(x) = \frac{x}{1+x}$ and $g(x) = f[f(x)]$, then $g'(x)$ is equal to

(a) $\frac{1}{(2x-3)^2}$ (b) $\frac{1}{(x+1)^2}$

(c) $\frac{1}{x^2}$ (d) $\frac{1}{(2x+1)^2}$

Sol : Delhi 2018

We have $f(x) = \frac{x}{1+x}$

and $g(x) = f[\,f(x)\,]$

$$g(x) = f\left(\frac{x}{x+1}\right)$$

$$= \frac{\frac{x}{1+x}}{1+\frac{x}{1+x}}$$

$$= \frac{x}{2x+1}$$

$$g(x) = \frac{x}{2x+1}$$

Differentiating both sides w.r.t. x, we get

$$g'(x) = \frac{(2x+1)1 - x(2)}{(2x+1)^2}$$

$$= \frac{1}{(2x+1)^2}$$

Thus (d) is correct option.

24. If $\sin(x+y) = \log(x+y)$, then $\frac{dy}{dx}$ is equal to

(a) 2 (b) -2

(c) 1 (d) -1

Sol : Delhi 2013, OD 2011

We have, $\sin(x+y) = \log(x+y)$

Differentiating w.r.t. x, we get

$$\cos(x+y)\left(1+\frac{dy}{dx}\right) = \frac{1}{x+1}\left(1+\frac{dy}{dx}\right)$$

$$\left[\cos(x+y) - \frac{1}{x+y}\right]\left(1+\frac{dy}{dx}\right) = 0$$

$$\frac{dy}{dx} = -1$$

Thus (d) is correct option.

25. If $x = e^{y+e^{y+e^{y+\dots}}}$, then $\frac{dy}{dx}$ is equal to

(a) $\frac{1}{x}$ (b) $\frac{1-x}{x}$

(c) $\frac{x}{1+x}$ (d) None of these

Sol : Comp 2007

We have $x = e^{y+e^{y+e^{y+\dots}}}$

$$x = e^{y+x}$$

Taking log both sides, we get

$$\log x = (x+y)\log e$$

$$\frac{1}{x} = \frac{dy}{dx} + 1$$

$$2ay = x^2$$

$$\frac{dy}{dx} = \frac{x}{a}$$

Thus (c) is correct option.

33. If $f(x) = \log_e[\log_e x]$, then $f'(e)$

(a) e^{-1} (b) e

(c) 1 (d) 0

Sol : Foreign 2008

We have $f(x) = \log_e[\log_e x]$

Differentiating wrt x we get

$$f'(x) = \frac{1}{\log_e x}\cdot\frac{1}{x}$$

or

$$f'(e) = \frac{1}{\log_e e}\cdot\frac{1}{e} = \frac{1}{e} = e^{-1} \quad [\log_e e = 1]$$

Thus (a) is correct option.

34. If $y = \log_e \sin x$, then $\frac{dy}{dx} =$

(a) $\operatorname{cosec} x$ (b) $\cot x$

(c) $-\cot x$ (d) $-\operatorname{cosec} x$

Sol : SQP 2018, Comp 2010

We have $\frac{dy}{dx} = \frac{1}{\sin x}\times\frac{d}{dx}(\sin x)$

$$= \cot x$$

Thus (b) is correct option.

35. If $y = x + x^2 + \frac{x^3}{\underline{|2}} + \frac{x^4}{\underline{|3}} + ...$ then $x\frac{dy}{dx}$ will be

(a) xe^x (b) $y(x+1)$

(c) $x(y+1)$ (d) $y\log(1+x)$

Sol : Comp 2010

We have $y = x + x^2 + \frac{x^3}{\underline{|2}} + \frac{x^4}{\underline{|3}} + ...$

$$= x\left(1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + ...\right)$$

$$y = xe^x$$

$$\frac{dy}{dx} = e^x(x+1)$$

$$x\frac{dy}{dx} = y(x+1)$$

Thus (b) is correct option.

36. Differentiation of $\cos x$ w.r.t. $\cot x$ is

(a) $\cos^3 x$ (b) $\sin^3 x$

(c) $\tan^3 x$ (d) None of these

Sol : Delhi 2017

Let $u = \cos x$ and $v = \cot x$

Then $\frac{du}{dx} = -\sin x$ and $\frac{dv}{dx} = -\operatorname{cosec}^2 x$

Now

$$\frac{d(\cos x)}{d(\cot x)} = \frac{du}{dv} = \frac{du}{dv} = \frac{-\sin x}{-\operatorname{cosec}^2 x} = \sin^3 x$$

Thus (b) is correct option.

37. Differential co-efficient of $\sin x$ wrt $\tan x$ is

(a) $\sin^3 x$ (b) $\cos^3 x$

(c) $\sec^3 x$ (d) None of the above

Sol : Foreign 2010

Let $u = \sin x$ and $v = \tan x$

Then $\frac{du}{dx} = \cos x$ and $\frac{dv}{dx} = \sec^2 x$

Now

$$\frac{d(\sin x)}{d(\tan x)} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{\cos x}{\sec^2 x} = \cos^3 x$$

Thus (b) is correct option.

38. Derivative of x^3 wrt x^2 is equal to

(a) $\frac{3}{2}x$ (b) $\frac{2}{3}x$

(c) $\frac{3}{2x}$ (d) $\frac{2}{3x}$

Sol : Delhi 2008, OD 2007

We have $\frac{d(x^3)}{d(x^2)} = \frac{\frac{d}{dx}(x^3)}{\frac{d}{dx}(x^2)} = \frac{3x^2}{2x} = \frac{3}{2}x$

Thus (a) is correct option.

39. nth derivative of e^x with respect to x is

(a) e^x (b) $e^x\cdot\underline{|n}$

(c) 0 (d) e^{x^n}

Sol : Comp 2018

We have $y = e^x$

$$y_1 = e^x$$

$$y_2 = e^x$$

$$y_n = e^x$$

Thus (a) is correct option.

Multiplying and Dividing by $\frac{9}{4}$ we get

$$\text{L.H.S.} = \frac{2\sin^2\frac{3x}{2}}{x^2 \times \frac{9}{4}} \times \frac{9}{4}$$

$$= \frac{9}{2}\left(\frac{\sin\frac{3x}{2}}{\frac{3x}{2}}\right)^2 \qquad \left[\lim_{x \to 0}\frac{\sin x}{x} = 1\right]$$

$$= \frac{9}{2}$$

Both (A) and (R) are true and (R) is the correct explanation of (A).

Thus (a) is correct option.

44. **Assertion (A) :** If $y = x^3 \cos x$, then

$$\frac{dy}{dx} = x^3 \sin x + 3x^2 \cos x$$

Reason (R) : $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$.

(a) Both (A) and (R) are true and (R) is the correct explanation of (A).

(b) Both (A) and (R) are true but (R) is not the correct explanation of (A).

(c) (A) is true but (R) is false.

(d) (A) is false but (R) is true

Sol : Delhi 2008

We have $\quad y = x^3 \cos x$

Differentiating both sides with respect to x using product rule, we get

$$\frac{dy}{dx} = \frac{d}{dx}(x^3 \cos x)$$

$$= \cos x \frac{d}{dx}x^3 + x^3 \frac{d}{dx}\cos x$$

$$= 3x^2 \cos x - \sin x \cdot x^3$$

Product rule, $$\frac{dy}{dx} = \frac{d}{dx}(uv)$$

$$= u\frac{dv}{dx} + v\frac{du}{dx}$$

Assertion is false but reason is true since coefficient of $\sin x$ should be -1.

Thus (d) is correct option.

45. **Assertion (A) :** $f(x)$ is defined as $f(x) = \begin{cases} x^3 - 3, x \leq 2 \\ x^2 + 1, x > 2 \end{cases}$ is continuous at $x = 2$.

Reason (R) : $f(2) = \lim_{x \to 2} f(x)$

(a) Both (A) and (R) are true and (R) is the correct explanation of (A).

(b) Both (A) and (R) are true but (R) is not the correct explanation of (A).

(c) (A) is true but (R) is false.

(d) (A) is false but (R) is true

Sol : OD 2012

We have $$f(x) = \begin{cases} x^3 - 3, & x \leq 2 \\ x^2 + 1, & x > 2 \end{cases}$$

Function $f(x)$ is continuous at $x = 2$, if and only if

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} f(x) = f(2)$$

Now, $$\lim_{x \to 2^+} f(x) = \lim_{h \to 0} f(2 + h)$$

$$= \lim_{h \to 0} (2 + h)^2 + 1$$

$$= \lim_{h \to 0} 4 + h^2 + 4h + 1$$

$$= 4 + 1$$

$$= 5$$

and $$\lim_{x \to 2^-} f(x) = \lim_{h \to 0} f(2 - h)$$

$$= \lim_{h \to 0} (2 - h)^3 - 3$$

$$= \lim_{h \to 0} 2^3 - h^3 - 6h^2 + 12h - 3$$

$$= 8 - 3$$

$$= 5$$

$$f(2) = 2^3 - 3$$

$$= 8 - 3$$

$$= 5$$

Since, $$f(2) = \lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x)$$

Therefore, given function is continuous at $x = 2$

Both (A) and (R) are true and (R) is the correct explanation of (A).

Thus (a) is correct option.

46. **Assertion (A) :** If $x = at^2$ and $y = 2at$, then $\left.\frac{d^2y}{dx^2}\right|_{t=2} = \frac{-1}{16a}$

Reason (R) : $\frac{d^2y}{dx^2} = \left(\frac{dy}{dt}\right)^2 \times \left(\frac{dt}{dx}\right)^2$

(a) Both (A) and (R) are true and (R) is the correct explanation of (A).

(b) Both (A) and (R) are true but (R) is not the correct explanation of (A).

(c) (A) is true but (R) is false.

(d) (A) is false but (R) is true

Sol : Foreign 2009, OD 2007

We have, $\quad x = at^2,\ y = 2at$

$$\frac{dx}{dt} = 2at \text{ and } \frac{dy}{dt} = 2a$$

Then, $\dfrac{dy}{dx} = \dfrac{d(e^{\sqrt{3x}})}{dx}$

$= \dfrac{3 \cdot e^{\sqrt{3x}}}{2 \cdot \sqrt{3x}}$

$= \dfrac{3e^{\sqrt{3x}}}{2\sqrt{3x}}$

51. If $y = \cos(\sqrt{3x})$, then find $\dfrac{dy}{dx}$.

Sol : SQP 2019

We have $y = \cos(\sqrt{3x})$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\{\cos(\sqrt{3x})\}$$
$$= -\sin(\sqrt{3x}) \cdot \frac{d}{dx}(\sqrt{3x})$$
$$= -\sin(\sqrt{3x}) \cdot \frac{1}{2\sqrt{3x}} \cdot 3$$
$$= -\frac{3\sin\sqrt{3x}}{2\sqrt{3x}}$$

52. If $f(x) = x + 1$, find $\dfrac{d}{dx}(fof)(x)$.

Sol : Delhi 2019, OD 2010

We have $f(x) = x + 1$

$$f(f(x)) = f(x) + 1$$
$$fof(x) = x + 1 + 1$$
$$= x + 2$$

Now, $\dfrac{d}{dx}[fof(x)] = \dfrac{d}{dx}(x + 2) = 1$

53. If $f(x) = x + 7$ and $g(x) = x - 7$, $x \in R$, then find the values of $\dfrac{d}{dx}(fog)x$.

Sol : Foreign 2019

We have $f(x) = x + 7$,

$$g(x) = x - 7,\ x \in R$$

Now, $(fog)(x) = f[g(x)]$

$$= f(x - 7)$$
$$= (x - 7) + 7$$
$$(fog)(x) = x$$

Differentiating w.r.t. x, we get

$$\frac{d}{dx}(fof)(x) = \frac{d}{dx}(x) = 1$$

54. If $y = x|x|$, find $\dfrac{dy}{dx}$ for $x < 0$.

Sol : Comp 2019

We have $y = x|x|$

When, $x < 0$, then $|x| = -x$. Thus

$$y = x(-x) = -x^2$$
$$\frac{dy}{dx} = -2x$$

55. Differentiate $\tan^{-1}\left(\dfrac{1 + \cos x}{\sin x}\right)$ with respect to x.

Sol : OD 2018, Delhi 2016

We have $y = \tan^{-1}\left(\dfrac{1 + \cos x}{\sin x}\right)$

$$= \tan^{-1}\left(\frac{2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}\right)$$
$$= \tan^{-1}\left(\frac{\cos\frac{x}{2}}{\sin\frac{x}{2}}\right)$$
$$= \tan^{-1}\left(\cot\frac{x}{2}\right)$$
$$= \frac{\pi}{2} - \cot^{-1}\left(\cot\frac{x}{2}\right)$$
$$= \frac{\pi}{2} - \frac{x}{2}$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 0 - \frac{1}{2} = -\frac{1}{2}$$

56. Differentiate $\tan^{-1}\left(\dfrac{\cos x - \sin x}{\cos x + \sin x}\right)$ with respect to x.

Sol : Comp 2018

Let $y = \tan^{-1}\left(\dfrac{\cos x - \sin x}{\cos x + \sin x}\right)$

$$= \tan^{-1}\left(\frac{1 - \tan x}{1 + \tan x}\right)$$
$$= \tan^{-1}\left(\frac{\tan\frac{\pi}{4} - \tan x}{1 + \tan\frac{\pi}{4}\tan x}\right)$$
$$= \tan^{-1}\left[\tan\left(\frac{\pi}{4} - x\right)\right]$$
$$= \frac{\pi}{4} - x$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = -1$$

57. Determine the value of k so that the function

$$f(x) = \begin{cases} kx^2, & \text{if } x \le 2 \\ 3, & \text{if } x > 2 \end{cases}$$

is continuous.

Sol : OD 2017

For the function to be continuous

$$f(2^-) = f(2) = f(2^+)$$

As $f(2^+) = 3$

$$f(2) = 3 = k(2^2)$$
$$k = \frac{3}{4}$$

$$\sin^{-1}x - \sin^{-1}y = 2\cot^{-1}a$$

Differentiate both sides, we have

$$\frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-y^2}}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}} \qquad \text{Hence Proved}$$

63. If $y = \sqrt{ax+b}$, prove that $y\left(\frac{d^2y}{dx^2}\right) + \left(\frac{dy}{dx}\right)^2 = 0$

Sol : OD 2023

We have $\quad y = \sqrt{ax+b}$

Squaring both sides,

$$y^2 = ax + b$$

Differentiating w.r.t x we have

$$2y\frac{dy}{dx} = a$$

Again, differentiating w.r.t x we have

$$2y\left(\frac{d^2y}{dx^2}\right) + 2\left(\frac{dy}{dx}\right)\left(\frac{dy}{dx}\right) = 0$$

$$y\left(\frac{d^2y}{dx^2}\right) + \left(\frac{dy}{dx}\right)^2 = 0 \qquad \text{Hence Proved.}$$

64. If $f(x) = \begin{cases} ax+b & ; \quad 0 < x \le 1 \\ 2x^2 - x; & 1 < x < 2 \end{cases}$ is a differentiable function in (0, 2), then find the values of a and b.

Sol : OD 2023

We have $\quad f(x) = \begin{cases} ax+b & ; \quad 0 < x \le 1 \\ 2x^2 - x; & 1 < x < 2 \end{cases}$

Function is differentiable and continuous at $x = 1$. Therefore

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1)$$

$$a(1) + b = 2(1)^2 - 1$$

$$a + b = 1$$

Differentiating $f(x)$ w.r.t x we have

$$f'(x) = \begin{cases} a & ; \quad 0 < x < 1 \\ 4x - 1 & ; \quad 1 < x < 2 \end{cases}$$

Now, $\quad f'(1^-) = f'(1^+)$

$$a = 4(1) - 1 = 3$$

$$b = -2$$

65. If the function f defined as

$$f(x) = \begin{cases} \frac{x^2-9}{x-3}, & x \ne 3 \\ k, & x = 3 \end{cases}$$

is continuous at $x = 3$, find the value of k.

Sol : OD 2020

We have $\quad f(x) = \begin{cases} \frac{x^2-9}{x-3}, & x \ne 3 \\ k, & x = 3 \end{cases}$

Since, $f(x)$ is continuous at $x = 3$, we can write

$$\lim_{x \to 3} f(x) = f(3)$$

$$\lim_{x \to 3} \frac{x^2-9}{x-3} = k$$

$$\lim_{x \to 3} \frac{(x-3)(x+3)}{(x-3)} = k$$

$$\lim_{x \to 3}(x+3) = k$$

$$3 + 3 = k$$

$$k = 6$$

66. If $x = a\cos\theta$; $y = b\sin\theta$, then find $\frac{d^2y}{dx^2}$.

Sol : OD 2020

We have, $\quad x = a\cos\theta,\ y = b\sin\theta$

$$\frac{dx}{d\theta} = -a\sin\theta,\ \frac{dy}{d\theta} = b\cos\theta$$

Now $\quad \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{b\cos\theta}{-a\sin\theta} = \frac{-b}{a}\cot\theta$

Again differentiating w.r.t x we have

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right) \\ &= \frac{d}{dx}\left(\frac{-b}{a}\cot\theta\right) \\ &= \frac{-b}{a}(-\operatorname{cosec}^2\theta)\frac{d\theta}{dx} \\ &= \frac{b}{a}\operatorname{cosec}^2\theta \times \frac{1}{(-a\sin\theta)} \qquad \left[\frac{d\theta}{dx} = \frac{1}{\frac{dx}{d\theta}}\right] \\ &= \frac{-b}{a^2}\operatorname{cosec}^3\theta \end{aligned}$$

67. Find the differential of $\sin^2 x$ w.r.t. $e^{\cos x}$.

Sol : OD 2020

Let $\quad y = \sin^2 x$ and $z = e^{\cos x}$

Differentiating y and z w.r.t, x, we get

$$\frac{dy}{dx} = 2\sin x\cos x$$

and $\quad \frac{dz}{dx} = e^{\cos x}(-\sin x) = -(\sin x)e^{\cos x}$

Now, $\quad \frac{dy}{dz} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}} = \frac{2\sin x\cos x}{-(\sin x)e^{\cos x}} = \frac{-2\cos x}{e^{\cos x}}$

Since, $f(x)$ is continuous at $x = 0$.

$$f(0) = LHL \Rightarrow -1 = k$$
$$k = -1$$

72. Find the value of k, so that the following function is continuous at $x = \pi$.

$$f(x) = \begin{cases} kx+1, & \text{if } x \le \pi \\ \cos x & \text{if } x > \pi \end{cases}$$

Sol : OD 2011

We have $f(x) = \begin{cases} kx+1, & \text{if } x \le \pi \\ \cos x & \text{if } x > \pi \end{cases}$

If $f(x)$ is continuous at $x = \pi$ then we have

$$\text{LHL} = \text{RHL} = f(\pi) \qquad ...(i)$$

$$\begin{aligned} \text{LHL} &= \lim_{x \to \pi^-} f(x) \\ &= \lim_{x \to \pi^-} (kx+1) \\ &= \lim_{h \to 0} [k(\pi - h) + 1] \\ &= \lim_{h \to 0} (k\pi - kh + 1) \\ &= k\pi + 1 \end{aligned}$$

and
$$\begin{aligned} \text{RHL} &= \lim_{x \to \pi^+} f(x) \\ &= \lim_{x \to \pi^+} \cos x \\ &= \lim_{h \to 0} \cos(\pi + h) \\ &= \cos \pi \\ &= -1 \end{aligned}$$

Now, from Eq. (i), we have

$$\text{LHL} = \text{RHL}$$
$$k\pi + 1 = -1 \Rightarrow k\pi = -2$$
$$k = -\frac{2}{\pi}$$

73. Discuss the continuity of the functions $f(x)$ at $x = \frac{1}{2}$, when $f(x)$ is defined as follows

$$f(x) = \begin{cases} \frac{1}{2} + x, & 0 \le x < \frac{1}{2} \\ 1, & x = \frac{1}{2} \\ \frac{3}{2} + x, & \frac{1}{2} < x \le 1 \end{cases}$$

Sol : Foreign 2011, Comp 2010

Here, we have to find LHL, RHL and $f(\frac{1}{2})$.

If $\text{LHL} = \text{RHL} = f\left(\frac{1}{2}\right)$,

then we say that $f(x)$ is continuous at $x = \frac{1}{2}$, otherwise $f(x)$ is discontinuous at $x = \frac{1}{2}$.

We have $f(x) = \begin{cases} \frac{1}{2} + x, & 0 \le x < \frac{1}{2} \\ 1, & x = \frac{1}{2} \\ \frac{3}{2} + x, & \frac{1}{2} < x \le 1 \end{cases}$

Now,
$$\begin{aligned} \text{LHL} &= \lim_{x \to \frac{1}{2}^-} f(x) \\ &= \lim_{x \to \frac{1}{2}^-} \left(\frac{1}{2} + x\right) \\ &= \lim_{h \to 0} \left(\frac{1}{2} + \frac{1}{2} - h\right) \\ &= \lim_{h \to 0} (1 - h) = 1 \end{aligned}$$

and
$$\begin{aligned} \text{RHL} &= \lim_{x \to \frac{1}{2}^+} f(x) \\ &= \lim_{x \to \frac{1}{2}^+} \left(\frac{3}{2} + x\right) \\ &= \lim_{h \to 0} \left(\frac{3}{2} + \frac{1}{2} + h\right) \\ &= \lim_{h \to 0} (2 + h) = 2 \end{aligned}$$

$$\text{LHL} \ne RHL \text{ at } x = \frac{1}{2}$$

Thus $f(x)$ is discontinuous at $x = \frac{1}{2}$.

74. Find $\frac{dy}{dx}$ at $x = 1,\ y = \frac{\pi}{4}$ if $\sin^2 y + \cos xy = K$.

Sol : Delhi 2017

We have, $\sin^2 y + \cos xy = K$

Differentiating w.r.t. x, we get

$$\begin{aligned} \frac{d}{dx}(\sin^2 y + \cos xy) &= \frac{d}{dx}(K) \\ \frac{d}{dx}(\sin^2 y) + \frac{d}{dx}(\cos xy) &= 0 \\ 2\sin y \cos y \frac{dy}{dx} + (-\sin xy)\frac{d}{dx}(xy) &= 0 \\ \sin 2y \frac{dy}{dx} - \sin xy\left(x\frac{dy}{dx} + y \cdot 1\right) &= 0 \\ \sin 2y \frac{dy}{dx} - x \sin xy \frac{dy}{dx} &= y \sin xy \\ \frac{dy}{dx} &= \frac{y \sin(xy)}{\sin 2y - x \sin(xy)} \end{aligned}$$

At $x = 1,\ y = \frac{\pi}{4}$

$$\begin{aligned} \left(\frac{dy}{dx}\right) &= \frac{\frac{\pi}{4}\sin\left(1 \cdot \frac{\pi}{4}\right)}{\sin\left(2 \cdot \frac{\pi}{4}\right) - 1 \sin\left(1 \cdot \frac{\pi}{4}\right)} \\ &= \frac{\frac{\pi}{4}\sin\left(\frac{\pi}{4}\right)}{\sin\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{2}\right)} \\ &= \frac{\frac{\pi}{4}\left(\frac{1}{\sqrt{2}}\right)}{1 - \frac{1}{\sqrt{2}}} = \frac{\pi}{4(\sqrt{2} - 1)} \end{aligned}$$

75. If $y = \sin^{-1}(6x\sqrt{1 - 9x^2})$, $-\frac{1}{3\sqrt{2}} < x < \frac{1}{3\sqrt{2}}$, then find $\frac{dy}{dx}$.

Sol : Delhi 2017

We have $y = \sin^{-1}(6x\sqrt{1 - 9x^2})$

or $y = \sin^{-1}(2 \cdot 3x\sqrt{1 - (3x)^2})$

$$\frac{d^2y}{dx^2} = \cos(\sin x)\cdot(-\sin x) + \cos x(-\sin(\sin x))\cdot\cos x$$

$$= \frac{1}{\cos x}\cdot\left(\frac{dy}{dx}\right)(-\sin x) - y\cos^2 x$$

$$\frac{d^2y}{dx^2} = -\tan x\frac{dy}{dx} - y\cos^2 x$$

$$\frac{d^2y}{dx^2} + \tan x\frac{dy}{dx} + y\cos^2 x = 0$$

Hence proved.

80. If $(x^2+y^2)^2 = xy$, find $\frac{dy}{dx}$.

Sol : Comp 2018, Delhi 2014

We have, $(x^2+y^2)^2 = xy$

Differentiating w.r.t. x, we get

$$2(x^2+y^2)\left[2x+2y\frac{dy}{dx}\right] = \left[x\frac{dy}{dx}+y\right]$$

$$4(x^2+y^2)\left(x+y\frac{dy}{dx}\right) = \left(y+x\frac{dy}{dx}\right)$$

$$4(x^2+y^2)x + 4(x^2+y^2)y\frac{dy}{dx} = y + x\frac{dy}{dx}$$

$$\frac{dy}{dx}[4(x^2+y^2)y - x] = y - 4x(x^2+y^2)$$

$$\frac{dy}{dx} = \frac{y-4x(x^2+y^2)}{4(x^2+y^2)y-x}$$

$$\frac{dy}{dx} = -\frac{[y-4x(x^2+y^2)]}{[x-4y(x^2+y^2)]}]$$

81. If $x = a(2\theta - \sin 2\theta)$ and $y = a(1-\cos 2\theta)$ find $\frac{dy}{dx}$ when $\theta = \frac{\pi}{3}$.

Sol : Delhi 2018

We have, $x = a(2\theta - \sin 2\theta)$

and $y = a(1-\cos 2\theta)$

Differentiating above both equation w.r.t. θ, we get

$$\frac{dx}{d\theta} = a(2-\cos 2\theta\cdot 2) = 2a(1-\cos 2\theta)$$

and $$\frac{dy}{d\theta} = a(0+2\sin 2\theta) = 2a\sin 2\theta$$

From Eq. (i), we get

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

$$= \frac{2a\sin 2\theta}{2a(1-\cos 2\theta)}$$

$$= \frac{\sin 2\theta}{1-\cos 2\theta}$$

$$= \frac{2\sin\theta\cos\theta}{2\sin^2\theta} \qquad \cos 2\theta = 1-2\sin^2\theta$$

$$= \cot\theta$$

Now, $$\left(\frac{dy}{dx}\right)_{\theta=\pi/3} = \cos\frac{\pi}{3} = \frac{1}{\sqrt{3}}$$

82. If $\sin y = x\cos(a+y)$, then show that

$$\frac{dy}{dx} = \frac{\cos^2(a+y)}{\cos a}$$

Also, show that $\frac{dy}{dx} = \cos a$, when $x = 0$.

Sol : SQP 2018, Delhi 2010

We have $\sin y = x\cos(a+y)$ ...(1)

$$x = \frac{\sin y}{\cos(a+y)}$$

Differentiating w.r.t. x, we get

$$\frac{dx}{dy} = \frac{\cos(a+y)\frac{d}{dy}(\sin y) - \sin y\frac{d}{dy}\cos(a+y)}{\cos^2(a+y)}$$

$$= \frac{\cos(a+y)\cos y + \sin y\sin(a+y)}{\cos^2(a+y)}$$

$$= \frac{\cos(a+y-y)}{\cos^2(a+y)}$$

$$= \frac{\cos a}{\cos^2(a+y)}$$

$$\frac{dy}{dx} = \frac{\cos^2(a+y)}{\cos a}$$

Substituting $x = 0$ in Eq. (1), we get $y = 0$

Now, $$\frac{dy}{dx} = \frac{\cos^2(a+0)}{\cos a}$$

$$= \frac{\cos^2 a}{\cos a}$$

or $$\frac{dy}{dx} = \cos a$$

Hence proved.

83. If $x = ae^t(\sin t + \cos t)$ and $y = ae^t(\sin t - \cos t)$, then prove that $\frac{dy}{dx} = \frac{x+y}{x-y}$.

Sol : OD 2019

We have $x = ae^t(\sin t + \cos t)$

and $y = ae^t(\sin t - \cos t)$

Differentiating above both equation w.r.t. t, we get

Now $$\frac{dx}{dt} = a[e^t(\cos t - \sin t) + e^t(\sin t + \cos t)]$$

$$= -y + x = x - y$$

$$\frac{dy}{dt} = a[e^t(\cos t + \sin t) + e^t(\sin t - \cos t)]$$

$$= x + y$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\frac{dy}{dx} = \frac{x+y}{x-y}$$

Hence proved.

84. Differentiate $x^{\sin x} + (\sin x)^{\cos x}$ with respect to x.

Differentiating w.r.t. x, we get

$$\frac{1}{v}\cdot\frac{dv}{dx} = \frac{x}{y}\frac{dy}{dx}+\log y$$

$$\frac{dv}{dx} = xy^{x-1}\frac{dy}{dx}+y^x\log y$$

Now, Eq. (1) becomes,

$$y\cdot x^{y-1}+x^y\cdot\log x\frac{dy}{dx}+xy^{x-1}\frac{dy}{dx}+y^x\log y = 0$$

$$\frac{dy}{dx}(x^y\log x+xy^{x-1}) = -y^x\log y-y\cdot x^{y-1}$$

$$\frac{dy}{dx} = \frac{-y^x\cdot\log y-y\cdot x^{y-1}}{x^y\cdot\log x+x\cdot y^{x-1}}$$

89. If $e^y(x+1)=1$, then show that $\frac{d^2y}{dx^2}=\left(\frac{dy}{dx}\right)^2$.

Sol : OD 2017

We have $e^y(x+1) = 1$

Taking log both sides, we get

$$\log[e^y(x+1)] = \log 1$$

$$\log e^y+\log(x+1) = 0$$

$$y+\log(x+1) = 0 \qquad [\log e^y = y]$$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx}+\frac{1}{x+1} = 0 \qquad ...(1)$$

Again, differentiating both sides w.r.t. x, we get

$$\frac{d^2y}{dx^2}-\frac{1}{(x+1)^2} = 0$$

$$\frac{d^2y}{dx^2}-\left(-\frac{dy}{dx}\right)^2 = 0 \qquad \text{from Eq. (1)}$$

$$\frac{d^2y}{dx^2}-\left(\frac{dy}{dx}\right)^2 = 0$$

$$\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2 \qquad \text{Hence proved.}$$

90. If $y=\tan^{-1}\left(\frac{a}{x}\right)+\log\sqrt{\frac{x-a}{x+a}}$, prove that

$$\frac{dy}{dx} = \frac{2a^3}{x^4-a^4}$$

Sol : Comp 2014

We have $y = \tan^{-1}\left(\frac{a}{x}\right)+\log\sqrt{\frac{x-a}{x+a}}$

$$= \tan^{-1}\left(\frac{a}{x}\right)+\log\left(\frac{x-a}{x+a}\right)^{1/2}$$

$$y = \tan^{-1}\left(\frac{a}{x}\right)+\frac{1}{2}[\log(x-a)-\log(x+a)]$$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{1+\frac{a^2}{x^2}}\cdot\left(\frac{-a}{x^2}\right)+\frac{1}{2}\left[\frac{1}{x-a}-\frac{1}{x+a}\right]$$

$$= \frac{-a}{x^2+a^2}+\frac{1}{2}\left[\frac{x+a-x+a}{(x-a)(x+a)}\right]$$

$$= \frac{-a}{x^2+a^2}+\frac{a}{x^2-a^2}$$

$$= \frac{-x^2a+a^3+x^2a+a^3}{(x^2+a^2)(x^2-a^2)}$$

$$\frac{dy}{dx} = \frac{2a^3}{x^4-a^4}$$

91. If $x=2\cos\theta-\cos2\theta$ and $y=2\sin\theta-\sin2\theta$ then prove that $\frac{dy}{dx}=\tan\left(\frac{3\theta}{2}\right)$.

Sol : Comp 2013, OD 2009

We have $x = 2\cos\theta-\cos2\theta$

and $y = 2\sin\theta-\sin2\theta$

Differentiating both sides w.r.t. θ, we get

$$\frac{dx}{d\theta} = -2\sin\theta+2\sin2\theta$$

and
$$\frac{dy}{d\theta} = 2\cos\theta-2\cos2\theta$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

$$= \frac{2\cos\theta-\cos2\theta}{2(-\sin\theta+\sin2\theta)}$$

$$= \frac{2\sin\left(\frac{2\theta+\theta}{2}\right)\sin\left(\frac{2\theta-\theta}{2}\right)}{2\left[\cos\left(\frac{\theta+2\theta}{2}\right)\sin\left(\frac{2\theta-\theta}{2}\right)\right]}$$

$$= \frac{\sin\left(\frac{3\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{3\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)}$$

$$= \tan\left(\frac{3\theta}{2}\right) \qquad \text{Hence proved.}$$

92. If $y=\log[x+\sqrt{x^2+1}]$, then prove that

$$(x^2+1)\frac{d^2y}{dx^2}+x\frac{dy}{dx} = 0.$$

Sol : Foreign 2011

We have $y = \log[x+\sqrt{x^2+1}]$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{x+\sqrt{x^2+1}}\frac{d}{dx}(x+\sqrt{x^2+1})$$

$$= \frac{1}{x+\sqrt{x^2+1}}\left(1+\frac{2x}{2\sqrt{x^2+1}}\right)$$

$$= \frac{1}{x+\sqrt{x^2+1}}\left(\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}}\right)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2+1}}$$

$$\left[\frac{dy}{dx}\right]^2 = \frac{1}{x^2+1}$$

$$(x^2+1)\left[\frac{dy}{dx}\right]^2 = 1$$

Again, Differentiating both sides w.r.t. x, we get

$$= a(\theta\sin\theta) \qquad ...(ii)$$

Now, $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$ From (i) and (ii)

$$= \frac{a\theta\sin\theta}{a\theta\cos\theta}$$

$$= \frac{\sin\theta}{\cos\theta}$$

$$\frac{dy}{dx} = \tan\theta$$

Now $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$

$$= \frac{d}{dx}(\tan\theta)$$

$$= \sec^2\theta\cdot\frac{d\theta}{dx}$$

$$= \frac{\sec^2\theta}{\frac{dx}{d\theta}}$$

$$= \frac{\sec^2\theta}{a\theta\cos\theta}$$

$$= \frac{\sec^3\theta}{a\theta}$$

97. If $x = a(\theta - \sin\theta)$ and $y = a(1+\cos\theta)$, then find $\dfrac{dy}{dx}$ at $\theta = \dfrac{\pi}{3}$.

Sol : SQP 2010

We have $x = a(\theta - \sin\theta)$

and $y = a(1+\cos\theta)$

Differentiating both sides w.r.t. θ, we get

$$\frac{dx}{d\theta} = a(1-\cos\theta) \text{ and } \frac{dy}{d\theta} = -a\sin\theta$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-a\sin\theta}{a(1-\cos\theta)}$$

$$\frac{dy}{dx} = \frac{-2a\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{a\times 2\sin^2\frac{\theta}{2}}$$

$$\frac{dy}{dx} = -\cot\frac{\theta}{2}$$

Substituting $\theta = \dfrac{\pi}{3}$, we get

$$\left[\frac{dy}{dx}\right]_{\theta=\frac{\pi}{3}} = -\cot\frac{\pi}{6} = -\sqrt{3}$$

Hence, $\dfrac{dy}{dx}$ at $\theta = \dfrac{\pi}{3}$ is $-\sqrt{3}$.

98. If $y = (\sin x - \cos x)^{(\sin x - \cos x)}$, $\frac{\pi}{4} < x < \frac{3\pi}{4}$, then find $\frac{dy}{dx}$.

Sol : OD 2009, Comp 2007

We have $y = (\sin x - \cos x)^{(\sin x - \cos x)}$, $\dfrac{\pi}{4} < x < \dfrac{3\pi}{4}$

Taking log both sides, we get

$$\log y = \log(\sin x - \cos x)^{(\sin x - \cos x)}$$

$$\log y = (\sin x - \cos x)\cdot\log(\sin x - \cos x)$$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = (\sin x - \cos x)\times\frac{d}{dx}\log(\sin x - \cos x) + \log(\sin x - \cos x)\times\frac{d}{dx}(\sin x - \cos x)$$

$$= (\sin x - \cos x)\cdot\frac{1}{(\sin x - \cos x)}\cdot\frac{d}{dx}(\sin x - \cos x) + \log(\sin x - \cos x)\cdot(\cos x + \sin x)$$

$$= (\cos x + \sin x) + (\cos x + \sin x)\log(\sin x - \cos x)$$

$$\frac{1}{y}\frac{dy}{dx} = (\cos x + \sin x)[1 + \log(\sin x - \cos x)]$$

$$\frac{dy}{dx} = y(\cos x + \sin x)[1 + \log(\sin x - \cos x)]$$

$$\frac{dy}{dx} = (\sin x - \cos x)^{(\sin x - \cos x)}\cdot(\cos x + \sin x)[1 + \log(\sin x - \cos x)]$$

99. If $y = (\cot^{-1}x)^2$, then show that

$$(x^2+1)^2\frac{d^2y}{dx^2} + 2x(2x^2+1)\frac{dy}{dx} = 2.$$

Sol : Delhi 2009

We have $y = (\cot^{-1}x)^2$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = 2\cot^{-1}x\cdot\frac{-1}{1+x^2}$$

$$\frac{dy}{dx} = -\frac{2\cot^{-1}x}{1+x^2}$$

$$(1+x^2)\frac{dy}{dx} = -2\cot^{-1}x$$

Again, differentiating both sides w.r.t. x, we get

$$(1+x^2)\cdot\frac{d}{dx}\left(\frac{dx}{dy}\right) + \frac{dy}{dx}\cdot\frac{d}{dx}(1+x^2) = \frac{d}{dx}(-2\cot^{-1}x)$$

$$(1+x^2)\cdot\frac{d^2y}{dx^2} + \frac{dy}{dx}\cdot 2x = \frac{2}{1+x^2}$$

$$(1+x^2)\frac{d^2y}{dx^2} + \frac{dy}{dx}\cdot 2x\cdot(1+x^2) = \frac{1}{1+x^2}\cdot(1+x^2)$$

$$(1+x^2)^2\frac{d^2y}{dx^2} + 2x(1+x^2)\frac{dy}{dx} = 2 \qquad \text{Hence proved.}$$

100. If $x = \tan\left(\dfrac{1}{a}\log y\right)$, then show that

$$(1+x^2)\frac{d^2y}{dx^2} + (2x-a)\frac{dy}{dx} = 0.$$

Sol : Comp 2009

We have $x = \tan\left(\dfrac{1}{a}\log y\right)$

$$\tan^{-1}x = \frac{1}{a}\log y$$

$$a\tan^{-1}x = \log y$$

Differentiating both sides w.r.t. x, we get

$$\left(\frac{dy}{dx}\right)^2 = \frac{1}{1-x^2}$$

$$(1-x^2)\left(\frac{dy}{dx}\right)^2 = 1$$

Differentiating again w.r.t. x, we get

$$-2x\left(\frac{dy}{dx}\right)^2 + (1-x^2)2\frac{dy}{dx}\frac{d^2y}{dx^2} = 1$$

$$-x\frac{dy}{dx} + (1-x^2)\frac{d^2y}{dx^2} = 1$$

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 1 \qquad \text{Hence Proved}$$

105. If $y = (\tan^{-1}x)^2$, then show that

$$(x^2+1)^2\frac{d^2y}{dx^2} + 2x(x^2+1)\frac{dy}{dx} = 2.$$

Sol : SQP 2012

We have $y = (\tan^{-1}x)^2$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = 2\tan^{-1}x\cdot\frac{1}{1+x^2}$$

$$\frac{dy}{dx} = \frac{2\tan^{-1}x}{1+x^2}$$

$$(1+x^2)\frac{dy}{dx} = 2\tan^{-1}x$$

Again, differentiating both sides w.r.t. x, we get

$$(1+x^2)\cdot\frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx}\cdot\frac{d}{dx}(1+x^2) = \frac{d}{dx}(2\tan^{-1}x)$$

$$(1+x^2)\cdot\frac{d^2y}{dx^2} + \frac{dy}{dx}\cdot 2x = \frac{2}{1+x^2}$$

$$(1+x^2)\frac{d^2y}{dx^2} + \frac{dy}{dx}\cdot 2x\cdot(1+x^2) = \frac{2}{1+x^2}\cdot(1+x^2)$$

$$(1+x^2)^2\frac{d^2y}{dx^2} + 2x(1+x^2)\frac{dy}{dx} = 2 \quad \text{Hence proved.}$$

106. If $x^y = e^{x-y}$, then prove that then prove that

$$\frac{dy}{dx} = \frac{\log x}{\{\log(ex)\}^2}$$

Sol : OD 2013, Delhi 2010

We have $x^y = e^{x-y}$

Taking log both sides, we get

$$y\log_e x = (x-y)\log_e e$$

$$y\log_e x = x-y$$

$$y(1+\log x) = x$$

$$y = \frac{x}{1+\log x}$$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{(1+\log x)\frac{d}{dx}(x) - x\frac{d}{dx}(1+\log x)}{(1+\log x)^2}$$

$$= \frac{1+\log x - x\cdot\frac{1}{x}}{(1+\log x)^2}$$

$$= \frac{1+\log x - 1}{(1+\log x)^2}$$

Hence, $$\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$$

Also, it can be written as

$$\frac{dy}{dx} = \frac{\log x}{(\log_e e + \log x)^2}$$

$$\frac{dy}{dx} = \frac{\log x}{\{\log(ex)\}^2} \qquad \text{Hence proved.}$$

107. If $y^x = e^{y-x}$, then prove that $\frac{dy}{dx} = \frac{(1+\log y)^2}{\log y}$.

Sol : OD 2013, Delhi 2010

We have $y^x = e^{y-x}$

Taking log both sides, we get

$$x\log_e y = (y-x)\log_e e$$

$$x\log y = y-x$$

$$x\log y + x = y$$

$$x(1+\log y) = y$$

$$x = \frac{y}{1+\log y}$$

Differentiating both sides w.r.t. y, we get

$$\frac{dx}{dy} = \frac{(1+\log y)\frac{d}{dy}(y) - y\frac{d}{dy}(1+\log y)}{(1+\log y)^2}$$

$$= \frac{1+\log y - y\cdot\frac{1}{y}}{(1+\log y)^2}$$

$$= \frac{1+\log y - 1}{(1+\log y)^2}$$

$$\frac{dx}{dy} = \frac{\log y}{(1+\log y)^2}$$

Hence, $$\frac{dy}{dx} = \frac{(1+\log y)^2}{\log y} \qquad \text{Hence proved.}$$

108. If $x = \sin t$ and $y = \sin pt$, then prove that $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2y = 0$.

Sol : Foreign 2016

We have $x = \sin t$ and $y = \sin pt$

Differentiating x and y separately w.r.t. t, we get

$$\frac{dx}{dt} = \cos t \text{ and } \frac{dy}{dt} = p\cos pt$$

Now $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{p\cos pt}{\cos t}$$

Now, differentiating above w.r.t. x, we get

$$\frac{d^2y}{dx^2} = p\frac{[\cos t(-\sin pt\cdot p) - \cos pt(-\sin t)]}{\cos^2 t}\cdot\frac{dt}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{p[\cos pt\cdot\sin t - \cos t\sin pt\cdot p]}{\cos^2 t}\cdot\frac{1}{\cos t}$$

113. If $y = e^x \sin x$, then prove that $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 2y = 0$.

Sol : Foreign 2010

We have, $y = e^x \sin x$ ...(1)

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = e^x \cdot \frac{d}{dx}(\sin x) + \sin x \cdot \frac{d}{dx}(e^x)$$

$$= e^x \cdot \cos x + \sin x \cdot e^x$$

$$\frac{dy}{dx} = e^x(\cos x + \sin x) \quad ...(3)$$

Again, differentiating both sides w.r.t. x, we get

$$\frac{d^2y}{dx^2} = e^x \cdot \frac{d}{dx}(\cos x + \sin x) + (\cos x + \sin x) \cdot \frac{d}{dx}(e^x)$$

$$= e^x(-\sin x + \cos x) + (\cos x + \sin x) \cdot e^x$$

$$= e^x[-\sin x + \cos x + \cos x + \sin x]$$

$$= 2\cos x\, e^x$$

$$\frac{d^2y}{dx^2} = 2\cos x\, e^x \quad ...(3)$$

Now, from eqs. (1), (2) and (3) we have

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 2e^x\cos x - 2e^x(\cos x + \sin x) + 2e^x \sin x$$

$$= 2e^x \cos x - 2e^x \cos x - 2e^x \sin x + 2e^x \sin x$$

$$= 0$$

Hence proved.

114. If $y = \cos^{-1}\left[\dfrac{3x + 4\sqrt{1-x^2}}{5}\right]$, then find $\dfrac{dy}{dx}$.

Sol : OD 2010

We have $y = \cos^{-1}\left[\dfrac{3x + 4\sqrt{1-x^2}}{5}\right]$

Substituting $x = \sin\theta$, then $\theta = \sin^{-1}x$ we have

$$y = \cos^{-1}\left[\frac{3\sin\theta + 4\sqrt{1-\sin^2\theta}}{5}\right]$$

$$= \cos^{-1}\left[\frac{3\sin\theta + 4\cos\theta}{5}\right]$$

$$= \cos^{-1}\left[\frac{3}{5}\sin\theta + \frac{4}{5}\cos\theta\right]$$

Let $\dfrac{3}{5} = \sin\alpha$, then $\dfrac{4}{5} = \cos\alpha$

Now $y = \cos^{-1}[\sin\alpha\sin\theta + \cos\alpha\cos\theta]$

$$= \cos^{-1}[\cos(\theta - \alpha)]$$

$$= \theta - \alpha$$

or $y = \sin^{-1}x - \alpha$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - 0$$

$$= \frac{1}{\sqrt{1-x^2}}$$

115. If $x\sin(a+y) + \sin a\cos(a+y) = 0$, then prove that

$$\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$$

Sol : OD 2013

We have $x\sin(a+y) + \sin a\cos(a+y) = 0$

$$x = \frac{-\sin a\cos(a+y)}{\sin(a+y)}$$

Differentiating both sides w.r.t. y, we get

$$\frac{dx}{dy} = \frac{-\left[\begin{array}{l}\sin(a+y)\frac{d}{dy}\{\sin a\cos(a+y)\} \\ \qquad - \sin a\cos(a+y)\frac{d}{dy}\{\sin(a+y)\}\end{array}\right]}{\sin^2(a+y)}$$

$$= \frac{\sin(a+y)\cdot\sin a\sin(a+y) + \sin a\cos(a+y)\cos(a+y)}{\sin^2(a+y)}$$

$$= \frac{\sin a}{\sin^2(a+y)}\{\sin^2(a+y) + \cos^2(a+y)\}$$

$$= \frac{\sin a}{\sin^2(a+y)} \cdot 1$$

$$\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$$

Hence proved.

116. Differentiate the following with respect to x. $\sin^{-1}\left[\dfrac{2^{x+1}\cdot 3^x}{1+(36)^x}\right]$

Sol : OD 2013, Delhi 2010

We have $y = \sin^{-1}\left[\dfrac{2^{x+1}\cdot 3^x}{1+(36)^x}\right]$

$$= \sin^{-1}\left[\frac{2\cdot 2^x\cdot 3^x}{1+(6^2)^x}\right]$$

$$= \sin^{-1}\left[\frac{2\cdot 6^x}{1+(6^x)^2}\right]$$

Substituting $6^x = \tan\theta \Rightarrow \theta = \tan^{-1}(6^x)$ we have

$$y = \sin^{-1}\left(\frac{2\cdot\tan\theta}{1+\tan^2\theta}\right)$$

$$= \sin^{-1}(\sin 2\theta) \qquad \sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}$$

$$= 2\theta$$

$$y = 2\tan^{-1}(6^x) \qquad \theta = \tan^{-1}(6^x)$$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{2}{1+(6^x)^2}\frac{d}{dx}(6^x)$$

$$= \frac{2}{1+(6^x)^2}\cdot 6^x \cdot \log 6$$

or $\dfrac{dy}{dx} = \left[\dfrac{2^{x+1}\cdot 3^x}{1+(36)^x}\right]\log 6$

117. Differentiate $\tan^{-1}\left[\dfrac{\sqrt{1+x^2}-1}{x}\right]$ w.r.t. x.

Sol : OD 2012

Let $y = \tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$

Substituting $x = \tan\theta \Rightarrow \theta = \tan^{-1}x$, we have

Sol : OD 2024

We have $f(x) = \begin{cases} ax+b, & x > -1 \\ bx^2-3, & x \leq -1 \end{cases}$

Since $f(x)$ is differentiable for all values of x, $f(x)$ must be continuous as well for all values of x.

So, $f(x)$ is continuous at $x = -1$

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^+} f(x) = f(-1)$$

$$\lim_{x \to -1^-}(bx^2 - 3) = \lim_{x \to -1^+}(ax+b) = b-3$$

$$b - 3 = -a + b = b - 3$$

$$b - 3 = -a + b$$

$$a = 3$$

Now, $f(x)$ is differentiable at $x = -1$

$$\text{LHD at } x = -1 = \text{RHD at } x = -1$$

$$\lim_{x \to -1^-} \frac{f(x) - f(-1)}{x - (-1)} = \lim_{x \to -1^+} \frac{f(x) - f(-1)}{x - (-1)}$$

$$\lim_{x \to -1^-} \frac{bx^2 - 3 - (b-3)}{x+1} = \lim_{x \to -1^+} \frac{ax + b - (b-3)}{x+1}$$

$$\lim_{x \to -1^-} \frac{bx^2 - b}{x+1} = \lim_{x \to -1^+} \frac{ax+3}{x+1}$$

$$\lim_{x \to -1^-} \frac{b^2(x^2-1)}{x+1} = \lim_{x \to -1^+} \frac{ax+3}{x+1}$$

$$\lim_{x \to -1^-} \frac{b(x-1)(x+1)}{(x+1)} = \lim_{x \to -1^+} \frac{3x+3}{x+1} \qquad (\text{as } a = 3)$$

$$\lim_{x \to -1^-} b(x-1) = \lim_{x \to -1^+} \frac{3(x+1)}{(x+1)}$$

$$\lim_{x \to -1^-} b(x-1) = \lim_{x \to -1^+} \frac{3(x+1)}{(x+1)}$$

$$b(-2) = 3$$

$$-2b = 3$$

$$b = -\frac{3}{2}$$

So, for given $f(x)$, $a = 3$, $b = -\frac{3}{2}$

122. Find $\frac{dy}{dx}$, if $(\cos x)^y = (\cos y)^x$

Sol : OD 2024

We have $(\cos x)^y = (\cos y)^x$

Taking log both sides, we get

$$\log(\cos x)^y = \log(\cos y)^x$$

$$y\log(\cos x) = x\log(\cos y)$$

Now, differentiate both sides with respect to x

$$\frac{d[y\log(\cos x)]}{dx} = \frac{d[x\log(\cos y)]}{dx}$$

$$\frac{dy}{dx} \cdot \text{logcos}\, x + \frac{d(\log(\cos x))}{dx} \cdot y$$

$$= \frac{dx}{dx}\log(\cos y) + \frac{d[\log(\cos y)]}{dx} \cdot x$$

$$\frac{dy}{dx}\log\cos x + \frac{1}{\cos x}\frac{d(\cos x)}{dx} \cdot y$$

$$= \log\cos y + \frac{1}{\cos y}\frac{d(\cos y)}{dx} \cdot x$$

$$\frac{dy}{dx} \cdot \log\cos x + \frac{1}{\cos}(-\sin x) \cdot y$$

$$= \log\cos y + \frac{1}{\cos y}(-\sin y)\frac{dy}{dx} \cdot x$$

$$\frac{dy}{dx} \cdot \log(\cos x) - \tan x.y$$

$$= \log\cos y - \tan y \cdot x \cdot \frac{dy}{dx}$$

$$\frac{dy}{dx}(\log(\cos x) + x\tan y) = \log\cos y + y\tan x$$

$$\frac{dy}{dx} = \frac{\log\cos y + y\tan x}{\log\cos x + x\tan y}$$

123. If $y = (\log x)^x + x^{\log x}$, then find $\frac{dy}{dx}$.

Sol : OD 2020

We have $y = (\log x)^x + x^{\log x}$

Let $u = (\log x)^x$ and $v = x^{\log x}$

Then, $y = u + v$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad ...(1)$$

Now, $u = (\log x)^x$

Taking log on both sides, we get

$$\log u = \log(\log x)^x$$

$$\log u = x\log(\log x)$$

Differentiating both sides w.r.t x, by chain rule of derivative, we get

$$\frac{1}{u}\frac{du}{dx} = x\frac{d}{dx}\log(\log x) + \log(\log x)\frac{d}{dx}(x)$$

$$\frac{1}{u}\frac{du}{dx} = \frac{x}{\log x} \times \frac{1}{x} + \log(\log x)$$

$$\frac{du}{dx} = u\left[\frac{1}{\log x} + \log(\log x)\right]$$

$$\frac{du}{dx} = (\log x)^x\left[\frac{1}{\log x} + \log(\log x)\right]$$

Again, $v = x^{\log x}$

Taking log on both sides, we get

$$\log v = \log x^{\log x}$$

$$= (\log x)(\log x)$$

$$\log v = (\log x)^2$$

$$= \lim_{h \to 0} \frac{\sin(a+1)h}{(a+1)h} \times (a+1) + 2\lim_{h \to 0} \frac{\sin h}{h}$$

$$= 1 \times (a+1) + 2 \times 1$$

and RHL $= \lim_{x \to 0^+} f(x)$

$$= \lim_{h \to 0} f(0+h)$$

$$= \lim_{h \to 0} \frac{\sqrt{1+b(0+h)}-1}{0+h}$$

$$= \lim_{h \to 0} \frac{\sqrt{1+bh}-1}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{1-bh}-1}{h} \times \frac{\sqrt{1+bh}+1}{\sqrt{1+bh}+1}$$

$$= \lim_{h \to 0} \frac{(1+bh)-1}{h(\sqrt{1+bh}+1)}$$

$$= \lim_{h \to 0} \frac{bh}{h(\sqrt{1+bh}+1)}$$

$$= \lim_{h \to 0} \frac{b}{(\sqrt{1+bh}+1)}$$

$$= \frac{b}{\sqrt{1+0}+1} = \frac{b}{2} \qquad \text{...(iii)}$$

From Eqs. (i), (ii) and (iii) we get

$$a+3 = \frac{b}{2} = 2 \qquad f(0) = 2$$

$$a+3 = 2 \text{ and } \frac{b}{2} = 2$$

$$a = -1 \text{ and } b = 4$$

126. Find the value of k, so that the which following function is continuous at $x = 0$.

$$f(x) = \begin{cases} \left(\dfrac{1-\cos 4x}{8x^2}\right), & \text{if } x \neq 0 \\ k, & x = 0 \end{cases}$$

Sol : Comp 2014, OD 2010

We have
$$f(x) = \begin{cases} \left(\dfrac{1-\cos 4x}{8x^2}\right), & \text{if } x \neq 0 \\ k, & \text{if } x = 0 \end{cases}$$

If $f(x)$ is continuous at $x = 0$ then we have

$$\text{LHL} = \text{RHL} = f(0) \qquad \text{...(i)}$$

Now, LHL $= \lim_{x \to 0^-} f(x)$

Substituting $x = 0 - h$; when $x \to 0^-$, then $h \to 0$, we have

$$\text{LHL} = \lim_{x \to 0^-} \frac{1-\cos 4x}{8x^2}$$

$$= \lim_{h \to 0} \frac{1-\cos 4(0-h)}{8(0-h)^2}$$

$$= \lim_{h \to 0} \frac{1-\cos 4h}{8h^2}$$

$$= \lim_{h \to 0} \frac{2\sin^2 2h}{8h^2}$$

$$= \lim_{h \to 0} \frac{\sin^2 2h}{4h^2}$$

$$= \lim_{h \to 0} \left(\frac{\sin 2h}{2h}\right)^2 = 1$$

Substituting this value in Eq. (i), we get

$$1 = f(0) \Rightarrow 1 = k \qquad f(0) = k$$

Hence, for $k = 1$, the given function $f(x)$ is continuous at $x = 0$.

Alternative :

We have
$$f(x) = \begin{cases} \left(\dfrac{1-\cos 4x}{8x^2}\right), & \text{if } x \neq 0 \\ k, & \text{if } x = 0 \end{cases}$$

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{1-\cos 4x}{8x^2}$$

$$= \lim_{x \to 0} \frac{2\sin^2 2x}{8x^2}$$

$$= \lim_{x \to 0} \left(\frac{\sin 2x}{2x}\right)^2 = 1$$

If $f(x)$ is continuous at $x = 0$ then we have

$$\lim_{x \to 0} f(x) = f(0) \Rightarrow 1 = k \Rightarrow k = 1 \qquad f(0) = k$$

127. If $f(x) = \begin{cases} \dfrac{1-\cos 4x}{x^2}, & \text{when } x < 0 \\ a, & \text{when } x = 0 \\ \dfrac{\sqrt{x}}{\sqrt{17+\sqrt{x}}-4}, & \text{when } x > 0 \end{cases}$ and f is continuous at $x = 0$, then find the value of a.

Sol : Foreign 2013

We have
$$f(x) = \begin{cases} \dfrac{1-\cos 4x}{x^2}, & \text{when } x < 0 \\ a, & \text{when } x = 0 \\ \dfrac{\sqrt{x}}{\sqrt{17+\sqrt{x}}-4}, & \text{when } x > 0 \end{cases}$$

If $f(x)$ is continuous at $x = 0$ then we have

$$\text{LHL} = \text{RHL} = f(0) \qquad \text{...(i)}$$

Now, $(\text{LHL})_{x=0} = \lim_{x \to 0^-} f(x)$

Substituting $x = 0 - h$; when $x \to 0^-$, then $h \to 0$, we have

$$(\text{LHL})_{x=0} = \lim_{x \to 0^-} \frac{1-\cos 4x}{x^2}$$

$$= \lim_{h \to 0} \frac{1-\cos 4(0-h)}{(0-h)^2}$$

$$= \lim_{h \to 0} \frac{1-\cos 4h}{h^2}$$

$$\frac{dy}{dx} = \frac{1-x}{x}$$

Thus (b) is correct option.

26. If x is measured in degrees, then $\frac{d}{dx}(\cos x)$ is equal to

(a) $-\sin x$ (b) $-\frac{180}{\pi}\sin x$

(c) $-\frac{\pi}{180}\sin x$ (d) $\sin x$

Sol : Comp 2009

$$\frac{d}{dx}(\cos x) = -\frac{\pi}{180}\sin x$$

Thus (c) is correct option.

27. If $f(x) = \log_e(\sin x)$, then $f'(e)$ is equal to

(a) e^{-1} (b) e

(c) 1 (d) 0

Sol : OD 2016

We have $f(x) = \log_e(\log_e x)$

$$f'(x) = \frac{1}{x\log_e(x)} = \frac{1}{e\log_e(e)} = \frac{1}{e}$$

28. The derivative of $\cos^3 x$ with respect to $\sin^3 x$ is

(a) $-\cot x$ (b) $\cot x$

(c) $\tan x$ (d) $-\tan x$

Sol : Foreign 2017, SQP 2014

Let $u = \cos^3 x$

and $v = \sin^3 x$

$$\frac{du}{dx} = -3\cos^2 x\sin x$$

and $$\frac{dv}{dx} = 3\sin^2 x\cos x$$

Now, $$\frac{du}{dv} = \frac{-3\cos^2 x\sin x}{3\sin^2 x\cos x} = -\cot x$$

Thus (a) is correct option.

29. If $y = \tan^{-1}\sqrt{\frac{1-\sin x}{1+\sin x}}$, then the value of $\frac{dy}{dx}$ at $x = \frac{\pi}{6}$ is

(a) $-\frac{1}{2}$ (b) $\frac{1}{2}$

(c) 1 (d) -1

Sol : Comp 2012

We have $$y = \tan^{-1}\sqrt{\frac{1-\sin x}{1+\sin x}}$$

$$= \tan^{-1}\sqrt{\frac{1-\cos\left(\frac{\pi}{2}-x\right)}{1+\cos\left(\frac{\pi}{2}-x\right)}}$$

$$= \tan^{-1}\left|\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right|$$

$$= \frac{\pi}{4}-\frac{x}{2}$$

$$\frac{dy}{dx} = -\frac{1}{2}$$

Thus (a) is correct option.

30. If $y = \log[\sin(x^2)]$, $0 < x < \frac{\pi}{2}$, then $\frac{dy}{dx}$ at $x = \frac{\sqrt{\pi}}{2}$ is

(a) 0 (b) 1

(c) $\pi/4$ (d) $\sqrt{\pi}$

Sol : Delhi 2013

We have $y = \log[\sin(x^2)]$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{\sin x^2}\cdot\cos x^2\cdot 2x = 2x\cot x^2$$

At $x = \frac{\sqrt{\pi}}{2}$, $$\frac{dy}{dx} = \frac{2\sqrt{\pi}}{2}\cot\left(\frac{\sqrt{\pi}}{2}\right)^2 = \sqrt{\pi}\cot\left(\frac{\pi}{4}\right) = \sqrt{\pi}$$

Thus (d) is correct option.

31. The derivative of $\log|x|$ is

(a) $\frac{1}{x}$, $x > 0$ (b) $\frac{1}{|x|}$, $x \neq 0$

(c) $\frac{1}{x}$, $x \neq 0$ (d) None of these

Sol : Delhi 2008 , Comp 2007

We have $y = \log|x|$

$$= \begin{cases}\log x, & x > 0\\ \log(-x), & x < 0\end{cases}$$

$$\frac{dy}{dx} = \begin{cases}\frac{1}{x}, & x > 0\\ \frac{1}{-x}(-1) = \frac{1}{x}, & x < 0\end{cases}$$

$$\frac{dy}{dx} = \frac{1}{x},\ x \neq 0$$

Thus (c) is correct option.

32. If $x = \frac{2at}{1+t^3}$ and $y = \frac{2at^2}{(1+t^3)^2}$, then $\frac{dy}{dx}$ is equal to

(a) ax (b) a^2x^2

(c) $\frac{x}{a}$ (d) $\frac{x}{2a}$

Sol : OD 2007

We have $$x = \frac{2at}{1+t^3}$$

and $$y = \frac{2at^2}{(1+t^3)^2}$$

Sol : SQP 2019

Let $y = x^{\sin x} + (\sin x)^{\cos x}$

$y = e^{\sin x \log x} + e^{\cos x \log \sin x}$ $\quad a^b = e^{\log a^b} = e^{b \log a}$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = e^{\sin x \log x}\frac{d}{dx}(\sin x \cdot \log x) + e^{\cos x \log \sin x}\frac{d}{dx}(\cos x \cdot \log \sin x)$$

$$= x^{\sin x}\{\log x \frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(\log x)\} + (\sin x)^{\cos x}\{\log \sin x \frac{d}{dx}(\cos x) + \cos x \cdot \frac{d}{dx}(\log \sin x)\}$$

$$= x^{\sin x}\left\{\cos x \cdot \log x + \frac{\sin x}{x}\right\} + (\sin x)^{\cos x}\left\{-\sin x \log(\sin x) + \cos x \times \frac{1}{\sin x} \times \cos x\right\}$$

$$= x^{\sin x}\left\{\cos x \cdot \log x + \frac{\sin x}{x}\right\} + (\sin x)^{\cos x}\left\{-\sin x \log(\sin x) + \frac{\cos^2 x}{\sin x}\right\}$$

85. If $\log(x^2 + y^2) = 2\tan^{-1}\left(\frac{y}{x}\right)$, show that $\frac{dy}{dx} = \frac{x+y}{x-y}$.

Sol : Delhi 2019

We have $\log(x^2 + y^2) = 2\tan^{-1}\left(\frac{y}{x}\right)$

Differentiating w.r.t. x, we get

$$\frac{1}{x^2+y^2}\left(2x + 2y\frac{dy}{dx}\right) = 2 \cdot \frac{1}{1 + \frac{y^2}{x^2}}\left(\frac{y' \cdot x - y}{x^2}\right)$$

$$\frac{2(x + y \cdot y')}{x^2 + y^2} = \frac{2x^2}{x^2+y^2}\left(\frac{y'x - y}{x^2}\right)$$

$$x + y \cdot y' = y'x - y$$

$$y'(x - y) = x + y$$

$$y' = \frac{x+y}{x-y}$$

$$\frac{dy}{dx} = \frac{x+y}{x-y}$$ Hence proved.

86. If $y = (\sin^{-1}x)^2$, prove that

$$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 2 = 0$$

Sol : Delhi 2019, OD 2012

We have, $y = (\sin^{-1}x)^2$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 2\sin^{-1}x \cdot \frac{1}{\sqrt{1-x^2}} \quad (1)$$

Again differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{(\sqrt{1-x^2})\left(\frac{2}{\sqrt{1-x^2}}\right) - \left(\frac{\frac{1}{2}(-2x)}{\sqrt{1-x^2}}\right) \cdot (2\sin^{-1}x)}{(\sqrt{1-x^2})^2}$$

$$= \frac{2 + \frac{2x\sin^{-1}x}{\sqrt{1-x^2}}}{1-x^2}$$

or $$(1-x^2)\frac{d^2y}{dx^2} = 2 + \frac{2x\sin^{-1}x}{\sqrt{1-x^2}} \quad (2)$$

From eq (1) and (2) we have

$$(1-x^2)\frac{d^2y}{dx^2} = 2 + xy$$

or $$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 2 = 0$$ Hence proved.

87. If $y = e^{\tan^{-1}x}$, prove that $(1+x^2)\frac{d^2y}{dx^2} + (2x-1)\frac{dy}{dx} = 0$

Sol : OD 2018

We have, $y = e^{\tan^{-1}x}$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = e^{\tan^{-1}x}\frac{d}{dx}(\tan^{-1}x)$$

$$\frac{dy}{dx} = e^{\tan^{-1}x} \times \frac{1}{(1+x^2)}$$

$$(1+x^2)\frac{dy}{dx} = e^{\tan^{-1}x} \quad ...(i)$$

Again differentiating both sides w.r.t. x, we get

$$(1+x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = e^{\tan^{-1}x} \times \frac{1}{(1+x^2)}$$

$$(1+x^2)\frac{d^2y}{dx^2} + (2x)\frac{dy}{dx} = \frac{dy}{dx} \quad \text{[from Eq. (i)]}$$

$$(1+x^2)\frac{d^2y}{dx^2} + (2x-1)\frac{dy}{dx} = 0$$ Hence proved.

88. If $x^y + y^x = a^b$, then find $\frac{dy}{dx}$.

Sol : OD 2017, Comp 2011

We have $x^y + y^x = a^b$

Let $x^y = u$ and $y^x = v$

Then, $u + v = a^b$

$$\frac{du}{dx} + \frac{dv}{dx} = 0 \quad ...(1)$$

Now, $u = x^y$

$$\log u = y\log x$$

Differentiating w.r.t. x, we get

$$\frac{1}{u}\frac{du}{dx} = \frac{y}{x} + \log x\frac{dy}{dx}$$

$$\frac{du}{dx} = u\frac{y}{x} + u\log x\frac{dy}{dx}$$

$$\frac{du}{dx} = y \cdot x^{y-1} + x^y \cdot \log x\frac{dy}{dx}$$

and $v = y^x$

$$\log v = x\log y$$

$$\frac{a}{(1+x^2)} = \frac{1}{y}\frac{dy}{dx}$$

$$(1+x^2)\frac{dy}{dx} = ay$$

Differentiating again both sides w.r.t. x, we get

$$(1+x^2)\cdot\frac{d}{dx}\left(\frac{dy}{dx}\right)+\frac{dy}{dx}\cdot\frac{d}{dx}(1+x^2) = \frac{d}{dx}(ay)$$

$$(1+x^2)\frac{d^2y}{dx^2}+\frac{dy}{dx}\cdot(2x) = a\cdot\frac{dy}{dx}$$

$$(1+x^2)\frac{d^2y}{dx^2}+2x\frac{dy}{dx}-a\frac{dy}{dx} = 0$$

$$(1+x^2)\frac{d^2y}{dx^2}+(2x-a)\frac{dy}{dx} = 0$$

Hence proved.

101. Prove that

$$\frac{d}{dx}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right] = \sqrt{a^2-x^2}.$$

Sol : Foreign 2011, Delhi 2008

We have $\frac{d}{dx}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]= \sqrt{a^2-x^2}$.

Let $y = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}$

$$\frac{dy}{dx} = \frac{x}{2}\times\frac{d}{dx}\sqrt{a^2-x^2}+\sqrt{a^2-x^2}\frac{d}{dx}\left(\frac{x}{2}\right)+\frac{a^2}{2}\frac{d}{dx}\sin^{-1}\frac{x}{a}$$

$$= \frac{x}{2}\cdot\frac{1}{2\sqrt{a^2-x^2}}\frac{d}{dx}(a^2-x^2)$$

$$+\sqrt{a^2-x^2}\cdot\frac{1}{2}+\frac{a^2}{2}\cdot\frac{1}{\sqrt{1-\frac{x^2}{a^2}}}\cdot\frac{d}{dx}\left(\frac{x}{a}\right)$$

$$= \frac{-x^2}{2\sqrt{a^2-x^2}}+\frac{\sqrt{a^2-x^2}}{2}+\frac{a^2}{2a}\cdot\frac{1}{\sqrt{\frac{a^2-x^2}{a^2}}}$$

$$= \frac{-x^2}{2\sqrt{a^2-x^2}}+\frac{\sqrt{a^2-x^2}}{2}+\frac{a^2}{2a}\times\frac{a}{\sqrt{a^2-x^2}}$$

$$= \frac{-x^2}{2\sqrt{a^2-x^2}}+\frac{\sqrt{a^2-x^2}}{2}+\frac{a^2}{2\sqrt{a^2-x^2}}$$

$$= \frac{-x^2+(a^2-x^2)+a^2}{2\sqrt{a^2-x^2}}$$

$$= \frac{2a^2-2x^2}{2\sqrt{a^2-x^2}}$$

$$= \frac{2(a^2-x^2)}{2\sqrt{a^2-x^2}}$$

$$= \sqrt{a^2-x^2}$$

$$= \text{RHS}$$

Hence proved.

102. If $y = \cos^{-1}\left[\frac{2x-3\sqrt{1-x^2}}{\sqrt{13}}\right]$, then find $\frac{dy}{dx}$.

Sol : OD 2008

We have $y = \cos^{-1}\left[\frac{2x-3\sqrt{1-x^2}}{\sqrt{13}}\right]$

Substituting $x = \sin\theta$, then $\theta = \sin^{-1}x$ we have

$$y = \cos^{-1}\left[\frac{2\sin\theta-3\sqrt{1-\sin^2\theta}}{\sqrt{13}}\right]$$

$$= \cos^{-1}\left[\frac{2\sin\theta-3\cos\theta}{\sqrt{13}}\right]$$

or $$y = \cos^{-1}\left[\frac{2}{\sqrt{13}}\sin\theta-\frac{3}{\sqrt{13}}\cos\theta\right]$$

Let $\frac{2}{\sqrt{13}} = \cos\alpha$, then $\frac{3}{\sqrt{13}} = \sin\alpha$

Now $$y = \cos^{-1}[\sin\theta\cos\alpha-\cos\theta\sin\alpha]$$

$$= \cos^{-1}[\sin(\theta-\alpha)]$$

$$= \cos^{-1}\cos\left[\frac{\pi}{2}-(\theta-\alpha)\right]$$

$$= \frac{\pi}{2}-\theta+\alpha$$

or $$y = \frac{\pi}{2}-\sin^{-1}x+\alpha$$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = 0-\frac{1}{\sqrt{1-x^2}}+0$$

$$= \frac{-1}{\sqrt{1-x^2}}$$

103. If $\sin y = x\sin(a+y)$, then prove that $\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$.

Sol : OD 2012

We have $\sin y = x\sin(a+y)$

$$x = \frac{\sin y}{\sin(a+y)}$$

Differentiating w.r.t. x, we get

$$\frac{dx}{dy} = \frac{\sin(a+y)\frac{d}{dy}\sin y-\sin y\frac{d}{dy}\sin(a+y)}{\sin^2(a+y)}$$

$$= \frac{\sin(a+y)\times\cos y-\sin y\times\cos(a+y)}{\sin^2(a+y)}$$

$$= \frac{\sin(a+y)\cos y-\cos(a+y)\sin y}{\sin^2(a+y)}$$

$$= \frac{\sin(a+y-y)}{\sin^2(a+y)}$$

$$\frac{dx}{dy} = \frac{\sin a}{\sin^2(a+y)}$$

$$\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$$

Hence Proved

104. If $y = \sin^{-1}x$, show that $(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx} = 0$.

Sol : Delhi 2012

We have, $y = \sin^{-1}x$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$= \lim_{h \to 0} (3a - ah + 1)$$

$$\text{LHL} = 3a + 1$$

and $\quad \text{RHL} = \lim_{x \to 3^+} f(x)$

$$= \lim_{x \to 3^+} (bx + 3)$$

$$= \lim_{h \to 0} [b(3 + h) + 3]$$

$$= \lim_{h \to 0} (3b + bh + 1)$$

$$\text{LHL} = 3b + 3$$

Now, from Eq. (i), we have

$$\text{LHL} = \text{RHL}$$

$$3a + 1 = 3b + 3$$

$$3a - 3b = 2$$

which is required result.

133. Find the value of a, if the following function $f(x)$ is continuous at $x = 2$. Also, discuss the continuous of $f(x)$ at $x = 3$.

$$f(x) = \begin{cases} 2x - 1, & x < 2 \\ a, & x = 2 \\ x + 1, & x > 2 \end{cases}$$

Comp 2007

Sol :

We have $\quad f(x) = \begin{cases} 2x - 1, & x < 2 \\ a, & x = 2 \\ x + 1, & x > 2 \end{cases}$

If $f(x)$ is continuous at $x = 2$ then we have

$$\text{LHL} = \text{RHL} = f(2) \qquad \text{...(i)}$$

Now, $\quad f(2) = a$

and $\quad \text{LHL} = \lim_{x \to 2^-} f(x)$

$$= \lim_{x \to 2^-} (2x - 1)$$

$$= \lim_{h \to 0} [2(2 - h) - 1]$$

$$= \lim_{h \to 0} [4 - 4h - 1]$$

$$= \lim_{h \to 0} [3 - 4h]$$

$$= 3$$

From Eq. (i), we have

$$\text{LHL} = f(2) \Rightarrow a = 3$$

Now, let us check the continuity at $x = 3$.

Consider, $\quad \lim_{x \to 3} f(x) = \lim_{x \to 3} (x + 1) \qquad f(x) = x + 1 \text{ for } x > 2$

$$= 4 = f(3)$$

Thus $f(x)$ is continuous at $x = 3$.

134. Find the values of a and b such that the function defined as follows is continuous.

$$f(x) = \begin{cases} x + 2, & x \le 2 \\ ax + b, & 2 < x < 5 \\ 3x - 2, & x \ge 5 \end{cases}$$

Foreign 2007

Sol :

We have $\quad f(x) = \begin{cases} x + 2, & x \le 2 \\ ax + b, & 2 < x < 5 \\ 3x - 2, & x \ge 5 \end{cases}$

If $f(x)$ is continuous at $x = 2$ and $x = 5$ then we have

$$(\text{LHL})_{x=2} = (\text{RHL})_{x=2} = f(2) \qquad \text{...(i)}$$

and $\quad (\text{LHL})_{x=5} = (\text{RHL})_{x=5} = f(5) \qquad \text{...(ii)}$

Now, let us calculate LHL and RHL at $x = 2$.

$$\text{LHL} = \lim_{x \to 2^-} f(x)$$

$$= \lim_{x \to 2^-} (x + 2)$$

$$= \lim_{h \to 0} (2 - h + 2)$$

$$= (2 + 2) = 4$$

and $\quad \text{RHL} = \lim_{x \to 2^+} f(x)$

$$= \lim_{x \to 2^+} (ax + b)$$

$$= \lim_{h \to 0} \{a(2 + h) + b\}$$

$$= \lim_{h \to 0} (2a + ah + b)$$

$$= 2a + b$$

From Eq. (i), we have

$$\text{LHL} = \text{RHL}$$

$$2a + b = 4 \qquad \text{...(iii)}$$

Now, we have to find LHL and RHL at $x = 5$.

$$\text{LHL} = \lim_{x \to 5^-} f(x)$$

$$= \lim_{x \to 5^-} (ax + b)$$

$$= \lim_{h \to 0} [a(5 - h) + b]$$

$$= \lim_{h \to 0} (5a - ah + b)$$

$$\text{LHL} = 5a + b$$

and $\quad \text{RHL} = \lim_{x \to 5^+} f(x)$

$$= \lim_{x \to 5^+} (3x - 2)$$

$$= \lim_{h \to 0} [3(5 + h) - 2]$$

$$= \lim_{h \to 0} [15 + 3h - 2]$$

$$= 15 + 0 - 2 = 13$$

Now, from Eq. (ii), we have

$$\text{LHL} = \text{RHL}$$

Sol : Delhi 2016, Comp 2014

Let $u = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$

Substituting $x = \tan\theta \Rightarrow \theta = \tan^{-1}x$, we have

$$\begin{aligned} u &= \tan^{-1}\left[\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right] \\ &= \tan^{-1}\left[\frac{\sqrt{\sec^2\theta}-1}{\tan\theta}\right] \\ &= \tan^{-1}\left[\frac{\sec\theta-1}{\tan\theta}\right] \\ &= \tan^{-1}\left[\frac{1-\cos\theta}{\sin\theta}\right] \\ &= \tan^{-1}\left[\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2}}\right] \\ &= \tan^{-1}\left(\tan\frac{\theta}{2}\right) \\ u &= \frac{\theta}{2} = \frac{1}{2}\tan^{-1}x \end{aligned}$$

Differentiating w.r.t. x, we get

$$\frac{du}{dx} = \frac{1}{2(1+x^2)}$$

Also, let $v = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Substituting $x = \tan\theta \Rightarrow \theta = \tan^{-1}x$, we get

$$\begin{aligned} v &= \sin^{-1}\left[\frac{2\tan\theta}{1+\tan^2\theta}\right] \\ &= \sin^{-1}[\sin 2\theta] \\ &= 2\theta \end{aligned}$$

or $v = 2\tan^{-1}x$

Differentiating w.r.t. x, we get

$$\frac{dv}{dx} = \frac{2}{1+x^2} \quad \text{...(ii)}$$

Now, $\frac{du}{dv} = \frac{du}{dx}\times\frac{dx}{dv} = \frac{1}{2(1+x^2)}\times\frac{(1+x^2)}{2}$

$$\frac{du}{dv} = \frac{1}{4}$$

143. If $x = a\sin 2t(1+\cos 2t)$ and $y = b\cos 2t(1-\cos 2t)$, then find the values of $\frac{dy}{dx}$ at $t = \frac{\pi}{4}$ and $t = \frac{\pi}{3}$.

Sol : Delhi 2016, OD 2014

We have $x = a\sin 2t(1+\cos 2t)$

and $y = b\cos 2t(1-\cos 2t)$

Differentiating x and y separately w.r.t. t, we get

$$\begin{aligned} \frac{dx}{dt} &= a\left[\sin 2t\frac{d}{dt}(1+\cos 2t) + (1+\cos 2t)\frac{d}{dt}(\sin 2t)\right] \\ &= a[\sin 2t\times(0-2\sin 2t) + (1+\cos 2t)(2\cos 2t)] \\ &= a(-2\sin^2 2t + 2\cos 2t + 2\cos^2 2t) \\ &= a[2(\cos^2 2t - \sin^2 2t) + 2\cos 2t] \qquad \cos^2 2\theta - \sin^2 2\theta = \cos 4\theta \\ &= a(2\cos 4t + 2\cos 2t) \\ &= 2a(\cos 4t + \cos 2t) \\ &= 2a\left[2\cos\left(\frac{4t+2t}{2}\right)\cdot\cos\left(\frac{4t-2t}{2}\right)\right] \\ &= 4a\cos 3t\cos t \end{aligned}$$

$$\begin{aligned} \frac{dy}{dt} &= b\left[\cos 2t\frac{d}{dt}(1-\cos 2t) + (1-\cos 2t)\frac{d}{dt}(\cos 2t)\right] \\ &= b[\cos 2t\times(0+2\sin 2t) + (1-\cos 2t)(-2\sin 2t)] \\ &= b(2\sin 2t\cos 2t - 2\sin 2t + 2\sin 2t\cos 2t) \\ &= 2b(2\sin 2t\cos 2t - \sin 2t) \\ &= 2b(\sin 4t - \sin 2t) \qquad 2\sin 2\theta\cos 2\theta = \sin 4\theta \\ &= 2b\left[2\cos\left(\frac{4t+2t}{2}\right)\sin\left(\frac{4t-2t}{2}\right)\right] \\ &= 4b\cos 3t\sin t \end{aligned}$$

Now, $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

$$= \frac{4b\cos 3t\sin t}{4a\cos 3t\cdot\cos t} = \frac{b}{a}\tan t$$

At $t = \frac{\pi}{4}$, $\frac{dy}{dx} = \frac{b}{a}\tan\frac{\pi}{4} = \frac{b}{a}$

At $t = \frac{\pi}{3}$, $\frac{dy}{dx} = \frac{b}{a}\tan\frac{\pi}{3} = \frac{\sqrt{3}\,b}{a}$

144. If $x\cos(a+y) = \cos y$, then prove that

$$\frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}.$$

Hence, show that

$$\sin a\frac{d^2y}{dx^2} + \sin 2(a+y)\frac{dy}{dx} = 0.$$

Sol : OD 2016

We have $x\cos(a+y) = \cos y$

$$x = \frac{\cos y}{\cos(a+y)}$$

Differentiating w.r.t. y, we get

$$\begin{aligned} \frac{dx}{dy} &= \frac{\cos(a+y)\frac{d}{dy}\cos y - \cos y\frac{d}{dy}\cos(a+y)}{\cos^2(a+y)} \\ &= \frac{\cos(a+y)\times(-\sin y) + \cos y\times\sin(a+y)}{\cos^2(a+y)} \\ &= \frac{\sin(a+y)\cos y - \cos(a+y)\sin y}{\cos^2(a+y)} \\ &= \frac{\sin(a+y-y)}{\cos^2(a+y)} \\ \frac{dx}{dy} &= \frac{\sin a}{\cos^2(a+y)} \\ \frac{dy}{dx} &= \frac{\cos^2(a+y)}{\sin a} \quad \text{...(i)} \end{aligned}$$

At $\theta = \frac{\pi}{6}$, $\left(\frac{d^2y}{dx^2}\right)_{\theta=\frac{\pi}{6}} = \frac{1}{3a(\cos\frac{\pi}{6})^4(\sin\frac{\pi}{6})}$

$$= \frac{1}{3a\left(\frac{\sqrt{3}}{2}\right)^4\left(\frac{1}{2}\right)}$$

$$= \frac{1}{3a\left(\frac{9}{16}\right)\left(\frac{1}{2}\right)} = \frac{32}{27a}$$

170. If $x = \sqrt{a^{\sin^{-1}t}}$ and $y = \sqrt{a^{\cos^{-1}t}}$, then show that $\frac{dy}{dx} = \frac{-y}{x}$.

Sol : OD 2012, Delhi 2008

We have $x = \sqrt{a^{\sin^{-1}t}}$ and $y = \sqrt{a^{\cos^{-1}t}}$

Now consider, $x = (a^{\sin^{-1}t})^{1/2}$

Differentiating both sides w.r.t. x, we get

$$\frac{dx}{dt} = \frac{1}{2}(a^{\sin^{-1}t})^{-1/2}\frac{d}{dt}(a^{\sin^{-1}t})$$

$$= \frac{1}{2}(a^{\sin^{-1}t})^{-1/2}a^{\sin^{-1}t}\log a\frac{d}{dt}(\sin^{-1}t)$$

$$= \frac{1}{2}(a^{\sin^{-1}t})^{-1/2}a^{\sin^{-1}t}\log a\cdot\frac{1}{\sqrt{1-t^2}}$$

$$= \frac{1}{2}(a^{\sin^{-1}t})^{1/2}\log a\cdot\frac{1}{\sqrt{1-t^2}}$$

$$\frac{dx}{dt} = \frac{\frac{1}{2}\sqrt{a^{\sin^{-1}t}}\cdot\log a}{\sqrt{1-t^2}} \quad ...(i)$$

Now, consider $y = (a^{\cos^{-1}t})^{1/2}$

Differentiating both sides w.r.t. t, we get

$$\frac{dy}{dt} = \frac{1}{2}(a^{\cos^{-1}t})^{-1/2}\frac{d}{dt}(a^{\cos^{-1}t})$$

$$= \frac{1}{2}(a^{\cos^{-1}t})^{-1/2}a^{\cos^{-1}t}\log a\frac{d}{dt}(\cos^{-1}t)$$

$$= \frac{1}{2}(a^{\cos^{-1}t})^{1/2}\log a\cdot\frac{(-1)}{\sqrt{1-t^2}}$$

$$\frac{dy}{dt} = \frac{-\frac{1}{2}\sqrt{a^{\cos^{-1}t}}\cdot\log a}{\sqrt{1-t^2}} \quad ...(ii)$$

Dividing Eq. (ii) by Eq. (i), we get

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{\left(\frac{-\frac{1}{2}\sqrt{a^{\cos^{-1}t}}\log a}{\sqrt{1-t^2}}\right)}{\left(\frac{\frac{1}{2}\sqrt{a^{\sin^{-1}t}}\log a}{\sqrt{1-t^2}}\right)}$$

$$= -\frac{\sqrt{a^{\cos^{-1}t}}}{\sqrt{a^{\sin^{-1}t}}} = -\frac{y}{x}$$

Hence proved.

171. If $y = x^{\sin x - \cos x} + \frac{x^2-1}{x^2+1}$, then find $\frac{dy}{dx}$.

Sol : Delhi 2012

We have $y = x^{\sin x-\cos x} + \frac{x^2-1}{x^2+1}$

Let $u = x^{\sin x-\cos x}$ and $v = \frac{x^2-1}{x^2+1}$

Then, the given equation becomes

$$y = u + v$$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad ...(i)$$

Consider $u = x^{\sin x-\cos x}$

Taking log both sides, we get

$$\log u = (\sin x - \cos x)\cdot\log x$$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{u}\frac{du}{dx} = (\sin x - \cos x)\cdot\frac{1}{x} + \log x\cdot(\cos x + \sin x)$$

$$\frac{du}{dx} = x^{\sin x-\cos x}\left[\frac{\sin x-\cos x}{x} + \log x\cdot(\cos x + \sin x)\right] \quad ...(ii)$$

Now, consider $v = \frac{x^2-1}{x^2+1} = 1 - \frac{2}{x^2+1}$

Differentiating both sides w.r.t. x, we get

$$\frac{dv}{dx} = 0 - \frac{(x^2+1)\frac{d}{dx}(2) - 2\frac{d}{dx}(x^2+1)}{(x^2+1)^2}$$

$$\frac{dv}{dx} = -\left[\frac{0-2\cdot 2x}{(x^2+1)^2}\right] = \frac{4x}{(x^2+1)^2} \quad ...(iii)$$

Substituting the values from Eqs. (ii) and (iii) to Eq. (i), we get

$$\frac{dy}{dx} = x^{\sin x-\cos x}\left[\frac{\sin x-\cos x}{x} + \log x(\cos x+\sin x)\right] + \frac{4x}{(x^2+1)^2}$$

172. If $x = a(\cos t + t\sin t)$ and $y = a(\sin t - t\cos t)$, then find $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$ and $\frac{d^2y}{dx^2}$.

Sol : Comp 2012

We have $x = a(\cos t + t\sin t)$

Differentiating both sides w.r.t. t, we get

$$\frac{dx}{dt} = a\left[-\sin t + \frac{d}{dt}(t)\cdot\sin t + t\frac{d}{dt}(\sin t)\right]$$

$$\frac{dx}{dt} = a(-\sin t + 1\cdot\sin t + t\cos t) = at\cos t \quad ...(i)$$

Also, given, $y = a(\sin t - t\cos t)$

Differentiating both sides w.r.t. t, we get

$$\frac{dy}{dt} = a[\cos t - \frac{d}{dt}(t)\cos t - t\frac{d}{dt}(\cos t)]$$

$$\frac{dy}{dt} = a(\cos t - \cos t\cdot 1 + t\sin t) = at\sin t \quad ...(ii)$$

Now, $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{at\sin t}{at\cos t} = \tan t$

[from Eqs. (i) and (ii)]

Again, differentiating both sides w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$\log u = x\log x \qquad \log m^n = n\log m$$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{u}\frac{du}{dx} = x\cdot\frac{d}{dx}(\log x) + \log x\cdot\frac{d}{dx}(x)$$

$$= x\cdot\frac{1}{x} + \log x\cdot 1$$

$$\frac{1}{u}\frac{du}{dx} = 1 + \log x$$

$$\frac{du}{dx} = u(1+\log x)$$

$$\frac{du}{dx} = x^x(1+\log x) \qquad ..(2)$$

Now, consider, $v = (\sin x)^x$. Taking log both sides, we get

$$\log v = \log(\sin x)^x$$

$$\log v = x\log(\sin x) \qquad \log m^n = n\log m$$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{v}\frac{dv}{dx} = x\cdot\frac{d}{dx}\log(\sin x) + \log(\sin x)\cdot\frac{d}{dx}(x)$$

$$= x\cdot\frac{1}{\sin x}\cdot\frac{d}{dx}(\sin x) + \log\sin x$$

$$= x\cdot\frac{1}{\sin x}\cdot\cos x + \log\sin x$$

$$\frac{1}{v}\frac{dv}{dx} = x\cot x + \log\sin x$$

$$\frac{dv}{dx} = v(x\cot x + \log\sin x)$$

$$\frac{dv}{dx} = (\sin x)^x(x\cot x + \log\sin x) \qquad ...(3)$$

Now, from eqs. (1), (2) and (3), we get

$$\frac{dy}{dx} = x^x(1+\log x) + (\sin x)^x(x\cot x + \log\sin x)$$

CASE BASED QUESTIONS

180. In apparels industries retailers have an interesting conundrum facing them. On one hand, consumers are more drawn to hot promotional deals than ever before. The result of this is that they sell more units (of product) for less money, and this adversely impacts comp store sales.

Arvind Fashions knows that the it can sell 1000 shirts when the price is ₹ 400 per shirt and it can sell 1500 shirts when the price is ₹ 200 a shirt. Assume demanded price to be linear, Determine

(i) the price function

(ii) the revenue function

(iii) the marginal revenue function.

Sol :

(i) the price function

Since the demanded price is assumed to be linear, let it be given by

$$p = a + bx$$

where p is the price per shirt and x is the quantity demanded at this price.

Since, $x = 1000$ when $p = 400$

and $x = 1500$ when $p = 200$

We have, $400 = a + 1000b$ and $200 = a + 1500b$

Solving these equations simultaneously for a and b, we obtain

$$a = 800 \text{ and } b = -\frac{2}{5}$$

Hence the demanded price is,

$$p = 800 - \frac{2x}{5}$$

(ii) the revenue function

The revenue function is

$$R(x) = px = \left(800 - \frac{2x}{5}\right)x$$

$$= 800x - \frac{2x^2}{5}$$

(iii) the marginal revenue function.

The marginal revenue function is,

$$MR = \frac{dR(x)}{dx} = 800 - \frac{4x}{5}$$

181. Hindustan Pencils Pvt. Ltd. is an Indian manufacturer of pencils, writing materials and other stationery items, established in 1958 in Bombay. The company makes writing implements under the brands Nataraj and Apsara, and claims to be the largest pencil manufacturer in India.

Hindustan Pencils manufactures x units of pencil in a given time, if the cost of raw material is square of

$$\text{RHL} = \frac{1}{2} \quad ...(iv)$$

Substituting the values from Eqs. (ii), (iii) and (iv) to Eq. (i), we get

$$a = \frac{1}{2} = a \Rightarrow a = \frac{1}{2}$$

130. If the following function $f(x)$ is continuous at $x = 1$, then find the values of a and b.

$$f(x) = \begin{cases} 3ax + b, & \text{if } x > 1 \\ 11, & \text{if } x = 1 \\ 5ax - 2b, & \text{if } x < 1 \end{cases}$$

Sol : Delhi 2007, OD 2010

We have $$f(x) = \begin{cases} 3ax + b, & \text{if } x > 1 \\ 11, & \text{if } x = 1 \\ 5ax - 2b, & \text{if } x < 1 \end{cases}$$

If $f(x)$ is continuous at $x = 1$ then we have

$$\text{LHL} = \text{RHL} = f(1) \quad ...(i)$$

Now, $$\text{LHL} = \lim_{x \to 1^-} f(x)$$
$$= \lim_{x \to 1^-} (5ax - 2b)$$
$$= \lim_{h \to 0} [5a(1-h) - 2b]$$
$$= \lim_{h \to 0} (5a - 5ah - 2b)$$
$$= 5a - 2b$$

and $$\text{RHL} = \lim_{x \to 1^+} (3ax + b)$$
$$= \lim_{h \to 0} [3a(1+h) + b]$$
$$= \lim_{h \to 0} (3a + 3ah + b)$$
$$= 3a + b$$

Also, given that $f(1) = 11$

Substituting these values in Eq. (i), we get

$$5a - 2b = 3a + b = 11$$

$$3a + b = 11 \quad ...(ii)$$

and $$5a - 2b = 11 \quad ...(iii)$$

Solving (ii) and (iii) we get $a = 3$ and $b = 2$.

131. Find the values of a and b such that following function $f(x)$ is a continuous function.

$$f(x) = \begin{cases} 5, & x \le 2 \\ ax + b, & 2 < x < 10 \\ 21, & x \ge 10 \end{cases}$$

Sol : Delhi 2007

We have $$f(x) = \begin{cases} 5, & x \le 2 \\ ax + b, & 2 < x < 10 \\ 21, & x \ge 10 \end{cases}$$

If $f(x)$ is continuous at $x = 2$ and $x = 10$ then we have

$$(\text{LHL})_{x=2} = (\text{RHL})_{x=2} = f(2) \quad ...(i)$$

and $$(\text{LHL})_{x=10} = (\text{RHL})_{x=10} = f(10) \quad ...(ii)$$

Now, let us calculate LHL and RHL at $x = 2$.

$$\text{LHL} = \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} 5 = 5$$

and $$\text{RHL} = \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (ax + b)$$
$$= \lim_{h \to 0} \{a(2+h) + b\}$$
$$= \lim_{h \to 0} (2a + ah + b)$$
$$= 2a + b$$

From Eq. (i), we have

$$\text{LHL} = \text{RHL}$$

$$2a + b = 5 \quad ...(iii)$$

Now, we have to find LHL and RHL at $x = 10$.

$$\text{LHL} = \lim_{x \to 10^-} f(x)$$
$$= \lim_{x \to 10^-} (ax + b)$$
$$= \lim_{h \to 0} [a(10 - h) + b]$$
$$= \lim_{h \to 0} (10a - ah + b)$$
$$\text{LHL} = 10a + b$$

and $$\text{RHL} = \lim_{x \to 10^+} f(x) = \lim_{x \to 10^+} 21 = 21$$

Now, from Eq. (ii), we have

$$\text{LHL} = \text{RHL}$$

$$10a + b = 21 \quad ...(iv)$$

Subtracting eq (3) from eq (4), we get

$$8a = 16 \Rightarrow a = 2$$

Substituting $a = 2$ in Eq. (iv), we get

$$20 + b = 21 \Rightarrow b = 1$$

Hence, $a = 2$ and $b = 1$.

132. Find the relationship between a and b, so that the following unction f defined by is continuous at $x = 3$.

$$f(x) = \begin{cases} ax + 1, & \text{if } x \le 3 \\ bx + 3, & \text{if } x > 3 \end{cases}$$

Sol : OD 2007

We have $$f(x) = \begin{cases} ax + 1, & \text{if } x \le 3 \\ bx + 3, & \text{if } x > 3 \end{cases}$$

If $f(x)$ is continuous at $x = 3$ then we have

$$(\text{LHL})_{x=3} = (\text{RHL})_{x=3} = f(3) \quad ...(i)$$

Now $$\text{LHL} = \lim_{x \to 3^-} f(x)$$
$$= \lim_{x \to 3^-} (ax + 1)$$
$$= \lim_{h \to 0} [a(3 - h) + 1]$$

$$5a + b = 13 \qquad ...(iv)$$

Subtracting eq. (iii) from Eq. (iv), we get

$$3a = 9 \Rightarrow a = 3$$

Substituting $a = 3$ in Eq. (iii), we get

$$2 \times 3 + b = 4 \Rightarrow b = -2$$

Hence, $a = 3$ and $b = -2$.

135. For what values of k, the following function is continuous at $x = 0$? Also, find whether the function is continuous at $x = 1$.

$$f(x) = \begin{cases} k(x^2+2), & \text{if } x \leq 0 \\ 3x+1, & \text{if } x > 0 \end{cases}$$

Sol : Delhi 2010, Comp 2010

We have $$f(x) = \begin{cases} k(x^2+2), & \text{if } x \leq 0 \\ 3x+1, & \text{if } x > 0 \end{cases}$$

If $f(x)$ is continuous at $x = 0$ then we must have

$$\text{LHL} = \text{RHL} = f(\pi) \qquad ...(i)$$

Now, $$\text{LHL} = \lim_{x \to 0^-} f(x)$$
$$= \lim_{x \to 0^-} k(x^2+2)$$
$$= \lim_{h \to 0} k[(0-h)^2+2]$$
$$= \lim_{h \to 0} k(h^2+2)$$
$$\text{LHL} = 2k$$

and $$\text{RHL} = \lim_{x \to 0^+} f(x)$$
$$= \lim_{x \to 0^+} (3x+1)$$
$$= \lim_{h \to 0} [3(0+h)+1]$$
$$\text{RHL} = 1$$

From Eq. (i), we have

$$\text{LHL} = \text{RHL}$$
$$2k = 1$$
$$k = \frac{1}{2}$$

Now, let us check the continuity of the given function $f(x)$ at $x = 1$.

Consider, $$\lim_{x \to 1} f(x) = \lim_{x \to 1} 3x+1 \qquad f(x) = 3x+1 \text{ for } x > 0$$
$$= 4 = f(1)$$

Thus $f(x)$ is continuous at $x = 1$

136. Find all points of discontinuity of f, where f is defined as follows.

$$f(x) = \begin{cases} |x|+3, & x \leq -3 \\ -2x, & -3 < x < 3 \\ 6x+2, & x \leq 3 \end{cases}$$

Sol : Delhi 2010, OD 2007

We have $$f(x) = \begin{cases} |x|+3, & x \leq -3 \\ -2x, & -3 < x < 3 \\ 6x+2, & x \leq 3 \end{cases}$$

or $$f(x) = \begin{cases} -x+3, & x \leq -3 \\ -2x, & -2 < x < 3 \\ 6x+2, & x \geq 3 \end{cases}$$

Continuity at $x = -3$

$$\text{LHL} = \lim_{x \to (-3)^-} f(x)$$
$$= \lim_{x \to (-3)^-} (-x+3)$$
$$= \lim_{h \to 0} (3+h+3)$$
$$= 3+3 = 6$$

and $$\text{RHL} = \lim_{x \to (-3)^+} f(x)$$
$$= \lim_{x \to (-3)^+} (-2x)$$
$$= \lim_{h \to 0} [-2(-3+h)]$$
$$= \lim_{h \to 0} (6-2h)$$
$$\text{RHL} = 6$$

Also value of $f(x)$ at $x = -3$,

$$f(-3) = = -(-3)+3$$
$$= 3+3 = 6$$
$$\text{LHL} = \text{RHL} = f(-3)$$

Thus $f(x)$ is continuous at $x = -3$. So, $x = -3$ is the point of continuity.

Continuity at $x = 3$

$$\text{LHL} = \lim_{x \to 3^-} f(x)$$
$$= \lim_{x \to 3^-} [-(2x)]$$
$$= \lim_{h \to 0} [-2(3-h)]$$
$$= \lim_{h \to 0} (-6+2h)$$
$$\text{LHL} = -6$$

and $$\text{RHL} = \lim_{x \to 3^+} f(x)$$
$$= \lim_{x \to 3^+} (6x+2)$$
$$= \lim_{h \to 0} [6(3+h)+2]$$
$$= \lim_{h \to 0} (18+6h+2)$$
$$\text{RHL} = 20$$

since $\text{LHL} \neq \text{RHL}$

Thus $f(x)$ is discontinuous at $x = 3$.

Now $\left[\dfrac{d^2y}{dt^2}\right]_{t=\frac{\pi}{4}} = -\sin\dfrac{\pi}{4} = -\dfrac{1}{\sqrt{2}}$

Again, $x = \cos t + \log\tan\dfrac{t}{2}$

Differentiate w.r.t. t we have

$$\begin{aligned}\frac{dx}{dt} &= -\sin t + \frac{1}{\tan\frac{t}{2}}\cdot\sec^2\frac{t}{2}\cdot\frac{1}{2}\\ &= -\sin t + \frac{\cos\frac{t}{2}}{2\cdot\sin\frac{t}{2}}\cdot\frac{1}{\cos^2\frac{t}{2}}\\ &= -\sin t + \frac{1}{2\cdot\sin\frac{t}{2}}\cdot\frac{1}{\cos\frac{t}{2}}\\ &= -\sin t + \frac{1}{\sin\frac{2\times t}{2}} \qquad 2\sin a\cos a = \sin 2a\\ &= -\sin t + \operatorname{cosec} t \qquad \text{...(ii)}\end{aligned}$$

Now, $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{\cos t}{\operatorname{cosec} t - \sin t}$ From (i) and (ii)

$$\begin{aligned}&= \frac{\cos t}{1-\sin^2 t}\cdot\sin t = \frac{\sin t\cdot\cos t}{\cos^2 t}\\ \frac{dy}{dx} &= \tan t\end{aligned}$$

Now $$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right)\\ &= \frac{d}{dx}(\tan t)\\ &= \sec^2 t\cdot\frac{dt}{dx}\\ &= \frac{\sec^2 t}{\frac{dx}{dt}}\\ &= \frac{\sec^2 t}{\operatorname{cosec} t - \sin t}\\ &= \frac{\sec^2 t\sin t}{1-\sin^2 t}\\ &= \frac{\sec^2 t\cdot\sin t}{\cos^2 t}\\ &= \sec^3 t\cdot\tan t\end{aligned}$$

$$\begin{aligned}\left[\frac{d^2y}{dx^2}\right]_{t=\frac{\pi}{4}} &= \sec^3\frac{\pi}{4}\cdot\tan\frac{\pi}{4}\\ &= 2\sqrt{2}\times 1 = 2\sqrt{2}\end{aligned}$$

140. If $x = a\sec^3\theta$ and $y = a\tan^3\theta$, find $\dfrac{d^2y}{dx^2}$ at $\theta = \dfrac{\pi}{3}$.

Sol : Comp 2018

We have $y = a\tan^3\theta$ and $x = a\sec^3\theta$

Differentiating w.r.t. θ, we get

$$\begin{aligned}\frac{dy}{d\theta} &= 3a\tan^2\theta\frac{d}{d\theta}(\tan\theta)\\ &= 3a\tan^2\theta\sec^2\theta\end{aligned}$$

and $$\begin{aligned}\frac{dx}{d\theta} &= 3a\sec^2\theta\frac{d}{d\theta}(\sec\theta)\\ &= 3a\sec^2\theta\sec\theta\tan\theta\end{aligned}$$

Noe $$\begin{aligned}\frac{dy}{dx} &= \frac{dy/d\theta}{dx/d\theta}\\ &= \frac{3a\tan^2\theta\sec^2\theta}{3a\sec^2\theta\sec\theta\tan\theta}\\ &= \frac{\tan\theta}{\sec\theta} = \sin\theta\end{aligned}$$

Again differentiating both sides w.r.t. x, we get

$$\begin{aligned}\frac{d}{dx}\left(\frac{dy}{dx}\right) &= \frac{d}{dx}(\sin\theta)\\ &= \frac{d}{d\theta}(\sin\theta)\frac{d\theta}{dx}\\ &= \cos\theta\frac{1}{3a\sec^3\theta\tan\theta}\\ &= \frac{\cos^5\theta}{3a\sin\theta}\end{aligned}$$

At $\theta = \dfrac{\pi}{3}$, $$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{\cos^5\frac{\pi}{3}}{3a\sin\frac{\pi}{3}}\\ &= \frac{\left(\frac{1}{2}\right)^5}{3a\left(\frac{\sqrt{3}}{2}\right)} = \frac{1\times 2}{2^5\times 3a\sqrt{3}}\\ &= \frac{1}{48\sqrt{3}\,a}\end{aligned}$$

141. If $y = x^x$, then prove that $\dfrac{d^2y}{dx^2} - \dfrac{1}{y}\left(\dfrac{dy}{dx}\right)^2 - \dfrac{y}{x} = 0$.

Sol : Delhi 2016, 2014

We have $y = x^x$

Taking log both sides, we get

$$\begin{aligned}\log y &= \log x^x\\ \log y &= x\log x\end{aligned}$$

Differentiating w.r.t. x, we get

$$\begin{aligned}\frac{1}{y}\frac{dy}{dx} &= x\frac{d}{dx}(\log x) + \log x\frac{d}{dx}(x)\\ \frac{1}{y}\frac{dy}{dx} &= x\times\frac{1}{x} + \log x\cdot 1\\ \frac{1}{y}\frac{dy}{dx} &= (1+\log x)\\ \frac{dy}{dx} &= y(1+\log x) \qquad \text{...(i)}\end{aligned}$$

Again, differentiating both sides w.r.t. x, we get

$$\begin{aligned}\frac{d^2y}{dx^2} &= y\frac{d}{dx}(1+\log x) + (1+\log x)\frac{dy}{dx}\\ &= y\times\frac{1}{x} + (1+\log x)\frac{dy}{dx}\\ &= \frac{y}{x} + (1+\log x)\frac{dy}{dx}\\ \frac{d^2y}{dx^2} &= \frac{y}{x} + \frac{1}{y}\left(\frac{dy}{dx}\right)\left(\frac{dy}{dx}\right) \qquad \text{[using Eq. (i)]}\\ \frac{d^2y}{dx^2} - \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - \frac{y}{x} &= 0\end{aligned}$$

Hence proved.

142. Differentiate $\tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$ w.r.t. $\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$, when $x \neq 0$.

Again, differentiating both sides of Eq. (i) w.r.t. x,

$$\frac{d^2y}{dx^2} = \frac{1}{\sin a}\frac{d}{dx}\cos^2(a+y)$$

$$= \frac{1}{\sin a} \times \frac{d}{dy}\cos^2(a+y) \times \frac{dy}{dx}$$

$$= \frac{1}{\sin a} \times 2\cos(a+y)[-\sin(a+y)] \times \frac{dy}{dx}$$

$$= -\frac{2\sin(a+y)\cos(a+y)}{\sin a} \times \frac{dy}{dx}$$

$$\frac{d^2y}{dx^2} = -\frac{\sin 2(a+y)}{\sin a}\frac{dy}{dx}$$

$$\sin a\frac{d^2y}{dx^2} + \sin 2(a+y)\frac{dy}{dx} = 0$$ Hence proved.

145. Find $\frac{dy}{dx}$, if $y = \sin^{-1}\left[\frac{6x - 4\sqrt{1-4x^2}}{5}\right]$.

Sol : OD 2016

We have $y = \sin^{-1}\left[\frac{6x - 4\sqrt{1-4x^2}}{5}\right]$

Substituting $x = \frac{1}{2}\sin\theta$ we have

$$y = \sin^{-1}\left[\frac{6 \times \frac{\sin\theta}{2} - 4\sqrt{1 - 4 \times \left(\frac{\sin\theta}{2}\right)^2}}{5}\right]$$

$$= \sin^{-1}\left(\frac{3\sin\theta - 4\sqrt{1-\sin^2\theta}}{5}\right)$$

$$= \sin^{-1}\left(\frac{3\sin\theta - 4\cos\theta}{5}\right)$$

$$= \sin^{-1}\left(\frac{3}{5}\sin\theta - \frac{4}{5}\cos\theta\right) \qquad ...(1)$$

Let $\cos\phi = \frac{3}{5}$, then $\sin\phi = \frac{4}{5}$

Now, Eq. (1) becomes

$$y = \sin^{-1}(\cos\phi\sin\theta - \sin\phi\cos\theta)$$

$$= \sin^{-1}[\sin(\theta - \phi)] = \theta - \phi$$

$$y = \sin^{-1}(2x) - \cos^{-1}\left(\frac{3}{5}\right)$$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(2x)^2}}\frac{d}{dx}(2x) - 0$$

$$= \frac{2}{\sqrt{1-4x^2}}$$

146. Find the values of a and b, if the following function is differentiable at $x = 1$.

$$f(x) = \begin{cases} x^2 + 3x + a, & x \le 1 \\ bx + 2, & x > 1 \end{cases}$$

Sol : Foreign 2016

We have $f(x) = \begin{cases} x^2 + 3x + a, & x \le 1 \\ bx + 2, & x > 1 \end{cases}$

If $f(x)$ is differentiable at $x = 1$.

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) \qquad ...(i)$$

Here, $\lim_{x \to 1^-} f(x) = \lim_{h \to 0}\frac{f(1-h) - f(1)}{-h}$

$$= \lim_{h \to 0}\frac{(1-h)^2 + 3(1-h) + a - (4+a)}{-h}$$

$$= \lim_{h \to 0}\frac{1 + h^2 - 2h + 3 - 3h + a - 4 - a}{-h}$$

$$= \lim_{h \to 0}\frac{h^2 - 5h}{-h}$$

$$= \lim_{h \to 0}(-h + 5)$$

$$= 5$$

and $\lim_{x \to 1^+} f(x) = \lim_{h \to 0}\frac{f(1+h) - f(1)}{h}$

$$= \lim_{h \to 0}\frac{b(1+h) + 2 - (4+a)}{h}$$

$$= \lim_{h \to 0}\frac{b + bh + 2 - 4 - a}{h}$$

$$= \lim_{h \to 0}\frac{bh + b - a - 2}{h}$$

If $\lim_{x \to 1^+} f(x)$ exist, $b - a - 2$ should be equal to 0, i.e.

$$b - a - 2 = 0 \qquad ...(ii)$$

Now, $\lim_{x \to 1^+} f(x) = \lim_{h \to 0}\frac{bh}{h}$

$$= \lim_{h \to 0} b = b$$

From Eq. (i), we have

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x)$$

$$5 = b$$

$$b = 5$$

Now, on substituting $b = 5$ in Eq. (ii), we get

$$5 - a - 2 = 0$$

$$a = 3$$

Hence, $a = 3$ and $b = 5$.

147. If $y = \tan^{-1}\frac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1+x^2} - \sqrt{1-x^2}}$, $x^2 \le 1$, then find $\frac{dy}{dx}$.

Sol : Delhi 2015

We have $y = \tan^{-1}\frac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1+x^2} - \sqrt{1-x^2}} \qquad (1)$

Substituting $x^2 = \sin\theta \Rightarrow \theta = \sin^{-1}x^2$ we have

$$\sqrt{1+x^2} = \sqrt{1+\sin\theta}$$

$$= \sqrt{\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$$

$$= \sqrt{\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right)^2}$$

or $\sqrt{1+x^2} = \left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right) \qquad (2)$

Sol : Foreign 2015, OD 2014

We have $f(x) = \begin{cases} x, & x < 1 \\ 2-x, & 1 \le x \le 2 \\ -2+3x-x^2, & x > 2 \end{cases}$

Differentiability at $x = 1$:

$$\text{LHD} = \lim_{h \to 0} \frac{f(1-h)-f(1)}{-h}$$
$$= \lim_{h \to 0} \frac{(1-h)-[2-(1)]}{-h}$$
$$= \lim_{h \to 0} \frac{-h}{-h} = 1$$
$$\text{RHD} = \lim_{h \to 0} \frac{f(1+h)-f(1)}{h}$$
$$= \lim_{h \to 0} \frac{2-(1+h)-(2-1)}{h}$$
$$= \lim_{h \to 0} \frac{-h}{h} = -1$$

Here LHD $\neq$ RHD

So, $f(x)$ is not differentiable at $x = 1$.

Differentiability at $x = 2$:

$$\text{LHD} = \lim_{h \to 0} \frac{f(2-h)-f(2)}{-h}$$
$$= \lim_{h \to 0} \frac{2-(2-h)-(2-2)}{-h}$$
$$= \lim_{h \to 0} \frac{h}{-h} = -1$$
$$\text{RHD} = \lim_{h \to 0} \frac{f(2+h)-f(2)}{h}$$
$$= \lim_{h \to 0} \frac{-2+3(2+h)-(2+h)^2-(2-2)}{h}$$
$$= \lim_{h \to 0} \frac{-2+6+3h-(4+h^2+4h)-0}{h}$$
$$= \lim_{h \to 0} \frac{-h^2-h}{h}$$
$$= \lim_{h \to 0} \frac{-h(h+1)}{h} = -(0+1) = -1$$

Here LHD = RHD

So, $f(x)$ is differentiable at $x = 2$.

Hence, $f(x)$ is not differentiable at $x = 1$, but it differentiable at $x = 2$.

151. For what value of λ, the following function is continuous at $x = 0$? Hence, check the differentiability of $f(x)$ at $x = 0$.

$$f(x) = \begin{cases} \lambda(x^2+2), & \text{if } x \le 0 \\ 4x+6, & \text{if } x > 0 \end{cases}$$

Sol : Comp 2015

We have $f(x) = \begin{cases} \lambda(x^2+2), & \text{if } x \le 0 \\ 4x+6, & \text{if } x > 0 \end{cases}$

Here, $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (4x+6)$

$$= \lim_{h \to 0} [4(0+h)+6]$$
$$= \lim_{h \to 0} (4h+6)$$
$$= 4 \times 0 + 6 = 6$$

Now $f(0) = \lambda(0^2+2) = 2\lambda$

If $f(x)$ is continuous at $x = 0$ then we have

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0) \qquad \text{...(i)}$$

Thus $2\lambda = 6 \Rightarrow \lambda = 3$

Now, given function becomes

$$f(x) = \begin{cases} 3(x^2+2), & \text{if } x \le 0 \\ 4x+6, & \text{if } x > 0 \end{cases}$$

Now, let us check the differentiability at $x = 0$.

$$\text{LHD} = \lim_{h \to 0} \frac{f(0-h)-f(0)}{-h}$$
$$= \lim_{h \to 0} \frac{3[(0-h)^2+2]-3(0+2)}{-h}$$
$$= \lim_{h \to 0} \frac{3[h^2+2]-6}{-h}$$
$$= \lim_{h \to 0} (-3h) = 0$$

and
$$\text{RHD} = \lim_{h \to 0} \frac{f(0+h)-f(0)}{h}$$
$$= \lim_{h \to 0} \frac{[4(0+h)+6]-3(0+2)}{h}$$
$$= \lim_{h \to 0} \frac{4h}{h} = 4$$

LHD $\neq$ RHD

So, $f(x)$ is not differentiable at $x = 0$.

152. If $y = \dfrac{x\cos^{-1}x}{\sqrt{1-x^2}} - \log\sqrt{1-x^2}$, then prove that $\dfrac{dy}{dx} = \dfrac{\cos^{-1}x}{(1-x^2)^{3/2}}$.

Sol : Delhi 2015, SQP 2012

We have, $y = \dfrac{x\cos^{-1}x}{\sqrt{1-x^2}} - \log\sqrt{1-x^2}$ ...(1)

Differentiating both sides of Eq. (1) w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x\cos^{-1}x}{\sqrt{1-x^2}}\right) - \frac{d}{dx}(\log\sqrt{1-x^2})$$

$$= \frac{\sqrt{1-x^2}\left[x \cdot \dfrac{(-1)}{\sqrt{1-x^2}} + \cos^{-1}x\right] - x\cos^{-1}x \cdot \dfrac{1}{2\sqrt{1-x^2}}(-2x)}{(\sqrt{1-x^2})^2} - \frac{1}{\sqrt{1-x^2}} \cdot \frac{(-2x)}{2\sqrt{1-x^2}}$$

$$= \frac{-x+\sqrt{1-x^2}\cos^{-1}x + \dfrac{x^2\cos^{-1}x}{\sqrt{1-x^2}}}{(\sqrt{1-x^2})^2} + \frac{x}{(\sqrt{1-x^2})^2}$$

Now $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

$= \frac{d}{dx}(\tan t)$

$= \sec^2 t \cdot \frac{dt}{dx}$

$= \frac{\sec^2 t}{\frac{dx}{dt}} = \frac{\sec^2 t}{a\operatorname{cosec} t - a\sin t}$

$= \frac{\sec^2 t}{a\frac{1}{\sin t} - a\sin t} = \frac{\sec^2 t\sin t}{a(1-\sin^2 t)}$

$= \frac{\sec^2 t \cdot \sin t}{a\cos^2 t} = \frac{\sec^2 t \cdot \sin t}{a\frac{1}{\sec t}\cos t}$

$= \frac{\sec^3 t \cdot \tan t}{a}$

$\left[\frac{d^2y}{dx^2}\right]_{t=\frac{\pi}{3}} = \frac{\sec^3\frac{\pi}{3} \cdot \tan\frac{\pi}{3}}{a}$

$= \frac{2^2 \times \sqrt{3}}{a} = \frac{8\sqrt{3}}{a}$

156. If $x^m y^n = (x+y)^{m+n}$, prove that $\frac{dy}{dx} = \frac{y}{x}$.

Sol : Foreign 2014

We have $x^m y^n = (x+y)^{m+n}$

Taking log both sides, we get

$$\log(x^m y^n) = \log(x+y)^{m+n}$$

$$\log(x^m) + \log(y^n) = (m+n)\log(x+y)$$

$$m\log x + n\log y = (m+n)\log(x+y)$$

Differentiating both sides w.r.t. x, we get

$$\frac{m}{x} + \frac{n}{y}\frac{dy}{dx} = \frac{m+n}{x+y}\left(1+\frac{dy}{dx}\right)$$

$$\frac{m}{x} + \frac{n}{y}\frac{dy}{dx} = \frac{m+n}{x+y} + \frac{m+n}{x+y}\frac{dy}{dx}$$

$$\frac{m}{x} - \frac{(m+n)}{x+y} = \left(\frac{m+n}{x+y} - \frac{n}{y}\right)\frac{dy}{dx}$$

$$\left[\frac{my+ny-nx-ny}{y(x+y)}\right]\frac{dy}{dx} = \frac{mx+my-mx-nx}{x(x+y)}$$

$$\frac{dy}{dx}\left[\frac{my-nx}{y}\right] = \frac{my-nx}{x}$$

Hence, $\frac{dy}{dx} = \frac{y}{x}$

157. Differentiate $\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$ w.r.t. $\cos^{-1}(2x\sqrt{1-x^2})$, when $x \neq 0$.

Sol : Delhi 2013

We have $u = \tan^{-1}\left[\frac{\sqrt{1-x^2}}{x}\right]$

Substituting $x = \cos\theta \Rightarrow \theta = \cos^{-1}x$ we have

Then, $u = \tan^{-1}\left[\frac{\sqrt{1-\cos^2\theta}}{\cos\theta}\right]$

$= \tan^{-1}\left[\frac{\sqrt{\sin^2\theta}}{\cos\theta}\right]$

$= \tan^{-1}\left[\frac{\sin\theta}{\cos\theta}\right]$

$= \tan^{-1}[\tan\theta]$

$= \theta$

$u = \cos^{-1}x$

Differentiating both sides w.r.t. x, we get

$$\frac{du}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

Again, let $v = \cos^{-1}(2x\sqrt{1-x^2})$

Substituting $x = \cos\theta \Rightarrow \theta = \cos^{-1}x$ we have

$v = \cos^{-1}[2\cos\theta\sqrt{1-\cos^2\theta}]$

$= \cos^{-1}[2\cos\theta\sin\theta]$

$= \cos^{-1}[\sin 2\theta]$

$= \cos^{-1}\left[\cos\left(\frac{\pi}{2} - 2\theta\right)\right]$

$= \frac{\pi}{2} - 2\theta$

$v = \frac{\pi}{2} - 2\cos^{-1}x$

Differentiating both sides w.r.t. x, we get

$$\frac{dv}{dx} = \frac{2}{\sqrt{1-x^2}}$$

Now, $\frac{du}{dv} = \frac{du}{dx} \times \frac{dx}{dv}$

$= -\frac{1}{\sqrt{1-x^2}} \times \frac{\sqrt{1-x^2}}{x} = -\frac{1}{2}$

158. Differentiate $\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$ w.r.t. $\sin^{-1}(2x\sqrt{1-x^2})$.

Sol : Delhi 2012, Foreign 2011

We have $u = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$

Substituting $x = \sin\theta \Rightarrow \theta = \sin^{-1}x$, we have

$u = \tan^{-1}\left[\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}\right]$

$= \tan^{-1}\left[\frac{\sin\theta}{\cos\theta}\right]$

$= \tan^{-1}(\tan\theta)$

$= \theta$

Thus $u = \sin^{-1}x$

Differentiating both sides w.r.t. x, we get

$$\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}} \quad ...(i)$$

Again, let $v = \sin^{-1}(2x\sqrt{1-x^2})$

Substituting $x = \sin\theta \Rightarrow \theta = \sin^{-1}x$ we have

162. If $x = a(\cos t + t\sin t)$ and $y = a(\sin t - t\cos t)$, then ind the value of $\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{4}$.

Sol : Foreign 2014

We have $y = a(\sin t - t\cos t)$

Differentiate w.r.t. t we have

$$\frac{dy}{dt} = a(\cos t - \cos t + t\sin t)$$
$$= a(t\sin t) \qquad ...(i)$$

Again, $x = a(\cos t + t\sin t)$

Differentiate w.r.t. t we have

$$\frac{dx}{dt} = a(-\sin t + \sin t + t\cos t)$$
$$= a(t\cos t) \qquad ...(ii)$$

Now, $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ From (i) and (ii)

$$= \frac{at\sin t}{at\cos t}$$
$$= \frac{\sin t}{\cos t}$$
$$\frac{dy}{dx} = \tan t$$

Now $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$
$$= \frac{d}{dx}(\tan t)$$
$$= \sec^2 t \cdot \frac{dt}{dx}$$
$$= \frac{\sec^2 t}{\frac{dx}{dt}} = \frac{\sec^2 t}{at\cos t}$$
$$= \frac{\sec^3 t}{at}$$
$$\left[\frac{d^2y}{dx^2}\right]_{t=\frac{\pi}{4}} = \frac{\sec^3\frac{\pi}{4}}{a\frac{\pi}{4}}$$
$$= \frac{(\sqrt{2})^3}{a\frac{\pi}{4}} = \frac{8\sqrt{2}}{a\pi}$$

Hence proved.

163. If $(\tan^{-1}x)^y + y^{\cot x} = 1$, then find $\frac{dy}{dx}$.

Sol : Foreign 2011

Let $u = (\tan^{-1}x)^y$ and $v = y^{\cot x}$

Then, given equation becomes

$$u + v = 1$$

Differentiating both sides w.r.t. x, we get

$$\frac{du}{dx} + \frac{dv}{dx} = 0 \qquad ...(i)$$

Now, $u = (\tan^{-1}x)^y$

Taking log both sides, we get

$$\log u = y\log(\tan^{-1}x)$$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{u}\frac{du}{dx} = \frac{d}{dx}(y)\cdot\log(\tan^{-1}x) + y\frac{d}{dx}(\log\tan^{-1}x)$$
$$\frac{1}{u}\frac{du}{dx} = \frac{dy}{dx}\log(\tan^{-1}x) + \frac{y}{(\tan^{-1}x)(1+x^2)}$$
$$\frac{du}{dx} = (\tan^{-1}x)^y\left[\frac{dy}{dx}\log(\tan^{-1}x) + \frac{y}{(\tan^{-1}x)(1+x^2)}\right] \quad .(ii)$$

Also, $v = y^{\cot x}$

Taking log both sides, we get

$$\log v = \cot x\log y$$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{v}\frac{dv}{dx} = \frac{d}{dx}(\cot x)\cdot\log y + \cot x\frac{d}{dx}(\log y)$$
$$\frac{1}{v}\frac{dv}{dx} = -\operatorname{cosec}^2 x\log y + \frac{\cot x}{y}\frac{dy}{dx}$$
$$\frac{dv}{dx} = y^{\cot x}\left[-\operatorname{cosec}^2 x\log y + \frac{\cot x}{y}\frac{dy}{dx}\right] \quad ...(iii)$$

Substituting values from Eqs. (ii) and (iii) in Eq. (i), we get

$$(\tan^{-1}x)^y\left[\frac{dy}{dx}\log(\tan^{-1}x) + \frac{y}{(\tan^{-1}x)(1+x^2)}\right] + y^{\cot x}\left[-\operatorname{cosec}^2 x\log y + \frac{\cot x}{y}\frac{dy}{dx}\right] = 0$$

$$\frac{dy}{dx}[(\tan^{-1}x)^y\log(\tan^{-1}x) + \cot x\cdot y^{\cot x - 1}]$$
$$= -\left[\frac{y}{1+x^2}(\tan^{-1}x)^{y-1} - y^{\cot x}\operatorname{cosec}^2 x\log y\right]$$
$$\frac{dy}{dx} = \frac{-\left[\frac{y}{1+x^2}(\tan^{-1}x)^{y-1} - y^{\cot x}\cdot\operatorname{cosec}^2 x\log y\right]}{[(\tan^{-1}x)^y\log(\tan^{-1}x) + \cot x\cdot y^{\cot x-1}]}$$

164. If $y = x\log\left(\frac{x}{a+bx}\right)$, then prove that

$$x^3\frac{d^2y}{dx^2} = \left(x\frac{dy}{dx} - y\right)^2.$$

Sol : Foreign 2013, OD 2008

We have $y = x\log\left(\frac{x}{a+bx}\right) \qquad ...(i)$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = x\frac{d}{dx}\log\left(\frac{x}{a+bx}\right) + \log\left(\frac{x}{a+bx}\right)\frac{d}{dx}(x)$$
$$= x\left(\frac{1}{\frac{x}{a+bx}}\right)\frac{d}{dx}\left(\frac{x}{a+bx}\right) + \log\left(\frac{x}{a+bx}\right)\cdot 1$$
$$= (a+bx)\left[\frac{(a+bx)(1) - x(b)}{(a+bx)^2}\right] + \log\left(\frac{x}{a+bx}\right)$$
$$= (a+bx)\left[\frac{a}{(a+bx)^2}\right] + \log\left(\frac{x}{a+bx}\right)$$
$$\frac{dy}{dx} = \frac{a}{a+bx} + \log\left(\frac{x}{a+bx}\right)$$
$$\frac{dy}{dx} = \frac{a}{a+bx} + \frac{y}{x} \qquad \text{[using Eq. (i)]}$$
$$\frac{dy}{dx} - \frac{y}{x} = \frac{a}{a+bx}$$

$$= \lim_{h \to 0} |3 + h - 3|$$
$$= \lim_{h \to 0} |h| = 0$$

and $\quad f(3) = |3 - 3| = 0$

Thus, $\quad \lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = f(3)$

Hence, $f(x)$ is continuous at $x = 3$.

Now, let us check the differentiability of $f(x)$ at $x = 3$.

$$\text{LHD } f'(3^-) = \lim_{h \to 0} \frac{f(3-h) - f(3)}{-h}$$
$$= \lim_{h \to 0} \frac{|3 - h - 3| - |3 - 3|}{-h}$$
$$= \lim_{h \to 0} \frac{|-h|}{-h}$$
$$= \lim_{h \to 0} \frac{h}{-h} = -1 \qquad |-x| = x, \text{if } x > 0$$

$$\text{RHD } f'(3^+) = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h}$$
$$= \lim_{h \to 0} \frac{|3 + h - 3| - |3 - 3|}{h}$$
$$= \lim_{h \to 0} \frac{|h|}{h}$$
$$= \lim_{h \to 0} \frac{h}{h} = 1 \qquad |x| = x, \text{if } x > 0$$

Since, $f'(3^-) \neq f'(3^+)$ at $x = 3$, $f(x)$ is not differentiable. Hence proved.

168. If $x = a\sin t$ and $y = a[\cos t + \log\tan\frac{t}{2}]$, then find $\frac{d^2y}{dx^2}$.

Sol : Comp 2013

We have $\quad x = a\sin t$

Differentiate w.r.t. t we have

$$\frac{dx}{dt} = a\cos t \qquad ...(i)$$

Again, $\quad y = a\left(\cos t + \log\tan\frac{t}{2}\right)$

Differentiate w.r.t. t we have

$$\frac{dy}{dt} = -a\sin t + \frac{a}{\tan\frac{t}{2}} \cdot \sec^2\frac{t}{2} \cdot \frac{1}{2}$$
$$= -a\sin t + \frac{a\cos\frac{t}{2}}{2 \cdot \sin\frac{t}{2}} \cdot \frac{1}{\cos^2\frac{t}{2}}$$
$$= -a\sin t + \frac{a}{2 \cdot \sin\frac{t}{2}} \cdot \frac{1}{\cos\frac{t}{2}}$$
$$= -a\sin t + \frac{a}{\sin\frac{2 \times t}{2}} \qquad 2\sin a\cos a = \sin 2a$$
$$= -a\sin t + a\operatorname{cosec} t$$
$$= a\operatorname{cosec} t - a\sin t \qquad ...(ii)$$

Now, $\quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \qquad$ From (i) and (ii)

$$= \frac{a\operatorname{cosec} t - a\sin t}{a\cos t}$$
$$= \frac{\frac{1}{\sin t} - \sin t}{\cos t}$$
$$= \frac{1 - \sin^2 t}{\sin t\cos t}$$
$$= \frac{\cos^2 t}{\sin t\cos t}$$
$$= \frac{\cos t}{\sin t}$$
$$\frac{dy}{dx} = \cot t$$

Now $\quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$
$$= \frac{d}{dx}(\cot t)$$
$$= -\operatorname{cosec}^2 t \cdot \frac{dt}{dx}$$
$$= -\frac{\operatorname{cosec}^2 t}{\frac{dx}{dt}}$$
$$= -\frac{\operatorname{cosec}^2 t}{a\cos t}$$

169. If $x = a\cos^3\theta$ and $y = a\sin^3\theta$, then find the value of $\frac{d^2y}{dx^2}$ at $\theta = \frac{\pi}{6}$.

Sol : OD 2013

We have $\quad x = a\cos^3\theta$ and $y = a\sin^3\theta$

Differentiating both sides of x and y w.r.t. θ, we get

$$\frac{dx}{d\theta} = 3a\cos^2\theta\frac{d}{d\theta}(\cos\theta)$$
$$= 3a\cos^2\theta \cdot (-\sin\theta)$$
$$= -3a\cos^2\theta \cdot \sin\theta$$

and $\quad \frac{dy}{d\theta} = 3a\sin^2\theta\frac{d}{d\theta}(\sin\theta)$
$$= 3a\sin^2\theta \cdot (\cos\theta)$$
$$= 3a\sin^2\theta \cdot \cos\theta$$

Now, $\quad \frac{dy}{dx} = \left(\frac{dy/d\theta}{dx/d\theta}\right)$
$$= \frac{3a\sin^2\theta \cdot \cos\theta}{-3a\cos^2\theta \cdot \sin\theta}$$
$$= -\tan\theta$$

Again, Differentiating both sides w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(-\tan\theta) = -\frac{d}{d\theta}(\tan\theta)\frac{d\theta}{dx}$$
$$= -\sec^2\theta \cdot \frac{d\theta}{dx} \qquad \frac{d\theta}{dx} = \frac{-1}{3a\cos^2\theta \cdot \sin\theta}$$
$$= -\sec^2\theta \cdot \left(\frac{-1}{3a\cos^2\theta \cdot \sin\theta}\right)$$
$$\frac{d^2y}{dx^2} = \frac{1}{3a\cos^4\theta \cdot \sin\theta}$$

$$= \frac{d}{dt}(\tan t)\frac{dt}{dx}$$
$$= \sec^2 t \frac{1}{dx/dt}$$
$$= \frac{\sec^2 t}{at\cos t} = \frac{\sec^3 t}{at} \qquad \text{[from Eqs. (i)]}$$

Also, $$\frac{d^2x}{dt^2} = \frac{d}{dt}(at\cos t)$$
$$= a\frac{d}{dt}(t\cos t)$$
$$= a\left[\frac{d}{dt}(t)\cdot\cos t + t\frac{d}{dt}(\cos t)\right]$$
$$= a[\cos t - t\sin t]$$

and $$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d}{dt}(at\sin t)$$
$$= a(\sin t + t\cos t)$$

173. If $x = \left(\cos t + \log\tan\frac{t}{2}\right)$ and $y = a\sin t$, find $\frac{d^2y}{dt^2}$ and $\frac{d^2y}{dx^2}$.

Sol : SQP 2012, OD 2007

We have $$y = a\sin t$$

Differentiate w.r.t. t we have

$$\frac{dy}{dt} = a\cos t \qquad ...(i)$$

Again $$\frac{d^2y}{dt^2} = -a\sin t$$

Again, $$x = \cos t + \log\tan\frac{t}{2}$$

Differentiate w.r.t. t we have

$$\frac{dx}{dt} = -\sin t + \frac{1}{\tan\frac{t}{2}}\cdot\sec^2\frac{t}{2}\cdot\frac{1}{2}$$
$$= -\sin t + \frac{\cos\frac{t}{2}}{2\cdot\sin\frac{t}{2}}\cdot\frac{1}{\cos^2\frac{t}{2}}$$
$$= -\sin t + \frac{1}{\sin 2\times\frac{t}{2}} \qquad 2\sin a\cos a = \sin 2a$$
$$= -\sin t + \operatorname{cosec} t \qquad ...(ii)$$

Now, $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{a\cos t}{\operatorname{cosec} t - \sin t} \qquad \text{From (i) and (ii)}$$
$$= \frac{a\cos t\cdot\sin t}{1-\sin^2 t}$$
$$= \frac{a\cos t\cdot\sin t}{\cos^2 t}$$
$$\frac{dy}{dx} = a\tan t$$

Now $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$
$$= \frac{d}{dx}(a\tan t)$$
$$= a\sec^2 t\cdot\frac{dt}{dx}$$
$$= \frac{a\sec^2 t}{\frac{dx}{dt}}$$
$$= \frac{a\sec^2 t}{\operatorname{cosec} t - \sin t}$$
$$= \frac{a\sec^2 t\sin t}{1-\sin^2 t}$$
$$= \frac{a\sec^2 t\cdot\sin t}{\cos^2 t}$$
$$= a\sec^3 t\cdot\tan t$$

174. Find $\frac{dy}{dx}$, when $y = x^{\cot x} + \frac{2x^2-3}{x^2+x+2}$.

Sol : OD 2007

We have $$y = x^{\cot x} + \frac{2x^2-3}{x^2+x+2}$$

Let $$u = x^{\cot x} \text{ and } v = \frac{2x^2-3}{x^2+x+2}$$

Then, given equation becomes

$$y = u + v$$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

Consider $y = x^{\cot x}$

Taking log both sides, we get

$$\log u = \cot x\log x$$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{u}\frac{du}{dx} = \cot x\cdot\frac{1}{x} - \operatorname{cosec}^2 x\cdot\log x$$
$$\frac{du}{dx} = u\left(\frac{\cot x}{x} - \operatorname{cosec}^2 x\cdot\log x\right)$$
$$= x^{\cot x}\left(\frac{\cot x}{x} - \operatorname{cosec}^2 x\cdot\log x\right) \qquad ...(ii)$$

Now, consider $v = \frac{2x^2-3}{x^2+x+2}$

Differentiating both sides w.r.t. x, we get

$$\frac{dv}{dx} = \frac{(x^2+x+2)(4x)-(2x^2-3)(2x+1)}{(x^2+x+2)^2}$$
$$= \frac{4x^3+4x^2+8x-4x^3-2x^2+6x+3}{(x^2+x+2)^2}$$
$$= \frac{2x^2+14x+3}{(x^2+x+2)^2} \qquad ...(iii)$$

Substituting the values from Eqs. (ii) and (iii) to Eq. (i), we get

$$\frac{dy}{dx} = x^{\cot x}\left(\frac{\cot x}{x} - \operatorname{cosec}^2 x\cdot\log x\right) + \frac{2x^2+14x+3}{(x^2+x+2)^2}$$

175. Differentiate $x^{x\cos x} + \frac{x^2+1}{x^2-1}$ w.r.t. x.

$$= \lim_{h \to 0} [2(2-h)^2 - (2-h)]$$
$$= \lim_{h \to 0} [2(2+h^2-4h) - (2-h)]$$
$$= \lim_{h \to 0} (8 + 2h^2 - 8h - 2 + h)$$
$$\lim_{x \to 2^-} f(x) = 8 - 2 = 6$$

Now $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (5x - 4)$

$$= \lim_{h \to 0} (5(2+h) - 4)$$
$$= \lim_{h \to 0} (10 + 5h - 4)$$
$$= \lim_{h \to 0} (6 + 5h)$$
$$\lim_{x \to 2^+} f(x) = 6$$

and $f(2) = 2(2)^2 - 2 = 8 - 2 = 6$

Since, $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = f(2)$, $f(x)$ is continuous at $x = 2$.

Differentiating at $x = 2$:

$$f'(2^-) = \lim_{h \to 0} \frac{f(2-h) - f(2)}{-h}$$
$$= \lim_{h \to 0} \frac{[2(2-h)^2 - (2-h)] - [8-2]}{-h}$$
$$= \lim_{h \to 0} \frac{2(4+h^2-4h) - (2-h) - 6}{-h}$$
$$= \lim_{h \to 0} \frac{8 + 2h^2 - 8h - 2 + h - 6}{h}$$
$$= \lim_{h \to 0} \frac{h(2h-7)}{-h} = \lim_{h \to 0} -(2h-7)$$
$$f'(2^-) = 7$$

and $f'(2^+) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$

$$= \lim_{h \to 0} \frac{[5(2+h) - 4] - [8-2]}{h}$$
$$= \lim_{h \to 0} \frac{(6+5h) - (6)}{h} = \lim_{h \to 0} \frac{5h}{h}$$
$$f'(2^+) = 5$$

Since $f'(2^-) \neq f'(2^+)$, $f(x)$ is not differentiating at $x = 2$. Hence, $f(x)$ is continuous at $x = 1$ and $x = 2$ but not differentiable at $x = 2$. Hence proved.

178. Find $\frac{dy}{dx}$, if $y = (\cos x)^x + (\sin x)^{1/x}$.

Sol : Delhi 2003

We have $y = (\cos x)^x + (\sin x)^{1/x}$

Let $u = (\cos x)^x$ and $v = (\sin x)^{1/x}$

Then, given equation becomes

$$y = u + v$$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad ...(1)$$

Consider, $u = (\cos x)^x$. Taking log both sides, we get

$$\log u = \log(\cos x)^x$$
$$\log u = x \log(\cos x) \qquad \log m^n = n \log m$$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{u}\frac{du}{dx} = x \cdot \frac{d}{dx}\log(\cos x) + \log(\cos x) \cdot \frac{d}{dx}(x)$$
$$= x \cdot \frac{1}{\cos x}(-\sin x) + \log \cos x \cdot 1$$
$$\frac{1}{u}\frac{du}{dx} = -x \tan x + \log(\cos x)$$
$$\frac{du}{dx} = u[-x \tan x + \log \cos x]$$
$$\frac{du}{dx} = (\cos x)^x[-x \tan x + \log \cos x] \quad ...(2)$$

Now, consider $v = (\sin x)^{1/x}$. Taking log both sides, we get

$$\log v = \log(\sin x)^{1/x}$$
$$\log v = \frac{1}{x}\log \sin x \qquad \log m^n = n \log m$$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{v} \cdot \frac{dv}{dx} = \frac{1}{x} \cdot \frac{d}{dx}(\log \sin x) + \log \sin x \cdot \frac{d}{dx}\left(\frac{1}{x}\right)$$
$$= \frac{1}{x} \cdot \frac{1}{\sin x} \cdot \cos x + \log \sin x\left(-\frac{1}{x^2}\right)$$
$$\frac{1}{v} \cdot \frac{dv}{dx} = \frac{\cot x}{x} - \frac{\log(\sin x)}{x^2}$$
$$\frac{dv}{dx} = v\left(\frac{\cot x}{x} - \frac{\log(\sin x)}{x^2}\right)$$
$$\frac{dv}{dx} = (\sin)^{1/x}\left[\frac{\cot x}{x} - \frac{\log(\sin x)}{x^2}\right] \quad ...(3)$$

Now, from eqs. (1), (2) and (3), we get

$$\frac{dy}{dx} = (\cos x)^x[-x \tan x + \log \cos x] + (\sin x)^{1/x}\left[\frac{\cot x}{x} - \frac{\log(\sin x)}{x^2}\right]$$

179. If $y = (x)^x + (\sin x)^x$, then find $\frac{dy}{dx}$.

Sol : OD 2002

We have $y = (x)^x + (\sin x)^x$

Let $u = (x)^x$ and $v = (\sin x)^x$

Now given equation becomes,

$$y = u + v$$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad ...(i)$$

Consider, $u = x^x$. Taking log both sides, we get

$$\log u = \log x^x$$

the pencils produced, cost of transportation is twice the number of pencils produced and the property tax costs ` 5000. Then,

(i) Find the cost function $C(x)$.

(ii) Find the cost of producing 21st pencil.

(iii) The marginal cost of producing 50 pencils.

Sol :

(i) Cost function $C(x)$,

The cost function is sum of variable cost and fixed cost.

Thus $$C(x) = x^2 + 2x + 5000$$

(ii) Find the cost of producing 21st pencil.

$$\begin{aligned} C(21) - C(20) &= \left[(21)^2 + 2 \times 21 + 5000\right] \\ &\quad - \left[(20)^2 + 2 \times 20 + 5000\right] \\ &= (21)^2 - (20)^2 + 21 \\ &= 41 + 21 = 62 \end{aligned}$$

The cost of producing 21st pencil is ` 62.

(iii) The marginal cost of producing 50 pencils.

$$MC = \frac{dC(x)}{dx} = 2x + 2$$

$$(MC)_{x=50} = 2 \times 50 + 2 = 102$$

The marginal cost of producing 50 pencils is ` 102.

182. Hindustan Times is an Indian English-language daily newspaper based in Delhi. It is the flagship publication of HT Media Limited, an entity controlled by Shobhana Baratia.

Monday, August 31, 2020

Hindustan Times

FIRST VOICE. LAST WORD.

India first to hit 80k cases in single day

Vaccine diplomacy in India's outreach plan

Diplomatic immunity

Won't borrow to meet GST deficit, some states say

It is estimated that t years from now, the circulation of a Hindustan Times newspaper will be $C(t) = 100t^2 + 400t + 5000$ (in Thousand).

(i) Derive an expression for the rate at which the circulation will be changing with respect to time t years from now.

(ii) At what rate will the circulation be changing with respect to time 5 years from now? Will the circulation be increasing or decreasing at that time?

(iii) By how much will the circulation actually change during the sixth year?

Sol :

We have $$C(t) = 100t^2 + 400t + 5000$$

(i) Expression for the rate at which the circulation will be changing

$$C'(t) = 200t + 400$$

(ii) The rate of change of the circulation 5 years from now

$$\begin{aligned} C'(5) &= 200 \times 5 + 400 \\ &= 1400 \text{ Thousand} \qquad \text{Increasing} \end{aligned}$$

(iii) The actual change in the circulation during the 6th year is

$$\begin{aligned} C(6) &= 100 \times 6^2 + 400 \times 6 + 5000 \\ &= 11000 \text{ Thousand} \\ C(5) &= 100 \times 5^2 + 400 \times 5 + 5000 \\ &= 9500 \text{ Thousand} \\ C(6) - C(5) &= 11000 - 9500 \\ &= 1500 \text{ Thousand newspaper} \end{aligned}$$

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CHAPTER 6

APPLICATION OF DERIVATIVES

OBJECTIVE QUESTIONS

1. The function $f(x) = \frac{x}{2} + \frac{2}{x}$ has a local minima at x equal to

(a) 2 (b) 1

(c) 0 (d) -2

Sol : OD 2024

We have $f(x) = \frac{x}{2} + \frac{2}{x}$

Differentiate both sides w.r.t. x we get

$$f'(x) = \frac{d}{dx}\left(\frac{x}{2} + \frac{2}{x}\right)$$

$$= \frac{d}{dx}\left(\frac{x}{2}\right) + \frac{d}{dx}\left(\frac{2}{x}\right)$$

$$f'(x) = \frac{1}{2} - \frac{2}{x^2} \quad ...(1)$$

Function has local minima at x if $f'(x) = 0$ and $f''(x) > 0$.

Substituting $f'(x) = 0$ in eq (1) we have

$$0 = \frac{1}{2} - \frac{2}{x^2}$$

$$\frac{2}{x^2} = \frac{1}{2}$$

$$x^2 = 4$$

$$x = \pm 2$$

Differentiating eq (1) w.r.t. x we have

$$f''(x) = \frac{d}{dx}\left(\frac{1}{2} - \frac{2}{x^2}\right)$$

$$= -\left(\frac{2(-2)}{x^3}\right)$$

$$f''(x) = \frac{4}{x^3}$$

At $x = -2$ we have

$$f''(-2) = \frac{4}{(-2)^3}$$

$$f''(-2) = \frac{-4}{8} < 0$$

Thus we have local maxima at $x = -2$

At $x = -2$,

$$f''(2) = \frac{4}{2^3}$$

$$= \frac{4}{8}$$

$$f''(2) = \frac{1}{2} > 0$$

We have local minima at $x = 2$.

2. Given a curve $y = 7x - x^3$ and x increases at the rate of 2 units per second. The rate at which the slope of the curve is changing, when $x = 5$ is

(a) -60 units/sec (b) 60 units/sec

(c) -70 units/sec (d) -140 units/sec

Sol : OD 2024

We have $y = 7x - x^3$

Differentiating above w.r.t x we have

$$\frac{dy}{dx} = \frac{d}{dx}(7x - x^3)$$

$$= 7x - 3x^2$$

$$m = \frac{dy}{dx} = 7 - 3x^2 \quad ...(1)$$

where m is the slope of $y = 7x - x^3$.

$$m = 7 - 3x^2$$

Since slope is changing, differentiating (1) w.r.t t

$$\frac{dm}{dt} = -6x\frac{dx}{dt} \quad ...(2)$$

Substituting $\frac{dx}{dt} = 2$ units/sec and $x = 5$ in eq (2) we have

$$\frac{dm}{dt} = -6(5)(2)$$

$$= -60$$

3. The function $f(x) = 2 - 3x$ is

(a) decreasing

(b) increasing

(c) neither decreasing nor increasing

(d) none of the above

(a) local maxima at $x=0$

(b) local minima at $x=0$

(c) point of inflexion at $x=0$

(d) none of the above

Sol : OD 2010

We have $\quad f(x) = x|x|$

$$f'(x) = x \cdot \frac{x}{|x|} + |x| = 2|x|$$

$$f'(x) = 0, \text{ only, if } |x| = 0, \text{ ie, if } x = 0$$

Since, $f'(x)$ does not change sign as we move from left to right through origin.
Hence, f has a point of inflexion at $x=0$.
Thus (c) is correct option.

11. Let $f(x) = x - \cos x$, $x \in R$ then f is

(a) a decreasing function

(b) an odd function

(c) an increasing function

(d) none of the above

Sol : SQP 2016, OD 2009

We have $\quad f(x) = x - \cos x$

Differentiating w.r.t x we have

$$f'(x) = 1 + \sin x \geq 0, \text{ for all } x \,\varepsilon\, R.$$

$$(-1 \leq \sin x \leq 1 \; 0 \leq 1 + \sin x \leq 2)$$

Hence, $f(x)$ is a decreasing function.
Thus (c) is correct option.

12. Let $f(x) = \tan x - 4x$, then in the interval $\left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$, $f(x)$ is

(a) a decreasing function

(b) an increasing function

(c) a constant function

(d) none of these

Sol : Foreign 2014

We have $\quad f(x) = \tan x - 4x$

Differentiating w.r.t x we have

$$f'(x) = \sec^2 x - 4 \leq 0 \text{ in } \left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$$

Hence, $f(x)$ is a decreasing function.
Thus (a) is correct option.

13. The radius of a cylinder is increasing at the rate of 3m/s and its altitude is decreasing at the rate of 4m/s . The rate of change of volume when radius is 4m and altitude is 6m, is

(a) $80\pi m^3/s$ (b) $144\pi m^3/s$

(c) $80 m^3/s$ (d) $64 m^3/s$

Sol : Delhi 2018

Let h and r be the height and radius of cylinder.

Given that, $\frac{dr}{dt} = 3\text{m/s}$ and $\frac{dh}{dt} = -4\text{m/s}$

Also, $\quad V = \pi r^2 h$

$$\frac{dV}{dt} = \pi\left[r^2 \frac{dh}{dt} + h \cdot 2r\frac{dr}{dt}\right]$$

At $r = 4$m and $h = 6$m we have

$$\frac{dV}{dt} = \pi[-64 + 144]$$

$$= 80\pi m^3/s$$

Thus (a) is correct option.

14. The function $f(x) = x^3$ has a

(a) local minima at $x=0$

(b) local maxima at $x=0$

(c) point of inflexion at $x=0$

(d) none of the above

Sol : OD 2015, Delhi 2009

We have $\quad f(x) = x^3$

Differentiating w.r.t x we have

$$f'(x) = 3x^2$$

$$f'(0) = 0$$

Similarly, $f''(0) = 0$ and $f'''(0) = 6$

Hence, f has a point of inflexion at $x=0$.
Thus (a) is correct option.

15. The minimum value of $f(x) = \sin x \cos x$ is

(a) $\frac{1}{2}$ (b) $-\frac{1}{2}$

(c) 0 (d) 5

Sol : Delhi 2007

We have $\quad f(x) = \sin x \cos x$

$$= \frac{1}{2}(\sin 2x)$$

Since, the minimum value of sin $2x$ is -1.

Hence, minimum value of $f(x)$ is $-\frac{1}{2}$.
Thus (b) is correct option.

16. The function $f(x) = 2 + 4x^2 + 6x^4 + 8x^6$ has

(a) only one maxima

(b) only one minima

Now letting $f'(x) = 0$ we have

$$2\cos x(1-3\sin x) = 0$$

$$\cos x = 0 \text{ or } \sin x = \frac{1}{3}$$

$$x = \frac{\pi}{2} \text{ or } x = \sin^{-1}\left(\frac{1}{3}\right)$$

At $x = 0$, $f(0) = 1+0+3 = 4$

At $x = \frac{2\pi}{3}$, $f\left(\frac{2\pi}{3}\right) = 1+2\sin\left(\frac{2\pi}{3}\right)+3\cos^2\left(\frac{2\pi}{3}\right)$

$$= 1+2\cdot\frac{\sqrt{3}}{2}+3\left(\frac{1}{2}\right)^2$$

$$= \frac{7}{4}+\sqrt{3}$$

At $x = \frac{\pi}{2}$, $f\left(\frac{\pi}{2}\right) = 1+2+3\cdot 0^2$

$$= 3$$

At $x = \sin^{-1}\frac{1}{3}$,

$$f(\sin^{-1}\tfrac{1}{3}) = 1+\frac{2}{3}+\frac{8}{3} = \frac{13}{3}$$

Hence, $f(x)$ is minimum at $x = \frac{\pi}{2}$.

Thus (a) is correct option.

21. If $a < 0$, the function $f(x) = e^{ax}+e^{-ax}$ is monotonically decreasing for all values of x where

(a) $x < 0$ (b) $x > 0$

(c) $x < 1$ (d) $x > 1$

Sol : SQP 2019

We have $f(x) = e^{ax}+e^{-ax}$

Differentiating w.r.t x we have

$$f'(x) = ae^{ax}-ae^{-ax}$$

$$= a\left(e^{ax}-\frac{1}{e^{ax}}\right) \text{ for all } x \in R.$$

Therefore, f is monotonically decreasing, if

$$f'(x) < 0$$

$$a\left(e^{\alpha x}-\frac{1}{e^{ax}}\right) < 0$$

$$e^{2ax} > 1 \ (a < 0)$$

$$2ax > 0$$

$$x > 0$$

Thus (b) is correct option.

22. The sides of an equilateral triangle are increasing at the rate of 2 cm/s. The rate at which the area increases, when the side is 10 cm, is

(a) $\sqrt{3}\,\text{cm}^2/\text{s}$ (b) $10\text{cm}^2/\text{s}$

(c) $10\sqrt{3}\,\text{cm}^2/\text{s}$ (d) $\frac{10}{\sqrt{3}}\text{cm}^2/\text{s}$

Sol : Foreign 2007

Let x is the length of each side of an equilateral triangle and A is its area, then

$$A = \frac{\sqrt{3}}{4}x^2$$

$$\frac{dA}{dt} = \frac{\sqrt{3}}{4}2x\frac{dx}{dt}$$

At $x = 10\,\text{cm}$, $\frac{dx}{dt} = 2\text{cm}$

$$\frac{dA}{dt} = \frac{\sqrt{3}}{4}\times 2\times 10\times 2$$

$$= 10\sqrt{3}\ \text{cm}^2/\text{s}$$

Thus (c) is correct option.

23. The function $f(x) = x^{1/x}$ is

(a) increasing in $(1,\infty)$

(b) decreasing in $(1,\infty)$

(c) increasing in $(1,e)$ and decreasing in (e,∞)

(d) decreasing in $(1,e)$ and increasing in (e,∞)

Sol : Comp 2017, OD 2014

We have $f(x) = x^{1/x}$

Differentiating w.r.t x we have

$$f'(x) = x^{1/x}\frac{d}{dx}\left(\frac{1}{x}\log x\right)$$

$$= x^{1/x}\left(\frac{1-\log x}{x^2}\right), x > 0$$

Thus, $f'(x) > 0$, iff $1-\log x > 0 \Rightarrow 0 < x < e$

and if $f'(x) < 0$, iff $1-\log x < 0 \Rightarrow x > e$

Hence, $f(x)$ is increasing in $(1,e)$ and decreasing in (e,∞).

Thus (c) is correct option.

24. The least, value of the function $f(x) = ax+\frac{b}{x}$, $a > 0$, $b > 0$, $x > 0$ is

(a) $\sqrt{ab}$ (b) $2\sqrt{\frac{a}{b}}$

(c) $2\sqrt{\frac{b}{a}}$ (d) $2\sqrt{ab}$

Sol : OD 2006

We have $f(x) = ax+\frac{b}{x}$, a, b, $x > 0$

$$f'(x) = a-\frac{b}{x^2}$$

For maxima or minima, putting $f'(x) = 0$ we have

$$x^2 = \frac{b}{a},\ x = \pm\sqrt{\frac{b}{a}}$$

Ch 7 : Integrals

Text Book | Worksheet | Solutions | Worksheet | Presentation

Ch 8 : Application of Integrals

Text Book | Worksheet | Solutions | Worksheet | Presentation

Ch 9 : Differential Equations

Text Book | Worksheet | Solutions | Worksheet | Presentation

Ch 10 : Vector Algebra

Text Book | Worksheet | Solutions | Worksheet | Presentation

Ch 11 : Three Dimensional Geometry

Text Book | Worksheet | Solutions | Worksheet | Presentation

Ch 12 : Linear Programming

Text Book | Worksheet | Solutions | Worksheet | Presentation

Ch 13 : Probability

Text Book | Worksheet | Solutions | Worksheet | Presentation

Class 10th

Maths	Science	Social Science	English LL
Info Tech.	Comp. App.	AI	English Comm.
हिंदी अ	हिंदी ब	संस्कृत	

Class 9th

Maths	Science	Social Science	English LL
Info Tech.	Comp. App.	AI	English Comm.
हिंदी अ	हिंदी ब	संस्कृत	

Class 8th

Maths	Science	Social Science	English
हिंदी	संस्कृत		

Class 7th

Maths	Science	Social Science	English
हिंदी	संस्कृत		

Class 6th

Maths	Science	Social Science	English
हिंदी	संस्कृत		

Class 5th

Maths English हिंदी EVS

Class 4th

Maths English हिंदी EVS

Class 3th

Maths English हिंदी EVS

Class 2th

Maths English हिंदी EVS

Class 1th

Maths English हिंदी EVS

Prep/KG 3rd Year

Maths English हिंदी EVS

Prep/KG 2nd Year

Maths English हिंदी EVS

Prep/KG 1st Year

Maths English हिंदी EVS

$$= \frac{1}{30\pi} \text{ ft/min}$$

Thus correct option is (b).

30. The length of the longest interval, in which $f(x) = 3\sin x - 4\sin^3 x$ is increasing is

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{2}$

(c) $\frac{3\pi}{2}$ (d) π

Sol : SQP 2020

We have $f(x) = 3\sin x - 4\sin^3 x = \sin 3x$

Since, $\sin x$ is increasing in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$$-\frac{\pi}{2} \le 3x \le \frac{\pi}{2}$$

$$-\frac{\pi}{6} \le x \le \frac{\pi}{6}$$

Thus, the length of interval is $\frac{\pi}{6} - \left(-\frac{\pi}{6}\right) = \frac{\pi}{3}$.

Thus correct option is (a).

31. Which of the following function is decreasing on $(0, \frac{\pi}{2})$?

(a) $\sin 2x$ (b) $\cos 3x$

(c) $\tan x$ (d) $\cos 2x$

Sol : Delhi 2015, OD 2007

We have $f(x) = \cos 2x$

$$f'(x) = -2\sin 2x < 0 \text{ in } \left(0, \frac{\pi}{2}\right)$$

So, $\cos 2x$ is decreasing in $\left(0, \frac{\pi}{2}\right)$.

Thus correct option is (d).

32. The minimum radius vector of the curve $\frac{a^2}{x^2} + \frac{b^2}{y^2} = 1$ is of length

(a) $a - b$ (b) $a + b$

(c) $2a + b$ (d) None of these

Sol : OD 2009

We have $\frac{a^2}{x^2} + \frac{b^2}{y^2} = 1.$

Let radius vector be r.

$$r^2 = x^2 + y^2$$

$$r^2 = \frac{a^2 y^2}{y^2 - b^2} + y^2 \qquad \frac{a^2}{x^2} + \frac{b^2}{y^2} = 1$$

$$\frac{d(r^2)}{dy} = \frac{-2yb^2a^2}{(y^2 - b^2)^2} + 2y$$

For minimum value of r, putting $\frac{d(r^2)}{dy} = 0$ we have

$$\frac{-2yb^2a^2}{(y^2 - b^2)^2} + 2y = 0$$

$$\frac{b^2a^2}{(y^2 - b^2)^2} - 1 = 0$$

$$b^2a^2 - (y^2 - b^2)^2 = 0$$

$$(y^2 - b^2)^2 = b^2a^2$$

$$y^2 - b^2 = ba$$

$$y^2 = b(a + b)$$

Substituting above result in equation we get

$$x^2 = a(a + b)$$

Thus $r^2 = (a + b)^2$

$$r = a + b$$

Thus correct option is (b).

33. The condition that $f(x) = ax^3 + bx^2 + cx + d$ has no extreme value is

(a) $b^2 > 3ac$ (b) $b^2 = 4ac$

(c) $b^2 = 3ac$ (d) $b^2 < 3ac$

Sol : Foreign 2010

We have $f(x) = ax^3 + bx^2 + cx + d$

Differentiating w.r.t. x, we get

$$f'(x) = 3ax^2 + 2bx + c$$

For extreme values putting $f'(x) = 0$ we have

$$3ax^2 + 2bx + c = 0$$

Since, it has no extreme value, we have

$$B^2 - 4AC < 0$$

$$(2b)^2 - 4 \times 3a \times c < 0$$

$$4b^2 - 12ac < 0$$

$$b^2 - 3ac < 0$$

$$b^2 < 3ac$$

Thus correct option is (d)

34. The maximum value of xe^{-x} is

(a) e (b) $1/e$

(c) $-e$ (d) $-1/e$

Sol : Comp 2017, Delhi 2007

We have $f(x) = xe^{-x}$

$$f'(x) = -xe^{-x} + e^{-x} = e^{-x}(1 - x)$$

For maxima or minima, putting $f'(x) = 0$ we have

$$e^{-x}(1 - x) = 0 \Rightarrow x = 1$$

Further, $f''(x) < 0$ at $x = 1$. Therefore, $f(x)$ attains its maxima at $x = 1$ and the maximum value is $\frac{1}{e}$.

Thus correct option is (b).

Sol : Foreign 2017, Delhi 2011

Let volume of sphere,

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\pi = 4\pi r^2 \frac{dr}{dt} \qquad \frac{dV}{dt} = \pi$$

$$\frac{dr}{dt} = \frac{1}{4r^2} \qquad ...(i)$$

Now, $$\frac{dS}{dt} = \frac{d}{dt}(4\pi r^2)$$

$$= 4\pi\left(2r\frac{dr}{dt}\right)$$

$$\left(\frac{dS}{dt}\right)_{r=1} = 4\pi\left(2\cdot 4\cdot \frac{1}{4}\right)$$

$$= 8\pi \text{ cm}^2/\text{s} \qquad \text{[from Eq. (i)]}$$

SHORT ANSWER QUESTIONS

42. Sand is pouring from a pipe at the rate of 15 cm^3/minute. The falling sand forms a cone on the ground such that the height of the cone is always one-third of the radius of the base. How fast is the height of the sand cone increasing at the instant when the height is 4 cm?

Sol : OD 2024

Let $r=$ radius; $h=$ height; $v=$ Volume of sand cone and $t=$ time.

Given, $h = 4$ cm; $\frac{dV}{dt} = 15\,\text{cm}^3/\min$ and $h = \frac{1}{3}r$

The height of the cone is always one-third of the radius of the base thus $r = 3h$.

Volume of cone $$V = \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3}\pi(3h)^2 h$$

$$V = 3\pi h^3$$

Differentiating both side w.r.t. t we have

$$\frac{dv}{dt} = 3\pi\, 3h^2\frac{dh}{dt}$$

$$15 = 3\pi\, 3(4)^2\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{5}{48}\,\text{cm}/\min$$

43. If the circumference of circle is increasing at the constant rate, prove that rate of change of area of circle is directly proportional to its radius.

Sol : OD 2023

Circumference $C = 2\pi r$

Differentiating w.r.t t we have

$$\frac{dC}{dt} = 2\pi\frac{dr}{dt}$$

Since circumference of circle is increasing at the constant rate $\frac{dC}{dt}$ is constant. Let k be that constant.

Let $2\pi\frac{dr}{dt} = k \Rightarrow \frac{dr}{dt} = \frac{k}{2\pi}$

Area, $$A = \pi r^2$$

$$\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$$

$$\frac{dA}{dt} = 2\pi r\left(\frac{k}{2\pi}\right) = rk$$

$$\frac{dA}{dt} \propto r$$

Rate of change of area of the circle is directly proportional to its radius.

44. Find the least value of the function

$$f(x) = ax + \frac{b}{x} \quad (a>0, b>0, x>0)$$

Sol : OD 2020

We have, $$f(x) = ax + \frac{b}{x} \quad (a>0, b>0, x>0)$$

$$f'(x) = a - \frac{b}{x^2}$$

and $$f''(x) = \frac{2b}{x^3}$$

For maxima and minima of $f(x)$

$$f'(x) = 0$$

$$a - \frac{b}{x^2} = 0$$

$$x^2 = \frac{b}{a}$$

$$x = \sqrt{\frac{b}{a}} \qquad [x>0]$$

Now $$f''\left(\sqrt{\frac{b}{a}}\right) = \frac{2b}{\left(\frac{b}{a}\right)^{3/2}}$$

$$= \frac{2a^{3/2}}{\sqrt{b}} > 0$$

So, $f(x)$ has least value at $x = \sqrt{\frac{b}{a}}$. Thus

$$f_{\min}(x) = f\left(\sqrt{\frac{b}{a}}\right)$$

$$= a\sqrt{\frac{b}{a}} + \frac{b}{\sqrt{\frac{b}{a}}}$$

$$= \sqrt{ab} + \sqrt{ab} = 2\sqrt{ab}$$

45. The total cost $C(x)$ associated with the production of x units of an item is given by

$$C(x) = 0.005x^3 - 0.02x^2 + 30x + 5000.$$

We have $\dfrac{da}{dt} = 2$ cm/s

Area of an equilateral triangle,

$$A = \frac{\sqrt{3}}{4}a^2$$

Differentiating both sides w.r.t. t, we get

$$\frac{dA}{dt} = \frac{\sqrt{3}}{4} \times 2a \times \frac{da}{dt}$$

$$= \frac{\sqrt{3}}{4} \times 2 \times 10 \times 2 \qquad [\text{given, } a = 10]$$

$$= 10\sqrt{3} \text{ cm}^2/\text{s}$$

Thus, the rate of area increasing is $10\sqrt{3}$ cm²/s.

50. Show that $y = \log(1+x) - \frac{2x}{2+x}$, $x > -1$ is an increasing function of x, throughout its domain.

Sol : OD 2012, Delhi 2007

We have $y = \log(1+x) - \dfrac{2x}{2+x}$.

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{1+x}(1) - \frac{(2+x)\cdot 2 - 2x\cdot 1}{(2+x)^2}$$

$$= \frac{1}{1+x} - \frac{4+2x-2x}{(2+x)^2}$$

$$= \frac{(2+x)^2 - 4(1+x)}{(1+x)(2+x)^2}$$

$$= \frac{4+x^2+4x-4-4x}{(1+x)(2+x)^2}$$

$$= \frac{x^2}{(1+x)(2+x)^2} \qquad \text{...(i)}$$

Now, x^2, $(2+x)^2$ are always positive, also $1+x>0$ for $x>-1$. Thus $\frac{dy}{dx} > 0$ for $x > -1$. Hence, function increases for $x > -1$.

51. If the radius of sphere is measured as 9 cm with an error of 0.03 cm, then find the approximate error in calculating its surface area.

Sol : OD 2011

Let S be the surface area and r be the radius of the sphere.

Given, $r = 9$ cm

Let $\triangle r$ be approximate error in radius $r = 0.03$ cm and $\triangle S$ be approximate error in surface area.

Surface area of sphere, $S = 4\pi r^2$

Differentiating both sides w.r.t. r, we get

$$\triangle S = 4\pi 2r \triangle r$$

$$= 8\pi r \triangle r$$

$$= 8\pi \times 9 \times 0.03 \qquad \triangle r = 0.03 \text{ cm}$$

$$= 72 \times 0.03\pi$$

$$\triangle S = 2.16\pi \text{ cm}^2/\text{cm}$$

Hence, approximate error in surface area is 216π cm²/cm.

52. Find the intervals in which the function

$$f(x) = \frac{3}{2}x^4 - 4x^3 - 45x^2 + 51 \text{ is}$$

(i) strictly increasing

(ii) strictly decreasing.

Sol : Foreign 2014

We have $f(x) = \dfrac{3}{2}x^4 - 4x^3 - 45x^2 + 51$

Differentiating both sides w.r.t. x, we get

$$f'(x) = 6x^3 - 12x^2 - 90x$$

For strictly increasing or strictly decreasing, put $f'(x) = 0$, we get

$$6x^3 - 12x^2 - 90x = 0$$

$$6x(x^2 - 2x - 15) = 0$$

$$6x(x+3)(x-5) = 0$$

$$x = 0, -3, 5$$

Now, we find intervals in which $f(x)$ is strictly increasing or strictly decreasing.

Interval	$f'(x) = 6(x+3)x(x-5)$	Sign of $f'(x)$
$x < -3$	$(-)(-)(-)$	$-$ ve
$-3 < x < 0$	$(+)(-)(-)$	$+$ ve
$0 < x < 5$	$(+)(+)(-)$	$-$ ve
$x > 5$	$(+)(+)(+)$	$+$ ve

A function $f(x)$ is said to be strictly increasing, if $f'(x) > 0$ and it is said to be strictly decreasing, if $f'(x) < 0$. So, the given function $f(x)$ is

(i) Strictly increasing in $(-3, 0)$ and $(5, \infty)$.

(ii) Strictly decreasing in $(-\infty, -3)$ and $(0, 5)$.

53. Find the intervals in which the function given by $f(x) = x^4 - 8x^3 + 22x^2 - 24x + 21$ is

(i) increasing

(ii) decreasing.

Sol : Comp 2012

We have $f(x) = x^4 - 8x^3 + 22x^2 - 24x + 21$

Differentiating both sides w.r.t. x, we get

$$f'(x) = 4x^3 - 24x^2 + 44x - 24$$

$$= 4(x^3 - 6x^2 + 11x - 6)$$

56. Find the intervals in which the function $f(x) = -2x^3 - 9x^2 - 12x + 1$ is

(i) strictly increasing

(ii) strictly decreasing.

Sol : OD 2019

We have $f(x) = -2x^3 - 9x^2 - 12x + 1$

Differentiating both sides w.r.t. x, we get

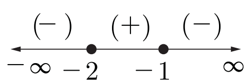
$$\begin{aligned} f'(x) &= -6x^2 - 18x - 12 \\ &= -6(x^2 + 3x + 2) \\ &= -6(x^2 + 2x + x + 2) \\ &= -6[x(x+2) + 1(x+2)] \\ &= -6(x+2)(x+1) \end{aligned}$$

Now, substituting $f'(x) = 0$ we have

$$-6(x+2)(x+1) = 0$$

$$x = -2, -1$$

The points, $x = -2$ and $x = -1$ divide the real line into their disjoint intervals $(-\infty, -2)$, $(-2, -1)$ and $(-1, \infty)$.



The nature of function in these intervals are given below

Intervals	Sign of $f'(x) = -6(x+2)(x+1)$	Nature of function
$(-\infty, -2)$	$(-)(-)(-) = (-) < 0$	Strictly decreasing
$(-2, -1)$	$(-)(+)(-) = (+) > 0$	Strictly increasing
$(-1, \infty)$	$(-)(+)(+) = (-) < 0$	Strictly decreasing

Hence, $f(x)$ is strictly increasing in the interval $(-2, -1)$ and $f(x)$ is strictly decreasing in the interval $(-\infty, -2) \cup (-1, \infty)$.

57. The length of a rectangle is decreasing at the rate of 5 cm/min and the width y is increasing at the rate of 4 cm/min. When $x = 8$ cm and $y = 6$ cm, find the rate of change of

(i) the perimeter.

(ii) area of rectangle.

Sol : OD 2017

Since length x rectangle is decreasing at the rate of 5 cm/min and the breadth y of rectangle is increasing at the rate of 4 cm/min, we have

$$\frac{dx}{dt} = -5 \text{ cm/min.} \qquad \text{...(i)}$$

$$\frac{dy}{dt} = 4 \text{ cm/min.} \qquad \text{...(ii)}$$

(i) Rate of change of the perimeter.

Perimeter of rectangle, $P = 2(x + y)$

Differentiating both sides w.r.t. x, we get

$$\begin{aligned} \frac{dP}{dt} &= 2\left(\frac{dx}{dt} + \frac{dy}{dt}\right) \\ &= 2(-5 + 4) \\ &= 2(-1) = -2 \text{ cm/min} \end{aligned}$$

(ii) Rate of change of area of rectangle.

Area of rectangle,

$$A = xy$$

Differentiating both sides w.r.t. t, we get

$$\frac{dA}{dt} = x \cdot \frac{dy}{dt} + y \cdot \frac{dx}{dt}$$

At $x = 8$ cm and $y = 6$ cm we have

$$\begin{aligned} \frac{dA}{dt} &= (8 \times 4) + [6 \times (-5)] \\ &= 32 - 30 = 2 \text{ cm/min} \end{aligned}$$

Hence, the area of rectangle is increasing at the rate 2 cm/min.

58. Find the intervals in which the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is

(i) strictly increasing

(ii) strictly decreasing.

Sol : Delhi 2014, OD 2011

We have $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$

Differentiating both sides w.r.t. x, we get

$$f'(x) = 12x^3 - 12x^2 - 24x$$

For strictly increasing or strictly decreasing, put $f'(x) = 0$, we get

$$\begin{aligned} 12x^3 - 12x^2 - 24x &= 0 \\ 12x(x^2 - x - 2) &= 0 \\ 12x[x^2 - 2x + x - 2] &= 0 \\ 12x(x+1)(x-2) &= 0 \end{aligned}$$

$$x = 0, -1 \text{ or } 2$$

Now, we find intervals in which $f(x)$ is strictly increasing or strictly decreasing.

As the ladder is pulled along the ground away from the wall at the rate of 2 m/s, we have

$$\frac{dx}{dt} = 2 \text{ m/s}$$

In right angled ΔABC, by Pythagoras theorem, we get

$$(AB)^2 + (BC)^2 = (AC)^2$$

$$x^2 + y^2 = 25 \quad ...(1)$$

$$(4)^2 + y^2 = 25$$

$$16 + y^2 = 25$$

$$y^2 = 9$$

$$y = \sqrt{9} = 3$$

Differentiating both sides of Eq. (1) w.r.t. t, we get

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$x\frac{dx}{dt} + y\frac{dy}{dt} = 0 \quad ...(2)$$

Substituting the values of x, y and $\frac{dx}{dt}$ in Eq. (2)

$$4 \times 2 + 3 \times \frac{dy}{dt} = 0$$

$$8 + 3 \times \frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = \frac{-8}{3} \text{ m/s}$$

Hence, height of the walls is decreasing at the rate of $\frac{8}{3}$ m/s.

NOTE : In a rate of change of a quantity, + ve sign shows that it is increasing and − ve sign shows that it is decreasing.

62. Find the intervals in which the function given by $f(x) = \sin x + \cos x$, $0 \le x \le 2\pi$ is
(i) increasing (ii) decreasing.

Sol : Delhi 2012

We have $f(x) = \sin x + \cos x$.

Differentiating both sides w.r.t. x, we get

$$f'(x) = \cos x - \sin x$$

Now, substituting $f'(x) = 0$ we have

$$\cos x - \sin x = 0$$

$$\tan x = 1,$$

$$x = \frac{\pi}{4}, \frac{5\pi}{4}, \text{ as } 0 \le x \le 2\pi$$

Now, we find the intervals in which $f(x)$ is strictly increasing or strictly decreasing.

For interval $0 < x < \frac{\pi}{4}$ at $x = \frac{\pi}{6}$,

$$f'(x) = \cos x - \sin x$$

$$= \frac{\sqrt{3}}{2} - \frac{1}{2} = \frac{\sqrt{3}-1}{2} \qquad + \text{ve}$$

For interval $\frac{\pi}{4} < x < \frac{5\pi}{6}$ at $x = \frac{\pi}{2}$,

$$f'(x) = \cos x - \sin x$$

$$= 0 - 1 = -1 \qquad - \text{ve}$$

For interval $\frac{5\pi}{4} < x < 2\pi$ at $x = \frac{3\pi}{2}$,

$$f'(x) = \cos x - \sin x$$

$$= 0 - (-1) = 1 \qquad + \text{ve}$$

Here $f'(x) > 0$ in $\left(0, \frac{\pi}{4}\right)$, $f'(x) < 0$ in $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$ and $f'(x) > 0$ in $\left(\frac{5\pi}{4}, 2\pi\right)$.

Since, $f(x)$ is a trigonometric function, so it is continuous at $x = 0$, $\frac{\pi}{4}$, $\frac{5\pi}{4}$ and 2π.

Hence, the function is

(i) increasing in $\left[0, \frac{\pi}{4}\right]$ and $\left[\frac{5\pi}{4}, 2\pi\right]$.

(ii) decreasing in $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$.

63. Sand is pouring from the pipe at the rate of 12 cm^3/z. The falling sand forms a cone on a ground in such a way that the height of cone is always one-sixth of radius of the base. How fast when the height is 4 cm?

Sol : Delhi 2011

Let V be the volume of cone, h be the height and r be the radius of base of the cone.

We have $$\frac{dV}{dt} = 12 \text{ cm}^3\text{/s} \quad ...(1)$$

Since height of cone is always one-sixth of radius of the base, we have

$$h = \frac{1}{6}r \text{ or } r = 6h \quad ...(2)$$

Volume of cone $$V = \tfrac{1}{3}\pi r^2 h \quad ...(3)$$

Substituting $r = 6h$ from Eq. (2) in Eq. (3), we get

$$V = \frac{1}{3}\pi(6h)^2 \cdot h$$

$$= \frac{\pi}{3} \cdot 36h^3$$

or $$V = 12\pi h^3$$

Differentiating both sides w.r.t. t, we get

$$\frac{dV}{dt} = 12\pi \times 3h^2 \cdot \frac{dh}{dt}$$

Substituting $\frac{dV}{dt} = 12$ cm^3/s and $h = 4$ cm, we get

$$12 = 12\pi \times 3 \times 16 \times \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{1}{\pi \times 3 \times 16}$$

$$\frac{dh}{dt} = \frac{1}{48\pi} \text{ cm/s}$$

Interval	$f'(x) = 6(x-2)(x-3)$	Sign of $f'(x)$
$x < 2$	$(-)(-)$	+ ve
$2 < x < 3$	$(+)(-)$	− ve
$x > 3$	$(+)(+)$	+ ve

A function $f(x)$ is said to be an strictly increasing function, if $f'(x) > 0$ and strictly decreasing, if $f'(x) < 0$.

Hence, given function is increasing on $(-\infty, 2]$ and $[3, \infty)$ and decreasing on $[2,3]$

67. Find the intervals in which the function $f(x) = 2x^3 - 9x^2 + 12x + 15$ is

(i) increasing

(ii) decreasing.

Sol : Foreign 2010, OD 2007

We have $f(x) = 2x^3 - 9x^2 + 12x + 15$

Differentiating both sides w.r.t. x, we get

$$f'(x) = 6x^2 - 18x + 12$$

Substituting $f'(x) = 0$, we have

$$6x^2 - 18x + 12 = 0$$

$$6(x^2 - 3x + 2) = 0$$

$$6(x-1)(x-2) = 0$$

$$x = 1,\ 2,$$

Now, we find intervals and check in which interval $f(x)$ is strictly increasing and strictly decreasing.

Interval	$f'(x) = 6(x-1)(x-2)$	Sign of $f'(x)$
$x < 1$	$(-)(-)$	+ ve
$1 < x < 2$	$(+)(-)$	− ve
$x > 2$	$(+)(+)$	+ ve

A function $f(x)$ is said to be an strictly increasing function, if $f'(x) > 0$ and strictly decreasing, if $f'(x) < 0$.

Hence, given function is

(i) increasing on intervals $(-\infty, 1]$ and $[2, \infty)$.

(ii) decreasing on intervals $[1,2]$.

68. Prove that $y = \dfrac{4\sin\theta}{2+\cos\theta} - \theta$ is increasing function in $\left(0, \dfrac{\pi}{2}\right)$.

Sol : OD 2016, 2011

We have $$y = \frac{4\sin\theta}{2+\cos\theta} - \theta \qquad \text{...(i)}$$

A function $y = f(x)$ is said to be an increasing function, if $\frac{dy}{dx} \geq 0$, for all values of x.

Differentiating both sides of Eq. (i) w.r.t. θ, we get

$$\frac{dy}{d\theta} = \frac{(2+\cos\theta) \times \frac{d}{d\theta}(4\sin\theta) - 4\sin\theta \times \frac{d}{d\theta}(2+\cos\theta)}{(2+\cos\theta)^2} - 1$$

$$= \frac{(2+\cos\theta)(4\cos\theta) - 4\sin\theta(0-\sin\theta)}{(2+\cos\theta)^2} - 1$$

$$= \frac{8\cos\theta + 4\cos^2\theta + 4\sin^2\theta - (2+\cos\theta)^2}{(2+\cos\theta)^2}$$

$$= \frac{8\cos\theta + 4(\cos^2\theta + \sin^2\theta) - (4 + \cos^2\theta + 4\cos\theta)}{(2+\cos\theta)^2}$$

$$= \frac{8\cos\theta + 4 - 4 - \cos^2\theta - 4\cos\theta}{(2+\cos\theta)^2}$$

$$= \frac{4\cos\theta - \cos^2\theta}{(2+\cos\theta)^2}$$

$$\frac{dy}{d\theta} = \frac{\cos\theta(4-\cos\theta)}{(2+\cos\theta)^2}$$

Now, as $\cos\theta > 0$, $\forall (0, \frac{\pi}{2})$ and $(2+\cos\theta)^2$ being a perfect square is always positive for all $\theta \in (0, \frac{\pi}{2})$.

Also, for $\theta \in \left(0, \frac{\pi}{2}\right)$, we know that $0 < \cos\theta < 1$

$$4 - \cos\theta > 0 \text{ for all } \theta \in \left(0, \frac{\pi}{2}\right)$$

Thus, we conclude that

$$\frac{\cos\theta(4-\cos\theta)}{(2+\cos\theta)^2} > 0,\ \forall\theta \in \left(0, \frac{\pi}{2}\right)$$

$$\frac{dy}{d\theta} > 0,\ \forall\theta \in \left(0, \frac{\pi}{2}\right)$$

Thus y is an increasing function in $\left(0, \frac{\pi}{2}\right)$. Hence proved.

69. Find the intervals in which the function $f(x) = \sin 3x - \cos 3x$, $0 < x < \pi$, is strictly increasing or strictly decreasing.

Sol : Delhi 2016

We have $f(x) = \sin 3x - \cos 3x$, $0 < x < \pi$

Differentiating both sides w.r.t. x, we get

$$f'(x) = 3\cos x + 3\sin 3x$$

Substituting $f'(x) = 0$, we get

$$\sin 3x = -\cos 3x$$

$$\tan 3x = -1$$

Since $\tan\theta$ is negative in IInd and IVth quadrants,

$$3x = \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}$$

$$x = \frac{\pi}{4}, \frac{7\pi}{12}, \frac{11\pi}{12}$$

Now, we find intervals and check in which intervals $f(x)$ is strictly increasing or strictly decreasing.

In interval $0 < x < \frac{\pi}{4}$ at test value $x = \frac{\pi}{6}$,

$$f'(x) = 3(\cos 3x + \sin 3x)$$

Sol : Delhi 2010

We have $\quad f(x) = \sin x - \cos x,\ 0 \leq x \leq 2\pi$

Differentiating both sides w.r.t. x, we get

$$f'(x) = \cos x + \sin x$$

Substituting $f'(x) = 0$, we get

$$\cos x + \sin x = 0$$

$$\sin x = -\cos x$$

$$\frac{\sin x}{\cos x} = -1$$

$$\tan x = -1$$

For $x \in [0, 2\pi]$,

Since $\tan\theta$ is negative in IInd and IVth quadrants,

$$\tan x = \tan\frac{3\pi}{4} \Rightarrow x = \frac{3\pi}{4}$$

or $$\tan x = \tan\frac{7\pi}{4} \Rightarrow x = \frac{7\pi}{4}$$

$$x = \frac{3\pi}{4}, \frac{7\pi}{4}$$

Now, we find the intervals in which $f(x)$ is strictly increasing or strictly decreasing.

In interval $0 < x < \frac{3\pi}{4}$ at test value $x = \frac{\pi}{2}$,

$$f'(x) = \cos x + \sin x$$

$$f'\left(\frac{\pi}{2}\right) = \cos\frac{\pi}{2} + \sin\frac{\pi}{2}$$

$$= 0 + 1 = 1 \qquad +\text{ive}$$

In interval $\frac{3\pi}{4} < x < \frac{7\pi}{4}$ at test value $x = \frac{5\pi}{6}$,

$$f'(x) = \cos x + \sin x$$

$$f'\left(\frac{5\pi}{6}\right) = \cos\frac{5\pi}{6} + \sin\frac{5\pi}{6}$$

$$= \cos\left(\pi - \frac{\pi}{6}\right) + \sin\left(\pi - \frac{\pi}{6}\right)$$

$$= -\cos\frac{\pi}{6} + \sin\frac{\pi}{6}$$

$$= \frac{-\sqrt{3}}{2} + \frac{1}{2} = \frac{-\sqrt{3}+1}{2} \qquad -\text{ive}$$

In interval $\frac{7\pi}{4} < x < 2\pi$ at test value $x = \frac{5\pi}{6}$,

$$f'(x) = \cos x + \sin x$$

$$f'\left(\frac{23\pi}{12}\right) = \cos\frac{23\pi}{12} + \sin\frac{23\pi}{12}$$

$$= \cos\left(2\pi - \frac{\pi}{12}\right) + \sin\left(2\pi - \frac{\pi}{12}\right)$$

$$= \cos\frac{\pi}{12} - \sin\frac{\pi}{12} > 0 \qquad +\text{ive}$$

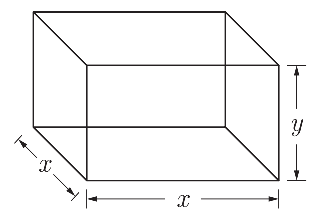
$$= \frac{-\sqrt{3}}{2} + \frac{1}{2} = \frac{-\sqrt{3}+1}{2} \qquad -\text{ive}$$

A function $f(x)$ is said to be strictly increasing in an interval when $f'(x) > 0$ and it is said to be strictly decreasing when $f'(x) < 0$. So, the given function $f(x)$ is strictly increasing in intervals $\left(0, \frac{3\pi}{4}\right)$ and $\left(\frac{7\pi}{4}, 2\pi\right)$ and it is strictly decreasing in the interval $\left(\frac{3\pi}{4}, \frac{7\pi}{4}\right)$.

73. An open tank with a square base and vertical sides is to be constructed from a metal sheet so as to hold a given quantity of water. Show that the cost of material will be least when depth of the tank is half of its width. If the cost is to be borne by nearby settled lower income families, for whom water will be provided.

Sol : OD 2018, OD 2010

Let x be the length of a side of square base and y be the length of vertical side. Also, let V be the given quantity of water.



Volume of cuboid, $\quad V = x^2 y \qquad$ ...(i)

Surface area, $\quad S = 4xy + x^2$

$$= 4x \cdot \frac{V}{x^2} + x^2 \qquad \text{[using Eq. (i)]}$$

$$S(x) = \frac{4V}{x} + x^2$$

Differentiating above both sides w.r.t. x, we get

$$S'(x) = \frac{-4V}{x^2} + 2x \qquad \text{...(ii)}$$

Substituting $S'(x) = 0$, we get

$$\frac{-4V}{x^2} + 2x = 0$$

$$\frac{-4V}{x^2} = -2x$$

$$x^3 = 2V$$

$$x = (2V)^{\frac{1}{3}}$$

Again differentiating both sides of Eq. (ii) w.r.t. x, we get

$$S''(x) = \frac{8V}{x^3} + 2$$

and $\quad S''((2V)^{1/3}) = \frac{8V}{2V} + 2 = 4 + 2 = 6 > 0$

Thus $S(x)$ is minimum when $x = (2V)^{\frac{1}{3}}$.

From Eq. (i), we get

$$y = \frac{V}{x^2} = \frac{\frac{x^3}{2}}{x^2} = \frac{x}{2} \qquad [x^3 = 2V]$$

Thus, the cost of material will be least when depth of the is half of its width.

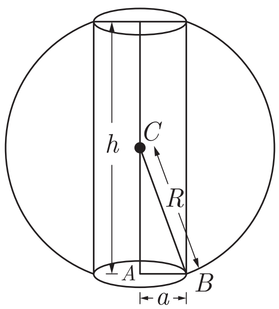
Least Cost $= 280 + 180 \times 4 = 280 \times 720 = Rs. 1000$

Hence, the cost of least expensive tank is Rs. 1000.

76. Prove that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$. Also, find the maximum volume.

Sol : OD 2019, 2014, 2012C, 201, Delhi 2013

Let h be the height and a be the radius of base of cylinder inscribed in the given sphere of radius (R).



In ΔABC using Pythagoras theorem,

$$AB^2 + AC^2 = BC^2$$

$$a^2 + \left(\frac{h}{2}\right)^2 = R^2$$

$$a^2 = R^2 - \frac{h^2}{4}$$

Volume of cylinder,

$$\begin{aligned} V &= \pi a^2 h \\ &= \pi h\left(R^2 - \frac{h^2}{4}\right) \\ &= \frac{\pi}{4}(4R^2 h - h^3) \end{aligned}$$

Differentiating both sides two times w.r.t. h, we get

$$\frac{dV}{dh} = \frac{\pi}{4}(4R^2 - 3h^2)$$

and $$\frac{d^2V}{dh^2} = \frac{\pi}{4}(-6h) = -\frac{3\pi h}{2} \qquad ...(i)$$

For maxima or minima, $\frac{dV}{dh} = 0$.

$$\begin{aligned} \frac{\pi}{4}(4R^2 - 3h^2) &= 0 \\ h^2 &= \frac{4}{3}R^2 \\ h &= \frac{2}{\sqrt{3}}R \end{aligned}$$

Substituting the value of h in Eq. (i), we get

$$\begin{aligned} \frac{d^2V}{dh^2} &= \frac{-3\pi}{2} \cdot \frac{2}{\sqrt{3}}R \\ &= -\sqrt{3}\pi R < 0 \end{aligned}$$

Thus V is maximum.

Hence, the required height of cylinder is $\frac{2R}{\sqrt{3}}$. Hence proved.

Now, maximum volume of cylinder,

$$\begin{aligned} V &= \pi h\left(R^2 - \frac{h^2}{4}\right) \\ &= \pi \frac{2R}{\sqrt{3}}\left(R^2 - \frac{1}{4} \cdot \frac{4}{3}R^2\right) \qquad h = \frac{2}{\sqrt{3}}R \\ &= \frac{2\pi R}{\sqrt{3}} \frac{(3R^2 - R^2)}{3} \\ &= \frac{4\pi R^3}{3\sqrt{3}} \text{ cu units} \end{aligned}$$

77. Find the point on the curve $y^2 = 4x$, which is nearest to the point $(2, -8)$.

Sol : OD 2019

Given, equation of curve is $y^2 = 4x$.

Let $P(x, y)$ be a point on the curve, which is nearest to point $A(2, -8)$.

Distance between the point A and P is given by

$$\begin{aligned} AP &= \sqrt{(x-2)^2 + (y+8)^2} \\ &= \sqrt{\left(\frac{y^2}{4} - 2\right)^2 + (y+8)^2} \\ &= \sqrt{\frac{y^4}{16} + 4 - y^2 + y^2 + 16y + 64} \\ &= \sqrt{\frac{y^4}{16} + 16y + 68} \end{aligned}$$

Let $$z = AP^2 = \frac{y^4}{16} + 16y + 68$$

Now, $$\frac{dz}{dy} = \frac{1}{16} \times 4y^3 + 16 = \frac{y^3}{4} + 16$$

For maximum or minimum value of z, put

$$\begin{aligned} \frac{dz}{dy} &= 0 \\ \frac{y^3}{4} + 16 &= 0 \\ y^3 + 64 &= 0 \\ (y+4)(y^2 - 4y + 16) &= 0 \end{aligned}$$

Here $y^2 - 4y + 16 = 0$ gives imaginary values of y.

Thus $$y = -4$$

Now, $$\frac{d^2z}{dy^2} = \frac{1}{4} \times 3y^2 = \frac{3}{4}y^2$$

For $y = -4$, $\frac{d^2z}{dy^2} = \frac{3}{4}(-4)^2 = 12 > 0$

Thus, z is minimum when $y = -4$.

Substituting $y = -4$ in equation of the curve $y^2 = 4x$; we obtain $x = 4$.

Hence, the point $(4, -4)$ on the curve $y^2 = 4x$ is nearest to the point $(2, -8)$.

78. Show that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius r is $\frac{4r}{3}$. Also, find the maximum volume in terms of volume of the sphere.

$A =$ area of rectangle + area of semicircular region

$$= 2x \times y + \frac{1}{2}\pi x^2$$

$$= 2x\left(\frac{10 - x(\pi+2)}{2}\right) + \frac{1}{2}\pi x^2 \qquad \text{[from Eq. (i)]}$$

$$A = 10x - x^2(\pi+2) + \frac{1}{2}\pi x^2$$

Differentiating both sides w.r.t x, we get

$$\frac{dA}{dx} = 10 - 2x(\pi+2) + \pi x$$

$$= 10 - 2x\pi - 4x + \pi x$$

$$= 10 - \pi x - 4x \qquad \text{...(ii)}$$

For maximum, $\frac{dA}{dx} = 0$

Thus $\quad 10 - \pi x - 4x = 0$

$$\pi x + 4x = 10$$

$$x(\pi+4) = 10$$

$$x = \frac{10}{\pi+4}$$

Again, on differentiating both sides of Eq. (ii), we get

$$\frac{d^2A}{dx^2} = -\pi - 4$$

$$\left.\frac{d^2A}{dx^2}\right|_{x=\frac{10}{\pi+4}} = -(\pi+4) < 0$$

Thus, area is maximum when $x = \frac{10}{\pi+4}$

Now, substituting the value of x in Eq. (i), we get

$$2y = 10 - (\pi+2) \times \frac{10}{\pi+4}$$

$$2y = 10\left[\frac{\pi+4-\pi-2}{\pi+4}\right] = \frac{20}{\pi+4}$$

$$y = \frac{10}{\pi+4}$$

Hence, length of window is $\frac{20}{\pi+4}$ m and width of window $\frac{10}{\pi+4}$ m, to admit maximum light through the whole opening.

80. Show that the surface area of a closed cuboid with square base and given volume is minimum, when it is a cube.

Sol : OD 2017

Let V be the fixed volume of a closed cuboid with length x, breadth x and height y.

Then, $\quad V = x \times x \times y$

$$y = \frac{V}{x^2} \qquad \text{...(i)}$$

Let S be the its surface area.

Then, $\quad S = 2(x^2 + xy + xy)$

$$S = 2(x^2 + 2xy)$$

$$= 2\left(x^2 + \frac{2V}{x}\right) \qquad \text{[using Eq. (i)]}$$

$$S = 2\left(x^2 + \frac{2V}{x}\right)$$

Differentiating both sides w.r.t x, we get

$$\frac{dS}{dx} = 2\left(2x - \frac{2V}{x^2}\right)$$

and $$\frac{d^2S}{dx^2} = \left(4 + \frac{8V}{x^3}\right)$$

Now, $$\frac{dS}{dx} = 0$$

$$2\left(2x - \frac{2V}{x^2}\right) = 0$$

$$2x = \frac{2V}{x^2}$$

$$x^3 = V$$

$$V = x^3$$

$$x \times x \times y = x^3 \Rightarrow y = x$$

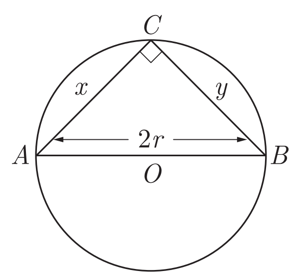
Also $$\left(\frac{d^2S}{dx^2}\right)_{x=y^{1/3}} = 4 + \frac{8V}{V} = 12 > 0$$

Thus S is minimum when length $= x$, breadth $= x$ and height $= x$, i.e it is cube.

81. AB is the diameter of a circle and C is any point of the circle. Show that the area of ΔABC is maximum, when it is an isosceles triangle.

Sol : OD 2017, Foreign 2014

Let $AC = x$, $BC = y$ and r be the radius of circle. Also, $\angle C = 90°$ [Angle made in semi-circle is 90°]



In ΔABC, we have

$$(AB)^2 = (AC)^2 + (BC)^2$$

$$(2r)^2 = (x)^2 + (y)^2$$

$$4r^2 = x^2 + y^2 \qquad \text{...(i)}$$

Area of ΔABC, $\quad A = \frac{1}{2}x \cdot y$

Squaring both sides, we get

$$A^2 = \frac{1}{4}x^2y^2$$

Let $A^2 = S$, then, $\quad S = \frac{1}{4}x^2y^2$

$$x^2 y = 1024$$

$$y = \frac{1024}{x^2}$$

Let C denotes the cost of the box.

$$C = 2x^2 \times 5 + 4xy \times 2.50$$

$$= 10x^2 + 10xy = 10x(x+y)$$

$$= 10x\left(x + \frac{1024}{x^2}\right)$$

$$= 10x^2 + \frac{1024}{x} \quad \text{...(i)}$$

Differentiating both side w.r.t. x, we get

$$\frac{dC}{dx} = 20x + 10240(-x)^{-2}$$

$$= 20x - \frac{10240}{x^2} \quad \text{...(ii)}$$

Now, $\frac{dC}{dx} = 0$

$$20x = \frac{10240}{x^2}$$

$$20x^3 = 10240$$

$$x^3 = 512 = 8^3 \Rightarrow x = 8$$

Again, differentiating Eq. (ii) w.r.t. x, we get

$$\frac{d^2C}{dx^2} = 20 - 10240(-2)\cdot\frac{1}{x^3}$$

$$= 20 + \frac{20480}{x^3} > 0$$

$$\left(\frac{d^2C}{dx^2}\right)_{x=8} = 20 + \frac{20480}{512} = 60 > 0$$

For $x = 8$, cost is minima and the corresponding least cost of the box

$$C(8) = 10\cdot 8^2 + \frac{10240}{8}$$

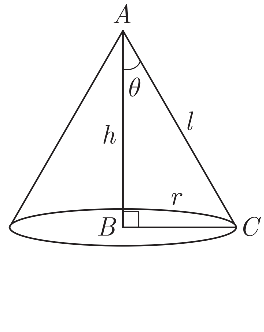
$$= 640 + 1280 = 1920 \quad \text{[using Eq. (i)]}$$

Hence, least cost of the box is Rs. 1920.

84. Show that the semi-vertical angle of the cone of the maximum volume and of given slant height is $\cos^{-1}\frac{1}{\sqrt{3}}$.

Sol : OD 2016; Delhi 2014

Let θ be the semi-vertical angle of the cone. It is clear that $\theta \in \left(0, \frac{\pi}{2}\right)$. Let r, h and l be the radius, height and the slant height of the cone, respectively.



Since, slant height of the cone is given, so consider it as constant.

Now, in ΔABC, $r = l\sin\theta$ and $h = l\cos\theta$

Let V be the volume of the cone, then we have

$$V = \frac{\pi}{3}r^2 h$$

$$= \frac{1}{3}\pi(l^2\sin^2\theta)(l\cos\theta)$$

$$V = \frac{1}{3}\pi l^3 \sin^2\theta\cos\theta$$

Differentiating above w.r.t. θ two times, we get

$$\frac{dV}{d\theta} = \frac{l^3\pi}{3}[\sin^2\theta(-\sin\theta) + \cos\theta(2\sin\theta\cos\theta)]$$

$$= \frac{l^3\pi}{3}(-\sin^3\theta + 2\sin\theta\cos^2\theta)$$

and $$\frac{d^2V}{d\theta^2} = \frac{l^3\pi}{3}(-3\sin^2\theta\cos\theta + 2\cos^3\theta - 4\sin^2\theta\cos\theta)$$

$$\frac{d^2V}{d\theta^2} = \frac{l^3\pi}{3}(2\cos^3\theta - 7\sin^2\theta\cos\theta)$$

For maxima or minima, $\frac{dV}{d\theta} = 0$

$$\frac{l^3\pi}{3}(-\sin^3\theta + 2\sin\theta\cos^2\theta) = 0$$

$$-\sin^3\theta + 2\sin\theta\cos^2\theta = 0$$

$$\sin^3\theta = 2\sin\theta\cos^2\theta$$

$$\tan^2\theta = 2$$

$$\tan\theta = \sqrt{2} \Rightarrow \theta = \tan^{-1}\sqrt{2}$$

Now, when $\theta = \tan^{-1}\sqrt{2}$, then $\tan^2\theta = 2$

$$\sin^2\theta = 2\cos^2\theta$$

Now, we have

$$\frac{d^2V}{d\theta^2} = \frac{l^3\pi}{3}(2\cos^3\theta - 14\cos^3\theta)$$

$$= -4\pi l^3\cos^3\theta < 0, \text{ for } \theta \in \left(0, \frac{\pi}{2}\right)$$

Thus V is maximum, when $\theta = \tan^{-1}\sqrt{2}$

or $$\theta = \cos^{-1}\frac{1}{\sqrt{3}} \qquad \cos\theta = \frac{1}{\sqrt{1+\tan^2\theta}} = \frac{1}{\sqrt{1+2}} = \frac{1}{\sqrt{3}}$$

Hence, for given slant height, the semi-vertical angle of the cone of maximum volume is $\cos^{-1}\frac{1}{\sqrt{3}}$.

Hence proved.

85. Prove that the least perimeter of an isosceles triangle in which a circle of radius r can be inscribed, is $6\sqrt{3}\,r$.

Sol : OD 2016

Let ABC be the isosceles triangle, with $AB = AC$.

$$r = \sqrt{\frac{k-6x^2}{4\pi}} \quad \text{...(i)}$$

Sum of the volumes,

$$V = \frac{4}{3}\pi r^3 + \frac{x}{3} \times x \times 2x$$

$$= \frac{4\pi r^3}{3} + \frac{2}{3}x^3 \quad \text{...(ii)}$$

$$V = \frac{4}{3}\pi\left(\frac{k-6x^2}{4\pi}\right)^{\frac{3}{2}} + \frac{2}{3}x^3$$

Differentiating both sides w.r.t. x, we get

$$\frac{dV}{dx} = \frac{4}{3}\pi \times \frac{3}{2}\left(\frac{k-6x^2}{4\pi}\right)^{\frac{1}{2}}\left(\frac{-12x}{4\pi}\right) + \frac{2}{3} \times 3x^2$$

$$= 2\pi\sqrt{\frac{k-6x^2}{4\pi}}\left(\frac{-3x}{\pi}\right) + 2x^2$$

$$= (-6x)\sqrt{\frac{k-6x^2}{4\pi}} + 2x^2$$

For maxima or minima, $\frac{dV}{dx} = 0$

$$(-6x)\sqrt{\frac{k-6x^2}{4\pi}} + 2x^2 = 0$$

$$2x^2 = 6x\sqrt{\frac{k-6x^2}{4\pi}}$$

$$x = 3\sqrt{\frac{k-6x^2}{4\pi}}$$

$$x = 3r \qquad \text{[using Eq. (i)]}$$

Again, differentiating $\frac{dV}{dx}$ w.r.t. x, we get

$$\frac{d^2V}{dx^2} = -6\frac{d}{dx}\left(x\sqrt{\frac{k-6x^2}{4\pi}}\right) + 4x$$

$$= -6\left(\sqrt{\frac{k-6x^2}{4\pi}} + x \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{\frac{k-6x^2}{4\pi}}}\left(\frac{-12x}{4\pi}\right)\right) + 4x$$

$$= -6\left(r - \frac{3x^2}{2\pi r}\right) + 4x$$

$$= -6r + \frac{9x^2}{\pi r} + 4x$$

Now, $$\left(\frac{d^2V}{dx^2}\right)_{x=3r} = -6r + \frac{9 \times 9r^2}{\pi r} + 12r$$

$$= 6r + \frac{18r}{\pi} > 0$$

Hence, V is minimum when x is equal to three times the radius of the sphere. Hence proved.

Now, substituting $r = \frac{x}{3}$ in Eq. (ii), we get

$$V_{\min} = \frac{4\pi}{3}\left(\frac{x}{3}\right)^3 + \frac{2}{3}x^3$$

$$= \frac{4\pi}{81}x^3 + \frac{2}{3}x^3$$

$$= \frac{2}{3}x^2\left(\frac{2\pi}{27} + 1\right)$$

$$= \frac{2}{3}x^3\left(\frac{44}{189} + 1\right)$$

$$= \frac{2}{3}x^3\left(\frac{233}{189}\right) = \frac{466}{567}x^3$$

87. Find the local maxima and local minima for the function $f(x) = \sin x - \cos x$, $0 < x < 2\pi$. Also, find the local maximum and local minimum values.

Sol : Delhi 2015

We have, $f(x) = \sin x - \cos x$, $0 < x < 2\pi$

Differentiating both sides w.r.t. x, we get

$$f'(x) = \cos x + \sin x \quad \text{...(i)}$$

For local maximum and local minimum, $f'(x) = 0$,

i.e. $$\cos x + \sin x = 0$$

$$\cos x = -\sin x$$

$$\tan x = -1$$

$$x = \pi - \frac{\pi}{4} \text{ or } 2\pi - \frac{\pi}{4}$$

$$x = \frac{3\pi}{4} \text{ or } \frac{7\pi}{4}$$

Again, differentiating both sides of Eq. (i) w.r.t. x, we get

$$f''(x) = -\sin x + \cos x$$

When $x = \frac{3\pi}{4}$, then

$$f''\left(\frac{3\pi}{4}\right) = -\sin\frac{3\pi}{4} + \cos\frac{3\pi}{4}$$

$$= -\sin\left(\pi - \frac{\pi}{4}\right) + \cos\left(\pi - \frac{\pi}{4}\right)$$

$$= -\sin\frac{\pi}{4} - \cos\frac{\pi}{4} < 0$$

When $x = \frac{7\pi}{4}$, then

$$f''\left(\frac{7\pi}{4}\right) = -\sin\frac{7\pi}{4} + \cos\frac{7\pi}{4}$$

$$= -\sin\left(2\pi - \frac{\pi}{4}\right) + \cos\left(2\pi - \frac{\pi}{4}\right)$$

$$= \sin\frac{\pi}{4} + \cos\frac{\pi}{4} > 0$$

Thus, $x = \frac{3\pi}{4}$ is a point of local maxima and $x = \frac{7\pi}{4}$ is a point of local minima.

Now, the local maximum value,

$$f\left(\frac{3\pi}{4}\right) = \sin\frac{3\pi}{4} - \cos\frac{3\pi}{4}$$

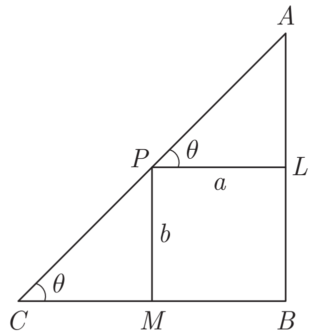
$$= \sin\left(\pi - \frac{\pi}{4}\right) - \cos\left(\pi - \frac{\pi}{4}\right)$$

$$= \sin\frac{\pi}{4} + \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$$

$$= \frac{2}{\sqrt{2}} = \sqrt{2}$$

and the local minimum value,

$$f\left(\frac{7\pi}{4}\right) = \sin\frac{7\pi}{4} - \cos\frac{7\pi}{4}$$



Let l be the length of the hypotenuse, then

$$l = AP + PC$$

$$l = a\sec\theta + b\,\mathrm{cosec}\,\theta,\ 0 < \theta < \frac{\pi}{2}$$

Differentiating both sides w.r.t. θ, we have

$$\frac{dl}{d\theta} = a\sec\theta\tan\theta - b\,\mathrm{cosec}\,\theta\cot\theta \qquad \text{...(i)}$$

For maxima or minima, put $\frac{dl}{d\theta} = 0$

$$a\sec\theta\tan\theta = b\,\mathrm{cosec}\,\theta\cot\theta$$

$$\frac{a\sin\theta}{\cos^2\theta} = \frac{b\cos\theta}{\sin^2\theta}$$

$$\tan\theta = \left(\frac{b}{a}\right)^{1/3}$$

Again, on differentiating both sides of Eq. (i) w.r.t. θ, we get

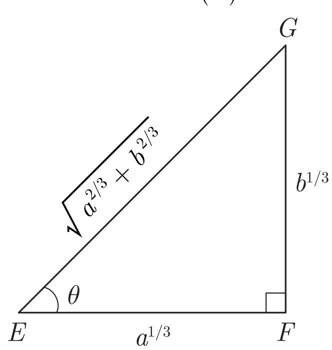
$$\frac{d^2 l}{d\theta^2} = a(\sec\theta\times\sec^2\theta + \tan\theta\times\sec\theta\tan\theta)$$
$$-b[\mathrm{cosec}\,\theta(-\mathrm{cosec}^2\theta) + \cot\theta(-\mathrm{cosec}\,\theta\cot\theta)]$$

$$= a\sec\theta(\sec^2\theta + \tan^2\theta) + b\,\mathrm{cosec}\,\theta(\mathrm{cosec}^2\theta + \cot^2\theta)$$

For $0 < \theta < \frac{\pi}{2}$, all trigonometric ratios are positive.

Also, $a > 0$ and $b > 0$. Thus $\frac{d^2 l}{d\theta^2}$ is positive.

Thus l is least when $\tan\theta = \left(\frac{b}{a}\right)^{\frac{1}{3}}$



Least value of l,

$$l = a\sec\theta + b\,\mathrm{cosec}\,\theta$$

$$= a\frac{\sqrt{a^{2/3}+b^{2/3}}}{a^{1/3}} + b\frac{\sqrt{a^{2/3}+b^{2/3}}}{b^{1/3}}$$

$$= \sqrt{a^{2/3}+b^{2/3}}\,(a^{2/3}+b^{2/3})$$

$$= (a^{2/3}+b^{2/3})^{3/2}$$ Hence proved.

In ΔEFG, $\tan\theta = \frac{b^{1/3}}{a^{1/3}}$,

$$\sec\theta = \frac{\sqrt{a^{2/3}+b^{2/3}}}{a^{1/3}}$$

and $\mathrm{cosec}\,\theta = \frac{\sqrt{a^{2/3}+b^{2/3}}}{b^{1/3}}$]

91. If the length of three sides of a trapezium other than the base are each equal to 10 cm, then find the area of the trapezium, when it is maximum.

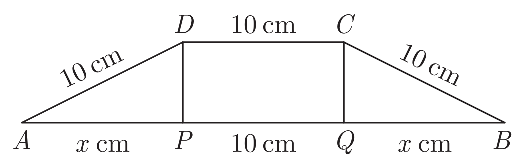
Sol : OD 2014, Foreign 2010, Delhi 2013

Let $ABCD$ be the given trapezium in which $AD = BC = CD = 10$ cm.

Let $AP = x$ cm

$$\Delta APD \cong \Delta BQC$$

$$QB = x \text{ cm}$$



In ΔAPD, using Pythagoras theorem

$$DP = \sqrt{10^2 - x^2}$$

Now, area of trapezium,

$$A = \frac{1}{2}\times(\text{sum of parallel sides})\times\text{height}$$

$$= \frac{1}{2}\times(2x+10+10)\times\sqrt{100-x^2}$$

$$= (x+10)\sqrt{100-x^2} \qquad \text{...(i)}$$

Differentiating both sides of Eq. (i) w.r.t. x, we get

$$\frac{dA}{dx} = (x+10)\frac{(-2x)}{2\sqrt{100-x^2}} + \sqrt{100-x^2}$$

$$= \frac{-x^2 - 10x + 100 - x^2}{\sqrt{100-x^2}}$$

$$= \frac{-2x^2 - 10x + 100}{\sqrt{100-x^2}} \qquad \text{...(ii)}$$

For maxima or minima, put $\frac{dA}{dx} = 0$

$$\frac{-2x^2 - 10x + 100}{\sqrt{100-x^2}} = 0$$

$$-2(x^2 + 5x - 50) = 0$$

$$-2(x+10)(x-5) = 0$$

$$x = 5 \text{ or } -10$$

$$18V^2 = 4\pi^2 r^6$$

$$9V^2 = 2\pi^2 r^6 \qquad ...(iv)$$

Again, differentiating Eq. (iii) w.r.t. r, we get

$$\frac{d^2}{dr^2}(S^2) = \frac{54V^2}{r^4} + 12\pi^2 r^2 > 0$$

At $r = \left(\frac{9V^2}{2\pi^2}\right)^{1/6}$, $\frac{d^2}{dr^2}(S^2) > 0$

So, S^2 or S is minimum, when

$$V^2 = 2\pi^2 r^6/9$$

Substituting $V^2 = 2\pi^2 r^6/9$ in Eq. (i), we get

$$2\pi^2 r^6 = \pi^2 r^4 h^2$$

$$2r^2 = h^2$$

$$h = \sqrt{2}\, r \Rightarrow \frac{h}{r} = \sqrt{2}$$

$$\cot\theta = \sqrt{2} \qquad \left[\text{from the figure, } \cot\theta = \frac{h}{r}\right]$$

$$\theta = \cot^{-1}\sqrt{2}$$

Hence, the semi-vertical of the right circular cone of given volume and least curved surface area is $\cot^{-1}\sqrt{2}$

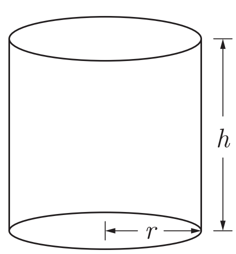
Hence proved.

NOTE : If square of any area is maximum (or minimum), then area is also maximum (or minimum).

94. Of all the closed right circular cylindrical cans of volume 128π cm^3, find the dimensions surface area.

Sol : SQP 2014

Let r cm be the radius of base and h cm be the height of the cylinder cane. Let its volume be V and S be its total surface area.



Now, $\quad V = 128\pi$ cm^3 [given]

$$\pi r^2 h = 128\pi \Rightarrow h = \frac{128}{r^2} \qquad ...(i)$$

Now, surface area of cylindrical cane,

$$S = 2\pi r^2 + 2\pi r h \qquad ...(ii)$$

$$= 2\pi r^2 + 2\pi r\left(\frac{128}{r^2}\right) \qquad \text{[using Eq. (i)]}$$

$$S = 2\pi r^2 + \frac{256\pi}{r} \qquad ...(iii)$$

Differentiating both sides of Eq. (iii) w.r.t. r, we get

$$\frac{dS}{dr} = 4\pi r - \frac{256\pi}{r^2} \qquad ...(iv)$$

For maxima or minima, putting $\frac{dS}{dr} = 0$ we have

$$4\pi r = \frac{256\pi}{r^2} \Rightarrow r^3 = \frac{256}{4} \Rightarrow r^3 = 64$$

Taking cube root on both sides, we get

$$r = (64)^{1/3} \Rightarrow r = 4 \text{ cm}$$

Again, differentiating Eq. (iv) w.r..t r, we get

$$\frac{d^2S}{dr^2} = 4\pi + \frac{512\pi}{r^3}$$

At $r = 4$, $\quad \frac{d^2S}{dr^2} = \frac{512\pi}{64} + 4\pi$

$$= 8\pi + 4\pi = 12\pi > 0$$

Thus, $\frac{d^2S}{dr^2} > 0$ at $r = 4$, so the surface area is minimum, when the radius of cylinder is 4 cm.

Substituting the value of r in Eq. (i), we get

$$h = \frac{128}{(4)^2} = \frac{128}{16} = 8 \text{ cm}$$

Hence, for the minimum surface area of cane, the dimension of the cylindrical can are $r = 4$ cm and $h = 8$ cm.

95. Show that a cylinder of a given volume which is open at the top has minimum total surface area, when its height is equal to the radius of its base.

Sol : Foreign 2014; Delhi 2011

Let r be the radius, h be the height, V be the volume and S be the total surface area of a right circular cylinder which is open at the top. Now, given that $V = \pi r^2 h$

$$h = \frac{V}{\pi r^2} \qquad ...(i)$$

Since cylinder is open at the top, therefore total surface area S,

$$S = 2\pi r h + \pi r^2$$

$$S = 2\pi r\left(\frac{V}{\pi r^2}\right) + \pi r^2 \qquad h = \frac{V}{\pi r^2}$$

$$= \frac{2V}{r} + \pi r^2$$

Differentiating both sides w.r.t. r, we get

$$\frac{dS}{dr} = -\frac{2V}{r^2} + 2\pi r$$

For maxima or minima, putting $\frac{dV}{dr} = 0$ we have

$$-\frac{2V}{r^2} + 2\pi r = 0 \Rightarrow V = \pi r^3$$

$$\pi r^2 h = \pi r^3 \qquad V = \pi r^2 h$$

Also, $\quad \frac{d^2S}{dr^2} = \frac{d}{dr}\left(\frac{dS}{dr}\right)$

From Eq. (ii), we get

$$h = \frac{6\pi r^2 - 2\pi r^2}{2\pi r} = 2r$$

Thus height is equal to the diameter of the base.

Also, $$\frac{d^2V}{dr^2} = \frac{d}{dr}\left(\frac{dV}{dr}\right) = \frac{d}{dr}\left(\frac{S - 6\pi r^2}{2}\right) = -6\pi r < 0$$

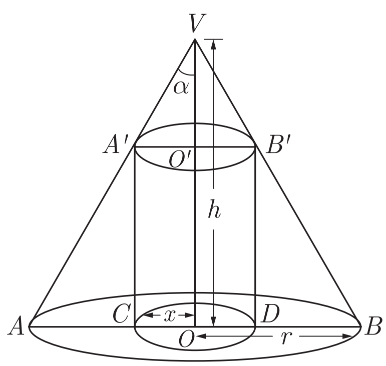
Thus V is maximum. Hence, V is maximum at $h = 2r$.

Hence proved.

98. Prove that radius of right circular cylinder of greatest curved surface area which can be inscribed in a given cone is half of that of the cone.

Sol : OD 2012

Let VAB be the cone of base radius r, height h and radius of base of the inscribed cylinder be x.



If the triangles are similar, then their sides are proportional.

Since $\Delta VOB \sim \Delta B'DB$,

$$\frac{VO}{B'D} = \frac{OB}{DB}$$

$$\frac{h}{B'D} = \frac{r}{r-x}$$

$$B'D = \frac{h(r-x)}{r}$$

Let C be the curved surface are of cylinder. Then,

$$C = 2\pi(OC)(B'D) = \frac{2\pi xh(r-x)}{r}$$

$$C = \frac{2\pi h}{r}(rx - x^2)$$

Differentiating both sides w.r.t. x, we get

$$\frac{dC}{dx} = \frac{2\pi h}{r}(r - 2x)$$

For maxima or minima, put $\frac{dC}{dx} = 0$

$$\frac{2\pi h}{r}(r-2x) = 0$$

$$r - 2x = 0 \Rightarrow r = 2x$$

$$x = \frac{r}{2}$$

Hence, radius of cylinder is half of that of cone.

Also, $$\frac{d^2C}{dx^2} = \frac{d}{dx}\left[\frac{2\pi h(r-2x)}{r}\right] = \frac{2\pi h}{r}(-2) = \frac{-4\pi h}{r} < 0 \text{ as } h,\ r > 0$$

Thus C is maximum or greatest.

Hence, C is greatest at $x = \frac{r}{2}$. Hence proved.

99. A open box with a square base is to be made out of a given quantity of cardboard of area C^2 sq units. Show that the maximum volume of box is $\frac{1}{6\sqrt{3}}C^3$ cu units.

Sol : OD 2012

Let x be the side of base and y be the height. Also, let V denotes its volume and S denotes its total surface area.

Surface area of open box is the area of its 5 faces.

Now, $$S = x^2 + 4xy$$

Given, $$x^2 + 4xy = C^2$$

$$y = \frac{C^2 - x^2}{4x} \quad ...(i)$$

Also, volume of the box is given by

$$V = x^2 y = x^2\left(\frac{C^2 - x^2}{4x}\right) \quad \text{[from Eq. (i)]}$$

$$V = \frac{xC^2 - x^3}{4}$$

Differentiating both sides w.r.t. x, we get

$$\frac{dV}{dx} = \frac{C^2 - 3x^2}{4}$$

For maxima or minima, putting $\frac{dV}{dx} = 0$ we have

$$\frac{C^2 - 3x^2}{4} = 0$$

$$C^2 = 3x^2$$

$$x = \frac{C}{\sqrt{3}}$$

Also, $$\frac{d^2V}{dx^2} = \frac{d}{dx}\left(\frac{dV}{dx}\right) = \frac{d}{dx}\left(\frac{C^2 - 3x^2}{4}\right)$$

$$4\pi^2 r^6 = 18\left(\frac{1}{3}\pi r^2 h\right)^2 \qquad V = \tfrac{1}{3}\pi r^2 h$$

$$= 18 \times \frac{1}{9}\pi^2 r^4 h^2$$

$$4\pi^2 r^6 = 2\pi^2 r^4 h^2$$

$$2r^2 = h^2$$

$$h = \sqrt{2}\, r$$

Hence, height $= \sqrt{2} \times$ (radius of base)

Also, $$\frac{d^2 Z}{dr^2} = \frac{d}{dr}\left(\frac{dZ}{dr}\right) = \frac{d}{dr}\left(4\pi^2 r^3 - \frac{18 V^2}{r^3}\right)$$

$$= 12\pi^2 r^2 + \frac{54 V^2}{r^4}$$

$$\frac{d^2 Z}{dr^2} = 12\pi^2 r^2 + \frac{54 V^2}{r^4} > 0$$

$\Rightarrow Z$ is minimum $\Rightarrow C$ is minimum.

Hence, curved surface area is least, when $h = \sqrt{2}\, r$.

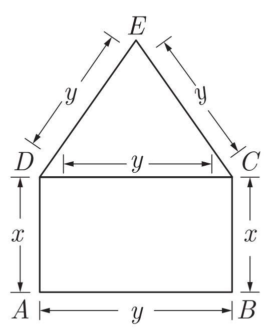
Hence proved.

NOTE : If C^2 is maximum/minimum, then C is also maximum/minimum.

102. A window has the shape of a rectangle surmounted by an equilateral triangle. If the perimeter of the window is 12 m, then find the produce the largest area of the window.

Sol : Delhi 2011

Let $ABCD$ be the rectangle which is surmounted by an equilateral ΔEDC.



Since perimeter of window is 12 m, we obtain

$$2x + 2y + y = 12$$

$$x = 6 - \frac{3}{2}y \qquad ...(i)$$

Let A denotes the combined area of the window.

Then, A = area of rectangle + area of equilateral triangle

$$A = xy + \frac{\sqrt{3}}{4}y^2$$

$$= y\left(6 - \frac{3}{2}y\right) + \frac{\sqrt{3}}{4}y^2 \qquad x = 6 - \tfrac{3}{2}y$$

$$A = 6y - \frac{3}{2}y^2 + \frac{\sqrt{3}}{4}y^2$$

Differentiating both sides w.r.t. y, we get

$$\frac{dA}{dy} = 6 - 3y + \frac{\sqrt{3}}{2}y$$

For maxima or minima, put $\frac{dA}{dy} = 0$

$$6 - 3y + \frac{\sqrt{3}}{2}y = 0$$

$$y\left(\frac{\sqrt{3}}{2} - 3\right) = -6$$

$$y = \frac{12}{6 - \sqrt{3}}$$

Now, $$\frac{d^2 A}{dy^2} = \frac{d}{dy}\left(\frac{dA}{dy}\right)$$

$$= \frac{d}{dy}\left(6 - 3y + \frac{\sqrt{3}}{2}y\right)$$

$$= -3 + \frac{\sqrt{3}}{2}$$

$$= \frac{-6 + \sqrt{3}}{2} < 0$$

Thus A is maximum.

Now, substituting $y = \frac{12}{6 - \sqrt{3}}$ in Eq. (i), we get

$$x = 6 - \frac{3}{2}\left(\frac{12}{6 - \sqrt{3}}\right)$$

$$= \frac{36 - 6\sqrt{3} - 18}{6 - \sqrt{3}}$$

$$x = \frac{18 - 6\sqrt{3}}{6 - \sqrt{3}}$$

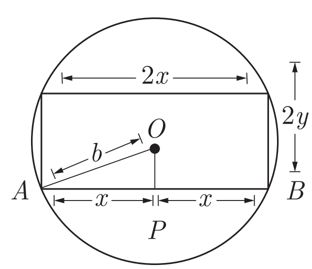
Hence, the area of the window is largest when the dimensions of the window are

$$x = \frac{18 - 6\sqrt{3}}{6 - \sqrt{3}} \text{ and } y = \frac{12}{6 - \sqrt{3}}$$

103. Show that of all the rectangles inscribed in a given fixed circle, the square has the maximum area.

Sol : OD 2011

Let $ABCD$ be the rectangle which is inscribed in a fixed circle whose centre is O and and radius b. Let $AB = 2x$ and $BC = 2y$.



Also $\quad \dfrac{d^2A}{dx^2} = \dfrac{d}{dx}\left(\dfrac{P-4x}{2}\right)$

$$= -\frac{4}{2} = -2 < 0$$

Thus A is maximum. Hence, area is maximum, when rectangle is a square.

Hence proved.

105. Show that of all the rectangles of given area, the square has the smallest perimeter.

Sol : Delhi 2011

Let x and y be the lengths of sides of a rectangle. Again, let A denotes its area and P be the perimeter. Now, area of rectangle,

$$A = xy$$

$$y = \frac{A}{x} \qquad \text{...(i)}$$

And $\quad P = 2(x+y)$

$$P = 2\left(x + \frac{A}{x}\right) \qquad y = \frac{A}{x}$$

Differentiating both sides w.r.t. x, we get

$$\frac{dP}{dx} = 2\left(1 - \frac{A}{x^2}\right)$$

For maxima or minima, putting $\dfrac{dP}{dx} = 0$ we have,

$$2\left(1 - \frac{A}{x^2}\right) = 0 \Rightarrow 1 = \frac{A}{x^2}$$

$$A = x^2$$

$$xy = x^2 \qquad A = xy$$

$$x = y$$

Also, $\quad \dfrac{d^2P}{dx^2} = \dfrac{d}{dx}\left[2\left(1 - \dfrac{A}{x^2}\right)\right]$

$$= 2\left(\frac{2A}{x^3}\right) = \frac{4A}{x^3} > 0$$

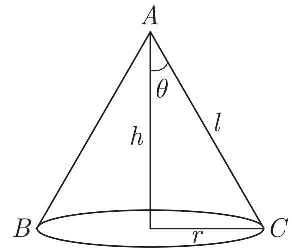
Here, x and A being the side and area of rectangle can never be negative. So, P is minimum.

Hence, perimeter of rectangle is minimum, when rectangle is a square. Hence proved.

106. Show that the semi-vertical angle of a right circular cone of maximum volume and given slant height is $\tan^{-1}\sqrt{2}$.

Sol : OD 2011

Let r be the radius of the base, h be the height, V be the volume, l be the slant height of the cone ABC and θ be the semi-vertical angle.



Now $\quad V = \dfrac{1}{3}\pi r^2 h \qquad \text{...(i)}$

Slant height $\quad l^2 = \sqrt{h^2 + r^2}$

or $\quad r^2 = l^2 - h^2$

Substituting above in (i) we have

$$V = \frac{1}{3}\pi(l^2 - h^2)h$$

Differentiating above w.r.t. h we have,

$$\frac{dV}{dh} = \frac{1}{3}\pi(l^2 - 3h^2)$$

For maxima or minima, putting $\dfrac{dV}{dh} = 0$ we have,

$$\frac{1}{3}\pi(l^2 - 3h^2) = 0$$

$$l^2 - 3h^2 = 0$$

$$l^2 = 3h^2$$

$$l = \sqrt{3}\,h$$

$$\frac{h}{l} = \frac{1}{\sqrt{3}}$$

From figure we have,

$$\frac{h}{l} = \cos\theta$$

where θ is semi vertical angle.

Hence $\quad \tan\theta = \dfrac{\sqrt{1-\cos^2\theta}}{\cos\theta}$

$$= \frac{\sqrt{1-\frac{1}{3}}}{\frac{1}{\sqrt{3}}} = \sqrt{2}$$

Therefore $\quad \tan\theta = \sqrt{2}$

or $\quad \theta = \tan^{-1}\sqrt{2}$

107. Find the point on the curve $y^2 = 2x$ which is a minimum distance from the point $(1, 4)$.

Sol : OD 2011, Comp 2008

The given equation of curve is $y^2 = 2x$ and the given point is $Q(1, 4)$.

Let $P(x, y)$ be any point on the curve.

Now, distance between points P and Q is given by

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

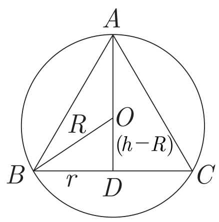
$$= \sqrt{(1-x)^2 + (4-y)^2}$$

109. Show that the volume of the largest cone that can be inscribed in a sphere of radius R is $\frac{8}{27}$ of the volume of the sphere.

Sol : Comp 2010, Delhi 2007

Let the centre of the sphere be O and radius be R. Let the height and radius of the variable cone inside the sphere be h and r respectively.

As per question figure is shown below.



Now in figure $OA = OB = R$

$AD = h,\ BD = r$

$OD = AD - OA = h - R$

In ΔOBD, by Pythagoras theorem, we have

$$OB^2 = OD^2 + BD^2$$

$$R^2 = (h-R)^2 + r^2$$

$$R^2 = h^2 - 2hR + R^2 + r^2$$

$$r^2 = 2hR - h^2 \qquad \text{...(i)}$$

Volume of V of the cone,

$$V = \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3}\pi(2hR - h^2)h \qquad \text{[from Eq. (i)]}$$

$$= \frac{1}{3}\pi(2Rh^2 - h^3) \qquad \text{...(ii)}$$

Differentiating both sides of Eq. (ii) w.r.t. h, we get

$$\frac{dV}{dh} = \frac{1}{3}\pi(4Rh - 3h^2) \qquad \text{...(iii)}$$

For maxima or minima, putting $\frac{dV}{dh} = 0$ we have

$$\frac{1}{3}\pi(4Rh - 3h^2) = 0$$

$$4Rh = 3h^2$$

$$4R = 3h$$

$$h = \frac{4R}{3} \qquad [\because h \neq 0]$$

Again, on differentiating Eq. (iii) w.r.t. h, we get

$$\frac{d^2V}{dh^2} = \frac{1}{3}\pi(4R - 6h)$$

At $h = \frac{4R}{3}$, $\left(\frac{d^2V}{dh^2}\right)_{h=\frac{4R}{3}} = \frac{1}{3}\pi\left(4R - 6\times\frac{4R}{3}\right)$

$$= \frac{\pi}{3}(4R - 8R)$$

$$= -\frac{4\pi R}{3} < 0$$

Thus V is maximum at $h = \frac{4R}{3}$.

Substituting the value of h in Eq. (ii), we get

$$V = \frac{1}{3}\pi\left[2R\left(\frac{4R}{3}\right)^2 - \left(\frac{4R}{3}\right)^3\right]$$

$$= \frac{\pi}{3}\left[\frac{32}{9}R^3 - \frac{64}{27}R^3\right]$$

$$= \frac{\pi}{3}R^3\left[\frac{32}{9} - \frac{64}{27}\right]$$

$$= \frac{\pi}{3}R^3\left[\frac{96-64}{27}\right]$$

$$= \frac{\pi}{3}R^3\left(\frac{32}{27}\right)$$

$$= \frac{8}{27}\times\left(\frac{4}{3}\pi R^3\right)$$

$$= \frac{8}{27}\times \text{(Volume of sphere)}$$

Hence, maximum volume of the cone is $\frac{8}{27}$ of the volume of the sphere.

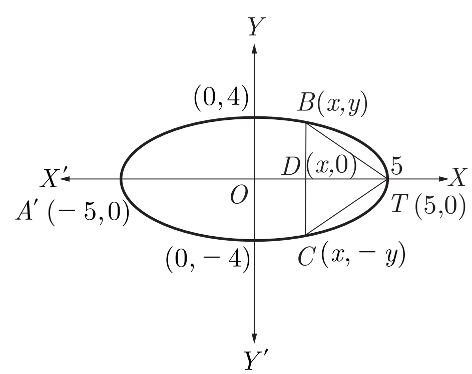
110. Find the maximum area of an isosceles triangle inscribed in the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$, with its vertex at one end of the major axis.

Sol : Comp 2010

We have $\frac{x^2}{25} + \frac{y^2}{16} = 1.$

Here, $a = 5,\ b = 4,\ a > b$. Thus major axis is along X-axis.

Let ΔBTC be the isosceles triangle which is inscribed in the ellipse and $OD = x$, $BC = 2y$ and $TD = 5 - x$.



Let A denotes the area of triangle. Then, we have

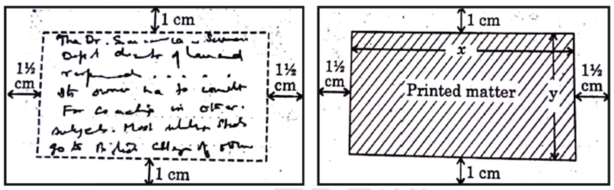
$$A = \frac{1}{2}\times \text{base}\times\text{height}$$

$$= \frac{1}{2}\times BC\times TD$$

$$= \frac{1}{2}\cdot 2y(5 - x)$$

CASE BASED QUESTIONS

112. A rectangular visiting card is to contain 24 sq. cm. of printed matter. The margins at the top and bottom of the card are to be 1 cm and the margins on the left and right are to be $1\frac{1}{2}$ cm as shown below :



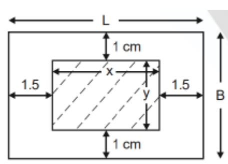
On the basis of the above information, answer the following questions:

(i) Write the expression for the area of the visiting card in terms of x.

(ii) Obtain the dimensions of the card of minimum area.

Sol : OD 2024

We have redrawn the given figure as below.



Area of printed matter,

$$xy = 24\,\text{cm}^2$$

$$y = \frac{24}{x}$$

(i) Area of visiting card,

$$\begin{aligned} L \times B &= (x+3)(y+2) \\ &= (x+3)\left(\frac{24}{x}+2\right) \\ &= 24+2x+\frac{72}{x}+6 \\ &= 2x+\frac{72}{x}+30 \end{aligned}$$

(ii) Dimensions of the card of minimum area

$$\begin{aligned} \frac{dA}{dx} &= \frac{d}{dx}\left(2x+\frac{72}{x}+30\right) \\ &= 2-\frac{72}{x^2} \end{aligned}$$

For maximum/minimum area, letting $\frac{dA}{dx}=0$ we have

$$\begin{aligned} 2-\frac{72}{x^2} &= 0 \\ 2x^2-72 &= 0 \\ x^2 &= 36 \Rightarrow x = \pm 6 \end{aligned}$$

As dimension cannot be negative we take $x=6$.

Now, $$\begin{aligned} \frac{d^2A}{dx^2} &= \frac{d}{dx}\left(2-\frac{72}{x^2}\right) \\ &= \frac{144}{x^3} \end{aligned}$$

Since $\frac{d^2A}{dx^2} > 0$ for $x=6$, thus area is minimum, when $x=6$.

Now $$y = \frac{24}{x} = \frac{24}{6} = 4$$

So, dimensions are $x = 6\,\text{cm}$ and $y = 4\,\text{cm}$

113. Engine displacement is the measure of the cylinder volume swept by all the pistons of a piston engine. The piston moves inside the cylinder bore.

The cylinder bore in the form of a circular cylinder open at the top is to be made from a metal sheet of area 75π cm^2.

Based on the above information, answer the following questions :

(i) If the radius of cylinder is r cm and height is h cm, then write the volume V of the cylinder in terms of radius r.

(ii) Find $\frac{dV}{dr}$

(iii) Find the radius of cylinder when its volume is maximum.

(iv) For maximum volume, $h > r$. State true or false and justify

Sol : OD 2023

(i) Volume V of the cylinder in terms of radius r

Given the radius, height and Volume of cylinder are r, h and V respectively.

Area of the cylinder,

$$\begin{aligned} A &= 75\pi\ \text{cm}^2 \\ \pi r^2 + 2\pi rh &= 75\pi \\ \pi r(r+2h) &= 75\pi \\ r+2h &= \frac{75}{r} \\ h &= \frac{1}{2}\left(\frac{75}{r}-r\right) \end{aligned}$$

Volume of the cylinder,

$$\begin{aligned} V &= \pi r^2 h \\ V &= \pi r^2\left(\frac{1}{2}\left(\frac{75}{r}-r\right)\right) \\ &= \frac{\pi}{2}(75r - r^3) \end{aligned}$$

(ii) $\frac{dV}{dr}$

$$V = \frac{\pi}{2}(75r - r^3)$$

$$S = 6x^2 + 4\pi y^2 \qquad ...(i)$$

(ii) we have,

$$V = \frac{4}{3}\pi y^3 + x \times 2x \times \frac{x}{3}$$

$$V = \frac{4}{3}\pi y^3 + \frac{2}{3}x^3$$

(iii) We have,

$$V = \frac{4}{3}\pi y^3 + \frac{2}{3}x^3 \qquad \text{[from part (ii)]}$$

$$= \frac{4}{3}\pi\left(\frac{S-6x^2}{4\pi}\right) + \frac{2}{3}x^3 \qquad [s = 6x^2 + 4\pi y^2]$$

$$\frac{dV}{dx} = \frac{1}{6\sqrt{\pi}} \times \frac{3}{2}(S-6x^2)^{\frac{1}{2}}(-12x) + \frac{2}{3} \times 3x^2$$

$$= -\frac{3}{\sqrt{\pi}}(S-6x^2)^{\frac{1}{2}} \times + 2x^2$$

For minimum, $\frac{dV}{dx} = 0$

$$-\frac{3}{\sqrt{\pi}}(S-6x^2)^{\frac{1}{2}}x + 2x^2 = 0$$

$$2x^2 = \frac{3x}{\sqrt{\pi}}(S-6x^2)^{\frac{1}{2}}$$

$$2x^2 = \frac{3x}{\sqrt{\pi}}(4\pi y^2)^{\frac{1}{2}} \qquad \text{[using Eq.(i)]}$$

$$2\sqrt{\pi}\,x = 3(4\pi y^2)^{\frac{1}{2}}$$

$$4\pi x^2 = 9 \times 4\pi y^2$$

$$x^2 = 9y^2$$

$$x = 3y$$

(iv) Minimum value of $V = \frac{4}{3}\pi y^3 + \frac{2}{3}x^3$

$$= \frac{4}{3}\pi\left(\frac{x}{3}\right)^3 + \frac{2}{3}x^3 \qquad [x = 3y]$$

$$= \frac{2}{3}x^3\left(1 + \frac{2\pi}{27}\right)$$

Now, when V is minimum,

$$S = 6x^2 + 4\pi y^2$$

$$= 6x^2 + 4\pi\left(\frac{x}{3}\right)^2 \qquad [x = 3y]$$

$$= 6x^2 + \frac{4}{9}\pi x^2$$

$$= 2x^2\left[3 + \frac{2}{9}\pi\right]$$

116. Minimum Support Price (MSP) is a form of market intervention by the Government of India to insure agricultural producers against any sharp fall in farm prices. The minimum support prices are announced by the Government of India at the beginning of the sowing season for certain crops on the basis of the recommendations of the Commission for Agricultural Costs and Prices (CACP). MSP is price fixed by Government of India to protect the producer - farmers - against excessive fall in price during bumper production years. The minimum support prices are a guarantee price for their produce from the Government. The major objectives are to support the farmers from distress sales and to procure food grains for public distribution. In case the market price for the commodity falls below the announced minimum price due to bumper production and glut in the market, government agencies purchase the entire quantity offered by the farmers at the announced minimum price.

The Government declare that farmers can get `300 per quintal for onions on 1st July and after that, the price will be dropped by `3 per quintal per extra day. Ramawatar has 80 quintal of onions in the field on 1st July and he estimated that crop is increasing at the rate of 1 quintal per day.

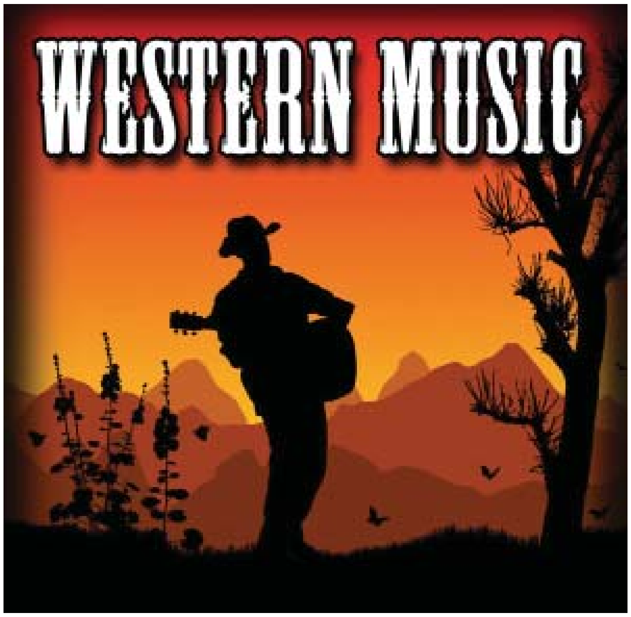
Based on the above information, answer the following questions.

(i) If x is the number of days after 1st July then find the price and quantity of onion in terms of x.

(ii) Find the expression for the revenue as a function of x.

(iii) Find the number of days after 1st July, when Ramawatar attain maximum revenue.

(iv) On which day should Ramawatar harvest the onions to maximum his revenue? What is this maximum revenue?



Western music is organised every year in the stadium that can hold 36000 spectators. With ticket price of `10, the average attendance has been 24000. Some financial expert estimated that price of a ticket should be determined by the function $p(x) = 15 - \frac{x}{3000}$, where x is the number of ticket sold.

Bases on the above information, answer of the following questions.

(i) Find the expression for total revenue R as a function of x.

(ii) Find the value of x for which revenue is maximum.

(iii) When the revenue is maximum, what will be the price of the ticket?

(iv) How many spectators should be present to maximum the revenue?

Sol :

(i) Let p be the price per ticket and x be the number of tickets sold.

Then, revenue function

$$R(x) = p \times x = \left(15 - \frac{x}{3000}\right)x$$

$$= 15x - \frac{x^2}{3000}$$

(ii) $$R(x) = 15x - \frac{x^2}{3000}$$

$$R'(x) = 15 - \frac{x}{1500}$$

For maxima/minima, put $R'(x) = 0$

$$15 - \frac{x}{1500} = 0$$

$$x = 22500$$

Also, $$R''(x) = -\frac{1}{1500} < 0$$

(iii) Maximum revenue will be at

$$\text{Price of ticket} = 15 - \frac{22500}{3000} = 15 - 7.5 = `7.5$$

(iv) Number of spectators will be equal to number of tickets sold.

Required number of spectators = 22500

119. A steel can, tin can, tin, steel packaging, or can is a container for the distribution or storage of goods, made of thin metal. Many cans require opening by cutting the "end" open; others have removable covers. They can store a broad variety of contents: food, beverages, oil, chemicals, etc.

A tin can manufacturer a cylindrical tin can for a company making sanitizer and disinfector. The tin can is made to hold 3 litres of sanitizer or disinfector.

Based on the above information, answer the following questions.

(i) If r be the radius and h be the height of the cylindrical tin can, find the surface area expressed as a function of r.

(ii) Find the radius that will minimize the cost of the material to manufacture the tin can.

(iii) Find the height that will minimize the cost of the material to manufacture the tin can.

or

(iv) If the cost of the material used to manufacture the tin can is `100/m^2 find the minimum cost. $\sqrt[3]{\frac{1500}{\pi}} \approx 7.8$

Now, $\frac{d^2R}{dx^2} = -400$ which is negative for all x.

Thus, $\frac{d^2R}{dx^2} < 0$, $R(x)$ is maximum when $x = 4$.

(i) Ticket price,

Thus ticket price, $p = 70 + 10x = 70 + 10 \times 4 = 110$

(ii) Income

Maximum Revenue,

$$R(4) = 21000 + 1600x - 200x^2$$
$$= 21000 + 1600 \times 4 - 200 \times 4^2$$
$$= 24200$$

121. RK Verma is production analysts of a ready-made garment company. He has to maximize the profit of company using data available. He find that $P(x) = -6x^2 + 120x + 25000$ (in Rupee) is the total profit function of a company where x denotes the production of the company.

Based on the above information, answer the following questions.

(i) Find the profit of the company, when the production is 3 units.
(ii) Find $P'(5)$
(iii) Find the interval in which the profit is strictly increasing.
(iv) Find the production, when the profit is maximum.

Sol :

We have $P(x) = -6x^2 + 120x + 25000$

$$P'(x) = -12x + 120$$

(i) At $x = 3$ we have,

$$P(3) = -6(3)^2 + 120(3) + 25000$$
$$= -54 + 360 + 25000$$
$$= 25306$$

(ii) $P'(5) = -12 \times 5 + 120$
$$= -60 + 120$$
$$= 60$$

(iii) For strictly increasing, $P'(x) > 0$

$$-12x + 120 > 0$$
$$120 > 12x$$
$$x < 10$$
$$x \in (0, 10)$$

(iv) For maximum profit we put $P'(x) = 0$, i.e

$$0 = -12x + 120$$
$$x = 10$$

Now $P''(x) = -12 < 0$

At $x = 10$, profit function is maximum.

122. The Indian toy industry is estimated to be worth US$1.5 billion, making up 0.5% of the global market share. The toy manufacturers in India can mostly be found in NCR, Mumbai, Karnataka, Tamil Nadu, and several smaller towns and cities across central states such as Chhattisgarh and Madhya Pradesh. The sector is fragmented with 90% of the market being unorganised. The toys industry has been predicted to grow to US$2-3 billion by 2024. The Indian toy industry only represents 0.5% of the global industry size indicating a large potential growth opportunity for Indian consumer product companies who will develop exciting innovations to deliver international quality standards at competitive prices.

Fisher Price is a leading toy manufacturer in India. Fisher Price produces x set per week at a total cost of $\frac{1}{25}x^2 + 3x + 100$. The produced quantity for his market is $x = 75 - 3p$ where p is the price set.

(i) Show that the maximum profit is obtained when about 30 toys are produced per week.
(ii) What is the price at maximum profit?

Now, $b = \dfrac{P-2l}{2}$

$$= \frac{P}{2} - l = \frac{P}{2} - \frac{P}{4} = \frac{P}{4}$$

$$(A)_{\max} = l \times b = \frac{P}{4} \times \frac{P}{4} = \frac{P^2}{16} \text{ sq. units.}$$

124. The use of electric vehicles will curb air pollution in the long run. The use of electric vehicles is increasing every year and estimated number of electric vehicles in use at any time t is given by the function

$$V(t) = t^3 - 3t^2 + 3t - 100$$

where t represents time and $t = 1, 2, 3,$ corresponds to year 2021, 2022, 2023 .... respectively

Based on the above information answer the following:

(i) Can the above function be used to estimate number of vehicles in the year 2020? Justify.

(ii) Find the estimated number of vehicles in the year 2040.

(iii) Prove that the function $V(t)$ is an increasing function.

Sol :

We have $V(t) = t^3 - 3t^2 + 3t - 100$

(i) No, the above function cannot be used to estimate number of vehicles in the year 2020 because for 2020 we have $t = 0$ and

$$V(0) = 0 - 0 + 0 - 100 = -100$$

which is not possible

(ii) Estimated number of vehicles in the year 2040

$$V(20) = (20)^3 - 3(20)^2 + 3(20) - 100$$

Therefore, the estimated number of vehicles in the year 2040 are 6760.

(iii) Function $V(t)$ is always increasing function.

$$V'(t) = 3t^2 - 6t + 3$$

$$= 3(t^2 - 2t + 1) = 3(t-1)^2 \geq 0.$$

125. Two metal rods, R_1 and R_2, of lengths 16 m and 12 m respectively, are insulated at both the ends. Rod R_1 is being heated from a specific point while rod R_2 is being cooled from a specific point.

The temperature (T) in Celsius within both rods fluctuates based on the distance (x) measured from either end. The temperature at a particular point along the rod is determined by the equations $T = (16 - x)x$ and $T = (x - 12)x$ for rods R_1 and R_2 respectively, where the distance x is measured in meters from one of the ends.

Based on the above information answer the following:

(i) Find the rate of change of temperature at the mid point of the rod that is being heated.

(ii) Find the minimum temperature attained by the rod that is being cooled.

Sol :

(i) The rod being heated is R_1 and the temperature at a particular point along the rod is

$$T = (16 - x)x$$

The rate of change of temperature at any distance from one end of R_1 is

$$\frac{dT}{dx} = \frac{d}{dx}(16 - x)x$$

$$= \frac{d}{dx}(16x - x^2)$$

$$= 16 - 2x$$

Since rod is 16 m long, the mid-point of the rod is at $x = 8$ m.

The rate of change of temperature at the mid point of R_1 is

$$\left[\frac{dT}{dx}\right]_{x=8} = 16 - 2(8)$$

$$= 0$$

(ii) The rod being cooled is R_2 the temperature at a particular point along the rod is

$$T = (x - 12)x$$

The rate of change of temperature at any distance x m is

$$\frac{dT}{dx} = \frac{d}{dx}(x - 12)x$$

$$= \frac{d}{dx}(x^2 - 12x)$$

$$= 2x - 12$$

Equates $\frac{dT}{dx}$ to 0 we get the critical point $x = 6$.

Now $\dfrac{d^2T}{dx^2} = 2$

Based on the above information answer the following:

(i) Is the function differentiable in the interval (0, 12)? Justify your answer.

(ii) If 6 is the critical point of the function, then find the value of the constant m.

(iii) Find the intervals in which the function is strictly increasing/strictly decreasing.

(iv) Find the points of local maximum/local minimum, if any, in the interval (0, 12) as well as the points of absolute maximum/absolute minimum in the interval [0, 12]. Also, find the corresponding local maximum/local minimum and the absolute maximum/absolute minimum values of the function.

Sol :

We have $f(x) = -0.1x^2 + mx + 98.6,$

(i) $f(x)$ being a polynomial function, is differentiable everywhere, hence, differentiable in (0, 12)

(ii) Value of the constant m

$$f'(x) = -0.2x + m$$

Since, 6 is the critical point, we have

$$f'(6) = 0$$

$$0 = -0.2 \times 6 + m$$

$$m = 1.2$$

(iii) Intervals in which the function is strictly increasing/strictly decreasing,

$$f(x) = -0.1x^2 + 1.2x + 98.6$$

$$f'(x) = -0.2x + 1.2$$

$$= -0.2(x-6)$$

In the Interval	$f'(x)$	Conclusion
(0,6)	+ve	f is strictly increasing in [0,6]
(6,12)	-ve	f is strictly decreasing in [6,12]

(iv) $f(x) = -0.1x^2 + 1.2x + 98.6$

$$f'(x) = -0.2x + 1.2,$$

$$f'(6) = 0,$$

$$f''(x) = -0.2$$

$$f''(6) = -0.2 < 0$$

Hence, by second derivative test 6 is a point of local maximum. The local maximum value

$$f(6) = -0.1 \times 6^2 + 1.2 \times 6 + 98.6$$

$$= 102.2$$

We have $f(0) = 98.6$

$$f(6) = 102.2$$

$$f(12) = 98.6$$

6 is the point of absolute maximum and the absolute maximum value of the function is 102.2.

0 and 12 both are the points of absolute minimum and the absolute minimum value of the function is 98.6.

128. In an elliptical sport field the authority wants to design a rectangular soccer field with the maximum possible area. The sport field is given by the graph of

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Based on the above information answer the following:

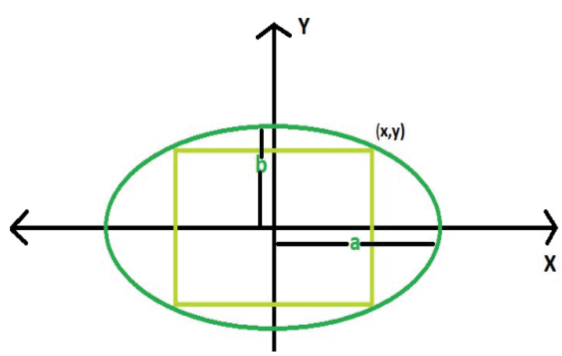
(i) If the length and the breadth of the rectangular field be $2x$ and $2y$ respectively, then find the area function in terms of x.

(ii) Find the critical point of the function.

(iii) Use first derivative test to find the length $2x$ and width $2y$ of the soccer field that maximize its area.

(iv) Use Second Derivative Test to find the length $2x$ and width $2y$ of the soccer field (in terms of a and b) that maximize its area.

Sol :



denoted by C_{max}. How long after administering the drug is C_{max}. attained? Show your work and give valid reasons.

(iii) Find the amount of drug in the bloodstream at the time when the effect of the drug is maximum.

Sol :

(i) Rate at which the amount of drug is changing in the blood stream 5 hours

We have $$C(t) = -t^3 + 4.5t^2 + 54t$$

$$C'(t) = -3t^2 + 9t + 54$$

$$C'(5) = 24\text{mg/hr}$$

(ii) Function $C(t)$ is strictly increasing in the interval (3, 4).

Now $$C'(t) = -3t^2 + 9t + 54$$

$$= -3(t^2 - 3t - 18)$$

$$= -3(t+3)(t-6)$$

For $t \in (3,4)$, $t+3$ is always positive and $t-6$ is always negative. So $C'(t)$ is always positive. Thus $C(t)$ is strictly increasing in the interval (3, 4).

(iii) How long after administering the drug is C_{max}. attained

Equates the derivative $C'(t)$ to 0 we have

$$-3t^2 + 9t + 54 = 0$$

$$-3(t^2 - 3t - 18) = 0$$

$$(t+3)(t-6) = 0$$

Thus critical points are s $t = 6$ hours and $t = -3$ hours.

Now again differentiating $C'(t)$ to get $C''(t)$ we have

$$C''(t) = -6t + 9$$

Substituting $t = 6$ we have

$$C''(6) = -27$$

Thus $C(t)$ attains its maximum at $t = 6$ hours as

$$C''(6) = (-27) < 0$$

Thus 6 hours after the drug is administered, C_{max} is attained.

(iv) Value of $C(t)_{max}$

$$C(6) = -(6)^3 + 4.5(6)^2 + 54(6)$$

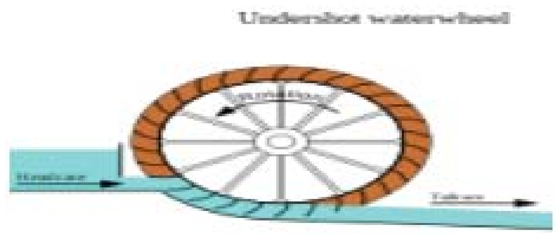
$$= 270 \text{ mg}$$

130. The proportion of a river's energy that can be obtained from an undershot water wheel is $E(x) = 2x^3 - 4x^2 + 2x$ units where x is the speed of the water wheel relative to the speed of the river.

Based on the above information answer the following:

(i) What is the speed of water wheel for maximum value of $E(x)$?

(ii) Find the maximum value of $E(x)$ in the interval $[0,1]$.



Sol :

(i) Speed of water wheel for maximum value of $E(x)$

We have $$E(x) = 2x^3 - 4x^2 + 2x \qquad ...(1)$$

Differentiating equation (1) w.r.t. x we have

$$E'(x) = 6x^2 - 8x + 2 \qquad ...(2)$$

For maximum or minimum value letting $E'(x) = 0$ we have

$$6x^2 - 8x + 2 = 0$$

$$3x^2 - 4x + 1 = 0$$

$$(3x-1)(x-1) = 0$$

i.e. $$x = \tfrac{1}{3} \text{ or } = 1$$

Differentiating equation (ii) w.r.t. x we have

$$E''(x) = 12x - 8$$

At $x = 1$ we have

$$E''(x) = 12(1) - 8$$

$$= 4 = +ve$$

At $x = \frac{1}{3}$ we have

$$E''(x) = 12\left(\frac{1}{3}\right) - 8$$

$$= -4 = -ve$$

Thus $E(x)$ has maximum value at $x = \frac{1}{3}$. Thus speed of water wheel for maximum value is $x = \frac{1}{3}$

(ii) Maximum value of $E(x)$ in the interval $[0,1]$

Maximum value of $E(x)$

$$E\left(\frac{1}{3}\right) = 2\left(\frac{1}{3}\right)^3 - 4\left(\frac{1}{3}\right)^2 + 2\left(\frac{1}{3}\right)$$

$$= \frac{2}{27} - \frac{4}{9} + \frac{2}{3}$$

$$= \frac{8}{27}$$

(ii) Maximum profit

$$p(-2) = 41 - 72(-2) - 18(-2)^2$$
$$= 113 \text{ units.}$$

133. The relation between the height of the plant (y in cm) with respect to its exposure to the sunlight is governed by the following equation $y = 4x - \frac{1}{2}x^2$, where x is the number of days exposed tot he sunlight, for $x \leq 3$.

Based on the above information answer the following:

(i) Find the rate of growth of the plant with respect to the number of days exposed to the sunlight.

(ii) Does the rate of growth of the plant increase or decrease in the first three days? What will be the height of the plant after 2 days?

Sol :

We have $\quad y = 4x - \frac{1}{2}x^2$

(i) Rate of growth of the plant with respect to the number of days exposed to sunlight is given by

$$\frac{dy}{dx} = 4 - x$$

(ii) Rate of growth of the plant in the first three days,

Let rate of growth be represented by the function

$$g(x) = \frac{dy}{dx}$$

Now, $\quad g'(x) = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

$$= -1 < 0 \qquad \text{Thus } g(x) \text{ decreases.}$$

So the rate of growth of the plant decreases for the first three days.

Height of the plant after 2 days is

$$y = 4 \times 2 - \frac{1}{2}(2)^2 = 6 \text{ cm.}$$

134. An architect designs an auditorium for a school for its cultural activities. The floor of the auditorium is rectangular in shape and has a fixed perimeter P.

Based on the above information, answer the following questions.

(i) If x and y represents the length and breadth of the rectangular region, then find the relation between the variable.

(ii) Find the area A of the rectangular region, as a function of x.

(iii) Find the value of y, for which the area of the floor is maximum.

Sol :

Given length of the rectangular auditorium is x.

Breadth of the rectangular auditorium is y.

Given perimeter of the rectangle is P.

(i) Relation between the variables

$$2x + 2y = P$$

(ii) Area of the floor,

$$A = \text{length} \times \text{breadth}$$
$$= x \times y$$
$$A = \left(\frac{P - 2x}{2}\right)x$$
$$= \frac{1}{2}(Px - 2x^2)$$

Similarly area of floor in terms of y,

$$A = \left(\frac{P - 2y}{2}\right)y$$
$$= \frac{1}{2}(Py - 2y^2) \qquad \text{...(1)}$$

Differentiating w.r.t. y we have

$$\frac{dA}{dy} = \frac{1}{2}(P - 4y) \qquad \text{...(2)}$$

For maximum or minimum value of A, letting $\frac{dA}{dx} = 0$ we have

$$P - 4y = 0$$
$$y = \frac{P}{4}$$

Differentiating eq (2) w.r.t. y we have

$$\frac{d^2A}{dx^2} = \frac{1}{2}(0 - 4)$$
$$= -2$$

At $y = \frac{P}{4}$ $\qquad \frac{d^2A}{dx^2} = -2$

$$= -ve$$

Thus area A is maximum at $y = \frac{P}{4}$ unit

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CHAPTER 7

INTEGRALS

OBJECTIVE QUESTIONS

1. The integral $\int \frac{dx}{\sqrt{9-4x^2}}$ is equal to :

(a) $\frac{1}{6}\sin^{-1}\left(\frac{2x}{3}\right)+c$ (b) $\frac{1}{2}\sin^{-1}\left(\frac{2x}{3}\right)+c$

(c) $\sin^{-1}\left(\frac{2x}{3}\right)+c$ (d) $\frac{3}{2}\sin^{-1}\left(\frac{2x}{3}\right)+c$

Sol : **OD 2024**

$$\begin{aligned} I &= \int \frac{1}{\sqrt{9-4x^2}}dx \\ &= \int \frac{1}{\sqrt{9(1-\frac{4}{3}x^2)}}dx \\ &= \int \frac{1}{3\sqrt{1-(\frac{2}{3}x)^2}}dx \\ &= \frac{1}{3}\int \frac{1}{\sqrt{1-(\frac{2}{3}x)^2}}dx \\ &= \frac{1}{3}\frac{\sin^{-1}(\frac{2}{3}x)}{\frac{2}{3}}+C \\ &= \frac{1}{2}\sin^{-1}\left(\frac{2}{3}x\right)+c \end{aligned}$$

Thus (b) is correct option.

2. The value of $\int_{\pi/4}^{\pi/2}\cot\theta\,\mathrm{cosec}^2\theta\,d\theta$ is

(a) $\frac{1}{2}$ (b) $-\frac{1}{2}$

(c) 0 (d) $-\frac{\pi}{8}$

Sol : **OD 2024**

$$I = \int_{\pi/4}^{\pi/2}\cot\theta\,\mathrm{cosec}^2\theta\,d\theta \qquad ...(i)$$

Let $\cot\theta = t$ then we have

$$\begin{aligned} -\mathrm{cosec}^2\theta\,d\theta &= dt \\ \mathrm{cosec}^2\theta\,d\theta &= -\,dt \end{aligned}$$

When $\theta = \frac{\pi}{4}$ $\quad t = \cot\frac{\pi}{4} = 1$

When $\theta = \frac{\pi}{2}$ $\quad t = \cot\frac{\pi}{2} = 0$

Hence, given integral become

$$\begin{aligned} I &= -\int_1^0 t\,dt \\ &= \int_0^1 t\,dt \qquad \left[\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx\right] \\ &= \left[\frac{t^2}{2}\right]_0^1 \\ &= \left(\frac{1}{2}-0\right) = \frac{1}{2} \end{aligned}$$

Thus (s) is correct option.

3. Anti-derivative of $\frac{\tan x - 1}{\tan x + 1}$ with respect to x is

(a) $\sec^2\left(\frac{\pi}{4}-x\right)+c$ (b) $-\sec^2\left(\frac{\pi}{4}-x\right)+c$

(c) $\log\left|\sec\left(\frac{\pi}{4}-x\right)\right|+c$ (d) $-\log\left|\sec\left(\frac{\pi}{4}-x\right)\right|+c$

Sol : **OD 2023**

$$\begin{aligned} I &= \int \frac{\tan x - 1}{\tan x + 1}dx \\ &= -\int \frac{1-\tan x}{1+\tan x}dx \\ &= -\int \tan\left(\frac{\pi}{4}-x\right)dx \\ &= \ln\left|\sec\left(\frac{\pi}{4}-x\right)\right|+c \end{aligned}$$

Thus (c) is correct option.

4. If $\frac{d}{dx}f(x) = 2x+\frac{3}{x}$ and $f(1)=1$, then $f(x)$ is

(a) $x^2+3\log|x|+1$ (b) $x^2+3\log|x|$

(c) $2-\frac{3}{x^2}$ (d) $x^2+3\log|x|-4$

Sol : **OD 2023**

We have $\qquad \frac{d}{dx}f(x) = 2x+\frac{3}{x}$

Integrating on both sides we have

$$\begin{aligned} \int \frac{d}{dx}f(x)\,dx &= \int\left(2x+\frac{3}{x}\right)dx \\ f(x) &= x^2+3\ln|x|+c \qquad ...(1) \end{aligned}$$

Since $f(1)=1$, substituting $x=1$ in above equation we have

$$\begin{aligned} f(1) &= 1^2+3\ln|1|+C \\ 1 &= 1+0+C \Rightarrow C=0 \end{aligned}$$

Thus $\qquad f(x) = x^2+3\ln|x|$

Thus (b) is correct option.

$$P = \frac{1}{4}$$

11. $\int \frac{x^{e-1}+e^{x-1}}{x^e+e^x}dx$ is equal to

(a) $\log(x^e+e^x)+C$ (b) $e\log(x^e+e^x)+C$

(c) $\frac{1}{e}\log(x^e+e^x)+C$ (d) None of these

Sol : **OD 2011**

We have $I = \int \frac{x^{e-1}+e^{x-1}}{x^e+e^x}dx$

Substituting $x^e+e^x = t \Rightarrow e(x^{e-1}+e^{x-1})\,dx = dt$

$$I = \frac{1}{e}\int\frac{dt}{t} = \frac{1}{e}\log t + C$$
$$= \frac{1}{e}\log(x^e+e^x)+C$$

Thus (c) is correct option.

12. $\int \frac{dx}{x(x^7+1)}$ is equal to

(a) $\log\left(\frac{x^7}{x^7+1}\right)+C$ (b) $\frac{1}{7}\log\left(\frac{x^7}{x^7+1}\right)+C$

(c) $\log\left(\frac{x^7+1}{x^7}\right)+C$ (d) $\frac{1}{7}\log\left(\frac{x^7+1}{x^7}\right)+C$

Sol : **Comp 2017, Delhi 2008**

We have $I = \int\frac{dx}{x(x^7+1)}$

Substituting $x^7 = t \Rightarrow dx = \frac{1}{7x^6}dt$ we have

$$I = \frac{1}{7}\int\frac{dt}{t(t+1)}$$
$$= \frac{1}{7}\int\left(\frac{1}{t}-\frac{1}{t+1}\right)dt$$
$$= \frac{1}{7}[\log t - \log(t+1)]+C$$
$$= \frac{1}{7}\log\left(\frac{t}{t+1}\right)+C$$
$$= \frac{1}{7}\log\left(\frac{x^7}{x^7+1}\right)+C$$

Thus (b) is correct option.

13. $\int_0^{\pi/2}\left|\cos\frac{x}{2}\right|dx$ is equal to

(a) 1 (b) -2

(c) $\sqrt{2}$ (d) 0

Sol : **Foreign 2009**

$$\int_0^{\pi/2}\left|\cos\left(\frac{x}{2}\right)\right|dx = \int_0^{\pi/2}\cos\left(\frac{x}{2}\right)dx$$
$$= 2\left[\sin\left(\frac{x}{2}\right)\right]_0^{\pi/2}]$$
$$= 2\left[\sin\frac{\pi}{4}-\sin 0\right]$$
$$= 2\sin\frac{\pi}{4}$$
$$= 2\times\frac{1}{\sqrt{2}} = \sqrt{2}$$

Thus (c) is correct option.

14. $3a\int_0^1\left(\frac{ax-1}{a-1}\right)^2dx$ is equal to

(a) $a-1+(a-1)^{-2}$ (b) $a+a^{-2}$

(c) $a-a^2$ (d) $a^2+\frac{1}{a^2}$

Sol : **SQP 2020**

$$3a\int_0^1\left(\frac{ax-1}{a-1}\right)^2dx = \frac{3a}{(a-1)^2}\left[\frac{(ax-1)^3}{3}\times\frac{1}{a}\right]_0^1$$
$$= \frac{1}{(a-1)^2}[(a-1)^3+1]$$
$$= (a-1)+(a-1)^{-2}$$

Thus (a) is correct option.

15. The value of $\int_0^1\frac{dx}{e^x+e}$ is

(a) $\frac{1}{e}\log\left(\frac{1+e}{2}\right)$ (b) $\log\left(\frac{1+e}{2}\right)$

(c) $\frac{1}{e}\log(1+e)$ (d) $\log\left(\frac{2}{1+e}\right)$

Sol : **Foreign 2018, Delhi 2010**

We have $I = \int_0^1\frac{dx}{e^x+e}$

$$= \int_0^1\frac{dx}{e^x\left(1+\frac{e}{e^x}\right)}$$

Substituting $1+\frac{e}{e^x} = t$

$$0-\frac{e}{e^x}dx = dt \Rightarrow \frac{1}{e^x}dx = -\frac{1}{e}dt \text{ we obtain}$$
$$I = -\frac{1}{e}\int_{1+e}^{2}\frac{1}{t}dt$$
$$= \frac{-1}{e}[\log t]_{1+e}^{2}$$
$$= \frac{-1}{e}[\log 2-\log(1+e)]$$
$$= \frac{-1}{e}\log\left(\frac{2}{1+e}\right)$$
$$= \frac{1}{e}\log\left(\frac{1+e}{2}\right)$$

Thus (a) is correct option.

16. The value of $\int_{-2}^{2}(x\cos x+\sin x+1)\,dx$ is

(a) 2 (b) 0

(c) -2 (d) 4

Sol : **Delhi 2015**

$$I = \int_{-2}^{2}(x\cos x+\sin x+1)\,dx$$
$$= \int_{-2}^{2}x\cos x\,dx+\int_{-2}^{2}\sin x\,dx+\int_{-2}^{2}1\,dx$$

(a) $2\int_0^a f(x)\,dx$ (b) 0

(c) 1 (d) -1

Sol : **Comp 2016**

We have $f(x) = -f(x)$

Thus $f(x)$ is odd function and for odd function

$$\int_{-a}^{a} f(x)\,dx = = 0$$

Thus (b) is correct option.

23. $\int_\alpha^\beta \phi(x)\,dx + \int_\beta^\alpha \phi(x)\,dx =$

(a) 1 (b) $2\int_\alpha^\beta \phi(x)\,dx$

(c) $-2\int_\beta^\alpha \phi(x)\,dx$ (d) 0

Sol : **OD 2012, Delhi 2010**

Using property $\int_\beta^\alpha \phi(x)\,dx = -\int_\alpha^\beta \phi(x)\,dx$ we have

$$\begin{aligned} I &= \int_\alpha^\beta \phi(x)\,dx + \int_\beta^\alpha \phi(x)\,dx \\ &= \int_\alpha^\beta \phi(x)\,dx - \int_\alpha^\beta \phi(x)\,dx \\ &= 0 \end{aligned}$$

Thus (d) is correct option.

24. $\int_0^1 (x)\,dx =$

(a) 0 (b) 1

(c) 2 (d) $\frac{1}{2}$

Sol : **Comp 2016**

We have
$$\begin{aligned} I &= \int_0^1 x dx \\ &= \frac{x^2}{2}\Big|_0^1 \\ &= \frac{1}{2} - \frac{0}{2} = \frac{1}{2} \end{aligned}$$

Thus (d) is correct option.

25. $\int \frac{dx}{\sqrt{x}} =$

(a) $\sqrt{x} + k$ (b) $2\sqrt{x} + k$

(c) $x + k$ (d) $\frac{2}{3}x^{3/2} + k$

Sol : **OD 2008**

We have
$$\begin{aligned} I &= \int \frac{dx}{\sqrt{x}} \\ &= \int x^{-1/2}\,dx \\ &= \frac{x^{-1/2+1}}{-1/2+1} + k \\ &= \frac{x^{1/2}}{1/2} + k \\ &= 2\sqrt{x} + k \end{aligned}$$

Thus (b) is correct option.

26. $\int_a^b x^5\,dx =$

(a) $b^5 - a^5$ (b) $\frac{b^6 - a^6}{6}$

(c) $\frac{a^6 - b^6}{6}$ (d) $a^5 - b^5$

Sol : **Delhi 2007**

Let,
$$\begin{aligned} I &= \int_a^b x^5\,dx \\ &= \frac{x^6}{6}\Big|_a^b \\ &= \frac{b^6}{6} - \frac{a^6}{6} \\ &= \frac{b^6 - a^6}{6} \end{aligned}$$

Thus (b) is correct option.

27. $\int_0^1 \frac{(\tan^{-1}x)^2}{1+x^2}\,dx =$

(a) 1 (b) $\frac{\pi^3}{64}$

(c) $\frac{\pi^2}{192}$ (d) None of these

Sol : **Delhi 2010**

Let, $I = \int_0^1 \frac{(\tan^{-1}x)^2}{1+x^2}\,dx$

Let $\tan^{-1}x = t \Rightarrow \frac{1}{1+x^2}\,dx = dt$

When $x = 0$, $t = 0$

and $x = 1$, $t = \frac{\pi}{4}$

Thus
$$\begin{aligned} I &= \int_0^1 \frac{(\tan^{-1}x)^2}{1+x^2}\,dx \\ &= \int_0^{\frac{\pi}{4}} t^2\,dt \\ &= \frac{t^3}{3}\Big|_0^{\pi/4} \\ &= \frac{1}{3}\left(\frac{\pi}{4}\right)^3 - \frac{0}{3} \\ &= \frac{\pi^3}{192} \end{aligned}$$

Thus (d) is correct option.

$$= \int \cot\theta \operatorname{cosec}\theta d\theta$$
$$= -\cos ec\,\theta + C$$

Thus (b) is correct option.

34. $\int \operatorname{cosec} x \cot x dx$ is

(a) $-\operatorname{cosec} x + c$ (b) $-\cot x$

(c) $\operatorname{cosec} x$ (d) None

Sol : **Comp 2008**

$$I = \int \operatorname{cosec} x \cot x dx$$
$$= -\frac{\operatorname{cosec} x}{\frac{d}{dx}(x)} + c$$
$$= -\operatorname{cosec} x + c$$

Thus (a) is correct option.

35. $\int \frac{\sin\sqrt{x}}{\sqrt{x}} dx$ is equal to

(a) $\cos\sqrt{x}$ (b) $-\cos\sqrt{x}$

(c) $2\cos\sqrt{x}$ (d) $-2\cos\sqrt{x}$

Sol : **OD 2007**

We have $I = \int \frac{\sin\sqrt{x}}{\sqrt{x}} dx$

Putting $\sqrt{x} = t \Rightarrow \frac{1}{2\sqrt{x}} dx = dt$ we have

$$I = \int 2\sin t dt$$
$$= -2\cos t + c$$
$$= -2\cos\sqrt{x} + c$$

Thus (d) is correct option.

36. $\int \log 2x\, dx$ is equal to

(a) $x\log x - 1$ (b) $x\log 2x + 1$

(c) $x\log 2x - x$ (d) $x\log 2x + 2x$

Sol : **Foreign 2018, OD 2013**

We have $I = \int \log 2x\, dx$

$$= \int \log 2x \cdot 1 dx$$
$$= \log 2x \cdot \int dx - \int \left\{\frac{1}{2x} \cdot 2 \int dx\right\} dx$$
$$= x\log 2x - \int \frac{1}{x} \cdot x dx$$
$$= x\log 2x - x + c$$

Thus (c) is correct option.

37. $\int \frac{\sin x + \cos x}{\sqrt{1 + \sin 2x}} dx$ is equal to

(a) $\sin x$ (b) x

(c) $\cos x$ (d) $\tan x$

Sol : **SQP 2020**

$$\int \frac{\sin x + \cos x}{\sqrt{1+\sin 2x}} dx = \int \frac{\sin x + \cos x}{\sqrt{(\sin^2 x + \cos^2 x) + 2\sin x \cos x}} dx$$
$$= \int \frac{\sin x + \cos x}{\sqrt{(\sin x + \cos x)^2}} dx$$
$$= \int \frac{\sin x + \cos x}{\sin x + \cos x} dx$$
$$= \int dx = x$$

Thus (b) is correct option.

38. The value of $\int_0^{\pi/2} \log \cos x\, dx$ is equal to the value of:

(a) $\int_0^{\pi/2} \log \sin x\, dx$ (b) $\int_0^{\pi/2} \log \sec x\, dx$

(c) $\int_0^{\pi/2} \log \cos x\, dx$ (d) $\int_0^{\pi/2} \log \tan x\, dx$

Sol : **OD 2012**

We have $I = \int_0^{\pi/2} \log \cos x\, dx$

Using property $\int_a^b f(x) \cdot dx = \int_a^b f(a+b-x) \cdot dx$ we get

$$I = \int_0^{\pi/2} \log \cos\left(\frac{x}{2} - x\right) dx$$
$$= \int_0^{\pi/2} \log \sin x\, dx$$

Thus (a) is correct option.

39. $I = \int_0^2 |1 - x|\, dx$ is equal to

(a) 0 (b) $\frac{1}{2}$

(c) 1 (d) None of these

Sol : **Delhi 2007**

We have $|1 - x| = 1 - x$, if $0 \leq x \leq 1$

$= x - 1$, if $1 < x \leq 2$

Hence
$$\int_0^2 |1-x|\, dx = \int_0^1 (1-x)\, dx + \int_1^2 (x-1)\, dx$$
$$= \left(x - \frac{x^2}{2}\right)_0^1 + \left(\frac{x^2}{2} - x\right)_1^2$$
$$= \left[\left(1 - \frac{1}{2}\right) - 0\right] + \left[\left(\frac{4}{2} - 2\right) - \left(\frac{1}{2} - 1\right)\right]$$
$$= 1$$

Thus (c) is correct option.

(c) (A) is true but (R) is false.

(d) (A) is false but (R) is true

Sol : OD 2010

$$\text{We have} \quad I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x\, dx$$

$$\text{Let} \quad f(x) = \sin^2 x$$

$$f(-x) = \sin^2(-x) = \sin^2 x = f(x)$$

Hence, $f(x)$ is an even function and for even function we have

$$\int_{-a}^{a} f(x)\, dx = 2\int_0^a f(x)\, dx$$

$$\text{Therefore,} \quad I = 2\int_0^{\frac{\pi}{2}} \sin^2 x\, dx$$

$$= 2\int_0^{\frac{\pi}{2}} \frac{1-\cos 2x}{2}\, dx$$

$$= x\Big|_0^{\frac{\pi}{2}} - \frac{\sin 2x}{2}\Big|_0^{\frac{\pi}{2}}$$

$$= \left(\frac{\pi}{2} - 0\right) - \frac{1}{2}(\sin\pi - \sin 0)$$

$$= \frac{\pi}{2}$$

Therefore both (A) and (R) are true and (R) is the correct explanation of (A).

Thus (a) is correct option.

45. **Assertion (A) :** $\int \log x dx = x(\log x - 1) + k$

Reason (R) :
$\int (uv)\, dx = u\int v dx - \int \left\{\frac{d}{dx}(u)\int v dx\right\} dx$.

(a) Both (A) and (R) are true and (R) is the correct explanation of (A).

(b) Both (A) and (R) are true but (R) is not the correct explanation of (A).

(c) (A) is true but (R) is false.

(d) (A) is false but (R) is true

Sol : SQP 2020, Foreign 2017

$$\text{We have} \quad I = x(\log x - 1) + k$$

$$= \int \underset{II}{1} \underset{I}{\log x} dx$$

Using integration by parts, we get

$$I = \log x \cdot \int dx - \int \left(\frac{d}{dx}\log x \int 1\, dx\right) dx$$

$$= \log x \cdot x - \int \left(\frac{1}{x} \times x\right) dx$$

$$= x \cdot \log x - x + k$$

$$= x(\log x - 1) + k$$

Formula of Integration by parts:

$$\int (uv)\, dx = u\int v dx - \int \left\{\frac{d}{dx}(u)\int v dx\right\} dx$$

Therefore both (A) and (R) are true and (R) is the correct explanation of (A).

Thus (a) is correct option.

46. **Assertion (A) :** $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x\, dx = 0$

Reason (R) : Here $\sin^7 x$ is odd function.

(a) Both (A) and (R) are true and (R) is the correct explanation of (A).

(b) Both (A) and (R) are true but (R) is not the correct explanation of (A).

(c) (A) is true but (R) is false.

(d) (A) is false but (R) is true

Sol : Delhi 2010

$$\text{We have} \quad I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x\, dx$$

$$\text{Let,} \quad f(x) = \sin^7 x$$

$$\text{Here} \quad f(-x) = \sin^7(-x) = (-\sin x)^7 = -\sin^7 x = -f(x)$$

Hence, $f(x)$ is odd function and we have

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x\, dx = 0$$

Both (A) and (R) are true and (R) is the correct explanation of (A).

Thus (a) is correct option.

47. **Assertion (A) :** $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\sin|x| + \cos|x|\right) dx = 2$

Reason (R) : $\int_{-a}^{a} f(x)\, dx = 2\int_0^a f(x)\, dx$,

where $f(x)$ is an even function.

(a) Both (A) and (R) are true and (R) is the correct explanation of (A).

(b) Both (A) and (R) are true but (R) is not the correct explanation of (A).

(c) (A) is true but (R) is false.

(d) (A) is false but (R) is true

Sol : Comp 2009

$$\text{Let,} \quad I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\sin|x| + \cos|x|\right) dx$$

(a) Both (A) and (R) are true and (R) is the correct explanation of (A).

(b) Both (A) and (R) are true but (R) is not the correct explanation of (A).

(c) (A) is true but (R) is false.

(d) (A) is false but (R) is true

Sol : **OD 2012**

As We know that

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx, \text{ Where } c\,\varepsilon\,(a.b)$$

So, given reason is true.

Now, $\int_{-\pi/2}^{\pi/2} |\sin x|\,dx \ \sin x,$

Since, $|\sin x|$ is an even faction

$$\text{so,} \quad \int_{-\pi/2}^{\pi/2} |\sin x|\,dx = 2\int_0^{\pi/2} |\sin x|\,dx$$

$$= 2\int_O^{\pi/2} |\sin x|\,dx$$

$$= 2\int_O^{\pi/2} \sin x\,dx$$

$$= -(\cos x)0^{\pi/2}$$

$$-2(0-1) = 2$$

Hence, both Assertion and reason are true bat reason is not correct explanation for assertion.

Thus (b) is correct option.

52. If $n > 1$, then

Assertion (A) : $\int_0^\infty \frac{dx}{1+x^n} = \int_0^1 \frac{dx}{(1-x^n)^{1/n}}$

Reason (R) : $\int_a^b f(x)\,dx = \int_a^b f(a+b+x)\,dx$

(a) Both (A) and (R) are true and (R) is the correct explanation of (A).

(b) Both (A) and (R) are true but (R) is not the correct explanation of (A).

(c) (A) is true but (R) is false.

(d) (A) is false but (R) is true

Sol : **Comp 2015, OD 2011**

As we know that,

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

So, given Reason is false.

Now, in L.H.S, put

$$x^n = \tan^2\theta$$

$$nx^{n-1}\,dx = 2\tan\theta\sec^2\theta\,d\theta$$

$$\text{So,} \quad \int_0^\infty \frac{dx}{1+x^n} = \frac{2}{n}\int_0^{\pi/2} \tan^{1-2+2/n}\,d\theta$$

$$= \frac{2}{n}\int_0^{\pi/2} \tan^{(2/n)-1}\theta\,d\theta$$

In RHS, substituting $x^n = \sin^2\theta$ we have

$$nx^{n-1}\,dx = 2\sin\theta\cos\theta\,d\theta$$

$$\text{So,} \quad \int_0^1 \frac{dx}{(1-x^n)^{1/n}} = \frac{2}{n}\int_0^{\pi/2} \frac{1}{\cos^{2/n}\theta}\cdot\sin^{2/n-1}\theta\cos\theta\,d\theta$$

$$= \frac{2}{n}\int_0^{\pi/2} \tan^{(2/n\,-1)}\theta\,d\theta$$

Hence, assertion is true but reason is false.

Thus (c) is correct option.

53. **Assertion (A) :** $\int_0^{2\pi} \sin^3 x\,dx = 0$

Reason (R) : $\sin^3 x$ is an odd function

(a) Both (A) and (R) are true and (R) is the correct explanation of (A).

(b) Both (A) and (R) are true but (R) is not the correct explanation of (A).

(c) (A) is true but (R) is false.

(d) (A) is false but (R) is true

Sol : **SQP 2016**

$$I = \int_0^{2\pi} \sin^3 x\,dx$$

$$= \int_0^{2\pi} (1-\cos^2 x)\sin x\,dx$$

Substituting $\cos x = t$ we have

$$\sin x\,dx = -\,dt$$

$$\text{Then} \quad I = \int_1^1 (1-t^2)(-\,dt) = 0$$

Thus assertion is true, Reason is true; Reason is not a correct explanation for Assertion.

Thus (b) is correct option.

54. **Assertion (A) :**

$$\int \sin 3x\cos 5x\,dx = \frac{-\cos 8x}{16} + \frac{\cos 2x}{4} + C$$

Reason (R) : $2\cos A\sin B = \sin(A+B) - \sin(A-B)$

(a) Assertion is true, Reason is true; Reason is a correct explanation for Assertion.

(b) Assertion is true, Reason is true; Reason is not a correct explanation for Assertion.

Ch 7 : Integrals

PDF Text Book | PDF Worksheet | PDF Solutions | DOC Worksheet | PPT Presentation

Ch 8 : Application of Integrals

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Ch 9 : Differential Equations

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Class 10th

Maths	Science	Social Science	English LL
Info Tech.	Comp. App.	AI	English Comm.
हिंदी अ	हिंदी ब	संस्कृत	

Class 9th

Maths	Science	Social Science	English LL
Info Tech.	Comp. App.	AI	English Comm.
हिंदी अ	हिंदी ब	संस्कृत	

Class 8th

Maths	Science	Social Science	English
हिंदी	संस्कृत		

Class 7th

Maths	Science	Social Science	English
हिंदी	संस्कृत		

Class 6th

Maths	Science	Social Science	English
हिंदी	संस्कृत		

Class 5th

Maths | English | हिंदी | EVS

Class 4th

Maths | English | हिंदी | EVS

Class 3th

Maths | English | हिंदी | EVS

Class 2th

Maths | English | हिंदी | EVS

Class 1th

Maths | English | हिंदी | EVS

Prep/KG 3rd Year

Maths | English | हिंदी | EVS

Prep/KG 2nd Year

Maths | English | हिंदी | EVS

Prep/KG 1st Year

Maths | English | हिंदी | EVS

59. Find $\int \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x} dx$.

Sol : **Foreign 2014**

We have
$$\begin{aligned} I &= \int \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x} dx \\ &= \int \frac{\sin^2 x}{\sin^2 x \cos^2 x} dx - \int \frac{\cos^2 x}{\sin^2 x \cos^2 x} dx \\ &= \int \sec^2 x\, dx - \int \operatorname{cosec}^2 x\, dx \\ &= \tan x + \cot x + C \end{aligned}$$

60. Find $\int \frac{\sin^6 x}{\cos^8 x} dx$.

Sol : **Foreign 2014**

We have
$$I = \int \frac{\sin^6 x}{\cos^8 x} dx = \int \tan^6 x \sec^2 x\, dx$$

Substituting $\tan x = t \Rightarrow \sec^2 x\, dx = dt$ we have
$$\begin{aligned} I &= \int t^6\, dt \\ &= \frac{t^7}{7} + C \\ &= \frac{\tan^7 x}{7} + C \end{aligned}$$

61. Evaluate $\int \cos^{-1}(\sin x)\, dx$.

Sol : **Foreign 2014**

We have
$$\begin{aligned} I &= \int \cos^{-1}(\sin x)\, dx \\ &= \int \cos^{-1}\left[\cos\left(\frac{\pi}{2} - x\right)\right] dx \\ &= \int \left(\frac{\pi}{2} - x\right) dx \qquad \cos^{-1}(\cos\theta) = \theta \\ &= \frac{\pi}{2}\int dx - \int x\, dx \\ &= \frac{\pi}{2}x - \frac{x^2}{2} + C \end{aligned}$$

62. Evaluate $\int (1-x)\sqrt{x}\, dx$.

Sol : **Delhi 2012, Comp 2010**

We have
$$\begin{aligned} I &= \int (1-x)\sqrt{x}\, dx \\ &= \int (\sqrt{x} - x\sqrt{x})\, dx \\ &= \int (x^{1/2} - x^{3/2})\, dx \\ &= \frac{2}{3}x^{3/2} - \frac{2}{5}x^{5/2} + C \end{aligned}$$

63. Evaluate $\int \frac{2}{1+\cos 2x} dx$.

Sol : **Foreign 2012**

We have
$$\begin{aligned} I &= \int \frac{2}{1+\cos 2x} dx \\ &= \int \frac{2}{2\cos^2 x} dx \qquad \cos 2\theta = 2\cos^2\theta - 1 \\ &= \int \sec^2 x dx \\ &= \tan x + C \end{aligned}$$

64. Write the value of $\int \frac{dx}{x^2+16}$.

Sol : **Foreign 2011**

We have
$$\begin{aligned} I &= \int \frac{dx}{x^2+16} \\ &= \int \frac{dx}{x^2+(4)^2} \qquad \int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C \\ &= \frac{1}{4}\tan^{-1}\frac{x}{4} + C \end{aligned}$$

65. Write the value of $\int \frac{2-3\sin x}{\cos^2 x} dx$.

Sol : **Delhi 2011**

We have
$$\begin{aligned} I &= \int \frac{2-3\sin x}{\cos^2 x} dx \\ &= \int \left(\frac{2}{\cos^2 x} - \frac{3\sin x}{\cos^2 x}\right) dx \\ &= \int (2\sec^2 x - 3\sec x \tan x)\, dx \\ &= 2\int \sec^2 x\, dx - 3\int \sec x \tan x\, dx \\ &= 2\tan x - 3\sec x + C \end{aligned}$$

66. Write the value of $\int \sec x(\sec x + \tan x)\, dx$.

Sol : **Delhi 2011**

We have
$$\begin{aligned} I &= \int \sec x(\sec x + \tan x)\, dx \\ &= \int (\sec^2 x + \sec x \tan x)\, dx \\ &= \int \sec^2 x\, dx + \int \sec x \tan x\, dx \\ &= \tan x + \sec x + C \end{aligned}$$

67. Evaluate $\int \frac{dx}{\sqrt{1-x^2}}$.

Sol : **OD 2011, Delhi 2008**

We have
$$\begin{aligned} I &= \int \frac{dx}{\sqrt{1-x^2}} \\ &= \int \frac{dx}{\sqrt{(1)^2 - x^2}} \\ &= \sin^{-1} x + C \qquad \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C \end{aligned}$$

68. Evaluate $\int \frac{e^{\tan^{-1}x}}{1+x^2} dx$.

Sol : **OD 2011**

We have
$$I = \int \frac{e^{\tan^{-1}x}}{1+x^2} dx$$

Substituting $\tan^{-1} x = t \Rightarrow \frac{1}{1+x^2} dx = dt$ we have
$$I = \int \frac{e^{\tan^{-1}x}}{1+x^2} dx = \int e^t\, dt$$

$$= \int \sec^2 x\, dx$$
$$= \tan x + C$$

77. Find $\int \frac{dx}{x^2 + 4x + 8}$.
Sol : **Delhi 2017**

We have
$$I = \int \frac{dx}{x^2 + 4x + 8}$$
$$= \int \frac{dx}{x^2 + 4x + 4 + 4}$$
$$= \int \frac{dx}{(x+2)^2 + (2)^2}$$
$$= \frac{1}{2} \tan^{-1}\left(\frac{x+2}{2}\right) + C$$

78. Find : $\int \frac{3 - 5\sin x}{\cos^2 x} dx$.
Sol : **Foreign 2018**

We have
$$I = \int \frac{3 - 5\sin x}{\cos^2 x} dx$$
$$= \int \left(\frac{3}{\cos^2 x} - \frac{5\sin x}{\cos^2 x}\right) dx$$
$$= 3\int \sec^2 x\, dx - 5\int \sec x \tan x\, dx$$
$$= 3\tan x - 5\sec x + C$$

79. Evaluate $\int_2^3 3^x\, dx$.
Sol : **Delhi 2017**

$$\int_2^3 3^x\, dx = \left(\frac{3^x}{\log 3}\right)_2^3$$
$$= \frac{1}{\log 3}[3^x]_2^3$$
$$= \frac{1}{\log 3}[3^3 - 2^2]$$
$$= \frac{1}{\log 3}(27 - 9)$$
$$= \frac{18}{\log 3}$$

80. Evaluate $\int_0^{\pi/4} \tan x\, dx$.
Sol : **Foreign 2014, Delhi 2013**

$$\int_0^{\pi/4} \tan x\, dx = [\log|\sec x|]_0^{\pi/4}$$
$$= \log\left|\sec\frac{\pi}{4}\right| - \log|\sec 0|$$
$$= \log|\sqrt{2}| - \log|1|$$
$$= \frac{1}{2}\log 2$$

81. Evaluate $\int_0^1 x\, e^{x^2}\, dx$.
Sol : **Foreign 2014**

We have
$$I = \int_0^1 x\, e^{x^2}\, dx$$

Substituting $x^2 = t \Rightarrow 2x\, dx = dt \Rightarrow dx = \frac{dt}{2x}$
When $x = 0$, then $t = 0$ and when $x = 1$, then $t = 1$,

Thus
$$I = \int_0^1 x\, e^t \frac{dt}{2x}$$
$$= \frac{1}{2}\int_0^1 e^t dt$$
$$= \frac{1}{2}[e^t]_0^1$$
$$= \frac{1}{2}[e^1 - e^0]$$
$$= \frac{1}{2}[e - 1]$$

82. Evaluate $\int_0^{\pi/4} \sin 2x\, dx$.
Sol : **Foreign 2014**

We have
$$I = \int_0^{\pi/4} \sin 2x\, dx$$
$$= \left[\frac{-\cos 2x}{2}\right]_0^{\pi/4}$$
$$= -\frac{1}{2}[\cos 2x]_0^{\pi/4}$$
$$= -\frac{1}{2}\left[\cos 2\frac{\pi}{4} - \cos 0\right]$$
$$= -\frac{1}{2}\left[\cos\frac{\pi}{2} - 1\right]$$
$$= -\frac{1}{2}[0 - 1] = \frac{1}{2}$$

83. Evaluate $\int_0^1 \frac{1}{\sqrt{1 - x^2}} dx$.
Sol : **Comp 2014, OD 2011**

We have
$$I = \int_0^1 \frac{1}{\sqrt{1 - x^2}} dx$$
$$= [\sin^{-1} x]_0^1 \qquad \int \frac{1}{\sqrt{1 - x^2}} dx = \sin^{-1} x + C$$
$$= \sin^{-1} 1 - \sin^{-1} 0$$
$$= \sin^{-1}\left(\sin\frac{\pi}{2}\right) - \sin^{-1}(\sin 0)$$
$$= \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

84. Evaluate $\int_1^2 \frac{x^3 - 1}{x^2} dx$.
Sol : **Comp 2014**

We have
$$I = \int_1^2 \frac{x^3 - 1}{x^2} dx$$
$$= \int_1^2 \left(x - \frac{1}{x^2}\right) dx$$
$$= \left[\frac{x^2}{2} + \frac{1}{x}\right]_1^2$$
$$= \left(\frac{(2)^2}{2} + \frac{1}{2}\right) - \left(\frac{(1)^2}{2} + \frac{1}{1}\right)$$
$$= \left(2 + \frac{1}{2}\right) - \left(\frac{1}{2} + 1\right) = 1$$

$$= [\log 2 - \log 1]$$
$$= \log 2 - 0$$
$$= \log 2$$

SHORT ANSWER QUESTIONS

93. Evaluate : $\int_0^\pi \frac{e^{\cos x}}{e^{\cos x} + e^{-\cos x}} dx$

Sol : **OD 2024**

Let $I = \int_0^\pi \frac{e^{\cos x}}{e^{\cos x} + e^{-\cos x}} dx$ ...(1)

Using $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ we have

$$I = \int_0^\pi \frac{e^{\cos(\pi - x)}}{e^{\cos(\pi - x)} + e^{-\cos(\pi - x)}} dx$$

$$I = \int_0^\pi \frac{e^{-\cos x}}{e^{-\cos x} + e^{\cos x}} dx \quad ...(2)$$

Adding equations (1) and (2), we get

$$2I = \int_0^\pi \frac{e^{\cos x}}{e^{\cos x} + e^{-\cos x}} dx + \int_0^\pi \frac{e^{-\cos x}}{e^{-\cos x} + e^{\cos x}} dx$$
$$= \int_0^\pi \frac{e^{\cos x} + e^{-\cos x}}{e^{\cos x} + e^{-\cos x}} dx$$
$$= \int_0^\pi dx$$
$$= [x]_0^\pi$$
$$2I = \pi$$
$$I = \frac{\pi}{2}$$

94. Find : $\int \frac{2x+1}{(x+1)^2(x-1)} dx$

Sol : **OD 2024**

We have $I = \int \frac{2x+1}{(x+1)^2(x-1)} dx$

Let $\frac{2x+1}{(x+1)^2(x-1)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x-1)}$

$$2x+1 = A(x+1)(x-1) + B(x-1) + C(x+1)^2$$

Substituting $x = 1$ we have

$$3 = C(1+1)^2 \Rightarrow C = \tfrac{3}{4}$$

Substituting $x = -1 = 1$ we have

$$-1 = B(-1-1)$$
$$-1 = -2B \Rightarrow B = \tfrac{1}{2}$$
$$B = \frac{1}{2}$$

Comparing coefficient of x^2 we have

$$A + C = 0 \Rightarrow A = -C = -\tfrac{3}{4}$$

Thus given integral becomes

$$I = \int \frac{-3}{4(x+1)} dx + \frac{1}{2}\int \frac{dx}{(x+1)^2} + \frac{3}{4}\int \frac{dx}{(x-1)}$$
$$= -\frac{3}{4}\log|(x+1)| + \frac{1}{2}\left[\frac{-1}{(x+1)}\right] + \frac{3}{4}\log|(x-1)| + C$$
$$= -\frac{3}{4}\log|(x+1)| - \frac{1}{2(x+1)} + \frac{3}{4}\log|(x-1)| + C$$
$$= \frac{3}{4}\log\left|\left(\frac{x-1}{x+1}\right)\right| - \frac{1}{2(x+1)} + C$$

95. Evaluate $\int_{\log\sqrt{2}}^{\log\sqrt{3}} \frac{1}{(e^x + e^{-x})(e^x - e^{-x})} dx$

Sol : **OD 2023**

Let $I = \int_{\log\sqrt{2}}^{\log\sqrt{3}} \frac{1}{(e^x + e^{-x})(e^x - e^{-x})} dx$

$$I = \int_{\log\sqrt{2}}^{\log\sqrt{3}} \frac{1}{(e^{2x} - e^{-2x})} dx$$
$$= \int_{\log\sqrt{2}}^{\log\sqrt{3}} \frac{e^{2x}}{(e^{4x} - 1)} dx$$

Substituting $t = e^{2x} \Rightarrow dt = 2e^{2x}\,dx$

$$e^{2(\log\sqrt{3})} = 3 \text{ and } e^{2(\log\sqrt{2})} = 2$$

Now $I = \int_2^3 \frac{1}{2(t^2 - 1)} dt$

$$= \frac{1}{4}\left[\ln\left|\frac{t-1}{t+1}\right|\right]_2^3$$
$$= \frac{1}{4}\left[\ln\frac{1}{2} - \ln\frac{1}{3}\right]$$
$$= \frac{1}{4}\ln\left(\frac{3}{2}\right)$$

96. Find $\int \frac{\sin^{-1}x}{(1-x^2)^{3/2}}$

Sol : **OD 2023**

We have $I = \int \frac{\sin^{-1}x}{(1-x^2)^{3/2}} dx$

$$= \int \frac{\sin^{-1}x}{(1-x^2)\sqrt{1-x^2}} dx$$

Let $t = \sin^{-1}x \Rightarrow x = \sin t$, then we have

$$dt = \frac{1}{\sqrt{1-x^2}} dx$$

Now substituting above we have

$$I = \int \frac{t}{1-\sin^2 t} dt$$
$$I = \int t\sec^2 t\,dt$$

102. Evaluate $\int \frac{dx}{\sin^2 x \cos^2 x}$.

Sol : Delhi 2014C; Foreign 2014

We have
$$I = \int \frac{dx}{\sin^2 x \cos^2 x}$$
$$= \int \frac{(\sin^2 x + \cos^2 x)}{\sin^2 x \cdot \cos^2 x} dx \qquad \sin^2\theta + \cos^2\theta = 1$$
$$= \int (\sec^2 x + \operatorname{cosec}^2 x)\, dx$$
$$= \int \sec^2 x\, dx + \int \operatorname{cosec}^2 x\, dx$$
$$= \tan x - \cot x + C$$

Alternative :

Dividing the numerator and denominator by $\cos^4 x$, we get
$$I = \int \frac{\sec^2 x \cdot \sec^2 x}{\tan^2 x} dx$$
$$= \int \frac{(1 + \tan^2 x) \cdot \sec^2 x}{\tan^2 x} dx$$
Substituting $\tan x = t \Rightarrow \sec^2 x\, dx = dt$ we have
$$I = \int \frac{1+t^2}{t^2} dt = \int 1\, dt + \int \frac{1}{t^2} dt$$
$$= t - \frac{1}{t} + C$$
Substituting $t = \tan x$ we obtain
$$I = \tan x - \cot x + C$$

103. Write the anti-derivative of $\left(3\sqrt{x} + \frac{1}{\sqrt{x}}\right)$.

Sol : Delhi 2014

$$I = \int \left(3\sqrt{x} + \frac{1}{\sqrt{x}}\right) dx$$
$$= 3\int \sqrt{x}\, dx + \int \frac{1}{\sqrt{x}} dx$$
$$= \int \left(3\sqrt{x} + \frac{1}{\sqrt{x}}\right) dx$$
$$= 3\int \sqrt{x}\, dx + \int \frac{1}{\sqrt{x}} dx$$
$$= 3\left(\frac{x^{1/2+1}}{1/2+1}\right) + \left[\frac{x^{-1/2+1}}{-1/2+1}\right] + C$$
$$= 2(x^{3/2} + x^{1/2}) + C$$

104. Given, $\int e^x (\tan x + 1) \sec x\, dx = e^x f(x) + C$. Write $f(x)$ satisfying above.

Sol : OD 2012; Foerign 2011

$$\int e^x (\tan x + 1) \sec x dx = e^x f(x) + C$$
$$\int e^x (\tan x \sec x + \sec x)\, dx = e^x f(x) + C$$
$$\int e^x (\sec x + \sec x \tan x)\, dx = e^x f(x) + C$$
It may be easily seen that $f(x) = \sec x$ and $f'(x) = \sec x \tan x$. Thus we may use
$$\int e^x [f(x) + f'(x)]\, dx = e^x f(x) + C$$
and given integral reduce to
$$e^x \cdot \sec x + C = e^x f(x) + C$$
Comparing both sides, we get
$$f(x) = \sec x$$

105. Write the value of $\int \frac{x + \cos 6x}{3x^2 + \sin 6x} dx$.

Sol : Comp 2012

We have
$$I = \int \frac{x + \cos 6x}{3x^2 + \sin 6x} dx$$
Substituting $3x^2 + \sin 6x = t$ we have
$$6x + 6\cos 6x = \frac{dt}{dx}$$
$$(x + \cos 6x)\, dx = \frac{dt}{6}$$
Thus
$$I = \int \frac{dt}{6t} = \frac{1}{6} \log|t| + C$$
$\int \frac{1}{x} dx = \log|x| + C$

Now Substituting $t = 3x^2 + \sin 6x$ we get
$$I = \frac{1}{6}[\log|(3x^2 + \sin 6x)|] + C$$

106. Write the value of $\int \frac{\sec^2 x}{\operatorname{cosec}^2 x} dx$.

Sol : Delhi 2012C, 2011

We have
$$I = \int \frac{\sec^2 x}{\operatorname{cosec}^2 x} dx = \int \frac{\frac{1}{\cos^2 x}}{\frac{1}{\sin^2 x}} dx$$
$$= \int \frac{\sin^2 x}{\cos^2 x} dx$$
$$= \int \tan^2 x\, dx \qquad \tan^2 x = \sec^2 x - 1$$
$$= \int (\sec^2 x - 1)\, dx$$
$$= \int \sec^2 x\, dx - \int 1\, dx$$
$$= \tan x - x + C$$

107. Evaluate $\int \frac{(\log x)^2}{x} dx$.

Sol : OD 2011

We have
$$I = \int \frac{(\log x)^2}{x} dx$$
Substituting $\log x = t \Rightarrow \frac{1}{x} dx = dt$ we have
$$I = \int \frac{(\log x)^2}{x} dx$$
$$= \int t^2\, dt$$
$$= \frac{t^3}{3} + C$$
Substituting $t = \log x$ we have

Substituting $\sin x = t \Rightarrow \cos x\,dx = dt$ we have

$$I = \int \frac{2}{t^2}\,dt$$
$$= 2\int \frac{dt}{t^2}$$
$$= \frac{2t^{-1}}{-1} + C$$

Substituting $t = \sin x$ we have

$$I = -2(\sin x)^{-1} + C$$
$$= \frac{-2}{\sin x} + C$$
$$= -2\operatorname{cosec} x + C$$

Alternative :

We have $$I = \int \frac{2\cos x}{\sin^2 x}\,dx$$
$$= 2\int \frac{1}{\sin x}\cdot\frac{\cos x}{\sin x}\,dx$$
$$= 2\int \operatorname{cosec} x \cot x\,dx$$
$$= 2(-\operatorname{cosec} x) + C$$
$$= -2\operatorname{cosec} x + C$$

114. Find : $\int \frac{\sec^2 x}{\sqrt{\tan^2 x + 4}}\,dx$.
Sol : **Delhi 2019**

We have $$I \int \frac{\sec^2 x}{\sqrt{\tan^2 x + 4}}\,dx$$

Substituting $\tan x = t$ we obtain

$$\sec^2 x = \frac{dt}{dx} \Rightarrow dx = \frac{dt}{\sec^2 x}$$

Thus $$I = \int \frac{\sec^2 x}{\sqrt{t^2+4}}\frac{dt}{\sec^2 x}$$
$$= \int \frac{dt}{\sqrt{t^2+2^2}} \qquad \int \frac{dx}{\sqrt{x^2+a^2}} = \log\left|x + \sqrt{x^2+a^2}\right| + C$$
$$= \log\left|t + \sqrt{t^2+4}\right| + C$$

Now substituting $t = \tan x$ we have

$$I = \log\left|\tan x + \sqrt{\tan^2 x + 4}\right| + C$$

115. Find : $\int \sqrt{1-\sin 2x}\,dx$, $\frac{\pi}{4} < x < \frac{\pi}{2}$.
Sol : **Delhi 2019, Comp 2010**

We have $$I = \int \sqrt{1-\sin 2x}\,dx$$

Using the facts $\sin^2 x + \cos^2 x = 1$ and $2\sin x\cos x = \sin 2x$ we have

$$I = \int \sqrt{\sin^2 x + \cos^2 x - 2\sin x\cos x}\,dx$$
$$= \int \sqrt{(\sin x - \cos x)^2}\,dx$$
$$= \int (\sin x - \cos x)\,dx$$

In the interval $\frac{\pi}{4} < x < \frac{\pi}{2}$ always $\sin x > \cos x$. Thus

$$I = \int (\sin x - \cos x)\,dx$$
$$= -\cos x - \sin x + C$$
$$= -(\cos x + \sin x) + C$$

116. Find : $\int \sin^{-1}(2x)\,dx$.
Sol : **Delhi 2019**

We have $$I = \int \sin^{-1}(2x)\,dx$$

Substituting $2x = y \Rightarrow x = \frac{y}{2} \Rightarrow dx = \frac{dy}{2}$ we have

$$I = \frac{1}{2}\int \sin^{-1}(y)\,dy$$

Here we will use integration by parts technique.

$$I = \frac{1}{2}\left[\sin^{-1}(y)\cdot y - \int \frac{1}{\sqrt{1-y^2}}\cdot y\,dy\right]$$
$$= \frac{1}{2}\left[y\sin^{-1}y + \frac{1}{2}\int -\frac{2y}{\sqrt{1-y^2}}\,dy\right]$$
$$= \frac{1}{2}\left[y\sin^{-1}y + \sqrt{1-y^2} + C\right] \qquad \int \frac{dy}{\sqrt{x}} = 2\sqrt{x} + C$$

Now substituting $y = 2x$ we have

$$I = \frac{1}{2}\left[2x\sin^{-1}2x + \sqrt{1-4x^2} + C\right]$$

117. Find the value of $\int \frac{\tan^2 x\cdot\sec^2 x}{1-\tan^6 x}\,dx$.
Sol : **Delhi 2019**

We have $$I = \int \frac{\tan^2 x\cdot\sec^2 x}{1-\tan^6 x}\,dx$$

Substituting $\tan^3 x = t \Rightarrow 3\tan^2 x\sec^2 x\,dx = dt$ we have

$$\tan^2 x\sec^2 dx = \frac{1}{3}\cdot dt$$

Thus $$I = \int \frac{1}{1-t^2}\frac{dt}{3}$$
$$= \frac{1}{3}\int \frac{dt}{1-t^2} \qquad \int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$$
$$= \frac{1}{3}\left[\frac{1}{2}\log\left|\frac{1+t}{1-t}\right| + C\right]$$
$$= \frac{1}{6}\log\left|\frac{1+t}{1-t}\right| + C$$

Now substituting the value of t, we get

$$\int \frac{\tan^2 x\sec^2 x}{1-\tan^6 x}\,dx = \frac{1}{6}\log\left|\frac{1+\tan^3 x}{1-\tan^3 x}\right| + C$$

118. Find the values of $\int \sin x\cdot\log\cos x\,dx$.
Sol : **Delhi 2019**

We have $$I = \int \sin x\cdot\log\cos x\,dx$$

Substituting $\cos x = t \Rightarrow -\sin x\,dx = dt$ we have

$$I = -\int \log t\,dt$$

124. Find $\int \frac{\sqrt{x}}{\sqrt{a^3 - x^3}}\,dx$.

Sol : **Delhi 2016, OD 2012**

We have
$$I = \int \frac{\sqrt{x}}{\sqrt{a^3-x^3}}\,dx$$
$$= \int \frac{\sqrt{x}}{\sqrt{(a^{3/2})^2 - (x^{3/2})^2}}\,dx$$

Substituting $x^{3/2} = a^{3/2}t$ we get
$$\frac{3}{2}x^{1/2}\,dx = a^{3/2}\,dt \Rightarrow \sqrt{x}\,dx = \frac{2}{3}a^{3/2}\,dt$$

Thus
$$I = \int \frac{\frac{2}{3}a^{3/2}}{\sqrt{(a^{3/2})^2 - (a^{3/2}t)^2}}\,dt$$
$$= \frac{2}{3}a^{3/2}\int \frac{dt}{a^{3/2}\sqrt{1-t^2}}$$
$$= \frac{2}{3}\int \frac{dt}{\sqrt{1-t^2}}$$
$$= \frac{2}{3}\sin^{-1}\left(\frac{t}{1}\right) + C \qquad \int \frac{dx}{a^2-x^2} = \sin^{-1}\left(\frac{x}{a}\right) + C$$
$$= \frac{2}{3}\sin^{-1}\left(\frac{x^{3/2}}{a^{3/2}}\right) + C \qquad t = \frac{x^{3/2}}{a^{3/2}}$$
$$= \frac{2}{3}\sin^{-1}\left(\sqrt{\frac{x^3}{a^3}}\right) + C$$

125. Evaluate $\int \frac{x\sin^{-1}x}{\sqrt{1-x^2}}\,dx$.

Sol : **Foreign 2016; Delhi 2012**

We have
$$I = \int \frac{x\sin^{-1}x}{\sqrt{1-x^2}}\,dx$$

Substituting $\sin^{-1}x = t$ we get
$$\frac{1}{\sqrt{1-x^2}}\,dx = dt$$

Thus
$$I = \int t\sin t\,dt$$

Using integration by parts, taking t as the first function and $\sin t$ as the second function, we get
$$I = \int \underset{I}{t}\,\underset{II}{\sin}\,t\,dt$$
$$= t\int \sin t\,dt - \int\left[\frac{d}{dt}(t)\cdot\int \sin t\,dt\right]dt$$
$$= -t\cos t - \int (1\times -\cos t)\,dt$$
$$= -t\cos t + \int \cos t\,dt$$
$$= -t\cos t + \sin t + C$$
$$= -t\sqrt{1-\sin^2 t} + \sin t + C$$

Substituting $t = \sin^{-1}x \Rightarrow x = \sin t$ we have
$$I = -\sin^{-1}x\sqrt{1-x^2} + x + C$$

126. Evaluate $\int \frac{\sin(x-a)}{\sin(x+a)}\,dx$.

Sol : **Foreign 2015; Delhi 2013**

We have
$$I = \int \frac{\sin(x-a)}{\sin(x+a)}\,dx$$

Substituting $x + a = t \Rightarrow dx = dt$ we have
$$I = \int \frac{\sin(t-a-a)}{\sin t}\,dt$$
$$= \int \frac{\sin(t-2a)}{\sin t}\,dt$$
$$= \int \frac{\sin t\cos 2a - \cos t\sin 2a}{\sin t}\,dt$$
$$= \int \cos 2a\,dt - \int \sin 2a\cdot\cot t\,dt$$
$$= \cos 2a[t] - \sin 2a[\log|\sin t|] + C_1$$
$$= (x+a)\cos 2a - \sin 2a\log|\sin(x+a)| + C_1$$
$$= x\cos 2a - \sin 2a\log|\sin(x+a)| + C,$$

where, $C = a\cos 2a + C_1$

127. Find $\int \frac{(x^2+1)e^x}{(x+1)^2}\,dx$.

Sol : **Delhi 2014**

We have
$$I = \int e^x\frac{(x^2+1)}{(x+1)^2}\,dx$$
$$= \int e^x\frac{(x^2+1+2x-2x)}{(x+1)^2}\,dx$$
$$= \int e^x\left(\frac{(x+1)^2-2x}{(x+1)^2}\right)dx$$
$$= \int e^x\left(1 - \frac{2x}{(x+1)^2}\right)dx$$
$$= \int e^x\,dx - 2\int e^x\cdot\frac{x}{(x+1)^2}\,dx$$
$$= e^x - 2\int e^x\left(\frac{x+1-1}{(x+1)^2}\right)dx$$
$$= e^x - 2\int e^x\left(\frac{1}{(x+1)} + \frac{(-1)}{(x+1)^2}\right)dx$$

Here $f(x) = \frac{1}{x+1}$ and $f'(x) = \frac{(-1)}{(x+1)^2}$

The above integrand is of the form $e^x[f(x) + f'(x)]$.

Using fact we $\int e^x[f(x) + f'(x)]\,dx = e^x f(x) + C$ have
$$I = e^x - 2e^x\frac{1}{(x+1)} + C$$
$$= e^x\left(\frac{x+1-2}{x+1}\right) + C$$
$$= e^x\left(\frac{x-1}{x+1}\right) + C$$

128. Find $\int \frac{x^3}{x^4+3x^2+2}\,dx$.

Sol : **Comp 2014**

We have
$$I = \int \frac{x^3}{x^4+3x^2+2}\,dx$$

Substituting $x^2 = t \Rightarrow 2x = \frac{dt}{dx} \Rightarrow x\,dx = \frac{dt}{2}$ we have
$$I = \frac{1}{2}\int \frac{t}{t^2+3t+2}\,dt$$

Substituting $t = x^3 \Rightarrow dt = 3x^2\,dx$ we have

$$I = \int \frac{dt}{3t(t+1)}$$
$$= \frac{1}{3}\int\left[\frac{1}{t} - \frac{1}{t+1}\right]dt$$
$$= \frac{1}{3}[\log|\,t\,| - \log|\,t+1\,|] + C$$
$$= \frac{1}{3}\log\left|\frac{t}{t+1}\right| + C$$
$$= \frac{1}{3}\log\left|\frac{x^3}{x^3+1}\right| + C \qquad t = x^5$$

134. Evaluate $\int \frac{dx}{x(x^3+8)}$.
Sol : **OD 2013**

We have $$I = \int \frac{dx}{x(x^3+8)}$$
$$= \int \frac{x^2}{x^3(x^3+8)}dx$$

Substituting $t = x^3 \Rightarrow dt = 3x^2\,dx$ we have

$$I = \int \frac{dt}{3t(t+8)}$$
$$= \frac{1}{3}\int \frac{1}{8}\left[\frac{1}{t} - \frac{1}{t+8}\right]dt$$
$$= \frac{1}{24}\int\left[\frac{1}{t} - \frac{1}{t+8}\right]dt$$
$$= \frac{1}{24}[\log|\,t\,| - \log|\,t+8\,|] + C$$
$$= \frac{1}{24}\log\left|\frac{t}{t+8}\right| + C$$
$$= \frac{1}{24}\log\left|\frac{x^3}{x^3+1}\right| + C \qquad t = x^5$$

135. Evaluate $\int \sin x \cdot \sin 2x \cdot \sin 3x\,dx$.
Sol : **Delhi 2012, Comp 2009**

$$I = \int \sin x \sin 2x \sin 3x\,dx$$
$$= \frac{1}{2}\int \sin x(2\sin 2x \sin 3x)\,dx$$
$$= \frac{1}{2}\int \sin x[\cos(2x-3x) - \cos(2x+3x)]\,dx$$
$$= \frac{1}{2}\int \sin x[\cos(-x) - \cos 5x]\,dx$$
$$= \frac{1}{2}\int \sin x(\cos x - \cos 5x)\,dx \qquad \cos(-x) = \cos x$$
$$= \frac{1}{2}\int \sin x\cos x\,dx - \frac{1}{2}\int \sin x \cos 5x\,dx$$
$$= \frac{1}{4}\int 2\sin x\cos x\,dx - \frac{1}{4}\int(2\sin x\cos 5x)\,dx$$
$$= \frac{1}{2}\int \sin 2x - \frac{1}{4}\int\{\sin(x+5x) + \sin(x-5x)\}\,dx$$
$$= \frac{1}{4}\int \sin 2x\,dx - \frac{1}{4}\int[\sin 6x + \sin(-4x)]\,dx$$
$$= \frac{1}{4}\int \sin 2x\,dx - \frac{1}{4}\int(\sin 6x - \sin 4x)\,dx$$
$$= \frac{-1}{4}\cdot\frac{\cos 2x}{2} - \frac{1}{4}\left[\frac{-\cos 6x}{6} + \frac{\cos 4x}{4}\right] + C$$
$$= \frac{-\cos 2x}{8} + \frac{\cos 6x}{24} - \frac{\cos 4x}{16} + C$$

136. Evaluate $\int \frac{\sqrt{1-\sin x}}{1+\cos x}e^{\frac{-x}{2}}\,dx$.
Sol : **Comp 2013**

We have $$I \int \frac{\sqrt{1-\sin x}}{1+\cos x}e^{\frac{-x}{2}}\,dx$$

Substituting $\frac{-x}{2} = t \Rightarrow dx = -2\,dt$ we have

$$I = \int \frac{\sqrt{1-\sin(-2t)}}{1+\cos(-2t)}\cdot e^t(-2\,dt)$$

Using the fact $\sin(-\theta) = -\sin\theta$ and $\cos(-\theta) = \cos\theta$ we have

$$I = -2\int e^t\frac{\sqrt{1+\sin 2t}}{1+\cos 2t}\,dt$$
$$= -2\int e^t\left(\frac{\sqrt{(\cos t + \sin t)^2}}{2\cos^2 t}\right)dt$$
$$= -\int e^t\left(\frac{\cos t + \sin t}{2\cos^2 t}\right)dt$$
$$= -\int e^t(\sec t + \tan t\sec t)\,dt$$

Now, let $f(t) = \sec t$, then $f'(t) = \sec t\tan t dt$.
Thus, the given integrand is of the form

$$I = \int e^t[f(t) + f'(t)]\,dt.$$

Using $\int e^t[f(t) + f'(t)]\,dt = e^t f(t) + C$ we have

$$I = -e^t\sec t + C$$
$$= -e^{-x/2}\sec\frac{x}{2} + C$$

137. Evaluate $\int\left(\frac{1+\sin x}{1+\cos x}\right)e^x\,dx$.
Sol : **Comp 2012**

We have $$I = \int\left(\frac{1+\sin x}{1+\cos x}\right)e^x\,dx$$

Using the fact $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$ and $1+\cos x = 2\cos^2\frac{x}{2}$ we have

$$I = \int\frac{1+2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\cdot e^x\,dx$$
$$= \int\left(\frac{1}{2}\sec^2\frac{x}{2} + \tan\frac{x}{2}\right)e^x\,dx$$
$$= \int e^x\left(\tan\frac{x}{2} + \frac{1}{2}\sec^2\frac{x}{2}\right)dx$$

Here, $f(x) = \tan\frac{x}{2}$ and $f'(x) = \frac{1}{2}\sec^2\frac{x}{2}$

Using the fact $\int e^x[f(x) + f'(x)]\,dx = e^x f(x) + C$ we have,

$$I = e^x\tan\frac{x}{2} + C$$

Sol : **OD 2014**

We have $$I = \int_2^4 \frac{x}{x^2+1}dx$$

Substituting $x^2+1 = t \Rightarrow 2x\,dx = dt \Rightarrow x\,dx = \frac{dt}{2}$

When $x = 2$, then $t = 2^2+1 = 5$

When $x = 4$, then $t = 4^2+1 = 17$.

Thus
$$\begin{aligned} I &= \int_5^{17}\frac{1}{t}\frac{dt}{2} \\ &= \frac{1}{2}\int_5^{17}\frac{1}{t}dt \\ &= \frac{1}{2}[\log|t|]_5^{17} \\ &= \frac{1}{2}[\log 17 - \log 5] \\ &= \frac{1}{2}\log\left(\frac{17}{5}\right) \qquad \log m - \log n = \log\frac{m}{n} \end{aligned}$$

143. Evaluate $\int_0^3 \frac{dx}{9+x^2}$.

Sol : **Delhi 2014**

We have
$$\begin{aligned} I &= \int_0^3 \frac{dx}{9+x^2} \\ &= \int_0^3 \frac{dx}{x^2+(3)^2} \\ &= \left[\frac{1}{3}\tan^{-1}\frac{x}{3}\right]_0^3 \\ &= \frac{1}{3}\left[\tan^{-1}\left(\frac{3}{3}\right) - \tan^{-1}(0)\right] \\ &= \frac{1}{3}[\tan^{-1}(1) - 0] \\ &= \frac{1}{3}\left(\frac{\pi}{4}\right) = \frac{\pi}{12} \end{aligned}$$

144. Evaluate $\int_0^1 \frac{\tan^{-1}x}{1+x^2}dx$.

Sol : **Comp 2014, OD 2009**

We have $$I = \int_0^1 \frac{\tan^{-1}x}{1+x^2}dx$$

Substituting $\tan^{-1}x = t \Rightarrow \frac{1}{1+x^2}dx = dt$

When $x = 0$, then $t = 0$

When $x = 1$, then $t = \pi/4$.

Thus
$$\begin{aligned} I &= \int_0^{\pi/4} t\,dt \\ &= \left[\frac{t^2}{2}\right]_0^{\pi/4} \\ &= \frac{1}{2}\left[\left(\frac{\pi}{4}\right)^2 - (0)^2\right] = \frac{\pi^2}{32} \end{aligned}$$

145. Evaluate $\int_{-\pi/4}^{\pi/4}\sin^3 x\,dx$.

Sol : **Comp 2010**

Use, the property $\int_{-a}^{a} f(x)\,dx = 0$, if $f(x)$ is an odd function.

We have $$I = \int_{-\pi/4}^{\pi/4}\sin^3 x\,dx$$

Consider, $$f(x) = \sin^3 x.$$

Then,
$$\begin{aligned} f(-x) &= \sin^3(-x) \\ &= (-\sin x)^3 \\ &= -\sin^3 x \\ &= -f(x) \end{aligned}$$

Thus $f(x)$ is an odd function.

Thus, the given integrand is an odd function.

$$I = 0$$

146. Write the value of the following integral $\int_{-\pi/2}^{\pi/2}\sin^5 x\,dx$.

Sol : **OD 2010**

Use, the property $\int_{-a}^{a} f(x)\,dx = 0$, if $f(x)$ is an odd function.

We have $$I = \int_{-\pi/2}^{\pi/2}\sin^5 x\,dx$$

Consider, $$f(x) = \sin^5 x.$$

Then,
$$\begin{aligned} f(-x) &= \sin^5(-x) \\ &= (-\sin x)^5 \\ &= -\sin^5 x = -f(x) \end{aligned}$$

Thus $f(x)$ is an odd function.

Thus, the given integrand is an odd function.

$$I = 0$$

147. Evaluate $\int_{-\pi}^{\pi}(1-x^2)\sin x \cdot \cos^2 x\,dx$.

Sol : **Delhi 2019**

We have $$I = \int_{-\pi}^{\pi}(1-x^2)\sin x\cos^2 x\,dx$$

Here
$$\begin{aligned} f(x) &= (1-x^2)\sin x\cos^2 x \\ f(-x) &= [1-(-x)^2]\sin(-x)\cos^2(-x) \\ &= (1-x^2)(-\sin x)\cos^2 x \\ &= -(1-x^2)\sin x\cos^2 x \\ &= -f(x) \end{aligned}$$

Thus $f(x)$ is odd function and for odd function

$$\int_{-a}^{a} f(x)\,dx = 0$$

Thus $$I = 0$$

148. Evaluate $\int_{-1}^{2}\frac{|x|}{x}dx$.

Sol : **Delhi 2019**

We have
$$\begin{aligned} I &= \int_{-1}^{2}\frac{|x|}{x}dx \\ &= \int_{-1}^{0}\frac{|x|}{x}dx + \int_0^2\frac{|x|}{x}dx \end{aligned}$$

Sol : **Delhi 2013**

For, $2 \le x < 5$, $|x-2| = (x-2)$

For, $2 \le x < 3$, $|x-3| = -(x-3)$

For, $3 \le x < 5$, $|x-3| = (x-3)$

and $2 \le x < 5$, $|x-5| = -(x-5)$

Thus $I = \int_2^5 [|x-2|+|x-3|+|x-5|]\, dx$

$$= \int_2^5 (x-2)\, dx + \int_2^3 (3-x)\, dx + \int_3^5 (x-3)\, dx - \int_2^5 (x-5)\, dx$$

$$= \left[\frac{(x-2)^2}{2}\right]_2^5 - \left[\frac{(x-3)^2}{2}\right]_2^3 + \left[\frac{(x-3)^2}{2}\right]_3^5 - \left[\frac{(x-5)^2}{2}\right]_2^5$$

$$= \left[\frac{3^2}{2}\right] + \left[\frac{1}{2}\right] + \left[\frac{2^2}{2}\right] + \left[\frac{3^2}{2}\right]$$

$$= \frac{9}{2} + \frac{1}{2} + 2 + \frac{9}{2} = \frac{23}{2}$$

154. Evaluate $\int_0^{2\pi} \frac{1}{1+e^{\sin x}}\, dx$.

Sol : **OD 2013**

We have $I = \int_0^{2\pi} \frac{dx}{1+e^{\sin x}}$ ...(i)

Using the fact $\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$ we have

$$I = \int_0^{2\pi} \frac{dx}{1+e^{\sin x(2\pi - x)}}$$

$$= \int_0^{2\pi} \frac{dx}{1+e^{-\sin x}}$$

$$I = \int_0^{2\pi} \frac{e^{\sin x}}{e^{\sin x}+1}\, dx \quad \text{...(ii)}$$

Adding Eqs. (i) and (ii), we get

$$I + I = \int_0^{2\pi} \frac{dx}{1+e^{\sin x}} + \int_0^{2\pi} \frac{e^{\sin x}}{1+e^{\sin x}}\, dx$$

$$2I = \int_0^{2\pi} \frac{(1+e^{\sin x})}{(1+e^{\sin x})}\, dx$$

$$= \int_0^{2\pi} 1\, dx$$

$$= [x]_0^{2\pi}$$

$$2I = 2\pi - 0 = 2\pi$$

$$I = \pi$$

155. Evaluate $\int_2^5 [|x|+|x-2|+|x-4|]\, dx$.

Sol : **Delhi 2013**

We have $I = \int_0^4 [|x|+|x-2|+|x-4|]\, dx$

For, $0 < x < 4$, $|x| = x$

For $0 < x \le 2$, $|x-2| = -(x-2)$

For $2 \le x < 4$, $|x-2| = (x-2)$

For $0 < x < 4$, $|x-4| = -(x-4)$

Now $I = \int_0^4 x\, dx + \int_0^2 (2-x)\, dx + \int_2^4 (x-2)\, dx + \int_0^4 (4-x)\, dx$

$$= \left[\frac{x^2}{2}\right]_0^4 + \left[2x - \frac{x^2}{2}\right]_0^2 + \left[\frac{x^2}{2} - 2x\right]_2^4 + \left[4x - \frac{x^2}{2}\right]_0^4$$

$$= (8) + [(4-2)-0] + [(8-8)-(2-4)] + \left[16 - \frac{16}{2}\right]$$

$$= 8 + 2 + 2 + (16-8) = 20$$

156. Evaluate $\int_1^3 [|x-1|+|x-2|+|x-3|]\, dx$.

Sol : **Delhi 2013**

For, $1 \le x < 3$, $|x-1| = (x-1)$

For, $1 \le x < 2$, $|x-2| = -(x-2)$

For, $2 \le x < 3$, $|x-2| = (x-2)$

and $1 \le x < 3$, $|x-3| = -(x-3)$

We have $I = \int_1^4 (|x-1|+|x-2|+|x-3|)\, dx$

$$= \int_1^3 (x-1)\, dx - \int_1^2 (x-2)\, dx + \int_2^3 (x-2)\, dx - \int_1^3 (x-3)\, dx$$

$$= \left[\frac{(x-1)^2}{2}\right]_1^3 - \left[\frac{(x-2)^2}{2}\right]_1^2 + \left[\frac{(x-2)^2}{2}\right]_2^3 - \left[\frac{(x-3)^2}{2}\right]_1^3$$

$$= \left[\frac{2^2}{2}\right] + \left[\frac{1}{2}\right] + \left[\frac{1}{2}\right] + \left[\frac{(-2)^2}{2}\right]$$

$$= 2 + \frac{1}{2} + \frac{1}{2} + 2 = 5$$

157. Evaluate $\int_0^{\pi} \frac{x \tan x}{\sec x \cdot \operatorname{cosec} x}\, dx$.

Sol : **Delhi 2011C, 2008, OD 2009**

We have $I = \int_0^{\pi} \frac{x \tan x}{\sec x \cdot \operatorname{cosec} x}\, dx$ ...(i)

Using the fact $\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$, we get

$$I = \int_0^{\pi} \frac{(\pi - x)\tan(\pi - x)}{\sec(\pi - x)\operatorname{cosec}(\pi - x)}\, dx$$

$$= \int_0^{\pi} \frac{(\pi - x)(-\tan x)}{-\sec x \operatorname{cosec} x}\, dx$$

$$= \int_0^{\pi} \frac{(\pi - x)\tan x}{\sec x \operatorname{cosec} x}\, dx \quad \text{...(ii)}$$

Adding Eqs. (i) and (ii), we get

$$2I = \int_0^{\pi} \frac{\pi \tan x}{\sec x \operatorname{cosec} x}\, dx$$

$$I = \frac{\pi}{2} \int_0^{\pi} \frac{\sin^2 x(\cos x)}{(\cos x)}\, dx$$

$$= \frac{\pi}{2} \int_0^{\pi} \sin^2 x\, dx$$

$$= \frac{\pi}{2} \int_0^{\pi} \left(\frac{1-\cos 2x}{2}\right) dx \quad \cos 2\theta = 1 - 2\sin^2\theta$$

$$= \frac{\pi}{4}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi}$$

Using the fact $\sin 2x = 2\sin x\cos x$ and $\cos 2x = 1-2\sin^2 x$ we have

$$I = \int e^x\left(\frac{2\sin 2x\cos 2x - 4}{2\sin^2 2x}\right)dx$$

$$= \int e^x\left(\frac{2\sin 2x\cos 2x}{2\sin^2 2x} - \frac{4}{2\sin^2 2x}\right)dx$$

$$= \int e^x(\cot 2x - 2\,\mathrm{cosec}^2 2x)\,dx$$

Here, $f(x) = \cot 2x$

$f'(x) = -2\,\mathrm{cosec}^2 2x$

Using the fact $\int e^x[f(x)+f'(x)]\,dx = e^x f(x) + C$ we have,

$$I = e^x\cot 2x + C$$

163. Evaluate $\int_1^4 (|x-1|+|x-2|+|x-4|)\,dx$.

Sol : **OD 2017, Comp 2011**

For, $1 \le x < 4$, $|x-1| = (x-1)$

For, $1 \le x < 2$, $|x-2| = -(x-2)$

For, $2 \le x < 4$, $|x-2| = (x-2)$

and $1 \le x < 4$, $|x-4| = -(x-4)$

We have $I = \int_1^4 (|x-1|+|x-2|+|x-4|)\,dx$

$$= \int_1^4 (x-1)\,dx - \int_1^2 (x-2)\,dx + \int_2^4 (x-2)\,dx - \int_1^4 (x-4)\,dx$$

$$= \left[\frac{(x-1)^2}{2}\right]_1^4 - \left[\frac{(x-2)^2}{2}\right]_1^2 + \left[\frac{(x-2)^2}{2}\right]_2^4 - \left[\frac{(x-4)^2}{2}\right]_1^4$$

$$= \left[\frac{3^2}{2}\right] + \left[\frac{1}{2}\right] + \left[\frac{2^2}{2}\right] + \left[\frac{(-3)^2}{2}\right]$$

$$= \frac{9}{2} + \frac{1}{2} + 2 + \frac{9}{2} = \frac{23}{2}$$

LONG ANSWER QUESTIONS

164. Find $\int \frac{x}{x^2+3x+2}\,dx$

Sol : **OD 2020**

We have $I = \int \frac{x}{x^2+3x+2}\,dx$

Let $x = A\frac{d}{dx}(x^2+3x+2) + B$

$x = (2x+3)A + B$

$x = 2Ax + (3A+B)$

Comparing both side we have

$2A = 1$ and $3A + B = 0$

$A = \frac{1}{2},\ B = -\frac{3}{2}$

Now $I = \frac{1}{2}\int\frac{(2x+3)\,dx}{x^2+3x+2} - \frac{3}{2}\int\frac{dx}{x^2+3x+2}$

$= \frac{1}{2}I_1 - \frac{3}{2}I_2$

where $I_1 = \int\frac{2x+3}{x^2+3x+2}\,dx$

and $I_2 = \int\frac{dx}{x^2+3x+2}$

Now, $I_1 = \int\frac{2x+3}{x^2+3x+2}\,dx$

Substituting $x^2+3x+2 = t \Rightarrow (2x+3)\,dx = dt$ we have

$$I_1 = \int\frac{dt}{t} = \log|t| + C_1$$

$$= \log|x^2+3x+2| + C_1$$

and $I_2 = \int\frac{dx}{x^2+3x+2} = \int\frac{dx}{(x+\frac{3}{2})^2 + 2 - \frac{9}{4}}$

$$= \int\frac{dx}{(x+\frac{3}{2})^2 - (\frac{1}{2})^2}$$

$$= \frac{1}{2\times\frac{1}{2}}\log\left|\frac{x+\frac{3}{2}-\frac{1}{2}}{x+\frac{3}{2}+\frac{1}{2}}\right| + C$$

$$= \log\left|\frac{x+1}{x+2}\right| + C_2$$

$$I = \frac{1}{2}\log|x^2+3x+2| + \frac{1}{2}C_1 - \frac{3}{2}\log\left|\frac{x+1}{x+2}\right| - \frac{3}{2}C_2$$

$$= \frac{1}{2}\log|x^2+3x+2| - \frac{3}{2}\log\left|\frac{x+1}{x+2}\right| + C$$

where $C = \frac{1}{2}C_1 - \frac{3}{2}C_2$

165. Find : $\int\frac{3x+5}{x^2+3x-18}\,dx$.

Sol : **Delhi 2019**

We have $I = \int\frac{3x+5}{x^2+3x-18}\,dx$ ...(1)

Now $3x+5 = A\frac{d}{dx}(x^2+3x-18) + B$

or $3x+5 = A(2x+3) + B$ ...(2)

Comparing the coefficient of x on both side, we have

$$2A = 3 \Rightarrow A = \frac{3}{2}$$

and comparing the constant on both side, we get

$$5 = 3A + B$$

$$B = 5 - 3A \Rightarrow B = 5 - 3\left(\frac{3}{2}\right) = \frac{1}{2}$$

Now $3x+5 = \frac{3}{2}(2x+3) + \frac{1}{2}$ ...(3

Now $I = \int\frac{\frac{3}{2}(2x+3)+\frac{1}{2}}{x^2+3x-18}\,dx$

$$2 = A(1+t^2) + (Bt+C)(1-t) \quad ...(1)$$

Substituting $t = 1$ in eq. (1), we get

$$2 = A(1+1) + 0 \Rightarrow A = 1 \quad ...(2)$$

Substituting $t = 0$ in eq. (2), we get

$$2 = A + C \Rightarrow 2 = 1 + C \Rightarrow C = 1$$

Substituting $t = -1$ in eq. (1), we get

$$2 = 2A + (-B+C)(2)$$

$$2 = 2 - 2B + 2$$

$$2B = 2 \Rightarrow B = 1$$

Thus $$\frac{2}{(1-t)(1+t^2)} = \frac{1}{1-t} + \frac{t+1}{1+t^2}$$

Now $$I = 1\int \frac{1}{1-t}dt + \int \frac{t+1}{1+t^2}dt$$

$$= \int \frac{1}{1-t}dt + \frac{1}{2}\int \frac{2t}{1+t^2}dt + \int \frac{1}{1+t^2}dt$$

$$= -\log|1-t| + \frac{1}{2}\log|1+t^2| + \tan^{-1}t + C$$

$$= -\log|1-\sin x| + \frac{1}{2}\log|1+\sin^2 x| + \tan^{-1}(\sin x) + C$$

$$= \tan^{-1}(\sin x) + \log\left|\frac{\sqrt{1+\sin^2 x}}{1-\sin x}\right| + C$$

Here we have used the fact $\log m - \log n = \log(\frac{m}{n})$ and $n\log m = \log m^n$

169. Find $\int \frac{2\cos x}{(1-\sin x)(1+\sin^2 x)}dx$.

Sol : **OD 2018, Foreign 2007**

We have $$I = \int \frac{2\cos x}{(1-\sin x)(1+\sin^2 x)}dx$$

Substituting $\sin x = t \Rightarrow \cos x\,dx = dt$ we have

$$I = \int \frac{2}{(1-t)(1+t^2)}dt$$

Now, let $$\frac{2}{(1-t)(1+t^2)} = \frac{A}{1-t} + \frac{Bt+C}{1+t^2}$$

$$2 = A(1+t^2) + (Bt+C)(1-t) \quad ...(1)$$

Substituting $t = 1$ in eq. (1), we get

$$2 = A(1+1) + 0 \Rightarrow A = 1 \quad ...(2)$$

Substituting $t = 0$ in eq. (2), we get

$$2 = A + C \Rightarrow 2 = 1 + C \Rightarrow C = 1$$

Substituting $t = -1$ in eq. (1), we get

$$2 = 2A + (-B+C)(2)$$

$$2 = 2 - 2B + 2$$

$$2B = 2 \Rightarrow B = 1$$

Thus $$\frac{2}{(1-t)(1+t^2)} = \frac{1}{1-t} + \frac{t+1}{1+t^2}$$

Now $$I = 1\int \frac{1}{1-t}dt + \int \frac{t+1}{1+t^2}dt$$

$$= \int \frac{1}{1-t}dt + \frac{1}{2}\int \frac{2t}{1+t^2}dt + \int \frac{1}{1+t^2}dt$$

$$= -\log|1-t| + \frac{1}{2}\log|1+t^2| + \tan^{-1}t + C$$

$$= -\log|1-\sin x| + \frac{1}{2}\log|1+\sin^2 x| + \tan^{-1}(\sin x) + C$$

$$= \tan^{-1}(\sin x) + \log\left|\frac{\sqrt{1+\sin^2 x}}{1-\sin x}\right| + C$$

Here we have used the fact $\log m - \log n = \log(\frac{m}{n})$ and $n\log m = \log m^n$

170. Find $\int \frac{4}{(x-2)(x^2+4)}dx$.

Sol : **Comp 2018**

We have $$I = \int \frac{4}{(x-2)(x^2+4)}dx$$

Now $$\frac{4}{(x-2)(x^2+4)} = \frac{A}{(x-2)} + \frac{Bx+C}{x^2+4}$$

$$4 = A(x^2+4) + (Bx+C)(x-2)$$

$$4 = x^2(A+B) + x(-2B+C) + 4A - 2C$$

Equating the coefficients of x^2, x and constant in both sides, we get

$$A + B = 0 \quad ...(1)$$

$$-2B + C = 0 \quad ...(2)$$

and $$4A - 2C = 4 \quad ...(3)$$

Solving eqs. (1), (2) and (3), we get

$$A = \frac{1}{2},\ B = -\frac{1}{2} \text{ and } C = -1$$

Thus $$\frac{4}{(x-2)(x^2+4)} = \frac{\frac{1}{2}}{(x-2)} + \frac{-\frac{1}{2}x-1}{x^2+4}$$

Now $$I = \int \frac{4}{(x-2)(x^2+4)}dx$$

$$= \int \frac{\frac{1}{2}}{(x-2)}dx + \int \frac{-\frac{1}{2}x-1}{x^2+4}dx$$

$$= \frac{1}{2}\int \frac{dx}{(x-2)} - \int \frac{x+2}{2(x^2+4)}dx$$

$$= \frac{1}{2}\log|x-2| - \frac{1}{4}\log|x^2+4| - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$$

171. Find $\int \frac{2x}{(x^2+1)(x^2+2)^2}dx$.

Sol : **Delhi 2017**

We have $$I = \int \frac{2x}{(x^2+1)(x^2+2)^2}dx$$

Substituting $x^2 = t \Rightarrow 2x\,dx = dt$ we have

$$I = \int \frac{dt}{(t+1)(t+2)^2}$$

$$= [x\sin x - \int \sin x\,dx]_{-2}^{2} + \int_{-2}^{2}\sin x\,dx + [x]_{-2}^{2}$$
$$= 2\sin 2 - 2\sin 2 - \int_{-2}^{2}\sin x\,dx + \int_{-2}^{2}\sin x\,dx + (2+2)$$
$$= 4$$

Thus (d) is correct option.

17. $\int_0^{\pi/2}\frac{\sin x - \cos x}{1+\sin x\cos x}dx$ is equal to

(a) 0 (b) $\frac{\pi}{4}$

(c) $\frac{\pi}{2}$ (d) π

Sol : **OD 2008**

We have
$$I = \int_0^{\pi/2}\frac{\sin x - \cos x}{1+\sin x\cos x}dx \quad ...(1)$$
$$= \int_0^{\pi/2}\frac{\sin\left(\frac{\pi}{2}-x\right)-\cos\left(\frac{\pi}{2}-x\right)}{1+\sin\left(\frac{\pi}{2}-x\right)\cos\left(\frac{\pi}{2}-x\right)}dx$$
$$= \int_0^{\pi/2}\frac{\cos x - \sin x}{1+\cos x\sin x}dx \quad ...(2)$$

Adding Eqs. (1) and (2), we get
$$2I = \int_0^{\pi/2}\frac{0}{1+\sin x\cos x}dx$$
$$= 0$$
$$I = 0$$

Thus (a) is correct option.

18. $\int_0^{\pi}\sqrt{\frac{1+\cos 2x}{2}}\,dx$ is equal to

(a) 0 (b) 2

(c) 4 (d) -2

Sol : **Foreign 2010, OD 2009**

$$\int_0^{\pi}\sqrt{\frac{1+\cos 2x}{2}}\,dx = \int_0^{\pi}\sqrt{\frac{2\cos^2 x}{2}}\,dx$$
$$= \int_0^{\pi}\sqrt{\cos^2 x}\,dx$$
$$= \int_0^{\pi}|\cos x|dx$$
$$= \int_0^{\pi/2}\cos x\,dx - \int_{\pi/2}^{\pi}\cos x\,dx$$
$$= [\sin x]_0^{\pi/2} - [\sin x]_{\pi/2}^{\pi}$$
$$= [1-0]-[0-1]$$
$$= 1+1$$
$$= 2$$

Thus (b) is correct option.

19. If $\int_0^a f(2a-x)\,dx = m$ and $\int_0^a f(x)\,dx = n$, then $\int_0^{2a} f(x)\,dx$ is equal to

(a) $2m+n$ (b) $m+2n$

(c) $m-n$ (d) $m+n$

Sol : **Comp 2018**

$$\int_0^{2a} f(x)\,dx = \int_0^a \{f(2a-x)+f(x)\}\,dx$$
$$= \int_0^a f(2a-x)\,dx + \int_0^a f(x)\,dx$$
$$= m+n$$

Thus (d) is correct option.

20. $\int\sqrt{1+\cos 2x}\,dx =$

(a) $\sqrt{2}\cos x + c$ (b) $\sqrt{2}\sin x + c$

(c) $-\cos x - \sin x + c$ (d) $\sqrt{2}\sin\frac{x}{2} + c$

Sol : **OD 2014, Comp 2012**

We have
$$I = \int\sqrt{1+\cos 2x}\,dx$$
$$= \int\sqrt{2\cos^2 x}\,dx$$
$$= \int\sqrt{2}\cos x dx$$

Thus,
$$I = \sqrt{2}\sin x + c$$

Thus (b) is correct option.

21. $\int\frac{xe^x}{(x+1)^2}dx =$

(a) $\frac{e^x}{(x+1)^2}+c$ (b) $\frac{-e^x}{x+1}+c$

(c) $\frac{e^x}{x+1}+c$ (d) $\frac{-e^x}{(x+1)^2}+c$

Sol : **Foreign 2007**

We have
$$I = \int\frac{xe^x}{(1+x)^2}dx$$

Adding and subtracting e^x in numerator
$$I = \int\frac{xe^x+e^x-e^x}{(1+x)^2}$$
$$= \int e^x\left[\frac{1}{1+x}-\frac{1}{(1+x)^2}\right]$$

Let,
$$f(x) = \frac{1}{1+x}$$
$$f'(x) = \frac{-1}{(1+x)^2}$$

Hence,
$$I = \frac{e^x}{1+x}+c$$

Thus (c) is correct option.

22. If $f(-x) = -f(x)$ then $\int_{-a}^{a} f(x)\,dx =$

40. $\int_{-\pi}^{\pi} \sin^3 x \cos^4 x dx$ is equal to

(a) 0 (b) $\frac{1}{2}$

(c) 1 (d) None of these

Sol : **Delhi 2018**

We have $f(x) = \sin^3 x \cos^4 x$

Hence, $f(-x) = \sin^3(-x) . \cos^4(-x)$

$= -\sin^3 x \cos^4 x$

$= -f(x)$

Since $f(x)$ is an odd function

Hence, $\int_{-\pi}^{\pi} \sin^3 x \cos^4 x dx = 0$

Thus (a) is correct option.

41. $\int_0^1 \frac{dx}{e^x + e^{-x}}$ is equal to

(a) $\frac{\pi}{4}$ (b) $\tan^{-1} e - \frac{\pi}{4}$

(c) $\cot^{-1} e - \frac{\pi}{4}$ (d) $\frac{\pi}{2}$

Sol : **OD 2014, Foreign 2008**

We have $\int_0^1 \frac{dx}{e^x + e^{-x}} = \int_0^1 \frac{e^x dx}{e^{2x} + 1}$

Substituting $e^x = t \Rightarrow e^x dx = dt$ we have

$$\int_0^1 \frac{dx}{e^x + e^{-x}} = \int_1^e \frac{dt}{t^2 + 1}$$

$$= (\tan^{-1} t)_1^e$$

$$= \tan^{-1} e - \frac{\pi}{4}$$

Thus (b) is correct option.

42. **Assertion (A) :** $\int \frac{dx}{\sqrt{a^2 - x^2}} = \tan^{-1}\frac{x}{a} + c$

Reason (R) : If we let $x = a\sin\theta$ then given function becomes constant.

(a) Both (A) and (R) are true and (R) is the correct explanation of (A).

(b) Both (A) and (R) are true but (R) is not the correct explanation of (A).

(c) (A) is true but (R) is false.

(d) (A) is false but (R) is true

Sol : **Comp 2017**

We have $I = \int \frac{dx}{\sqrt{a^2 - x^2}}$

Let $x = a\sin\theta$

Differentiating both sides with respect to x, we get

$$dx = a\cos\theta \cdot d\theta$$

Substituting above in given integral we have

$$I = \int \frac{a\cos\theta \cdot d\theta}{\sqrt{a^2 - a^2\sin^2\theta}}$$

$$= \int \frac{a\cos\theta \cdot d\theta}{a\sqrt{1 - \sin^2\theta}}$$

$$= \int \frac{a\cos\theta \cdot d\theta}{a\cos\theta}$$

$$= \int d\theta$$

$$= \theta + c$$

$$= \sin^{-1}\left(\frac{x}{a}\right) + c$$

Therefore (A) is false but (R) is true

Thus (d) is correct option.

43. **Assertion (A) :** $\int_2^4 \frac{dx}{x} = \log 2$

Reason (R) : $\int_a^b f(x) dx = F(b) - F(a)$

(a) Both (A) and (R) are true and (R) is the correct explanation of (A).

(b) Both (A) and (R) are true but (R) is not the correct explanation of (A).

(c) (A) is true but (R) is false.

(d) (A) is false but (R) is true

Sol : **Delhi 2018**

Let, $I = \int_2^4 \frac{dx}{x}$

$= \log x \big|_2^4$

$= \log 4 - \log 2$

$= \log\frac{4}{2}$

$= \log 2$

Therefore both (A) and (R) are true and (R) is the correct explanation of (A).

Thus (a) is correct option.

44. **Assertion (A) :** $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x dx = \frac{\pi}{2}$

Reason (R) : Here $f(x)$ is even function if $f(-x) = f(x)$: $\int_{-a}^{a} f(x) dx = 2\int_0^a f(x) dx$.

(a) Both (A) and (R) are true and (R) is the correct explanation of (A).

(b) Both (A) and (R) are true but (R) is not the correct explanation of (A).

$$= -\int (\log t) \cdot 1\, dt$$
$$= -\left[\log t \int 1\, dt - \int \left[\frac{d}{dt}(\log t)\int 1\, dt\right] dt\right]$$
$$= -\left[(\log t) \cdot t - \int \frac{1}{t} \cdot t dt\right]$$
$$= -t \cdot \log t + t + C$$

Now substituting the value of t, we get

$$I = -\cos x \log \cos x + \cos x + C$$

119. Find $\int \sqrt{3 - 2x - x^2}\, dx$.

Sol : **OD 2019**

We have
$$I = \int \sqrt{3 - 2x - x^2}\, dx$$
$$= \int \sqrt{-(x^2 + 2x - 3)}\, dx$$
$$= \int \sqrt{-(x^2 + 2x + 1 - 4)}\, dx$$
$$= \int \sqrt{-((x^2 + 1)^2 - 2^2)}\, dx$$
$$= \int \sqrt{2^2 - (x + 1)^2}\, dx$$

Substituting $x + 1 = t \Rightarrow dx = dt$ we have
$$I = \int \sqrt{2^2 - t^2}\, dt$$
$$= \frac{1}{2}\left[t\sqrt{2^2 - t^2} + 2^2 \sin^{-1}\left(\frac{t}{2}\right)\right] + C$$

Now substituting the original value of t, we get
$$I = \frac{1}{2}\left[(x + 1)\sqrt{3 - 2x - x^2} + 4\sin^{-1}\left(\frac{x + 1}{2}\right)\right] + C$$

120. Find $\int \frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x}\, dx$.

Sol : **OD 2019, Delhi 2017**

We have
$$I = \int \left(\frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x}\right) dx$$
$$= \int \left(\frac{\sin^3 x}{\sin^2 x \cos^2 x} + \frac{\cos^3 x}{\sin^2 x \cos^2 x}\right) dx$$
$$= \int \left(\frac{\sin x}{\cos^2 x} + \frac{\cos x}{\sin^2 x}\right) dx$$
$$= \int [(\tan x \cdot \sec x) + (\cot x \cdot \operatorname{cosec} x)]\, dx$$
$$= \int \sec x \cdot \tan x\, dx + \int \cot x \cdot \operatorname{cosec} x\, dx$$
$$= \sec x + (-\operatorname{cosec} x) + C$$
$$= \sec x - \operatorname{cosec} x + C$$

121. Find $\int \frac{x - 3}{(x - 1)^3} e^x\, dx$.

Sol : **OD 2019**

We have
$$I = \int \frac{(x - 3)}{(x - 1)^3} e^x\, dx$$
$$= \int \frac{e^x (x - 1 - 2)}{(x - 1)^3}\, dx$$
$$= \int e^x \left\{\frac{(x - 1)}{(x - 1)^3} - \frac{2}{(x - 1)^3}\right\} dx$$
$$= \int e^x \left\{\frac{1}{(x - 1)^2} - \frac{2}{(x - 1)^3}\right\} dx$$
$$= \int e^x \cdot \{f(x) + f'(x)\}\, dx,$$

where $f(x) = \frac{1}{(x - 1)^2}$ and $f'(x) = \frac{-2}{(x - 1)^3}$

Thus
$$I = e^x \cdot f(x) + C$$
$$= e^x \cdot \frac{1}{(x - 1)^2} + C$$
$$= \frac{e^x}{(x - 1)^2} + C$$

122. Find $\int \frac{x - 5}{(x - 3)^3} e^x\, dx$.

Sol : **OD 2019**

We have
$$I = \int \frac{(x - 5)}{(x - 3)^3} \cdot e^x\, dx$$
$$= \int \frac{(x - 3 - 2)}{(x - 3)^3} \cdot e^x\, dx$$
$$= \int e^x \left\{\frac{(x - 3)}{(x - 3)^2} - \frac{2}{(x - 3)^2}\right\} dx$$
$$= \int e^x \left\{\frac{1}{(x - 3)^2} - \frac{2}{(x - 3)^3}\right\} dx$$
$$= \int e^x \{f(x) + f'(x)\}\, dx,$$

where $f(x) = \frac{1}{(x - 3)^2}$ and $f'(x) = -\frac{2}{(x - 3)^{-3}}$

Thus using the fact $\int e^x [f(x) + f'(x)]\, dx = e^x f(x) + C$ we have
$$I = e^x f(x) + C$$
$$= \frac{e^x}{(x - 3)^2} + C$$

123. Find $\int \frac{dx}{5 - 8x - x^2}$.

Sol : **OD 2017**

We have
$$I = \int \frac{dx}{5 - 8x - x^2}$$
$$= \int \frac{dx}{5 - 2 \cdot 4 \cdot x - x^2 - (4)^2 + (4)^2}$$
$$= \int \frac{dx}{5 + 16 - [x^2 + (4)^2 + 2 \cdot 4 \cdot x]}$$
$$= \int \frac{dx}{21 - (x + 4)^2}$$
$$= \int \frac{dx}{(\sqrt{21})^2 - (x + 4)^2}$$

Using the fact $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\log\left|\frac{a + x}{a - x}\right| + C$ we have
$$I = \frac{1}{2\sqrt{21}}\log\left|\frac{\sqrt{21} + x + 4}{\sqrt{21} - x - 4}\right| + C$$

Now $$\frac{1}{(t+1)(t+2)^2} = \frac{A}{t+1} + \frac{B}{t+2} + \frac{C}{(t+2)^2}$$

$$\frac{1}{(t+1)(t+2)^2} = \frac{A(t+2)^2 + B(t+1)(t+2) + C(t+1)}{(t+1)(t+2)^2}$$

$$1 = A(t+2)^2 + B(t+1)(t+2) + C(t+1) \quad (1)$$

Substituting $t = -1$ in eq. (1), we get

$$1 = A(-1+2)^2 + 0 + 0 \Rightarrow 1 = A$$

Substituting $t = -2$ in eq. (i), we get

$$1 = 0 + 0 + C(-2+1) \Rightarrow 1 = -C \Rightarrow C = -1$$

Comparing the coefficient of x^2 of both side in eq (1) we have

$$0 = A + B$$

Thus $A = -B = -1$

Now $$\frac{1}{(t+1)(t+2)^2} = \frac{1}{t+1} - \frac{1}{t+2} - \frac{1}{(t+2)^2}$$

$$I = \int \frac{1}{t+1}dt - \int \frac{1}{t+2}dt - \int \frac{1}{(t+2)^2}dt$$

$$= \log|t+1| - \log|t+2| - \frac{(t+2)^{-1}}{-1} + C$$

$$= \log|t+1| - \log|t+2| + \frac{1}{(t+2)} + C$$

$$= \log|x^2+1| - \log|x^2+2| + \frac{1}{(x^2+2)} + C$$

172. Find $\int \frac{2x}{(x^2+1)(x^4+4)}dx$.

Sol : **Delhi 2017**

We have $$I = \int \frac{2x}{(x^2+1)(x^4+4)}dx$$

Substituting $x^2 = t \Rightarrow 2x\,dx = dt$

$$I = \int \frac{dt}{(t+1)(t^2+4)}$$

Now, $$\frac{1}{(t+1)(t^2+4)} = \frac{A}{t+1} + \frac{Bt+C}{t^2+4}$$

$$1 = A(t^2+4) + (Bt+C)(t+1) \quad (1)$$

Substituting $t = -1$ in eq. (1), we get

$$1 = A(1^2+4) + 0 \Rightarrow 1 = 5A \Rightarrow A = \tfrac{1}{5}$$

Comparing the coefficient of t^2, t and constant term both side in eq (1) we obtain

$$A + B = 0 \quad ...(2)$$

$$B + C = 0 \quad ...(3)$$

$$4A + C = 1 \quad ...(4)$$

Substituting $A = \frac{1}{5}$ in (2) we get

$$B = -A = -\tfrac{1}{5}$$

Substituting $B = -\frac{1}{5}$ in (3) we get

$$C = -B = \tfrac{1}{5}$$

Thus, $$\frac{1}{(t+1)(t^2+4)} = \frac{\frac{1}{5}}{t+1} + \frac{-\frac{1}{5}t + \frac{1}{5}}{t^2+4}$$

$$= \frac{\frac{1}{5}}{t+1} - \frac{\frac{1}{5}t}{t^2+4} + \frac{\frac{1}{5}}{t^2+4}$$

Now, $$I = \int \frac{dt}{(t+1)(t^2+4)}$$

$$= \int \frac{\frac{1}{5}}{t+1}dt - \frac{\frac{1}{5}t}{t^2+4}dt + \int \frac{\frac{1}{5}}{t^2+4}dt$$

$$= \frac{1}{5}\int \frac{1}{t+1}dt - \frac{1}{5}\int \frac{t}{t^2+4}dt - \frac{1}{5}\int \frac{1}{t^2+4}dt$$

$$= \frac{1}{5}\log|t+1| - \frac{1}{5}\left[\frac{1}{2}\log|t^2+4| - \frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right)\right] + C$$

$$= \frac{1}{5}\log|x^2+1| - \frac{1}{5}\left[\frac{1}{2}\log|x^4+4| - \frac{1}{2}\tan^{-1}\left(\frac{x^2}{2}\right)\right] + C$$

173. Find $\int \frac{\cos\theta}{(4+\sin^2\theta)(5-4\cos^2\theta)}d\theta$.

Sol : **OD 2017, Comp 2015**

We have $$I = \int \frac{\cos\theta}{(4+\sin^2\theta)(5-4\cos^2\theta)}d\theta$$

$$= \int \frac{\cos\theta}{(4+\sin^2\theta)[5-4(1-\sin^2\theta)]}d\theta$$

$$= \int \frac{\cos\theta}{(4+\sin^2\theta)(5-4+4\sin^2\theta)}d\theta$$

$$= \int \frac{\cos\theta}{(4+\sin^2\theta)(1+4\sin^2\theta)}d\theta$$

Substituting $\sin\theta = t \Rightarrow \cos\theta\, d\theta = dt$ we have

$$I = \int \frac{dt}{(4+t^2)(1+4t^2)} \quad ...(i)$$

Now $$\frac{1}{(4+t^2)(1+4t^2)} = \frac{A}{4+t^2} + \frac{B}{1+4t^2}$$

$$1 = A(1+4t^2) + B(4+t^2) \quad (1)$$

Substituting $t = 0$ in eq (1) we have

$$1 = A + 4B \quad (2)$$

Substituting $t = 1$ in eq (1) we have

$$1 = 5A + 5B \quad (3)$$

Solving eqs. (2) and (3), we get

$$A = -\frac{1}{15} \text{ and } B = \frac{4}{15}$$

Substituting $A = -\frac{1}{15}$ and $B = \frac{4}{15}$ in eq. (ii), we get

Thus $$\frac{1}{(4+t^2)(1+4t^2)} = \frac{-\frac{1}{15}}{4+t^2} + \frac{\frac{4}{15}}{1+4t^2}$$

$$= \frac{-1}{15(4+t^2)} + \frac{4}{15(1+4t^2)}$$

Now $$I = \int \frac{dt}{(4+t^2)(1+4t^2)}$$

$$= \int \frac{-1}{15(4+t^2)} + \frac{4}{15(1+4t^2)}$$

$$= \frac{-1}{15}\int \frac{1}{4+t^2}dt + \frac{4}{15}\int \frac{1}{1+4t^2}dt$$

$$= (-1)\int\left[\sqrt{1-x^2} + \int \frac{3x-2}{\sqrt{1-x^2}}\right]dx$$

$$= (-1)\left[\int \sqrt{1-x^2}\,dx + \int \frac{3x-2}{\sqrt{1-x^2}}\,dx\right]$$

Let $\quad I = (-1)[I_1 + I_2]$

or $\quad -I = I_1 + I_2 \quad \ldots(1)$

Now, $\quad I_1 = \int \sqrt{1-x^2}\,dx$

Using $\int \sqrt{a^2 - x^2}\,dx = \frac{1}{2}\left[x\sqrt{a^2-x^2} + a^2\sin^{-1}\left(\frac{x}{a}\right)\right] + C$

$$I_1 = \frac{1}{2}[x\sqrt{1-x^2} + \sin^{-1}x] + C_1 \quad \ldots(2)$$

Now, $\quad I_2 = \int \frac{3x-2}{\sqrt{1-x^2}}\,dx$

$$= \int \frac{3x}{\sqrt{1-x^2}}\,dx - 2\int \frac{dx}{\sqrt{1-x^2}}$$

$$= -\frac{3}{2}\int \frac{-2x}{\sqrt{1-x^2}}\,dx - 2\int \frac{dx}{\sqrt{1-x^2}}$$

$$= -\frac{3}{2} \times 2\sqrt{1-x^2} - 2\sin^{-1}x + C_2$$

$$= -3\sqrt{1-x^2} - 2\sin^{-1}x + C_2 \quad \ldots(3)$$

From eqs. (1), (2) and (3), we have

$$-I = \frac{1}{2}[x\sqrt{1-x^2} + \sin^{-1}(x)] + C_1$$
$$-3\sqrt{1-x^2} - 2\sin^{-1}(x) + C_2$$

$$-I = \frac{1}{2}x\sqrt{1-x^2} - \frac{3}{2}\sin^{-1}x - 3\sqrt{1-x^2} + C_1 + C_2$$

or $\quad I = \frac{3}{2}\sin^{-1}x - \frac{x}{2}\sqrt{1-x^2} + 3\sqrt{1-x^2} + C$

where, $\quad C = -C_1 - C_2$

182. Evaluate $\int (3-2x)\sqrt{2+x-x^2}\,dx$.

Sol : **OD 2015**

We have $\quad I = \int (3-2x)\sqrt{2+x-x^2}\,dx$

Given integral is the form of $\int (px+q)\sqrt{ax^2+bx+c}\,dx$

Now $\quad (3-2x) = A\frac{d}{dx}(2+x-x^2) + B$

$$3-2x = A(1-2x) + B \quad \ldots(i)$$

$$3-2x = A - 2Ax + B$$

$$3-2x = A + B - 2Ax$$

Comparing the coefficients of x and constant terms, we get

$$-2A = -2 \Rightarrow A = 1$$

and $\quad A + B = 3 \Rightarrow 1 + B = 3 \Rightarrow B = 2$

Thus, $\quad 3-2x = (1-2x) + 2$

$$3-2x = \frac{d}{dx}(2+x-x^2) + 2$$

Now, given integral becomes

$$I = \int (1-2x)\sqrt{2+x-x^2}\,dx + 2\int \sqrt{2+x-x^2}\,dx$$

Let $\quad I = I_1 + I_2 \quad \ldots(ii)$

Now $\quad I_1 = \int (1-2x)\sqrt{2+x-x^2}\,dx$

Substituting $2+x-x^2 = t$ we get

$$(1-2x)\,dx = dt$$

Thus $\quad I_1 = \int \sqrt{t}\,dt$

$$= \frac{2}{3}(t)^{\frac{3}{2}} + C_1$$

$$= \frac{2}{3}(2+x-x^2)^{\frac{3}{2}} + C_1$$

and $\quad I_2 = \int \sqrt{2+x-x^2}\,dx$

$$= \int \sqrt{-(x^2-x-2)}\,dx$$

$$= \int \sqrt{-(x^2 - 2\times\tfrac{1}{2}x + \tfrac{1}{4} - \tfrac{1}{4} - 2)}\,dx$$

$$= \int \sqrt{-\{(x-\tfrac{1}{2})^2 - \tfrac{9}{4}\}}\,dx$$

$$= \int \sqrt{\tfrac{9}{4} - (x-\tfrac{1}{2})^2}\,dx$$

$$= \int \sqrt{(\tfrac{3}{2})^2 - (x-\tfrac{1}{2})^2}\,dx$$

Using $\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}\left[x\sqrt{a^2-x^2} + a^2\sin^{-1}\left(\frac{x}{a}\right) + C\right.$,

$$I_2 = \frac{1}{2}\left[(x-\tfrac{1}{2})\sqrt{\tfrac{9}{4}-(x-\tfrac{1}{2})^2} + \tfrac{9}{4}\sin^{-}\left(\frac{x-\frac{1}{2}}{\frac{3}{2}}\right)\right] + C_2$$

Now $\quad I = I_1 + I_2$

$$= \frac{2}{3}(2+x-x^2)^{3/2} + \frac{(2x-1)}{2}\sqrt{2+x-x^2}$$
$$\left. + \frac{9}{4}\sin^{-1}\left(\frac{2x-1}{3}\right) + C\right]$$

where, $\quad C = C_1 + C_2$.

183. Find $\int \frac{\log|x|}{(x+1)^2}\,dx$.

Sol : **OD 2015, Delhi 2010**

We have $\quad I = \int \frac{\log|x|}{(x+1)^2}\,dx$

$$= \int \log|x| \cdot \frac{1}{(x+1)^2}\,dx$$

Using integration by parts technique

$$\int \underset{\text{I}}{u} \cdot \underset{\text{II}}{v}dx = \left[u\int v dx - \int\left\{\frac{d}{dx}u \cdot \int v dx\right\}dx\right]$$

and choosing its function with the help of ILATE procedure we have

$$I = \log|x| \cdot \int \frac{dx}{(x+1)^2} - \int\left[\frac{d}{dx}\log|x| \cdot \int \frac{dx}{(x+1)^2}\right]dx$$

$$= \log|x| \cdot \frac{(-1)}{x+1} + \int \frac{1}{x(x+1)}\,dx$$

$$I_2 = \log\left|(x-\tfrac{9}{2})+\sqrt{(x-\tfrac{9}{2})^2-(\tfrac{1}{2})^2}\right|+C_1$$
$$= \log\left|x-\tfrac{9}{2}+\sqrt{x^2-9+20}\right|+C_2 \quad ...(3)$$

Substituting the values of I_1 and I_2 from Eqs. (2) and (3) in eq. (1), we get

$$I = 3[2\sqrt{x^2-9x+20}+C_1] + 34\left[\log\left|x-\tfrac{9}{2}+\sqrt{x^2-9x+20}\right|+C_2\right]$$
$$I = 6\sqrt{x^2-9x+20}+34\log\left|x-\tfrac{9}{2}+\sqrt{x^2-9x+20}\right|+C$$

where, $C = \dfrac{C_1}{2}-\dfrac{C_2}{2}$.

213. Prove that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$, hence evaluate
$$I = \int_0^\pi \frac{x\sin x}{1+\cos^2 x}dx.$$

Sol : **Delhi 2019**

We have $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

Consider RHS $\int_0^a f(a-x)\,dx$

Substituting $t = a-x$, then $dt = -dx$

Also, when $x = 0$, then $t = a$ and when $x = a$, then $t = 0$

Thus RHS $\int_0^a f(a-x)\,dx = -\int_a^0 f(t)\,dt$
$$= \int_0^a f(t)\,dt$$
$$= \text{LHS} \quad \text{Hence proved.}$$

Now, we have $I = \int_0^\pi \dfrac{x\sin x}{1+\cos^2 x}dx \quad ...(i)$
$$= \int_0^\pi \frac{(\pi-x)\sin(\pi-x)\,dx}{1+\cos^2(\pi-x)}$$
$$= \int_0^\pi \frac{(\pi-x)\sin x\,dx}{1+\cos^2 x}$$
$$= \pi\int_0^\pi \frac{\sin x\,dx}{1+\cos^2 x} - \int_0^\pi \frac{x\sin x\,dx}{1+\cos^2 x}$$
$$= \pi\int_0^\pi \frac{\sin x\,dx}{1+\cos^2 x} - I$$
$$2I = \pi\int_0^\pi \frac{\sin x}{1+\cos^2 x}dx$$
$$I = \frac{\pi}{2}\int_0^\pi \frac{\sin x}{1+\cos^2 x}dx$$

Substituting $\cos x = t \Rightarrow -\sin x\,dx = dt$

Also, when $x = 0$, then $t = 1$ and when $x = \pi$, then $t = -1$

Thus $I = -\dfrac{\pi}{2}\int_1^{-1}\dfrac{dt}{1+t^2}$
$$= \frac{\pi}{2}\int_{-1}^1 \frac{dt}{1+t^2}$$
$$= \frac{\pi}{2}[\tan^{-1}t]_{-1}^1$$
$$= \frac{\pi}{2}[\tan^{-1}(1)-\tan^{-1}(-1)]$$
$$= \frac{\pi}{2}\left[\frac{\pi}{4}-\left(-\frac{\pi}{4}\right)\right] = \frac{\pi}{4}\left[\frac{\pi}{2}\right] = \frac{\pi^2}{4}$$
$$= \frac{\pi}{2}\left[\frac{\pi}{4}-\left(-\frac{\pi}{4}\right)\right] = \frac{\pi}{4}\left[\frac{\pi}{2}\right] = \frac{\pi^2}{4}$$

214. Prove that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$. and hence evaluate $\int_0^{\pi/2}\dfrac{x}{\sin x+\cos x}dx$.

Sol : **OD 2019, Delhi 2007**

We have $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

Consider RHS $\int_0^a f(a-x)\,dx$

Substituting $t = a-x$, then $dt = -dx$

Also, when $x = 0$, then $t = a$ and when $x = a$, then $t = 0$

Thus RHS $\int_0^a f(a-x)\,dx = -\int_a^0 f(t)\,dt$
$$= \int_0^a f(t)\,dt$$
$$= \text{LHS} \quad \text{Hence proved.}$$

Now, $I = \int_0^{\pi/2}\dfrac{x}{(\sin x+\cos x)}dx \quad ...(1)$
$$= \int_0^{\pi/2}\frac{(\frac{\pi}{2}-x)}{\sin(\frac{\pi}{2}-x)+\cos(\frac{\pi}{2}-x)}dx$$
$$I = \int_0^{\pi/2}\frac{(\frac{\pi}{2}-x)}{(\cos x+\sin x)}dx$$
$$= \int_0^{\pi/2}\frac{(\frac{\pi}{2}-x)}{(\sin x+\cos x)}dx \quad ...(2)$$

Adding Eqs. (1) and (2), we get
$$2I = \frac{\pi}{2}\int_0^{\pi/2}\frac{dx}{\sin x+\cos x}$$
$$I = \frac{\pi}{4}\int_0^{\pi/2}\frac{dx}{\sin x+\cos x}$$
$$= \frac{\pi}{4}\int_0^{\pi/2}\frac{dx}{\left[\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}+\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right]}$$
$$= \frac{\pi}{4}\int_0^{\pi/2}\frac{\sec^2\frac{x}{2}}{1-\tan^2\frac{x}{2}+2\tan\frac{x}{2}}dx$$

Substituting $t = \tan\dfrac{x}{2} \Rightarrow \dfrac{1}{2}\sec^2\dfrac{x}{2}dx = dt$

and $x = 0 \Rightarrow t = 0$ and $x = \dfrac{\pi}{2} \Rightarrow t = 1$
$$I = \frac{\pi}{4}\int_0^1\frac{2dt}{(1-t^2+2t)}$$
$$= \frac{\pi}{4}\int_0^1\frac{dt}{[(\sqrt{2})^2-(t-1)^2]}dt$$
$$= \frac{\pi}{2}\cdot\frac{1}{2\sqrt{2}}\log\left|\frac{\sqrt{2}+(t-1)}{\sqrt{2}-(t-1)}\right|_0^1$$
$$= \frac{\pi}{4\sqrt{2}}\log\left|\frac{\sqrt{2}+1}{\sqrt{2}-1}\right|$$

$$= \frac{1}{\sqrt{2}}\left[\log 1 - \log\frac{\sqrt{2}-1}{\sqrt{2}+1}\right]$$

$$= -\frac{1}{\sqrt{2}}\log\left[\frac{\sqrt{2}-1}{\sqrt{2}+1}\times\frac{\sqrt{2}+1}{\sqrt{2}+1}\right] \qquad \log 1 = 0$$

$$= \frac{-1}{\sqrt{2}}\log\frac{2-1}{(\sqrt{2}+1)^2}$$

$$= \frac{-1}{\sqrt{2}}\log\frac{1}{(\sqrt{2}+1)^2}$$

$$2I = \frac{2}{\sqrt{2}}\log(\sqrt{2}+1)$$

Thus $I = \frac{1}{\sqrt{2}}\log(\sqrt{2}+1)$ Hence proved.

224. Evaluate $\int_0^1 \frac{x^4+1}{x^2+1}dx$.

Sol : **Foreign 2011**

We have $I = \int_0^1 \frac{x^4+1}{x^2+1}dx$

$$= \int_0^1 \frac{x^4-1+1+1}{x^2+1}dx$$

$$= \int_0^1 \frac{x^4-1+2}{x^2+1}dx$$

$$= \int_0^1 \frac{(x^2-1)(x^2+1)+2}{x^2+1}dx$$

$$= \int_0^1 \left[\frac{(x^2-1)(x^2+1)}{x^2+1} + \frac{2}{x^2+1}\right]dx$$

$$= \int_0^1 \left[x^2 - 1 + \frac{2}{x^2+1}\right]dx$$

$$= \left[\frac{x^3}{3} - x + 2\tan^{-1}x\right]_0^1$$

$$= \frac{1}{3} - 1 + 2\tan^{-1}1 - 0$$

$$= -\frac{2}{3} + 2\times\frac{\pi}{4}$$

or $I = \frac{3\pi-4}{6}$

225. Evaluate $\int_0^{\pi/2}\frac{x+\sin x}{1+\cos x}dx$.

Sol : **OD 2011, Comp 2008**

We have $I = \int_0^{\pi/2}\frac{x+\sin x}{1+\cos x}dx$

Using the fact

$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$ and $1+\cos x = 2\cos^2\frac{x}{2}$ we have

$$I = \int_0^{\pi/2}\frac{x + 2\sin\frac{x}{2}\cdot\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}dx$$

$$= \frac{1}{2}\int_0^{\pi/2}\underset{I}{x}\,\underset{II}{\sec^2}\frac{x}{2}dx + \int_0^{\pi/2}\tan\frac{x}{2}dx$$

$$= \frac{1}{2}\left\{\left[x\int\sec^2\frac{x}{2}dx\right]_0^{\pi/2} - \int_0^{\pi/2}\left[\frac{d}{dx}(x)\int\left(\sec^2\frac{x}{2}dx\right)\right]dx\right\} + \int_0^{\pi/2}\tan\frac{x}{2}dx$$

$$= \frac{1}{2}\left\{\left[x\cdot\frac{\tan\frac{x}{2}}{\frac{1}{2}}\right]_0^{\pi/2} - \int_0^{\pi/2}\frac{\tan\frac{x}{2}}{\frac{1}{2}}dx\right\} + \int_0^{\pi/2}\tan\frac{x}{2}dx$$

[using integration by parts]

$$= \left[x\cdot\tan\frac{x}{2}\right]_0^{\pi/2} - \int_0^{\pi/2}\tan\frac{x}{2}dx + \int_0^{\pi/2}\tan\frac{x}{2}dx$$

$$= \frac{\pi}{2}\cdot\tan\frac{\pi}{4} - 0$$

$$I = \frac{\pi}{2} \qquad \left[\tan\frac{\pi}{4} = 1\right]$$

226. Evaluate $\int_1^2\frac{5x^2}{x^2+4x+3}dx$.

Sol : **OD 2011**

We have $I = \int_1^2\frac{5x^2}{x^2+4x+3}dx$

$$= 5\int_1^2\frac{x^2}{x^2+4x+3}dx$$

$$= 5\int_1^2\frac{x^2+4x+3-4x-3}{x^2+4x+3}dx$$

$$= 5\int_1^2\left(1 - \frac{4x+3}{x^2+4x+3}\right)dx$$

$$= 5\int_1^2 dx - 5\int_1^2\frac{4x+3}{x^2+4x+3}dx$$

$$I = 5[x]_1^2 - 5\int_1^2\frac{4x+3}{(x+3)(x+1)}dx \quad \text{...(i)}$$

Now $\frac{4x+3}{(x+3)(x+1)} = \frac{A}{x+3} + \frac{B}{x+1}$

$$4x+3 = A(x+1) + B(x+3)$$

Substituting $x = -1$ we get,

$$-4+3 = 0 + B2 \Rightarrow B = -\tfrac{1}{2}$$

Substituting $x = -3$ we get,

$$-4\times 3 + 3 = A(-2) + 0 \Rightarrow A = \tfrac{9}{2}$$

Thus $I = 5(2-1) - 5\int_1^2\left(\frac{\frac{9}{2}}{x+3} + \frac{-\frac{1}{2}}{x+1}\right)dx$

$$= 5 - 5\left[\frac{9}{2}\log|x+3| - \frac{1}{2}\log|x+1|\right]_1^2$$

$$= 5 - 5\left[\left(\frac{9}{2}\log 5 - \frac{1}{2}\log 3\right) - \left(\frac{9}{2}\log 4 - \frac{1}{2}\log 2\right)\right]$$

$$= 5 - 5\left[\frac{9}{2}(\log 5 - \log 4) - \frac{1}{2}(\log 3 - \log 2)\right]$$

$$= 5 - 5\left[\frac{9}{2}\log\frac{5}{4} - \frac{1}{2}\log\frac{3}{2}\right] \qquad \log m - \log n = \log\frac{m}{n}$$

$$= 5 - \frac{45}{2}\log\frac{5}{4} + \frac{5}{2}\log\frac{3}{2}$$

227. Evaluate $\int_0^1\frac{\log|1+x|}{1+x^2}dx$.

Sol : **Foreign 2011**

We have $I = \int_0^1\frac{\log|1+x|}{1+x^2}dx$

Substituting $x = \tan\theta$ we have $dx = \sec^2\theta d\theta$

Now $$\frac{1}{(t+1)(t+2)^2} = \frac{A}{t+1} + \frac{B}{t+2} + \frac{C}{(t+2)^2}$$

$$\frac{1}{(t+1)(t+2)^2} = \frac{A(t+2)^2 + B(t+1)(t+2) + C(t+1)}{(t+1)(t+2)^2}$$

$$1 = A(t+2)^2 + B(t+1)(t+2) + C(t+1) \quad (1)$$

Substituting $t = -1$ in eq. (1), we get

$$1 = A(-1+2)^2 + 0 + 0 \Rightarrow 1 = A$$

Substituting $t = -2$ in eq. (i), we get

$$1 = 0 + 0 + C(-2+1) \Rightarrow 1 = -C \Rightarrow C = -1$$

Comparing the coefficient of x^2 of both side in eq (1) we have

$$0 = A + B$$

Thus $A = -B = -1$

Now $$\frac{1}{(t+1)(t+2)^2} = \frac{1}{t+1} - \frac{1}{t+2} - \frac{1}{(t+2)^2}$$

$$I = \int \frac{1}{t+1}dt - \int \frac{1}{t+2}dt - \int \frac{1}{(t+2)^2}dt$$

$$= \log|t+1| - \log|t+2| - \frac{(t+2)^{-1}}{-1} + C$$

$$= \log|t+1| - \log|t+2| + \frac{1}{(t+2)} + C$$

$$= \log|x^2+1| - \log|x^2+2| + \frac{1}{(x^2+2)} + C$$

172. Find $\int \frac{2x}{(x^2+1)(x^4+4)}dx$.

Sol : **Delhi 2017**

We have $$I = \int \frac{2x}{(x^2+1)(x^4+4)}dx$$

Substituting $x^2 = t \Rightarrow 2x\,dx = dt$

$$I = \int \frac{dt}{(t+1)(t^2+4)}$$

Now, $$\frac{1}{(t+1)(t^2+4)} = \frac{A}{t+1} + \frac{Bt+C}{t^2+4}$$

$$1 = A(t^2+4) + (Bt+C)(t+1) \quad (1)$$

Substituting $t = -1$ in eq. (1), we get

$$1 = A(1^2+4) + 0 \Rightarrow 1 = 5A \Rightarrow A = \tfrac{1}{5}$$

Comparing the coefficient of t^2, t and constant term both side in eq (1) we obtain

$$A + B = 0 \quad ...(2)$$

$$B + C = 0 \quad ...(3)$$

$$4A + C = 1 \quad ...(4)$$

Substituting $A = \frac{1}{5}$ in (2) we get

$$B = -A = -\tfrac{1}{5}$$

Substituting $B = -\frac{1}{5}$ in (3) we get

$$C = -B = \tfrac{1}{5}$$

Thus, $$\frac{1}{(t+1)(t^2+4)} = \frac{\frac{1}{5}}{t+1} + \frac{-\frac{1}{5}t + \frac{1}{5}}{t^2+4}$$

$$= \frac{\frac{1}{5}}{t+1} - \frac{\frac{1}{5}t}{t^2+4} + \frac{\frac{1}{5}}{t^2+4}$$

Now, $$I = \int \frac{dt}{(t+1)(t^2+4)}$$

$$= \int \frac{\frac{1}{5}}{t+1}dt - \frac{\frac{1}{5}t}{t^2+4}dt + \int \frac{\frac{1}{5}}{t^2+4}dt$$

$$= \frac{1}{5}\int \frac{1}{t+1}dt - \frac{1}{5}\int \frac{t}{t^2+4}dt - \frac{1}{5}\int \frac{1}{t^2+4}dt$$

$$= \frac{1}{5}\log|t+1| - \frac{1}{5}\left[\frac{1}{2}\log|t^2+4| - \frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right)\right] + C$$

$$= \frac{1}{5}\log|x^2+1| - \frac{1}{5}\left[\frac{1}{2}\log|x^4+4| - \frac{1}{2}\tan^{-1}\left(\frac{x^2}{2}\right)\right] + C$$

173. Find $\int \frac{\cos\theta}{(4+\sin^2\theta)(5-4\cos^2\theta)}d\theta$.

Sol : **OD 2017, Comp 2015**

We have $$I = \int \frac{\cos\theta}{(4+\sin^2\theta)(5-4\cos^2\theta)}d\theta$$

$$= \int \frac{\cos\theta}{(4+\sin^2\theta)[5-4(1-\sin^2\theta)]}d\theta$$

$$= \int \frac{\cos\theta}{(4+\sin^2\theta)(5-4+4\sin^2\theta)}d\theta$$

$$= \int \frac{\cos\theta}{(4+\sin^2\theta)(1+4\sin^2\theta)}d\theta$$

Substituting $\sin\theta = t \Rightarrow \cos\theta\,d\theta = dt$ we have

$$I = \int \frac{dt}{(4+t^2)(1+4t^2)} \quad ...(i)$$

Now $$\frac{1}{(4+t^2)(1+4t^2)} = \frac{A}{4+t^2} + \frac{B}{1+4t^2}$$

$$1 = A(1+4t^2) + B(4+t^2) \quad (1)$$

Substituting $t = 0$ in eq (1) we have

$$1 = A + 4B \quad (2)$$

Substituting $t = 1$ in eq (1) we have

$$1 = 5A + 5B \quad (3)$$

Solving eqs. (2) and (3), we get

$$A = -\frac{1}{15} \text{ and } B = \frac{4}{15}$$

Substituting $A = -\frac{1}{15}$ and $B = \frac{4}{15}$ in eq. (ii), we get

Thus $$\frac{1}{(4+t^2)(1+4t^2)} = \frac{-\frac{1}{15}}{4+t^2} + \frac{\frac{4}{15}}{1+4t^2}$$

$$= \frac{-1}{15(4+t^2)} + \frac{4}{15(1+4t^2)}$$

Now $$I = \int \frac{dt}{(4+t^2)(1+4t^2)}$$

$$= \int \frac{-1}{15(4+t^2)} + \frac{4}{15(1+4t^2)}$$

$$= \frac{-1}{15}\int \frac{1}{4+t^2}dt + \frac{4}{15}\int \frac{1}{1+4t^2}dt$$

and $\quad A+2C = 1 \quad ...(5)$

Substituting $A=\frac{3}{5}$ in (3) and (5) we get $C=\frac{1}{5}$ and $B=\frac{2}{5}$.

Thus $\quad \dfrac{x^2+x+1}{(x^2+1)(x+2)} = \dfrac{3}{5}\cdot\dfrac{1}{(x+2)}+\dfrac{1}{5}\dfrac{(2x+1)}{(x^2+1)}$

Now $\quad I = \dfrac{3}{5}\int\dfrac{dx}{x+2}+\dfrac{1}{5}\int\dfrac{(2x+1)}{x^2+2}dx$

$$= \frac{3}{5}\int\frac{dx}{x+2}+\frac{1}{5}\int\frac{2x}{x^2+1}+\frac{1}{5}\int\frac{dx}{x^2+1}$$

$$= \frac{3}{5}\log|x+2|+\frac{1}{5}\log|x^2+1|+\frac{1}{5}\tan^{-1}x+C$$

177. Find $\int\dfrac{(2x-5)e^{2x}}{(2x-3)^3}dx$.

Sol : **OD 2016**

We have $\quad I = \int\dfrac{(2x-5)e^{2x}}{(2x-3)^3}dx$

$$= \int\frac{(2x-3-2)e^{2x}}{(2x-3)^3}dx$$

$$= \int\frac{e^{2x}}{(2x-3)^2}dx-2\int\frac{e^{2x}}{(2x-3)^3}dx$$

$$= \int e^{2x}(2x-3)^{-2}dx-2\int e^{2x}(2x-3)^{-3}dx$$

Using integration by parts technique we get

$$I = (2x-3)^{-2}\int e^{2x}dx-\int\left\{\frac{d}{dx}(2x-3)^{-2}\int e^{2x}dx\right\}dx \\ -2\int e^{2x}(2x-3)^{-3}dx$$

$$= (2x-3)^{-2}\frac{e^{2x}}{2}-\int -2(2x+3)^{-3}\times 2\times\frac{e^{2x}}{2}dx \\ -2\int e^{2x}(2x-3)^{-3}dx$$

$$= \frac{e^{2x}(2x-3)^{-2}}{2}+2\int e^{2x}(2x-3)^{-3}dx \\ -2\int e^{2x}(2x-3)^{-3}dx$$

$$= \frac{e^{2x}(2x-3)^{-2}}{2}+C$$

178. Find $\int(2x+5)\sqrt{10-4x-3x^2}\,dx$.

Sol : **Foreign 2016**

We have $\quad I = \int(2x+5)\sqrt{10-4x-3x^2}\,dx$

Given integral is the form of $\int(px+q)\sqrt{ax^2+bx+c}\,dx$

Thus $\quad 2x+5 = A\dfrac{d}{dx}(10-4x-3x^2)+B$,

where A and B are constants,

$$2x+5 = A(-4-6x)+B \qquad ...(i)$$

$$2x+5 = -6Ax+(B-4A)$$

Comparing the coefficient of x and the constant term, we get

$$-6A = 2 \text{ and } -6A = A \Rightarrow A = -\frac{1}{3}$$

and $\quad B-4A = 5 \Rightarrow B = 5+4A = 5+4\left(\dfrac{-1}{3}\right) = \dfrac{11}{3}$

Thus, $\quad (2x+5) = \dfrac{-1}{3}(-4-6x)+\dfrac{11}{3}$

Now, $\quad I = \dfrac{-1}{3}\int(-4-6x)\sqrt{10-4x-3x^2}\,dx$

$$+\frac{11}{3}\int\sqrt{10-4x-3x^2}\,dx$$

Let $\quad I = \dfrac{-1}{3}I_1+\dfrac{11}{3}I_2 \quad ...(ii)$

Now $\quad I_1 = \int(-4-6x)\sqrt{10-4x-3x^2}\,dx$

Substituting $10-4x-3x^2 = t \Rightarrow (-4-6x)\,dx = dt$

$$I_1 = \int\sqrt{t}\,dt = \frac{2}{3}t^{3/2}+C_1$$

$$= \frac{2}{3}(10-4x-3x^2)^{3/2}+C_1 \qquad ...(iii)$$

Now $\quad I_2 = \int\sqrt{10-4x-3x^2}\,dx$

$$= \sqrt{3}\int\sqrt{-\left(x^2+\frac{4}{3}x-\frac{10}{3}\right)}dx$$

$$= \sqrt{3}\int\sqrt{-\left(x^2+2\cdot\frac{2}{3}x+\left(\frac{2}{3}\right)^2-\left(\frac{2}{3}\right)^2-\frac{10}{3}\right)}dx$$

$$= \sqrt{3}\int\sqrt{-\left[\left(x+\frac{2}{3}\right)^2-\frac{34}{9}\right]}dx$$

$$= \sqrt{3}\int\sqrt{\frac{34}{9}-\left(x+\frac{2}{3}\right)^2}dx$$

$$= \sqrt{3}\int\sqrt{\left(\frac{\sqrt{34}}{3}\right)^2-\left(x+\frac{2}{3}\right)^2}dx$$

$$= \frac{\sqrt{3}}{2}\left[\left(x+\frac{2}{3}\right)\sqrt{\left(\frac{\sqrt{34}}{3}\right)^2-\left(x+\frac{2}{3}\right)^2}\right. \\ \left.+\frac{34}{9}\sin^{-1}\left(\frac{\left(x+\frac{2}{3}\right)}{\left(\frac{\sqrt{34}}{3}\right)}\right)+C_2\right]$$

$$= \frac{\sqrt{3}}{2}\left[\left(x+\frac{2}{3}\right)\sqrt{\frac{34}{9}-\left(x+\frac{2}{3}\right)^2}\right. \\ \left.+\frac{34}{9}\sin^{-1}\left(\frac{3x+2}{\sqrt{34}}\right)+C_2\right] \qquad ...(iv)$$

Now, from Eqs. (ii), (iii) and (iv), we get

$$I = \frac{-2}{9}(10-4x-3x^2)^{\frac{3}{2}}+ \\ +\frac{11\sqrt{3}}{6}\left[\left(x+\frac{2}{3}\right)\sqrt{\frac{34}{9}-\left(x+\frac{2}{3}\right)^2}+\frac{34}{9}\sin^{-1}\left(\frac{3x+2}{\sqrt{34}}\right)\right]+C$$

where, $\quad C = \dfrac{-C_1}{3}+\dfrac{11}{3}C_2$

179. Find $\int\dfrac{(x^2+1)(x^2+4)}{(x^2+3)(x^2-5)}dx$.

Sol : **Foreign 2016, OD 2007**

We have $\quad I = \int\dfrac{(x^2+1)(x^2+4)}{(x^2+3)(x^2-5)}dx$

Substituting $x^2 = y$ we get

$$= (-1)\int\left[\sqrt{1-x^2} + \int \frac{3x-2}{\sqrt{1-x^2}}\right]dx$$

$$= (-1)\left[\int \sqrt{1-x^2}\,dx + \int \frac{3x-2}{\sqrt{1-x^2}}\,dx\right]$$

Let $\quad I = (-1)[I_1 + I_2]$

or $\quad -I = I_1 + I_2 \qquad ...(1)$

Now, $\quad I_1 = \int \sqrt{1-x^2}\,dx$

Using $\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}\left[x\sqrt{a^2-x^2} + a^2\sin^{-1}\left(\frac{x}{a}\right)\right] + C$

$$I_1 = \frac{1}{2}[x\sqrt{1-x^2} + \sin^{-1}x] + C_1 \qquad ...(2)$$

Now, $\quad I_2 = \int \frac{3x-2}{\sqrt{1-x^2}}\,dx$

$$= \int \frac{3x}{\sqrt{1-x^2}}\,dx - 2\int \frac{dx}{\sqrt{1-x^2}}$$

$$= -\frac{3}{2}\int \frac{-2x}{\sqrt{1-x^2}}\,dx - 2\int \frac{dx}{\sqrt{1-x^2}}$$

$$= -\frac{3}{2} \times 2\sqrt{1-x^2} - 2\sin^{-1}x + C_2$$

$$= -3\sqrt{1-x^2} - 2\sin^{-1}x + C_2 \qquad ...(3)$$

From eqs. (1), (2) and (3), we have

$$-I = \frac{1}{2}[x\sqrt{1-x^2} + \sin^{-1}(x)] + C_1$$
$$-3\sqrt{1-x^2} - 2\sin^{-1}(x) + C_2$$

$$-I = \frac{1}{2}x\sqrt{1-x^2} - \frac{3}{2}\sin^{-1}x - 3\sqrt{1-x^2} + C_1 + C_2$$

or $\quad I = \frac{3}{2}\sin^{-1}x - \frac{x}{2}\sqrt{1-x^2} + 3\sqrt{1-x^2} + C$

where, $\quad C = -C_1 - C_2$

182. Evaluate $\int (3-2x)\sqrt{2+x-x^2}\,dx$.

Sol : **OD 2015**

We have $\quad I = \int (3-2x)\sqrt{2+x-x^2}\,dx$

Given integral is the form of $\int (px+q)\sqrt{ax^2+bx+c}\,dx$

Now $\quad (3-2x) = A\frac{d}{dx}(2+x-x^2) + B$

$$3-2x = A(1-2x) + B \qquad ...(i)$$

$$3-2x = A - 2Ax + B$$

$$3-2x = A + B - 2Ax$$

Comparing the coefficients of x and constant terms, we get

$$-2A = -2 \Rightarrow A = 1$$

and $\quad A + B = 3 \Rightarrow 1 + B = 3 \Rightarrow B = 2$

Thus, $\quad 3-2x = (1-2x) + 2$

$$3-2x = \frac{d}{dx}(2+x-x^2) + 2$$

Now, given integral becomes

$$I = \int (1-2x)\sqrt{2+x-x^2}\,dx + 2\int \sqrt{2+x-x^2}\,dx$$

Let $\quad I = I_1 + I_2 \qquad ...(ii)$

Now $\quad I_1 = \int (1-2x)\sqrt{2+x-x^2}\,dx$

Substituting $2+x-x^2 = t$ we get

$$(1-2x)\,dx = dt$$

Thus $\quad I_1 = \int \sqrt{t}\,dt$

$$= \frac{2}{3}(t)^{\frac{3}{2}} + C_1$$

$$= \frac{2}{3}(2+x-x^2)^{\frac{3}{2}} + C_1$$

and $\quad I_2 = \int \sqrt{2+x-x^2}\,dx$

$$= \int \sqrt{-(x^2-x-2)}\,dx$$

$$= \int \sqrt{-(x^2 - 2 \times \tfrac{1}{2}x + \tfrac{1}{4} - \tfrac{1}{4} - 2)}\,dx$$

$$= \int \sqrt{-\{(x-\tfrac{1}{2})^2 - \tfrac{9}{4}\}}\,dx$$

$$= \int \sqrt{\tfrac{9}{4} - (x-\tfrac{1}{2})^2}\,dx$$

$$= \int \sqrt{(\tfrac{3}{2})^2 - (x-\tfrac{1}{2})^2}\,dx$$

Using $\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}\left[x\sqrt{a^2-x^2} + a^2\sin^{-1}\left(\frac{x}{a}\right) + C\right.$,

$$I_2 = \frac{1}{2}\left[(x-\tfrac{1}{2})\sqrt{\tfrac{9}{4}-(x-\tfrac{1}{2})^2} + \tfrac{9}{4}\sin^{-}\left(\frac{x-\frac{1}{2}}{\frac{3}{2}}\right)\right] + C_2$$

Now $\quad I = I_1 + I_2$

$$= \frac{2}{3}(2+x-x^2)^{3/2} + \frac{(2x-1)}{2}\sqrt{2+x-x^2}$$
$$+ \frac{9}{4}\sin^{-1}\left(\frac{2x-1}{3}\right) + C\Big]$$

where, $\quad C = C_1 + C_2$.

183. Find $\int \frac{\log|x|}{(x+1)^2}\,dx$.

Sol : **OD 2015, Delhi 2010**

We have $\quad I = \int \frac{\log|x|}{(x+1)^2}\,dx$

$$= \int \log|x| \cdot \frac{1}{(x+1)^2}\,dx$$

Using integration by parts technique

$$\int \underset{\text{I}}{u} \cdot \underset{\text{II}}{v}dx = \left[u\int v dx - \int\left\{\frac{d}{dx}u \cdot \int v dx\right\}dx\right]$$

and choosing its function with the help of ILATE procedure we have

$$I = \log|x| \cdot \int \frac{dx}{(x+1)^2} - \int\left[\frac{d}{dx}\log|x| \cdot \int \frac{dx}{(x+1)^2}\right]dx$$

$$= \log|x| \cdot \frac{(-1)}{x+1} + \int \frac{1}{x(x+1)}\,dx$$

$$= \frac{1}{2}I_1 - \frac{9}{2}I_2, \text{ (say)} \qquad ...(i)$$

Now $\quad I_1 = \int (2x+3)\sqrt{x^2+3x-18}\,dx.$

Substituting $x^2+3x-18 = t$ we get

$$(2x+3)\,dx = dt$$

Thus
$$I_1 = \int t^{1/2}\,dt$$
$$= \frac{2}{3}t^{3/2} + C_1$$
$$= \frac{2}{3}(x^2+3x-18)^{3/2} + C_1 \qquad t = x^2+3x-18$$

and
$$I_2 = \int \sqrt{x^2+3x-18}\,dx$$
$$= \int \sqrt{\left(x+\frac{3}{2}\right)^2 - 18 - \frac{9}{4}}\,dx$$
$$= \int \sqrt{\left(x+\frac{3}{2}\right)^2 - \frac{81}{4}}\,dx$$
$$= \int \sqrt{\left(x+\frac{3}{2}\right)^2 - \left(\frac{9}{2}\right)^2}\,dx$$

Using
$$\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + C$$

$$I_2 = \frac{(x+\frac{3}{2})}{2}\sqrt{x^2+3x-18}$$
$$-\frac{81}{8}\log\left|\left(x+\frac{3}{2}\right)+\sqrt{x^2+3x-18}\right| + C_2$$
$$= \frac{2x+3}{4}\sqrt{x^2+3x-18}$$
$$-\frac{81}{8}\log\left|\frac{2x+3}{2}+\sqrt{x^2+3x-18}\right| + C_2$$

Substituting the values of I_1 and I_2 in eq. (i), we get

$$I = \frac{1}{2}\left[\frac{2}{3}(x^2+3x-18)^{3/2} + C_1\right]$$
$$-\frac{9}{2}\left[\frac{2x+3}{4}\sqrt{x^2+3x-18}\right.$$
$$\left.-\frac{81}{8}\log\left|\frac{2x+3}{2}+\sqrt{x^2+3x-18}\right| + C_2\right]$$
$$I = \frac{1}{2}(x^2+3x-18)^{3/2} - \frac{9}{8}(2x+3)\sqrt{x^2+3x-18}$$
$$+\frac{729}{16}\log\left|\frac{2x+3}{2}+\sqrt{x^2+3x-18}\right| + C$$

where, $\quad C = \dfrac{C_1}{2} - \dfrac{9C_2}{2}.$

187. Evaluate $\int \frac{x+2}{\sqrt{x^2+5x+6}}\,dx.$

Sol : **OD 2014**

We have
$$I = \int \frac{x+2}{\sqrt{x^2+5x+6}}\,dx$$

Now
$$x+2 = A\frac{d}{dx}(x^2+5x+6) + B$$
$$x+2 = A(2x+5) + B$$

Equating the coefficients of x and constant terms from both sides, we get

$$2A = 1 \;\Rightarrow\; A = \tfrac{1}{2}$$

and $5A + B = 2 \;\Rightarrow\; 5\times\frac{1}{2} + B = 2 \Rightarrow B = -\frac{1}{2}$

Thus
$$I = \int \frac{\frac{1}{2}(2x+5) - \frac{1}{2}}{\sqrt{x^2+5x+6}}\,dx$$
$$= \frac{1}{2}\int \frac{2x+5}{\sqrt{x^2+5x+6}}\,dx - \frac{1}{2}\int \frac{1}{\sqrt{x^2+5x+6}}\,dx$$
$$= \frac{1}{2}I_1 - \frac{1}{2}I_2 \text{ (say)} \qquad ...(1)$$

Now,
$$I_1 = \int \frac{2x+5}{\sqrt{x^2+5x+6}}\,dx$$

Substituting $x^2+5x+6 = t \Rightarrow (2x+5)\,dx = dt$ we get

$$I_1 = \int \frac{1}{\sqrt{t}}\,dt = 2\sqrt{t} + C_1$$
$$= 2\sqrt{x^2+5x+6} + C_1 \qquad ...(2)$$

and
$$I_2 = \int \frac{1}{\sqrt{x^2+5x+6}}\,dx$$
$$= \int \frac{1}{\sqrt{x^2+2\times\frac{5}{2}\times x+6+\frac{25}{4}-\frac{25}{4}}}\,dx$$
$$= \int \frac{1}{\sqrt{(x+\frac{5}{4})^2+6-\frac{25}{4}}}\,dx$$
$$= \int \frac{1}{\sqrt{(x+\frac{5}{2})^2-(\frac{1}{2})^2}}\,dx$$

Using the fact $\int \frac{dx}{\sqrt{x^2-a^2}} = \log\left|x+\sqrt{x^2-a^2}\right| + C,$

$$I_2 = \log\left|\left(x+\frac{5}{2}\right)+\sqrt{\left(x+\frac{5}{2}\right)^2-\left(\frac{1}{2}\right)^2}\right| + C_1$$
$$= \log\left|x+\frac{5}{2}+\sqrt{x^2+5x+6}\right| + C_2 \qquad ...(3)$$

Substituting the values of I_1 and I_2 from Eqs. (2) and (3) in eq. (1), we get

$$I = \frac{1}{2}[2\sqrt{x^2+5x+6} + C_1]$$
$$-\frac{1}{2}\left[\log\left|x+\frac{5}{2}+\sqrt{x^2+5x+6}\right| + C_2\right]$$
$$= \sqrt{x^2+5x+6} + \frac{C_1}{2}$$
$$-\frac{1}{2}\log\left|x+\frac{5}{2}+\sqrt{x^2+5x+6}\right| - \frac{C_2}{2}$$
$$I = \sqrt{x^2+5x+6} - \frac{1}{2}\log\left|x+\frac{5}{2}+\sqrt{x^2+5x+6}\right| + C$$

188. Evaluate $\int (3x-2)\sqrt{x^2+x+1}\,dx.$

Sol : **Foreign 2014**

We have
$$I = \int (3x-2)\sqrt{x^2+x+1}\,dx$$

Here, integrand is of the form $(px-q)\sqrt{ax^2+bx+c}$, so first write $3x-2$ as

$$3x-2 = A\frac{d}{dx}(x^2+x+1) + B$$
$$3x-2 = A(2x+1) + B$$

$$3x+1 = A(x^2+4x+3)+B(x+3) + C(x^2+1+2x)$$

$$3x+1 = (A+C)x^2+(4A+B+2C)x + 3A+3B+C$$

Comparing like powers of x^2 from both sides, we get

$$A+C = 0$$

Substituting $C=-2$ we get $A=2$.

Thus $\dfrac{3x+1}{(x+1)^2(x+3)} = \dfrac{2}{(x+1)} - \dfrac{1}{(x+1)^2} - \dfrac{2}{(x+3)}$

Now $I = \displaystyle\int \frac{3x+1}{(x+1)^2(x+3)}dx$

$$= \int \frac{2}{(x+1)}dx - \int \frac{1}{(x+1)^2}dx - \int \frac{2}{(x+3)}dx$$

$$= 2\log|x+1| - \frac{(x+1)^{-2+1}}{(-2+1)} - 2\log|x+3| + C$$

$$= 2\log\left|\frac{x+1}{x+3}\right| + \frac{1}{(x+1)} + C \qquad \log m - \log n = \log\frac{m}{n}$$

191. Evaluate $\displaystyle\int \frac{2x^2+1}{x^2(x^2+4)}dx$.

Sol : Delhi 2013

We have $I = \displaystyle\int \frac{2x^2+1}{x^2(x^2+4)}dx$

Substituting $x^2=t$ we have

$$\int \frac{2x^2+1}{x^2(x^2+4)}dx = \frac{2t+1}{t(t+4)}$$

or $\dfrac{2t+1}{t(t+4)} = \dfrac{A}{t} + \dfrac{B}{t+4}$

$$2t+1 = A(t+4)+Bt$$

Substituting $t=0$ we have

$$1 = A(0+4) \Rightarrow A = \tfrac{1}{4}$$

Substituting $t=-4$ we have

$$-2\times4+1 = 0+B(-4) \Rightarrow B = \tfrac{7}{4}$$

Thus $\dfrac{2t+1}{t(t+4)} = \dfrac{\frac{1}{4}}{t} + \dfrac{\frac{7}{4}}{t+4}$

$$= \frac{1}{4x^2} + \frac{7}{4(x^2+4)} \qquad t=x^2$$

Thus $I = \displaystyle\int \frac{2x^2+1}{x^2(x^2+4)}dx$

$$= \int\left[\frac{1}{4x^2} + \frac{7}{4(x^2+4)}\right]dx$$

$$= \frac{1}{4}\int\frac{dx}{x^2} + \frac{7}{4}\int\frac{dx}{x^2+4}$$

$$= -\frac{1}{4x} + \frac{7}{4}\times\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$$

$$= -\frac{1}{4x} + \frac{7}{8}\tan^{-1}\left(\frac{x}{2}\right) + C$$

192. Evaluate $\displaystyle\int \frac{x^2+1}{(x^2+4)(x^2+25)}dx$.

Sol : Delhi 2013, SQP 2012

We have $I = \displaystyle\int \frac{x^2+1}{(x^2+4)(x^2+25)}dx$

Substituting $x^2=t$ we have

$$\int \frac{x^2+1}{(x^2+4)(x^2+25)}dx = \frac{t+1}{(t+4)(t+25)}$$

or $\dfrac{t+1}{(t+4)(t+25)} = \dfrac{A}{t+4} + \dfrac{B}{t+25}$

$$t+1 = A(t+25)+B(t+4)$$

Substituting $t=-4$ we have

$$-4+1 = A(-4+25)+0$$

$$-3 = A21 \Rightarrow A = -\tfrac{1}{7}$$

Substituting $t=-25$ we have

$$-25+1 = 0+B(-25+4)$$

$$-24 = -B21 \Rightarrow B = \tfrac{24}{21} = \tfrac{8}{7}$$

Thus $\dfrac{t+1}{(t+4)(t+25)} = \dfrac{-\frac{1}{7}}{t+4} + \dfrac{\frac{8}{7}}{t+25}$

$$= \frac{-\frac{1}{7}}{x^2+4} + \frac{\frac{8}{7}}{x^2+25} \qquad t=x^2$$

Thus $I = \displaystyle\int \frac{x^2+1}{(x^2+4)(x^2+25)}dx$

$$= \int\left[\frac{-\frac{1}{7}}{x^2+4} + \frac{\frac{8}{7}}{x^2+25}\right]dx$$

$$= -\frac{1}{7}\int\frac{1}{x^2+4}dx + \frac{8}{7}\int\frac{1}{x^2+25}dx$$

$$= -\frac{1}{7}\times\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + \frac{8}{7}\times\frac{1}{5}\tan^{-1}\left(\frac{x}{5}\right) + C$$

$$= -\frac{1}{14}\tan^{-1}\left(\frac{x}{2}\right) + \frac{8}{35}\tan^{-1}\left(\frac{x}{5}\right) + C$$

193. Evaluate $\displaystyle\int \frac{x+2}{\sqrt{x^2+2x+3}}dx$.

Sol : OD 2013

We have $I = \displaystyle\int \frac{x+2}{\sqrt{x^2+2x+3}}dx$

Now $x+2 = A\dfrac{d}{dx}(x^2+2x+3)+B$

$$x+2 = A(2x+2)+B$$

Equating the coefficients of x and constant terms from both sides, we get

$$2A = 1 \Rightarrow A = \tfrac{1}{2}$$

and $2A+B=2 \Rightarrow 2\times\frac{1}{2}+B=2 \Rightarrow B=1$

Thus $I = \displaystyle\int \frac{\frac{1}{2}(2x+2)+1}{\sqrt{x^2+2x+3}}dx$

Sol : Comp 2012

We have $I = \int \frac{x^2}{(x\sin x + \cos x)^2}dx$

$$I = \int \frac{x\cos x}{(x\sin x+\cos x)^2} \cdot x\sec x\, dx \quad ...(1)$$

Substituting $x\sin x + \cos x = t$ we get

$$(x\cos x + \sin x - \sin x)\,dx = dt$$

$$x\cos x\,dx = dt$$

Let $I_1 = \int \frac{x\cos x}{(x\sin x+\cos x)^2}dx$

$$= \frac{dt}{t^2} = \frac{-1}{t} = \frac{-1}{x\sin x + \cos x}$$

[Substituting $t = x\sin x + \cos x$]

Now, integrating eq. (1) by parts, we get

$$I = \int x\sec x \cdot \frac{x\cos x}{(x\underset{I}{\sin}\, x + \underset{II}{\cos}\, x)^2}dx$$

$$= x\sec x \cdot \frac{(-1)}{x\sin x + \cos x}$$

$$-\int (1\cdot \sec x + x\sec x\tan x)\cdot \frac{-dx}{x\sin x+\cos x}$$

$$= \frac{-x\sec x}{x\sin x + \cos x} + \int \sec x\left(1 + \frac{x\sin x}{\cos x}\right)\frac{dx}{x\sin x + \cos x}$$

$$= \frac{-x\sec x}{x\sin x+\cos x} + \int \sec^2 x\,dx$$

$$= \frac{-x\sec x}{x\sin x + \cos x} + \tan x + C$$

197. Evaluate $\int e^{2x}\sin x\,dx$.

Sol : Foreign 2011

We have $I \int e^{2x}\cdot \sin x dx$

Using integration by parts, we get

$$I = \int \underset{II}{e^{2x}}\underset{I}{\sin}\, x dx \quad ...(i)$$

$$= \sin x\int e^{2x}\,dx - \int\left\{\frac{d}{dx}\sin x\int e^{2x}\,dx\right\}dx$$

$$= \frac{\sin x\cdot e^{2x}}{2} - \int \cos x\cdot \frac{e^{2x}}{2}dx$$

Again integration by parts, we get

$$I = \frac{e^{2x}\sin x}{2} - \frac{1}{2}\int \underset{II}{e^{2x}}\underset{I}{\cos}\, x dx$$

$$= \frac{e^{2x}\sin x}{2} - \frac{1}{2}\left[\cos x\int e^{2x}\,dx - \int\left\{\frac{d\cos x}{dx}\int e^{2x}\,dx\right\}dx\right]$$

$$= \frac{e^{2x}\sin x}{2} - \frac{1}{2}\left[\frac{e^{2x}\cos x}{2} - \int(-\sin x)\int e^{2x}\,dxdx\right]$$

$$= \frac{e^{2x}\sin x}{2} - \frac{e^{2x}\cos x}{4} - \frac{1}{2}\int(\sin x)\frac{e^{2x}}{2}dx$$

$$= \frac{e^{2x}\sin x}{2} - \frac{e^{2x}\cos x}{4} - \frac{1}{4}\int e^{2x}\sin x dx$$

$$I = \frac{e^{2x}\sin x}{2} - \frac{e^{2x}\cos x}{4} - \frac{1}{4}I$$

$$I + \frac{1}{4}I = \frac{e^{2x}}{4}(2\sin x - \cos x)$$

$$\frac{5}{4}I = \frac{e^{2x}}{4}(2\sin x - \cos x)$$

$$I = \frac{e^{2x}}{5}(2\sin x - \cos x) + C$$

198. Evaluate $\int \frac{3x+5}{\sqrt{x^2-8x+7}}dx$.

Sol : Foreign 2011, Delhi 2008

We have $I = \int \frac{3x+5}{\sqrt{x^2-8x+7}}dx$

Now $3x+5 = A\frac{d}{dx}(x^2-8x+7) + B$

$$3x+5 = A(2x-8) + B$$

Equating the coefficients of x and constant terms from both sides, we get

$$2A = 3 \Rightarrow A = \tfrac{3}{2}$$

and $-8A + B = 5 \Rightarrow -8\times\frac{3}{2} + B = 5 \Rightarrow B = 17$

Thus $I = \int \frac{\frac{3}{2}(2x-8)+17}{\sqrt{x^2-8x+7}}dx$

$$= \frac{3}{2}\int\frac{2x-8}{\sqrt{x^2-8x+7}}dx + 17\int\frac{1}{\sqrt{x^2-8x+7}}dx$$

$$= \frac{3}{2}I_1 + 17I_2 \text{ (say)} \quad ...(1)$$

Now, $I_1 = \int \frac{2x-8}{\sqrt{x^2-8x+7}}dx$

Substituting $x^2 - 8x + 7 = t \Rightarrow (2x-8)\,dx = dt$ we get

$$I_1 = \int\frac{1}{\sqrt{t}}dt = 2\sqrt{t} + C_1$$

$$= 2\sqrt{x^2-8x+7} + C_1 \quad ...(2)$$

and $I_2 = \int \frac{1}{\sqrt{x^2-8x+7}}dx$

$$= \int\frac{1}{\sqrt{x^2 - 2\times 4\times x + 16 - 9}}dx$$

$$= \int \frac{1}{\sqrt{(x-4)^2 - 3^2}}dx$$

Using the fact $\int\frac{dx}{\sqrt{x^2-a^2}} = \log\left|x + \sqrt{x^2-a^2}\right| + C$,

$$I_2 = \log\left|(x-4) + \sqrt{(x-4)^2 - 3^2}\right| + C_1$$

$$= \log\left|x - 4 + \sqrt{x^2-8x+7}\right| + C_2 \quad ...(3)$$

Substituting the values of I_1 and I_2 from Eqs. (2) and (3) in eq. (1), we get

$$I = \frac{3}{2}[2\sqrt{x^2-8x+7} + C_1]$$

$$+17\left[\log\left|x-4+\sqrt{x^2-8x+7}\right| + C_2\right]$$

$$I = 3\sqrt{x^2-8x+7} + 17\log\left|x-4+\sqrt{x^2-8x+7}\right| + C$$

199. Evaluate $\int \frac{2x}{(x^2+1)(x^2+3)}dx$.

Using integration by parts technique

$$\int \underset{\mathrm{I}}{u} \cdot \underset{\mathrm{II}}{v} dx = \left[u\int v dx - \int \left\{\frac{d}{dx}u \cdot \int v dx\right\} dx\right]$$

and choosing its function with the help of ILATE procedure we have

$$I = \int \underset{I}{\log(\log x)} \cdot \underset{II}{1} dx + \int \frac{1}{(\log x)^2} dx$$

$$= \log(\log x)\int 1\, dx - \int \left[\frac{d}{dx}\log(\log x)\int 1\, dx\right] dx + \int \frac{1}{(\log x)^2} dx$$

$$= \log(\log x) \cdot x - \int \frac{1}{\log x} \cdot \frac{1}{x} \cdot x\, dx + \int \frac{1}{(\log x)^2} dx$$

$$= x\log(\log x) - \int \underset{I}{(\log x)} \underset{II}{1} dx + \int \frac{1}{(\log x)^2} dx$$

Again, applying integration by parts in the middle integral, we get

$$I = x\log(\log x) - [(\log x)^{-1}]\int 1\, dx - \int \left\{\frac{d}{dx}(\log x)^{-1}\int 1\, dx\right\} dx\Big] + \int \frac{1}{(\log x)^2} dx + C$$

$$= x\log(\log x) - \left[\frac{x}{\log x} - \int -(\log x)^{-1} \cdot \frac{1}{x} \cdot x\, dx\right] + \int \frac{1}{(\log x)^2} dx + C$$

$$= x\log(\log x) - \frac{x}{\log x} - \int \frac{1}{(\log x)^2} dx + \int \frac{1}{(\log x)^2} dx + C$$

$$= x\log(\log x) - \frac{x}{\log x} + C$$

203. Evaluate $\int \frac{x+2}{\sqrt{(x-2)(x-3)}} dx$. **OD 2010**

Sol :

We have $I = \int \frac{x+2}{\sqrt{(x-2)(x-3)}} dx$

$$= \int \frac{x+2}{\sqrt{x^2-5x+6}} dx$$

Now $x+2 = A\frac{d}{dx}(x^2-5x+6) + B$

$$x+2 = A(2x-5) + B$$

Equating the coefficients of x and constant terms from both sides, we get

$$2A = 1 \;\Rightarrow\; A = \tfrac{1}{2}$$

and $-5A + B = 2 \;\Rightarrow\; -5 \times \frac{1}{2} + B = 2 \Rightarrow B = \frac{9}{2}$

Thus $I = \int \frac{\frac{1}{2}(2x-5) + \frac{9}{2}}{\sqrt{x^2-5x+6}} dx$

$$= \frac{1}{2}\int \frac{2x-5}{\sqrt{x^2-5x+6}} dx + \frac{9}{2}\int \frac{1}{\sqrt{x^2-5x+6}} dx$$

$$= \frac{1}{2}I_1 + \frac{9}{2}I_2 \text{ (say)} \qquad ...(1)$$

Now, $I_1 = \int \frac{2x-5}{\sqrt{x^2-5x+6}} dx$

Substituting $x^2 - 5x + 6 = t \Rightarrow (2x-5)\,dx = dt$ we get

$$I_1 = \int \frac{1}{\sqrt{t}} dt = 2\sqrt{t} + C_1$$

$$= 2\sqrt{x^2-5x+6} + C_1 \qquad ...(2)$$

and $I_2 = \int \frac{1}{\sqrt{x^2-5x+6}} dx$

$$= \int \frac{1}{\sqrt{x^2 - 2 \times \frac{5}{2} \times x + \frac{25}{4} - \frac{25}{4} + 6}} dx$$

$$= \int \frac{1}{\sqrt{\left(x-\frac{5}{2}\right)^2 - \frac{1}{4}}} dx$$

$$= \int \frac{1}{\sqrt{\left(x-\frac{5}{2}\right)^2 - \left(\frac{1}{2}\right)^2}} dx$$

Using the fact $\int \frac{dx}{\sqrt{x^2-a^2}} = \log\left|x + \sqrt{x^2-a^2}\right| + C$,

$$I_2 = \log\left|\left(x-\tfrac{5}{2}\right) + \sqrt{\left(x-\tfrac{5}{2}\right)^2 - \left(\tfrac{1}{2}\right)^2}\right| + C_1$$

$$= \log\left|x - \tfrac{5}{2} + \sqrt{x^2-5x+6}\right| + C_2 \qquad ...(3)$$

Substituting the values of I_1 and I_2 from Eqs. (2) and (3) in eq. (1), we get

$$I = \frac{1}{2}[2\sqrt{x^2-5x+6} + C_1] + \frac{9}{2}\left[\log\left|x-\tfrac{5}{2}+\sqrt{x^2-5x+6}\right| + C_2\right]$$

$$I = \sqrt{x^2-5x+6} + \frac{9}{2}\log\left|x-\tfrac{5}{2}+\sqrt{x^2-5x+6}\right| + C$$

$$I = \sqrt{(x-2)(x-3)} + \frac{9}{2}\log\left|x-\tfrac{5}{2}+\sqrt{(x-2)(x-3)}\right| + C$$

204. Evaluate $\int \frac{1}{\sin^4 x + \sin^2 x\cos^2 x + \cos^4 x} dx$. **OD 2014, Delhi 2010**

Sol :

We have $I = \int \frac{1}{\sin^4 x + \sin^2 x\cos^2 x + \cos^4 x} dx$

Dividing numerator and denominator by $\cos^4 x$,

$$I = \int \frac{\sec^4 x}{\tan^4 x + \tan^2 x + 1} dx$$

$$= \int \frac{(\sec^2 x)(\sec^2 x)}{\tan^4 x + \tan^2 x + 1} dx$$

Substituting $\tan x = t \Rightarrow \sec^2 x\, dx = dt$ we have

$$\sec^2 x = 1 + \tan^2 x = 1 + t^2$$

Thus $I = \int \frac{1+t^2}{t^4+t^2+1} dt$

$$= \int \frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}+1} dt$$

$$= \int \frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}-2+2+1} dt$$

$$= \int \frac{1+\frac{1}{t^2}}{\left(t-\frac{1}{t}\right)^2+3} dt$$

Again, substituting $u = t - \frac{1}{t}$ we have

$$-4 = -3B \Rightarrow B = \frac{4}{3}$$

Substituting $t = -1$, we get

$$-1 = 3A \Rightarrow A = -\frac{1}{3}$$

Thus, $$\frac{t}{(t+4)(t+9)} = \frac{-1}{3(t+1)} + \frac{4}{3(t+4)}$$
$$= \frac{-1}{3(x^2+1)} + \frac{4}{3(x^2+4)}$$

Now $$I = \int \frac{x^2}{(x^2+4)(x^2+9)}dx$$
$$= \int\left(\frac{-1}{3(x^2+1)} + \frac{4}{3(x^2+4)}\right)dx$$
$$= \int \frac{-1}{3(x^2+1)}dx + \int \frac{4}{3(x^2+4)}dx$$
$$= -\frac{1}{3}\int \frac{dx}{x^2+1} + \frac{4}{3}\int \frac{dx}{x^2+4}$$

Using the fact $\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$ we have

$$I = -\frac{1}{3}\cdot\tan^{-1}x + \frac{4}{3}\cdot\frac{1}{2}\cdot\tan^{-1}\left(\frac{x}{2}\right) + C$$
$$= -\frac{1}{3}\tan^{-1}x + \frac{2}{3}\tan^{-1}\frac{x}{2} + C$$

208. Find $\int \frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}dx,\ x \in [0,1]$.

OD 2014

Sol :

We have $$I = \int \frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}dx$$

Using the fact $\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x} = \frac{\pi}{2}$ we have

$$\cos^{-1}\sqrt{x} = \frac{\pi}{2} - \sin^{-1}\sqrt{x}$$

Thus $$I = \int \frac{\sin^{-1}\sqrt{x}\left(\frac{\pi}{2} - \sin^{-1}\sqrt{x}\right)}{\frac{\pi}{2}}dx$$
$$= \int \frac{2\sin^{-1}\sqrt{x} - \frac{\pi}{2}}{\frac{\pi}{2}}$$
$$= \frac{2}{\pi}\int\left(2\sin^{-1}\sqrt{x} - \frac{\pi}{2}\right)dx$$
$$= \frac{\pi}{4}\int \sin^{-1}\sqrt{x}\,dx - \int 1\,dx$$
$$= \frac{4}{\pi}\int \sin^{-1}\sqrt{x}\,dx - x$$
$$I = \frac{4}{\pi}I_1 - x$$

where, $$I_1 = \int \sin^{-1}\sqrt{x}\,dx$$

Substituting $\sqrt{x} = t$ we have

$$x = t^2 \text{ and } dx = 2t\,dt$$

Thus $$I_1 = \int \sin^{-1}t\,2t\,dt = 2\int \underset{I}{\sin^{-1}t}\cdot \underset{II}{t\,dt}$$

Now using integration by part technique we have

$$I_1 = 2\left[\sin^{-1}t\int t\,dt - \int\left\{\frac{d}{dt}(\sin^{-1}t)\int t\,dt\right\}dt\right]$$
$$= 2\left[\sin^{-1}t\cdot\frac{t^2}{2} - \int\frac{1}{\sqrt{1-t^2}}\cdot\frac{t^2}{2}dt\right]$$
$$= t^2\sin^{-1}t + \int\frac{(1-t^2)-1}{\sqrt{1-t^2}}dt$$
$$= t^2\sin^{-1}t + \int\sqrt{1-t^2}\,dt - \int\frac{1}{\sqrt{1-t^2}}dt$$
$$= t^2\sin^{-1}t + \frac{t\sqrt{1-t^2}}{2} + \frac{1}{2}\sin^{-1}t - \sin^{-1}t + C_1$$
$$= \left(t^2 - \frac{1}{2}\right)\sin^{-1}t + \frac{1}{2}t\sqrt{1-t^2} + C_1$$
$$= \frac{1}{2}[(2x-1)\sin^{-1}\sqrt{x} + \sqrt{x}\sqrt{1-x}] + C_1$$
$$= \frac{1}{2}[(2x-1)\sin^{-1}\sqrt{x} + \sqrt{x} + \sqrt{x-x^2}] + C_1$$

Substituting the value of I_1 in eq. (i), we get

$$I = \frac{2}{\pi}[(2x-1)\sin^{-1}\sqrt{x} + \sqrt{x-x^2}] - x + C,$$

where, $$C = \frac{4}{\pi}C_1$$

209. Find $\int \frac{x^2+x+1}{(x+1)^2(x+2)}dx$.

Comp 2014, OD 2010

Sol :

We have, $$I = \int\frac{x^2+x+1}{(x+1)^2(x+2)}dx$$

Now $$\frac{x^2+x+1}{(x+1)^2(x+2)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x+2}$$

$$x^2+x+1 = A(x+1)(x+2) + B(x+2) + C(x+1)^2$$

Substituting $x = -1$ in above equation we have

$$1-1+1 = 0 + B(-1+2) + 0 \Rightarrow B = 1$$

Substituting $x = -2$ in above equation we have

$$4-2+1 = 0+0+C(-2+1)^2 \Rightarrow C = 3$$

Comparing the coefficients of x^2 powers from both sides, we get

$$A + C = 1,$$

Substituting $C = 3$ in above we obtain $A = -2$.

Thus $$\frac{x^2+x+1}{(x+1)^2(x+2)} = \frac{-2}{x+1} + \frac{1}{(x+1)^2} + \frac{3}{x+2}$$

Now, $$I = \int\frac{x^2+x+1}{(x+1)^2(x+2)}dx$$
$$= -2\int\frac{1}{x+1}dx + \int\frac{dx}{(x+1)^2} + 3\int\frac{dx}{(x+2)}$$
$$= -2\log|x+1| - \frac{1}{x+1} + 3\log|x+2| + C$$

210. Find $\int \frac{\sqrt{x^2+1}\left(\log|x^2+1| - 2\log|x|\right)}{x^4}dx$.

$$I_2 = \log\left|(x-\tfrac{9}{2}) + \sqrt{(x-\tfrac{9}{2})^2 - (\tfrac{1}{2})^2}\right| + C_1$$

$$= \log\left|x - \tfrac{9}{2} + \sqrt{x^2 - 9 + 20}\right| + C_2 \quad ...(3)$$

Substituting the values of I_1 and I_2 from Eqs. (2) and (3) in eq. (1), we get

$$I = 3[2\sqrt{x^2 - 9x + 20} + C_1] + 34\left[\log\left|x - \tfrac{9}{2} + \sqrt{x^2 - 9x + 20}\right| + C_2\right]$$

$$I = 6\sqrt{x^2 - 9x + 20} + 34\log\left|x - \tfrac{9}{2} + \sqrt{x^2 - 9x + 20}\right| + C$$

where, $C = \dfrac{C_1}{2} - \dfrac{C_2}{2}$.

213. Prove that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$, hence evaluate
$$I = \int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx.$$

Sol : **Delhi 2019**

We have $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

Consider RHS $\int_0^a f(a-x)\,dx$

Substituting $t = a - x$, then $dt = -dx$

Also, when $x = 0$, then $t = a$ and when $x = a$, then $t = 0$

Thus RHS $\int_0^a f(a-x)\,dx = -\int_a^0 f(t)\,dt$

$$= \int_0^a f(t)\,dt$$

$= \text{LHS}$ Hence proved.

Now, we have
$$I = \int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx \quad ...(i)$$
$$= \int_0^\pi \frac{(\pi - x)\sin(\pi - x)\,dx}{1+\cos^2(\pi - x)}$$
$$= \int_0^\pi \frac{(\pi - x)\sin x\,dx}{1+\cos^2 x}$$
$$= \pi\int_0^\pi \frac{\sin x\,dx}{1+\cos^2 x} - \int_0^\pi \frac{x\sin x\,dx}{1+\cos^2 x}$$
$$= \pi\int_0^\pi \frac{\sin x\,dx}{1+\cos^2 x} - I$$
$$2I = \pi\int_0^\pi \frac{\sin x}{1+\cos^2 x}\,dx$$
$$I = \frac{\pi}{2}\int_0^\pi \frac{\sin x}{1+\cos^2 x}\,dx$$

Substituting $\cos x = t \Rightarrow -\sin x\,dx = dt$

Also, when $x = 0$, then $t = 1$ and when $x = \pi$, then $t = -1$

Thus
$$I = -\frac{\pi}{2}\int_1^{-1} \frac{dt}{1+t^2}$$
$$= \frac{\pi}{2}\int_{-1}^{1} \frac{dt}{1+t^2}$$
$$= \frac{\pi}{2}[\tan^{-1} t]_{-1}^{1}$$
$$= \frac{\pi}{2}[\tan^{-1}(1) - \tan^{-1}(-1)]$$
$$= \frac{\pi}{2}\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = \frac{\pi}{4}\left[\frac{\pi}{2}\right] = \frac{\pi^2}{4}$$
$$= \frac{\pi}{2}\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = \frac{\pi}{4}\left[\frac{\pi}{2}\right] = \frac{\pi^2}{4}$$

214. Prove that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$. and hence evaluate $\int_0^{\pi/2} \dfrac{x}{\sin x + \cos x}\,dx$.

Sol : **OD 2019, Delhi 2007**

We have $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

Consider RHS $\int_0^a f(a-x)\,dx$

Substituting $t = a - x$, then $dt = -dx$

Also, when $x = 0$, then $t = a$ and when $x = a$, then $t = 0$

Thus RHS $\int_0^a f(a-x)\,dx = -\int_a^0 f(t)\,dt$

$$= \int_0^a f(t)\,dt$$

$= \text{LHS}$ Hence proved.

Now,
$$I = \int_0^{\pi/2} \frac{x}{(\sin x + \cos x)}\,dx \quad ...(1)$$
$$= \int_0^{\pi/2} \frac{(\frac{\pi}{2} - x)}{\sin(\frac{\pi}{2} - x) + \cos(\frac{\pi}{2} - x)}\,dx$$
$$I = \int_0^{\pi/2} \frac{(\frac{\pi}{2} - x)}{(\cos x + \sin x)}\,dx$$
$$= \int_0^{\pi/2} \frac{(\frac{\pi}{2} - x)}{(\sin x + \cos x)}\,dx \quad ...(2)$$

Adding Eqs. (1) and (2), we get
$$2I = \frac{\pi}{2}\int_0^{\pi/2} \frac{dx}{\sin x + \cos x}$$
$$I = \frac{\pi}{4}\int_0^{\pi/2} \frac{dx}{\sin x + \cos x}$$
$$= \frac{\pi}{4}\int_0^{\pi/2} \frac{dx}{\left[\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} + \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right]}$$
$$= \frac{\pi}{4}\int_0^{\pi/2} \frac{\sec^2\frac{x}{2}}{1 - \tan^2\frac{x}{2} + 2\tan\frac{x}{2}}\,dx$$

Substituting $t = \tan\dfrac{x}{2} \Rightarrow \dfrac{1}{2}\sec^2\dfrac{x}{2}\,dx = dt$

and $x = 0 \Rightarrow t = 0$ and $x = \dfrac{\pi}{2} \Rightarrow t = 1$

$$I = \frac{\pi}{4}\int_0^1 \frac{2\,dt}{(1 - t^2 + 2t)}$$
$$= \frac{\pi}{4}\int_0^1 \frac{dt}{[(\sqrt{2})^2 - (t-1)^2]}\,dt$$
$$= \frac{\pi}{2}\cdot\frac{1}{2\sqrt{2}}\log\left|\frac{\sqrt{2} + (t-1)}{\sqrt{2} - (t-1)}\right|_0^1$$
$$= \frac{\pi}{4\sqrt{2}}\log\left|\frac{\sqrt{2}+1}{\sqrt{2}-1}\right|$$

Using the fact $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ we have

$$I = \int_0^{\pi} \frac{(\pi - x)}{1+\sin\alpha\sin(\pi - x)}\,dx$$

$$= \int_0^{\pi} \frac{(\pi - x)}{1+\sin\alpha\sin x}\,dx \qquad \text{...(ii)}$$

Adding Eqs. (i) and (ii), we get

$$2I = \int_0^{\pi} \frac{\pi}{1+\sin\alpha\sin x}\,dx$$

$$= \pi\int_0^{x} \frac{dx}{1+\sin\alpha\sin x}$$

$$= \pi\int_0^{\pi} \frac{dx}{1+\sin\alpha\sin x}$$

$$= \pi\int_0^{\pi} \frac{dx}{1+\sin\alpha\left(\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)}$$

$$= \pi\int_0^{\pi} \frac{(1+\tan^2\frac{x}{2})\,dx}{(1+\tan^2\frac{x}{2})+2\sin\alpha\tan\frac{x}{2}}$$

$$= \pi\int_0^{\pi} \frac{(\sec^2\frac{x}{2})\,dx}{\tan^2\frac{x}{2}+2\sin\alpha\cdot\tan\frac{x}{2}+1}$$

Substituting $\tan\frac{x}{2} = t \Rightarrow \sec^2\frac{x}{2}\cdot\frac{1}{2}dx = dt$

or $\quad \sec^2\frac{x}{2}\,dx = 2\,dt$

Also, when $x=0$, then $t=0$ and when $x=\pi$, then $t\to\infty$

Thus
$$2I = \pi\int_0^{\infty}\frac{2\,dt}{t^2+2\sin\alpha\cdot t+1}$$

$$I = \pi\int_0^{\infty}\frac{dt}{t^2+2\sin\alpha\cdot t+\sin^2\alpha+\cos^2\alpha}$$

$$= \pi\int_0^{\infty}\frac{dt}{(t+\sin\alpha)^2+(\cos\alpha)^2}$$

$$= \frac{\pi}{\cos\alpha}\left[\tan^{-1}\left(\frac{t+\sin\alpha}{\cos\alpha}\right)\right]_0^{\infty}$$

$$= \frac{\pi}{\cos\alpha}[\tan^{-1}(\infty)-\tan^{-1}(\tan\alpha)]$$

$$= \frac{\pi}{\cos\alpha}\left[\frac{\pi}{2}-\alpha\right]$$

219. Evaluate $\int_{-\pi}^{\pi}(\cos ax - \sin bx)^2\,dx$.

Sol : **Delhi 2015**

$$I = \int_{-\pi}^{\pi}(\cos ax-\sin bx)^2\,dx$$

$$= \int_{-\pi}^{\pi}(\cos^2 ax+\sin^2 bx-2\cos ax\sin bx)\,dx$$

$$= \int_{-\pi}^{\pi}(\cos^2 ax+\sin^2 bx)\,dx - 2\int_{-\pi}^{\pi}\cos ax\sin bx\,dx$$

$$= I_1 - I_2$$

Now I_1 is an even function thus

$$I_1 = \int_{-\pi}^{\pi}(\cos^2 ax+\sin^2 bx)\,dx$$

$$= 2\int_0^{\pi}(\cos^2 ax+\sin^2 bx)\,dx$$

$$= 2\int_0^{\pi}\left(\frac{1+\cos 2ax}{2}+\frac{1-\cos 2bx}{2}\right)dx$$

$$= \int_0^{\pi}(1+\cos 2ax+1-\cos 2bx)\,dx$$

$$= \int_0^{\pi}(2+\cos 2ax-\cos 2bx)\,dx$$

$$= \left(2x+\frac{\sin 2ax}{2a}-\frac{\sin 2bx}{2b}\right)_0^{\pi}$$

$$= 2\pi+\frac{\sin 2a\pi}{2a}-\frac{\sin 2b\pi}{2b}$$

and I_2 is an odd function, thus

$$I_2 = 2\int_{-\pi}^{\pi}(\cos ax\sin bx)\,dx$$

$$= 0$$

Thus $\quad I = I_1 - I_2 = 2\pi+\frac{2a\pi}{2a}-\frac{\sin 2b\pi}{2b}$

If $f(x)$ is even, $\quad \int_{-a}^{a} f(x) = 2\int_0^{a} f(a)$

If $f(x)$ is odd then, $\quad \int_{-a}^{a} f(x) = 0$

220. Find $\int_0^{\pi/4}\frac{dx}{\cos^3 x\sqrt{2\sin 2x}}$.

Sol : **OD 2015**

We have
$$I = \int_0^{\pi/4}\frac{dx}{\cos^3 x\sqrt{2\sin 2x}} \qquad \sin 2x = 2\sin x\cos x$$

$$= \int_0^{\pi/4}\frac{dx}{\cos^3 x\sqrt{2\,(2\sin x\cos x)}}$$

$$= \frac{1}{2}\int_0^{\pi/4}\frac{dx}{\cos^3 x\cos^{1/2}\sin^{1/2}x}$$

$$= \frac{1}{2}\int_0^{\pi/4}\frac{dx}{\cos^{7/2}x\sin^{1/2}x}$$

$$= \frac{1}{2}\int_0^{\pi/4}\frac{\sec^4 x}{\cos^{\frac{7}{2}-4}\sin^{1/2}x}\,dx$$

$$= \frac{1}{2}\int_0^{\pi/4}\frac{\sec^4 x}{\cos^{-1/2}x\sin^{1/2}x}\,dx$$

$$= \frac{1}{2}\int_0^{\pi/4}\frac{\sec^2 x(1+\tan^2 x)}{\tan^{1/2}x}\,dx \qquad \sec^2 x-\tan^2 x = 1$$

Substituting $\tan x = t \Rightarrow \sec^2 x\,dx = dt$

When $x=0$, then $t=\tan 0 = 0$

When $x=\frac{\pi}{4}$, then $t=\tan\frac{\pi}{4}=1$

$$I = \frac{1}{2}\int_0^1\left(\frac{1+t^2}{t^{1/2}}\right)dt$$

$$= \frac{1}{2}\int_0^1(t^{-1/2}+t^{3/2})\,dt$$

$$= \frac{1}{2}\left[2t^{-1/2}+\frac{2}{5}t^{5/2}\right]_0^1$$

$$= \left[t^{1/2}+\frac{1}{5}t^{5/2}\right]_0^1$$

$$= (1)^{1/2}+\frac{1}{5}(1)^{5/2}-0$$

$$= 1+\frac{1}{5} = \frac{6}{5}$$

$$= \frac{1}{\sqrt{2}}\left[\log 1 - \log\frac{\sqrt{2}-1}{\sqrt{2}+1}\right]$$

$$= -\frac{1}{\sqrt{2}}\log\left[\frac{\sqrt{2}-1}{\sqrt{2}+1} \times \frac{\sqrt{2}+1}{\sqrt{2}+1}\right] \qquad \log 1 = 0$$

$$= \frac{-1}{\sqrt{2}}\log\frac{2-1}{(\sqrt{2}+1)^2}$$

$$= \frac{-1}{\sqrt{2}}\log\frac{1}{(\sqrt{2}+1)^2}$$

$$2I = \frac{2}{\sqrt{2}}\log(\sqrt{2}+1)$$

Thus $\quad I = \frac{1}{\sqrt{2}}\log(\sqrt{2}+1)$ Hence proved.

224. Evaluate $\int_0^1 \frac{x^4+1}{x^2+1}dx$.

Sol : **Foreign 2011**

We have
$$I = \int_0^1 \frac{x^4+1}{x^2+1}dx$$

$$= \int_0^1 \frac{x^4-1+1+1}{x^2+1}dx$$

$$= \int_0^1 \frac{x^4-1+2}{x^2+1}dx$$

$$= \int_0^1 \frac{(x^2-1)(x^2+1)+2}{x^2+1}dx$$

$$= \int_0^1 \left[\frac{(x^2-1)(x^2+1)}{x^2+1} + \frac{2}{x^2+1}\right]dx$$

$$= \int_0^1 \left[x^2 - 1 + \frac{2}{x^2+1}\right]dx$$

$$= \left[\frac{x^3}{3} - x + 2\tan^{-1}x\right]_0^1$$

$$= \frac{1}{3} - 1 + 2\tan^{-1}1 - 0$$

$$= -\frac{2}{3} + 2 \times \frac{\pi}{4}$$

or
$$I = \frac{3\pi - 4}{6}$$

225. Evaluate $\int_0^{\pi/2} \frac{x+\sin x}{1+\cos x}dx$.

Sol : **OD 2011, Comp 2008**

We have
$$I = \int_0^{\pi/2}\frac{x+\sin x}{1+\cos x}dx$$

Using the fact

$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$ and $1+\cos x = 2\cos^2\frac{x}{2}$ we have

$$I = \int_0^{\pi/2}\frac{x + 2\sin\frac{x}{2}\cdot\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}dx$$

$$= \frac{1}{2}\int_0^{\pi/2}\underset{I}{x}\,\underset{II}{\sec^2}\frac{x}{2}dx + \int_0^{\pi/2}\tan\frac{x}{2}dx$$

$$= \frac{1}{2}\left\{\left[x\int\sec^2\frac{x}{2}dx\right]_0^{\pi/2} - \int_0^{\pi/2}\left[\frac{d}{dx}(x)\int\left(\sec^2\frac{x}{2}dx\right)\right]dx\right\} + \int_0^{\pi/2}\tan\frac{x}{2}dx$$

$$= \frac{1}{2}\left\{\left[x\cdot\frac{\tan\frac{x}{2}}{\frac{1}{2}}\right]_0^{\pi/2} - \int_0^{\pi/2}\frac{\tan\frac{x}{2}}{\frac{1}{2}}dx\right\} + \int_0^{\pi/2}\tan\frac{x}{2}dx$$

[using integration by parts]

$$= \left[x\cdot\tan\frac{x}{2}\right]_0^{\pi/2} - \int_0^{\pi/2}\tan\frac{x}{2}dx + \int_0^{\pi/2}\tan\frac{x}{2}dx$$

$$= \frac{\pi}{2}\cdot\tan\frac{\pi}{4} - 0$$

$$I = \frac{\pi}{2} \qquad \left[\tan\frac{\pi}{4} = 1\right]$$

226. Evaluate $\int_1^2 \frac{5x^2}{x^2+4x+3}dx$.

Sol : **OD 2011**

We have
$$I = \int_1^2\frac{5x^2}{x^2+4x+3}dx$$

$$= 5\int_1^2\frac{x^2}{x^2+4x+3}dx$$

$$= 5\int_1^2\frac{x^2+4x+3-4x-3}{x^2+4x+3}dx$$

$$= 5\int_1^2\left(1 - \frac{4x+3}{x^2+4x+3}\right)dx$$

$$= 5\int_1^2 dx - 5\int_1^2\frac{4x+3}{x^2+4x+3}dx$$

$$I = 5[x]_1^2 - 5\int_1^2\frac{4x+3}{(x+3)(x+1)}dx \quad \text{...(i)}$$

Now
$$\frac{4x+3}{(x+3)(x+1)} = \frac{A}{x+3} + \frac{B}{x+1}$$

$$4x+3 = A(x+1) + B(x+3)$$

Substituting $x = -1$ we get,

$$-4+3 = 0 + B2 \Rightarrow B = -\tfrac{1}{2}$$

Substituting $x = -3$ we get,

$$-4\times 3 + 3 = A(-2) + 0 \Rightarrow A = \tfrac{9}{2}$$

Thus
$$I = 5(2-1) - 5\int_1^2\left(\frac{\frac{9}{2}}{x+3} + \frac{-\frac{1}{2}}{x+1}\right)dx$$

$$= 5 - 5\left[\frac{9}{2}\log|x+3| - \frac{1}{2}\log|x+1|\right]_1^2$$

$$= 5 - 5\left[\left(\frac{9}{2}\log 5 - \frac{1}{2}\log 3\right) - \left(\frac{9}{2}\log 4 - \frac{1}{2}\log 2\right)\right]$$

$$= 5 - 5\left[\frac{9}{2}(\log 5 - \log 4) - \frac{1}{2}(\log 3 - \log 2)\right]$$

$$= 5 - 5\left[\frac{9}{2}\log\frac{5}{4} - \frac{1}{2}\log\frac{3}{2}\right] \qquad \log m - \log n = \log\frac{m}{n}$$

$$= 5 - \frac{45}{2}\log\frac{5}{4} + \frac{5}{2}\log\frac{3}{2}$$

227. Evaluate $\int_0^1 \frac{\log|1+x|}{1+x^2}dx$.

Sol : **Foreign 2011**

We have
$$I = \int_0^1\frac{\log|1+x|}{1+x^2}dx$$

Substituting $x = \tan\theta$ we have $dx = \sec^2\theta\, d\theta$

$$= \frac{1}{30}[-\log 4^{-1}] \qquad \log m^n = n\log m$$

$$= \frac{1}{30}\log 4$$

231. Evaluate $\int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16\sin 2x}\,dx$.

Sol : **Foreign 2014; Delhi 2014, 2011**

We have
$$I = \int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16\sin 2x}\,dx$$
$$= \int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16(1 + \sin 2x - 1)}\,dx$$
$$= \int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16[1 - (1 - \sin 2x)]}\,dx$$
$$= \int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16[1 - (\cos^2 x + \sin^2 x - 2\sin x\cos x)]}\,dx$$
$$I = \int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16[1 - (\cos x - \sin x)^2]}\,dx$$

Substituting $\cos x - \sin x = t$

$$(-\sin x - \cos x)\,dx = dt$$
$$(\sin x + \cos x)\,dx = -\,dt$$

When $x = 0$, then $t = \cos 0 - \sin 0 = 1$ and when $x = \frac{\pi}{4}$, then

$$t = \cos\frac{\pi}{4} - \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 0.$$

Thus
$$I = \int_0^1 \frac{-\,dt}{9 + 16(1 - t^2)}$$
$$I = \int_0^1 \frac{dt}{9 + 16(1 - t^2)}$$
$$= \int_0^1 \frac{dt}{25 - 16t^2}$$
$$= \frac{1}{16}\int_0^1 \frac{dt}{\left(\frac{5}{4}\right)^2 - t^2}$$
$$= \frac{1}{2 \times \frac{5}{4} \times 16}\left[\log\left|\frac{5 + 4t}{5 - 4t}\right|\right]_0^1$$
$$= \frac{1}{40}\left[\log\left|\frac{5 + 4}{5 - 4}\right| - \log\left|\frac{5}{5}\right|\right]$$
$$= \frac{1}{40}\left[\log\left(\frac{9}{1}\right) - \log\left(\frac{5}{5}\right)\right]$$
$$= \frac{1}{40}(\log 9 - \log 1) = \frac{1}{40}(\log 9)$$
$$I = \frac{1}{40}\log(3)^2$$
$$= \frac{2}{40}\log 3 \qquad \log a^n = n\log a$$
$$I = \frac{1}{20}\log 3$$

232. Evaluate $\int_0^{\pi} \frac{x}{a^2\cos^2 x + b^2\sin^2 x}\,dx$.

Sol : **Foreign 2014; OD 2009**

We have
$$I = \int_0^{\pi} \frac{x}{a^2\cos^2 x + b^2\sin^2 x}\,dx \qquad \text{...(i)}$$

Using the fact $\int_0^a f(x)\,dx = \int_0^a f(a - x)\,dx$ we have

$$I = \int_0^{\pi} \frac{(\pi - x)}{a^2\cos^2(\pi - x) + b^2\sin^2(\pi - x)}\,dx$$
$$= \int_0^{\pi} \frac{(\pi - x)}{a^2\cos^2 x + b^2\sin^2 x}\,dx \qquad \text{...(ii)}$$

Adding Eqs. (i) and (ii), we get

$$2I = \int_0^{\pi} \frac{(x + \pi - x)}{a^2\cos^2 x + b^2\sin^2 x}\,dx$$
$$= \pi\int_0^{\pi} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x}$$

Here, $a^2\cos^2(\pi - x) + b^2\sin^2(\pi - x) = a^2\cos^2 x + b^2\sin^2 x$

Using the fact

$$\int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx,\ \text{if } f(2a - x) = f(x)$$

We have
$$2I = 2\pi\int_0^{\pi} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x}$$
$$= 2\pi\int_0^{\pi} \frac{\sec^2 x}{a^2 + b^2\tan^2 x}\,dx$$

Substituting $\tan x = t \Rightarrow \sec^2 x\,dx = dt$

When $x = 0$, then $t = \tan 0 = 0$ and

when $x = \frac{\pi}{2}$, then $t = \tan\frac{\pi}{2} = \infty$.

$$I = \pi\int_0^{\infty} \frac{dt}{a^2 + b^2t^2}$$
$$= \pi\int_0^{\infty} \frac{dt}{a^2 + (bt)^2}$$
$$= \frac{\pi}{b^2}\int_0^{\infty} \frac{dt}{\left(\frac{a}{b}\right)^2 + t^2}$$
$$I = \frac{\pi}{ab}\left[\tan^{-1}\frac{bt}{a}\right]_0^{\infty} \qquad \int\frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$
$$= \frac{\pi}{ab}[\tan^{-1}\infty - \tan^{-1}0]$$
$$= \frac{\pi}{ab}\left[\frac{\pi}{2} - 0\right] = \frac{\pi^2}{2ab}$$

233. Evaluate $\int_0^{\pi} \frac{x\tan x}{\sec x + \tan x}\,dx$.

Sol : **Foreign 2014; Delhi 2014C, 2010, 2008; OD 2008**

We have
$$I = \int_0^{\pi} \frac{x\tan x}{\sec x + \tan x}\,dx \qquad \text{...(i)}$$

Using the fact $\int_0^a f(x)\,dx = \int_0^a f(a - x)\,dx$ we have

$$I = \int_0^{\pi} \frac{(\pi - x)\tan(\pi - x)}{\sec(\pi - x) + \tan(\pi - x)}\,dx$$
$$= \int_0^{\pi} \frac{-(\pi - x)\tan x}{-\sec x - \tan x}\,dx$$
$$= \int_0^{\pi} \frac{(\pi - x)\tan x}{\sec x + \tan x}\,dx \qquad \text{...(ii)}$$

Adding Eqs. (i) and (ii), we get

$$2I = \int_0^{\pi} \frac{\pi\tan x}{\sec x + \tan x}\,dx$$
$$= \pi\int_0^{\pi/2} \frac{\tan x}{\sec x + \tan x}\,dx + \pi\int_{\pi/2}^{\pi} \frac{\tan x}{\sec x + \tan x}\,dx$$

$$= \frac{1}{4}\log 2 - \frac{1}{2}\left(\frac{\pi}{2} - \frac{\pi}{4} - \frac{1}{2}\right)$$

$$= \frac{1}{4}\log 2 - \frac{1}{2}\left(\frac{\pi}{4} - \frac{1}{2}\right)$$

$$= \frac{1}{4}\log 2 - \frac{\pi}{8} + \frac{1}{4}$$

237. Evaluate $\int_0^{\pi/2} 2\sin x \cos x \tan^{-1}(\sin x)\, dx$.

Sol : **Delhi 2011**

We have $I = \int_0^{\pi/2} 2\sin x \cos x \tan^{-1}(\sin x)\, dx$

Substituting $\sin x = t \Rightarrow \cos x\, dx = dt$

When $x = 0$, then $t = \sin 0 = 0$

when $x = \frac{\pi}{2}$, then $t = \sin\frac{\pi}{2} = 1$.

$$I = 2\int_0^1 t \times \tan^{-1} t\, dt$$

Using integration by parts technique

$$\int \underset{\mathrm{I}}{u} \cdot \underset{\mathrm{II}}{v} dx = \left[u\int v dx - \int\left\{\frac{d}{dx}u \cdot \int v dx\right\}dx\right]$$

and choosing $\tan^{-1} t$ as 1st function and t as 2nd function, we get

$$I = 2\left[\frac{t^2}{2} \times \tan^{-1} t\right]_0^1 - 2\int_0^1 \frac{1}{1+t^2} \times \frac{t^2}{2}\, dt$$

$$= 2\left[\frac{t^2}{2} \times \tan^{-1} t\right]_0^1 - \int_0^1 \frac{t^2}{1+t^2}\, dt$$

$$= 2 \times \frac{1}{2} \times \tan^{-1}(1) - \int_0^1 \frac{1+t^2-1}{1+t^2}\, dt$$

$$= 1 \times \frac{\pi}{4} - \int_0^1 \left(\frac{1+t^2}{1+t^2} - \frac{1}{1+t^2}\right) dt$$

$$= \frac{\pi}{4} - \int_0^1 \left(1 - \frac{1}{1+t^2}\right) dt$$

$$= \frac{\pi}{4} - [t - \tan^{-1} t]_0^1$$

$$= \frac{\pi}{4} - 1 + \tan^{-1} 1$$

$$= \frac{\pi}{4} - 1 + \frac{\pi}{4} = \frac{2\pi}{4} - 1 \qquad 1 = \tan\frac{\pi}{4}$$

$$I = \left(\frac{\pi}{2} - 1\right)$$

CASE BASED QUESTIONS

238. Commodity prices are primarily determined by the forces of supply and demand in the market. For example, if the supply of oil increases, the price of one barrel decreases. Conversely, if demand for oil increases (which often happens during the summer), the price rises. Gasoline and natural gas fall into the energy commodities category.

The price p (dollars) of each unit of a particular commodity is estimated to be changing at the rate

$$\frac{dp}{dx} = \frac{-135x}{\sqrt{9+x^2}}$$

where x (hundred) units is the consumer demand (the number of units purchased at that price). Suppose 400 units $(x = 4)$ are demanded when the price is \$30 per unit.

(i) Find the demand function $p(x)$.

(ii) At what price will 300 units be demanded? At what price will no units be demanded?

(iii) How many units are demanded at a price of \$20 per unit?

Sol :

(i) $$p(x) = \int \frac{dp}{dx}\, dx$$

Let $u = 9 + x^2$ then we have $du = 2x\, dx$ or $x dx = \frac{1}{2} du$

Thus $$p(x) = \int \frac{-135x}{\sqrt{9+x^2}}\, dx$$

$$= \int \frac{-135}{u^{1/2}}\left(\frac{1}{2}\right) du$$

$$= \frac{-135}{2}\int u^{-1/2}\, du$$

$$= \frac{-135}{2}\left(\frac{u^{1/2}}{\frac{1}{2}}\right) + C$$

$$= -135\sqrt{9+x^2} + C$$

Since $p = 30$ when $x = 4$, we have

$$30 = -135\sqrt{9+4^2} + C$$

$$C = 30 + 135\sqrt{25} = 705$$

So, $$p(x) = -135\sqrt{9+x^2} + 705$$

(ii) When 300 units are demanded, $x = 3$ and the corresponding price is

$$p(3) = -135\sqrt{9+3^2} + 705$$

$$= \$132.24 \text{ per unit}$$

No units are demanded when $x = 0$ and the corresponding price is

In a certain chemical reaction, a substance is converted into another substance at a rate proportional to the square of the amount of the first substance present at any time t. Initially $(t=0)$ 50 g of the first substance was present; 1 hr later, only 10 g of it remained.

(i) Find an expression that gives the amount of the first substance present at any time t.

(ii) What is the amount present after 2 hr?

Sol :

Since rate is proportional to the square of the amount of the first substance present at any time t, we are led to the differential equation

$$\frac{dQ}{dt} = kQ^2$$

The differential equation is separable. Separating the variables and integrating, we obtain

$$\int \frac{dQ}{Q^2} = \int k\,dt$$

and $$-\frac{1}{Q} = kt + C$$

Therefore, $$Q = -\frac{1}{kt+C}$$

Now, $Q = 50$ when $t = 0$, therefore, $50 = -\frac{1}{C}$ and $C = -\frac{1}{50}$.

Therefore, $$Q = -\frac{1}{kt - \frac{1}{50}}$$

Since $Q = 10$ when $t = 1$,

$$10 = -\frac{1}{k - \frac{1}{50}}$$

$$10\left(k - \frac{1}{50}\right) = -1$$

$$10k - \frac{1}{5} = -1$$

$$10k = -1 + \frac{1}{5} = -\frac{4}{5}$$

$$k = -\frac{4}{50} = -\frac{2}{25}$$

Therefore, $$Q(t) = \frac{1}{\frac{2}{25}t + \frac{1}{50}}$$

$$= \frac{1}{\frac{4t+1}{50}} = \frac{50}{4t+1}$$

and $$Q(2) = \frac{50}{8+1} \approx 5.56 \text{ grams}$$

241. Bata India is the largest retailer and leading manufacturer of footwear in India and is a part of the Bata Shoe Organization. Incorporated as Bata Shoe Company Private Limited in 1931, the company was set up initially as a small operation in Konnagar (near Calcutta) in 1932. In January 1934,

The manager of BATA show room at Jaipur determines that the price p (dollars) for each pair of a popular brand of sports sneakers is changing at the rate of

$$p'(x) = \frac{-300x}{(x^2+9)^{3/2}}$$

when x (hundred) pairs are demanded by consumers. When the price is \$75 per pair, 400 pairs $(x=4)$ are demanded by consumers.

(i) Find the demand (price) function $p(x)$.

(ii) At what price will 500 pairs of sneakers be demanded? At what price will no sneakers be demanded?

(iii) How many pairs will be demanded at a price of \$90 per pair?

Sol :

(i) $$p(x) = \int p'(x)\,dx$$

$$= \int \frac{-300x}{(x^2+9)^{3/2}}\,dx$$

Let $u = x^2 + 9$, then we have $\frac{du}{dx} = 2x$, or $\frac{1}{2}du = x\,dx$.

Thus $$p(x) = -300\int \frac{1}{(x^2+9)^{3/2}}x\,dx$$

$$= -150\int u^{-3/2}\,du$$

$$= \frac{300}{\sqrt{x^2+9}} + C$$

When the price is \$75, 4 hundred pair are demanded, so

$$75 = \frac{300}{\sqrt{(4)^2+9}} + C$$

or $$C = 15$$

So, $$p(x) = \frac{300}{\sqrt{x^2+9}} + 15$$

(ii) When $x = 5$ hundred,

$$p(5) = \frac{300}{\sqrt{(5)^2+9}} + 15$$

$$= \$66.45 \text{ per pair}$$

When $x = 0$ hundred

Let $u = e^{0.01t} + 1$

Then $\dfrac{du}{dt} = 0.01 e^{0.01t}$

or $100\,du = e^{0.01t}\,dt$

$$= -\,0.01\int \frac{1}{(e^{0.01t}+1)^2}\,e^{0.01t}\,dt$$

$$= -\int \frac{1}{u^2}\,du$$

$$= \frac{1}{e^{0.01t}+1} + C$$

When the shot is initially administered, $t = 0$ and

$$0.5 = \frac{1}{e^0+1} + C \text{ or } C = 0$$

So, $C(t) = \dfrac{1}{e^{0.01t}+1}$

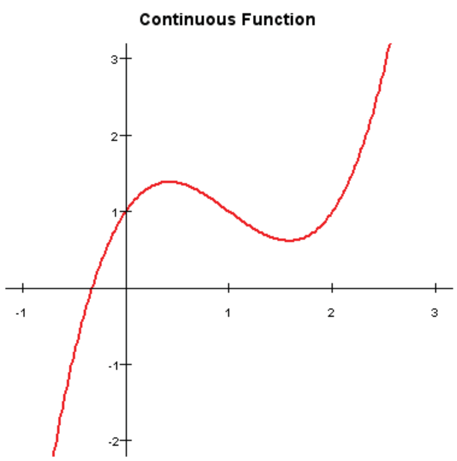
(ii) After one hour, when $t = 60$ minutes, the concentration is

$$C(60) = \frac{1}{e^{0.01(60)}+1} \approx 0.3543\text{ mg/cm}^3$$

After three hours, when $t = 180$ minutes, the concentration is

$$C(180) = \frac{1}{e^{0.01(180)}+1} \approx 0.1419\text{ mg/cm}^3$$

244. In mathematics, a continuous function is a function such that a continuous variation of the argument induces a continuous variation of the value of the function. This means that there are no abrupt changes in value, known as discontinuities.



Let $f(x)$ be a continuous function defined on $[a, b]$, then

$$\int_a^b f(x)\,dx = \int_0^b f(a+b-x)\,dx$$

On the basis of above information, answer the following questions.

(i) Evaluate $\displaystyle\int_1^2 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{3-x}}\,dx$

(ii) Evaluate $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \log\tan x\,dx$

iii) Evaluate $\displaystyle\int_a^b \frac{x^{\frac{1}{n}}}{x^{\frac{1}{n}}+(a+b-x)^{\frac{1}{n}}}\,dx]$

(iv) Evaluate $\displaystyle\int_a^b \frac{f(x)}{f(x)+f(a+b-x)}\,dx$

Sol :

(i) Let, $$I = \int_1^2 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{3-x}}\,dx \quad \text{...(i)}$$

$$I = \int_1^2 \frac{\sqrt{3-x}}{\sqrt{3-x}+\sqrt{x}}\,dx \quad \text{...(ii)}$$

Adding Eqs. (i) and (ii), we get

$$2I = \int_1^2 1\,dx = [x]_1^2 = (2-1) = 1$$

$$I = \frac{1}{2}$$

(ii) Let, $$I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \log(\tan x)\,dx \quad \text{...(i)}$$

$$I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \log\tan\left(\frac{\pi}{2}-x\right)dx$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \log\tan\left(\frac{\pi}{2}-x\right)dx$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \log\cot x\,dx \quad \text{...(ii)}$$

Adding Eqs. (i) and (ii), we get

$$2I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \log[(\tan x)(\cot x)]\,dx$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 0\,dx = 0 \quad [\log 1 = 0]$$

$$I = 0$$

(iii) Let, $$I = \int_a^b \frac{x^{1/n}}{x^{1/n}+(a+b-x)^{1/n}}\,dx \quad \text{...(i)}$$

$$I = \int_a^b \frac{(a+b-x)^{1/n}}{(a+b-x)^{1/n}+x^{1/n}}\,dx \quad \text{...(ii)}$$

Adding Eqs. (i) and (ii), we get

$$2I = \int_a^b 1\,dx = [x]_a^b$$

$$2I = b-a$$

$$I = \frac{b-a}{2}$$

(iv) Let,

$$I = \int_a^b \frac{f(x)}{f(x) + f(a+b-x)} dx \qquad ...(i)$$

$$= \int_a^b \frac{f(a+b-x)}{f(a+b-x) + f(a+b-(a+b-x)} dx$$

$$= \int_a^b \frac{f(a+b-x)}{f(a+b-x) + f(x)} dx \qquad ...(ii)$$

Adding Eqs. (i) and (ii), we get

$$2I = \int_a^b 1\, dx = [x]_a^b = b - a$$

$$I = \frac{1}{2}(b - a)$$

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CHAPTER 8

APPLICATION OF INTEGRALS

OBJECTIVE QUESTIONS

1. The area of enclosed by $y=3x-5,\ y=0,\ x=3$ and $x=5$ is

(a) 12 sq units (b) 13 sq units

(c) $13\frac{1}{2}$ sq units (d) 14 sq units

Sol : OD 2017

The region is bounded by the curves $y=3x-5,\ y=0$, $x=3$ and $x=5$.

Required area,

$$\begin{aligned}A &= \int_3^5 (3x-5)\,dx\\ &= \left[\frac{3x^2}{2}-5x\right]_3^5\\ &= \left(\frac{75}{2}-25\right)-\left(\frac{27}{2}-15\right)\\ &= \frac{75}{2}-25-\frac{27}{2}+15\\ &= \frac{48}{2}-10 = 14 \text{ sq units}\end{aligned}$$

Thus (d) is correct option.

2. The area bounded by $y=\log x$, x-axis and ordinates $x=1,\ x=2$ is

(a) $\frac{1}{2}(\log 2)^2$ (b) $\log(2/e)$

(c) $\log(4/e)$ (d) $\log 4$

Sol : Delhi 2018

Required area,

$$\begin{aligned}A &= \int_1^2 \log x\,dx\\ &= [x\log x - x]_1^2\\ &= 2\log 2 - 1\\ &= \log 4 - \log e\\ &= \log\left(\frac{4}{e}\right)\end{aligned}$$

Thus (c) is correct option.

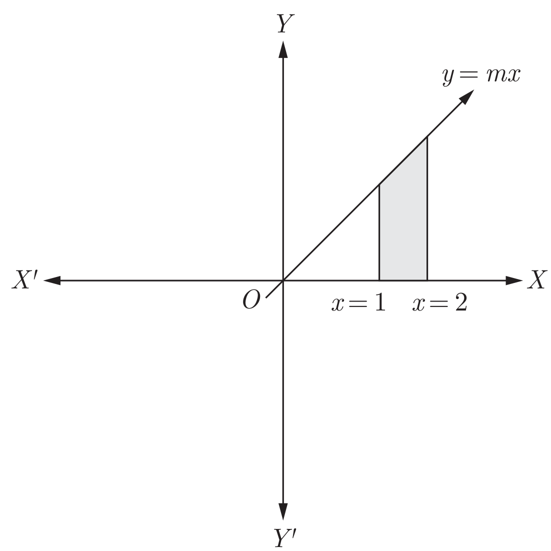
3. The area of the region bounded by the lines $y=mx$, $x=1,\ x=2$ and x-axis is 6 sq units, then m is equal to

(a) 3 (b) 1

(c) 2 (d) 4

Sol : Delhi 2015

Given, equation of line is $y=mx$ and bounded by $x=1,\ x=2$ and X-axis.



$$\begin{aligned}\text{Required area,}\quad A &= \int_1^2 mx\,dx\\ 6 &= m\left[\frac{x^2}{2}\right]_1^2\\ 6 &= m\left(\frac{4}{2}-\frac{1}{2}\right)\\ 6 &= m\frac{3}{2}\\ m &= 4\end{aligned}$$

Thus (d) is correct option.

4. Area of a curve $xy=4$, bounded by the lines $x=1$ and $x=3$ and x-axis will be

(a) $\log 12$ (b) $\log 64$

(c) $\log 81$ (d) $\log 27$

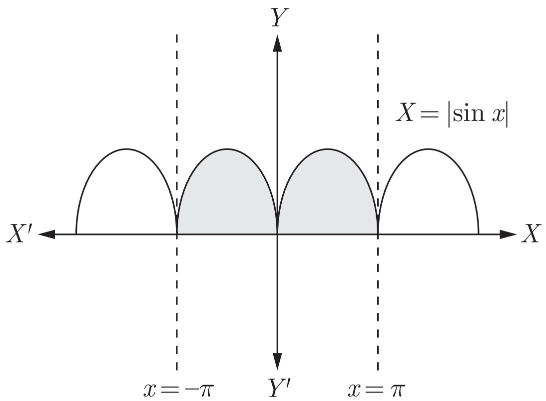
7. The area bounded by $y = |\sin x|$, x-axis and the lines $|x| = \pi$ is

(a) 2 sq units (b) 3 sq units

(c) 4 sq units (d) None of these

Sol : OD 2010

The region bounded by $y = |\sin x|$, x-axis and the lines $|x| = \pi$ is shown shaded in figure below.



Required area

$$A = 2\int_0^{\pi} \sin x \, dx$$
$$= 2[-\cos x]_0^{\pi} \text{ sq unit}$$
$$= 2[1 + 1] = 4 \text{ sq unit}$$

Thus (c) is correct option.

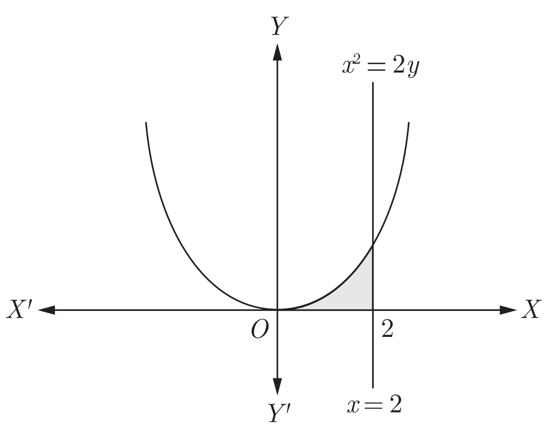
8. The area bounded by the curve $y = \frac{1}{2}x^2$, the x-axis and the ordinate $x = 2$ is

(a) $\frac{1}{3}$ sq unit (b) $\frac{2}{3}$ sq unit

(c) 1 sq unit (d) $\frac{4}{3}$ sq unit

Sol : Delhi 2014, OD 2010

The area bounded by the curve $y = \frac{1}{2}x^2$, the x-axis and the ordinate $x = 2$ is shown shaded in figure below.



Required area,

$$A = \int_0^2 y \, dx$$
$$= \int_0^2 \frac{x^2}{2} dx \text{ sq units}$$
$$= \left[\frac{x^3}{6}\right]_0^2 = \frac{4}{3} \text{ sq units}$$

Thus (d) is correct option.

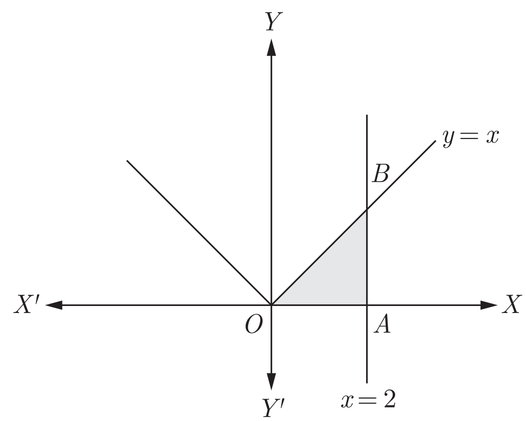
9. Area of the region satisfying $x \le 2$, $y \le |x|$ and $x \ge 0$ is

(a) 4 sq units (b) 1 sq unit

(c) 2 sq units (d) None of these

Sol : Foreign 2006

Area of the region satisfying $x \le 2$, $y \le |x|$ and $x \ge 0$ is shown shaded in figure below.



Required area

$$A = \int_0^2 x dx \text{ sq units}$$
$$= \left[\frac{x^2}{2}\right]_0^2 = 2 \text{ sq units}$$

Thus (c) is correct option.

10. The area bounded by the parabola $y^2 = 8x$ and its latusrectum is

(a) 16/3 sq units

(b) 32/3 sq units

(c) 8/3 sq units

(d) 64/3 sq units

Sol : OD 2010

The area bounded by the parabola $y^2 = 8x$ and its latusrectum is shown shaded in figure below.

$$= m\left(\frac{4}{2} - \frac{1}{2}\right)$$
$$= m\left(2 - \frac{1}{2}\right)$$
$$= \frac{3}{2} \text{ m sq units}$$

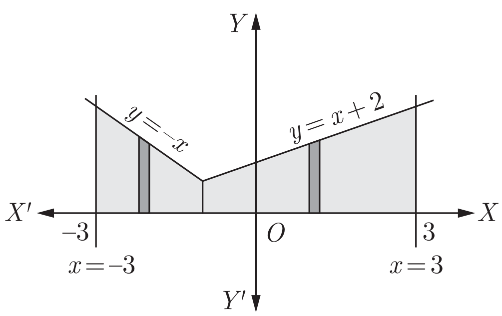
14. Using integration, find the area of the region bounded by the curves $y = |x+1| + 1$, $x = -3$, $x = 3$ and $y = 0$.

Sol : Delhi 2014

Given, curves are

$$y = |x+1| + 1$$
$$= \begin{cases} (x+1)+1, & \text{if } x+1 \geq 0 \\ -(x+1)+1, & \text{if } x+1 < 0 \end{cases} \quad ...(i)$$
$$= \begin{cases} x+2, & \text{if } x \geq -1 \\ -x, & \text{if } x < -1 \end{cases}$$
$$x = -3 \quad ...(ii)$$
$$x = 3 \quad (iii)$$
and, $$y = 0 \quad ...(iv)$$

We have sketch the region bounded by above line in following figure.



Required area is the area of region $ABCDEA$.

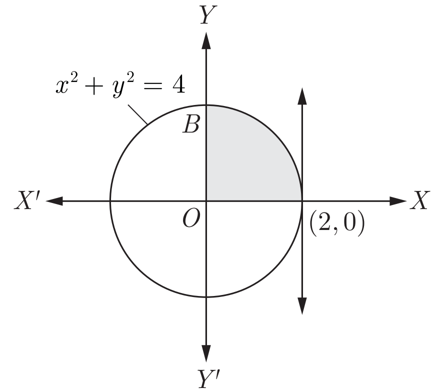
$$A = \int_{-3}^{-1} (-x)\,dx + \int_{-1}^{3} (x+2)\,dx$$
$$= -\left[\frac{x^2}{2}\right]_{-3}^{-1} + \left[\frac{x^2}{2} + 2x\right]_{-1}^{3}$$
$$= -\frac{1}{2}(1-9) + \left[\left(\frac{9}{2} + 6\right) - \left(\frac{1}{2} - 2\right)\right]$$
$$= 4 + \frac{21}{2} + \frac{3}{2}$$
$$= 16 \text{ sq units.}$$

Hence, the required area is 16 sq units.

15. Find the area lying in the first quadrant and bounded by the circle $x^2 + y^2 = 4$ and the lines $x = 0$ and $x = 2$.

Sol : Comp 2015, OD 2013

The area bounded by the circle $x^2 + y^2 = 4$ and the lines $x = 0$ and $x = 2$ in the first quadrant is shown in the figure by shaded region.



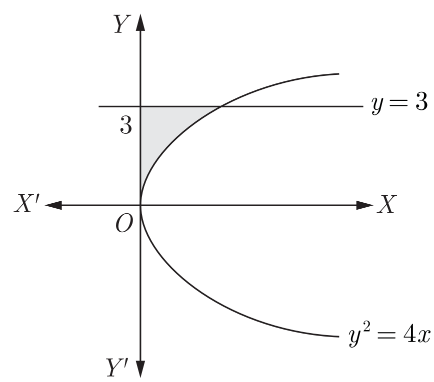
Required area

$$A = \int_0^2 |y|\,dx$$
$$= \int_0^2 \sqrt{4 - x^2}\,dx$$
$$= \left[\frac{x}{2}\sqrt{4 - x^2} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_0^2$$
$$= 0 + 2\sin^{-1}(1) - 0$$
$$= 2 \times \frac{\pi}{2}$$
$$= \pi \text{ sq units}$$

16. Find the area of the region bounded by the curve $y^2 = 4x$, y-axis and the line $y = 3$.

Sol : Foreign 2010

The area bounded by the curve $y^2 = 4x$, y-axis and $y = 3$ is shown in the figure by shaded region.



Area of the shaded region,

$$A = \int_0^3 |x|\,dy$$
$$= \int_0^3 \frac{y^2}{4}\,dy$$
$$= \frac{1}{4}\left[\frac{y^3}{3}\right]_0^3$$
$$= \frac{1}{12}(3^3 - 0)$$

Ch 7 : Integrals

PDF Text Book | PDF Worksheet | PDF Solutions | DOC Worksheet | PPT Presentation

Ch 8 : Application of Integrals

PDF Text Book | PDF Worksheet | PDF Solutions | DOC Worksheet | PPT Presentation

Ch 9 : Differential Equations

PDF Text Book | PDF Worksheet | PDF Solutions | DOC Worksheet | PPT Presentation

Ch 10 : Vector Algebra

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Ch 11 : Three Dimensional Geometry

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Ch 12 : Linear Programming

PDF Text Book | PDF Worksheet | PDF Solutions | DOC Worksheet | PPT Presentation

Ch 13 : Probability

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Class 10th

Maths | Science | Social Science | English LL

Info Tech. | Comp. App. | AI | English Comm.

हिंदी अ | हिंदी ब | संस्कृत

Class 9th

Maths | Science | Social Science | English LL

Info Tech. | Comp. App. | AI | English Comm.

हिंदी अ | हिंदी ब | संस्कृत

Class 8th

Maths | Science | Social Science | English

हिंदी | संस्कृत

Class 7th

Maths | Science | Social Science | English

हिंदी | संस्कृत

Class 6th

Maths | Science | Social Science | English

हिंदी | संस्कृत

Class 5th

Maths | English | हिंदी | EVS

Class 4th

Maths | English | हिंदी | EVS

Class 3th

Maths | English | हिंदी | EVS

Class 2th

Maths | English | हिंदी | EVS

Class 1th

Maths | English | हिंदी | EVS

Prep/KG 3rd Year

Maths | English | हिंदी | EVS

Prep/KG 2nd Year

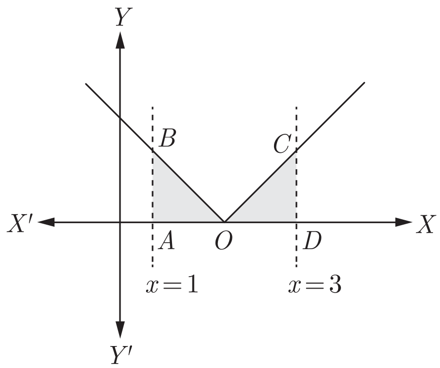
Maths | English | हिंदी | EVS

Prep/KG 1st Year

Maths | English | हिंदी | EVS

Sol : Delhi 2018

The area of the region bounded by the curve $y = |x - 2|,$, $x = 1$, $x = 3$ and x-axis is shown in the figure by shaded region.



Area of the shaded region,

$$\begin{aligned}A &= \int_1^3 |x-2|\,dx \\ &= \int_1^2 -(x-2)\,dx + \int_2^3 (x-2)\,dx \\ &= \int_1^2 (2-x)\,dx + \int_2^3 (x-2)\,dx \\ &= \left[2x - \frac{x^2}{2}\right]_1^2 + \left[\frac{x^2}{2} - 2x\right]_2^3 \\ &= 2 - \frac{3}{2} - \frac{3}{2} + 2 \\ &= 1 \text{ sq unit.}\end{aligned}$$

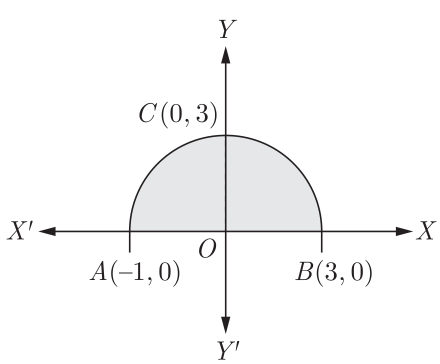
21. Find the area bounded by $y = -x^2 + 2x + 3$ and $y = 0$.

Sol : SQP 2019, OD 2017

We have $\quad y = -x^2 + 2x + 3$

Substituting $y = 0$ and then solving we get $x = -1$ and $x = 3$. Thus intersection points of given curves with x-axis are $(-1, 0)$ and $(3, 0)$.

The area bounded by the curve $y = -x^2 + 2x + 3$ and x-axis is shown in the figure by shaded region.



Area of the shaded region,

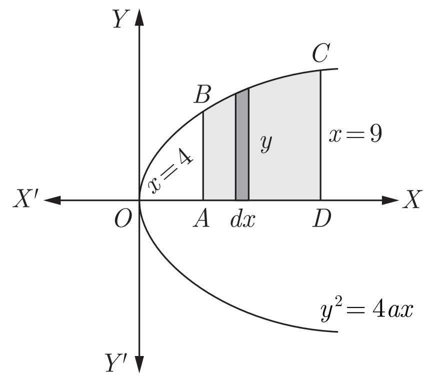
$$\begin{aligned}A &= \int_{-1}^3 (-x^2 + 2x + 3)\,dx \\ &= \left[-\frac{x^3}{3} + \frac{2x^2}{2} + 3x\right]_{-1}^3 \\ &= \left[-9 + 9 + 9\left(\frac{1}{3} + 1 - 3\right)\right] \\ &= \frac{32}{3} \text{ sq units}\end{aligned}$$

22. Find the area of the region bounded by the parabola $y^2 = 4ax$, its axis and two ordinates $x = 4$ and $x = 9$ in first quadrant.

Sol : Comp 2008

Given equation of parabola is $y^2 = 4ax$. Its axis is $y = 0$ and vertex is (0,0).

The area bounded by the parabola $y^2 = 4ax$, its axis and two ordinates $x = 4$ and $x = 9$ is shown in the figure by shaded region.



Area of the shaded region,

$$\begin{aligned}A &= \int_4^9 y\,dx \\ &= \int_4^9 \sqrt{4ax}\,dx \\ &= 2\sqrt{a}\int_4^9 \sqrt{x}\,dx \\ &= 2\sqrt{a}\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_4^9 \\ &= 2\sqrt{a} \times \frac{2}{3}\left[(9)^{\frac{3}{2}} - (4)^{\frac{3}{2}}\right] \\ &= \frac{4}{3}\sqrt{a}\left[(3^2)^{\frac{3}{2}} - (2^2)^{\frac{3}{2}}\right] \\ &= \frac{4}{3}\sqrt{a}[27 - 8] \\ &= \frac{4}{3}\sqrt{a}(19) = \frac{76}{3}\sqrt{a} \text{ sq units}\end{aligned}$$

Hence, the required area is $\frac{76\sqrt{a}}{3}$ sq units.

23. Find the area bounded by the line $y = x$, the x-axis and the lines $x = -1$ and $x = 2$.

Sol : OD 2009

The area bounded by the line $y = x$, the x-axis and the lines $x = -1$ and $x = 2$ is shown in the figure by shaded region.

$$= -2 \times \frac{2}{3}\left[(4-x)^{\frac{3}{2}}\right]_0^4$$

$$= -\frac{4}{3}\left[(4-4)^{\frac{3}{2}} - (4-0)^{\frac{3}{2}}\right]$$

$$= -\frac{4}{3}\left[0 - (4)^{\frac{3}{2}}\right]$$

$$= \frac{4}{3} \times (4)^{\frac{3}{2}}$$

$$= \frac{4}{3} \times (2^2)^{\frac{3}{2}}$$

$$= \frac{4}{3} \times 8 = \frac{32}{3} \text{ sq units}$$

Hence,the required area is $\frac{32}{3}$ sq units.

ALTERNATIVE METHOD :

Substituting $y = 0$ in given equation $x = 4 - y^2$ we get $y = 2$ and $y = -2$. Thus given curve intersect the $y-$ axis at $y = 2$ and $y = -2$.

Area of the shaded region,

$$A = \int_{-2}^{2} x dy$$

$$= 2\int_0^2 x dy$$

$$= 2\int_0^2 (4 - y^2)\, dy$$

$$= 2\left[4y - \frac{y^3}{3}\right]_0^2$$

$$= 2\left[4 \times 2 - \frac{2^3}{3}\right] - 2[0]$$

$$= 2\left[8 - \frac{8}{3}\right]$$

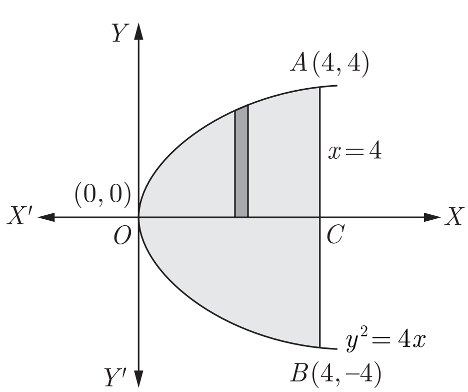
$$= 2 \times 8\left[1 - \frac{1}{3}\right]$$

$$= 2 \times 8 \times \frac{2}{3} = \frac{32}{3}$$

26. Find the area of region bounded by the curve $y^2 = 4x$ and the line $x = 4$.

Sol : OD 2017, Delhi 2014

The given curve $y^2 = 4x$ is a parabola which is of the form of $Y^2 = 4aX$ having vertex (0,0) and given line is $x = 4$. We have sketch the curves as shown in figure.



It is clear from the figure that, the region for which we have to find area is $OBCAO$. Also, the region $OCAO$ is symmetrical about x- axis.

Now, let us find the intersection point of curve and line.

Substituting $x = 4$ in parabola $y^2 = 4x$ we get $y = \pm 4$. Thus, line and curve intersect at two points (4,4) and $(4, -4)$. So, coordinates of point A are (4,4) as it is in I quadrant.

Now, area of bounded region $OABCO$ is the required area which is shown shaded in figure below.

$$A = \int_{-4}^{4} y dx$$

$$= 2\int_0^4 y dx$$

$$= 2\int_0^4 2\sqrt{x}\, dx$$

$$= 4\int_0^4 x^{\frac{1}{2}}\, dx$$

$$= 4\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^4$$

$$= 4 \cdot \frac{2}{3}\left[x^{\frac{3}{2}}\right]_0^4$$

$$= \frac{8}{3}\left[(4)^{\frac{3}{2}} - 0\right]$$

$$= \frac{8}{3} \times (2^2)^{\frac{3}{2}}$$

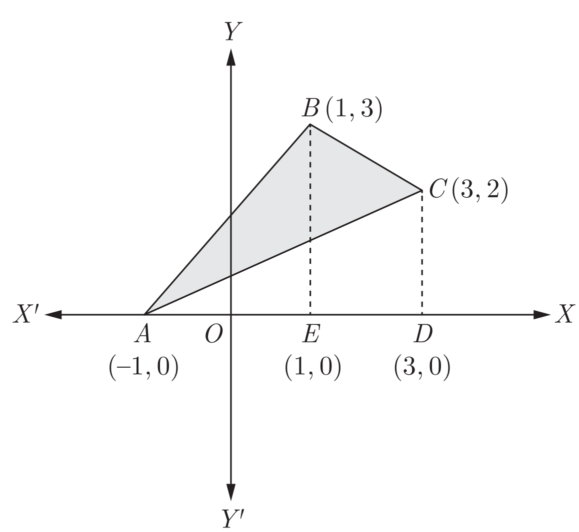
$$= \frac{8}{3} \times 8 = \frac{64}{3} \text{ sq units}$$

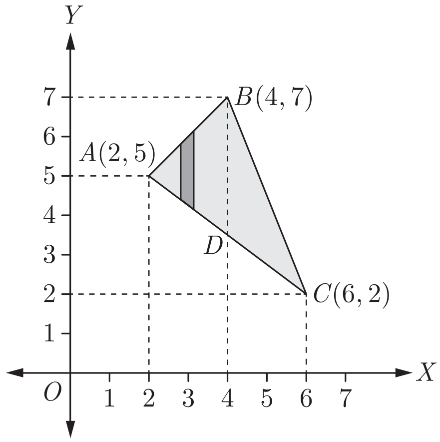
Hence, the required area is $\frac{64}{3}$ sq units.

27. Using integration, find the area of the region bounded by the triangle whose vertices are $(-1,0)$, $(1,3)$ and $(3,2)$.

Sol : OD 2009

Let $A(-1,0)$, $B(1,3)$ and $(3,2)$ be the vertices of the ΔABC.





Equation of the line AB is given by

$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1} \Rightarrow \frac{y-5}{7-5} = \frac{x-2}{4-2}$$

or $\quad y = x+3 \qquad$ ...(i)

Similarly, equation of the line BC is

$$\frac{y-7}{2-7} = \frac{x-4}{6-4} \Rightarrow y = \frac{1}{2}(-5x+34)$$

or $\quad y = -\frac{5x}{2}+17 \qquad$ ...(ii)

Equation of line AC is

$$\frac{y-5}{2-5} = \frac{x-2}{6-2}$$

or $\quad y = \frac{1}{4}(-3x+26)$

$$y = \frac{-3}{4}x+\frac{13}{2} \qquad \text{...(iii)}$$

Now, required area,

= (Area under line segment AB)
+ (Area of under line segment BC)
− (Area under line segment AC)

Here we considered the small strip area trip dx parallel to y axis. Thus

$$A = \int_2^4 (x+3)\,dx + \int_4^6 \left(-\frac{5x}{2}+17\right)dx - \int_2^6 \left(-\frac{3}{4}x+\frac{13}{2}\right)dx$$

$$= \left[\frac{x^2}{2}+3x\right]_2^4 + \left[-\frac{5x^2}{4}+17x\right]_4^6 - \left[-\frac{3x^2}{8}+\frac{13}{2}x\right]_2^6$$

$$= 12+9-14 = 7 \text{ sq units}$$

30. Sketch the graph of $y=|x+3|$ and evaluate the area under the curve $y=|x+3|$ above x-axis and between $x=-6$ to $x=0$.

Sol : OD 2011

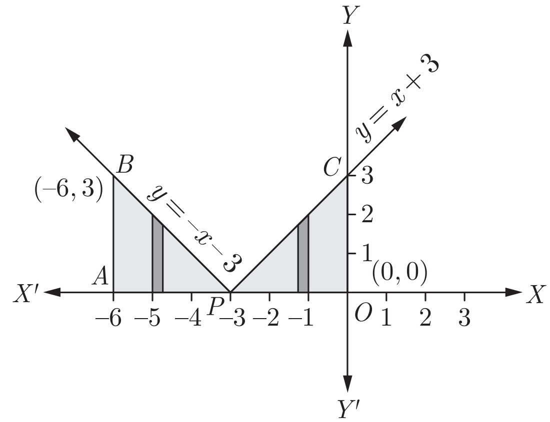
The given $y=|x+3|$ can be written as

$$y = |x+3| = \begin{cases} x+3, & if\ x+3 \geq 0 \\ -(x+3), & if\ x+3 < 0 \end{cases}$$

$$y = |x+3| = \begin{cases} x+3, & if\ x \geq -3 \\ -x-3, & if\ x < -3 \end{cases}$$

So, we have $y=x+3$ for $x \geq -3$ and $y=-x-3$ for $x<-3$.

A sketch of $y=|x+3|$ is shown below:



Here, $y=x+3$ is the straight line which cuts X and Y-axis at $(-3,0)$ and $(0,3)$, respectively. Thus $y=x+3$ for $x \geq -3$ is the parts of line which lies on the right side of $x=-3$.

Similarly, $y=-x-3$, $x<-3$ is the part of line $y=-x-3$, which lies on left side of $x=-3$.

Clearly, required area

= Area of region $ABPA$ + Area of region $PCOP$

$$= \int_{-6}^{-3}(-x-3)\,dx + \int_{-3}^{0}(x+3)\,dx$$

$$= \left[\frac{x^2}{2}-3x\right]_{-6}^{-3} + \left[\frac{x^2}{2}+3x\right]_{-3}^{0}$$

$$= \left[\left(-\frac{9}{2}+9\right)-(-18+18)\right]\left[0-\left(\frac{9}{2}-9\right)\right]$$

$$= \left(-\frac{9}{2}-\frac{9}{2}\right)+(9+9)$$

$$= 18-9 = 9 \text{ sq units}$$

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CHAPTER 9

DIFFERENTIAL EQUATIONS

OBJECTIVE QUESTIONS

1. The general solution of the differential equation $\frac{dy}{dx} = e^{x+y}$ is :

(a) $e^x + e^{-y} = c$ (b) $e^{-x} + e^{-y} = c$

(c) $e^{x+y} = c$ (d) $2e^{x+y} = c$

Sol : OD 2024

We have
$$\frac{dy}{dx} = e^{x+y}$$
$$= e^x e^y$$
$$\frac{dy}{e^y} = e^x\,dx$$
$$e^{-y}\,dy = e^x\,dx$$

Integrating both sides, we get
$$-e^{-y} = e^x + k$$
$$e^x + e^{-y} = c$$

Thus (a) is correct option.

2. Degree of the differential equation $\sin x + \cos\left(\frac{dy}{dx}\right) = y^2$ is

(a) 2 (b) 1

(c) not defined (d) 0

Sol : OD 2023

We have $\sin x + \cos\left(\frac{dy}{dx}\right) = y^2$

The order of the differential equation is 1.

The given differential equation cannot be written as a polynomial equation. The degree of the differential equation is not defined.

Thus (c) is correct option.

3. The integrating factor of the differential equation $(1-y^2)\frac{dx}{dy} + yx = ay$, $(-1 < y < 1)$ is

(a) $\frac{1}{y^2-1}$ (b) $\frac{1}{\sqrt{y^2-1}}$

(c) $\frac{1}{1-y^2}$ (d) $\frac{1}{\sqrt{1-y^2}}$

Sol : OD 2023

We have $(1-y^2)\frac{dx}{dy} + yx = ay$ $(-1 < y < 1)$
$$\frac{dx}{dy} + \frac{yx}{1-y^2} = \frac{ay}{1-y^2}$$
$$\frac{dx}{dy} + \left(\frac{y}{1-y^2}\right)x = \frac{ay}{1-y^2}$$

The given equation in form of $\frac{dx}{dy} + Px = Q$. Thus
$$\text{I.F.} = e^{\int P dy}$$
$$= e^{\int\left(\frac{y}{1-y^2}\right)dy}$$
$$= e^{\int -\frac{1}{2}\left(\frac{-2y}{1-y^2}\right)dy}$$
$$= e^{-\frac{1}{2}\log(1-y^2)}$$
$$= e^{\log\frac{1}{\sqrt{1-y^2}}}$$
$$= e^{\log\frac{1}{\sqrt{1-y^2}}}$$

Thus (d) is correct option.

1. Order of the equation $\left(1 + 5\frac{dy}{dx}\right)^{3/2} = 10\frac{d^3y}{dx^3}$ is

(a) 2 (b) 3

(c) 1 (d) 0

Sol : Delhi 2017

We have
$$\left(1 + 5\frac{dy}{dx}\right)^{3/2} = 10\frac{d^3y}{dx^3}$$

Squaring both sides, we get
$$\left(1 + 5\frac{dy}{dx}\right)^{3} = 100\left(\frac{d^3y}{dx^3}\right)^2$$
$$1 + 125\left(\frac{dy}{dx}\right)^3 + 15\frac{dy}{dx}\left(1 + 5\frac{dy}{dx}\right) = 100\left(\frac{d^3y}{dx^3}\right)^2$$

Clearly, the order of highest derivative occurring in the differential equation is 3. Hence, the order of given differential equation is 3.

Thus (b) is correct option.

2. The order and degree of the differential equation $y = x\frac{dy}{dx} + \frac{2}{\frac{dy}{dx}}$ are

(a) 1, 2 (b) 1, 3

(c) 2, 1 (d) 1, 1

(c) $e^{-y} = -e^{-x} - e^{x} - x^2 + C$

(d) $e^{y} = e^{-x} + e^{x} + x^2 + C$

Sol : OD 2012, Comp 2010

We have
$$\begin{aligned} \frac{dy}{dx} &= e^y(e^x + e^{-x} + 2x) \\ \frac{dy}{e^y} &= dx(e^x + e^{-x} + 2x) \end{aligned}$$

Integrating both sides, we have
$$\begin{aligned} \int \frac{dy}{e^y} &= \int dx(e^x + e^{-x} + 2x) \\ e^{-y} &= e^x - e^{-x} + x^2 + C \\ e^{-y} &= e^{-x} - e^x - x^2 + C \end{aligned}$$

Thus (b) is correct option.

9. Solution of the differential equation $xdy - ydx = 0$ represents a

(a) parabola (b) circle

(c) hyperbola (d) straight line

Sol : Foreign 2009

We have
$$\begin{aligned} x\,dy &= y\,dx \\ \frac{dy}{y} &= \frac{dx}{x} \\ \int \frac{dy}{y} &= \int \frac{dx}{x} \\ \log_e y &= \log_e x + \log_e C \\ y &= Cx \end{aligned}$$

which is a straight line.

10. The solution of $\frac{dy}{dx} = \frac{ax+g}{by+f}$ represents a circle, when

(a) $a = b$ (b) $a = -b$

(c) $a = -2b$ (d) $a = 2b$

Sol : Foreign 2018

We have
$$\begin{aligned} \frac{dy}{dx} &= \frac{ax+g}{by+f} \\ (by+f)\,dy &= (ax+g)\,dx \end{aligned}$$

Integrating both sides, we have
$$\begin{aligned} \frac{by^2}{2} + fy &= \frac{ax^2}{2} + gx + C \\ ax^2 - by^2 + 2gx - 2fy + C &= 0 \end{aligned}$$

which represents a circle, if $a = -b$.

Thus (b) is correct option.

11. An integrating factor of the deferential equation
$$x\frac{dy}{dx} + y\log x = xe^x x^{-\frac{1}{2}\log x} (x > 0) \text{ is}$$

(a) $x^{\log x}$ (b) $(\sqrt{x})^{\log x}$

(c) $(\sqrt{e})^{(\log x)^2}$ (d) e^{x^2}

Sol : Delhi 2010, OD 2007

We have
$$\begin{aligned} x\frac{dy}{dx} + y\log x &= xe^x x^{-\frac{1}{2}\log x} \\ \frac{dy}{dx} + y\frac{1}{x}\log x &= e^x x^{-(1/2)\log x} \\ \text{IF} &= e^{\frac{1}{x}\log x\,dx} \\ &= e^{\frac{(\log x)^2}{2}} \\ &= (\sqrt{e})^{(\log x)^2} \end{aligned}$$

Thus (c) is correct option.

12. Integrating factor (IF) of the following differential equation is
$$\frac{dy}{dx} - \frac{3x^2y}{1+x^3} = \frac{\sin^2(x)}{1+x}$$

(a) e^{1+x^3} (b) $\log(1+x^3)$

(c) $1+x^3$ (d) $\frac{1}{1+x^3}$

Sol : OD 2017, Delhi 2015

We have
$$\frac{dy}{dx} - \frac{3x^2y}{1+x^3} = \frac{\sin^2 x}{1+x}$$

From the given equation,
$$\begin{aligned} P &= -\frac{3x^2}{1+x^3},\ Q = \frac{\sin^2 x}{1+x} \\ \text{IF} &= e^{\int P\,dx} = e^{\int \frac{-3x^2}{1+x^3}dx} \end{aligned}$$

Substituting $1 + x^3 = t \Rightarrow 3x^2\,dx = dt$ we get
$$\begin{aligned} \text{IF} &= e^{\int -\frac{1}{t}dt} = e^{-\log t} = e^{-\log}(1+x^3) \\ &= e^{\log}(1+x^3)^{-1} \end{aligned}$$

Hence, $\text{IF} = (1+x^3)^{-1} = \frac{1}{1+x^3}$

Thus (d) is correct option.

13. The order of the DE $\left[1 + \left(\frac{dy}{dx}\right)\right]^{3/2} = \frac{d^2y}{dx^2}$ is

(a) 1 (b) 2

(c) 3 (d) 4

Sol : OD 2007

Given differential equation can be rewritten as
$$\left[1 + \frac{dy}{dx}\right]^3 = \left(\frac{d^2y}{dx^2}\right)^2$$

Here, degree is 2.

Thus (b) is correct option.

14. The degree of the differential equation $\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^2 = x\sin\left(\frac{d^2y}{dx^2}\right)$

$$\log\left(\frac{y}{cx}\right) = x$$

$$y = cxe^x$$

Thus (a) is correct option.

20. The order of the differential equation of all conics whose centre lie at the origin is given by

(a) 2 (b) 3

(c) 4 (d) 5

Sol : Foreign 2015

The general equation of all conics whose centre lie at the origin is

$$ax^2 + 2hxy + by^2 = 1$$

Since, it has three arbitrary constants. Hence, order of the differential equation obtained is 3.

Thus (b) is correct option.

21. The order of the differential equation of all circles of radius a is

(a) 2 (b) 3

(c) 4 (d) 1

Sol : Comp 2012

The general equation of all circles of radius a is

$$(x-h)^2 + (y-k)^2 = a^2.$$

Since, it has two arbitrary constants.

Hence, order of the differential equation obtained is 2.

Thus (a) is correct option.

22. Solution of $(x^2 + xy)\,dy = (x^2 + y^2)\,dx$ is

(a) $\log x = \log(x-y) + \frac{y}{x} + c$

(b) $\log x = 2\log(x-y) + \frac{y}{x} + c$

(c) $\log x = \log(x-y) + \frac{x}{y} + c$

(d) none of the above

Sol : SQP 2020

We have $(x^2 + xy)\,dy = (x^2 + y^2)\,dx$

$$\frac{dy}{dx} = \frac{x^2 + y^2}{x^2 + xy}$$

This is homogeneous equation. Thus putting $y = vx$ we have

$$v + x\frac{dv}{dx} = \frac{1+v^2}{1+v}$$

$$x\frac{dv}{dx} = \frac{1+v^2}{1+v} - v = \frac{1-v}{1+v}$$

$$\left(-1 + \frac{2}{1-v}\right)dv = \frac{dx}{x}$$

Integrating both sides, we get

$$-\frac{y}{x} - 2\log\left(1 - \frac{y}{x}\right) + c = \log x$$

$$\frac{y}{x} + 2\log(x-y) + c = \log x$$

Thus (b) is correct option.

23. If m and n are the order and degree of the differential equation

$$\left(\frac{d^2y}{dx^2}\right)^5 + 4\frac{\left(\frac{d^2y}{dx^2}\right)^3}{\frac{d^3y}{dx^3}} + \frac{d^3y}{dx^3} = x^2 - 1, \text{ then}$$

(a) $m = 3, n = 2$ (b) $m = 3, n = 3$

(c) $m = 3, n = 5$ (d) $m = 3, n = 1$

Sol : SQP 2016, OD 2009

Given differential equation can be rewritten as

$$\left(\frac{d^2y}{dx^2}\right)^5\frac{d^3y}{dx^3} + 4\left(\frac{d^2y}{dx^2}\right)^3 + \left(\frac{d^3y}{dx^3}\right)^2 = (x^2-1)\frac{d^2y}{dx^2}$$

Here, order $m = 3$

degree $n = 2$

Thus (a) is correct option.

24. The solution of the differential equation $2x\frac{dy}{dx} - y = 3$ represents

(a) straight line (b) circle

(c) parabola (d) ellipse

Sol : Delhi 2010

We have $2x\frac{dy}{dx} = y + 3$

$$2\frac{dy}{y+3} = \frac{dx}{x}$$

$$2\log(y+3) = \log x + \log c$$

$$(y+3)^2 = cx$$

which is an equation of parabola.

Thus (c) is correct option.

25. The integrating factor of the differential equation $\frac{dy}{dx}(x\log x) + y = 2\log x$ is given by

(a) e^x (b) $\log x$

(c) $\log\log x$ (d) x

Sol : Delhi 2014

The given differential equation can be rewritten as

$$\frac{dy}{dx} + \frac{y}{x\log x} = \frac{2}{x}$$

It is of the form $\frac{dy}{dx} + Py = Q$. Here

30. The number of solutions of $y' = \frac{y+1}{x-1}, y(1) = 2$ is

(a) zero (b) one

(c) two (d) infinite

Sol : Delhi 2010

We have
$$\frac{dy}{dx} = \frac{y+1}{x-1}$$
$$\frac{dy}{y+1} = \frac{dx}{x-1}$$
Integrating, we get
$$\log(y+1) = \log(x-1) + \log c$$
$$y+1 = c(x-1)$$
For $x = 1$, $y = 2$
$$3 = c(0)$$
$$3 = 0 \qquad \text{which is not possible.}$$
Hence, given equation has zero solution.

Thus (a) is correct option.

31. The general solution of the differential equation $\frac{dy}{dx} + \sin\frac{x+y}{2} = \sin\frac{x-y}{2}$ is

(a) $\log\tan\left(\frac{y}{2}\right) = c - 2\sin x$

(b) $\log\tan\left(\frac{y}{4}\right) = c - 2\sin\frac{x}{2}$

(c) $\log\tan\left(\frac{y}{2} + \frac{\pi}{4}\right) = c - 2\sin x$

(d) $\log\tan\left(\frac{y}{4} + \frac{\pi}{4}\right) = c - 2\sin\left(\frac{x}{2}\right)$

Sol : Comp 2008

We have
$$\frac{dy}{dx} + \sin\frac{x+y}{2} = \sin\frac{x-y}{2}$$
$$\frac{dy}{dx} = \sin\left(\frac{x-y}{2}\right) - \sin\left(\frac{x+y}{2}\right)$$
$$= -2\cos\frac{x}{2}\sin\frac{y}{2}$$
$$\int\frac{1}{2}\operatorname{cosec}\frac{y}{2}\,dy = -\int\cos\frac{x}{2}\,dx$$
$$\log\tan\frac{y}{4} = -\frac{\sin\frac{x}{2}}{\frac{1}{2}}x + c$$
$$\log\tan\left(\frac{y}{4}\right) = c - 2\sin\frac{x}{2}$$
Thus (b) is correct option.

32. Solution of the equation, $x^2\frac{d^2y}{dx^2} = \log x$, when $x = 1, y = 0$ and $\frac{dy}{dx} = -1$

(a) $y = \frac{1}{2}(\log x)^2 + \log x$

(b) $y = \frac{1}{2}(\log x)^2 - \log x$

(c) $y = -\frac{1}{2}(\log x)^2 + \log x$

(d) $y = -\frac{1}{2}(\log x)^2 - \log x$

Sol : OD 2009

We have
$$x^2\frac{d^2y}{dx^2} = \log x$$
$$\frac{d^2y}{dx^2} = \frac{\log x}{x^2}$$
$$\frac{dy}{dx} = -\frac{\log x}{x} + \int\frac{1}{x^2}dx + c$$
$$\frac{dy}{dx} = -\frac{\log x}{x} - \frac{1}{x} + c$$
Thus (d) is correct option.

33. Which of the following is a solution of the differential equation $\left(\frac{dy}{dx}\right)^2 - x\frac{dy}{dx} + y = 0$

(a) $y = 2x - 4$ (b) $y = 2x^2 - 4$

(c) $y = 2$ (d) $y = 2x$

Sol : Delhi 2015, OD 2011

We have
$$\left(\frac{dy}{dx}\right)^2 - x\frac{dy}{dx} + y = 0$$
Taking option (a) we have
$$y = 2x - 4$$
$$\frac{dy}{dx} = 2$$
Substituting above in given equation we get
$$(2)^2 - x(2) + 2x - 4 = 0$$
$$0 = 0$$
Thus (a) is correct option.

34. The differential equation of all non-horizontal lines in a plane is

(a) $\frac{d^2y}{dx^2} = 0$ (b) $\frac{d^2x}{dy^2} = 0$

(c) $\frac{dy}{dx} = 0$ (d) $\frac{dx}{dy} = 0$

Sol : SQP 2015

The general equation of all non-horizontial lines in a plane is
$$ax + by = 1, \qquad \text{where } a \neq 0$$
Differentiating w.r.t x we have
$$a\frac{dx}{dy} + b = 0$$
$$a\frac{dx}{dy} + b = 0$$
$$a\frac{d^2x}{dy^2} = 0$$

Sol : Foreign 2012, Delhi 2008

We have $(x^2 - xy)\,dy = (xy + y^2)\,dx$

$$\frac{dy}{dx} = \frac{xy + y^2}{x^2 - xy}$$

This is homogeneous equation. Thus putting $y = vx$ we have

$$v + x\frac{dv}{dx} = \frac{v + v^2}{1 - v}$$

$$x\frac{dv}{dx} = \frac{v + v^2}{1 - v} - v = \frac{2v^2}{1 - v}$$

$$\frac{1 - v}{v^2}\,dv = 2\frac{dx}{x}$$

$$\left(\frac{1}{v^2} - \frac{1}{v}\right)dv = 2\frac{dx}{x}$$

$$-\frac{1}{v} - \log v = 2\log x + \text{constant}$$

$$-\frac{x}{y} - \log\frac{y}{x} = \log x^2 + \log c_1$$

$$-\frac{x}{y} = \log\left(\frac{y}{x}\cdot x^2\cdot c_1\right) = \log c_1 xy$$

$$xy = \frac{1}{c_1}e^{-x/y} = ce^{-x/y} \quad \left(\text{take}, c_1 = \frac{1}{c}\right)$$

Thus (b) is correct option.

41. The solution of the differential equation $\frac{dy}{dx} + \frac{y}{x} = x^2$ is

(a) $x + y = \frac{x^2}{2} + c$ (b) $x - y = \frac{1}{3}x^3 + c$

(c) $xy = \frac{1}{4}x^4 + c$ (d) $y - x = \frac{1}{4}x^4 + c$

Sol : Comp 2009

We have $\frac{dy}{dx} + \frac{y}{x} = x^2$

Here, $P = \frac{1}{x}$ and $Q = x^2$

Now
$$\text{IF} = e^{\int\frac{1}{x}dx} = e^{\log x} = x$$

Solution is,
$$yx = \int x^2\cdot x\,dx = \frac{x^4}{4} + c$$

Thus (c) is correct option.

42. If $dy/dx = e^{-2y}$ and $y = 0$, when $x = 5$, then the value of x, when $y = 3$ is

(a) e^5 (b) $e^6 + 1$

(c) $\frac{e^6 + 9}{2}$ (d) $\log_e 6$

Sol : Delhi 2017, OD 2015

We have $\frac{dy}{dx} = e^{-2y}$

$$\int e^{2y}\,dy = \int dx + c$$

$$\frac{e^{2y}}{2} = x + c \qquad (1)$$

When $x = 5$ and $y = 0$ we have

$$\frac{1}{2} = 5 + c$$

$$c = -\frac{9}{2}$$

Thus eq(1) becomes,

$$\frac{e^{2y}}{2} = x - \frac{9}{2}$$

When $y = 3$, $\frac{e^6}{2} = x - \frac{9}{2}$

$$x = \frac{9 + e^6}{2}$$

Thus (c) is correct option.

43. Equation to the curve through (2, 1) whose slope at the point (x, y) is $\frac{x^2 + y^2}{2xy}$, is

(a) $2(x^2 - y^2) = 3x$ (b) $2(y^2 - x^2) = 6y$

(c) $x(x^2 - y^2) = 6$ (d) none of these

Sol : SQP 2018

We have $\frac{dy}{dx} = \frac{x^2 + y^2}{2xy}$

This is homogeneous equation. Thus putting $y = vx$ we have

$$v + x\frac{dv}{dx} = \frac{1 + v^2}{2v}$$

$$v + x\frac{dv}{dx} = \frac{1 + v^2}{2v}$$

$$x\frac{dv}{dx} = \frac{1 + v^2}{2v} - v$$

$$\frac{2v}{1 - v^2}\,dv = \frac{dv}{dx}$$

$$\log(1 - v^2) + \log x = \log c$$

$$\log x\left(1 - \frac{y^2}{x^2}\right) = \log c$$

$$x^2 - y^2 = cx \qquad (1)$$

It passes through (2, 1), we get,

$$4 - 1 = 2c \Rightarrow c = \frac{3}{2}$$

Thus eq(1) becomes,

$$2(x^2 - y^2) = 3x$$

Thus (a) is correct option.

$$\frac{dy}{dx} = -\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2}\,dy + \sqrt{1-y^2}\,dx = 0$$

49. The solution of $e^{dy/dx} = x+1, y(0) = 3$, is

(a) $y = x\log x - x + 2$

(b) $y = (x+1)\log(x+1) - x + 3$

(c) $y = (x+1)\log(x+1) + x + 3$

(d) $y = x\log x + x + 3$

Sol : Delhi 2010

We have $\quad e^{dy/dx} = x+1$

$$\frac{dy}{dx} = \log(x+1)$$

$$y = (x+1)\log(x+1) - x + c$$

At $x = 0$, $y = 3$ then $c = 3$.

Thus $\quad y = (x+1)\log(x+1) - x + 3$

Thus (b) is correct option.

50. Order of the differential equation of the family of all concentric circles centred at (h, k), is

(a) 2 (b) 3

(c) 1 (d) 4

Sol : OD 2008

We have $\quad (x-h)^2 + (y-k)^2 = r^2$

Here only one arbitrary constant r.

So, order of differential equation is 1.

Thus (c) is correct option.

51. Assertion : The general solution of $\frac{dy}{dx} + y = 1$ is $ye^x = e^x + c$

Reason : The number of arbitrary constants is in the general solution of the differential equation is equal to the order of differential equation.

(a) Assertion is true, reason is true, reason is a correct explanation for assertion.

(b) Assertion is true,reason is true, reason is not a correct explanation for assertion.

(c) Assertion is true, reason is false.

(d) Assertion is false, reason is true.

Sol : Comp 2015, OD 2007

Given that, $\quad \frac{dy}{dx} + y = 1$

$$\frac{dy}{dx} = 1 - y$$

$$\frac{dy}{1-y} = dx$$

$$\int \frac{dy}{1-y} = \int dx$$

$$-\log(1-y) = x$$

$$1 - y = e^{-x}$$

$$ye^x = e^x + c$$

Since, order of differential equation is the number of arbitrary Constants, so given Reason is true.

Hence, Both Assertion and reason are true but reason is not a correct explanation for assertion.

Thus (b) is correct option.

52. Assertion : The elimination of four arbitrary constants in $y = (c_1 + c_2 + c_3 e^{c_4})x$ results into a differential equation of the first order $x\frac{dy}{dx} = y$

Reason : Elimination of n arbitrary constants requires in general a differential equation of the n^{th} order.

(a) Assertion is true, reason is true, reason is a correct explanation for assertion.

(b) Assertion is true,reason is true, reason is not a correct explanation for assertion.

(c) Assertion is true, reason is false.

(d) Assertion is false, reason is true.

Sol : Delhi 2007

Here given Reason is standard property.

So, given Reason is true.

Let $\quad c_1 + c_2 + c_3 e^{c_4} = \text{A (constant)}$

Then, $\quad y = \text{A}x$

Differentiating, we get

$$\frac{dy}{dx} = \text{A}$$

So given equation can be expressed as

$$y = x\frac{dy}{dx}$$

$$x\frac{dy}{dx} = y$$

Hence, both assertion and reason are true and reason is correct explanation for assertion.

Thus (a) is correct option.

53. Assertion: The equation of curve passing through (3, 9) which satisfies differential equation $\frac{dy}{dx} = x + \frac{1}{x^2}$ is $6xy = 3x^3 + 29x - 6$

Reason: The solution of differential equation $\left(\frac{dy}{dx}\right)^2 - \left(\frac{dy}{dx}\right)(e^x + e^{-x}) + 1 = 0$ is $y = c_1 e^x + c_2 e^{-x}$

(a) Assertion is true, Reason is true; Reason is a correct explanation for Assertion.

Sol : OD 2019

We have $$y = ae^{2x} + 5 \qquad ...(1)$$

Differentiating w.r.t. x, we get

$$y' = ae^{2x} \cdot 2$$

$$ae^{2x} = \frac{y'}{2}$$

$$y - 5 = \frac{y'}{2} \qquad y = ae^{2x} + 5$$

$$2y - 10 = y',$$

$$y' - 2y + 10 = 0$$

which is the required equation.

60. Write the sum of the order and degree of the differential equation

$$\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^3 + x^4 = 0.$$

Sol : Foreign 2015, Comp 2009

We have $$\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^3 + x^4 = 0.$$

Here, we see that the highest order derivative is $\frac{d^2y}{dx^2}$, whose degree is 2.

Sum of the order and degree $= 2 + 2 = 4$

61. Write the differential equation obtained by eliminating the arbitrary constant C is the equation representing the family of curves $xy = C\cos x$.

Sol : Delhi 2015

We have $$xy = C\cos x \qquad ...(1)$$

Differentiating both sides w.r.t. x, we get

$$1 \cdot y + x\frac{dy}{dx} = C(-\sin x)$$

$$y + x\frac{dy}{dx} = -\left(\frac{xy}{\cos x}\right)\sin x \qquad xy = C\cos x$$

$$y + x\frac{dy}{dx} + xy\tan x = 0$$

62. Write the degree of the differential equation

$$\left(\frac{dy}{dx}\right)^4 + 3x\frac{d^2y}{dx^2} = 0.$$

Sol : Delhi 2013

We have $$\left(\frac{dy}{dx}\right)^4 + 3x\frac{d^2y}{dx^2} = 0$$

Here, highest order derivative is $\frac{d^2y}{dx^2}$, whose degree is one. So, the degree of differential equation is 1.

63. Write the degree of the differential equation

$$x\left(\frac{d^2y}{dx^2}\right)^3 + y\left(\frac{dy}{dx}\right)^4 + x^3 = 0$$

Sol : Delhi 2013

We have $$x\left(\frac{d^2y}{dx^2}\right)^3 + y\left(\frac{dy}{dx}\right)^4 + x^3 = 0$$

Here, highest order derivative is $\frac{d^2y}{dx^2}$, whose degree is three. So, the degree of differential equation is 3.

64. Write the degree of the differential equation

$$\left(\frac{dy}{dx}\right)^4 + 3y\frac{d^2y}{dx^2} = 0.$$

Sol : Foreign 2013

We have $$\left(\frac{dy}{dx}\right)^4 + 3y\frac{d^2y}{dx^2} = 0.$$

Here, highest order derivative is $\frac{d^2y}{dx^2}$, whose degree is one. So, the degree of differential equation is 1.

65. Write the differential equation representing the family of curves $y = mx$, where m is the arbitrary constant.

Sol : OD 2013

We have $$y = mx. \qquad ...(1)$$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = m$$

Substituting $m = \frac{dy}{dx}$ in Eq. (1), we get

$$y = x\frac{dy}{dx}$$

which is the required differential equation.

66. What is the degree of the following differential equation?

$$5x\left(\frac{dy}{dx}\right)^2 - \frac{d^2y}{dx^2} - 6y = \log x$$

Sol : Delhi 2010

Here, highest order derivative is $\frac{d^2y}{dx^2}$, whose degree is one. So, the degree of differential equation is 1.

67. Write the sum of the order and degree of the differential equation $\frac{d}{dx}\left\{\left(\frac{dy}{dx}\right)^3\right\} = 0$.

Sol : OD 2015

The degree of the differential equation is the degree of the highest order derivative, when differential coefficients are made free from radicals and fractions sign.

We have $$\frac{d}{dx}\left\{\left(\frac{dy}{dx}\right)^3\right\} = 0$$

$$3\left(\frac{dy}{dx}\right)^{3-1}\frac{d}{dx}\left(\frac{dy}{dx}\right) = 0$$

$$3\left(\frac{dy}{dx}\right)^2\frac{d^2y}{dx^2} = 0$$

Here, order is 2 and degree is 1.

Sum of the order and degree $= 2 + 1 = 3$

or $$\frac{d^2V}{dr^2} = -\frac{2}{r}\frac{dV}{dr}$$

Thus, the required differential equation is

$$\frac{d^2V}{dr^2} + \frac{2}{r}\frac{dV}{dr} = 0.$$

74. Form the differential equation representing the family of curves $y = e^{2x}(a + bx)$, where 'a' and 'b' are arbitrary constants.

Sol : Delhi 2019, OD 2017

We have, $$y = e^{2x}(a + bx) \quad ...(1)$$

Differentiating above equation wrt x, we have

$$\begin{aligned}\frac{dy}{dx} &= e^{2x}\frac{d}{dx}(a+bx) + (a+bx)\frac{d}{dx}e^{2x}\\ &= e^{2x}(b) + (a+bx)2\cdot e^{2x}\\ &= b\cdot e^{2x} + 2\cdot e^{2x}(a+bx)\\ &= be^{2x} + 2\cdot y\end{aligned}$$

or $$y' = 2y + be^{2x} \quad ...(2)$$

Again differentiating eq (2) wrt x, we get

$$y'' = 2y' + 2be^{2x} \quad ...(3)$$

Multiplying Eq. (2) by 2 and then subtracting from Eq. (3), we get

$$\begin{aligned}y'' - 2y' &= 2y' - 4y\\ y'' &= 2y' + 2y' - 4y\\ y'' - 4y' + 4y &= 0,\end{aligned}$$

which is the required equation.

75. Find the differential equation of the family of curves $y = Ae^{2x} + Be^{-2x}$, where A and B are arbitrary constants.

Sol : OD 2019

We have $$y = A\cdot e^{2x} + B\cdot e^{-2x} \quad ...(1)$$

Differentiating above equation wrt x, we have

$$\frac{dy}{dx} = 2A\cdot e^{2x} - 2B\cdot e^{-2x} \quad ...(2)$$

Again differentiating eq. (2) w.r.t. x, we get

$$\begin{aligned}\frac{d^2y}{dx^2} &= 4Ae^{2x} + 4Be^{-2x}\\ &= 4(Ae^{2x} + Be^{-2x})\\ \frac{d^2y}{dx^2} &= 4y \Rightarrow \frac{d^2y}{dx^2} - 4y = 0,\end{aligned}$$

which is the required equation.

76. Find the differential equation representing the family of curves $y = ae^{bx+5}$, where a and b are arbitrary constants.

Sol : OD 2018

We have, $$y = ae^{bx+5}$$

Differentiating above equation wrt x, we have

$$\begin{aligned}\frac{dy}{dx} &= ae^{bx+5}\cdot b\\ &= y\cdot b\end{aligned}$$

Again, differentiating above wrt x, we get

$$\begin{aligned}\frac{d^2y}{dx^2} &= b\frac{dy}{dx}\\ &= \left(\frac{1}{y}\cdot\frac{dy}{dx}\right)\left(\frac{dy}{dx}\right) \quad \text{[using Eq. (1)]}\\ \frac{d^2y}{dx^2} &= \frac{1}{y}\left(\frac{dy}{dx}\right)^2\end{aligned}$$

or $$y\left(\frac{d^2y}{dx^2}\right) - \left(\frac{dy}{dx}\right)^2 = 0,$$

which is the required differentiating equation.

77. Find the integrating factor of DE $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right)\frac{dx}{dy} = 1$.

Sol : Delhi 2015

We have $$\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right)\frac{dx}{dy} = 1$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\\ \frac{dy}{dx} + \frac{y}{\sqrt{x}} &= \frac{e^{-2\sqrt{x}}}{\sqrt{x}}\end{aligned}$$

which is linear differential equation of the form

$$\frac{dy}{dx} + Py = Q, \text{ were } P = \frac{1}{\sqrt{x}} \text{ and } Q = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}.$$

Integrating Factor, $$\begin{aligned}\text{IF} &= e^{\int Pdx}\\ &= e^{\int\frac{1}{\sqrt{x}}dx} = e^{2\sqrt{x}}\end{aligned}$$

78. Write the integrating factor of the following differential equation.

$$(1 + y^2) + (2xy - \cot y)\frac{dy}{dx} = 0.$$

Sol : OD 2015, SQP 2011

We have $$(1 + y^2) + (2xy - \cot y)\frac{dy}{dx} = 0.$$

The above equation can be rewritten as

$$\begin{aligned}(\cot y - 2xy)\frac{dy}{dx} &= 1 + y^2\\ \frac{\cot y - 2xy}{(1+y^2)} &= \frac{dx}{dy}\\ \frac{dx}{dy} &= \frac{\cot y}{1+y^2} - \frac{2xy}{1+y^2}\\ \frac{dx}{dy} + \frac{2y}{1+y^2}\cdot x &= \frac{\cot y}{1+y^2}\end{aligned}$$

$$= \sec x + \int (\sec^2 x - 1)\, dx$$

or $$y(\sec x + \tan x) = \sec x + \tan x - x + C$$

83. Solve the following differential equation.

$$e^x \tan y\, dx + (1 - e^x)\sec^2 y\, dy = 0$$

Sol : Delhi 2011

We have $$e^x \tan y\, dx + (1 - e^x)\sec^2 y\, dy = 0$$

which can be rewritten as

$$e^x \tan y\, dx = (e^x - 1)\sec^2 y\, dy$$

$$\frac{\sec^2 y}{\tan y} dy = \frac{e^x}{e^x - 1} dx$$

$$\int \frac{\sec^2 y}{\tan y} dy = \int \frac{e^x}{e^x - 1} dx$$

$$\tan|\tan y| = \log|e^x - 1| + C$$

$\int \frac{f'(x)}{f(x)} dx = \log|f(x)| + C$

$$\log|\tan y| - \log|e^x - 1| = C$$

$$\log\left|\frac{\tan y}{e^x - 1}\right| = C$$

$$\frac{\tan y}{e^x - 1} = e^C$$

$$\tan y = e^C(e^x - 1)$$

$$\tan y = C_1(e^x - 1)$$

$$y = \tan^{-1}[C_1(e^x - 1)]$$

LONG ANSWER QUESTIONS

84. Solve the following differential equation:

$x^2 dy + y(x + y) dx = 0$

Sol : OD 2024

We have $$x^2 dy + y(x + y) dx = 0$$

$$x^2 dy = -y(x + y) dx$$

$$\frac{dy}{dx} = \frac{-y(x + y)}{x^2} \qquad ...(1)$$

Substituting $y = vx$ we have

$$\frac{dy}{dx} = v + \frac{xdv}{dx}$$

Substituting this in equation (1) we have,

$$v + \frac{xdv}{dx} = \frac{-vx(x + vx)}{x^2}$$

$$= \frac{-vx^2(1 + v)}{x^2}$$

$$= -v(1 + v)$$

$$v + \frac{xdv}{dx} = -v - v^2$$

$$\frac{xdv}{dx} = -2v - v^2$$

$$-\frac{dv}{2v + v^2} = \frac{dx}{x}$$

$$-\int \frac{dv}{v^2 + 2v} = \int \frac{dx}{x}$$

$$\int \frac{dx}{x} = -\int \frac{dv}{(v + 1)^2 - 1}$$

$$\log|x| = -\int \frac{dv}{(v + 1)^2 - 1}$$

$$= -\frac{1}{2}\log\left|\frac{v + 1 - 1}{v + 1 + 1}\right| + c$$

$$= -\frac{1}{2}\log\left|\frac{v}{v + 2}\right| + c$$

$$= -\frac{1}{2}\log\left|\frac{y/x}{y/x + 2}\right| + c$$

$$= -\frac{1}{2}\log\left|\frac{y}{y + 2x}\right| + c$$

85. Find the general solution of the differential equation: $(xy - x^2)dy = y^2 dx$

Sol : Delhi 2017

We have $$(xy - x^2)dy = y^2 dx$$

Above equation can be written as

$$\frac{dy}{dx} = \frac{y^2}{(xy - x^2)}$$

$$= \frac{\left(\frac{y}{x}\right)^2}{\left(\frac{y}{x} - 1\right)}$$

which is a homogeneous differential equation as

$$\frac{dy}{dx} = F\left(\tfrac{y}{x}\right)$$

Now substituting $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ we have

$$v + x\frac{dv}{dx} = \frac{v^2 x^2}{vx^2 - x^2}$$

$$x\frac{dv}{dx} = \frac{v^2}{v^2 - 1} - v$$

$$x\frac{dv}{dx} = \frac{v^2 - v(v - 1)}{v - 1}$$

$$x\frac{dv}{dx} = \frac{v}{v - 1}$$

$$\left(\frac{v - 1}{v}\right)dv = \frac{dx}{x}$$

$$\int\left(1 - \frac{1}{v}\right)dv = \int \frac{dx}{x}$$

$$v - \ln v = \ln x + \ln C$$

where C is an arbitrary constant.

$$e^y - 2 = -\frac{1}{x+1}$$
$$e^y = \left(2 - \frac{1}{x+1}\right)$$
$$y = \log\left(2 - \frac{1}{x+1}\right)$$

which is the required solution.

89. Solve the following differential equation. $xdy - ydx = \sqrt{x^2+y^2}\,dx$, given that $y = 0$ when $x = 1$.

Sol : Delhi 2019, OD 2011

We have, $$x\,dy - y\,dx = \sqrt{x^2+y^2}\,dx$$
$$(y + \sqrt{x^2+y^2})\,dx = x\,dy$$
$$\frac{dy}{dx} = \frac{y}{x} + \sqrt{1 + \frac{y^2}{x^2}} \quad ...(1)$$

which is a homogeneous differential equation as $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$.

Substituting $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ in eq (1) we have

$$v + x\frac{dv}{dx} = v + \sqrt{1+v^2}$$
$$x\frac{dv}{dx} = \sqrt{1+v^2} \Rightarrow \frac{dv}{\sqrt{1+v^2}} = \frac{dx}{x}$$

Integrating both sides, we have

$$\int \frac{dv}{\sqrt{1+v^2}} = \int \frac{dx}{x}$$
$$\log\left|v + \sqrt{1+v^2}\right| = \log|x| + C$$
$$\log\left|\frac{y}{x} + \sqrt{1+\frac{y^2}{x^2}}\right| = \log|x| + C \qquad v = \frac{y}{x}$$
$$\log\left|\frac{y+\sqrt{x^2+y^2}}{x}\right| - \log|x| = C$$
$$\log\frac{\left|\frac{y+\sqrt{x^2+y^2}}{x}\right|}{x} = C$$
$$\frac{y+\sqrt{x^2+y^2}}{x^2} = e^C \qquad \log y = x \Rightarrow y = e^x$$
$$y + \sqrt{x^2+y^2} = x^2 \cdot e^C$$
$$y + \sqrt{x^2+y^2} = Ax^2 \quad ...(3)$$

where $A = e^C$

Now, as $y = 0$, when $x = 1$

$$0 + \sqrt{1^2+0^2} = A \cdot 1 \Rightarrow A = 1$$

Substituting the value of A, in Eq. (3), we get

$$y + \sqrt{x^2+y^2} = x^2,$$

which is the required solution.

90. Solve the differential equation

$$(1+x^2)\frac{dy}{dx} + 2xy - 4x^2 = 0,$$

subject to the initial condition $y(0) = 0$.

Sol : Delhi 2019

We have, $$(1+x^2)\frac{dy}{dx} + 2xy - 4x^2 = 0$$
$$\frac{dy}{dx} + \frac{2x}{1+x^2}y = \frac{4x^2}{1+x^2}$$

which is the equation of the form

$$\frac{dy}{dx} + Py = Q,$$

where $P = \frac{2x}{1+x^2}$ and $Q = \frac{4x^2}{1+x^2}$

Now, $$\text{IF} = e^{\int \frac{2x}{1+x^2}dx}$$
$$= e^{\log(1+x^2)} = 1+x^2$$

Using $y \cdot (IF) = \int Q \cdot (IF)\,dx + C$ the general solution is

$$y \cdot (1+x^2) = \int (1+x^2)\frac{4x^2}{(1+x^2)}dx + C$$
$$(1+x^2)\,y = \int 4x^2\,dx + C$$
$$(1+x^2)\,y = \frac{4x^3}{3} + C$$
$$y = \frac{4x^3}{3(1+x^2)} + C(1+x^2)^{-1} \quad ...(1)$$

Now, $$y(0) = 0$$
$$0 = \frac{4 \cdot 0^3}{3(1+0^2)} + C(1+0^2)^{-1}$$
$$C = 0$$

Substituting the value of C Eq. (1), we get

$$y = \frac{4x^3}{3(1+x^2)},$$

which is the required solution.

91. Solve the differential equation

$$\frac{dy}{dx} - \frac{2xy}{1+x^2}y = x^2 + 2$$

Sol : Delhi 2019

We have, $$\frac{dy}{dx} - \frac{2x}{1+x^2}y = x^2 + 2 \quad ...(1)$$

This is linear differential equation with

$$P = \frac{-2x}{1+x^2} \text{ and } Q = x^2 + 2$$

Thus $$\text{IF} = e^{\int \frac{-2x}{x^2+1}dx} = e^{-\int \frac{2x}{x^2+1}dx}$$
$$= e^{-\log(x^2+1)}$$
$$= \frac{1}{x^2+1}$$

Also, given that $y = 0$, when $x = \frac{\pi}{3}$.

Substituting $y = 0$ and $x = \frac{\pi}{3}$ in Eq. (1), we get

$$0 \times \sec^2 \frac{\pi}{3} = \sec \frac{\pi}{3} + C$$

$$0 = 2 + C \Rightarrow C = -2$$

Substituting the value of C in Eq. (1), we get

$$y \sec^2 x = \sec x - 2$$

$$y = \cos x - 2\cos^2 x$$

which is the required particular solution of the given differential equation.

95. Solve the differential equation

$$(x^2 - y^2)\,dx + 2xy\,dy = 0.$$

Sol : Foreign 2018

We have $\quad (x^2 - y^2)\,dx + 2xydy = 0,$

which can be re-written as

$$(x^2 - y^2)\,dx = -2xy\,dy$$

$$\frac{dy}{dx} = \frac{x^2 - y^2}{-2xy} = \frac{y^2 - x^2}{2xy}$$

$$= \frac{(\frac{y}{x})^2 - 1}{2(\frac{y}{x})}$$

In RHS, degree of numerator and denominator is same. It is a homogeneous differential equation and can be written as

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

Substituting $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ in eq (1) we have

$$v + x\frac{dv}{dx} = \frac{v^2 - 1}{2v}$$

$$x\frac{dv}{dx} = \frac{v^2 - 1}{2v} - v$$

$$= \frac{v^2 - 1 - 2v^2}{2v}$$

or $\quad x\frac{dv}{dx} = \frac{-v^2 - 1}{2v}$

$$\frac{2v}{v^2 + 1}dv = -\frac{dx}{x}$$

$$\int \frac{2v}{v^2 + 1}dv = -\int \frac{dx}{x}$$

$$\log|v^2 + 1| = -\log|x| + \log C$$

$$\log\left|\frac{y^2}{x^2} + 1\right| = -\log|x| + \log C$$

$$\log\left|\frac{y^2 + x^2}{x^2}\right| = \log C$$

$$\frac{y^2 + x^2}{x} = C \Rightarrow y^2 + x^2 = Cx,$$

which is the required solution.

96. Find the particular solution of the differential equation $(1 + x^2)\frac{dy}{dx} + 2xy = \frac{1}{1 + x^2}$, given that $y = 0$, when $x = 1$.

Sol : OD 2018 C; Foreign 2011

We have $\quad (x^2 + 1)\frac{dy}{dx} + 2xy = \frac{1}{x^2 + 1}$

$$\frac{dy}{dx} + \frac{2x}{x^2 + 1}y = \frac{1}{(x^2 + 1)^2}$$

which is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$, here $P = \frac{2x}{x^2 + 1}$ and $Q = \frac{1}{(x^2 + 1)^2}$

Now, integrating factor,

$$\text{IF} = e^{\int Pdx}$$

$$= e^{\int \frac{2x}{x^2+1}dx}$$

$$= e^{\log|x^2 + 1|}$$

$$= x^2 + 1$$

Using $y \cdot (IF) = \int Q \cdot (IF)\,dx + C$ the general solution is

$$y(x^2 + 1) = \int \frac{1}{(x^2 + 1)^2} \times (x^2 + 1)\,dx + C$$

$$y(x^2 + 1) = \int \frac{1}{x^2 + 1}dx + C$$

$$y(x^2 + 1) = \tan^{-1}x + C \qquad ...(1)$$

when $x = 1$, then $y = 0$

$$0 = \tan^{-1}1 + C \Rightarrow C = \frac{-\pi}{4}$$

Now, $\quad y(x^2 + 1) = \tan^{-1}x - \frac{\pi}{4}$ [from Eq. (1)]

which is the required differential equation.

97. Show that the family of curves for which $\frac{dy}{dx} = \frac{x^2 + y^2}{2xy}$, is given by $x^2 - y^2 = cx$.

Sol : Delhi 2017, OD 2007

We have $\quad \frac{dy}{dx} = \frac{x^2 + y^2}{2xy}$

In RHS, degree of numerator and denominator is same. It is a homogeneous differential equation and can be written as

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

$$= \frac{1 + (\frac{y}{x})^2}{2(\frac{y}{x})}$$

Substituting $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ in eq (1) we have

$$v + x\frac{dv}{dx} = \frac{1 + v^2}{2v}$$

$$x\frac{dv}{dx} = \int \frac{1 + v^2}{2v} - v$$

Also, given that at $x=\frac{\pi}{2}$; $y=1$

Substituting $x=\frac{\pi}{2}$ and $y=1$ in Eq. (1), we get

$$1 = 1+C\cdot\frac{2}{\pi} \Rightarrow C=0$$

Substituting the value of C in Eq. (1), we get

$$y = \sin x$$

which is the required solution of given differential equation.

100. Solve the differential equation

$$(\tan^{-1}x-y)\,dx = (1+x^2)\,dy.$$

Sol : OD 2017

We have, $$(\tan^{-1}x-y)\,dx = (1+x^2)\,dy$$

$$\frac{dy}{dx} = \frac{\tan^{-1}x-y}{(1+x^2)}$$

$$\frac{dy}{dx} = \frac{\tan^{-1}x}{1+x^2}-\frac{1}{1+x^2}y$$

$$\frac{dy}{dx}+\frac{1}{1+x^2}y = \frac{\tan^{-1}x}{1+x^2} \qquad ...(1)$$

which is a linear differential equation of the form $\frac{dy}{dx}+Py=Q$, here $P=\frac{1}{1+x^2}$ and $Q=\frac{\tan^{-1}x}{1+x^2}$

Now, $$\text{IF} = e^{\int P\,dx}=e^{\int\frac{\tan^{-1}x}{1+x^2}dx}=e^{\tan^{-1}x}$$

Using $y\cdot(IF)=\int Q\cdot(IF)\,dx+C$ the general solution is

$$y\cdot e^{\tan^{-1}x} = \int\frac{\tan^{-1}x}{1+x^2}\cdot e^{\tan^{-1}x}\,dx+C$$

Substituting $\tan^{-1}x = t \Rightarrow \frac{1}{1+x^2}dx=dt$ we have

$$ye^{\tan^{-1}x} = \int t\cdot e^t\,dt+C$$

$$= t\cdot e^t-\int 1\cdot e^t\,dt+C$$

$$= t\cdot e^t-e^t+C$$

$$= \tan^{-1}x\cdot e^{\tan^{-1}x}-e^{\tan^{-1}x}+C$$

or $$ye^{\tan^{-1}x} = (\tan^{-1}x-1)\,e^{\tan^{-1}x}+C$$

101. Find the general solution of the differential equation

$$\frac{dy}{dx}-y = \sin x$$

Sol : OD 2017

We have $$\frac{dy}{dx}-y = \sin x$$

which is a linear differential equation of the form

$$\frac{dy}{dx}+Py = Q,\text{ where } P=-1 \text{ and } Q=\sin x$$

$$\text{IF} = e^{\int P\,dx}=e^{\int(-1)\,dx}=e^{-x}$$

Using $y\cdot(IF)=\int Q\cdot(IF)\,dx+C$ the general solution is

$$y\cdot e^{-x} = \int e^{-x}\sin x\,dx+C \qquad ...(1)$$

Let $$I = \int \underset{II}{e^{-x}}\underset{I}{\sin}\,x\cdot dx \qquad ...(2)$$

By using the method of integration by parts, we get

$$I = \sin x\frac{e^{-x}}{(-1)}-\int\cos x\frac{e^{x}}{(-1)}dx$$

$$=-\sin x\,e^{-x}+\int\underset{II}{e^{-x}}\underset{I}{\cos}\,x\,dx$$

Again, by using integration by parts, we get

$$I=-\sin x\,e^{-x}+\cos x\frac{e^{-x}}{(-1)}-\int(-\sin x)\frac{e^{-x}}{(-1)}dx$$

$$=-\sin x\,e^{-x}-\cos x\,e^{-x}-\int e^{-x}\sin x\,dx$$

$$=-\sin x\,e^{-x}-\cos x\,e^{-x}-I \qquad \text{[from Eq. (2)]}$$

$$2I =-e^{-x}(\sin x+\cos x)$$

$$I =-\frac{e^x}{2}(\sin x+\cos x)$$

Thus from Eq. (1), we get

$$y\cdot e^{-x} =-\frac{e^{-x}}{2}(\sin x+\cos x)+C$$

$$y =-\frac{1}{2}(\sin x+\cos x)+Ce^x$$

102. Find the general solution of the following differential equation

$$(1+y^2)+(x-e^{\tan^{-1}y})\frac{dy}{dx} = 0$$

Sol : Delhi 2016

We have $$(1+y^2)+(x-e^{\tan^{-1}y})\frac{dy}{dx} = 0$$

$$(1+y^2)\frac{dx}{dy}+x-e^{\tan^{-1}y} = 0$$

or $$\frac{dy}{dx}+\frac{1}{(1+y^2)}x = \frac{e^{\tan^{-1}y}}{1+y^2}$$

It is linear differential equation of the form

$$\frac{dx}{dy}+Px = Q.$$

Here, $$P = \frac{1}{1+y^2}\text{ and } Q=\frac{e^{\tan^{-1}y}}{1+y^2}$$

Now, integrating factor,

$$\text{IF} = e^{\int P\,dy}$$

Using $x\cdot(IF)=\int Q\cdot(IF)\,dy+C$ the general solution is

$$x\times e^{\tan^{-1}y} = \int\frac{e^{\tan^{-1}y}}{1+y^2}\times e^{\tan^{-1}y}\,dy+C$$

$$xe^{\tan^{-1}y} = \int\frac{e^{2\tan^{-1}y}}{1+y^2}dy+C \qquad ...(1)$$

Substituting $\tan^{-1}y=t\Rightarrow\frac{1}{1+y^2}dy=dt$ in Eq. (1), we get

$$x\frac{dv}{dx} = \frac{1-v}{1+v} - v = \frac{1-v-v-v^2}{1+v}$$
$$= \frac{1-2v-v^2}{1+v}$$
$$\frac{1+v}{v^2+2v-1}\,dv = -\frac{1}{x}\,dx$$

Integrating both sides, we have

$$\int \frac{1+v}{v^2+2v-1}\,dv = -\int \frac{1}{x}\,dx$$
$$\frac{1}{2}\log\left| v^2+2v-1 \right| = -\log\left| x \right| + \log C$$
$$\frac{1}{2}\log\left| v^2+2v-1 \right| + \log\left| x \right| = \log C$$
$$\log\left| v^2+2v-1 \right| + 2\log\left| x \right| = 2\log C$$
$$\log\left| \frac{y^2}{x^2} + \frac{2y}{x} - 1 \right| + \log x^2 = \log C^2$$
$$\log\left(\frac{y^2}{x^2} + \frac{2y}{x} - 1\right)x^2 = \log C^2$$
$$y^2 + 2xy - x^2 = C^2$$
$$y^2 + 2xy - x^2 = C_1 \text{ where } C_1 = C^2$$

106. Solve the following differential equation

$$y^2\,dx + (x^2 - xy + y^2)\,dy = 0$$

Sol : Foreign 2016

We have, $y^2\,dx + (x^2 - xy + y^2)\,dy = 0$

$$\frac{dy}{dx} = \frac{-y^2}{x^2 - xy + y^2} \quad ...(1)$$

This is homogeneous differential equation.

Substituting $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ in eq (1) we have

$$v + x\frac{dv}{dx} = \frac{-v^2x^2}{x^2 - vx^2 + v^2x^2}$$
$$v + x\frac{dv}{dx} = \frac{-v^2}{1-v+v^2}$$
$$x\frac{dv}{dx} = \frac{-v^2}{1-v+v^2} - v$$
$$x\frac{dv}{dx} = \frac{-v-v^3}{1-v+v^2}$$
$$\frac{1-v+v^2}{v(1+v^2)}\,dv = -\frac{1}{x}\,dx$$

Integrating both sides, we have

$$\int \frac{1+v^2}{v(1+v^2)}\,dv - \int \frac{v}{v(1+v^2)}\,dv = -\int \frac{1}{x}\,dx$$
$$\int \frac{1}{v}\,dv - \int \frac{1}{1+v^2}\,dv = -\int \frac{1}{x}\,dx$$
$$\log\left| v \right| - \tan^{-1}v = -\log\left| x \right| + \log C$$
$$\log\left| \frac{vx}{C} \right| = \tan^{-1}v$$
$$\left| \frac{vx}{C} \right| = e^{\tan^{-1}v}$$
$$\left| \frac{y}{C} \right| = e^{\tan^{-1}(y/x)}$$

or $$\left| y \right| = Ce^{\tan^{-1}(y/x)},$$

which is the required solution.

107. Solve the following differential equation

$$(\cot^{-1}y + x)\,dy = (1+y^2)\,dx$$

Sol : Foreign 2016

We have, $$(\cot^{-1}y + x)\,dy = (1+y^2)\,dx$$
$$\frac{dx}{dy} = \frac{\cot^{-1}y + x}{1+y^2}$$
$$\frac{dx}{dy} + \left(-\frac{1}{1+y^2}\right)x = \frac{\cot^{-1}y}{1+y^2}$$

This is a linear differential equation of the form $\frac{dx}{dy} + Px = Q$, here $P = \frac{-1}{1+y^2}$ and $Q = \frac{\cot^{-1}y}{1+y^2}$.

$$\text{IF} = e^{-\int\frac{1}{1+y^2}dy} = e^{\cot^{-}y}$$

Using $x \cdot (IF) = \int Q \cdot (IF)\,dy + C$ the general solution is

$$xe^{\cot^{-1}y} = \int \frac{\cot^{-1}y}{(1+y^2)} e^{\cot^{-1}y}\,dy + C \quad ...(1)$$

Substituting $\cot^{-1}y = t \Rightarrow \frac{1}{1+y^2}dy = -\,dt$ in Eq. (1), we get

$$xe^{\cot^{-1}y} = -\int te^t\,dt + C$$
$$= -\,e^t(t-1) + C$$
$$xe^{\cot^{-1}y} = e^{\cot^{-1}y}(1 - \cot^{-1}y) + C$$

which is the required solution.

108. Solve the following differential equation.

$$x\frac{dy}{dx} + y - x + xy\cot x = 0,\; x \neq 0$$

Sol : OD 2015, Comp 2011, Foreign 2012

We have $$x\frac{dy}{dx} + y - x + xy\cot x = 0,\; x \neq 0$$
$$x\frac{dy}{dx} + y(1 + x\cot x) = x$$
$$\frac{dy}{dx} + y\left(\frac{1+x\cot x}{x}\right) = 1$$
$$\frac{dy}{dx} + y\left(\frac{1}{x} + \cot x\right) = 1$$

which is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$,

where, $P = \frac{1}{x} + \cot x$ and $Q = 1$.

$$\text{IF} = e^{\int P\,dx}$$

Also, given that at $x = 0$, $y = 1$.

Substituting $x = 0$ and $y = 1$ in Eq. (1), we get

$$(1+1)(2+\sin 0) = C \Rightarrow C = 4$$

Substituting $C = 4$ in Eq. (1), we get

$$1 + y = \frac{4}{2+\sin x}$$

$$y = \frac{4}{2+\sin x} - 1$$

$$= \frac{4-2-\sin x}{2+\sin x}$$

$$= \frac{2-\sin x}{2+\sin x}$$

Now, at $x = \frac{\pi}{2}$, $\quad y\left(\frac{\pi}{2}\right) = \dfrac{2-\sin\frac{\pi}{2}}{2+\sin\frac{\pi}{2}}$

$$y\left(\frac{\pi}{2}\right) = \frac{1}{3}$$

111. Find the particular solution of the differential equation

$$\frac{dy}{dx} = \frac{x(2\log|x|+1)}{\sin y + y\cos y},$$

given that $y = \frac{\pi}{2}$, when $x = 1$.

Sol : Delhi 2014

We have $\quad \dfrac{dy}{dx} = \dfrac{x(2\log|x|+1)}{\sin y + y\cos y}$

Separating the variables, we get

$$(\sin y + y\cos y)\,dy = x(2\log|x|+1)\,dx$$

$$\sin y\,dy + y\cos y\,dy = 2x\log|x|dx + x\,dx$$

Integrating both sides, we have

$$\int \sin y dy + \int y\cos y dy = 2\int x\log|x|\,dx + \int x dx$$

$$-\cos y + \int y\cos y dy = 2\int x\log|x|\,dx + \frac{x^2}{2}$$

We find the integral as below.

$$\int y\cos y dy = y\int \cos y dy - \int\left\{\frac{d}{dy}(y)\int \cos y dy\right\}dy$$

$$= y\sin y - \int \sin y dy$$

$$= y\sin y + \cos y$$

$$\int x\log|x|\,dx = \log|x|\int x dx - \int\left\{\frac{d(\log|x|)}{dx}\int x dx\right\}dx$$

$$= \frac{x^2}{2}\log|x| - \int\left\{\frac{1}{x}\cdot\frac{x^2}{2}\right\}dx$$

$$= \frac{x^2}{2}\log|x| - \frac{1}{2}\int x dx$$

$$= \frac{x^2}{2}\log|x| - \frac{x^2}{4}$$

Now substituting all integrals we have

$$-\cos y + y\sin y + \cos y = 2\left[\frac{x^2}{2}\log|x| - \frac{x^2}{4}\right] + \frac{x^2}{2} + C$$

$$y\sin y = x^2\log|x| - \frac{x^2}{2} + \frac{x^2}{2} + C$$

$$y\sin y = x^2\log|x| + C \qquad ...(1)$$

Also, given that $y = \frac{\pi}{2}$, when $x = 1$.

Substituting $y = \frac{\pi}{2}$ and $x = 1$ in Eq. (1), we get

$$\frac{\pi}{2}\sin\left(\frac{\pi}{2}\right) = (1)^2\log(1) + C$$

$$C = \frac{\pi}{2}$$

Substituting the value of C in Eq. (1), we get

$$y\sin y = x^2\log|x| + \frac{\pi}{2}$$

which is the required particular solution.

112. Solve the following differential equation

$$(x^2-1)\frac{dy}{dx} + 2xy = \frac{2}{x^2-1},\ x \neq 1$$

Sol : Delhi 2014; OD 2014C, 2010

We have $\quad (x^2-1)\dfrac{dy}{dx} + 2xy = \dfrac{2}{x^2-1}$

$$\frac{dy}{dx} + \frac{2x}{x^2-1}y = \frac{1}{(x^2-1)^2}$$

which is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$, here $P = \frac{2x}{x^2-1}$ and $Q = \frac{1}{(x^2-1)^2}$

Now, integrating factor,

$$\text{IF} = e^{\int P dx}$$

$$= e^{\int \frac{2x}{x^2-1}dx}$$

$$= e^{\log|x^2-1|}$$

$$= x^2 - 1$$

Using $y\cdot(IF) = \int Q\cdot(IF)\,dx + C$ the general solution is

$$y(x^2-1) = \int \frac{1}{(x^2-1)^2}\times(x^2-1)\,dx + C$$

$$= \int \frac{1}{x^2-1}dx + C$$

$$y(x^2-1) = \log\left|\frac{x-1}{x+1}\right| + C$$

113. Find the particular solution of the differential equation

$$e^x\sqrt{1-y^2}\,dx + \frac{y}{x}dy = 0,$$

given that $y = 1$, when $x = 0$.

Sol : Delhi 2014

We have $\quad e^x\sqrt{1-y^2}\,dx + \dfrac{y}{x}dy = 0$

$$e^x\sqrt{1-y^2}\,dx = \frac{-y}{x}dy$$

Separating the variables, we get

$$\int \cos v\, dv = \int \frac{dx}{x}$$
$$\sin v = \log|x| + C$$
$$\sin\left(\frac{y}{x}\right) = \log|x| + C \qquad \left[\text{put } v = \frac{y}{x}\right]$$

which is the required solution of given differential equation.

116. Find the particular solution of the differential equation

$$x\frac{dy}{dx} - y + x\operatorname{cosec}\frac{y}{x} = 0$$

Sol : OD 2014C, 2011

We have
$$x\frac{dy}{dx} - y + x\operatorname{cosec}\left(\frac{y}{x}\right) = 0$$
$$\frac{dy}{dx} - \frac{y}{x} + \operatorname{cosec}\left(\frac{y}{x}\right) = 0$$
$$\frac{dy}{dx} = \frac{y}{x} - \operatorname{cosec}\left(\frac{y}{x}\right) \qquad ...(1)$$

which is a homogeneous differential equation as

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$$

Substituting $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ in eq (1) we have

$$v + x\frac{dv}{dx} = \frac{vx}{x} - \operatorname{cosec}\left(\frac{vx}{x}\right)$$
$$v + x\frac{dv}{dx} = v - \operatorname{cosec} v$$
$$x\frac{dv}{dx} = -\operatorname{cosec} v$$
$$\frac{dv}{\operatorname{cosec} v} = \frac{-dx}{x}$$

Integrating both sides, we have

$$\int \frac{dv}{\operatorname{cosec} v} = \int -\frac{dx}{x}$$
$$\int \sin v\, dv = \int -\frac{dx}{x}$$
$$-\cos v = -\log|x| + C$$
$$-\cos\frac{y}{x} = -\log|x| + C$$
$$\cos\frac{y}{x} = (\log|x| - C) \qquad ...(2)$$

Also, given that $x = 1$ and $y = 0$.
Substituting above values in Eq. (2), we get

$$\cos 0 = \log|1| - C$$
$$1 = 0 - C \Rightarrow C = -1$$
$$\cos\frac{y}{x} = \log|x| + 1 \qquad [\text{from Eq. (2)}]$$

which is required particular solution of given differential equation.

117. Find the particular solution of the differential equation

$$\frac{dy}{dx} = 1 + x + y + xy,$$

given that $y = 0$ when $x = 1$.

Sol : OD 2014

We have
$$\frac{dy}{dx} = 1 + x + y + xy$$
$$\frac{dy}{dx} = 1(1 + x) + y(1 + x)$$
$$\frac{dy}{dx} = (1 + x)(1 + y) \qquad ...(1)$$

Separating the variables, we get

$$\frac{1}{(1 + y)}dy = (1 + x)\,dx \qquad ...(2)$$

Integrating both sides of above equation, we have

$$\int \frac{1}{1 + y}dy = \int (1 + x)\,dx$$
$$\log|1 + y| = x + \frac{x^2}{2} + C \qquad ...(3)$$

Also, given that $y = 0$, when $x = 1$.
Substituting $x = 1$, $y = 0$ in Eq. (3), we get

$$\log[1 + 0] = 1 + \frac{1}{2} + C \Rightarrow C = -\frac{3}{2}$$

Now, substituting the value of C in Eq. (3), we get

$$\log|1 + y| = x + \frac{x^2}{2} - \frac{3}{2}$$

which is the required particular solution of given differential equation.

118. Solve the differential equation

$$(1 + x^2)\frac{dy}{dx} + y = e^{\tan^{-1}x}.$$

Sol : OD 2014

We have
$$(1 + x^2)\frac{dy}{dx} + y = e^{\tan^{-1}x}$$
$$\frac{dy}{dx} + \frac{1}{1 + x^2}y = \frac{e^{\tan^{-1}x}}{1 + x^2}$$

which is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$, here $P = \frac{1}{1 + x^2}$ and $Q = \frac{e^{\tan^{-1}x}}{1 + x^2}$

Now, integrating factor,

$$\text{IF} = e^{\int P dx}$$
$$= e^{\int \frac{1}{1 + x^2}dx}$$
$$= e^{\tan^{-1}x}$$

Using $y \cdot (IF) = \int Q \cdot (IF)\,dx + C$ the general solution is

$$ye^{\tan^{-1}x} = \int \frac{e^{\tan^{-1}x}}{(1 + x^2)} \times e^{\tan^{-1}x}\,dx + C$$

$$= 2\Big[-\frac{1}{2}\log|x| + \int \frac{1}{x^2}dx\Big]$$

or $\qquad y\log|x| = -\frac{2}{x}\log|x| - \frac{2}{x} + C,$

which is the required solution.

122. Solve the differential equation $\frac{dy}{dx} + y\cot x = 2\cos x$, given that $y = 0$, when $x = \frac{\pi}{2}$.

Sol : Foreign 2014

We have $\qquad \frac{dy}{dx} + y\cot x = 2\cos x$

which is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q$$

Were, $\quad P = \cot x$ and $Q = 2\cos x$

$$\begin{aligned} \text{IF} &= e^{\int P\,dx} \\ &= e^{\int \cot x\,dx} \\ &= e^{\log|\sin x|} \\ \text{IF} &= \sin x \end{aligned}$$

Using $y\cdot(IF) = \int Q\cdot(IF)\,dx + C$ the general solution is

$$\begin{aligned} y\sin x &= \int 2\sin x\cos x\,dx + C \\ y\sin x &= \int \sin 2x\,dx + C \\ y\sin x &= -\frac{\cos 2x}{2} + C \qquad ...(1) \end{aligned}$$

Also, given that $y = 0$, when $x = \frac{\pi}{2}$.

Substituting $x = \frac{\pi}{2}$ and $y = 0$ in Eq. (1), we get

$$\begin{aligned} 0\sin\frac{\pi}{2} &= -\frac{\cos\left(2\frac{\pi}{2}\right)}{2} + C \\ C - \frac{\cos\pi}{2} &= 0 \\ C + \frac{1}{2} &= 0 \\ C &= -\frac{1}{2} \end{aligned}$$

Substituting the value of C in Eq. (1), we get

$$\begin{aligned} y\sin x &= -\frac{\cos 2x}{2} - \frac{1}{2} \\ 2y\sin x + \cos 2x + 1 &= 0 \end{aligned}$$

which is the required equation.

123. Solve the differential equation $(x^2 - yx^2)\,dy + (y^2 + x^2y^2)\,dx = 0$, given that $y = 1$, when $x = 1$.

Sol : Foreign 2014

We have $\;(x^2 - yx^2)\,dy + (y^2 + x^2y^2)\,dx = 0$

$$\begin{aligned} x^2(1-y)\,dy + y^2(1+x^2)\,dx &= 0 \\ -x^2(1-y)\,dy &= y^2(1+x^2)\,dx \\ x^2(y-1)\,dy &= y^2(1+x^2)\,dx \\ \frac{y-1}{y^2}dy &= \frac{1+x^2}{x^2}dx \end{aligned}$$

Integrating both sides, we have

$$\begin{aligned} \int\frac{y-1}{y^2}dy &= \int\frac{1+x^2}{x^2}dx \\ \int\frac{1}{y}dy - \int\frac{1}{y^2}dy &= \int\frac{1}{x^2}dx + \int 1\,dx \\ \log|y| + \frac{1}{y} &= \frac{-1}{x} + x + C \qquad ...(1) \end{aligned}$$

Also, given that $y = 1$, when $x = 1$

Substituting $y = 1$ and $x = 1$ in Eq. (1), we get

$$\begin{aligned} \log|1| + 1 &= -1 + 1 + C \\ C &= 1 \end{aligned}$$

Substituting the value of C in Eq. (1), we get

$$\log|y| + \frac{1}{y} = -\frac{1}{x} + x + 1$$

which is the required solution.

124. Solve the following differential equation :

$x\cos\left(\frac{y}{x}\right)(y\,dx + x\,dy) = y\sin\left(\frac{y}{x}\right)(x\,dy - y\,dx)$.

Or

Solve the following differential equation.

$$\left(x\cos\frac{y}{x} + y\sin\frac{y}{x}\right)y - \left(y\sin\frac{y}{x} - x\cos\frac{y}{x}\right)x\frac{dy}{dx} = 0$$

Sol : Comp 2013, Foreign 2010

We have

$$\left(x\cos\frac{y}{x} + y\sin\frac{y}{x}\right)y - \left(y\sin\frac{y}{x} - x\cos\frac{y}{x}\right)x\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{\left[x\cos\frac{y}{x} + y\sin\frac{y}{x}\right]\cdot y}{\left(y\sin\frac{y}{x} - x\cos\frac{y}{x}\right)\cdot x} \qquad ...(1)$$

which is a homogeneous differential equation as

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$$

Substituting $y = vx \;\Rightarrow\; \frac{dy}{dx} = v + x\frac{dv}{dx}$ in eq (1) we have

$$\begin{aligned} v + x\frac{dv}{dx} &= \frac{(x\cos v + vx\sin v)\cdot vx}{(vx\sin v - x\cos v)\cdot x} \\ v + x\frac{dv}{dx} &= \frac{v\cos v + v^2\sin v}{v\sin v - \cos v} \\ x\frac{dv}{dx} &= \frac{v\cos v + v^2\sin v}{v\sin v - \cos v} - v \\ &= \frac{v\cos v + v^2\sin v - v^2\sin v + v\cos v}{v\sin v - \cos v} \end{aligned}$$

$$y \cdot \sin x = \int (2x + x^2 \cdot \cot x) \sin x \, dx + C$$
$$= 2\int x \sin x \, dx + \int x^2 \cos x \, dx + C$$
$$= 2\int x \sin x \, dx + x^2 \sin x - \int 2x \sin x \, dx + C$$
$$y \cdot \sin x = x^2 \sin x + C \quad ...(1)$$

Substituting $x = \frac{\pi}{2}$ and $y = 0$ in Eq. (1), we get

$$0 \cdot \sin\frac{\pi}{2} = \left(\frac{\pi}{2}\right)^2 \cdot \sin\frac{\pi}{2} + C \Rightarrow C = -\frac{\pi^2}{4}$$

Substituting $C = \frac{-\pi^2}{4}$ in Eq. (1), we get

$$y \cdot \sin x = x^2 \sin x - \frac{\pi^2}{4}$$
$$y = x^2 - \frac{\pi^2}{4}\operatorname{cosec} x$$

128. Solve the following differential equation

$$\frac{dy}{dx} + y\cot x = 4x\operatorname{cosec} x,$$

given that $y = 0$, when $x = \frac{\pi}{2}$.

Sol : Delhi 2012C; OD 2012; Foreign 2011

We have $\quad \frac{dy}{dx} + y \cot x = 4x \operatorname{cosec} x$

which is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$, here $P = \cot x$ and $Q = 4x \operatorname{cosec} x$

$$\text{IF} = e^{\int P\, dx} = e^{\int \cot x \, dx}$$
$$= e^{\log|\sin x|} = \sin x$$

Using $y \cdot (IF) = \int Q \cdot (IF)\, dx + C$ the general solution is

$$y\sin x = \int 4x \operatorname{cosec} x \cdot \sin x \, dx + C$$
$$= \int 4x \cdot \frac{1}{\sin x} \cdot \sin x \, dx + C$$
$$= \int 4x \, dx + C$$

or $\quad y \sin x = 2x^2 + C \quad ...(1)$

Also, given that $y = 0$, when $x = \frac{\pi}{2}$.

Substituting $y = 0$ and $x = \frac{\pi}{2}$ in Eq. (1), we get

$$0 = 2 \times \frac{\pi^2}{4} + C \Rightarrow C = \frac{-\pi^2}{2}$$

Substituting $C = -\frac{\pi^2}{2}$ in Eq. (1), we get

$$y \sin x = 2x^2 - \frac{\pi^2}{2}$$
$$y = 2x^2 \operatorname{cosec} x - \frac{\pi^2}{2}\operatorname{cosec} x$$

which is the required solution.

129. Find the particular solution of the differential equation

$$xy\frac{dy}{dx} = (x+2)(y+2) \quad y = -1 \text{ when } x = 1.$$

Sol : Delhi 2012

We have, $\quad xy\frac{dy}{dx} = (x+2)(y+2)$

Separating the variables, we get

$$\frac{y\, dy}{y+2} = \frac{x+2}{x}dx$$
$$\left(\frac{y+2-2}{y+2}\right)dy = \left(1 + \frac{2}{x}\right)dx$$
$$\left(1 - \frac{2}{y+2}\right)dy = \left(1 + \frac{2}{x}\right)dx$$

Integrating both sides, we have

$$\int\left(1 - \frac{2}{y+2}\right)dy = \int\left(1 + \frac{2}{x}\right)dx$$
$$y - 2\log|y+2| = x + 2\log|x| + C \quad ...(1)$$

Given that $y = -1$, when $x = 1$

Substituting $x = 1$ and $y = -1$ in Eq. (1), we get

$$-1 - 2\log(1) = 1 + 2\log|1| + C$$
$$-1 = 1 + C \Rightarrow C = -2$$

Substituting $C = -2$ in Eq. (1), we get

$$y - 2\log|y+2| = x + 2\log|x| - 2$$

which is required particular solution.

130. Solve the following differential equation

$$2x^2\frac{dy}{dx} - 2xy + y^2 = 0$$

Sol : Delhi 2012

We have $\quad 2x^2\frac{dy}{dx} - 2xy + y^2 = 0$

$$\frac{dy}{dx} = \frac{y}{x} - \frac{y^2}{2x^2} \quad ...(1)$$

which is a homogeneous differential equation as

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$$

Substituting $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ in eq (1) we have

$$v + x\frac{dv}{dx} = v - \frac{v^2}{2}$$
$$x\frac{dv}{dx} = \frac{-v^2}{2} \Rightarrow \frac{2dv}{v^2} = -\frac{1}{x}dx$$

Integrating both sides, we have

$$2\int v^{-2}\, dv = -\log|x| + C$$
$$\frac{2v^{-1}}{-1} = -\log|x| + C$$
$$\frac{-2}{v} = -\log|x| + C$$
$$\frac{-2x}{y} = -\log|x| + C$$
$$-2x = y(-\log|x| + C)$$

$$dy = \frac{\cot x - 2xy}{1+x^2}dx$$

$$\frac{dy}{dx} + \frac{2xy}{1+x^2} = \frac{\cot x}{1+x^2}$$

which is a linear differential equation of 1st order and is of the form

$$\frac{dy}{dx} + Py = Q$$

Here, $P = \frac{2x}{1+x^2}$ and $Q = \frac{\cot x}{1+x^2}$

$$\text{IF} = e^{\int \frac{2x}{1+x^2}dx} = e^{\log|1+x^2|} = 1+x^2$$

Using $y \cdot (IF) = \int Q \cdot (IF)\, dx + C$ the general solution is

$$y(1+x^2) = \int \frac{\cot x}{1+x^2} \times (1+x^2)\, dx + C$$

$$y(1+x^2) = \int \cot x\, dx + C$$

$$y(1+x^2) = \log|\sin x| + C$$

Dividing both sides by $(1+x^2)$, we get

$$y = \frac{\log|\sin x|}{1+x^2} + \frac{C}{1+x^2}$$

which is the required solution.

134. Find the particular solution of the following differential equation $x\frac{dy}{dx} - y + x\sin\frac{y}{x} = 0$, given that when $x = 2$, $y = \pi$.

Sol : OD 2012

We have, $$x\frac{dy}{dx} - y + x\sin\left(\frac{y}{x}\right) = 0$$

$$\frac{dy}{dx} - \frac{y}{x} + \sin\left(\frac{y}{x}\right) = 0 \qquad ...(1)$$

which is a homogeneous differential equation as

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$$

Substituting $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ in eq (1),

$$v + x\frac{dv}{dx} - v + \sin v = 0$$

$$x\frac{dv}{dx} + \sin v = 0$$

$$\operatorname{cosec} v\, dv + \frac{dx}{x} = 0$$

Integrating both sides, we have

$$\int \operatorname{cosec} v\, dv + \int \frac{dx}{x} = \log C$$

$$\log|\operatorname{cosec} v - \cot v| + \log x = \log C$$

$$x(\operatorname{cosec} v - \cot v) = C$$

$$x\left[\operatorname{cosec}\left(\frac{y}{x}\right) - \cot\left(\frac{y}{x}\right)\right] = C \qquad ...(2)$$

Substituting $x = 2$ and $y = \pi$ in Eq. (2) we get

$$2\left[\operatorname{cosec}\left(\frac{\pi}{2}\right) - \cot\left(\frac{\pi}{2}\right)\right] = C \Rightarrow C = 2$$

Substituting $C = 2$ in Eq. (2), we get

$$x\left[\operatorname{cosec}\left(\frac{y}{x}\right) - \cot\left(\frac{y}{x}\right)\right] = 2$$

which is the required particular solution.

135. Solve the following differential equation :

$$\left[x\sin^2\left(\frac{y}{x}\right) - y\right]dx + x\, dy = 0$$

Sol : Foreign 2011

We have $$\left[x\sin^2\left(\frac{y}{x}\right) - y\right]dx + x\, dy = 0$$

$$\frac{dy}{dx} = \frac{y}{x} - \sin^2\left(\frac{y}{x}\right) \quad ...(1)$$

which is a homogeneous differential equation as

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$$

Substituting $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ in eq (1) we have

$$v + x\frac{dv}{dx} = v - \sin^2 v$$

$$x\frac{dv}{dx} = -\sin^2 v$$

$$\operatorname{cosec}^2 v\, dv = -\frac{dx}{x}$$

Integrating both sides, we have

$$\int \operatorname{cosec}^2 v\, dv + \int \frac{dx}{x} = 0$$

$$-\cot v + \log|x| = C$$

$$-\cot\left(\frac{y}{x}\right) + \log|x| = C \qquad \left[v = \frac{y}{x}\right]$$

$$y = x \cdot \cot^{-1}(\log x - C)$$

which is the required solution.

136. Solve the following differential equation.

$$(1+y^2)(1+\log|x|)\,dx + x\, dy = 0$$

Sol : Delhi 2011

We have $$(1+y^2)(1+\log|x|)\,dx + x\, dy = 0$$

Separating the variables, we get

$$\frac{1+\log|x|}{x}dx = -\frac{dy}{1+y^2}$$

Integrating both sides, we have

$$\int \frac{1+\log|x|}{x}dx = -\int \frac{dy}{1+y^2}$$

$$\int \frac{1}{x}dx + \int \frac{\log|x|}{x}dx = -\int \frac{dy}{1+y^2}$$

$$\log|x| + I_1 + K = -\tan^{-1}y \qquad ...(1)$$

139. Solve the following differential equation.

$$x\,dy - (y + 2x^2)\,dx = 0$$

Sol : OD 2011

We have
$$x\,dy - (y + 2x^2)\,dx = 0$$
$$xdy = (y + 2x^2)\,dx$$
$$\frac{dy}{dx} = \frac{(y + 2x^2)}{x}$$
$$\frac{dy}{dx} = \frac{y}{x} + 2x$$
$$\frac{dy}{dx} - \frac{y}{x} = 2x$$
which is a linear differential equation of the form
$$\frac{dy}{dx} + Py = Q.$$
were, $P = \frac{-1}{x}$ and $Q = 2x$
$$\text{IF} = e^{\int P\,dy} = e^{\int -\frac{1}{x}dx}$$
$$= e^{-\log|x|} = e^{\log x^{-1}}$$
or
$$\text{IF} = x^{-1} = \frac{1}{x}$$
Using $y \cdot (IF) = \int Q \cdot (IF)\,dx + C$ the general solution is
$$y \times \frac{1}{x} = \int \left(2x \times \frac{1}{x}\right)dx$$
$$\frac{y}{x} = \int 2\,dx \Rightarrow \frac{y}{x} = 2x + C$$
$$y = 2x^2 + Cx$$
which is the required solution.

140. Solve the following differential equation :

$$x\,dy + (y - x^3)\,dx = 0$$

Sol : OD 2011

We have
$$x\,dy + (y - x^3)\,dx = 0$$
$$xdy = -(y - x^3)\,dx$$
$$\frac{dy}{dx} = -\frac{(y - x^3)}{x}$$
$$\frac{dy}{dx} = -\frac{y}{x} + x^2$$
$$\frac{dy}{dx} + \frac{y}{x} = x^2$$
which is a linear differential equation of the form
$$\frac{dy}{dx} + Py = Q.$$
were, $P = \frac{1}{x}$ and $Q = x^2$
$$\text{IF} = e^{\int P\,dy} = e^{\int \frac{1}{x}dx}$$
$$= e^{\log|x|} = x$$
or
$$\text{IF} = x$$
Using $y \cdot (IF) = \int Q \cdot (IF)\,dx + C$ the general solution is
$$y \times x = \int (x^2 \times x)\,dx$$
$$yx = \int x^3\,dx$$
$$yx = \frac{x^3}{4} + C$$
$$y = \frac{x^3}{4} + \frac{C}{x}$$

141. Find the particular solution of the differential equation

$$(1 + e^{2x})\,dy + (1 + y^2)\,e^x\,dx = 0$$

given that $y = 1$, when $x = 0$.

Sol : Foreign 2011

We have
$$(1 + e^{2x})\,dy + (1 + y^2)\,e^x\,dx = 0$$
Separating the variables, we get
$$\frac{dy}{1 + y^2} = \frac{-e^x}{1 + e^{2x}}\,dx$$
Integrating both sides, we have
$$\int \frac{dy}{1 + y^2} = -\int \frac{e^x}{1 + e^{2x}}\,dx$$
Substituting $e^x = t \Rightarrow e^x\,dx = dt$ in RHS, we get
$$\tan^{-1} y = -\int \frac{1}{1 + t^2}\,dt$$
$$\tan^{-1} y = -\tan^{-1} t + C$$
$$\tan^{-1} y = -\tan^{-1}(e^x) + C \qquad ...(1)$$
Also, given that $y = 1$, when $x = 0$.
Substituting above values in Eq. (1), we get
$$\tan^{-1} 1 = -\tan^{-1}(e^0) + C$$
$$\tan^{-1} 1 = -\tan^{-1} 1 + C$$
$$2\tan^{-1} 1 = C$$
$$2\tan^{-1}\left(\tan\frac{\pi}{4}\right) = C$$
$$C = 2 \times \frac{\pi}{4} = \frac{\pi}{2}$$
Substituting $C = \frac{\pi}{2}$ in Eq. (1), we get
$$\tan^{-1} y = -\tan^{-1} e^x + \frac{\pi}{2}$$
$$y = \tan\left[\frac{\pi}{2} - \tan^{-1}(e^x)\right] = \cot[\tan^{-1}(e^x)]$$
$$= \cot\left[\cot^{-1}\left(\frac{1}{e^x}\right)\right]$$
$$y = \frac{1}{e^x}$$
which is the required solution.

Using $y\cdot(IF)=\int Q\cdot(IF)\,dx+C$ the general solution is

$$\begin{aligned} y(x^2+1) &= \int\frac{\sqrt{x^2+4}}{x^2+1}(x^2+1)\,dx \\ &= \int\sqrt{x^2+4}\,dx \\ &= \int\sqrt{x^2+(2)^2}\,dx \\ &= \frac{x}{2}\sqrt{x^2+4}+\frac{4}{2}\log\left|x+\sqrt{x^2+4}\right|+C \end{aligned}$$

or $\quad y(x^2+1)=\frac{x}{2}\sqrt{x^2+4}+2\log\left|x+\sqrt{x^2+4}\right|+C$

which is the required solution.

145. Solve the following differential equation.

$$(x^3+x^2+x+1)\frac{dy}{dx}=2x^2+x$$

Sol : OD 2010, Delhi 2007

We have $\quad (x^3+x^2+x+1)\frac{dy}{dx}=2x^2+x$

Separating the variables, we get

$$dy=\frac{2x^2+x}{x^3+x^2+x+1}dx$$

Integrating both sides, we get

$$\begin{aligned} \int dy &= \int\frac{2x^2+x}{x^3+x^2+x+1}dx \\ y &= \int\frac{2x^2+x}{x^2(x+1)+1(x+1)}dx+C \\ y &= \int\frac{2x^2+x}{(x+1)(x^2+1)}dx \qquad ...(1) \end{aligned}$$

Using partial fractions method,

Now $$\frac{2x^2+x}{(x+1)(x^2+1)}=\frac{A}{x+1}+\frac{Bx+C}{x^2+1} \qquad ...(2)$$

$$\begin{aligned} \frac{2x^2+x}{(x+1)(x^2+1)} &= \frac{A(x^2+1)+(Bx+C)(x+1)}{(x+1)(x^2+1)} \\ 2x^2+x &= A(x^2+1)+(Bx+C)(x+1) \\ 2x^2+x &= A(x^2+1)+B(x^2+x)+C(x+1) \end{aligned}$$

Substituting $x=-1$ we have

$$2-1=A(1+1)+0\ \Rightarrow A=\tfrac{1}{2}$$

Comparing the coefficients of x^2 we have

$$A+B=2\ \Rightarrow \tfrac{1}{2}+B=2\Rightarrow B=\tfrac{3}{2}$$

Comparing the constant terms from sides, we get

$$A+C=0\ \Rightarrow C=-A=-\tfrac{1}{2}$$

Thus $$\frac{2x^2-x}{(x+1)(x^2+1)}=\frac{\frac{1}{2}}{x+1}+\frac{\frac{3}{2}x-\frac{1}{2}}{x^2+1}$$

Integrating both sides, we have

$$\begin{aligned} y &= \int\frac{2x^2+x}{(x+1)(x^2+1)}dx \\ &= \frac{1}{2}\int\frac{dx}{x+1}+\frac{3}{2}\int\frac{x}{x^2+1}dx-\frac{1}{2}\int\frac{dx}{x^2+1} \\ &= \frac{1}{2}\int\frac{dx}{x+1}+\frac{3}{4}\int\frac{2x}{x^2+1}dx-\frac{1}{2}\int\frac{dx}{x^2+1} \\ y &= \frac{1}{2}\log|x+1|+\frac{3}{4}\log|x^2+1|-\frac{1}{2}\tan^{-1}x+C \end{aligned}$$

which is the required solution.

146. Find the particular solution of the differential equation $(x-y)\frac{dy}{dx}=x+2y$, given that when $x=1,\ y=0$.

Sol : OD 2017, Foreign 2013

We have, $$\begin{aligned} (x-y)\frac{dy}{dx} &= x+2y \\ \frac{dy}{dx} &= \frac{x+2y}{x-y} \qquad ...(1) \end{aligned}$$

This is a homogeneous differential equation.

Substituting $y=vx\ \Rightarrow\ \frac{dy}{dx}=v+x\frac{dv}{dx}$ in eq (1) we have

$$\begin{aligned} v+x\frac{dv}{dx} &= \frac{x+2vx}{x-vx} \\ x\frac{dv}{dx} &= \frac{1+2v}{1-v}-v \\ &= \frac{1+2v-v+v^2}{1-v} \\ x\frac{dv}{dx} &= \frac{1+v+v^2}{1-v} \\ \frac{1-v}{v^2+v+1}dv &= \frac{1}{x}dx \end{aligned}$$

Integrating both sides, we have

$$\begin{aligned} \int\frac{1-v}{v^2+v+1}dv &= \int\frac{1}{x}dx \\ I &= \log x+C \end{aligned}$$

Now $$\begin{aligned} I &= \int\frac{1-v}{v^2+v+1}dv \\ &= \int\frac{-\frac{1}{2}(2v+1)-\frac{3}{2}}{v^2+v+1}dv \end{aligned}$$

$$\begin{aligned} -\frac{1}{2}\int\frac{(2v+1)\,dv}{v^2+v+1}+\frac{3}{2}\int\frac{dv}{v^2+v+1} &= \int\frac{1}{x}dx \\ &= -\frac{1}{2}\log(v^2+v+1)+\frac{3}{2}\int\frac{dv}{(v+\frac{1}{2})^2+\frac{3}{4}} \\ &= -\frac{1}{2}\log(v^2+v+1)+\frac{3}{2}\int\frac{dv}{(v+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2} \\ &= -\frac{1}{2}\log(v^2+v+1)+\frac{3}{2}\frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\frac{(v+\frac{1}{2})}{\frac{\sqrt{3}}{2}} \\ &= -\frac{1}{2}\log(v^2+v+1)+\sqrt{3}\tan^{-1}\frac{(2v+1)}{\sqrt{3}} \\ &= -\frac{1}{2}\log\left(\frac{y^2}{x^2}+\frac{y}{x}+1\right)+\sqrt{3}\tan^{-1}\frac{(\frac{2y}{x}+1)}{\sqrt{3}} \end{aligned}$$

Thus substituting I we have general solution, Substituting this value of C in Eq. (2), we get

$$-\cot\left(\frac{y}{x}\right)+\log|x| = -1$$

$$1+\log|x|-\cot\left(\frac{y}{x}\right) = 0$$

which is the required particular solution of given differential equation.

149. Solve the differential equation

$$\frac{dy}{dx}-3y\cot x = \sin 2x,$$

given $y=2$ when $x=\frac{\pi}{2}$.

Sol : OD 2015

We have, $\frac{dy}{dx}-3y\cot x = \sin 2x$ ...(1)

This is a linear differential equation of the form $\frac{dy}{dx}+Py=Q$, where $P=-3\cot x$ and $Q=\sin 2x$.

$$\begin{aligned}\text{IF} &= e^{\int P\,dx} = e^{-3\int \cot x\,dx}\\ &= e^{-3\log|\sin x|} = e^{\log|\sin x|^{-3}}\\ &= |\sin x|^{-3}\end{aligned}$$

Using $y\cdot(IF)=\int Q\cdot(IF)\,dx+C$ the general solution is

$$\begin{aligned}y\cdot(\sin x)^{-3} &= \int(\sin x)^{-3}(\sin 2x)\,dx+C\\ &= \int\frac{2\sin x\cos x}{\sin^3 x}dx+C\\ y\cdot(\sin x)^{-3} &= \int\frac{2\cos x}{\sin^2 x}dx+C \quad ...(1)\end{aligned}$$

Substituting $\sin x=t\Rightarrow\cos x\,dx=dt$ in Eq. (1), we get

$$\begin{aligned}y\cdot(\sin x)^{-3} &= 2\int\frac{dt}{t^2}+C = 2\times\frac{t^{-1}}{-1}+C\\ y(\sin x)^{-3} &= -\frac{2}{t}+C\\ y(\sin x)^{-3} &= \frac{-2}{\sin x}+C\\ y &= -2\sin^2 x+C\sin^3 x \quad ...(2)\end{aligned}$$

Substituting $x=\frac{\pi}{2}$ and $y=2$ in Eq. (2), we get

$$\begin{aligned}2 &= -2\sin^2\frac{\pi}{2}+C\sin^3\frac{\pi}{2}\\ 2 &= -2\cdot 1+C\cdot 1\\ C &= 4\\ y &= -2\sin^2 x+4\sin^3 x,\end{aligned}$$

which is required particular solution.

150. Find the particular solution of the differential equation $(\tan^{-1}y-x)\,dy=(1+y^2)\,dx$, given that $x=1$ when $y=0$.

Sol : OD 2015

We have, $(\tan^{-1}y-x)\,dy = (1+y^2)\,dx$

$$\begin{aligned}\frac{dx}{dy} &= \frac{\tan^{-1}y-x}{1+y^2}\\ \frac{dx}{dy}+\frac{1}{1+y^2}x &= \frac{\tan^{-1}y}{1+y^2}\end{aligned}$$

This is a linear differential equation of the form $\frac{dx}{dy}+Px=Q$, here $P=\frac{1}{1+y^2}$ and $Q=\frac{\tan^{-1}y}{1+y^2}$.

$$\text{IF} = e^{\int\frac{1}{1+y^2}dy} = e^{\tan^{-1}y}$$

Using $x\cdot(IF)=\int Q\cdot(IF)\,dy+C$ the general solution is

$$xe^{\tan^{-1}y} = \int\frac{\tan^{-1}y}{(1+y^2)}e^{\tan^{-1}y}dy+C \quad ...(1)$$

Substituting $\tan^{-1}y=t\Rightarrow\frac{1}{1+y^2}dy=dt$ in Eq. (1), we get

$$\begin{aligned}xe^{\tan^{-1}y} &= \int te^t\,dt+C\\ &= e^t(t-1)+C\\ xe^{\cot^{-1}y} &= e^{\tan^{-1}y}(\tan^{-1}y-1)+C\end{aligned}$$

It is given that $x=1$, when $y=0$.

Therefore, we have $1\cdot e^0=e^0(0-1)+C$

$$1 = -1+C\Rightarrow C=2$$

Hence, the required solution is

$$xe^{\tan^{-1}y} = e^{\tan^{-1}y}(\tan^{-1}y-1)+2$$

151. Show that the differential equation $\frac{dy}{dx}=\frac{y^2}{xy-x^2}$ is homogeneous and also solve it.

Sol : OD 2015

We have $\frac{dy}{dx} = \frac{y^2}{xy-x^2}$ ...(1)

Let $F(x,y) = \frac{y^2}{xy-x^2}$

Now, replacing x by λx and y by λy, we get

$$\begin{aligned}F(\lambda x,\lambda y) &= \frac{\lambda^2y^2}{\lambda^2(xy-x^2)}\\ &= \lambda^0\frac{y^2}{xy-x^2} = \lambda^0F(x,y)\end{aligned}$$

which is a homogeneous differential equation as

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$$

Substituting $y=vx \Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$ in eq (1) we have

From Eq. (1), we get

$$\frac{dy}{dx} = -\frac{3xy+y^2}{x^2+xy} \quad ...(1)$$

$$\frac{dy}{dx} = -\frac{3\frac{y}{x}+\frac{y^2}{x^2}}{1+\frac{y}{x}} \quad ...(1)$$

which is a homogeneous differential equation as

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$$

Substituting $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ in eq (1) we have

$$v + x\frac{dv}{dx} = -\frac{3v+v^2}{1+v}$$

$$x\frac{dv}{dx} = -\left(\frac{3v+v^2+v+v^2}{1+v}\right)$$

$$x\frac{dv}{dx} = -\left(\frac{2v^2+4v}{1+v}\right)$$

$$\frac{(1+v)\,dv}{2(v^2+2v)} = -\frac{dx}{x}$$

Integrating both sides, we get

$$\int \frac{1+v}{2(v^2+2v)}dv = -\int \frac{dx}{x}$$

Again, substituting $v^2 + 2v = z \Rightarrow (2v + 2dv) = dz$

$$(1+v)\,dv = \frac{dz}{2}$$

Then, Eq. (2) becomes,

$$\int \frac{1}{2} \times \frac{dz}{2z} = -\int \frac{dx}{x}$$

$$\frac{1}{4}\log|z| = -\log|x| + \log|C|$$

$$\frac{1}{4}[\log|z| + 4\log|x|] = \log|C|$$

$$\log|zx^4| = 4\log|C|$$

$$zx^4 = C^4 \Rightarrow zx^4 = C_1,$$

where $C_1 = C^4$

$$x^4(v^2+2v) = C_1 \qquad z = v^2 + 2v$$

$$x^4\left(\frac{y^2}{x^2} + \frac{2y}{x}\right) = C_1 \qquad [\text{put } v = \frac{y}{x}] \; ...(3)$$

Also, given that $y = 1$ for $x = 1$.

Substituting $x = 1$ and $y = 1$ in Eq. (3), we get

$$1\left(\frac{1}{1} + \frac{2}{1}\right) = C_1 \Rightarrow C_1 = 3$$

Thus substituting $C_1 = 3$ in Eq. (3), we get

$$x^4\left(\frac{y^2}{x^2} + \frac{2y}{x}\right) = 3$$

$$y^2x^2 + 2yx^3 = 3$$

which is the required particular solution.

155. Find the particular solution of the following differential equation given that $y = 0$, when $x = 1$..

$$(x^2 + xy)\,dy = (x^2 + y^2)\,dx.$$

Sol : Delhi 2013

We have, $(x^2 + xy)\,dy = (x^2 + y^2)\,dx$

$$\frac{dy}{dx} = \left(\frac{x^2+y^2}{x^2+xy}\right) \quad ...(1)$$

which is a homogeneous differential equation. Substituting $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ in eq (1) we have

$$v + x\frac{dv}{dx} = \left(\frac{x^2+v^2x^2}{x^2+x\cdot xv}\right)$$

$$x\frac{dv}{dx} = \frac{1+v^2}{1+v} - v$$

$$= \frac{1+v^2-v-v^2}{1+v} = \frac{1-v}{1+v}$$

$$\left(\frac{1+v}{1-v}\right)dv = \frac{1}{x}dx$$

Integrating both sides, we have

$$\int\left(\frac{1+v}{1-v}\right)dv = \int \frac{1}{x}dx$$

$$\int\left[-1 + \frac{2}{1-v}\right]dv = \log|x| + \log C$$

$$-v - 2\log(1-v) = \log|x| + \log C$$

$$-v = 2\log(1-v) + \log|x| + \log C$$

$$-v = \log(1-v)^2 + \log\{C|x|\}$$

$$-v = \log\{C|x|(1-v)^2\}$$

$$C|x|(1-v)^2 = e^{-v}$$

$$C|x|\left(1-\frac{y}{x}\right)^2 = e^{-y/x} \quad ...(2)$$

Substituting $x = 1$ and $y = 0$ in Eq. (2), we get

$$C\cdot 1(1-0) = e^0 \Rightarrow C = 1$$

Thus, the required solution is

$$|x|\left(1-\frac{y}{x}\right)^2 = e^{-y/x}$$

$$(x-y)^2 = |x|e^{-y/x}$$

which is the required particular solution.

156. Show that the differential equation

$$x\frac{dy}{dx}\sin\left(\frac{y}{x}\right) + x - y\sin\left(\frac{y}{x}\right) = 0$$

is homogeneous. Find the particular solution of this differential equation, given that $x = 1$, when $y = \frac{\pi}{2}$.

Sol : Delhi 2013

We have $x\frac{dy}{dx}\sin\left(\frac{y}{x}\right) = y\sin\left(\frac{y}{x}\right) - x$

any time t. Initially $(t=0)$ 50 g of the first substance was present; 1 hr later, only 10 g of it remained.

(i) Find an expression that gives the amount of the first substance present at any time t.

(ii) What is the amount present after 2 hr?

Sol :

Since rate is proportional to the square of the amount of the first substance present at any time t, we are led to the differential equation

$$\frac{dQ}{dt} = kQ^2$$

The differential equation is separable. Separating the variables and integrating, we obtain

$$\int \frac{dQ}{Q^2} = \int k\,dt$$

and $$-\frac{1}{Q} = kt + C$$

Therefore, $$Q = -\frac{1}{kt+C}$$

Now, $Q=50$ when $t=0$, therefore, $50 = -\frac{1}{C}$ and $C = -\frac{1}{50}$.

Therefore, $$Q = -\frac{1}{kt - \frac{1}{50}}$$

Since $Q = 10$ when $t=1$,

$$10 = -\frac{1}{k - \frac{1}{50}}$$

$$10\left(k - \frac{1}{50}\right) = -1$$

$$10k - \frac{1}{5} = -1$$

$$10k = -1 + \frac{1}{5} = -\frac{4}{5}$$

$$k = -\frac{4}{50} = -\frac{2}{25}$$

Therefore, $$Q(t) = \frac{1}{\frac{2}{25}t + \frac{1}{50}}$$

$$= \frac{1}{\frac{4t+1}{50}} = \frac{50}{4t+1}$$

and $$Q(2) = \frac{50}{8+1} \approx 5.56 \text{ grams}$$

159. A kite is a tethered heavier-than-air or lighter-than-air craft with wing surfaces that react against the air to create lift and drag forces. A kite consists of wings, tethers and anchors. Kites often have a bridle and tail to guide the face of the kite so the wind can lift it.

Radha has a fond of flying kites. Today after taking math exam she is flying kite to release the exam stress. Her kite is flying along the curve having differential equation $\frac{dy}{dx} + 2y = \sin x$.

Based on the above information answer the following:

(i) What type of differential equation is it ? Find its order and degree.

(ii) Find the general solution of the differential equation along which kite is flying.

Sol :

We have $$\frac{dy}{dx} + 2y = \sin x$$

(i) Type of differential equation

From $\frac{dy}{dx} + Py = Q$ we get that this is a linear differential equation of the type where

$$P = P(x) = 2$$

$$Q = Q(x) = \sin x$$

It degree is 1 and order is also .1

(ii) General solution:

Here $$\text{IF} = e^{\int 2dx}$$

$$= e^{2x}$$

$$ye^{2x} = \int \sin x e^{2x}\,dx + c$$

$$ye^{2x} = I + c$$

Now $$I = \int \sin x e^{2x}\,dx$$

$$= \frac{e^{2x}}{2}\sin x - \frac{1}{2}\int e^{2x}\cos x dx$$

$$= \frac{e^{2x}}{2}\sin x - e^{2x}\frac{1}{4}\cos x - \frac{1}{4}\int e^{2x}\sin x dx$$

$$= \frac{e^{2x}}{2}\sin x - \frac{1}{4}e^{2x}\cos x - \frac{1}{4}I$$

$$I + \frac{1}{4}I = \frac{e^{2x}}{2}\sin x - \frac{1}{4}e^{2x}\cos x$$

$$\frac{5}{4}I = \frac{e^{2x}}{2}\sin x - \frac{1}{4}e^{2x}\cos x$$

Based on the above information, answer the following questions:
(i) What are the values of P and Q respectively?
(ii) What is the value of I.F.?
(iii) Find the solution of given equation.
Sol :

We have $\cos^2 x\dfrac{dy}{dx}+y = \tan x$
Dividing on both side by $\cos^2 x$, we have

$$\frac{dy}{dx}+\frac{1}{\cos^2 x}y = \frac{\tan x}{\cos^2 x}$$

$$\frac{dy}{dx}+\sec^2 xy = \tan x\cdot\sec^2 x$$

(i) Comparing this differential equation with

$$\frac{dy}{dx}+Py = Q$$

we have $P = \sec^2 x$

and $Q = \tan x\sec^2 x$

(ii) I.F. (Integrating Factor)

$$\begin{aligned}\text{IF} &= e^{\int Pdx}\\ &= e^{\int \sec^2 x.dx}\\ &= e^{\tan x}\\ &= e^{\tan x}\end{aligned}$$

(iii) Solution of given equation

$$\begin{aligned}y(\text{IF}) &= \int Q(\text{IF})\,dx+C\\ y(e^{\tan x}) &= \int \tan x\sec^2 xe^{\tan x}+C\end{aligned}$$

Substituting $\tan x=t$ we have

$$\sec^2 x.dx = dt$$

Thus $y\,e^{\tan x} = \int e^t t dt$
Integrating by part, we have

$$\begin{aligned}y\,e^{\tan x} &= t\int e^t\,dt-\int\left(\frac{dt}{dt}\int e^t\,dt\right)dt\\ &= te^t-\int e^t\,dt+C\\ &= te^t-e^t+C\\ &= (t-1)e^t+C\\ y\,e^{\tan x} &= (\tan x-1)e^{\tan x}+C\end{aligned}$$

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CHAPTER 10

VECTOR ALGEBRA

OBJECTIVE QUESTIONS

1. The position vectors of points P and Q are $\vec{p}$ and $\vec{q}$ respectively. The point R divides line segment PQ in the ratio 3 : 1 and S is the mid-point of line segment PR. The position vector of S is :

(a) $\dfrac{\vec{p}+3\vec{q}}{4}$ (b) $\dfrac{\vec{p}+3\vec{q}}{8}$

(c) $\dfrac{5\vec{p}+3\vec{q}}{4}$ (d) $\dfrac{5\vec{p}+3\vec{q}}{8}$

Sol : OD 2024

Position vector of point P,

$$\overrightarrow{OP} = \vec{p}$$

Position vector of point Q,

$$\overrightarrow{OQ} = \vec{q}$$

Point R divides line segment PQ in the ratio 3 : 1.
Thus position vector of point R

$$\overrightarrow{OS} = \frac{3\vec{q}+\vec{p}}{3+1} = \frac{3\vec{q}+\vec{p}}{4}$$

Thus position vector of point S

$$\overrightarrow{OS} = \frac{\overrightarrow{OP}+\overrightarrow{OR}}{2}$$

$$= \frac{p+\left(\dfrac{3\vec{q}+\vec{p}}{4}\right)}{2}$$

$$= \frac{5\vec{p}+3\vec{q}}{8}$$

Thus (d) is correct option.

2. Unit vector along $\overrightarrow{PQ}$, where coordinates of P and Q respectively are (2, 1, –1) and (4, 4, –7), is

(a) $2\hat{i}+3\hat{j}-6\hat{k}$ (b) $-2\hat{i}-3\hat{j}+6\hat{k}$

(c) $-\dfrac{2\hat{i}}{7}-\dfrac{3\hat{j}}{7}+\dfrac{6\hat{k}}{7}$ (d) $\dfrac{2\hat{i}}{7}+\dfrac{3\hat{j}}{7}-\dfrac{6\hat{k}}{7}$

Sol : OD 2023

We have $\overrightarrow{PQ} = \overrightarrow{OQ}-\overrightarrow{OP}$

$$= \left(4\hat{i}+4\hat{j}-7\hat{k}\right)-\left(2\hat{i}+\hat{j}-\hat{k}\right)$$

$$= 2\hat{i}+3\hat{j}-6\hat{k}$$

$$\left|\overrightarrow{PQ}\right| = \sqrt{2^2+3^2+(-6)^2} = 7$$

Unit vector along $\overrightarrow{PQ}$ is

$$\hat{a} = \frac{\overrightarrow{PQ}}{\left|\overrightarrow{PQ}\right|} = \frac{2\hat{i}}{7}+\frac{3\hat{j}}{7}-\frac{6\hat{k}}{7}$$

Thus (d) is correct option.

3. If in ΔABC, $\overrightarrow{BA} = 2\vec{a}$ and $\overrightarrow{BC} = 3\vec{b}$, then $\overrightarrow{AC}$ is

(a) $2\vec{a}+3\vec{b}$ (b) $2\vec{a}-3\vec{b}$

(c) $3\vec{b}-2\vec{a}$ (d) $-2\vec{a}-3\vec{b}$

Sol : Delhi 2015

By triangle law of addition,

$$\overrightarrow{BA}+\overrightarrow{AC} = \overrightarrow{BC}$$

$$\overrightarrow{AC} = \overrightarrow{BC}-\overrightarrow{BA}$$

$$= 3\vec{b}-2\vec{a}$$

Thus (c) is correct option.

4. If $\left|\vec{a}\times\vec{b}\right| = \sqrt{3}$ and $\vec{a}\cdot\vec{b} = -3$, then angle between $\vec{a}$ and $\vec{g}$ is

(a) $\dfrac{2\pi}{3}$ (b) $\dfrac{\pi}{6}$

(c) $\dfrac{\pi}{3}$ (d) $\dfrac{5\pi}{6}$

Sol : OD 2023

We have $\left|\vec{a}\times\vec{b}\right| = \sqrt{3}$

and $\vec{a}\cdot\vec{b} = -3$

$$\vec{a}||\vec{b}|\sin\theta = \sqrt{3}$$

and $\vec{a}||\vec{b}|\cos\theta = -3$

Dividing both equations, we get

$$\tan\theta = -\frac{1}{\sqrt{3}}$$

$$\theta = \frac{5\pi}{6}$$

Thus (d) is correct option.

5. If $\hat{i},\hat{j},\hat{k}$ are unit vectors along three mutually perpendicular directions, then

$$|\vec{a}-\vec{b}|^2 = 7$$

$$|\vec{a}|^2+|\vec{b}|^2-2\vec{a}\cdot\vec{b} = 7$$

$$(\sqrt{1+4+9})^2+|\vec{b}|^2-2|\vec{b}|^2 = 7$$

$$14-|\vec{b}|^2 = 7$$

$$|\vec{b}|^2 = 7$$

$$|\vec{b}| = \sqrt{7}$$

Thus (a) is correct option.

5. If $\vec{a}\cdot\vec{b} = -|\vec{a}||\vec{b}|$, then the angle between a and b is

(a) 45° (b) 180°

(c) 90° (d) 60°

Sol : Comp 2018

We have $\vec{a}\cdot\vec{b} = -|\vec{a}||\vec{b}|$

$$|\vec{a}||\vec{b}|\cos\theta = -|\vec{a}||\vec{b}|$$

$$\cos\theta = -1 \Rightarrow \theta = 180^\circ$$

Thus (b) is correct option.

6. Suppose $\vec{a} = \lambda\hat{i}-7\hat{j}+3\hat{k}$, $\vec{b} = \lambda\hat{i}+\hat{j}+2\lambda\hat{k}$.
If the angle between $\vec{a}$ and $\vec{b}$ is greater than 90°, then λ satisfies the inequality

(a) $-7<\lambda<1$ (b) $\lambda>1$

(c) $1<\lambda<7$ (d) $-5<\lambda<1$

Sol : SQP 2020

We have $\vec{a} = \lambda\hat{i}-7\hat{j}+3\hat{k}$

$$\vec{b} = \lambda\hat{i}+\hat{j}+2\lambda\hat{k}$$

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$$

$$= \frac{\lambda^2-7+6\lambda}{\sqrt{\lambda^2+49+9}\sqrt{\lambda^2+1+4\lambda^2}} <$$

Since angle θ is greater than 90° i.e. $\cos\theta<0$, we have

$$(\lambda+7)(\lambda-1) < 0$$

$$-7 < \lambda < 1$$

Thus (a) is correct option.

7. If $\vec{x}$ and $\vec{y}$ are unit vectors and $\vec{x}\cdot\vec{y}=0$, then

(a) $|\vec{x}+\vec{y}|=1$ (b) $|\vec{x}+\vec{y}|=\sqrt{3}$

(c) $|\vec{x}+\vec{y}|=2$ (d) $|\vec{x}+\vec{y}|=\sqrt{2}$

Sol : OD 2011, Delhi 2007

We have $|\vec{x}| = |\vec{y}|=1$ and $\vec{x}\cdot\vec{y}=0$

$$|\vec{x}\cdot\vec{y}|^2 = |\vec{x}|^2+|\vec{y}|^2+2(\vec{x}\cdot\vec{y})$$

$$|\vec{x}+\vec{y}|^2 = 1+1+0=2$$

$$|\vec{x}+\vec{y}| = \sqrt{2}$$

Thus (a) is correct option.

8. The projection of $\vec{a}=3\hat{i}-\hat{j}+5\hat{k}$ on $\vec{b}=2\hat{i}+3\hat{j}+\hat{k}$ is

(a) $\frac{8}{\sqrt{35}}$ (b) $\frac{8}{\sqrt{39}}$

(c) $\frac{8}{\sqrt{14}}$ (d) $\sqrt{14}$

Sol : Comp 2017

The projection of $\vec{a}$ on $\vec{b}$,

$$\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = \frac{(3\hat{i}-\hat{j}+5\hat{k})\cdot(2\hat{i}+3\hat{j}+\hat{k})}{\sqrt{2^2+3^2+1^2}}$$

$$= \frac{8}{\sqrt{14}}$$

Thus (c) is correct option.

9. If $\hat{a}$, $\hat{b}$ and $\hat{c}$ are unit vectors such that $\vec{a}+\vec{b}+\vec{c}=0$. Then, which one of the following is correct?

(a) $\vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{c}\times\vec{a}=0$

(b) $\vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{c}\times\vec{a}\neq 0$

(c) $\vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{a}\times\vec{c}=0$

(d) $\vec{a}\times\vec{b}$, $\vec{b}\times\vec{c}$, $\vec{c}\times\vec{a}$ are mutually perpendicular.

Sol : SQP 2019

We have $\vec{a}+\vec{b}+\vec{c} = 0$

Taking cross product of both sides, we get

$$(\vec{a}+\vec{b}+\vec{c})\times\vec{a} = 0\times\vec{a}$$

$$\vec{a}\times\vec{a}+\vec{b}\times\vec{a}+\vec{c}\times\vec{a} = 0$$

$$0+\vec{b}\times\vec{a}+\vec{c}\times\vec{a} = 0$$

$$-\vec{a}\times\vec{b}+\vec{c}\times\vec{a} = 0$$

$$\vec{a}\times\vec{b} = \vec{c}\times\vec{a}$$

Similarly, $\vec{b}\times\vec{c} = \vec{c}\times\vec{a}$

$$\vec{a}\times\vec{b}=\vec{b}\times\vec{c} = \vec{c}\times\vec{a}\neq 0$$

Thus (b) is correct option.

10. If $|\vec{a}\times\vec{b}|^2+|\vec{a}\cdot\vec{b}|^2=144$ and $|\vec{a}|=4$, then $|\vec{b}|$ is equal to

(a) 16 (b) 8

(c) 3 (d) 12

Sol : OD 2015, Comp 2011

We have $|\vec{a}\times\vec{b}|^2+|\vec{a}\cdot\vec{b}|^2=144$

(c) $(\vec{a}\cdot\hat{j})\hat{i}+(\vec{a}\cdot\hat{k})\hat{j}+(\vec{a}\cdot\hat{i})\hat{k}$

(d) $(\vec{a}\cdot\vec{a})(\hat{i}+\hat{j}+\hat{k})$

Sol : Delhi 2018, OD 2012

Let, $\vec{a} = x\hat{i}+y\hat{j}+z\hat{k}$

$$\vec{a}\cdot\hat{i} = x\hat{i}\cdot\hat{i} = x$$

Similarly, $y = \vec{a}\cdot\hat{j}$ and $z = \vec{a}\cdot\hat{k}$

$$\vec{a} = (\vec{a}\cdot\hat{i})\hat{i}+(\vec{a}\cdot\hat{j})\hat{j}+(\vec{a}\cdot\hat{k})\hat{k}$$

Thus (b) is correct option.

18. $\hat{i}\cdot(\hat{j}\times\hat{k})+\hat{j}\cdot(\hat{k}\times\hat{i})+\hat{k}\cdot(\hat{i}\times\hat{j})$ is equal to

(a) 0 (b) -3

(c) -1 (d) 3

Sol : Delhi 2013

$$\begin{aligned}\hat{i}\cdot(\hat{j}\times\hat{k})+\hat{j}\cdot(\hat{k}\times\hat{i})+\hat{k}\cdot(\hat{i}\times\hat{j}) &= \hat{i}\cdot\hat{i}+\hat{j}\cdot\hat{j}+\hat{k}\cdot\hat{k}\\ &= 1+1+1 = 3\end{aligned}$$

Thus (d) is correct option.

19. If the position vectors of the vertices A, B, C of a triangle ABC are $7\hat{j}+10\hat{k}, -\hat{i}+6\hat{j}+6\hat{k}$ and $-4\hat{i}+9\hat{j}+6\hat{k}$ respectively, then triangle is

(a) equilateral

(b) isosceles

(c) scalene

(d) right angled and isosceles also

Sol : Foreign 2015

We have $\overrightarrow{AB} = -\hat{i}-\hat{j}-4\hat{k}$,

$$\overrightarrow{BC} = -3\hat{i}+3\hat{j}$$

and $\overrightarrow{CA} = 4\hat{i}-2\hat{j}-4\hat{k}$

Now $|\overrightarrow{AB}| = \sqrt{(-1)^2+(-1)^2+(-4)^2} = 3\sqrt{2}$

$$|\overrightarrow{BC}| = \sqrt{(-3)^2+3^2} = 3\sqrt{2}$$

and $|\overrightarrow{CA}|\ \sqrt{4^2+(-2)^2+(-4)^2} = 6$

Now, $|\overrightarrow{AB}|^2+|\overrightarrow{BC}|^2 = |\overrightarrow{CA}|^2$

$$18+18 = 36$$

$$36 = 36$$

Hence, $\triangle ABC$ is right angled and isosceles also.

Thus (d) is correct option.

20. If $\vec{a} = \hat{i}+\hat{j}, \vec{b} = 2\hat{j}-\hat{k}$ and $\vec{r}\times\vec{a} = \vec{b}\times\vec{a}$, $\vec{r}\times\vec{b} = \vec{a}\times\vec{b}$, then $\frac{\vec{r}}{|\vec{r}|}$ is equal to

(a) $\frac{1}{\sqrt{11}}(\hat{i}+3\hat{j}-\hat{k})$ (b) $\frac{1}{\sqrt{11}}(\hat{i}-3\hat{j}+\hat{k})$

(c) $\frac{1}{\sqrt{3}}(\hat{i}-\hat{j}+\hat{k})$ (d) none of these

Sol : Comp 2010, OD 2007

Since, $\vec{r}\times\vec{a}+\vec{r}\times\vec{b} = \vec{b}\times\vec{a}+\vec{a}\times\vec{b} = \vec{0}$

$$\vec{r}\times(\vec{a}+\vec{b}) = \vec{0}$$

Thus $\vec{r}$ is parallel to $\vec{a}+\vec{b}$ and we get

$$\vec{r} = t(\vec{a}+\vec{b}), \quad \text{where } t \text{ is some scalar}$$

$$= t(\hat{i}+3\hat{j}-\hat{k})$$

$$|\vec{r}| = t\sqrt{(1+9+1)} = \sqrt{11}\,t$$

$$\frac{\vec{r}}{|\vec{r}|} = \frac{\hat{i}+3\hat{j}-\hat{k}}{\sqrt{11}} = \frac{1}{\sqrt{11}}(\hat{i}+3\hat{j}-\hat{k})$$

Thus (a) is correct option.

21. If $\vec{a}$ is perpendicular to $\vec{b}$ and $\vec{p}$ is a non-zero vector such that $p\vec{r}+(\vec{r}\cdot\vec{b})\vec{a} = \vec{c}$, then $\vec{r}$ is equal to

(a) $\frac{\vec{c}}{p}-\frac{(\vec{b}\cdot\vec{c})\vec{a}}{p^2}$ (b) $\frac{\vec{a}}{p}-\frac{(\vec{c}\cdot\vec{a})\vec{b}}{p^2}$

(c) $\frac{\vec{b}}{p}-\frac{(\vec{a}\cdot\vec{b})\vec{c}}{p^2}$ (d) $\frac{\vec{c}}{p^2}-\frac{(\vec{b}\cdot\vec{c})\vec{a}}{p}$

Sol : Foreign 2013

We have $p\vec{r}+(\vec{r}\cdot\vec{b})\vec{a} = \vec{c}$ ...(1)

$$p(\vec{r}\cdot\vec{b})+(\vec{r}\cdot\vec{b})(\vec{a}\cdot\vec{b}) = \vec{c}\cdot\vec{b}$$

Since $\vec{a}$ is perpendicular to $\vec{b}$, thus $\vec{a}\cdot\vec{b} = 0$ and we get

$$p(\vec{r}\cdot\vec{b}) = \vec{c}\cdot\vec{b}$$

$$\vec{r}\cdot\vec{b} = \frac{\vec{c}\cdot\vec{b}}{p}$$

From Eq. (1), we get

$$p\vec{r}+\frac{(\vec{c}\cdot\vec{b})}{p}\vec{a} = \vec{c}$$

$$\vec{r} = \frac{\vec{c}}{p}-\frac{\vec{c}\cdot\vec{b}}{p^2}\vec{a}$$

Thus (a) is correct option.

22. Value of a for which $2\hat{i}-\hat{j}+\hat{k}, \hat{i}+2\hat{j}-3\hat{k}$ and $3\hat{i}+a\hat{j}+5\hat{k}$ are coplanar, is

(a) 4 (b) 2

(c) -4 (d) -2

(a) 5 (b) $5\sqrt{2}$

(c) $5\sqrt{3}$ (d) 2

Sol : OD 2012, Delhi 2010

$$\left[\vec{a}+\vec{b}+\vec{c}\right]^2$$
$$=\vec{a}^2+\vec{b}^2+\vec{c}^2+2\vec{a}\cdot\vec{b}+2\vec{b}\cdot\vec{c}+2\vec{c}\cdot\vec{a}$$
$$=a^2+b^2+c^2+\vec{a}\cdot(\vec{b}+\vec{c})+\vec{b}\cdot(\vec{c}+\vec{a})+\vec{c}\cdot(\vec{a}+\vec{b})$$
$$=9+16+25+0+0+0$$
$$=50$$
$$|\vec{a}+\vec{b}+\vec{c}|=5\sqrt{2}$$

Thus (b) is correct option.

29. The area of a parallelogram whose adjacent sides are $\hat{i}-2\hat{j}+3\hat{k}$ and $2\hat{i}+\hat{j}-4\hat{k}$, is

(a) $5\sqrt{3}$ (b) $10\sqrt{3}$

(c) $5\sqrt{6}$ (d) $10\sqrt{6}$

Sol : Foreign 2017

We have $$|\vec{a}\times\vec{b}|=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\1&-2&3\\2&1&-4\end{vmatrix}$$
$$=|\hat{i}(8-3)-\hat{j}(-4-6)+\hat{k}(1+4)|$$
$$=|5\hat{i}+10\hat{j}+5\hat{k}|$$
$$=\sqrt{25+100+25}=5\sqrt{6}$$

Required area $=|\vec{a}\times\vec{b}|=5\sqrt{6}$

Thus (c) is correct option.

30. Let a,b,c be distinct non-negative numbers. If the vectors $a\hat{i}+a\hat{j}+c\hat{k},\hat{i}+\hat{k},c\hat{i}+c\hat{j}+b\hat{k}$ lie in a plane, then c is

(a) the AM of a and b (b) the GM of a and b

(c) the HM of a and b (d) equal to zero

Sol : SQP 2019

Since, the given vectors are coplanar.

$$\begin{vmatrix}a&a&c\\1&0&1\\c&c&b\end{vmatrix}=0$$
$$a(0-c)-a(b-c)+c(c-0)=0$$
$$0-ac-ab+ac+c^2-0=0$$
$$-ab+c^2=0$$
$$ab=c^2$$

Thus c is a GM between a and b.

Thus (b) is correct option.

31. The projection of vector $\hat{i}-2\hat{j}+\hat{k}$ on the $4\hat{i}-4\hat{j}+7\hat{k}$ is

(a) $\frac{\sqrt{6}}{19}$ (b) $\frac{19}{6}$

(c) $\frac{9}{19}$ (d) $\frac{19}{9}$

Sol : OD 2008

We have $\vec{a}=\hat{i}-2\hat{j}+\hat{k}$

and $\vec{b}=4\hat{i}-4\hat{j}+7\hat{k}$

Projection of $\vec{a}$ on $\vec{b}$,

$$\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}=\frac{(\hat{i}-2\hat{j}+\hat{k})\cdot(4\hat{i}-4\hat{j}+7\hat{k})}{\sqrt{16+16+49}}$$
$$=\frac{19}{9}$$

Thus (d) is correct option.

32. If $\vec{a}=-3\hat{i}+7\hat{j}+5\hat{k},\vec{b}=-3\hat{i}+7\hat{j}-3\hat{k}$ and $\vec{c}=7\hat{i}-5\hat{j}-3\hat{k}$ are the three coterminous edges of a parallelepiped, then its volume is

(a) 108 (b) 210

(c) 272 (d) 308

Sol : Delhi 2011, OD 2010

$$\text{Required volume}=\left\|\begin{matrix}-3&7&5\\-3&7&-3\\7&-5&-3\end{matrix}\right\|$$
$$=|(-3)(-21-15)-7(9+21)+5(15-49)|$$
$$=|108-210-170|$$
$$=272$$

33. If $\vec{a}=\hat{i}+\hat{j}+\hat{k},\vec{b}=4\hat{i}+3\hat{j}+4\hat{k}$ and $\vec{c}=\hat{i}+\alpha\hat{j}+\beta\hat{k}$ are linearly dependent vectors and $|\vec{c}|=\sqrt{3}$, then

(a) $\alpha=1,\beta=-1$ (b) $\alpha=1,\beta=\pm1$

(c) $\alpha=-1,\beta=\pm1$ (d) $\alpha=\pm1,\beta=1$

Sol : Comp 2017

Since, vectors $\vec{a},\vec{b}$ and $\vec{c}$ are linearly dependent.

$$\begin{vmatrix}1&1&1\\4&3&4\\1&\alpha&\beta\end{vmatrix}=0$$
$$(3\beta-4\alpha)-(4\beta-4)+(4\alpha-3)=0$$
$$3\beta-4\alpha-4\beta+4+4\alpha-3=0$$
$$-\beta+1=0$$
$$\beta=1$$

Now, $$|\vec{c}|=\sqrt{1+\alpha^2+\beta^2}$$
$$\sqrt{3}=\sqrt{1+\alpha^2+\beta^2}$$
$$3=1+\alpha^2+\beta^2$$

Ch 7 : Integrals

PDF	PDF	PDF	DOC	PPT
Text Book	Worksheet	Solutions	Worksheet	Presentation

Ch 8 : Application of Integrals

PDF	PDF	PDF	DOC	PPT
Text Book	Worksheet	Solutions	Worksheet	Presentation

Ch 9 : Differential Equations

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Class 10th

Maths | Science | Social Science | English LL

Info Tech. | Comp. App. | AI | English Comm.

हिंदी अ | हिंदी ब | संस्कृत

Class 9th

Maths | Science | Social Science | English LL

Info Tech. | Comp. App. | AI | English Comm.

हिंदी अ | हिंदी ब | संस्कृत

Class 8th

Maths | Science | Social Science | English

हिंदी | संस्कृत

Class 7th

Maths | Science | Social Science | English

हिंदी | संस्कृत

Class 6th

Maths | Science | Social Science | English

हिंदी | संस्कृत

Class 5th

Maths | English | हिंदी | EVS

Class 4th

Maths | English | हिंदी | EVS

Class 3th

Maths | English | हिंदी | EVS

Class 2th

Maths | English | हिंदी | EVS

Class 1th

Maths | English | हिंदी | EVS

Prep/KG 3rd Year

Maths | English | हिंदी | EVS

Prep/KG 2nd Year

Maths | English | हिंदी | EVS

Prep/KG 1st Year

Maths | English | हिंदी | EVS

39. **Assertion (A) :** The vectors

$$\vec{a} = 6\hat{i} + 2\hat{j} - 8\hat{k}$$
$$\vec{b} = 10\hat{i} - 2\hat{j} - 6\hat{k}$$
$$\vec{c} = 4\hat{i} - 4\hat{j} + 2\hat{k}$$

represent the sides of a right-angled triangle.

Reason (R) : Three non-zero vectors of which none of two are collinear forms a triangle if their resultant is zero vector or sum of any two vectors is equal to the third.

(a) Both Assertion (A) and Reason (R) are true and the Reason (R) is the correct explanation of the Assertion (A).

(b) Both Assertion (A) and Reason (R) are true and Reason (R) is not the correct explanation of the Assertion (A).

(c) Assertion (A) is true, but Reason (R) is false.

(d) Assertion (A) is false, but Reason (R) is true. 19. Assertion (A) : The vectors

Sol : OD 2024

We have
$$\vec{a} = 6\hat{i} + 2\hat{j} - 8\hat{k}$$
$$\vec{b} = 10\hat{i} - 2\hat{j} - 6\hat{k}$$
$$\vec{c} = 4\hat{i} - 4\hat{j} + 2\hat{k}$$

Let ABC be a triangle such that

$$\overrightarrow{AB} = |\vec{a}| = \sqrt{6^2 + 2^2 + (-8)^2} = \sqrt{104}$$
$$|\overrightarrow{BC}| = |\vec{b}| = \sqrt{10^2 + (-2)^2 + (-6)^2} = \sqrt{140}$$
$$|\overrightarrow{AC}| = |\vec{c}| = \sqrt{4^2 + (-4)^2 + 2^2} = \sqrt{36} = 6$$

As, we can observe that

$$AB^2 + AC^2 = 104 + 36$$
$$= 140$$
$$= BC^2$$

So, ΔABC is a right-angled triangle

Also, $\vec{a} = 6\hat{i} + 2\hat{j} - 8\hat{k}$

and $\vec{c} = 4\hat{i} - 4\hat{j} + 2\hat{k}$

Thus sum of two vectors $\vec{a}$ and $\vec{c}$ is equal to third vector $\vec{b}$.

Hence, both Assertion (A) and Reason (R) are true and Reason (R) is not the correct explanation of the Assertion (A).

Thus (b) is correct option.

40. **Assertion (A) :** Area of the parallelogram whose adjacent sides are $\hat{i} + \hat{j} - \hat{k}$ and $2\hat{i} - j + \hat{k}$ is $3\sqrt{2}$ square units.

Reason (R) : area of the parallelogram whose adjacent sides are represented by the vectors $\vec{a}$ and $\vec{b}$ is $|\vec{a} - \vec{b}|$

(a) Both Assertion (A) and Reason (R) are true and the Reason (R) is the correct explanation of the Assertion (A).

(b) Both Assertion (A) and Reason (R) are true and Reason (R) is not the correct explanation of the Assertion (A).

(c) Assertion (A) is true, but Reason (R) is false.

(d) Assertion (A) is false, but Reason (R) is true. 19. Assertion (A) : The vectors

Sol : Delhi 2009

Area of the parallelogram whose adjacent side are represented by the vectors $\vec{a}$ is $\vec{b}$ is $|\vec{a}\times\vec{b}|$. So, given reason is false.

Let $\vec{a} = \hat{i} + \hat{j} - \hat{k}$ and $\vec{b} = 2\hat{i} - \hat{j} + \hat{k}$

then
$$\vec{a}\times\vec{b} = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}$$
$$= \hat{i}(1-1) - \hat{j}(1+2) + \hat{k}(-1-2)$$
$$= 3\hat{j} - 3\hat{k}$$

now
$$|\vec{a}\times\vec{b}| = |-3\hat{j} - 3\hat{k}|$$
$$= \sqrt{(-3)^2 + (-3)^2}$$
$$= \sqrt{9+9}$$
$$= \sqrt{18}$$
$$= 3\sqrt{2} \text{ square units}$$

Hence, Assertion is true; reason is false,

Thus (c) is correct option.

VERY SHORT ANSWER QUESTIONS

41. The vector equation of a line which passes through the points $(3, 4, -7)$ and $(1, -1, 6)$ is ....

Sol : OD 2020

Any line passing through the points $\vec{a}$ and $\vec{b}$ has vector equation

$$\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a}).$$

Vector equation of the line passing through the points $\vec{a} = 3\hat{i} + 4\hat{j} - 7\hat{k}$ and $\vec{b} = \hat{i} - \hat{j} + 6\hat{k}$ is given by

$$\vec{r} = 3\hat{i} + 4\hat{j} - 7\hat{k} + \lambda\left[\left(\hat{i} - \hat{j} + 6\hat{k}\right) - \left(3\hat{i} + 4\hat{j} - 7\hat{k}\right)\right]$$
$$\vec{r} = 3\hat{i} + 4\hat{j} - 7\hat{k} + \lambda\left(-2\hat{i} - 5\hat{j} + 13\hat{k}\right)$$

50. If vectors $\vec{a}$ and $\vec{b}$ are two unit vectors such that $\vec{a}+\vec{b}$ is also a unit vector, then find the angle between $\vec{a}$ and $\vec{b}$.

Sol : Delhi 2014

We have $\vec{a}+\vec{b} = \vec{c}$ (1)

where $|\vec{a}| = 1$ $|\vec{b}| = 1$ and $|\vec{c}| = 1$

Squaring both side of eq (1)we have

$$\begin{aligned}|\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}|^2\cos\theta &= |\vec{c}|^2\\ 1+1+2\times1\times1\cos\theta &= 1\\ 2+2\cos\theta &= 1\\ 2\cos\theta &= -1\\ \cos\theta &= -\frac{1}{2}\\ \cos\theta &= \cos\frac{2\pi}{3}\end{aligned}$$

Thus $\theta = \frac{2\pi}{2}$.

51. Find the value of $[\hat{i},\hat{k},\hat{j}]$.

Sol : OD 2018, SQP 2016

$$\begin{aligned}|\hat{i}\ \hat{k}\ \hat{j}| &= \hat{i}\cdot(\hat{k}\times\hat{j}) = -\hat{i}\cdot(\hat{j}\times\hat{k})\\ &= -[\hat{i}\ \hat{j}\ \hat{k}] = -\begin{vmatrix}1&0&0\\0&1&0\\0&0&1\end{vmatrix} = -1\end{aligned}$$

52. For what value of λ are the vectors $\hat{i}+2\lambda\hat{j}+\hat{k}$ and $2\hat{i}+\hat{j}-3\hat{k}$ perpendicular?

Sol : OD 2015

We have $\vec{a} = \hat{i}+2\lambda\hat{j}+\hat{k}$

and $\vec{b} = 2\hat{i}+\hat{j}-3\hat{k}$

Since, vectors are perpendicular.

$$\begin{aligned}\vec{a}\cdot\vec{b} &= 0\\ (\hat{i}+2\lambda\hat{j}+\hat{k})\cdot(2\hat{i}+\hat{j}-3\hat{k}) &= 0\\ 2+2\lambda-3 &= 0\\ 2\lambda-1 &= 0\\ \lambda &= \tfrac{1}{2}\end{aligned}$$

53. If $|\vec{a}| = \sqrt{3}$, $|\vec{b}| = 2$ and angle between $\vec{a}$ and $\vec{b}$ is $60°$, then find $\vec{a}\cdot\vec{b}$.

Sol : Foreign 2011

We know that,

$$\vec{a}\cdot\vec{b} = |\vec{a}|\cdot|\vec{b}|\cos\theta$$

Substituting $|\vec{a}| = \sqrt{3}$, $|\vec{b}| = 2$ and $\theta = 60°$, we get

$$\begin{aligned}\vec{a}\cdot\vec{b} &= \sqrt{3}\times 2\cos 60°\\ &= \frac{1}{2}\times 2\sqrt{3} = \sqrt{3} \qquad \cos 60° = \tfrac{1}{2}\\ \vec{a}\cdot\vec{b} &= \sqrt{3}\end{aligned}$$

54. Find the value of λ, if the vectors $2\hat{i}+\lambda\hat{j}+3\hat{k}$ and $3\hat{i}+2\hat{j}-4\hat{k}$ are perpendicular to each other.

Sol : Foreign 2010

We have $\vec{a} = 2\hat{i}+\lambda\hat{j}+3\hat{k}$

and $\vec{b} = 3\hat{i}+2\hat{j}-4\hat{k}$

Since, vectors are perpendicular.

$$\begin{aligned}\vec{a}\cdot\vec{b} &= 0\\ (2\hat{i}+\lambda\hat{j}+3\hat{k})\cdot(3\hat{i}+2\hat{j}-4\hat{k}) &= 0\\ 6+2\lambda-12 &= 0\\ 2\lambda-6 &= 0\\ \lambda-3 &= 0\\ \lambda &= 3\end{aligned}$$

55. If $|\vec{a}| = 2$, $|\vec{b}| = 3$ and $\vec{a}\cdot\vec{b} = 3$, then find the projection of $\vec{b}$ on $\vec{a}$.

Sol : Delhi 2010

We have $|\vec{a}| = 2$, $|\vec{b}| = 3$ and $\vec{a}\cdot\vec{b} = 3$

Projection of $\vec{b}$ and $\vec{a}$ is given by,

$$\begin{aligned}\frac{\vec{b}\cdot\vec{a}}{|\vec{a}|} &= \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|} \qquad \vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}\\ &= \frac{3}{2}\end{aligned}$$

56. If $\vec{a}$ and $\vec{b}$ are two vectors, such that $|\vec{a}\cdot\vec{b}| = |\vec{a}\times\vec{b}|$, then find the angle between $\vec{a}$ and $\vec{b}$.

Sol : Comp 2010, OD 2008

We have $|\vec{a}\cdot\vec{b}| = |\vec{a}\times\vec{b}|$

If θ is the angle between $\vec{a}$ and $\vec{b}$ the we have

$$\begin{aligned}|\vec{a}||\vec{b}|\cos\theta &= |\vec{a}||\vec{b}|\sin\theta\\ \cos\theta &= \sin\theta\\ \tan\theta &= 1\\ \tan\theta &= \tan\frac{\pi}{4} \qquad 1 = \tan\tfrac{\pi}{4}\\ \theta &= \frac{\pi}{4}\end{aligned}$$

So, the angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{4}$.

65. If $\vec{a}\cdot\vec{a}=0$ and $\vec{a}\cdot\vec{b}=0$, then what can be concluded about the vector $\vec{b}$?

Sol : Foreign 2011

We have,
$$\vec{a}\cdot\vec{a} = 0$$
$$|\vec{a}|^2 = 0$$
$$|\vec{a}| = 0 \quad ...(1)$$
and
$$\vec{a}\cdot\vec{b} = 0$$
$$|\vec{a}||\vec{b}|\cos\theta = 0 \quad ...(2)$$

From eq (1) and eq (2), it may be concluded that $\vec{b}$ is either zero or non-zero perpendicular vector.

66. Write the projection of vector $\hat{i}-\hat{j}$ on the vector $\hat{i}+\hat{j}$.

Sol : OD 2011

We have $\vec{a} = \hat{i}-\hat{j}$

and $\vec{b} = \hat{i}+\hat{j}$

Now, the projection of $\vec{a}$ and $\vec{b}$ is given by
$$\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = \frac{(\hat{i}-\hat{j})\cdot(\hat{i}+\hat{j})}{\sqrt{1^2+1^2+0^2}}$$
$$= \frac{1-1}{\sqrt{2}} = 0$$

67. If $\vec{a}$ and $\vec{b}$ are two vectors, such that $|\vec{a}|=2$, $|\vec{b}|=1$ and $\vec{a}\cdot\vec{b}=1$, then find $(3\vec{a}-5\vec{b})\cdot(2\vec{a}+7\vec{b})$.

Sol : Delhi 2007

We have $|\vec{a}|=2,\ |\vec{b}|=1$ and $\vec{a}\cdot\vec{b}=1 \quad ...(1)$

Now,
$$(3\vec{a}-5\vec{b})\cdot(2\vec{a}+7\vec{b}) = 6\vec{a}\cdot\vec{a}+21\vec{a}\cdot\vec{b}-10\vec{b}\cdot\vec{a}-35\vec{b}\cdot\vec{b}$$
$$= 6|\vec{a}|^2+21\vec{a}\cdot\vec{b}-10\vec{a}\cdot\vec{b}-35|\vec{b}|^2$$
$$= 6|\vec{a}|^2+11\vec{a}\cdot\vec{b}-35|\vec{b}|^2$$
$$= 6(2)^2+11(1)-35(1)^2 \quad \text{[from Eq. (1)]}$$
$$= 24+11-35 = 0$$

Hence, $(3\vec{a}-5\vec{b})\cdot(2\vec{a}+7\vec{b}) = 0$

68. Find a unit vector in the direction of vector $\vec{a}=2\hat{i}+3\hat{j}+6\hat{k}$.

Sol : OD 2009, Delhi 2007

We have
$$\vec{a} = 2\hat{i}+3\hat{j}+6\hat{k}$$
$$|\vec{a}| = \sqrt{2^2+3^2+6^2}$$
$$= \sqrt{49} = 7$$

Required unit vector in the direction of vector $\vec{a}$,
$$\hat{a} = \frac{\vec{a}}{|\vec{a}|}$$
$$= \frac{2\hat{i}+3\hat{j}+6\hat{k}}{7}$$
$$= \frac{2}{7}\hat{i}+\frac{3}{7}\hat{j}+\frac{6}{7}\hat{k}$$

69. If $\vec{a}=7\hat{i}+\hat{j}-4\hat{k}$ and $\vec{b}=2\hat{i}+6\hat{j}+3\hat{k}$, then find the projection of $\vec{a}$ on $\vec{b}$.

Sol : Delhi 2015

We have $\vec{a} = 7\hat{i}+\hat{j}-4\hat{k}$

and $\vec{b} = 2\hat{i}+6\hat{j}+3\hat{k}$

Now, the projection of $\vec{a}$ and $\vec{b}$ is given by
$$\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = \frac{(7\hat{i}+\hat{j}-4\hat{k})\cdot(2\hat{i}+6\hat{j}+3\hat{k})}{\sqrt{2^2+6^2+3^2}}$$
$$= \frac{14+6-12}{\sqrt{49}} = \frac{8}{7}$$

70. Find the projection of the vector $\hat{i}+3\hat{j}+7\hat{k}$ on the vector $2\hat{i}-3\hat{j}+6\hat{k}$.

Sol : Delhi 2014

We have $\vec{a} = \hat{i}+3\hat{j}+7\hat{k}$

and $\vec{b} = 2\hat{i}-3\hat{j}+6\hat{k}$

Now, the projection of $\vec{a}$ and $\vec{b}$ is given by
$$\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = \frac{(\hat{i}+3\hat{j}+7\hat{k})\cdot(2\hat{i}-3\hat{j}+6\hat{k})}{\sqrt{2^2+3^2+6^2}}$$
$$= \frac{2-9+42}{\sqrt{49}}$$
$$= \frac{35}{7} = 5$$

71. If $|\vec{a}|=8$, $|\vec{b}|=3$ and $|\vec{a}\times\vec{b}|=12$, find the angle between $\vec{a}$ and $\vec{b}$.

Sol : Comp 2014, OD 2010

Let θ be the angle between $\vec{a}$ and $\vec{b}$.

We have $|\vec{a}| = 8,\ |\vec{b}| = 3$ and $|\vec{a}\times\vec{b}| = 12$

If θ is the angle between $\vec{a}$ and $\vec{b}$, the we have
$$|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\theta$$
$$12 = |\vec{a}||\vec{b}|\sin\theta$$
$$\sin\theta = \frac{12}{|\vec{a}||\vec{b}|} = \frac{12}{8\times 3}$$
$$\sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$$

Hence, the required angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{6}$.

72. Write the projection of the vector $\vec{a}=2\hat{i}-\hat{j}+\hat{k}$ on the vector $\vec{b}=\hat{i}+2\hat{j}+2\hat{k}$.

$(2,3,-5)$ and having Dr's $(1,1,1)$ is

$$\vec{r} = 2\hat{i} + 3\hat{j} - 5\hat{k} + \lambda(\hat{i} + \hat{j} + \hat{k})$$

79. Given $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$, $\vec{b} = 3\hat{i} - \hat{k}$ and $\vec{c} = 2\hat{i} + \hat{j} - 2\hat{k}$. Find a vector $\vec{d}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$ and $\vec{c} \cdot \vec{d} = 3$.

Sol : OD 2024

We have $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$

$$\vec{b} = 3\hat{i} - \hat{k}$$

Vector which is perpendicular to both $\vec{a}$ and $\vec{b}$ must be parallel to $\vec{a} \times \vec{b}$.

Thus
$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 3 & 0 & -1 \end{vmatrix}$$

$$= \hat{i}(1) - \hat{j}(2-3) + 3\hat{k}$$

$$\vec{a} \times \vec{b} = \hat{i} + 5\hat{j} + 3\hat{k}$$

Since $\vec{d}$ is parallel to $\vec{a} \times \vec{b}$, then

$$\vec{d} = \lambda(\vec{a} \times \vec{b})$$

$$= \lambda(\hat{i} + 5\hat{j} + 3\hat{k})$$

Now $\vec{C} = 2\hat{i} + \hat{j} - 2\hat{k}$

Since $\vec{c} \cdot \vec{d} = 3$ we have

$$\vec{c} \cdot \vec{d} = (2\hat{i} + \hat{j} - 2\hat{k}) \cdot (\lambda)(\hat{i} + 5\hat{j} + 3\hat{k})$$

$$3 = 2\lambda + 5\lambda - 6\lambda$$

$$3 = \lambda$$

Thus $\vec{d} = 3(\hat{i} + 5\hat{j} + 3\hat{k})$

Hence, the vector $\vec{d}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$ is given by $3(\hat{i} + 5\hat{j} + 3\hat{k})$.

80. If $\vec{a}$, $\vec{b}$, $\vec{c}$ are three non-zero unequal vectors such that $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c}$, then find the angle between $\vec{a}$ and $\vec{b} - \vec{c}$.

Sol : OD 2023

We have $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c}$

$$\vec{a} \cdot (\vec{b} - \vec{c}) = 0$$

$$\vec{a} \perp (\vec{b} - \vec{c}) \text{ or } \vec{b} = \vec{c}$$

But it's given that $\vec{a}$, $\vec{b}$, $\vec{c}$ are three non-zero unequal vectors. Therefore $\vec{a} \perp (\vec{b} - \vec{c})$ and $\vec{b} \neq \vec{c}$

Hence, the angle between $\vec{a}$ and $\vec{b} - \vec{c}$ is $\frac{\pi}{2}$.

81. If $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{b} = 2\hat{i} + 4\hat{j} - 5\hat{k}$ represent two adjacent sides of a parallelogram, find unit vectors parallel to the diagonals of the parallelogram.

Sol : OD 2020

We have $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$

and $\vec{b} = 2\hat{i} + 4\hat{j} - 5\hat{k}$

Diagonals of the parallelogram whose adjacent sides are $\vec{a}$ and $\vec{b}$ are given by

$$\vec{p} = \vec{a} + \vec{b} \text{ and } \vec{q} = \vec{a} - \vec{b}$$

Now, $\vec{p} = (\hat{i} + 2\hat{j} + 3\hat{k}) + (2\hat{i} + 4\hat{j} - 5\hat{k})$

$$= 3\hat{i} + 6\hat{j} - 2\hat{k}$$

and $\vec{q} = (\hat{i} + 2\hat{j} + 3\hat{k}) - (2\hat{i} + 4\hat{j} - 5\hat{k})$

$$= -\hat{i} - 2\hat{j} + 8\hat{k}$$

$$\hat{p} = \frac{\vec{p}}{|\vec{p}|} = \frac{3\vec{i} + 6\vec{j} - 2\hat{k}}{\sqrt{9 + 36 + 4}}$$

$$= \frac{3\hat{i} + 6\hat{j} - 2\hat{k}}{7} = \frac{3}{7}\hat{i} + \frac{6}{7}\hat{j} - \frac{2}{7}\hat{k}$$

and
$$\hat{q} = \frac{\vec{q}}{|\vec{q}|} = \frac{-\hat{i} - 2\hat{j} + 8\hat{k}}{\sqrt{1 + 4 + 64}}$$

$$= \frac{-\hat{i} - 2\hat{j} + 8\hat{k}}{\sqrt{69}}$$

$$= \frac{-1}{\sqrt{69}}\hat{i} - \frac{-2}{\sqrt{69}}\hat{j} + \frac{8}{\sqrt{69}}\hat{k}$$

82. If $\vec{a} = 2\hat{i} + 3\hat{j} + \hat{k}$, $\vec{b} = \hat{i} - 2\hat{j} + \hat{k}$ and $\vec{c} = -3\hat{i} + \hat{j} + 2\hat{k}$, find $[\vec{a}\,\vec{b}\,\vec{c}]$.

Sol : Delhi 2019, OD 2011

We have $\vec{a} = 2\hat{i} + 3\hat{j} + \hat{k}$,

$$\vec{b} = \hat{i} - 2\hat{j} + \hat{k}$$

and $\vec{c} = -3\hat{i} + \hat{j} + 2\hat{k}$

Now
$$[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 3 & 1 \\ 1 & -2 & 1 \\ -3 & 1 & 2 \end{vmatrix}$$

$$= 2(-4-1) - 3(2+3) + 1(1-6)$$

$$= -10 - 15 - 5 = -30$$

83. If $|\vec{a}| = 2$, $|\vec{b}| = 7$ and $\vec{a} \times \vec{b} = 3\hat{i} + 2\hat{j} + 6\hat{k}$, find the angle between $\vec{a}$ and $\vec{b}$.

Sol : OD 2019

We have, $\vec{a} \times \vec{b} = 3\hat{i} + 2\hat{j} + 6\hat{k}$

Now, $|\vec{a} \times \vec{b}| = \sqrt{3^2 + 2^2 + 6^2} = \sqrt{49} = 7$

$$\cos\theta = \frac{20}{60} = \frac{1}{3}$$

$$\theta = \cos^{-1}\left(\frac{1}{3}\right)$$

88. Let $\vec{a} = 4\hat{i} + 5\hat{j} - \hat{k}, \vec{b} = \hat{i} - 4\hat{j} + 5\hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j} - \hat{k}$. Find a vector $\vec{d}$ which is perpendicular to both $\vec{c}$ and $\vec{b}$ and $\vec{d} \cdot \vec{a} = 21$.

Sol : OD 2018

We have, $\vec{a} = 4\hat{i} + 5\hat{j} - \hat{k}$,

$$\vec{b} = \hat{i} - 4\hat{j} + 5\hat{k}$$

and $\vec{c} = 3\hat{i} + \hat{j} - \hat{k}$

Since, $\vec{d}$ is perpendicular to both $\vec{c}$ and $\vec{b}$.

$$\vec{d} = \lambda(\vec{c} \times \vec{d})$$

$$= \lambda\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & -1 \\ 1 & -4 & 5 \end{vmatrix}$$

$$= \lambda[\hat{i}(5-4) - \hat{j}(15+1) + \hat{k}(-12-1)]$$

$$\vec{d} = \lambda(\hat{i} - 16\hat{j} - 13\hat{k}) \quad \text{...(1)}$$

Since $\vec{d} \cdot \vec{a} = 21$ we have

$$\lambda(\hat{i} - 16\hat{j} - 13\hat{k}) \cdot (4\hat{i} + 5\hat{j} - \hat{k}) = 21$$

$$\lambda(4 - 80 + 13) = 21$$

$$\lambda(-63) = 21$$

$$\lambda = \frac{-1}{3}$$

Now, from Eq. (1), we get

$$\vec{d} = -\frac{1}{3}(\hat{i} - 16\hat{j} - 13\hat{k})$$

89. Find the magnitude of each of the two vectors $\vec{a}$ and $\vec{b}$, having the same magnitude such that the angle between them is $60°$ and their scalar product is $\frac{9}{2}$.

Sol : OD 2018

Given, two vectors $\vec{a}$ and $\vec{b}$ such that $|\vec{a}| = |\vec{b}|$, $\vec{a} \cdot \vec{b} = \frac{9}{2}$ and angle between them is $60°$.

If θ is angle between $\vec{a}$ and $\vec{b}$ then we have

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta,$$

Substituting values we obtain

$$\frac{9}{2} = |\vec{a}| \cdot |\vec{a}|\cos 60°$$

$$\frac{1}{2} \cdot |\vec{a}|^2 = \frac{9}{2} \qquad \cos 60° = \frac{1}{2}$$

$$|\vec{a}|^2 = 9$$

$$|\vec{a}| = 3 \quad \text{magnitude cannot be negative]}$$

Thus, $|\vec{a}| = |\vec{b}| = 3$

90. Find the position vector of a point which divides the join of points with position vectors $\vec{a} - 2\vec{b}$ and $2\vec{a} + \vec{b}$ externally in the ratio 2 1.

Sol : Delhi 2016

Given position vectors are

$$\overrightarrow{OA} = \vec{a} - 2\vec{b} \text{ and}$$

$$\overrightarrow{OB} = 2\vec{a} + \vec{b}$$

Let $\overrightarrow{OC}$ be the position vector of a point C which divides the join of points, with position vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$, externally in the ratio 2 1.

Using externally section formula we have

$$\overrightarrow{OC} = \frac{2\overrightarrow{OB} - 1\overrightarrow{OA}}{2-1}$$

$$= \frac{2(2\vec{a} + \vec{b}) - 1(\vec{a} - 2\vec{b})}{1}$$

$$= 4\vec{a} + 2\vec{b} - \vec{a} + 2\vec{b}$$

$$= 3\vec{a} + 4\vec{b}$$

91. If $\vec{a} = 4\hat{i} - \hat{j} + \hat{k}$ and $\vec{b} = 2\hat{i} - 2\hat{j} + \hat{k}$, then find a unit vector parallel to the vector $\vec{a} + \vec{b}$.

Sol : OD 2016, Foreign 2014

We have $\vec{a} = 4\hat{i} - \hat{j} + \hat{k}$,

$$\vec{b} = 2\hat{i} - 2\hat{j} + \hat{k}$$

Now, $\vec{a} + \vec{b} = (4\hat{i} - \hat{j} + \hat{k}) + (2\hat{i} - 2\hat{j} + \hat{k})$

$$= 6\hat{i} - 3\hat{j} + 2\hat{k}$$

and $|\vec{a} + \vec{b}| = \sqrt{(6)^2 + (-3)^2 + (2)^2}$

$$= \sqrt{36 + 9 + 4}$$

$$= \sqrt{49} = 7 \text{ units}$$

The unit vector parallel to the vector $\vec{a} + \vec{b}$ is

$$\frac{\vec{a} + \vec{b}}{|\vec{a} + \vec{b}|} = \frac{6\hat{i} - 3\hat{j} + 2\hat{k}}{7}$$

92. Find a vector in the direction of vector $2\hat{i} - 3\hat{j} + 6\hat{k}$ which has magnitude of 21 units.

Sol : Foreign 2014

We have $\vec{a} = 2\hat{i} - 3\hat{j} + 6\hat{k}$

Now $|\vec{a}| = \sqrt{2^2 + (-3)^2 + 6^2}$

$$= \sqrt{4 + 9 + 36}$$

$$= \sqrt{49} = 7 \text{ units}$$

The unit vector in the direction of the given vector $\vec{a}$ is

$$9\hat{a} = 9 \times \frac{1}{3}(\hat{i} - 2\hat{j} + 3\hat{k})$$

$$= 3\hat{i} - 6\hat{j} + 6\hat{k}$$

97. Write the unit vector in the direction of vector $\overrightarrow{PQ}$, where P and Q are the points (1,3,0) and (4,5,6), respectively.

Sol : Foreign 2014

Given points are $P(1,3,0)$ and $Q(4,5,6)$.

Here, $x_1 = 1,\ y_1 = 3,\ z_1 = 0$

and $x_2 = 4,\ y_2 = 5,\ z_2 = 6$

So, position vector,

$$\overrightarrow{PQ} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$

$$= (4-1)\hat{i} + (5-3)\hat{j} + (6-0)\hat{k}$$

$$= 3\hat{i} + 2\hat{j} + 6\hat{k}$$

Magnitude of given vector

$$\left|\overrightarrow{PQ}\right| = \sqrt{3^2 + 2^2 + 6^2}$$

$$= \sqrt{9 + 4 + 36}$$

$$= \sqrt{49} = 7 \text{ units.}$$

Hence, the unit vector in the direction of $\overrightarrow{PQ}$ is

$$\frac{\overrightarrow{PQ}}{\left|\overrightarrow{PQ}\right|} = \frac{3\hat{i} + 2\hat{j} + 6\hat{k}}{7}$$

$$= \frac{3}{7}\hat{i} + \frac{2}{7}\hat{j} + \frac{6}{7}\hat{k}$$

98. If a unit vector $\vec{a}$ makes angle $\frac{\pi}{3}$ with $\hat{i}$, $\frac{\pi}{4}$ with $\hat{j}$ and an acute angle θ with $\hat{k}$, then find the value of θ with $\hat{k}$, then find the value of θ.

Sol : Delhi 2013, Comp 2009

If a line/vector makes angles α, β and γ with the x-axis, y-axis and z-axis, respectively, then we have

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

Here, we have $\alpha = \frac{\pi}{3}$, $\beta = \frac{\pi}{4}$ and $\gamma = \theta$

$$\cos^2\frac{\pi}{3} + \cos^2\frac{\pi}{4} + \cos^2\theta = 1$$

$$\left(\frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \cos^2\theta = 1$$

$$\frac{1}{4} + \frac{1}{2} + \cos^2\theta = 1$$

$$\cos^2\theta = 1 - \frac{1}{4} - \frac{1}{2} = \frac{1}{4} \Rightarrow \cos\theta = \pm\frac{1}{2}$$

Since θ is an acute angle, therefore $\cos\theta = \frac{1}{2}$ and

$$\theta = \frac{\pi}{3}$$

99. Write a unit vector in the direction of the sum of vectors $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{b} = -\hat{i} + \hat{j} + 3\hat{k}$.

Sol : Foreign 2013

We have $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$ and

$$\vec{b} = -\hat{i} + \hat{j} + 3\hat{k}$$

Sum of two vectors,

$$\vec{a} + \vec{b} = (2\hat{i} - \hat{j} + 2\hat{k}) + (-\hat{i} + \hat{j} + 3\hat{k})$$

$$= \hat{i} + 5\hat{k}$$

Required unit vector,

$$\frac{\vec{a} + \vec{b}}{|\vec{a} + \vec{b}|} = \frac{\hat{i} + 5\hat{k}}{\sqrt{1^2 + 5^2}} = \frac{\hat{i} + 5\hat{k}}{\sqrt{1 + 25}}$$

$$= \frac{\hat{i} + 5\hat{k}}{\sqrt{26}} = \frac{1}{\sqrt{26}}\hat{i} + \frac{5\hat{k}}{\sqrt{26}}$$

100. P and Q are two points with position vectors $3\vec{a} - 2\vec{b}$ and $\vec{a} + \vec{b}$, respectively. Write the position vector of a point R which divides the line segment PQ in the ratio 2 1 externally.

Sol : OD 2013

Given position vectors are

$$\overrightarrow{OP} = 3\vec{a} - 2\vec{b} \text{ and}$$

$$\overrightarrow{OQ} = \vec{a} + \vec{b}$$

Let $\overrightarrow{OR}$ be the position vector of a point R which divides the join of points, with position vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$, externally in the ratio 2 1.

Using externally section formula we have

$$\overrightarrow{OR} = \frac{2\overrightarrow{OQ} - 1\overrightarrow{OP}}{2-1}$$

$$= \frac{2(\vec{a} + \vec{b}) - 1(3\vec{a} - 2\vec{b})}{1}$$

$$= 2\vec{a} + 2\vec{b} - 3\vec{a} + 2\vec{b}$$

$$= -\vec{a} + 4\vec{b}$$

101. L and M are two points with position vectors $2\vec{a} - \vec{b}$ and $\vec{a} + 2\vec{b}$, respectively. Write the position vector of a point N which divides the line segment LM in the ratio 2 1 externally.

Sol : Comp 2013

Given position vectors are

$$\overrightarrow{OL} = 2\vec{a} - \vec{b} \text{ and}$$

$$\overrightarrow{OM} = \vec{a} + 2\vec{b}$$

Let $\overrightarrow{ON}$ be the position vector of a point N which divides the join of points, with position vectors $\overrightarrow{OL}$ and $\overrightarrow{OM}$, externally in the ratio 2 1.

The unit vector in the direction of the given vector $\vec{a}$ is

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{1}{3}(2\hat{i} - \hat{j} + 2\hat{k})$$
$$= \frac{2}{3}\hat{i} - \frac{1}{3}\hat{j} + \frac{2}{3}\hat{k}$$

The vector of magnitude equal to 6 units and in the direction of $\vec{a}$ is given by

$$6\hat{a} = 6\left(\frac{2}{3}\hat{i} - \frac{1}{3}\hat{j} + \frac{2}{3}\hat{k}\right)$$
$$= 4\hat{i} - 2\hat{j} + 4\hat{k}$$

107. Find the vector of vector of mid-point of the line segment AB, where A is point $(3, 4, -2)$ and B is point $(1, 2, 4)$.

Sol : Delhi 2010

Mid-point of the position vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ is given by

$$\frac{\vec{a} + \vec{b}}{2} = \frac{(a_1 + b_1)\hat{i} + (a_2 + b_2)\hat{j} + (a_3 + b_3)\hat{k}}{2}.$$

Here points are $A(3, 4, -2)$ and $B(1, 2, 4)$ whose position vectors are

$$\vec{a} = 3\hat{i} + 4\hat{j} - 2\hat{k} \text{ and } \vec{b} = \hat{i} + 2\hat{j} + 4\hat{k}.$$

Now, position vector of mid-point of vector joining points $A(3, 4, -2)$ and $B(1, 2, 4)$ is

$$\frac{\vec{a} + \vec{b}}{2} = \frac{(3\hat{i} + 4\hat{j} - 2\hat{k}) + (\hat{i} + 2\hat{j} + 4\hat{k})}{2}$$
$$= \frac{4\hat{i} + 6\hat{j} + 2\hat{k}}{2}$$
$$= 2\hat{i} + 3\hat{j} + \hat{k}$$

108. Find the unit vector in the direction of the sum of vectors $2\hat{i} + 3\hat{j} - \hat{k}$ and $4\hat{i} - 3\hat{j} + 2\hat{k}$.

Sol : Foreign 2010, Comp 2007

We have $\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$ and

$$\vec{b} = 4\hat{i} - 3\hat{j} + 2\hat{k}$$

Sum of two vectors,

$$\vec{a} + \vec{b} = (2\hat{i} + 3\hat{j} - \hat{k}) + (4\hat{i} - 3\hat{j} + 2\hat{k})$$
$$= 6\hat{i} + \hat{k}$$

Required unit vector

$$\frac{\vec{a} + \vec{b}}{|\vec{a} + \vec{b}|} = \frac{6\hat{i} + \hat{k}}{\sqrt{6^2 + 1^2}} = \frac{6\hat{i} + \hat{k}}{\sqrt{36 + 1}}$$
$$= \frac{6\hat{i} + \hat{k}}{\sqrt{37}} = \frac{6}{\sqrt{37}}\hat{i} + \frac{\hat{k}}{\sqrt{37}}$$

109. Write a vector of magnitude 9 units in the direction of vector $-2\hat{i} + \hat{j} + 2\hat{k}$.

Sol : OD 2010

We have $\vec{a} = -2\hat{i} + \hat{j} + 2\hat{k}$

Now
$$|\vec{a}| = \sqrt{2^2 + (1)^2 + 2^2}$$
$$= \sqrt{4 + 1 + 4}$$
$$= \sqrt{9} = 3 \text{ units}$$

The unit vector in the direction of the given vector $\vec{a}$ is

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{1}{3}(-2\hat{i} + \hat{j} + 2\hat{k})$$

The vector of magnitude equal to 9 units and in the direction of $\vec{a}$ is given by

$$3\hat{a} = \frac{9}{3}(-2\hat{i} + \hat{j} + 2\hat{k})$$
$$= 3(-2\hat{i} + \hat{j} + 2\hat{k})$$
$$= -6\hat{i} + 3\hat{j} + 6\hat{k}$$

110. Write a vector of magnitude 15 units in the direction of vector $\hat{i} - 2\hat{j} + 2\hat{k}$.

Sol : Delhi 2010

We have $\vec{a} = \hat{i} - 2\hat{j} + 2\hat{k}$

Now
$$|\vec{a}| = \sqrt{1^2 + 2^2 + 2^2}$$
$$= \sqrt{9} = 3 \text{ units}$$

The unit vector in the direction of the given vector $\vec{a}$ is

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{1}{3}(\hat{i} - 2\hat{j} + 2\hat{k})$$

The vector of magnitude equal to 15 units and in the direction of $\vec{a}$ is given by

$$3\hat{a} = \frac{15}{3}(\hat{i} - 2\hat{j} + 2\hat{k})$$
$$= 5(\hat{i} - 2\hat{j} + 2\hat{k})$$
$$= 5\hat{i} - 10\hat{j} + 10\hat{k}$$

111. What is the cosine of angle which the vector $\sqrt{2}\,\hat{i} + \hat{j} + \hat{k}$ makes with Y-axis?

Sol : Comp 2010

We have $a^2 = \sqrt{2}\,\hat{i} + \hat{j} + \hat{k}$

Unit vector in the direction of $\vec{a}$ is

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{\sqrt{2}\,\hat{i} + \hat{j} + \hat{k}}{\sqrt{2 + 1^2 + 1^2}}$$
$$= \frac{\sqrt{2}\,\hat{i} + \hat{j} + \hat{k}}{\sqrt{4}}$$
$$= \frac{\sqrt{2}\,\hat{i} + \hat{j} + \hat{k}}{2}$$

117. If $\hat{a}$, $\hat{b}$ and $\hat{c}$ are mutually perpendicular unit vectors, then find the value of $|2\hat{a}+\hat{b}+\hat{c}|$.

Sol : OD 2015

Given, $\hat{a}$, $\hat{b}$ and $\hat{c}$ are mutually perpendicular unit vector, i.e.

$$\hat{a}\cdot\hat{b} = \hat{b}\cdot\hat{c} = \hat{c}\cdot\hat{a} = 0 \qquad ...(1)$$

and $$|\hat{a}| = |\hat{b}| = |\hat{c}| = 1 \qquad ...(2)$$

Now, $$|2\hat{a}+\hat{b}+\hat{c}|^2 = (2\hat{a}+\hat{b}+\hat{c})\cdot(2\hat{a}+\hat{b}+\hat{c})$$

$$= 4(\hat{a}\cdot\hat{a}) + 2(\hat{a}\cdot\hat{b}) + 2(\hat{a}\cdot\hat{c}) + 2(\hat{b}\cdot\hat{a})$$
$$+(\hat{b}\cdot\hat{b}) + (\hat{b}\cdot\hat{c}) + 2(\hat{c}\cdot\hat{a}) + (\hat{c}\cdot\hat{b}) + (\hat{c}\cdot\hat{c})$$

$$= 4(|\hat{a}|^2 + 2(0) + 2(0) + 2(0) + |\hat{b}|^2 + (0)$$
$$+2(0) + (0) + |\hat{c}|^2$$

$$= 4(1) + 1 + 1$$

$$= 6$$

Thus $|2\hat{a}+\hat{b}+\hat{c}| = \sqrt{6}$ as length is always positive.

118. Write a unit vector perpendicular to both the vectors $\vec{a} = \hat{i}+\hat{j}+\hat{k}$ and $\vec{b} = \hat{i}+\hat{j}$.

Sol : OD 2015

First, determine perpendicular vectors of $\vec{a}$ and $\vec{b}$, i.e. $\vec{a}\times\vec{b}$. Further, determine perpendicular unit vector by using formula $\dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$.

We have $$\vec{a} = \hat{i}+\hat{j}+\hat{k}$$

and $$\vec{b} = \hat{i}+\hat{j}$$

Now $$\vec{a}\times\vec{b} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 1 & 1\\ 1 & 1 & 0\end{vmatrix}$$

$$= \hat{i}(0-1) - \hat{j}(0-1) + \hat{k}(1-1)$$

$$= -\hat{i}+\hat{j}$$

Unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ is given by

$$\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|} = \frac{-\hat{i}+\hat{j}}{\sqrt{(-1)^2+(1)^2}}$$

$$= \frac{-\hat{i}+\hat{j}}{\sqrt{2}} = \frac{-\hat{i}}{\sqrt{2}} + \frac{\hat{j}}{\sqrt{2}}$$

119. Find the area of a parallelogram whose adjacent sides represented by the vectors $2\hat{i}-3\hat{k}$ and $4\hat{i}+2\hat{k}$.

Sol : Foreign 2015, OD 2007

We have $$\vec{a} = 2\hat{i}-3\hat{k}$$

and $$\vec{b} = 4\hat{j}+2\hat{k}$$

where $\vec{a}$ and $\vec{b}$ are sides of a parallelogram

Area of parallelogram is give by

$$|\vec{a}\times\vec{b}| = |(2\hat{i}-3\hat{k})\times(4\hat{j}+2\hat{k})|$$

$$= \left\|\begin{matrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 0 & -3\\ 0 & 4 & 2\end{matrix}\right\|$$

$$= |\hat{i}(0+12) - \hat{j}(4+0) + \hat{k}(8-0)|$$

$$= |12\hat{i} - 4\hat{j} + 8\hat{k}|$$

$$= \sqrt{12^2 + (-4)^2 + (8)^2}$$

$$= \sqrt{144+16+64}$$

$$= \sqrt{224} = 4\sqrt{14} \text{ sq units}$$

120. If $\vec{a}$ and $\vec{b}$ are perpendicular vectors, $|\vec{a}+\vec{b}| = 13$ and $|\vec{a}| = 5$, then find the value of $|\vec{b}|$.

Sol : OD 2014

We have $$|\vec{a}+\vec{b}| = 13$$

$$|\vec{a}| = 5$$

Now, $(\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b}) = \vec{a}\cdot\vec{a} + \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{a} + \vec{b}\cdot\vec{b}$

$$|\vec{a}+\vec{b}|^2 = |\vec{a}|^2 + 0 + 0 + |\vec{b}|^2$$

$$(13)^2 = (5)^2 + |\vec{b}|^2$$

$$169 = 25 + |\vec{b}|^2$$

$$169 - 25 = |\vec{b}|^2$$

$$144 = |\vec{b}|^2 \Rightarrow |\vec{b}| = 12$$

as length is always positive.

121. If $\vec{a}$ and $\vec{b}$ are two unit vectors such that $\vec{a}+\vec{b}$ is also a unit vector, then find the angle between $\vec{a}$ and $\vec{b}$.

Sol : Delhi 2014

Since $\vec{a}$, $\vec{b}$ and $\vec{a}+\vec{b}$ are unit vector, we have

$$|\vec{a}| = 1, |\vec{b}| = 1 \text{ and } |\vec{a}+\vec{b}| = 1$$

Now, $$|\vec{a}+\vec{b}|^2 = (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})$$

$$1 = \vec{a}\cdot\vec{a} + \vec{b}\cdot\vec{a} + \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{b}$$

$$1 = |\vec{a}|^2 + 2\vec{a}\cdot\vec{b} + |\vec{b}|^2$$

$$1 = 1 + 2\vec{a}\cdot\vec{b} + 1$$

$$2\vec{a}\cdot\vec{b} = -1$$

$$|\vec{a}||\vec{b}|\cos\theta = -\frac{1}{2} \qquad \vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

$$\cos\theta = -\frac{1}{2} \qquad |\vec{a}| = |\vec{b}| = 1$$

Since projection of $\vec{a}$ on $\vec{b}$ is 4 unit, thus

$$\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = \frac{(\lambda\hat{i}+\hat{j}+4\hat{k})\cdot(2\hat{i}+6\hat{j}+3\hat{k})}{\sqrt{(2)^2+(6)^2+(3)^2}}$$

$$4 = \frac{(\lambda\hat{i}+\hat{j}+4\hat{k})\cdot(2\hat{i}+6\hat{j}+3\hat{k})}{\sqrt{(2)^2+(6)^2+(3)^2}}$$

$$4 = \frac{2\lambda+6+12}{\sqrt{49}}$$

$$4 = \frac{2\lambda+18}{7}$$

$$28 = 2\lambda+18$$

$$2\lambda = 10 \Rightarrow \lambda = 5$$

128. Write the angle between vectors $\vec{a}$ and $\vec{b}$ with magnitude $\sqrt{3}$ and 2 respectively, having $\vec{a}\cdot\vec{b} = \sqrt{6}$.

Sol : OD 2011

Let θ be the angle between $\vec{a}$ and $\vec{b}$, then use the following formula

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$$

We have $|\vec{a}| = \sqrt{3}$, $|\vec{b}| = 2$ and $\vec{a}\cdot\vec{b} = \sqrt{6}$.

If θ is the angle between $\vec{a}$ and $\vec{b}$, then angle between $\vec{a}$ and $\vec{b}$ is given by

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$$

$$= \frac{\sqrt{6}}{\sqrt{3}\times 2}$$

$$= \frac{\sqrt{3}\times\sqrt{2}}{\sqrt{3}\times 2}$$

or $$\cos\theta = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}} = \cos\frac{\pi}{4} \qquad \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

Thus $\theta = \frac{\pi}{4}$

129. If the sum of two unit vectors $\hat{a}$ and $\hat{b}$ is a unit vector , show that the magnitude of their difference is $\sqrt{3}$.

Sol : Delhi 2019, OD 2012

Let $\vec{c} = \vec{a}+\vec{b}$. Then, according to given condition $\vec{c}$ is a unit vector, i.e. $|\vec{c}| = 1$.

Now $$\vec{c} = \hat{a}+\hat{b}$$

$$|\vec{c}| = |\hat{a}+\hat{b}|$$

$$1 = |\hat{a}+\hat{b}|$$

$$|\hat{a}+\hat{b}|^2 = 1$$

$$(\hat{a}+\hat{b})\cdot(\hat{a}+\hat{b}) = 1$$

$$|\hat{a}|^2+2\hat{a}\cdot\hat{b}+|\hat{b}|^2 = 1$$

$$1+2\hat{a}\cdot\hat{b}+1 = 1$$

$$2\hat{a}\cdot\hat{b} = -1 \qquad ...(2)$$

Now $$|\hat{a}-\hat{b}|^2 = (\hat{a}-\hat{b})\cdot(\hat{a}-\hat{b})$$

$$= |\hat{a}|^2 - 2\hat{a}\cdot\hat{b}+|\hat{b}|^2$$

$$= 1-(-1)+1 \qquad \text{[from Eq. (1)]}$$

$$= 3$$

$$|\hat{a}-\hat{b}| = \sqrt{3} \qquad \text{Hence proved.}$$

As magnitude is always positive.

130. If $\vec{a}\times\vec{b} = \vec{c}\times\vec{d}$ and $\vec{a}\times\vec{c} = \vec{b}\times\vec{d}$, then show that $\vec{a}-\vec{d}$ is parallel to $\vec{b}-\vec{c}$, where $\vec{a}\neq\vec{d}$ and $\vec{b}\neq\vec{c}$.

Sol : Foreign 2016, Delhi 2009

We have, $$\vec{a}\times\vec{b} = \vec{c}\times\vec{d} \qquad ...(1)$$

and $$\vec{a}\times\vec{c} = \vec{b}\times\vec{d} \qquad ...(2)$$

Subtracting eq (2) from eq (1), we get

$$(\vec{a}\times\vec{b})-(\vec{a}\times\vec{c}) = (\vec{c}\times\vec{d})-(\vec{b}\times\vec{d})$$

$$(\vec{a}\times\vec{b})-(\vec{a}\times\vec{c})+(\vec{b}\times\vec{d})-(\vec{c}\times\vec{d}) = 0$$

$$\vec{a}\times(\vec{b}-\vec{c})+(\vec{b}-\vec{c})\times\vec{d} = \vec{0}$$

$$\vec{a}\times(\vec{b}-\vec{c})-\vec{d}\times(\vec{b}-\vec{c}) = \vec{0}$$

$$(\vec{a}-\vec{d})\times(\vec{b}-\vec{c}) = \vec{0}$$

Thus, we have that cross-product of vectors $\vec{a}-\vec{d}$ and $\vec{b}-\vec{c}$ is a zero vector, so $\vec{a}-\vec{d}$ is parallel to $\vec{b}-\vec{c}$.

131. If $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$, find $(\vec{r}\times\hat{i})\cdot(\vec{r}\times\hat{j})+xy$.

Sol : Delhi 2015

We have $$\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$$

Now, $$\vec{r}\times\hat{i} = (x\hat{i}+y\hat{j}+z\hat{k})\times\hat{i}$$

$$= x(\hat{i}\times\hat{i})+y(\hat{j}\times\hat{i})+z(\hat{k}\times\hat{i})$$

$$= x\cdot 0+y(-\hat{k})+z(\hat{j})$$

$$= -y\hat{k}+z\hat{j}$$

and $$(\vec{r}\times\hat{j}) = (x\hat{i}+y\hat{j}+z\hat{k})\times\hat{j}$$

$$= x(\hat{i}\times\hat{j})+y(\hat{j}\times\hat{j})+z(\hat{k}\times\hat{j})$$

$$= x\hat{k}+y\cdot 0+z(-\hat{i})$$

$$= x\hat{k}-z\hat{i}$$

$$(\vec{r}\times\hat{i})\cdot(\vec{r}\times\hat{j}) = (-y\hat{k}+z\hat{j})\cdot(x\hat{k}-z\hat{i})$$

$$= -yx+yz\cdot 0+0\cdot zx-z^2\cdot 0$$

$$= -xy$$

$$(\vec{r}\times\hat{i})\cdot(\vec{r}\times\hat{j})+xy = -xy+xy = 0$$

$$|\overrightarrow{AD}| = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{5}{2}\right)^2}$$
$$= \frac{1}{2}\sqrt{9+25} = \frac{\sqrt{34}}{2}$$
$$= \frac{\sqrt{17}\times\sqrt{2}}{2} = \sqrt{\frac{17}{2}} \text{ units}$$

136. If $\hat{i}+\hat{j}+\hat{k}$, $2\hat{i}+5\hat{j}$, $3\hat{i}+2\hat{j}-3\hat{k}$ and $\hat{i}-6\hat{j}-\hat{k}$ respectively, are the position vectors of points A, B, C and D, then find the angle between the straight lines AB and CD. Find whether $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are collinear or not.

Sol : Delhi 2019

We have
$$\overrightarrow{OA} = (\hat{i}+\hat{j}+\hat{k}),$$
$$\overrightarrow{OB} = (2\hat{i}+5\hat{j}),$$
$$\overrightarrow{OC} = (3\hat{i}+2\hat{j}-3\hat{k})$$
and
$$\overrightarrow{OD} = (\hat{i}-6\hat{j}-\hat{k})$$

Here,
$$\overrightarrow{AB} = (2-1)\hat{i} + (5-1)\hat{j} + (0-1)\hat{k}$$
$$= \hat{i}+4\hat{j}-\hat{k},$$
$$\overrightarrow{CD} = (1-3)\hat{i} + (-6-2)\hat{j} + (-1-(-3))\hat{k}$$
$$= \hat{i}+4\hat{j}-\hat{k},$$
$$|\overrightarrow{AB}| = \sqrt{1^2+4^2+(-1)^2} = \sqrt{18}$$
$$= \sqrt{9\times 2} = 3\sqrt{2},$$
and
$$|\overrightarrow{CD}| = \sqrt{(-2)^2+(-8)^2+2^2}$$
$$= \sqrt{72} = \sqrt{36\times 2} = 6\sqrt{2}$$

Angle between $\overrightarrow{AB}$ and $\overrightarrow{CD}$ is given by

$$\cos\theta = \frac{\overrightarrow{AB}\cdot\overrightarrow{CD}}{|\overrightarrow{AB}|\cdot|\overrightarrow{CD}|} \quad ....(1)$$
$$= \frac{(\hat{i}+4\hat{j}-\hat{k})\cdot(-2\hat{i}-8\hat{j}+2\hat{k})}{3\sqrt{2}\times 6\sqrt{2}}$$
$$= \frac{1(-2)+4(-8)+(-1)(2)}{3\times 6\times 2} = -1$$
$$\cos\theta = -1 \Rightarrow \theta = 180^\circ = \pi$$

So angle between $\overrightarrow{AB}$ and $\overrightarrow{CD}$ is π. Also, since angle between $\overrightarrow{AB}$ and $\overrightarrow{CD}$ is 180°, they are in opposite directions.

Since, $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are parallel to the same line m, they are collinear.

137. Let $\vec{a} = \hat{i}+\hat{j}+\hat{k}$, $\vec{b} = 4\hat{i}-2\hat{j}+3\hat{k}$ and $\vec{c} = \hat{i}-2\hat{j}+\hat{k}$. Find a vector of magnitude 6 units, which is parallel to the vector $2\vec{a}-\vec{b}+3\vec{c}$.

Sol : OD 2010

First, find the vector $2\vec{a}-\vec{b}+3\vec{c}$, then find a unit vector in the direction of $2\vec{a}-\vec{b}+3\vec{c}$. After this, the unit vector is multiplying by 6.

We have
$$\vec{a} = \hat{i}+\hat{j}+\hat{k},$$
$$\vec{b} = 4\hat{i}-2\hat{j}+3\hat{k}$$
and
$$\vec{c} = \hat{i}-2\hat{j}+\hat{k}$$

Now $\vec{d} = 2\vec{a}-\vec{b}+3\vec{c}$
$$= 2(\hat{i}+\hat{j}+\hat{k}) - (4\hat{i}-2\hat{j}+3\hat{k}) + +3(\hat{i}-2\hat{j}+\hat{k})$$
$$= 2\hat{i}+2\hat{j}+2\hat{k}-4\hat{i}+2\hat{j}-3\hat{k}+3\hat{i}-6\hat{j}+3\hat{k}$$
$$= i-2\hat{j}+2\hat{k}$$

Now, unit vector $\hat{d}$ in the direction of $\vec{d}$ is $\frac{\vec{d}}{|\vec{d}|}$

$$\hat{d} = \frac{2\vec{a}-\vec{b}+3\vec{c}}{|2\vec{a}-\vec{b}+3\vec{c}|}$$
$$= \frac{\hat{i}-2\hat{j}+2\hat{k}}{\sqrt{(1)^2+(-2)^2+(2)^2}}$$
$$= \frac{\hat{i}-2\hat{j}+2\hat{k}}{\sqrt{9}}$$
$$= \frac{\hat{i}-2\hat{j}+2\hat{k}}{3}$$
$$= \frac{1}{3}\hat{i} - \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}$$

Hence, vector of magnitude 6 units parallel to the vector $2\vec{a}-\vec{b}+3\vec{c}$ is given by

$$6\hat{d} = 6\left(\frac{1}{3}\hat{i} - \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}\right)$$
$$= 2\hat{i}-4\hat{j}+4\hat{k}$$

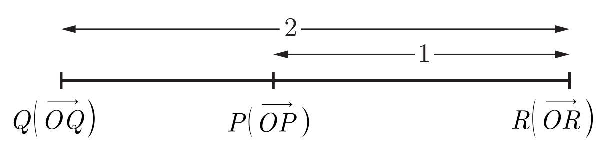
138. Find the position of a point R, which divides the line joining two points P and Q whose position vectors are $2\vec{a}+\vec{b}$ and $\vec{a}-3\vec{b}$ respectively, externally in the ratio 1 2. Also, show that P is the mid-point of line segment RQ.

Sol : Delhi 2010, OD 2008

Position vector of P, $\overrightarrow{OP} = 2\vec{a}+\vec{b}$

Position vector of Q, $\overrightarrow{OQ} = \vec{a}-3\vec{b}$

Let $\overrightarrow{OR}$ be the position vector of point R, which divides PQ in the ratio 1 2 externally. as shown below.



Sol : Delhi 2017, 2013C, 2011

If three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are mutually perpendicular to each other, then $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{c}=\vec{c}\cdot\vec{a}=0$ and if all three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are equally inclined with the vector $(\vec{a}+\vec{b}+\vec{c})$, that means each vector $\vec{a}$, $\vec{b}$ and $\vec{c}$ makes equal angle with $(\vec{a}+\vec{b}+\vec{c})$ by using formula $\cos\theta=\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$

We have $\quad |\vec{a}|=|\vec{b}|=|\vec{c}|=\lambda$ (say) ...(1)

and $\quad \vec{a}\cdot\vec{b}=0,\ \vec{b}\cdot\vec{c}=0$ and $\vec{c}\cdot\vec{a}=0$ ...(2)

Now,

$$\begin{aligned}|\vec{a}+\vec{b}+\vec{c}|^2 &= |\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})\\ &= \lambda^2+\lambda^2+\lambda^2+2(0+0+0)\\ &= 3\lambda^2\end{aligned}$$

$|\vec{a}+\vec{b}+\vec{c}|=\sqrt{3}\lambda$ as length cannot be negative.

Suppose $(\vec{a}+\vec{b}+\vec{c})$ is inclined at angles θ_1, θ_2 and θ_3 respectively with vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$, then

$$\begin{aligned}(\vec{a}+\vec{b}+\vec{c})\cdot\vec{a} &= |\vec{a}+\vec{b}+\vec{c}||\vec{a}|\cos\theta_1 \qquad \vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\theta\\ |\vec{a}|^2+\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c} &= \sqrt{3}\lambda\times\lambda\cos\theta_1\\ \lambda^2+0+0 &= \sqrt{3}\lambda^2\cos\theta_1 \qquad \text{from eq (1) and (2)}\\ (\vec{a}\cdot\vec{b}\cdot\vec{c})\cdot\vec{b} &= |\vec{a}+\vec{b}+\vec{c}||\vec{b}|\cos\theta_2\\ \vec{a}\cdot\vec{b}+|\vec{b}|^2+\vec{c}\cdot\vec{b} &= \sqrt{3}\lambda\cdot\lambda\cos\theta_2\\ 0+\lambda^2+0 &= \sqrt{3}\lambda^2\cos\theta_2 \qquad \text{from eq (1) and (2)}\\ \cos\theta_2 &= \frac{1}{\sqrt{3}}\end{aligned}$$

Similarly,

$$\begin{aligned}(\vec{a}+\vec{b}+\vec{c})\cdot\vec{c} &= |\vec{a}+\vec{b}+\vec{c}||\vec{c}|\cos\theta_3\\ \cos\theta_1 &= \frac{1}{\sqrt{3}}\end{aligned}$$

Thus, $\quad \cos\theta_1=\cos\theta_2=\cos\theta_3=\dfrac{1}{\sqrt{3}}$

Hence, it is proved that $(\vec{a}+\vec{b}+\vec{c})$ is equally inclined with the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$.

142. Using vectors, find the area of the ΔABC, whose vertices are $A(1,2,3)$, $B(2,-1,4)$ and $C(4,\ 5,\ -1)$.

Sol : Delhi 2017; OD 2013

We have a traingle whose vertices are $A(1,2,3)$, $B(2,-1,4)$ and $C(4,5,-1)$.

Position vectors of the vectors A, B and C of ΔABC are

$$\begin{aligned}\overrightarrow{OA} &= \hat{i}+2\hat{j}+3\hat{k},\\ \overrightarrow{OB} &= 2\hat{i}-\hat{j}+4\hat{k}\end{aligned}$$

and $\quad \overrightarrow{OC}=4\hat{i}+5\hat{j}-\hat{k}$

Now,

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{OB}-\overrightarrow{OA}\\ &= (2\hat{i}-\hat{j}+4\hat{k})-(\hat{i}+2\hat{j}+3\hat{k})\\ &= \hat{i}-3\hat{j}+\hat{k}\end{aligned}$$

and

$$\begin{aligned}\overrightarrow{AC} &= \overrightarrow{OC}-\overrightarrow{OA}\\ &= (4\hat{i}+5\hat{j}-\hat{k})-(\hat{i}+2\hat{j}+3\hat{k})\\ &= (3\hat{i}+3\hat{j}-4\hat{k})\end{aligned}$$

Now,

$$\begin{aligned}\overrightarrow{AB}\times\overrightarrow{AC} &= \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -3 & 1\\ 3 & 3 & -4\end{vmatrix}\\ &= \hat{i}(12-3)-\hat{j}(-4-3)+\hat{k}(3+9)\\ &= 9\hat{i}+7\hat{j}+12\hat{k}\end{aligned}$$

and

$$\begin{aligned}|\overrightarrow{AB}\times\overrightarrow{AC}| &= \sqrt{(9)^2+(7)^2+(12)^2}\\ &= \sqrt{81+49+144}=\sqrt{274}\end{aligned}$$

Area of ΔABC is given by

$$\begin{aligned}\Delta &= \frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|\\ &= \frac{1}{2}\sqrt{274}\ \text{sq units}\end{aligned}$$

143. Let $\vec{a}=\hat{i}+\hat{j}+\hat{k}$, $\vec{b}=\hat{i}$ and $\vec{c}=c_1\vec{i}+c_2\hat{j}+c_3\hat{k}$, then

(a) Let $c_1=1$ and $c_2=2$, find c_3 which makes $\vec{a}$, $\vec{b}$ and $\vec{c}$ coplanar.

(b) If $c_2=-1$ and $c_3=1$, show that no value of c_1 can make $\vec{a}$, $\vec{b}$ and $\vec{c}$ coplanar.

Sol : Delhi 2017

We have

$$\begin{aligned}\vec{a} &= \hat{i}+\hat{j}+\hat{k}\\ \vec{b} &= \hat{i}+0\cdot\hat{j}+0\cdot\hat{k}\end{aligned}$$

and $\quad \vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$

The given vectors are coplanar iff $|\vec{a}\,\vec{b}\,\vec{c}|=0$

$$\begin{vmatrix}1 & 1 & 1\\ 1 & 0 & 0\\ c_1 & c_2 & c_3\end{vmatrix}=0 \qquad ...(1)$$

(a) If $c_1=1$ and $c_2=2$ from eq (1), we get

$$\begin{vmatrix}1 & 1 & 1\\ 1 & 0 & 0\\ 1 & 2 & c_3\end{vmatrix}=0$$

$$1\begin{vmatrix}0 & 0\\ 2 & c_3\end{vmatrix}-1\begin{vmatrix}1 & 0\\ 1 & c_3\end{vmatrix}+1\begin{vmatrix}1 & 0\\ 1 & 2\end{vmatrix}=0$$

$$-1(c_3-0)+1(2-0)=0$$

$$= 0\hat{i} - \hat{j} - \hat{k} - (4\hat{i} + 5\hat{j} + \hat{k})$$
$$= -4\hat{i} - 6\hat{j} - 2\hat{k}$$
$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$$
$$= 3\hat{i} + 9\hat{j} + 4\hat{k} - (4\hat{i} + 5\hat{j} + \hat{k})$$
$$= -\hat{i} + 4\hat{j} + 3\hat{k}$$

and $\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA}$

$$= -4\hat{i} + 4\hat{j} + 4\hat{k} - (4\hat{i} + 5\hat{j} + \hat{k})$$
$$= -8\hat{i} - \hat{j} + 3\hat{k}$$

Four points are coplanar, if $\left[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}\right] = 0$. Thus

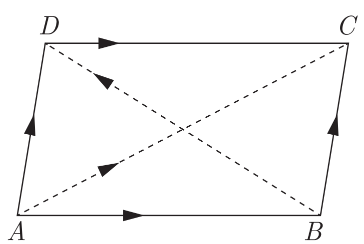
$$\left[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}\right] = \begin{vmatrix} -4 & -6 & -2 \\ -1 & 4 & 3 \\ -8 & -1 & 3 \end{vmatrix}$$
$$= -4(12+3) + 6(-3+24) - 2(1+32)$$
$$= -60 + 126 - 66 = 0$$

Hence, the four points A, B, C and D are coplanar.

147. The two adjacent sides of a parallelogram are $2\hat{i} - 4\hat{j} - 5\hat{k}$ and $2\hat{i} + 2\hat{j} + 3\hat{k}$. Find the two unit vectors parallel to its diagonals. Using the diagonal vectors, find the area of the parallelogram.

Sol : OD 2016

Let $ABCD$ be the given parallelogram with $\overrightarrow{AB} = 2\hat{i} - 4\hat{j} - 5\hat{k}$ and $\overrightarrow{AD} = 2\hat{i} + 2\hat{j} + 3\hat{k}$. As per question we have shown figure below.



Using parallelogram law of addition diagonal $\overrightarrow{AC}$ is given by

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{AD}$$
$$= 2\hat{i} - 4\hat{j} - 5\hat{k} + 2\hat{i} + 2\hat{j} + 3\hat{k}$$
$$= 4\hat{i} - 2\hat{j} - 2\hat{k}$$

and the diagonal $\overrightarrow{BD}$ is given by

$$\overrightarrow{BD} = \overrightarrow{BC} + \overrightarrow{BA}$$
$$= \overrightarrow{AD} - \overrightarrow{AB}$$
$$= 2\hat{i} + 2\hat{j} + 3\hat{k} - (2\hat{i} - 4\hat{j} - 5\hat{k})$$
$$= 6\hat{j} + 8\hat{k}$$

Now, the unit vector along $\overrightarrow{AC}$ is given by

$$\frac{\overrightarrow{AC}}{|\overrightarrow{AC}|} = \frac{4\hat{i} - 2\hat{j} - 2\hat{k}}{\sqrt{16+4+4}}$$
$$= \frac{4\hat{i} - 2\hat{j} - 2\hat{k}}{\sqrt{24}}$$
$$= \frac{4\hat{i} - 2\hat{j} - 2\hat{k}}{2\sqrt{6}}$$
$$= \frac{1}{\sqrt{6}}(2\hat{i} - \hat{j} - \hat{k})$$

and the unit vector along $\overrightarrow{BD}$ is given by

$$\frac{\overrightarrow{BD}}{|\overrightarrow{BD}|} = \frac{6\hat{j} + 8\hat{k}}{\sqrt{36+64}}$$
$$= \frac{6\hat{j} + 8\hat{k}}{10}$$
$$= \frac{1}{5}(3\hat{j} + 4\hat{k})$$

Here, $\overrightarrow{AC} \times \overrightarrow{BD} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -2 & -2 \\ 0 & 6 & 8 \end{vmatrix}$

$$= \hat{i}(-16+12) - \hat{j}(32-0) + \hat{k}(24-0)$$
$$= -4\hat{i} - 32\hat{j} + 24\hat{k}$$

and $|\overrightarrow{AC} \times \overrightarrow{BD}| = \sqrt{(-4)^2 + (-32)^2 + (24)^2}$

$$= \sqrt{4^2(1 + 8^2 + 6^2)}$$
$$= 4\sqrt{1+64+36} = 4\sqrt{101}$$

Now, area of parallelogram $ABCD$ is give by

$$\tfrac{1}{2}|\overrightarrow{AC} \times \overrightarrow{BD}| = \frac{1}{2} \times 4\sqrt{101}$$
$$= 2\sqrt{101} \text{ sq units}$$

148. If $\vec{a} = \hat{i} + 2\hat{j} + \hat{k}$, $\vec{b} = 2\hat{i} + \hat{j}$ and $\vec{c} = 3\hat{i} - 4\hat{j} - 5\hat{k}$, then find a unit vector perpendicular to both of the vectors $(\vec{a} - \vec{b})$ and $(\vec{c} - \vec{b})$.

Sol : OD 2015

We have $\vec{a} = \hat{i} + 2\hat{j} + \hat{k}$,

$$\vec{b} = 2\hat{i} + \hat{j}$$

and $\vec{c} = 3\hat{i} - 4\hat{j} - 5\hat{k}$

Now, $\vec{a} - \vec{b} = (\hat{i} + 2\hat{j} + \hat{k}) - (2\hat{i} + \hat{j})$

$$= -\hat{i} + \hat{j} + \hat{k}$$

and $\vec{c} - \vec{b} = (3\hat{i} - 4\hat{j} - 5\hat{k}) - (2\hat{i} + \hat{j})$

$$= \hat{i} - 5\hat{j} - 5\hat{k}$$

151. Vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are such that $\vec{a}+\vec{b}+\vec{c}=\vec{0}$ and $|\vec{a}|=3$, $|\vec{b}|=5$ and $|\vec{c}|=7$. Find the angle between $\vec{a}$ and $\vec{b}$.

Sol : Delhi 2014, 2008; OD 2008

We have
$$\vec{a}+\vec{b}+\vec{c} = 0$$
$$\vec{a}+\vec{b} = -\vec{c}$$
$$(\vec{a}+\vec{b})^2 = (-\vec{c})^2$$
$$(\vec{a}+\vec{b})(\vec{a}+\vec{b}) = (-\vec{c})\cdot(-\vec{c})$$
$$\vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{a}+\vec{b}\cdot\vec{b} = \vec{c}\cdot\vec{c}$$
$$|\vec{a}|^2+2\vec{a}\cdot\vec{b}+|\vec{b}|^2 = |\vec{c}|^2$$
$$|\vec{a}|^2+2|\vec{a}||\vec{b}|\cos\theta+|\vec{b}|^2 = |\vec{c}|^2$$

Substituting $|\vec{a}|=3$, $|\vec{b}|=5$ and $|\vec{c}|=7$ in above we get
$$3^2+2\times3\times5\times\cos\theta+5^2 = 7^2$$
$$9+30\cos\theta+25 = 49$$
$$30\cos\theta = 49-9-25=15$$
$$\cos\theta = \frac{15}{30}=\frac{1}{2}$$
$$\theta = \cos^{-1}\left(\frac{1}{2}\right)$$
$$= \frac{\pi}{3}$$

152. The scalar product of the vector $\vec{a}=\hat{i}+\hat{j}+\hat{k}$ with a unit vector along the sum of vectors $\vec{b}=2\hat{i}+4\hat{j}-5\hat{k}$ and $\vec{c}=\lambda\hat{i}+2\hat{j}+3\hat{k}$ is equal to one. Find the value of λ and hence, find the unit vector along $\vec{b}+\vec{c}$.

Or

The scalar product of vector $\hat{i}+\hat{j}+\hat{k}$ with the unit vector along the sum of vectors $2\hat{i}+4\hat{j}-5\hat{k}$ and $\lambda\hat{i}+2\hat{j}+3\hat{k}$ is equal to one. Find the value of λ

Sol : OD 2014, Delhi 2009, Foreign 2008

First, determine the unit vector of $\vec{b}+\vec{c}$, i.e. $\frac{\vec{b}+\vec{c}}{|\vec{b}+\vec{c}|}$. Further put $\vec{a}\cdot\frac{(\vec{b}+\vec{c})}{|\vec{b}+\vec{c}|}=1$ and then determine the value of λ.

We have
$$\vec{a} = \hat{i}+\hat{j}+\hat{k},$$
$$\vec{b} = 2\hat{i}+4\hat{j}-5\hat{k}$$
and
$$\vec{c} = \lambda\hat{i}+2\hat{j}+3\hat{k}.$$

Now,
$$\vec{b}+\vec{c} = 2\hat{i}+4\hat{j}-5\hat{k}+\lambda\hat{i}+2\hat{j}+3\hat{k}$$
$$= (2+\lambda)\hat{i}+6\hat{j}-2\hat{k}$$
$$|\vec{b}+\vec{c}| = \sqrt{(2+\lambda)^2+(6)^2+(-2)^2}$$
$$= \sqrt{4+\lambda^2+4\lambda+36+4}$$
$$= \sqrt{\lambda^2+4\lambda+44}$$

Since scalar product of $(\hat{i}+\hat{j}+\hat{k})$ with unit vector $\vec{b}+\vec{c}$ is 1.
$$(\hat{i}+\hat{j}+\hat{k})\cdot\frac{\vec{b}+\vec{c}}{|\vec{b}+\vec{c}|} = 1$$
$$(\hat{i}+\hat{j}+\hat{k})\cdot\frac{(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{\lambda^2+4\lambda+44}} = 1$$
$$\frac{1(2+\lambda)+1(6)+1(-2)}{\sqrt{\lambda^2+4\lambda+44}} = 1$$
$$\frac{(2+\lambda)+6-2}{\sqrt{\lambda^2+4\lambda+44}} = 1$$
$$\lambda+6 = \sqrt{\lambda^2+4\lambda+44}$$
$$(\lambda+6)^2 = \lambda^2+4\lambda+44$$
$$\lambda^2+36+12\lambda = \lambda^2+4\lambda+44$$
$$8\lambda = 8 \Rightarrow \lambda = 1$$

Hence, the value of λ is 1.

Now, the unit vector along $\vec{b}+\vec{c}$ is,
$$\frac{\vec{b}+\vec{c}}{|\vec{b}+\vec{c}|} = \frac{(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{\lambda^2+4\lambda+44}}$$
$$= \frac{(2+1)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{(1)^2+4(1)+44}}$$
$$= \frac{3\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{1+4+44}}$$
$$= \frac{3\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{49}}$$
$$= \frac{3}{7}\hat{i}+\frac{6}{7}\hat{j}-\frac{2}{7}\hat{k}$$

153. Find the vector $\vec{p}$ which is perpendicular to both $\vec{\alpha}=4\hat{i}+5\hat{j}-\hat{k}$ and $\vec{\beta}=\hat{i}-4\hat{j}+5\hat{k}$ and $\vec{p}\cdot\vec{q}=21$, where $\vec{q}=3\hat{i}+\hat{j}-\hat{k}$.

Sol : Comp 2014

We have
$$\vec{\alpha} = 4\hat{i}+3\hat{j}-\hat{k},$$
$$\vec{\beta} = \hat{i}-4\hat{j}+5\hat{k}$$
and
$$\vec{q} = 3\hat{i}+\hat{j}-\hat{k}$$

Since vector $\vec{p}$ is perpendicular to α and β we have
$$\vec{p} = \lambda(\vec{\alpha}\times\vec{\beta}) \qquad ...(1)$$

Now,
$$\vec{\alpha}\times\vec{\beta} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 4 & 5 & -1\\ 1 & -4 & 5\end{vmatrix}$$
$$= \hat{i}(25-4)-\hat{j}(20+1)+\hat{k}(-16-5)$$
$$= \hat{i}(21)-\hat{j}(21)+\hat{k}(-21)$$

Also, let $\vec{b} = \hat{i} - \hat{j} + \hat{k},$

$\vec{c} = 2\hat{i} + \hat{j} - 3\hat{k}$

and $\vec{d} = \hat{i} + \hat{j} + \hat{k}$

We have, $\vec{a} \cdot \vec{b} = 4,\ \vec{a} \cdot \vec{c} = 0$ and $\vec{a} \cdot \vec{b} = 2$

Now, $\vec{a} \cdot \vec{b} = 4$

$$(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \cdot (\hat{i} - \hat{j} + \hat{k}) = 4$$

$$a_1 - a_2 + a_3 = 4 \qquad ...(1)$$

$$\vec{a} \cdot \vec{c} = 0$$

$$2a_1 + a_2 - 3a_2 = 0 \qquad ...(2)$$

and $\vec{a} \cdot \vec{d} = 2$

$$(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \cdot (\hat{i} + \hat{j} + \hat{k}) = 2$$

$$a_1 + a_2 + a_3 = 2 \qquad ...(3)$$

Subtracting eq (3) from eq (1), we obtain

$$-2a_2 = 2 \Rightarrow a_2 = -1$$

Substituting $a_2 = -1$ in eq. (2) and (3), we get

$$2a_1 - 3a_3 = 1 \qquad ...(4)$$

$$a_1 + a_3 = 3 \qquad ...(5)$$

Multiplying Eq. (5) by 3 and then adding with Eq. (4), we get

$$5a_1 = 1 + 9 = 10 \Rightarrow a_1 = 2$$

Substituting $a_1 = 2$ in Eq. (5), we get $a_3 = 1$

Hence, the vector is $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$.

157. Find the values of λ for which the angle between the vectors $\vec{a} = 2\lambda^2\hat{i} + 4\lambda\hat{j} + \hat{k}$ and $\vec{b} = 7\hat{i} + 2\hat{j} + \lambda\hat{k}$ is obtuse.

Sol : OD 2013

We have $\vec{a} = 2\lambda^2\hat{i} + 4\lambda\hat{j} + \hat{k}$

and $\vec{b} = 7\hat{i} - 2\hat{j} + \lambda\hat{k}$

If θ is the obtuse angle between the vectors, then we have

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$$

$$= \frac{14\lambda^2 + 8\lambda + \lambda}{\sqrt{4\lambda^4 + 16\lambda^2 + 1}\sqrt{49 + 4 + \lambda^2}}$$

Since θ is an obtuse angle, $\cos\theta < 0$ an we have

$$\frac{14\lambda^2 - 7\lambda}{\sqrt{4\lambda^4 + 16\lambda^2 + 1}\sqrt{53 + \lambda^2}} < 0$$

$$14\lambda^2 - 7\lambda < 0$$

$$2\lambda^2 - \lambda < 0$$

$$\lambda(2\lambda - 1) < 0$$

Either $\lambda < 0,\ 2\lambda - 1 > 0$ or $\lambda > 0,\ 2\lambda - 1 < 0$

Either $\lambda < 0,\ \lambda > \frac{1}{2}$ or $\lambda > 0,\ \lambda < \frac{1}{2}$

Clearly, first option is impossible. Therefore

$$\lambda > 0,\ \lambda < \frac{1}{2}$$

$$0 < \lambda < \frac{1}{2}$$

$$\lambda \in \left(0, \frac{1}{2}\right)$$

158. If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three vectors such that each one is perpendicular to the vector obtained by sum of the order two and $|\vec{a}| = 3$, $|\vec{b}| = 4$ and $|\vec{c}| = 5$, then prove that $|\vec{a} + \vec{b} + \vec{c}| = 5\sqrt{2}$.

Or

If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three vectors, such that $|\vec{a}| = 3$, $|\vec{b}| = 4$ and $|\vec{c}| = 5$ and each one of these is perpendicular to the sum of other two, then find $|\vec{a} + b + \vec{c}|$.

Sol : Foreign 2013, OD 2011, Delhi 2010

We have $\vec{a} \perp (\vec{b} + \vec{c}),\ \vec{b} \perp (\vec{c} + \vec{a}),\ \vec{c} \perp (\vec{a} + \vec{b})$

and $|\vec{a}| = 3,\ |\vec{b}| = 4,\ |\vec{c}| = 5$

Now, $|\vec{a} + \vec{b} + \vec{c}|^2 = (\vec{a} + \vec{b} + \vec{c}) \cdot (\vec{a} + \vec{b} + \vec{c})$

$$= \vec{a} \cdot \vec{a} + \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{a} + \vec{b} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a} + \vec{c} \cdot \vec{b} + \vec{c} \cdot \vec{c}$$

$$= |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + \vec{a} \cdot (\vec{b} + \vec{c}) + \vec{b} \cdot (\vec{a} + \vec{c}) + \vec{c} \cdot (\vec{a} + \vec{b})$$

$$= |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 0 + 0 + 0$$

$$= 3^2 + 4^2 + 5^2$$

$$= 9 + 16 + 25$$

$$= 50$$

or $|\vec{a} + \vec{b} + \vec{c}|^2 = 50$

Thus $|\vec{a} + \vec{b} + \vec{c}| = 5\sqrt{2}$ as length cannot be negative.

Since $\vec{a} \perp (\vec{b} + \vec{c})$, therefore $\vec{a} \cdot (\vec{b} \cdot \vec{c}) = 0$ Similarly, $\vec{b} \cdot (\vec{a} + \vec{c}) = 0$ and $\vec{c} \cdot (\vec{a} + \vec{b}) = 0$.

159. If $\vec{a} = 3\hat{i} - \hat{j}$ and $\vec{b} = 2\hat{i} + \hat{j} - 3\hat{k}$, then express $\vec{b}$ in the form $\vec{b} = \vec{b}_1 + \vec{b}_2$, where $\vec{b}_1 || \vec{a}$ and $\vec{b}_2 \perp \vec{a}$.

Sol : Comp 2013

We have $\vec{a} = 3\hat{i} - \hat{j}$

and $\vec{b} = 2\hat{i} + \hat{j} - 3\hat{k}$

162. If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three vectors, such that $|\vec{a}| = 5$, $|\vec{b}| = 12$, $|\vec{c}| = 13$ and $\vec{a}+\vec{b}+\vec{c} = 0$, then find the value of $\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}$.

Sol : Delhi 2012

We have $\quad |\vec{a}| = 5, |\vec{b}| = 12, |\vec{c}| = 13$

and $\quad \vec{a}+\vec{b}+\vec{c} = \vec{0}$

Consider $\quad \vec{a}+\vec{b}+\vec{c} = \vec{0}$

$$(\vec{a}+\vec{b}+\vec{c})^2 = (\vec{0})^2$$

$$(\vec{a}+\vec{b}+\vec{c})\cdot(\vec{a}+\vec{b}+\vec{c}) = \vec{0}\cdot\vec{0}$$

$$\vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}+\vec{b}\cdot\vec{a}+\vec{b}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}$$
$$+\vec{c}\cdot\vec{b}+\vec{c}\cdot\vec{c} = 0$$

$$|\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}) = 0$$

$$5^2+12^2+13^2+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}) = 0$$

$$25+144+169+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}) = 0$$

$$169+169+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}) = 0$$

$$169+(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}) = 0$$

$$\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a} = -169$$

163. Let $\vec{a} = \hat{i}+4\hat{j}+2\hat{k}$, $\vec{b} = 3\hat{i}-2\hat{j}+7\hat{k}$ and $\vec{c} = 2\hat{i}-\hat{j}+4\hat{k}$. Find a vector $\vec{p}$, which is perpendicular to both $\vec{a}$ and $\vec{b}$ and $\vec{p}\cdot\vec{c} = 18$.

Sol : OD 2012, Comp 2010

We have $\quad \vec{a} = \hat{i}+4\hat{j}+2\hat{k}$,

$$\vec{b} = 3\hat{i}-2\hat{j}+7\hat{k}$$

and $\quad \vec{c} = 2\hat{i}-\hat{j}+4\hat{k}$

Let $\vec{p} = x\hat{i}+y\hat{j}+z\hat{k}$ where $\vec{p}$ is perpendicular to both $\vec{a}$ and $\vec{b}$. Thus

$$\vec{p}\cdot\vec{a} = 0$$

$$(x\hat{i}+y\hat{j}+z\hat{k})\cdot(\hat{i}+4\hat{j}+2\hat{k}) = 0$$

$$x+4y+2z = 0 \qquad ...(1)$$

and $\quad \vec{p}\cdot\vec{b} = 0$

$$(x\hat{i}+y\hat{j}+z\hat{k})\cdot(3\hat{i}-2\hat{j}+7\hat{k}) = 0$$

$$3x-2y+7z = 0 \qquad ...(2)$$

Since $\vec{p}\cdot\vec{c} = 18$ we have

$$(x\hat{i}+y\hat{j}+z\hat{k})\cdot(2\hat{i}-\hat{j}+4\hat{k}) = 18$$

$$2x-y+4z = 18 \qquad ...(3)$$

Multiplying Eq. (1) by 3 and subtracting it from Eq. (2), we get

$$-14y+z = 0 \qquad ...(4)$$

Now, multiplying Eq. (1) by 2 and subtracting it from Eq. (3), we get

$$-9y = 18 \Rightarrow y = -2$$

Substituting $y = -2$ in Eq. (4), we get

$$-14(-2)+z = 0$$

$$28+z = 0 \Rightarrow z = -28$$

Substituting $y = -2$ and $z = -28$ in Eq. (1), we get

$$x+4(-2)+2(-28) = 0$$

$$x-8-56 = 0 \Rightarrow x = 64$$

Hence, the required vector is

$$\vec{p} = x\hat{i}+y\hat{j}+z\hat{k}$$

i.e. $\quad \vec{p} = 64\hat{i}-2\hat{j}-28\hat{k}$

164. Find a unit vector perpendicular to each of the vectors $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$, where $\vec{a} = 3\hat{i}+2\hat{j}+2\hat{k}$ and $\vec{b} = \hat{i}+2\hat{j}-2\hat{k}$.

Sol : Delhi 2011

We have $\quad \vec{a} = 3\hat{i}+2\hat{j}+2\hat{k}$

and $\quad \vec{b} = \hat{i}+2\hat{j}-2\hat{k}$

Now, $\quad \vec{a}+\vec{b} = 3\hat{i}+2\hat{j}+2\hat{k}+(\hat{i}+2\hat{j}-2\hat{k})$

$$= 4\hat{i}+4\hat{j}+0\hat{k}$$

and $\quad \vec{a}-\vec{b} = 3\hat{i}+2\hat{j}+2\hat{k}-(\hat{i}+2\hat{j}-2\hat{k})$

$$= 2\hat{i}+0\hat{j}+4\hat{k}$$

Now, a vector $\vec{c}$ perpendicular to $(\vec{a}+\vec{b})$ and $(\vec{a}-\vec{b})$ is given by

$$\vec{c} = (\vec{a}+\vec{b})\times(\vec{a}-\vec{b})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 4 & 0 \\ 2 & 0 & 4 \end{vmatrix}$$

$$= \hat{i}(16-0)-\hat{j}(16-0)+\hat{k}(0-8)$$

$$= 16\hat{i}-16\hat{j}-8\hat{k}$$

$$\vec{c} = 8(2\hat{i}-2\hat{j}-\hat{k})$$

and unit vector $\hat{c}$ along $(\vec{a}+\vec{b})\times(\vec{a}-\vec{b})$ is given by

$$\hat{c} = \frac{8(2\hat{i}-2\hat{j}-\hat{k})}{8\sqrt{2^2+2^2+1}}$$

$$= \frac{2\hat{i}-2\hat{j}-\hat{k}}{\sqrt{9}}$$

Based on the above information, answer the following questions.

(i) What is the distance between House and ATM ?

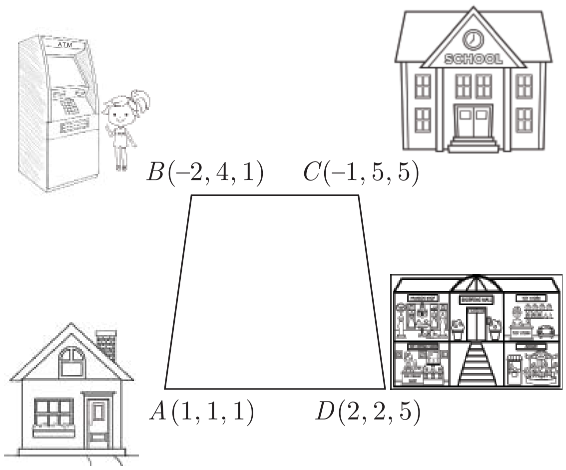
(ii) What is the distance between ATM and school?

(iii) What is the total distance travelled by Lavanya?

(iv) What is the extra distance travelled by Lavanya in reaching the shopping mall?

Sol :

We can redraw the given diagram as below.



(i) $\overrightarrow{AB} = (-2\hat{i}+4\hat{j}+\hat{k})-(\hat{i}+\hat{j}+\hat{k})$

$= -3i+3j$

$\overrightarrow{AB} = \sqrt{(-3)^2+3^2} = \sqrt{9+9}\ \sqrt{18}$

$= 3\sqrt{2}$

Distance between House (A) and ATM (B) is $3\sqrt{2}$ units.

(ii) $\overrightarrow{BC} = (-\hat{i}+5\hat{j}+5\hat{k})-(2\hat{i}+4\hat{j}+\hat{k})$

$= \hat{i}+\hat{j}+4\hat{k}$

$|\overrightarrow{BC}| = \sqrt{1^2+1^2+4^2} = \sqrt{1+1+16}$

$= \sqrt{18}\,s = 3\sqrt{2}$

Distance between ATM (B) and School (C) is $3\sqrt{2}$ units.

(iii) $\overrightarrow{CD} = (2\hat{i}+2\hat{j}+5\hat{k})-(-\hat{i}+5\hat{j}+5\hat{k})$

$= (3\hat{i}-3\hat{j})$

$|\overrightarrow{CD}| = \sqrt{3^2+(-3)^2} = \sqrt{9+9}$

$= 3\sqrt{2}$

Distance between School (C) and Shopping mall (D) is $3\sqrt{2}$ units.

Total distance travelled by Lavanya

$= |\overrightarrow{AB}|+|\overrightarrow{BC}|+|\overrightarrow{CD}|$

$= (3\sqrt{2}+3\sqrt{2}+3\sqrt{2})$ units.

$= 9\sqrt{2}$

(iv) Distance between house and shopping mall is $|\overrightarrow{AD}|$

Now, $|\overrightarrow{AD}| = \hat{i}+\hat{j}+4\hat{k}$

$|\overrightarrow{AD}| = \sqrt{1^2+1^2+4^2} = \sqrt{1+1+16}$

$= \sqrt{18} = 3\sqrt{2}$

Thus, extra distance travelled by Lavanya is reaching shopping mall $= (9\sqrt{2}-3\sqrt{2})$ units $= 6\sqrt{2}$ units.

168. If two vectors are represented by the two sides of a triangle taken in order, then their sum is represented by the third side of the triangle taken in opposite order and this is known as triangle law of vector addition.

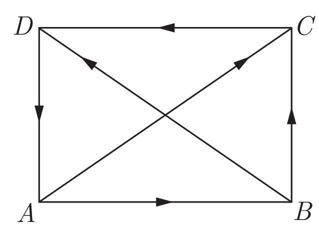
Based on the above information, answer the following questions.

(i) If $\hat{p}$, $\hat{q}$, $\hat{r}$ are the vectors represented by the side of a triangle taken in order, then find $\vec{q}+\vec{r}$.

(ii) If $ABCD$ is a parallelogram and AC and BD are its diagonals, then find $\overrightarrow{AC}+\overrightarrow{BD}$.

(iii) If $ABCD$ is a parallelogram, where $\overrightarrow{AB} = 2\vec{a}$ and $\overrightarrow{BC} = 2\vec{b}$, then find $\overrightarrow{AC}-\overrightarrow{BD}$.

(iv) If $ABCD$ is a quadrilateral, whose diagonals are $\overrightarrow{AC}$ and $\overrightarrow{BD}$, then find $\overrightarrow{BA}+\overrightarrow{CD}$.



Sol :

(i) Let OAB be a triangle such that

$\overrightarrow{AO} = -\vec{p}, \overrightarrow{AB} = \vec{q}, \overrightarrow{BO} = \vec{r}$

Now, $\vec{q}+\vec{r} = \overrightarrow{AB}+\overrightarrow{BO}$

$= \overrightarrow{AO} = -\vec{p}$

(ii) From the triangle law of vector addition,

$\overrightarrow{AC}+\overrightarrow{BD} = \overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{BC}+\overrightarrow{CD}$

(iv) Direction of the ring getting pulled

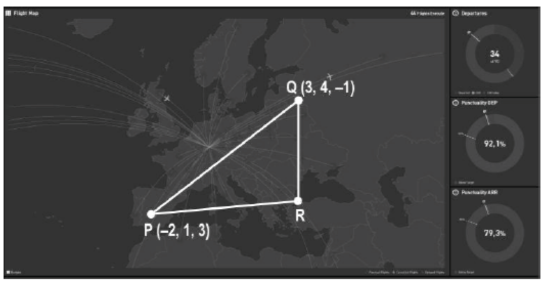
$$\overrightarrow{F} = -\hat{i} + \hat{j}$$

$$\theta = \pi - \tan^{-1}\left(\frac{1}{1}\right)$$

$$= \pi - \frac{\pi}{4} = \frac{3\pi}{4};$$

where θ is the angle made by the resultant force with the $+ve$ direction of the x- axis.

170. The flight path of two airplanes in a flight simulator game are shown below. The coordinates of the airports P and Q are given.



Airplane 1 flies directly from P to Q.
Airplane 2 has a layover at R and then flies to Q.
The path of aeroplane 2 from P to R can be represented by the vector $5\hat{i} + \hat{j} - 2\hat{k}$.
Assume that the flight path is straight and fuel is consumed uniformly throughout the flight Based on the above information answer the following:

(i) Find the vector that represents the flight path of Airplane 1.
(ii) Write the vector representing the path of Airplane 2 from R to Q. Show your steps.
(iii) What is the angle between the flight paths of Airplane 1 and Airplane 2 just after take off?
(iv) Consider that Airplane 1 started the flight with a full fuel tank. Find the position vector of the point where a third of the fuel runs out if the entire fuel is required for the flight.

Sol :

(i) Vectors for points P and Q are as follows

$$\overrightarrow{OP} = -2\hat{i} + \hat{j} + 3\hat{k}$$

$$\overrightarrow{OQ} = 3\hat{i} + 4\hat{j} - \hat{k}$$

Vector representing the flight path of Airplane 1 as:

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$

$$= \left(3\hat{i} + 4\hat{j} - \hat{k}\right) - \left(-2\hat{i} + \hat{j} + 3\hat{k}\right)$$

$$= 5\hat{i} + 3\hat{j} - 4\hat{k}$$

(ii) Vector representing the flight path from R to Q is

$$\overrightarrow{RQ} = \overrightarrow{PQ} - \overrightarrow{PR}$$

$$= \left(5\hat{i} + 3\hat{j} - 4\hat{k}\right) - \left(5\hat{i} + \hat{j} - 2\hat{k}\right)$$

$$= 2\hat{j} - 2\hat{k}$$

(iii) Let θ be the angle between the vectors representing the flight paths of Airplane 1. then we have

$$\cos\theta = \frac{\overrightarrow{PQ} \cdot \overrightarrow{PR}}{\left|\overrightarrow{PQ}\right|\left|\overrightarrow{PR}\right|}$$

$$= \frac{\left(5\hat{i} + 3\hat{j} - 4\hat{k}\right) \cdot \left(5\hat{i} + \hat{j} - 2\hat{k}\right)}{\sqrt{50} \cdot \sqrt{30}}$$

$$= \frac{18}{5\sqrt{15}}$$

Angle between the flight paths,

$$\theta = \cos^{-1}\left(\frac{18}{5\sqrt{15}}\right)$$

(iv) Considers a point S which divides PQ internally in the ratio 1:2.

Position vector of point S,

$$\overrightarrow{OS} = \frac{1\overrightarrow{OQ} + 2\overrightarrow{OP}}{1+2}$$

$$= \frac{1\left(3\hat{i} + 4\hat{j} - \hat{k}\right) + 2\left(-2\hat{i} + \hat{j} + 3\hat{k}\right)}{1+2}$$

$$= \frac{3\hat{i} + 4\hat{j} - \hat{k} - 4\hat{i} + 2\hat{j} + 6\hat{k}}{3}$$

$$= \frac{-\hat{i} + 6\hat{j} + 5\hat{k}}{3}$$

$$= -\tfrac{1}{3}\hat{i} + 2\hat{j} + \tfrac{5}{3}\hat{k}$$

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CHAPTER 11

THREE DIMENSIONAL GEOMETRY

OBJECTIVE QUESTIONS

1. The angle which the line $\frac{x}{1}=\frac{y}{-1}=\frac{z}{0}$ makes with the positive direction of y-axis is:

(a) $\frac{5\pi}{6}$ (b) $\frac{3\pi}{4}$

(c) $\frac{5\pi}{4}$ (d) $\frac{7\pi}{4}$

Sol : OD 2024

We have $\frac{x}{1}=\frac{y}{-1}=\frac{z}{0}$

Direction ratio of y-axis is (0, 1, 0) and direction ratio of the given line is $(1,-1,0)$

$$\cos\theta = \frac{(0)(1)+(1)(-1)+(0)(0)}{\sqrt{0^2+1^2+0^2}\sqrt{1^2+(-1)^2+0^2}}$$

$$\cos\theta = \frac{-1}{\sqrt{2}}$$

$$\theta = \cos^{-1}\left(\frac{-1}{\sqrt{2}}\right)$$

$$\theta = \frac{3\pi}{4}$$

Thus (b) is correct option.

2. The Cartesian equation of the line passing through the point $(1,-3,2)$ and parallel to the line : $\vec{r}=(2+\lambda)\hat{i}+\lambda\hat{j}+(2\lambda-1)\hat{k}$ is

(a) $\frac{x-1}{2}=\frac{y+3}{0}=\frac{z-2}{-1}$

(b) $\frac{x+1}{1}=\frac{y-3}{1}=\frac{z+2}{2}$

(c) $\frac{x+1}{2}=\frac{y-3}{0}=\frac{z+2}{-1}$

(d) $\frac{x-1}{1}=\frac{y+3}{1}=\frac{z-2}{2}$

Sol : OD 2024

Since line passes through the point $A(1,-3,2)$, position vector of the point is $\vec{a}=\hat{i}-3\hat{j}+2\hat{k}$

Also, the required line is parallel to the line

$$\vec{r} = (2+\lambda)\hat{i}+\lambda\hat{j}+(2\lambda-1)\hat{k}$$

or $$\vec{r} = 2\hat{i}-\hat{k}+\lambda(\hat{i}+\hat{j}+2\hat{k})$$

It is parallel to the vector

$$\vec{b} = \hat{i}+\hat{j}+2\hat{k}$$

The vector equation of the line passing through $A(\vec{a})$ and parallel to $\vec{b}$ is $\vec{r}=\vec{a}+\lambda\vec{b}$ where λ is a scalar.

The required vector equation of the line is

$$\vec{r} = (\hat{i}-3\hat{j}+2\hat{k})+\lambda(\hat{i}+\hat{j}+2\hat{k})$$

and required Cartesian equation of the above line is

$$\frac{x-1}{1} = \frac{y+3}{1}=\frac{z-2}{2}$$

Thus (d) is correct option.

3. Equation of line passing through origin and making 30°, 60° and 90° with x, y, z axes respectively is :

(a) $\frac{2x}{\sqrt{3}}=\frac{y}{2}=\frac{z}{0}$ (b) $\frac{2x}{\sqrt{3}}=\frac{2y}{1}=\frac{z}{0}$

(c) $2x=\frac{2y}{\sqrt{3}}=\frac{z}{1}$ (d) $\frac{2x}{\sqrt{3}}=\frac{2y}{1}=\frac{z}{1}$

Sol : OD 2023

Direction cosines is given by

$$(\cos\alpha,\cos\beta,\cos\gamma) \equiv (\cos 30°,\cos 60°,\cos 90°)$$

$$\equiv \left(\frac{\sqrt{3}}{2},\frac{1}{2},\ 0\right)$$

Equation of the line is

$$\frac{x-0}{\frac{\sqrt{3}}{2}} = \frac{y-0}{\frac{1}{2}} = \frac{z-0}{0}$$

$$\frac{2x}{\sqrt{3}} = \frac{2y}{1} = \frac{z}{0}$$

Thus (b) is correct option.

4. The lines $\frac{x-2}{1}=\frac{y-3}{1}=\frac{4-z}{k}$ and $\frac{x-1}{k}=\frac{y-4}{2}=\frac{z-5}{-2}$ are mutually perpendicular, if the value of k is

(a) $-\frac{2}{3}$ (b) $\frac{2}{3}$

(c) -2 (d) 2

Sol : OD 2020

We have, $$\frac{x-2}{1} = \frac{y-3}{1}=\frac{4-z}{k}$$

or $$\frac{x-2}{1} = \frac{y-3}{1}=\frac{z-4}{-k} \quad L_1$$

and $$\frac{x-1}{k} = \frac{y-4}{2}=\frac{z-5}{-k}$$

or $$\frac{x-1}{k} = \frac{y-4}{2}=\frac{z-5}{-2} \quad L_2$$

Since, the lines L_1 and L_2 are perpendicular, we have

L_1: $\dfrac{1-x}{3} = \dfrac{y-2}{2\alpha} = \dfrac{z-3}{2}$ and

L_2: $\dfrac{x-1}{3\alpha} = \dfrac{y-1}{1} = \dfrac{6-z}{5}$

(a) $\dfrac{-10}{7}$ (b) $\dfrac{10}{7}$

(c) $\dfrac{-10}{11}$ (d) $\dfrac{10}{11}$

Sol : Comp 2014, Delhi 2012

Given lines can be rewritten as

L_1: $\dfrac{x-1}{-3} = \dfrac{y-2}{3\alpha} = \dfrac{z-3}{2}$ and

L_2: $\dfrac{x-1}{3\alpha} = \dfrac{y-1}{1} = \dfrac{z-6}{-5}$

Since, lines are perpendicular.

$$a_1a_2 + b_1b_2 + c_1c_2 = 0$$

$$(-3)(3\alpha) + 2\alpha(1) + 2(-5) = 0$$

$$-9\alpha + 2\alpha - 10 = 0$$

$$\alpha = -\frac{10}{7}$$

Thus (a) is correct option.

11. Which of the following triplets gives the direction cosines of a line?

(a) $<1,1,1>$ (b) $<1,-1,1>$

(c) $<1,-1,-1>$ (d) $<\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}>$

Sol : SQP 2020

For direction cosines of a line $l^2 + m^2 + n^2$ must be equal to 1.

$$l^2 + m^2 + n^2 = 1$$

This is possible when

$$l = \frac{1}{\sqrt{3}},\ m = \frac{1}{\sqrt{3}} \text{ and } n = \frac{1}{\sqrt{3}}$$

$$l^2 + m^2 + n^2 = \left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2 = \frac{3}{3} = 1$$

Thus (d) is correct option.

12. If α, β, γ are the angles which a half ray makes with the positive directions of the axes, then $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$ is equal to

(a) 2 (b) 1

(c) 0 (d) -1

Sol : OD 2007

Since, $$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

Hence, $$1 - \sin^2\alpha + 1 - \sin^2\beta + 1 - \sin^2\gamma = 1$$

$$\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$$

Thus (a) is correct option.

13. The foot of the perpendicular from $(0, 2, 3)$ to the line $\dfrac{x+3}{5} = \dfrac{y=1}{2} = \dfrac{z+4}{3}$ is

(a) $(-2,3,4)$ (b) $(2,-1,3)$

(c) $(2,3,-1)$ (d) $(3,2,-1)$

Sol : Delhi 2009

Let N be the foot of the perpendicular from the point $(0, 2, 3)$ on the given line

$$\frac{x+3}{5} = \frac{y-1}{2} = \frac{z+4}{3} = \lambda \text{ (say)} \qquad ...(1)$$

Any point on the line is $P(5\lambda - 3,\ 2\lambda + 1,\ 3\lambda - 4)$. If this point is N, then direction ratios of NP are $<5\lambda - 3 - 0,\ 2\lambda + 1 - 2,\ 3\lambda - 4 - 3>$

i.e. $<5\lambda - 3,\ 2\lambda - 1,\ 3\lambda - 7>$

Since, PN is perpendicular to line (1), we have

$$5(5\lambda - 3) + 2(2\lambda - 1) + 3(3\lambda - 7) = 0$$

$$38\lambda - 38 = 0$$

$$\lambda = 1$$

Substituting $\lambda = 1$ we get the required point as $(2,3,-1)$.

Thus (c) is correct option.

14. A straight line which makes an angle of 60° with each of y and z axes, inclined with x-axis at an angle of

(a) 30° (b) 45°

(c) 75° (d) 60°

Sol : Comp 2017, Foreign 2015

$$\cos^2\alpha + \cos^2 60^\circ + \cos^2 60^\circ = 1$$

$$\cos^2\alpha = 1 - \frac{1}{4} - \frac{1}{4} = \frac{1}{2}$$

$$\alpha = 45^\circ$$

Thus (b) is correct option.

15. The points $(5,\ 2,\ 4)$, $(6,-1,2)$ and $(8,-7,k)$ are collinear, if k is equal to

(a) -2 (b) 2

(c) 3 (d) -1

Sol : OD 2013

Since, points are collinear, we have

$$\begin{vmatrix} 5 & 2 & 4 \\ 6 & -1 & 2 \\ 8 & -7 & k \end{vmatrix} = 0$$

$$8(4+4) + 7(10-24) + k(-5-12) = 0$$

$$64 - 98 - 17k = 0$$

$$k = -2$$

Thus (a) is correct option.

Sol : Foreign 2009

If line makes equal angles with the axes, then

$$l = m = n = \pm \frac{1}{\sqrt{3}}$$

Since, the line lies in the octant $OXYZ$, therefore we take +ve sign.

i.e. $$l = m = n = \frac{1}{\sqrt{3}}$$

Thus (a) is correct option.

23. If $A(1,2,3)$, $B(0,1,2)$ and $C(2,1,0)$ are vertices of a triangle then the length of the median through A is

(a) $\sqrt{5}$ (b) $2\sqrt{5}$

(c) 5 (d) 10

Sol : SQP 2017

Mid point of BC is $\left(\frac{2+0}{2}, \frac{1+1}{2}, \frac{0+2}{2}\right)$, ie, $D(1,1,1)$.

Hence, Length of median,

$$\begin{aligned} AD &= \sqrt{(1-1)^2 + (2-1)^2 + (3-1)^2} \\ &= \sqrt{1+4} \\ &= \sqrt{5} \end{aligned}$$

Thus (a) is correct option.

24. The distance between (4, 3, 7) and $(1, -1, -5)$ is

(a) 13 (b) 15

(c) 12 (d) 5

Sol : OD 2018, Delhi 2015

Distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by distance formula

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

Here, $$x_1 = 4,\ y_1 = 3,\ z_1 = 7$$

$$x_2 = 1,\ y_2 = -1,\ z_2 = -5$$

Using distance formula,

$$\begin{aligned} d &= \sqrt{(4-1)^2 + (3-(-1))^2 + (7-(-5))^2} \\ &= \sqrt{3^2 + 4^2 + 12^2} \\ &= \sqrt{9 + 16 + 144} \\ &= \sqrt{144 + 25} \\ &= \sqrt{169} \\ &= 13 \end{aligned}$$

Thus (a) is correct option.

25. The direction cosines of z-axis are

(a) (0, 0, 0) (b) (1, 0, 0)

(c) (0, 0, 1) (d) (0, 1, 0)

Sol : Delhi 2011, Comp 2010

Any vector $\vec{a}$ along z axis can be written as $\left(0\hat{i} + 0\hat{j} + 1\hat{k}\right)$

Direction cosine is coefficient of $\hat{i}, \hat{j}, \hat{k}$ components. Therefore,

Direction cosine of $z = (0,0,1)$

26. Let l_1, m_1, n_1 and l_2, m_2, n_2 be the direction cosines of two st- lines. Both the lines are perpendicular to each other, if

(a) $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$

(b) $l_1 l_2 + m_1 m_2 + n_1 n_2 = 1$

(c) $\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$

(d) $\frac{l_1}{l_2} + \frac{m_1}{m_2} + \frac{n_1}{n_2} = 0$

Sol : Delhi 2010

The angle between two lines having direction cosine l_1, m_1, n_1 and l_2, m_2, n_2 is given by,

$$\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$$

If both lines are perpendicular then $\theta = 90°$.

Hence, $$\cos 90° = l_1 l_2 + m_1 m_2 + n_1 n_2$$

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$$

Thus (a) is correct option.

27. Let a, b, c be the direction ratios of a line then direction cosines are

(a) $\frac{a}{\sqrt{\Sigma a^2}}, \frac{b}{\sqrt{\Sigma a^2}}, \frac{c}{\sqrt{\Sigma a^2}}$ (b) $, \frac{1}{\sqrt{\Sigma a^2}}, \frac{1}{\sqrt{\Sigma a^2}}$

(c) $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ (d) $, \frac{b}{\sqrt{\Sigma b^2}}, \frac{c}{\sqrt{\Sigma c^2}}$

Sol : Foreign 2018

Any three numbers a, b, c proportional to the direction cosine of a line are called direction ratios of the line.

If l, m, n, are the direction cosine of line and a, b, c are direction ratios, then

$$l = \frac{a}{\sqrt{a^2 + b^2 + c^2}} = \frac{a}{\sqrt{\Sigma a^2}}$$

$$m = \frac{b}{\sqrt{a^2 + b^2 + c^2}} = \frac{b}{\sqrt{\Sigma a^2}}$$

$$n = \frac{c}{\sqrt{a^2 + b^2 + c^2}} = \frac{c}{\sqrt{\Sigma a^2}}$$

Thus (a) is correct option.

28. A line is passing through (α, β, γ) and its direction cosines are l, m, n then the equations of the line are

(a) $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$

Therefore, $l = \frac{1}{\sqrt{35}}$, $m = \frac{3}{\sqrt{35}}$ and $n = \frac{5}{\sqrt{35}}$

Thus (a) is correct option.

33. The direction ratios of two straight lines are l, m, n and l_1, m_1, n_1. The lines will be perpendicular to each other if

(a) $\frac{l}{l_1} = \frac{m}{m_1} = \frac{n}{n_1}$ (b) $\frac{l}{l_1} + \frac{m}{m_1} + \frac{n}{n_1} = 0$

(c) $ll_1 + mm_1 + nn_1 = 0$ (d) $ll_1 + mm_1 + nn_1 = 1$

Sol : OD 2013

Direction ratio of line L_1 is (l, m, n)

Vector parallel to L_1 is $\vec{p} = l\hat{i} + m\hat{j} + n\hat{k}$

Direction ratio of line L_2 is (l_1, m_1, n_1)

Vector parallel to L_2 is $\vec{q} = l_1\hat{i} + m_1\hat{j} + n_1\hat{k}$

Since $\vec{p} \perp \vec{q}$, Therefore,

$$\vec{p} \cdot \vec{q} = 0$$

$$\vec{p} \cdot \vec{q} = (l\hat{i} + m\hat{j} + n\hat{k}) \cdot (l_1\hat{i} + m_1\hat{j} + n_1\hat{k})$$

$$= ll_1 + mm_1 + nn_1$$

$$= 0$$

Thus (c) is correct option.

34. A line passing through $(2, -1, 3)$ and its direction ratios are $3, -1, 2$. The equation of the line is

(a) $\frac{x+2}{3} = \frac{y-1}{-1} = \frac{z+3}{2}$

(b) $\frac{x-2}{3} = \frac{y+1}{-1} = \frac{z-3}{2}$

(c) $\frac{x-3}{2} = \frac{y+1}{-1} = \frac{z-2}{3}$

(d) $\frac{x-3}{2} = \frac{y+1}{-1} = \frac{z-2}{3}$

Sol : Foreign 2014, OD 2012

Line passing through point (a, b, c) with direction ratios l, m, n is given by

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}$$

Here, $a = 2,\ b = -1,\ c = 3$

$$l = 3,\ m = -1,\ n = 2$$

Therefore, required line L:

$$\frac{x-2}{3} = \frac{y+1}{-1} = \frac{z-3}{2}$$

Thus (b) is correct option.

35. The lines $\frac{x-1}{l} = \frac{y+2}{m} = \frac{z-4}{n}$ and $\frac{x+3}{2} = \frac{y-4}{3} = \frac{z}{6}$ are parallel to each other if

(a) $2l = 3m = n$ (b) $3l = 2m = n$

(c) $2l + 3m + 6n = 0$ (d) $lmn = 36$

Sol : Delhi 2010

Two lines are parallel to each other when their direction ratio are proportional to each other.

Hence, $\frac{l}{2} = \frac{m}{3} = \frac{n}{6}$

$$3l = 2m = n$$

Thus (b) is correct option.

36. The coordinates of the midpoint of the line segment joining the points (2, 3, 4) and $(8, -3, 8)$ are

(a) (10, 0, 12) (b) (5, 6, 0)

(c) (6, 5, 0) (d) (5, 0, 6)

Sol : OD 2008

We have $A = (2, 3, 4)$ B $(8, -3, 8)$

Mid point of line segment joining (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$P = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

Here, $x_1 = 2$ $y_1 = 3$ $z_1 = 4$

$$x_2 = 8 \quad y_2 = -3 \quad z_2 = 8$$

$$P = \left(\frac{2+8}{2}, \frac{3-3}{2}, \frac{4+8}{2}\right)$$

$$P = (5, 0, 6)$$

Thus (d) is correct option.

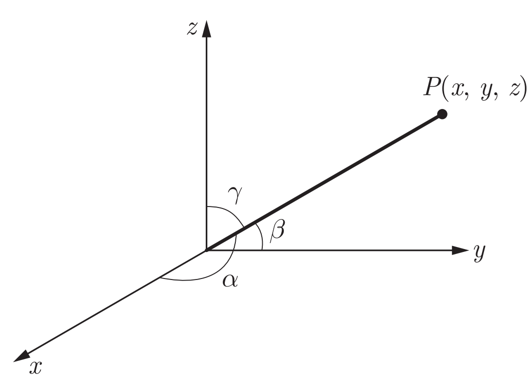
37. The direction cosines of the $y-$ axis are

(a) (0, 0, 0) (b) (1, 0, 0)

(c) (0, 1, 0) (d) (0, 0, 1)

Sol : Comp 2018, Foreign 2015

If α, β and γ are the angles which a vector $\overrightarrow{OP}$ makes with the positive direction of coordinate axes OX, OY, OZ respectively, then $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are known as the direction of cosines of $\overrightarrow{OP}$ and generally denoted by $l_1 m$ and n respectively.



$$x\hat{i}+y\hat{j}+z\hat{k} = 3\hat{i}+4\hat{j}-7\hat{k}+\lambda(-2\hat{i}-5\hat{j}+13\hat{k})$$
$$(x-3)\hat{i}+(y-4)\hat{j}+(z+7)\hat{k} = \lambda(-2\hat{i}-5\hat{j}+13\hat{k})$$

43. **Assertion:**: If a line makes angles α, β, γ with positive direction of the coordinate axes, then $\sin^2\alpha+\sin^2\beta+\sin^2\gamma=2$.

Reason : The sum of squares of the direction cosines of a line is 1.

(a) Assertion is true, Reason is true; Reason is a correct explanation for Assertion.

(b) Assertion is true, Reason is true; Reason is not a correct explanation for Assertion.

(c) Assertion is true; Reason is false.

Sol : OD 2023

If a line makes angles α, β, γ with positive direction of the coordinate axes, then

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma = 1$$
$$(1-\sin^2\alpha)+(1-\sin^2\beta)+(1-\sin^2\gamma) = 1$$
$$3-\sin^2\alpha-\sin^2\beta-\sin^2\gamma = 1$$
$$2-\sin^2\alpha-\sin^2\beta-\sin^2\gamma = 0$$
$$\sin^2\alpha+\sin^2\beta+\sin^2\gamma = 2$$

Thus both (A) and (R) are true and (R) is the correct explanation of (A).

Thus (a) is correct option.

44. **Assertion:** The pair of lines given by $\vec{r}=\hat{i}-\hat{j}+\lambda(2\hat{i}+k)$ and $\vec{r}=2\hat{i}-k+\mu(\hat{i}+\hat{j}-\hat{k})$ intersect.

Reason : Two lines intersect each other, if they are not parallel and shortest distance $=0$.

(a) Assertion is true, Reason is true; Reason is a correct explanation for Assertion.

(b) Assertion is true, Reason is true; Reason is not a correct explanation for Assertion.

(c) Assertion is true; Reason is false.

(d) Assertion is false; Reason is true.

Sol : Delhi 2011

Here, $\vec{a}_1 = \hat{i}-\hat{j}$, $\vec{b}_1=2\hat{i}+\hat{k}$

$\vec{a}_2 = 2\hat{i}-\hat{k}$, $\vec{b}_2=\hat{i}+\hat{j}-\hat{k}$

$\vec{b}_1 \neq \lambda\vec{b}_2$, for any scalar λ

Given lines are not parallel.

$$\vec{a}_2-\vec{a}_1 \; (2\hat{i}-\hat{k})-(\hat{i}-\hat{j}) = \hat{i}+\hat{j}-\hat{k}$$

$$\vec{b}_1\times\vec{b}_2 = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 0 & 1\\ 1 & 1 & -1\end{vmatrix} = -\hat{i}+3\hat{j}+2\hat{k}$$

$$\vec{b}_1\times\vec{b}_2 = \sqrt{(-1)^2+(3)^2+(2)^2}$$
$$= \sqrt{1+9+4}$$
$$= \sqrt{14}$$

$$\text{SD} = \left|\frac{(\vec{a}_2-\vec{a}_1)\cdot(\vec{b}_1\times\vec{b}_2)}{|(\vec{b}_1\times\vec{b}_2)|}\right|$$
$$= \left|\frac{(\hat{i}+\hat{j}-\hat{k})\cdot(-\hat{i}+3\hat{j}+2\hat{k})}{\sqrt{14}}\right|$$
$$= \left|\frac{-1+3-2}{\sqrt{14}}\right| = 0$$

Thus assertion is true, Reason is true; Reason is a correct explanation for Assertion.

Thus (a) is correct option.

VERY SHORT ANSWER QUESTIONS

45. If the equation of a line is $x=ay+b$, $z=cy+d$,then find the direction ratios of the line and a point on the line.

Sol : OD 2023

Given equation of the line is $x=ay+b$, $z=cy+d$

$$\frac{x-b}{a} = y,\ \frac{z-d}{c}=y$$
$$\frac{x-b}{a} = \frac{y-0}{1} = \frac{z-d}{c}$$

Direction ratios are $(a, 1, c)$ and a point on the given line is $(b, 0, d)$.

46. The line of shortest distance between two skew lines is .... to both the lines.

Sol : OD 2020

For skew lines, the line of the shortest distance will be perpendicular to both the lines and it is unique also.

47. If a line makes angles 90°, 135°, 45° with then x, y and z axes respectively, find the direction consines.

Sol : Delhi 2019

Let l, m and n be direction cosines of the lines.

We have $\alpha = 90^\circ$, $\beta = 135^\circ$ and $\gamma = 45^\circ$

Now $l = \cos\alpha = \cos 90^\circ = 0$,

$$m = \cos\beta = \cos 135^\circ = \frac{-1}{\sqrt{2}}$$

and $n = \cos\gamma = \cos 45^\circ = \dfrac{1}{\sqrt{2}}$

is the required equation of the given line in Cartesian form.

53. If a line makes angles α, β, γ with the position direction of coordinates axes, then write the value of $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$.

Sol : Delhi 2015

If a line makes angles α, β and γ with the x-axis, y-axis and z-axis, respectively, then we have

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$
$$1 - \sin^2\alpha + 1 - \sin^2\beta + 1 - \sin^2\gamma = 1$$
$$3 - \sin^2\alpha - \sin^2\beta - \sin^2\gamma = 1$$
$$-\sin^2\alpha - \sin^2\beta - \sin^2\gamma = 1 - 3 = -2$$
$$\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$$

54. Write the distance of a point $P(a, b, c)$ from x-axis.

Sol : Delhi 2014, Comp 2010

Let any point on x-axis be $Q(x,0,0)$. If we want to find out the distance of point $P(a, b, c)$ from x-axis, we have to take $x = a$.

Using distance formula for points $P(a, b, c)$ and $Q(x,0,0)$ we have

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$
$$= \sqrt{(a-a)^2 + (0-b)^2 + (0-c)^2}$$
$$= \sqrt{0 + b^2 + c^2}$$
$$= \sqrt{b^2 + c^2}$$

55. Write the equation of the straight line through the point (α,β,γ) and parallel to Z-axis.

Sol : OD 2014

The vector equation of a line parallel to z-axis is

$$\vec{m} = 0\hat{i} + 0\hat{j} + \hat{k}.$$

The line passing through the point $A(\alpha,\beta,\gamma)$ whose position vector is $\vec{r}_1 = \alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}$ and is parallel to the vector $\vec{m} = (0\hat{i} + 0\hat{j} + \hat{k})$ is given by

$$\vec{r} = \vec{r}_1 + \lambda\vec{m}$$
$$= (\alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}) + \lambda(0\hat{i} + 0\hat{j} + \hat{k})$$
$$= (\alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}) + \lambda(\hat{k})$$

56. Find the direction cosines of the line $\frac{4-x}{2} = \frac{y}{6} = \frac{1-z}{3}$.

Sol : Delhi 2013

We have $\frac{4-x}{2} = \frac{y}{6} = \frac{1-z}{3}$

Rewriting the given equation in standard form, we get

$$\frac{x-4}{-2} = \frac{y}{6} = \frac{z-1}{-3}$$

Here, DR's of the line are $(-2, 6, -3)$.

Now $\sqrt{(-2)^2 + 6^2 + (-3)^2} = \sqrt{49} = 7$

Thus direction cosines of the line are $\frac{-2}{7}$, $\frac{6}{7}$ and $\frac{-3}{7}$

57. Write the vector equation of a line passing through point $(1, -1, 2)$ and parallel to the line whose equation is $\frac{x-3}{1} = \frac{y-1}{2} = \frac{z+1}{-2}$.

Sol : OD 2013

Vector equation of a line passing through a point with position vector $\vec{a}$ and parallel to a given vector $\vec{b}$ is

$\vec{r} = \vec{a} + \lambda\vec{b}$, where $\lambda \in R$.

Vector for point $(1, -1, 2)$ is

$$\vec{a} = \hat{i} - \hat{j} + 2\hat{k}$$

and vector for line $\frac{x-3}{1} = \frac{y-1}{2} = \frac{z+1}{-2}$ is

$\vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$. DR's are 1, 2 and -2

Required vector equation of line is

$\vec{r} = (\hat{i} - \hat{j} + 2\hat{k}) + \lambda(\hat{i} + 2\hat{j} - 2\hat{k})$, where $\lambda \in R$.

58. Find the Cartesian equation of the line which passes through the point $(-2, 4, -5)$ and is parallel to the line $\frac{x+3}{6} = \frac{4-y}{5} = \frac{z+8}{6}$.

Sol : Delhi 2013, OD 2011

We have $\frac{x+3}{3} = \frac{4-y}{5} = \frac{z+8}{6}$

or $\frac{x+3}{3} = \frac{y-4}{-5} = \frac{z+8}{6}$

DR of above line are $(3, -5, 6)$. If two lines are parallel, then direction ratios of both lines are proportional.

The required equation of the line passing through $(-2, 4, -5)$ having DR's $(3, -5, 6)$ is

$$\frac{x+2}{3} = \frac{y-4}{-5} = \frac{z+5}{6}$$

59. If a line has direction ratios $(2, -1, -2)$, then what are its direction cosines?

Sol : Delhi 2012

If (a, b, c) are DR's of a line then direction cosines of line is given by

$$l = \pm\frac{a}{\sqrt{a^2 + b^2 + c^2}}$$
$$m = \pm\frac{b}{\sqrt{a^2 + b^2 + c^2}}$$
$$n = \pm\frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

Here DR's of the line are $(2, -1, -2)$.

Ch 7 : Integrals

PDF	PDF	PDF	DOC	PPT
Text Book	Worksheet	Solutions	Worksheet	Presentation

Ch 8 : Application of Integrals

PDF	PDF	PDF	DOC	PPT
Text Book	Worksheet	Solutions	Worksheet	Presentation

Ch 9 : Differential Equations

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Text Book	Worksheet	Solutions	Worksheet	Presentation

Ch 10 : Vector Algebra

PDF	PDF	PDF	DOC	PPT
Text Book	Worksheet	Solutions	Worksheet	Presentation

Ch 11 : Three Dimensional Geometry

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Text Book	Worksheet	Solutions	Worksheet	Presentation

Ch 12 : Linear Programming

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Text Book	Worksheet	Solutions	Worksheet	Presentation

Ch 13 : Probability

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Text Book	Worksheet	Solutions	Worksheet	Presentation

Class 10th

- Maths
- Science
- Social Science
- English LL
- Info Tech.
- Comp. App.
- AI
- English Comm.
- हिंदी अ
- हिंदी ब
- संस्कृत

Class 9th

- Maths
- Science
- Social Science
- English LL
- Info Tech.
- Comp. App.
- AI
- English Comm.
- हिंदी अ
- हिंदी ब
- संस्कृत

Class 8th

- Maths
- Science
- Social Science
- English
- हिंदी
- संस्कृत

Class 7th

- Maths
- Science
- Social Science
- English
- हिंदी
- संस्कृत

Class 6th

- Maths
- Science
- Social Science
- English
- हिंदी
- संस्कृत

Class 5th

Maths | English | हिंदी | EVS

Class 4th

Maths | English | हिंदी | EVS

Class 3th

Maths | English | हिंदी | EVS

Class 2th

Maths | English | हिंदी | EVS

Class 1th

Maths | English | हिंदी | EVS

Prep/KG 3rd Year

Maths | English | हिंदी | EVS

Prep/KG 2nd Year

Maths | English | हिंदी | EVS

Prep/KG 1st Year

Maths | English | हिंदी | EVS

$$\frac{x-3}{-5} = \frac{y+4}{7} = \frac{z-3}{2} = \lambda \text{ (let)}$$

or $x = -5\lambda + 3,\ y = 7\lambda - 4$ and $z = 2\lambda + 3$

Now,

$$x\hat{i} + y\hat{j} + z\hat{k} = (-5\lambda + 3)\,\hat{i} + (7\lambda - 4)\,\hat{j} + (2\lambda + 3)\,\hat{k}$$

$$= 3\hat{i} - 4\hat{j} + 3\hat{k} + \lambda(-5\hat{i} + 7\hat{j} + 2\hat{k})$$

$$\vec{r} = (3\hat{i} - 4\hat{j} + 3\hat{k}) + \lambda(-5\hat{i} + 7\hat{j} + 2\hat{k})$$

which is the required equation of line in vector form.

66. The equation of a line is

$$5x - 3 = 15y + 7 = 3 = 3 - 10z$$

Write the direction cosines of the line.

Sol : OD 2015

We have $5x - 3 = 15y + 7 = 3 - 10z$ ...(1)

Here coefficients of x, y and z are 5, 15 and 10. LCM $(5, 15, 10) = 30$. Thus dividing by 30 we have eq. (1) becomes

$$\frac{5x-3}{30} = \frac{15y+7}{30} = \frac{3-10z}{30}$$

$$\frac{5(x-\frac{3}{5})}{30} = \frac{15(y+\frac{7}{15})}{30} = \frac{-10(z-\frac{3}{10})}{30}$$

$$\frac{x-\frac{3}{5}}{6} = \frac{y+\frac{7}{15}}{2} = \frac{z-\frac{3}{10}}{-3} \quad ...(2)$$

The standard form of equation is given as

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c} \quad ...(3)$$

Comparing the above standard equation with Eq. (2), we get 6, 2, -3 are the direction ratios of the given line.

Now $\sqrt{6^2 + 2^2 + (-3)^2} = \sqrt{49} = 7$

Now, the direction cosines of given line are $\frac{6}{7}, \frac{2}{7}, \frac{-3}{7}$.

67. Write the vector equation of the following line

$$\frac{x-5}{3} = \frac{y+4}{7} = \frac{6-z}{2}.$$

Sol : Delhi 2010, OD 2008

We have $\frac{x-5}{3} = \frac{y+4}{7} = \frac{6-z}{2}$

Rewriting the given equation in standard form, we get

$$\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{-2} = \lambda \text{ (let)}$$

or $x = 3\lambda + 5,\ y = 7\lambda - 4$ and $z = -2\lambda + 6$

Now,

$$x\hat{i} + y\hat{j} + z\hat{k} = (3\lambda + 5)\,\hat{i} + (7\lambda - 4)\,\hat{j} + (-2\lambda + 6)\,\hat{k}$$

$$= 5\hat{i} - 4\hat{j} + 6\hat{k} + \lambda(3\hat{i} + 7\hat{j} - 2\hat{k})$$

$$\vec{r} = (5\hat{i} - 4\hat{j} + 6\hat{k}) + \lambda(3\hat{i} + 7\hat{j} - 2\hat{k})$$

which is the required equation of line in vector form.

68. Find the vector equation of the line passing through the point $A(1, 2, -1)$ and parallel to the line $5x - 25 = 14 - 7y = 35z$.

Sol : Delhi 2017

We have $5x - 25 = 14 - 7y = 35z$.

$$\frac{x-5}{\frac{1}{5}} = \frac{2-y}{\frac{1}{7}} = \frac{z}{\frac{1}{35}}$$

$$\frac{x-5}{\frac{1}{5}} = \frac{y-2}{-\frac{1}{7}} = \frac{z}{\frac{1}{35}}$$

Direction ratio of the given line are $\frac{1}{5}, -\frac{1}{7}, \frac{1}{35}$.

Direction ratio of a line parallel to the given line are $\frac{1}{5}, -\frac{1}{7}, \frac{1}{35}$.

The required equation of a line passing through the point $A(1, 2, -1)$ and parallel to the given line is

$$\frac{x-1}{\frac{1}{5}} = \frac{y-2}{-\frac{1}{7}} = \frac{z+1}{\frac{1}{35}}.$$

69. Find the direction cosines of the line

$$\frac{x+2}{2} = \frac{2y-7}{6} = \frac{5-z}{6}$$

Also, find the vector equation of the line through the point $A(-1, 2, 3)$ and parallel to the given line.

Sol : Comp 2014, Delhi 2012

We have $\frac{x+2}{2} = \frac{2y-7}{6} = \frac{5-z}{6}$

This equation can be written as

$$\frac{x+2}{2} = \frac{y-7/2}{3} = \frac{z-5}{-6}$$

So, direction ratios of line are $(2, 3, -6)$.

Now $\sqrt{(2^2 + 3^2 + (-6)^2} = \sqrt{49} = 7$

Thus direction cosines of given line are $\left(\frac{2}{7}, \frac{3}{7}, \frac{-6}{7}\right)$.

Here, DR's of a line parallel to given line are $(2, 3, -6)$. So, the required equation of line passes through the point $A(-1, 2, 3)$ and parallel to given line is

$$\frac{x+1}{2} = \frac{y-2}{3} = \frac{z-3}{-6}.$$

70. The x-coordinate of point on the line joining the points $P(2, 2, 1)$ and $Q(5, 1, -2)$ is 4. Find its z-coordinate.

Sol : OD 2017

The equation of the line joining the points $P(2, 2, 1)$ and $Q(5, 1, -2)$ is

$$\frac{x-2}{5-2} = \frac{y-2}{1-2} = \frac{z-1}{-2-1}$$

$$\frac{x-2}{3} = \frac{y-2}{-1} = \frac{z-1}{-3}$$

DR's of PR is $(12-4,\ 4-2,\ 5-(-6))$

$$=(8,\ 2,\ 11)$$

Equation of PR,

$$\frac{x-4}{8}=\frac{y-2}{2}=\frac{z+6}{11}$$

DR's of QS is $(11-5,\ 9-(-3),\ -2-1)$

$$=(6,\ 12,\ -3)$$

Equation of QS,

$$\frac{x-5}{6}=\frac{y+3}{12}=\frac{z-1}{-3}$$

Let the point of intersection of PR and QS be A. i.e. A lies on both the lines PR and QS.

From PQ we have

$$\frac{x-4}{8}=\frac{y-2}{2}=\frac{z+6}{11}=k$$

$$(x,y,z)=(8k+4, 2k+2, 11k-6) \quad ...(1)$$

From QS we have

$$\frac{x-5}{6}=\frac{y+3}{12}=\frac{z-1}{-3}=s$$

$$(x,y,z)=(6s+5, 12s-3, -3s+1) \quad ...(2)$$

Equating (1) and (2) to get A we get

$$8k+4=6s+5$$

$$8k-6s-1=0 \quad ...(3)$$

$$2k+2=12s-3$$

$$2k-12s+5=0 \quad ...(4)$$

$$11k-6=-3s+1$$

$$11k+3s-7=0 \quad ...(5)$$

Solving eq (3) and (4) we have $k=\frac{1}{2}$ and $s=\frac{1}{2}$. Substituting $k=\frac{1}{2}$ in 2 we get $A(8,\ 3,\ -\frac{1}{2})$ which is the point of intersection of the diagonals of the parallelogram $PQRS$.

74. Find the vector and Cartesian equations of the line which is perpendicular to the lines with equations

$$\frac{x+2}{1}=\frac{y-3}{2}=\frac{z+1}{4}$$

and $$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$$

and passes through the point (1, 1, 1). Also, find the angle between the given lines.

Sol : OD 2020

Any line through the point (1, 1, 1) is given by

$$\frac{x-1}{a}=\frac{y-1}{b}=\frac{z-1}{c} \quad ...(1)$$

where a, b and c are the direction ratios of line (1). Now, the line (1) is perpendicular to the lines

$$\frac{x+2}{1}=\frac{y-3}{2}=\frac{z+1}{4} \quad (2)$$

and $$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4} \quad (3)$$

DR's of these two lines are (1, 2, 4) and (2, 3, 4), respectively.

If two lines having DR's (a_1, b_1, c_1) and (a_2, b_2, c_2) are perpendicular, then $a_1a_2+b_1b_2+c_1c_2=0$.

Since Line (1) is perpendicular to line (2) and (3) we have

$$a+2b+4c=0 \quad ...(2)$$

and $$2a+3b+4c=0 \quad ...(3)$$

By cross-multiplication method, we get

$$\frac{a}{8-12}=\frac{b}{8-4}=\frac{c}{3-4}$$

$$\frac{a}{-4}=\frac{b}{4}=\frac{c}{-1}$$

Thus DR's of line (1) are –4, 4, –1

The required cartesian equation of line (1) is

$$\frac{x-1}{-4}=\frac{y-1}{4}=\frac{z-1}{-1}$$

and vector equation is

$$\vec{r}=\hat{i}+\hat{j}+\hat{k}+\lambda(-4\hat{i}+4\hat{j}-\hat{k})$$

Again, let θ be the angle between the given lines (2) and (3). Then we have

$$\cos\theta=\frac{1\times2+2\times3+4\times4}{\sqrt{1+4+16}\sqrt{4+9+16}}$$

$$=\frac{24}{\sqrt{21}\sqrt{29}}$$

$$=\frac{24}{\sqrt{609}}$$

$$\theta=\cos^{-1}\left(\frac{24}{\sqrt{609}}\right)$$

75. Find the value of λ, so that the lines

$$\frac{1-x}{3}=\frac{7y-14}{\lambda}=\frac{z-3}{2} \text{ and } \frac{7-7x}{3\lambda}=\frac{y-5}{1}=\frac{6-z}{5}$$

are at right angles. Also, find whether the lines are intersecting or not.

Sol : Delhi 2019, Foreign 2014

Writing the given line in standard form as

$$\frac{x-1}{-3}=\frac{y-2}{\frac{\lambda}{7}}=\frac{z-3}{2}=r_1 \text{ (let)} \quad ...(1)$$

and $$\frac{x-1}{\frac{-3\lambda}{7}}=\frac{y-5}{1}=\frac{z-6}{-5}=r_2 \text{ (let)} \quad ...(2)$$

Two lines with DR's a_1, b, c_1 and a_2, b_2, c_2 are perpendicular if

$$= \sqrt{4+1} = \sqrt{5}$$

Now, the shortest distance between the given lines,

$$d = \frac{|(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1)|}{|\vec{b}_1 \times \vec{b}_2|}$$

$$= \frac{|(2\hat{i} - \hat{j}) \cdot (-3\hat{i} + 2\hat{k})|}{\sqrt{5}}$$

$$= \frac{|-6|}{\sqrt{5}} = \frac{6}{\sqrt{5}} \text{ units}$$

78. Find the shortest distance between the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5}$.

Sol : Delhi 2018

We have $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$

and $\frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5}$

Comparing the given equations of lines with standard form we have

$$\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$$

and $\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$.

we get $x_1 = 1,\ y_1 = 2,\ z_1 = 3;\ a_1 = 2,\ b_1 = 3,\ c_1 = 4$

and $x_2 = 2,\ y_2 = 4,\ z_2 = 5;\ a_2 = 3,\ b_2 = 4,\ c_2 = 5$

Now $\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \begin{vmatrix} 2-1 & 4-2 & 5-3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}$

$$= \begin{vmatrix} 1 & 2 & 2 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}$$

$$= 1(15-16) - 2(10-12) + 2(8-9)$$

$$= -1 + 4 - 2 = 1$$

Now, $\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}$

$$= \sqrt{(3\times 5 - 4\times 4)^2 + (4\times 3 - 5\times 2)^2 + (2\times 4 - 3\times 3)^2}$$

$$= \sqrt{(15-16)^2 + (12-10)^2 + (8-9)^2}$$

$$= \sqrt{(-1)^2 + (2)^2 + (-1)^2}$$

$$= \sqrt{1+4+1} = \sqrt{6}$$

$$\text{SD} = \frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}}$$

$$= \frac{1}{\sqrt{6}} \text{ units,}$$

which is the required shortest distance.

79. Find the vector and Cartesian equations of the line through the point $(1, 2, -4)$ and perpendicular to the lines

$$\vec{r} = (8\hat{i} - 19\hat{j} + 10\hat{k}) + \lambda(3\hat{i} - 16\hat{j} + 7\hat{k})$$

and $\vec{r} = (15\hat{i} + 29\hat{j} + 5\hat{k}) + \mu(3\hat{i} + 8\hat{j} - 5\hat{k})$.

Sol : Delhi 2016, Delhi 2015

We have $\vec{r} = (8\hat{i} - 19\hat{j} + 10\hat{k}) + \lambda(3\hat{i} - 16\hat{j} + 7\hat{k})$

and $\vec{r} = (15\hat{i} + 29\hat{j} + 5\hat{k}) + \mu(3\hat{i} + 8\hat{j} - 5\hat{k})$

Comparing with $\vec{r} = \vec{a} + \lambda\vec{b}$ we get

$$\vec{b}_1 = 3\hat{i} - 16\hat{j} + 7\hat{k}$$

and $\vec{b}_2 = 3\hat{i} + 8\hat{j} - 5\hat{k}$

Now $\vec{b} = \vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -16 & 7 \\ 3 & 8 & -5 \end{vmatrix}$

$$= \hat{i}(80-56) - \hat{j}(-15-21) + \hat{k}(24+48)$$

$$= 24\hat{i} + 36\hat{j} + 72\hat{k} = 12(2\hat{i} + 3\hat{j} + 6\hat{k})$$

Since, the required line is perpendicular to the given lines. So, it is parallel to $\vec{b}_1 \times \vec{b}_2$. Now, the equation of a line passing through the point $(1, 2, -4)$ and parallel to $24\hat{i} + 36\hat{j} + 72\hat{k}$ or $(2\hat{i} + 3\hat{j} + 6\hat{k})$ is

$$\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$$

which is required vector equation of a line.

For Cartesian equation, substituting $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, we get

$$x\hat{i} + y\hat{j} + z\hat{k} = (1+2\lambda)\hat{i} + (2+3\lambda)\hat{j} + (-4+6\lambda)\hat{k}$$

Comparing the coefficients of $\hat{i}$, $\hat{j}$ and $\hat{k}$, we get

$$x = 1 + 2\lambda,\ y = 2 + 3\lambda$$

and $z = -4 + 6\lambda$

Thus $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z+4}{6}$

which is the required Cartesian equation of a line.

Alternative :

Let the equation of line passing through $(1, 2, -4)$ is

$$\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda_1(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) \quad ...(1)$$

Since, the line (1) is perpendicular to the given lines

$$\vec{r} = (8\hat{i} - 19\hat{j} + 10\hat{k}) + \lambda(3\hat{i} - 16\hat{j} + 7\hat{k})$$

and $\vec{r} = (15\hat{i} + 29\hat{j} + 5\hat{k}) + \mu(3\hat{i} + 8\hat{j} - 5\hat{k})$

Therefore, we have

$$(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) \cdot (3\hat{i} - 16\hat{j} + 7\hat{k}) = 0$$

$$3b_1 - 16b_2 + 7b_3 = 0 \quad ...(2)$$

82. Show that the line lines

$$\vec{r} = (\hat{i} + \hat{j} - \hat{k}) + \lambda(3\hat{i} - \hat{j}) \text{ and}$$

$$\vec{r} = (4\hat{i} - \hat{k}) + \mu(2\hat{i} + 3\hat{k})$$

intersect. Also, find their point of intersection.

Sol : Delhi 2014

We have $\vec{r} = (\hat{i} + \hat{j} - \hat{k}) + \lambda(3\hat{i} - \hat{j})$

and $\vec{r} = (4\hat{i} - \hat{k}) + \mu(2\hat{i} + 3\hat{k})$

Above lines can be rewritten as

$$\vec{r} = (3\lambda + 1)\hat{i} + (1 - \lambda)\hat{j} - \hat{k} \qquad ...(1)$$

and $$\vec{r} = (4 + 2\mu)\hat{i} + 0\hat{i} + (3\mu - 1)\hat{k} \qquad ...(2)$$

Clearly, any point on line (1) is of the form $P(3\lambda + 1,\ 1 - \lambda,\ -1)$ and any point on line (2) is of the form $Q(4 + 2\mu,\ 0,\ 3\mu - 1)$

If line (1) and (2) intersect, then these points must coincide for some λ and μ.

Consider, $3\lambda + 1 = 4 + 2\mu$

$$2\lambda - 2\mu = 3 \qquad ...(3)$$

$$1 - \lambda = 0 \qquad ...(4)$$

and $$3\mu - 1 = -1 \qquad ...(5)$$

From Eq. (4), we get $\lambda = 1$ and put the value of λ in Eq. (3), we get

$$3(1) - 2\mu = 3$$

$$-2\mu = 3 - 3 \Rightarrow \mu = 0$$

Substituting the value of μ in Eq. (5), we get

$$3(0) - 1 = -1 \Rightarrow 0 - 1 = -1$$

$-1 = -1$, which is true

Hence, both lines intersect each other.

The point of intersection of both lines can be obtained by putting $\lambda = 1$ in coordinates of P. So, the point of intersection is $(3 + 1, 1 - 1, -1)$, i.e. $(4, 0, -1)$.

83. Find the angle between the lines

$$\vec{r} = 2\hat{i} - 5\hat{j} + \hat{k} + \lambda(3\hat{i} + 2\hat{j} + 6\hat{k})$$

and $$\vec{r} = 7\hat{i} - 6\hat{j} - 6\hat{k} + \mu(\hat{i} + 2\hat{j} + 2\hat{k}).$$

Sol : Foreign 2014, OD 2008

If vector form of lines are $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \lambda\vec{b}_2$, then angle between them is $\cos\theta = \dfrac{|\vec{b}_1 \cdot \vec{b}_2|}{|\vec{b}_1||\vec{b}_2|}$.

We have $\vec{r} = (2\hat{i} - 5\hat{j} + \hat{k}) + \lambda(3\hat{i} + 2\hat{j} + 6\hat{k}) \qquad ...(1)$

and $\vec{r} = (7\hat{i} - 6\hat{j} - 6\hat{k}) + \mu(\hat{i} + 2\hat{j} + 2\hat{k}) \qquad ...(2)$

Comparing Eqs. (1) and (2) with vector form of equation of line, i.e. $\vec{r} = \vec{a} + \lambda\vec{b}$, we get

$$\vec{a}_1 = 2\hat{i} - 5\hat{j} + \hat{k},\ \vec{b}_1 = 3\hat{i} + 2\hat{j} + 6\hat{k}$$

and $$\vec{a}_2 = 7\hat{i} - 6\hat{j} - 6\hat{k},\ \vec{b}_2 = \hat{i} + 2\hat{j} + 2\hat{k}$$

Angle between two lines,

$$\begin{aligned}\cos\theta &= \frac{|\vec{b}_1 \cdot \vec{b}_2|}{|\vec{b}_1||\vec{b}_2|} \\ &= \left|\frac{(3\hat{i} + 2\hat{j} + 6\hat{k}) \cdot (\hat{i} + 2\hat{j} + 2\hat{k})}{\sqrt{(3)^2 + (2)^2 + (6)^2} \cdot \sqrt{(1)^2 + (2)^2 + (2)^2}}\right| \\ &= \left|\frac{3 + 4 + 12}{\sqrt{49} \times \sqrt{9}}\right|\end{aligned}$$

or $$\cos\theta = \left|\frac{19}{7 \times 3}\right| \Rightarrow \cos\theta = \frac{19}{21}$$

Hence, the angle between given two lines is $\theta = \cos^{-1}\left(\frac{19}{21}\right)$.

84. Show that the lines $\dfrac{x+1}{3} = \dfrac{y+3}{5} = \dfrac{z+5}{7}$ and $\dfrac{x-2}{1} = \dfrac{y-4}{3} = \dfrac{z-6}{5}$ intersect. Also, find their point intersection.

Sol : Delhi 2014

We have $\dfrac{x+1}{3} = \dfrac{y+3}{5} = \dfrac{z+5}{7} = \lambda$ (let) $\qquad ...(1)$

and $\dfrac{x-2}{1} = \dfrac{y-4}{3} = \dfrac{z-6}{5} = \mu$ (let) $\qquad ...(2)$

Then, any point on line (1) is of the form

$$P(3\lambda - 1, 5\lambda - 3, 7\lambda - 5) \qquad ...(3)$$

and any point on line (2) is of the form

$$Q(u + 2, 3\mu + 4, 5\mu + 6) \qquad ...(4)$$

If lines (1) and (2) intersect, then these points must coincide for some λ and μ.

Consider, $3\lambda - 1 = \mu + 2$

$$5\lambda - 3 = 3\mu + 4$$

and $$7\lambda - 5 = 5\mu + 6$$

$$3\lambda - \mu = 3 \qquad ...(5)$$

$$5\lambda - 3\mu = 7 \qquad ...(6)$$

and $$7\lambda - 5\mu = 11 \qquad ...(7)$$

Multiplying Eq. (5) by 3 and then subtracting Eq, (6), we get

$$9\lambda - 3\mu - 5\lambda + 3\mu = 9 - 7$$

$$4\lambda = 2 \Rightarrow \lambda = \frac{1}{2}$$

Substituting the value of λ in Eq. (5), we get

Sol : Foreign 2014, Delhi 2008

We have $\vec{r} = (\hat{i}+\hat{j}) + \lambda(2\hat{i}-\hat{j}+\hat{k})$ ...(1)

and $\vec{r} = (2\hat{i}+\hat{j}-\hat{k}) + \mu(3\hat{i}-5\hat{j}+2\hat{k})$ ...(2)

Comparing above equations with vector equation $\vec{r} = \vec{a} + \lambda\vec{b}$ we get

$$\vec{a}_1 = \hat{i}+\hat{j},\ \vec{b}_1 = 2\hat{i}-\hat{j}+\hat{k}$$

and $\vec{a}_2 = 2\hat{i}+\hat{j}-\hat{k},\ \vec{b}_2 = 3\hat{i}-5\hat{j}+2\hat{k}$

Now, $\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 3 & -5 & 2 \end{vmatrix}$

$$= \hat{i}(-2+5) - \hat{j}(4-3) + \hat{k}(-10+3)$$

$$\vec{b}_1 \times \vec{b}_2 = 3\hat{i}-\hat{j}-7\hat{k} \quad ...(4)$$

and $|\vec{b}_1 \times \vec{b}_2| = \sqrt{(3)^2+(-1)^2+(-7)^2}$

$$= \sqrt{9+1+49} = \sqrt{59} \quad ...(5)$$

Also, $\vec{a}_2 - \vec{a}_1 = (2\hat{i}+\hat{j}-\hat{k}) - (\hat{i}+\hat{j})$

$$= \hat{i}-\hat{k} \quad ...(6)$$

Shortest distance between two lines is given by

$$d = \left|\frac{(\vec{b}_1 \times \vec{b}_2)\cdot(\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|}\right| \quad ...(3)$$

$$= \left|\frac{(3\hat{i}-\hat{j}-7\hat{k})\cdot(\hat{i}-\hat{k})}{\sqrt{59}}\right|$$

$$= \left|\frac{3-0+7}{\sqrt{59}}\right| = \frac{10}{\sqrt{59}}$$

Hence, required shortest distance is $\frac{10}{\sqrt{59}}$ units.

88. Find the shortest distance between the tow lines whose vector equations are

$$\vec{r} = (\hat{i}+2\hat{j}+3\hat{k}) + \lambda(\hat{i}-3\hat{j}+2\hat{k}) \text{ and}$$

$$\vec{r} = (4\hat{i}+5\hat{j}+6\hat{k}) + \mu(2\hat{i}+3\hat{j}+\hat{k}).$$

Sol : Delhi 2014

We have $\vec{r} = (\hat{i}+2\hat{j}+3\hat{k}) + \lambda(\hat{i}-3\hat{j}+2\hat{k})$ (1)

and $\vec{r} = (4\hat{i}+5\hat{j}+6\hat{k}) + \mu(2\hat{i}+3\hat{j}+\hat{k}).$ (2)

Comparing above equations with vector equation $\vec{r} = \vec{a} + \lambda\vec{b}$ we get

$$\vec{a}_1 = \hat{i}+2\hat{j}+3\hat{k},\ \vec{b}_1 = \hat{i}-3\hat{j}+2\hat{k}$$

and $\vec{a}_2 = 4\hat{i}+5\hat{j}+6\hat{k},\ \vec{b}_2 = 2\hat{i}+3\hat{j}+\hat{k}$

Now, $\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 2 \\ 2 & 3 & 1 \end{vmatrix}$

$$= \hat{i}(-3-6) - \hat{j}(1-4) + \hat{k}(3+6)$$

$$= -9\hat{i}+3\hat{j}+9\hat{k}$$

$$\vec{b}_1 \times \vec{b}_2 = -9\hat{i}+3\hat{j}+9\hat{k} \quad ...(4)$$

and $|\vec{b}_1 \times \vec{b}_2| = \sqrt{(-9)^2+(3)^2+(9)^2}$

$$= \sqrt{171} \quad ...(5)$$

Also, $\vec{a}_2 - \vec{a}_1 = (4\hat{i}+5\hat{j}+6\hat{k}) - (\hat{i}+2\hat{j}+3\hat{k})$

$$= 3\hat{i}+3\hat{j}+3\hat{k} \quad ...(6)$$

Shortest distance between two lines is given by

$$d = \left|\frac{(\vec{b}_1 \times \vec{b}_2)\cdot(\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|}\right| \quad ...(3)$$

$$= \left|\frac{(-9\hat{i}+3\hat{j}+9\hat{k})\cdot(3\hat{i}+3\hat{j}+3\hat{k})}{\sqrt{171}}\right|$$

$$= \left|\frac{-27+9+27}{\sqrt{171}}\right| = \frac{9}{\sqrt{171}}$$

Hence, required shortest distance is $\frac{9}{\sqrt{171}}$ units.

89. Find the shortest distance between the following lines. $\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1},\ \frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}.$

Sol : Foreign 2014; Delhi 2008

We have $\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$ ...(1)

and $\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$ ...(2)

Comparing above equations with one point form of equation of line, i.e.

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c},$$

we get $a_1 = 1,\ b_1 = -2,\ c_1 = 1,\ x_1 = 3,\ y_1 = 5,\ z_1 = 7$

and $a_2 = 7,\ b_2 = -6,\ c_2 = 1,\ x_2 = -1,\ y_2 = -1,\ z_2 = -1$

Shortest distance between two lines is given by

$$d = \frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2+(a_1b_2-a_2b_1)^2}}$$

$$d = \frac{\begin{vmatrix} -1-3 & -1-5 & -1-7 \\ 1 & -2 & 1 \\ 7 & -6 & 1 \end{vmatrix}}{\sqrt{(-2-6)^2+(7-1)^2+(-6+14)^2}}$$

$$= \frac{\begin{vmatrix} -4 & -6 & -8 \\ 1 & -2 & 1 \\ 7 & -6 & 1 \end{vmatrix}}{\sqrt{(4)^2+(6)^2+(8)^2}}$$

$$\frac{x-\frac{1}{6}}{1} = \frac{y+\frac{1}{3}}{2} = \frac{z-1}{3}$$

Here, DR's of the line are (1,2,3).

Now $\sqrt{(1)^2+(2)^2+(3)^2} = \sqrt{14}$

Direction cosines of the line are $\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}$

The equation of a line passing through $(2,-1,-1)$ and parallel to the given line is

$$\frac{x-2}{1} = \frac{y+1}{2} = \frac{z+1}{3} = \lambda \text{ (say)}$$

$$x = 2+\lambda,\ y = -1+2\lambda \text{ and } z = -1+3\lambda$$

Now,

$$x\hat{i}+y\hat{j}+z\hat{k} = (2+\lambda)\,\hat{i}+(-1+2\lambda)\,\hat{j}+(-1+3\lambda)\,\hat{k}$$

$$\vec{r} = (2\hat{i}-\hat{j}-\hat{k})+\lambda(\hat{i}+2\hat{j}+3\hat{k})$$

which is the required equation of line in vector form.

93. Find the shortest distance between the two lines whose vector equations are

$$\vec{r} = (6\hat{i}+2\hat{j}+2\hat{k})+\lambda(\hat{i}-2\hat{j}+2\hat{k})$$

and $\vec{r} = (-4\hat{i}-\hat{k})+\mu(3\hat{i}-2\hat{j}-2\hat{k})$.

Sol : Comp 2013, Foreign 2011

We have $\vec{r} = (6\hat{i}+2\hat{j}+2\hat{k})+\lambda(\hat{i}-2\hat{j}+2\hat{k})$ (1)

and $\vec{r} = (-4\hat{i}-\hat{k})+\mu(3\hat{i}-2\hat{j}-2\hat{k})$ (2)

Comparing above equations with vector equation $\vec{r} = \vec{a}+\lambda\vec{b}$ we get

$$\vec{a}_1 = 6\hat{i}+2\hat{j}+2\hat{k},\ \vec{b}_1 = \hat{i}-2\hat{j}+2\hat{k}$$

and $\vec{a}_2 = -4\hat{i}-\hat{k},\ \vec{b}_2 = 3\hat{i}-2\hat{j}-2\hat{k}$

Now, $$\vec{b}_1\times\vec{b}_2 = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -2 & 2\\ 3 & -2 & -2\end{vmatrix}$$

$$= \hat{i}(4+4)-\hat{j}(-2-6)+\hat{k}(-2+6)$$

$$= 8\hat{i}+8\hat{j}+4\hat{k}$$

$$\vec{b}_1\times\vec{b}_2 = 8\hat{i}+8\hat{j}+4\hat{k} \quad ...(4)$$

and $$|\vec{b}_1\times\vec{b}_2| = \sqrt{(8)^2+(8)^2+(4)^2}$$

$$= \sqrt{144} = 12 \quad ...(5)$$

Also, $$\vec{a}_2-\vec{a}_1 = (-4\hat{i}-\hat{k})-(6\hat{i}+2\hat{j}+2\hat{k})$$

$$= -10\hat{i}-2\hat{j}-3\hat{k} \quad ...(6)$$

Shortest distance between two lines is given by

$$d = \left|\frac{(\vec{b}_1\times\vec{b}_2)\cdot(\vec{a}_2-\vec{a}_1)}{|\vec{b}_1\times\vec{b}_2|}\right| \quad ...(3)$$

$$= \left|\frac{(8\hat{i}+8\hat{j}+4\hat{k})\cdot(-10\hat{i}-2\hat{j}-3\hat{k})}{12}\right|$$

$$= \left|\frac{-80-16-12}{12}\right| = \frac{108}{12} = 9$$

Hence, required shortest distance is 9 units.

94. Show that the lines

$$\vec{r} = 3\hat{i}+2\hat{j}-4\hat{k}+\lambda(\hat{i}+2\hat{j}+2\hat{k});$$

$$\vec{r} = 5\hat{i}-2\hat{j}+\mu(3\hat{i}+2\hat{j}+6\hat{k})$$

are intersecting. Hence, find their point of intersection.

Sol : OD 2013

We have $$\vec{r} = 3\hat{i}+2\hat{j}-4\hat{k}+\lambda(\hat{i}+2\hat{j}+2\hat{k});$$

$$\vec{r} = 5\hat{i}-2\hat{j}+\mu(3\hat{i}+2\hat{j}+6\hat{k})$$

Above lines can be rewritten as

$$\vec{r} = (3+\lambda)\,\hat{i}+(2+2\lambda)\,\hat{j}-2(2-\lambda)\,\hat{k} \quad (1)$$

and $$\vec{r} = (5+3\mu)\,\hat{i}+(-2+2\mu)\,\hat{j}+(6\mu)\,\hat{k} \quad (2)$$

Clearly, any point on line (1) is of the form $P(3+\lambda,\ 2+2\lambda,\ -4+2\lambda)$ and any point on line (2) is of the form $Q(5+3\mu,\ -2+2\mu,\ 6\mu)$

If line (1) and (2) intersect, then these points must coincide for some λ and μ.

Thus $$3+\lambda = 5+3\mu \quad (3)$$

$$2+2\lambda = -2+2\mu \quad ...(4)$$

$$-4+2\lambda = 6\mu \quad ...(5)$$

Subtracting eq (4) from (5) we get

$$-4-2 = 6\mu+2-2\mu$$

$$-6 = 4\mu+2 \Rightarrow \mu = -2$$

Substituting $\mu = -2$ in eq (5) we have

$$-4+2\lambda = 6\times(-2) = -12$$

$$2\lambda = -8 \Rightarrow \lambda = -4$$

Substituting $\lambda = -4$ and $\mu = -2$ in eq (3) we have

$$3-4 = 5+3\times(-2)$$

$$-1 = -1, \text{ which is true}$$

Hence, both lines intersect each other.

The point of intersection of both lines can be obtained by putting $\lambda = -4$ in coordinates of P. So, the point of intersection is $(3-4, 2-8, -4-8)$, i.e. $(-1, -6, -112)$.

95. Using vectors, show that the points $A(-2,3,5)$, $B(7,0,-1)$, $C(-3,-2,-5)$ and $D(3,4,7)$ are such that AB and CD intersect at the point $P(1,2,3)$.

Sol : Comp 2012

The vector equation of line AB is

In Eqs. (4) and (5), by cross-multiplication, we get

$$\frac{a}{10-6} = \frac{b}{-9-5} = \frac{c}{2+6}$$
$$\frac{a}{4} = \frac{b}{-14} = \frac{c}{8}$$
$$\frac{a}{2} = \frac{b}{-7} = \frac{c}{4} = \lambda \text{ (say)}$$
$$a = 2\lambda,\ b = -7\lambda \text{ and } c = 4\lambda$$

Substituting the values of a, b and c in Eq. (3), we get

$$\frac{x+1}{2\lambda} = \frac{y-3}{-7\lambda} = \frac{z+2}{4\lambda}$$
$$\frac{x+1}{2} = \frac{y-3}{-7} = \frac{z+2}{4} \qquad ...(6)$$

98. Find the angle between following pair of lines.

$$\frac{-x+2}{-2} = \frac{y-1}{7} = \frac{z+3}{-3} \text{ and } \frac{x+2}{-1} = \frac{2y-8}{4} = \frac{z-5}{4}$$

and check whether the lines are parallel or perpendicular.

Sol : Delhi 2011

We have $\dfrac{-x+2}{-2} = \dfrac{y-1}{7} = \dfrac{z+3}{-3}$

and $\dfrac{x+2}{-1} = \dfrac{2y-8}{4} = \dfrac{z-5}{4}$

Above equations can be written as

$$\frac{x-2}{2} = \frac{y-1}{7} = \frac{z+3}{-3} \qquad ...(1)$$

and
$$\frac{x+2}{-1} = \frac{y-4}{2} = \frac{z-5}{4} \qquad ...(2)$$

Comparing Eqs. (1) and (2) with one point form of equations of line

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}, \text{ we get}$$
$$a_1 = 2,\ b_1 = 7,\ c_1 = -3$$

and
$$a_2 = -1,\ b_2 = 2,\ c_2 = 4$$

Angle between two lines is given by

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\cdot\sqrt{a_2^2+b_2^2+c_2^2}}$$
$$= \frac{(2)(-1)+(7)(2)+(-3)(4)}{\left[\sqrt{(2)^2+(7)^2+(-3)^2}\cdot\sqrt{(-1)^2+(2)^2+(4)^2}\right]}$$
$$\cos\theta = \frac{-2+14-12}{\sqrt{62}\times\sqrt{21}} = \frac{0}{\sqrt{62}\times\sqrt{21}} = 0$$
$$\cos\theta = \cos\frac{\pi}{2}$$
$$\theta = \frac{\pi}{2}$$

Hence, the angle between them is $\frac{\pi}{2}$. Therefore, the given pair of lines are perpendicular to each other.

99. Find the shortest distance between lines whose vector equations are

$$\vec{r} = (1-t)\hat{i} + (t-2)\hat{j} + (3-2t)\hat{k}$$
$$\vec{r} = (s+1)\hat{i} + (2s-1)\hat{j} - (2s+1)\hat{k}.$$

Sol : OD 2011

We have $\vec{r} = (1-t)\hat{i} + (t-2)\hat{j} + (3-2t)\hat{k}$ ...(1)

and $\vec{r} = (s+1)\hat{i} + (2s-1)\hat{i} - (2s+1)\hat{k}$ ...(2)

Firstly, we convert both equations in the vector form as

$$\vec{r} = \vec{a} + \lambda\vec{b} \qquad ...(3)$$

So, Eq. (1) can be written as

$$\vec{r} = (\hat{i} - 2\hat{j} + 3\hat{k}) + t(-\hat{i} + \hat{j} - 2\hat{k}) \qquad ...(4)$$

and Eq. (2) can be written as

$$\vec{r} = (\hat{i} - \hat{j} - \hat{k}) + s(\hat{i} + 2\hat{j} - 2\hat{k}) \qquad ...(5)$$

From Eqs. (3), (4) and (5), we get

$$\vec{a}_1 = \hat{i} - 2\hat{j} + 3\hat{k},\ \vec{b}_1 = -\hat{i} + \hat{j} - 2\hat{k}$$
$$\vec{a}_2 = \hat{i} - \hat{j} - \hat{k},\ \vec{b}_2 = \hat{i} + 2\hat{j} - 2\hat{k}$$

Now,
$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & -2 \\ 1 & 2 & -2 \end{vmatrix}$$
$$= \hat{i}(-2+4) - \hat{j}(2+2) + \hat{k}(-2-1)$$
$$\vec{b}_1 \times \vec{b}_2 = 2\hat{i} - 4\hat{j} - 3\hat{k}$$
$$|\vec{b}_1 \times \vec{b}_2| = \sqrt{(2)^2 + (-4)^2 + (-3)^2}$$
$$= \sqrt{4+16+9} = \sqrt{29}$$

Also,
$$\vec{a}_2 - \vec{a}_1 = (\hat{i} - \hat{j} - \hat{k}) - (\hat{i} - 2\hat{j} + 3\hat{k})$$
$$= \hat{j} - 4\hat{k}$$

Shortest distance between the lines,

$$d = \left|\frac{(\vec{b}_1 \times \vec{b}_2)\cdot(\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|}\right|$$
$$= \left|\frac{(2\hat{i} - 4\hat{j} - 3\hat{k})\cdot(\hat{j} - 4\hat{k})}{\sqrt{29}}\right|$$
$$= \left|\frac{0-4+12}{\sqrt{29}}\right| = \frac{8}{\sqrt{29}}$$

Thus
$$d = \frac{8\sqrt{29}}{29} \text{ units}$$

100. Find shortest distance between the lines

$$\vec{r} = (\hat{i} + 2\hat{j} + \hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k}) \text{ and}$$
$$\vec{r} = (2\hat{i} - \hat{j} - \hat{k}) + \mu(2\hat{i} + \hat{j} + 2\hat{k}).$$

Since, $QP \perp AB$ we have

$$a_1a_2 + b_1b_2 + c_1c_2 = 0 \quad ...(1)$$

$$2(2\lambda) - 3(-3\lambda - 1) + 8(8\lambda - 10) = 0$$

$$4\lambda + 9\lambda + 3 + 64\lambda - 80 = 0$$

$$77\lambda - 77 = 0 \Rightarrow \lambda = 1$$

Thus foot of perpendicular is

$$P = (2+1,\ -3-1,\ 8-10)$$

$$= (3,\ -4,\ -2)$$

Now, equation of perpendicular QP, where $Q(1,\ 0,\ 0)$ and $P(3,\ -4,\ -2)$, is

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

i.e $$\frac{x-1}{3-1} = \frac{y-0}{-4-0} = \frac{z-0}{-2-0}$$

$$\frac{x-3}{2} = \frac{y}{-4} = \frac{z}{-2}$$

or $$\frac{x-3}{-1} = \frac{y}{2} = \frac{z}{2}$$

Now, length of perpendicular $QP =$ distance between points $Q(1,\ 0,\ 0)$ and $P(3,\ -4,\ -2)$

$$= \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$$

$$= \sqrt{(3-1)^2 + (-4-0)^2 + (-2-0)^2}$$

$$= \sqrt{4+16+4} = \sqrt{24}$$

Hence, length of perpendicular is $\sqrt{24}$.

103. Find the points on the line $\frac{x+2}{3} = \frac{y+1}{2} = \frac{z-3}{2}$ at a distance of 5 units from the point $P(1,3,3)$.

Sol : OD 2010

Given equation of line is

$$\frac{x+2}{3} = \frac{y+1}{2} = \frac{z-3}{2} = \lambda \text{ (say)}$$

$$x = 3\lambda - 2,\ y = 2\lambda - 1,\ z = 2\lambda + 3$$

So, we have a point on the line is

$$Q(3\lambda - 2, 2\lambda - 1, 2\lambda + 3) \quad ...(1)$$

Now, given that distance between two points $P(1,3,3)$ and $Q(3\lambda - 2, 2\lambda - 1, 2\lambda + 3)$ is 5 units, i.e. $PQ = 5$.

$$\sqrt{(3\lambda - 2 - 1)^2 + (2\lambda - 1 - 3)^2 + (2\lambda + 3 - 3)^2} = 5$$

$$\sqrt{(3\lambda - 3)^2 + (2\lambda - 4)^2 + (2\lambda)^2} = 5$$

Squaring both sides, we get

$$(3\lambda - 3)^2 + (2\lambda - 4)^2 + (2\lambda)^2 = 25$$

$$9\lambda^2 + 9 - 18\lambda + 4\lambda^2 + 16 - 16\lambda + 4\lambda^2 = 25$$

$$17\lambda^2 - 34\lambda = 0 \Rightarrow 17\lambda(\lambda - 2) = 0$$

Either $17\lambda = 0$ or $\lambda - 2 = 0$

Thus $\lambda = 0$ or 2. Substituting $\lambda = 0$ and $\lambda = 2$ in Eq. (1), we get the required point as $(-2, -1, 3)$ or $(4,3,7)$

104. Find the coordinates of the foot of perpendicular and the length of the perpendicular drawn from the point $P(5,4,2)$ to the line $\vec{r} = -\hat{i} + 3\hat{j} + \hat{k} + \lambda(2\hat{i} + 3\hat{j} - \hat{k})$. Also, find the image of P in this line.

Sol : OD 2012

We have $$\vec{r} = -\hat{i} + 3\hat{j} + \hat{k} + \lambda(2\hat{i} + 3\hat{j} - \hat{k})$$

$$= (2\lambda - 1)\hat{i} + (3\lambda + 3)\hat{j} + (-\lambda + 1)\hat{k}$$

So, any point on above line is the form

$$(2\lambda - 1,\ 3\lambda + 3,\ -\lambda + 1)$$

Let foot of the perpendicular drawn from point $P(5,4,2)$ to the given line be $T(2\lambda - 1,\ 3\lambda + 3,\ -\lambda + 1)$.

Now, DR's of line PT is

$$(2\lambda - 1 - 5,\ 3\lambda + 3 - 4,\ -\lambda + 1 - 2)$$

or $$(2\lambda - 6,\ 3\lambda - 1,\ -\lambda - 1).$$

Since, PT is perpendicular to given line, using $a_1a_2 + b_1b_2 + c_1c_2 = 0$ we have

$$2(2\lambda - 6) + 3(3\lambda - 1) - 1(-\lambda - 1) = 0$$

$$4\lambda - 12 + 9\lambda - 3 + \lambda + 1 = 0$$

$$14\lambda - 14 = 0 \Rightarrow \lambda = 1$$

Coordinate of foot of perpendicular is

$T(2-1,\ 3+3,\ -1+1)$ or $T(1,\ 6,\ 0)$.

Let $Q(x,y,z)$ be the image of a point P with respect to the given line So, point T is the mid-point of PQ.

Coordinates of $T =$ Coordinates of mid-point of PQ

$$(1,\ 6,\ 0) = \left(\frac{x+5}{2}, \frac{y+4}{2}, \frac{z+2}{2}\right)$$

Equating the corresponding co-ordinates, we get

$$x + 5 = 1 \times 2 = 2 \Rightarrow x = -3$$

$$y + 4 = 6 \times 2 = 12 \Rightarrow y = 8$$

$$z + 2 = 0 \times 2 = 0 \Rightarrow z = -2$$

Hence, coordinates of the foot of perpendicular is $T(1,\ 6,\ 0)$ and image of the point P is $Q(-3,\ 8,\ -2)$

Length of perpendicular i.e. distance between $P(5,4,2)$ and $T(1,\ 6,\ 0)$,

$$d = \sqrt{(5-1)^2 + (4-6)^2 + (2-0)^2}$$

$$= \sqrt{4^2 + 2^2 + 2^2}$$

$$= \sqrt{24} = 2\sqrt{6} \text{ units.}$$

Substituting $\lambda = 1$ in Eq. (2), we get

$$Q = (1, 2+1, 3+2) = (1, 3, 5)$$

Let image of a point P be $T(x, y, z)$. Then, Q will be the mid-point of PT.

By using mid-point formula,

$$Q = \text{mid-point of } P(1,6,3) \text{ and } T(x,y,z)$$

$$= \left(\frac{x+1}{2}, \frac{y+6}{2}, \frac{z+3}{2}\right)$$

But $Q = (1, 3, 5)$

$$\left(\frac{x+1}{2}, \frac{y+6}{2}, \frac{z+3}{2}\right) = (1, 3, 5)$$

Equating corresponding coordinates, we get

$$\frac{x+1}{2} = 1,\ \frac{y+6}{2} = 3,\ \frac{z+3}{2} = 5$$

$$x = 2-1,\ y = 6-6,\ z = 10-3$$

$$x = 1,\ y = 0,\ z = 7$$

Coordinates of $T = (x, y, z) = (1, 0, 7)$

Hence, coordinates of image of point $P(1, 6, 3)$ is $T(1, 0, 7)$.

Now, equation of line joining $P(1, 6, 3)$ and $T(1, 0, 7)$ is

$$\frac{x-1}{1-1} = \frac{y-6}{0-6} = \frac{z-3}{7-3}$$

$$\frac{x-1}{0} = \frac{y-6}{-6} = \frac{z-3}{4}$$

Also, length of segment PT

$$= \sqrt{(1-1)^2 + (6-0)^2 + (3-7)^2}$$

$$= \sqrt{0+36+16} = \sqrt{52} \text{ units}$$

107. Write the vector equations of following lines and hence find the distance between them.

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z+4}{6},$$

$$\frac{x-3}{4} = \frac{y-3}{6} = \frac{z+5}{12}$$

Sol : Foreign 2010, Comp 2009

Given equations of lines are

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z+4}{6}$$

and
$$\frac{x-3}{4} = \frac{y-3}{6} = \frac{z+5}{12}$$

Now, the vector equation of given lines are

$$\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k}) \quad ...(1)$$

and
$$\vec{r} = (3\hat{i} + 3\hat{j} - 5\hat{k}) + \mu(4\hat{i} + 6\hat{j} + 12\hat{k}) \quad ...(2)$$

Here, $\vec{a}_1 = \hat{i} + 2\hat{j} - 4\hat{k},\ \vec{b}_1 = 2\hat{i} + 3\hat{j} + 6\hat{k}$

and $\vec{a}_2 = 3\hat{i} + 3\hat{j} - 5\hat{k},\ \vec{b}_2 = 4\hat{i} + 6\hat{j} + 12\hat{k}$

Now, $\vec{a}_2 - \vec{a}_1 = (3\hat{i} + 3\hat{j} - 5\hat{k}) - (\hat{i} + 2\hat{j} - 4\hat{k})$

$$= 2\hat{i} + \hat{j} - \hat{k} \quad ...(3)$$

and
$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 6 \\ 4 & 6 & 12 \end{vmatrix}$$

$$= \hat{i}(36-36) - \hat{j}(24-24) + \hat{k}(12-12)$$

$$= 0\hat{i} - 0\hat{j} + 0\hat{k} = \vec{0}$$

$$\vec{b}_1 \times \vec{b}_2 = \vec{0},$$

i.e. vector $\vec{b}_1$ is parallel to $\vec{b}_2$.

Thus, two lines are parallel.

$$\vec{b} = (2\hat{i} + 3\hat{j} + 6\hat{k}) \quad ...(4)$$

Since, the two lines are parallel, we use the formula for shortest distance between two parallel lines

$$d = \left|\frac{\vec{b} \times (\vec{a}_2 - \vec{a}_1)}{|\vec{b}|}\right|$$

$$d = \left|\frac{(2\hat{i} + 3\hat{j} + 6\hat{k}) \times (2\hat{i} + \hat{j} - \hat{k})}{\sqrt{(2)^2 + (3)^2 + (6)^2}}\right| \quad ...(5)$$

[from Eqs. (3) and (4)]

Now, $(2\hat{i} + 3\hat{j} + 6\hat{k}) \times (2\hat{i} + \hat{j} - \hat{k})$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 6 \\ 2 & 1 & -1 \end{vmatrix}$$

$$= \hat{i}(-3-6) - \hat{j}(-2-12) + \hat{k}(2-6)$$

$$= -9\hat{i} + 14\hat{j} - 4\hat{k}$$

From Eqs. (5), we get

$$d = \left|\frac{-9\hat{i} + 14\hat{j} - 4\hat{k}}{\sqrt{49}}\right|$$

$$= \frac{\sqrt{(-9)^2 + (14)^2 + (-4)^2}}{7}$$

$$= \frac{\sqrt{81+196+16}}{7} \text{ units}$$

$$= \frac{\sqrt{293}}{7} \text{ units}$$

108. The points $A(4, 5, 10)$, $B(2, 3, 4)$ and $C(1, 2, -1)$ are three vertices of parallelogram $ABCD$. Find the vector equations of sides AB and BC and also find coordinates of point D.

Sol : Delhi 2009

The vector equation of a side of a parallelogram, when two points are given, is $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$. Also, diagonals of a parallelogram intersect each other at mid-point.

Given points are $A(4, 5, 10)$, $B(2, 3, 4)$ and $C(1, 2, -1)$

and the distance measured is in kilometers.

Based on the above information, answer the following questions.

(i) What is the path of the rocket? Which of the following points lie on the path of the rocket?

(ii) At what distance will the rocket be from the starting point (0, 0, 0) in seconds?

(iii) At certain instant of time, if the rocket is above sea level, where equation of surface of sea is given by $3x - y + 4z = 2$ and position of rocket at that instant of time is $(1, -2, 2)$, then find the image of position of rocket in the sea.

Sol :

(i) Eliminating t form the given equations, we get equations of path as,

$$\frac{x}{2} = \frac{y}{-4} = \frac{z}{4}$$

or
$$\frac{x}{1} = \frac{y}{-2} = \frac{z}{2}$$

Thus, the path of the rocket represent a straight line. Since, only $(1, -2, 2)$ satisfy the equation of path of rocket therefore $(1, -2, 2)$ lie on the path of rocket.

(ii) For $t = 10\,\text{sec}$, we have $x = 20,\ y = -40,\ z = 40$

Now, required distance

$$\begin{aligned}\sqrt{x^2+y^2+z^2} &= \sqrt{20^2+(-40)^2+(40)^2}\\ &= \sqrt{400+1600+1600}\\ &= \sqrt{3600}\\ &= 60\,\text{km}\end{aligned}$$

(iv) Let Q be the image of point $P(1, -2, 2)$ in the plane $3x - y + 4z = 2$. Then equation of PQ is

$$\frac{x-1}{3} = \frac{y+2}{-1} = \frac{z-2}{4}$$

Let the coordinate of Q be $(3r+1,\ -r-2, 4r+2)$.

Let R be the mid-point of PQ. Then, coordinate of R are $\left(\frac{3r+2}{2}, \frac{-r-4}{2}, \frac{4r+4}{2}\right)$ or $\left(\frac{3}{2}r+1, \frac{-r}{2}-2, 2r+2\right)$

Since, R lies on $\quad 3x - y + 4z = 2$

$$\begin{aligned}3\left(\frac{3}{2}r+1\right) - \left(\frac{-r}{2}-2\right) + 4(2r+2) &= 2\\ \frac{9r}{2}+3+\frac{r}{2}+2+8r+8 &= 2\\ 13r+13 &= 2\\ r &= -\frac{11}{13}\end{aligned}$$

Hence, the coordinate of Q are

$$\left(\frac{-133}{13}+1,\ \frac{11}{13}-2,\ \frac{-44}{13}+2\right) \text{ i.e.,} \left(\frac{-20}{13}, \frac{-15}{13}, \frac{-18}{13}\right)$$

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CHAPTER 12

LINEAR PROGRAMMING

OBJECTIVE QUESTIONS

1. The objective function $Z = ax + by$ of an LPP has maximum value 42 at (4, 6) and minimum value 19 at (3, 2). Which of the following is true ?

(a) $a = 9, b = 1$ (b) $a = 5, b = 2$

(c) $a = 3, b = 5$ (d) $a = 5, b = 3$

Sol : OD 2023

Objective function $Z = ax + by$ has maximum value 42 at (4, 6) and minimum value 19 at (3, 2).

We can use these two points to form a system of two equations in two unknowns a and b as follows :

$$42 = 4a + 6b$$

and $$19 = 3a + 2b$$

Solving above equations, we get $a = 3$ and $b = 5$.

Thus (c) is correct option.

2. The corner points of the feasible region of a linear programming problem are (0,4), (8,0) and $\left(\frac{20}{3}, \frac{4}{3}\right)$. If $Z = 30x + 24y$ is the objective function, then (maximum value of Z – minimum value of Z) is equal to :

(a) 40 (b) 96

(c) 120 (d) 144

Sol : OD 2023

We have $Z = 30x + 24y$

Corner Point	Corresponding value of Z
(0, 4)	96 (minimum)
(8, 0)	240 (maximum)
$\left(\frac{20}{3}, \frac{4}{3}\right)$	232

maximum value of Z – minimum value of

$$Z = 240 - 96 = 144$$

Thus (c) is correct option.

3. The graph of the inequality $2x + 3y > 6$ is

(a) half plane that contains the origin

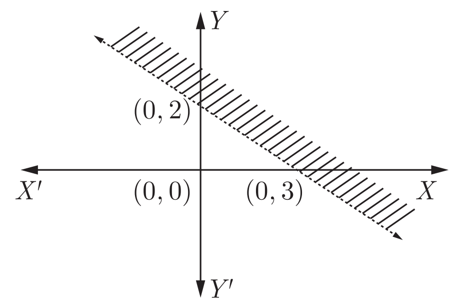
(b) half plane that neither contains the origin nor the points of the line $2x + 3y = 6$

(c) whole XOY –plane excluding the points on the line $2x + 3y = 6$

(d) entire XOY plane

Sol : OD 2020

The inequality $2x + 3y > 6$ represent half plane that neither contains the origin nor the points of the line $2x + 3y = 6$



Thus (b) is correct option.

4. The corner points of the feasible region determined by the system of linear constraints are $O(0, 0)$, $A(3, 0)$, $B(2, 3)$ and $C(0, 6)$. The objective function is $Z = 7x + 4y$.

Compare the quantity in Column A and Column B.

Column A	Column b
Maximum of Z	30

(a) The quantity in column A is greater

(b) The quantity in column B is greater.

(c) The two quantities are equal

(d) The relationship can not be determined on the basis of the information supplied

Sol : Delhi 2017

The values of objective function at corner points are given below

Corner points	$Z = 7x + 4y$
$O(0,0)$	$Z = 0 + 0 = 0$
$A(3,0)$	$Z = 7 \times 3 + 0 = 21$
$B(2,3)$	$Z = 7 \times 2 + 4 \times 3 = 26$
$C(0,6)$	$Z = 7 \times 0 + 4 \times 6 = 24$

At point $A(2,0)$, $p = 10+0 = 10$

At point $B(0,5)$, $p = 0+15 = 15$.

Hence, maximum value of P is 15.
Thus (c) is correct option.

9. Which of the term is not used in a linear programming problem ?

(a) Optimal solution

(b) Feasible solution

(c) Concave region

(d) Objective function

Sol : Foreign 2018

Concave region term is not used in a linear programming problem.
Thus (b) is correct option.

10. The optimal value of the objective function is attained at the point is

(a) given by intersection of inequations with axes only

(b) given by intersection of inequations with X-axis only

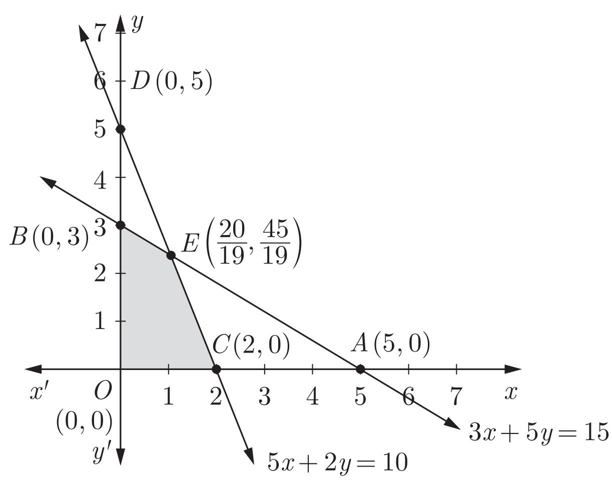
(c) given by corner points of the feasible region

(d) None of the above

Sol : SQP 2013, Comp 2007

(c) The optimal value of the objective function is attained at the points given by corner points of the feasible region.

11. For the feasible solution shown in the figure, the corner points of feasible region are



(a) (5, 0), (2, 0), ($\frac{20}{19}$, $\frac{45}{19}$), (0, 3)

(b) (0, 0), (2, 0), ($\frac{20}{19}$, $\frac{45}{19}$), (0, 3)

(c) (0, 5), (2, 0), ($\frac{20}{19}$, $\frac{45}{19}$), (0, 3)

(b) (0, 0), (2, 0), (0, 5), (5, 0), (0, 3)

Sol : OD 2008

Corner points are the corner of bounded region.
Thus (b) is correct option.

12. Which of the following statements is correct?

(a) Every linear programming problem admits an optimal solution

(b) A linear programming problem admits a unique solution

(c) If a linear programming problem admits two optimal solutions, then it has an infinite number of optimal solutions

(d) A linear programming problem admits two optimal solutions

Sol : OD 2013

If a linear programming problem admits two optimal solutions, then it has an infinite number of optimal solutions
Thus (c) is correct option.

13. Objective function of a linear programming problem is

(a) a constraint

(b) a function to be optimized

(c) a relation between the variables

(d) none of the above

Sol : Delhi 2015

Objective function of linear programming problem is a function to be optimized.
Thus (b) is correct option.

14. Solution set of the inequality $x \geq 0$ is

(a) half plane on the left of y-axis

(b) half plane on the right of y-axis excluding the points of y-axis

(c) half plane on the right of y-axis including the points on y-axis

(d) none of the above

Sol : Foreign 2008, OD 2007

Solution set of the given inequality is $\{(x,y)\colon x \geq 0\}$ ie, the set of all points whose abscissa are non-negative. All these points lie either on y-axis or on the right of y-axis.
Thus (c) is correct option.

Line $x+2y=100$			
x	0	100	Point $(0,0)$ is false for $x+2y \geq 100$. So, the region is away from the origin.
y	50	0	

(ii) Region corresponding to $2x-y \leq 0$:

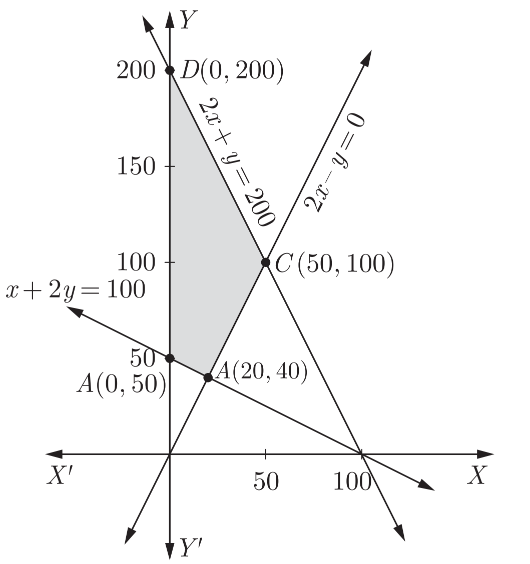
Line $2x-y=0$			
x	0	0	Point $(0,5)$ is true for $2x-y \leq 0$. So, the region is towards the y axis.
y	0	0	

(iii) Region corresponding to $2x+y \leq 200$:

Line $2x+y=200$			
x	0	100	Point $(0,0)$ is true for $2x+y \leq 200$. So, the region is towards the origin.
y	200	0	

Since $x \geq 0$, $y \geq 0$ the feasible region lies in the first quadrant.

Now we draw all line on the graph and find the common area. Clearly, feasible region is $ABCDA$. Solving equations $2x-y=0$ and $2x+y=200$, we get $C(50, 100)$. The corner points of the feasible region are $A(0, 50)$, $B(20, 40)$, $C(50,100)$ and $D(0, 200)$.



Since, the feasible region is a bounded region, we can check the profit function at all the vertices to find the maxima. The values of Z at these points are as follows

Corner point	$Z=x+2y$
$A(0, 50)$	$Z=0+2\times 50=100$
$B(20, 40)$	$Z=20+2\times 40=100$
$C(50,100)$	$Z=50+2\times 200=250$
$D(0, 200)$	$Z=0+2\times 200=400$

Thus the minimum value of Z is 400 at the point $D(0, 200)$.

17. Maximize $Z=3x+4y$, subject to the constraints; $x+y \leq 4$, $x \geq 0$, $y \geq 0$.

Sol : Foreign 2018

We have the following LPP,

Maximize $Z=3x+4y$

Subject to $x+y \leq 4$ (i)

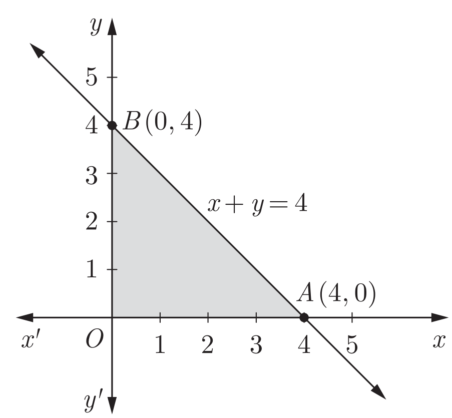
and $x \geq 0$, $y \geq 0$

Region corresponding to $x+y \leq 4$:

Line $x+y=4$			
x	0	4	Point $(0,0)$ is true for $x+y \leq 4$. So, the region is towards the origin.
y	4	0	

Since $x \geq 0$, $y \geq 0$ the feasible region lies in the first quadrant.

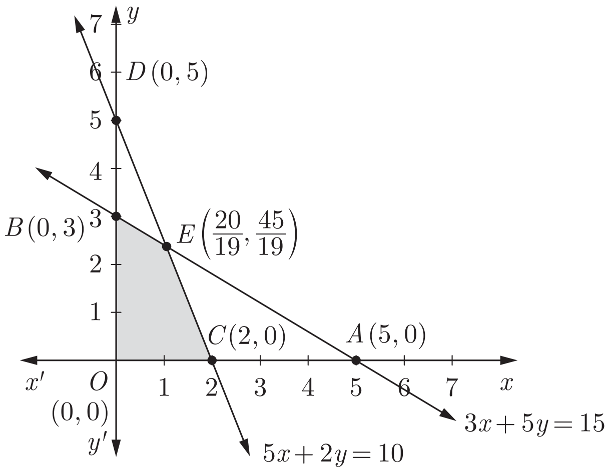
Now we draw all line on the graph and find the common area. Clearly, feasible region is $OABO$ which is bounded The corner points of the feasible region are $O(0,0)$, $A(4,0)$ and $B(0,4)$.



Since, the feasible region is a bounded region, we can check the objective function at all the corner to find the maxima. The values of objective function Z at these points are as follows.

Corner points	$Z=3x+4y$
$O(0,0)$	$Z=0+0=0$
$A(4,0)$	$Z=3\times 4+4\times 0=12$
$B(0,4)$	$Z=3\times 0+4\times 4=16$

The maximum value of Z is 16 at $B(0,4)$ and the optimal solution is $x=0$, $y=4$.



Since, the feasible region is a bounded region, we can check the objective function at all the corner to find the maxima. The values of objective function Z at these points are as follows.

Corner	$Z = 5x + 3y$
$O(0,0)$	$Z = 0 + 0 = 0$
$B(0,3)$	$Z = 5\times 0 + 3\times 3 = 9$
$E(\frac{20}{19},\frac{45}{19})$	$Z = 5\times\frac{20}{19} + 3\times\frac{45}{19} = 12.37$
$C(2,0)$	$= 5\times 2 + 3\times 0 = 10$

Hence the maximum value of Z is 12.37 is at $E(\frac{20}{19},\frac{45}{19})$ and the optimal solution is $x = \frac{20}{19}$ and $y = \frac{45}{19}$.

20. Minimize $Z = 3x + 5y$ such that $x + 3y \geq 3$, $x + y \geq 2$, x, $y \geq 0$.

Sol : SQP 2012

We have the following LPP,

Minimize $Z = 3x + 5y$

Subject to $x + 3y \geq 3$ (i)

$x + y \geq 2$ (ii)

and $x \geq 0,\ y \geq 0$

(i) Region corresponding to $x + 3y \geqslant 3$:

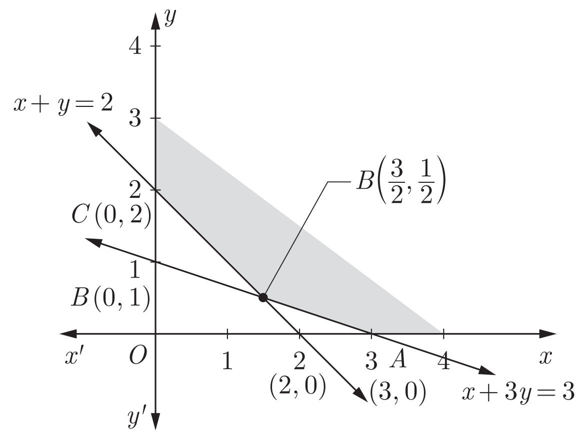
Line $x+3y=3$			
x	0	3	Point (0,0) is false for $x + 3y \geqslant 3$. So, the region is away from the origin.
y	1	0	

(ii) Region corresponding to $x + y \geqslant 2$:

Line $x+y=2$			
x	0	2	Point (0,0) is false for $x + y \geqslant 2$. So, the region is away from the origin.
y	2	0	

Since $x \geq 0$, $y \geq 0$ the feasible region lies in the first quadrant.

Now we draw all line on the graph and find the common area. Clearly, feasible region is ABC which is open and unbounded region. Solving equations $x + 3y = 3$ and $x + y = 2$, we get $B(\frac{3}{2},\frac{1}{2})$. The corner points of the feasible region are $A(3,0)$, $B(\frac{3}{2},\frac{1}{2})$ and $C(0,2)$.



The values of objective function Z at these points are as follows.

Corner points	$Z = 3x + 5y$
$A(3,0)$	$Z = 3\times 3 + 0 = 9$
$B(\frac{3}{2},\frac{1}{2})$	$Z = 3\times\frac{3}{2} + 5\times\frac{1}{2} = 7$
$C(0,2)$	$Z = 3\times 0 + 5\times 2 = 10$

Hence the minimum value of Z is 7 at $B(\frac{3}{2},\frac{1}{2})$.

21. Maximize $Z = 3x + 2y$ subject to $x + 2y \leq 0$, $3x + y \leq 15$, x, $y \geq 0$.

Sol : Delhi 2017, OD 2011

We have the following LPP,

Maximize $Z = 3x + 2y$

Subject to $x + 2y \leq 10$ (i)

$3x + y \leq 15$ (ii)

and $x \geq 0,\ y \geq 0$

(i) Region corresponding to $x + 2y \leq 10$:

Line $x+2y=10$			
x	0	10	Point (0,0) is true for $x + 2y \leq 10$. So, the region is towards the origin.
y	5	0	

(ii) Region corresponding to $3x + y \leq 15$:

$$x + 2y \geq 6 \quad \text{(ii)}$$

and $\quad x \geq 3,\ y \geq 0$

(i) Region corresponding to $x + y \geq 5$:

Line $x+y=5$			
x	0	5	Point (0,0) is false for $x+y \geq 5$. So, the region is away from the origin.
y	5	0	

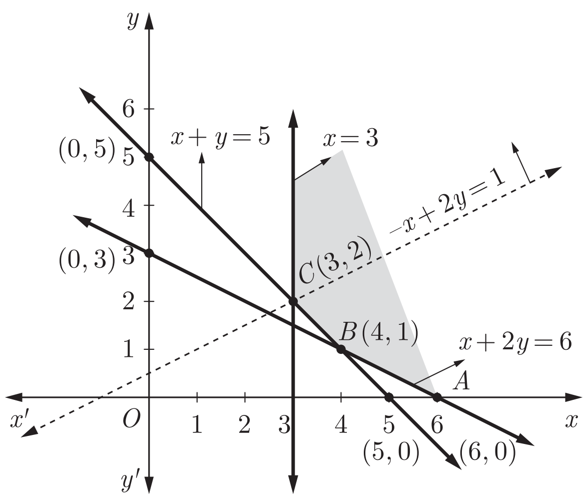
(ii) Region corresponding to $x + 2y \geq 6$:

Line $x+2y=6$			
x	0	3	Point (0,0) is false for $x+2y \geq 6$. So, the region is away from the origin.
y	6	0	

(iii) Region corresponding to $x \geq 3$ is away from the origin.

Since $x \geq 3$, $y \geq 0$ the feasible region lies in the first quadrant and away from the origin and other side of the line $x = 3$.

Now we draw all line on the graph and find the common area. Clearly, feasible region is CBA. The corner points of the feasible region are $A(6,0)$, $B(4,1)$ and $C(3,2)$.



The values of objective function Z at these points are as follows.

Corner points	$Z = -x + 2y$
$A(6,0)$	$Z = -6 + 2 \times 0 = -6$
$B(4,1)$	$Z = -4 + 2 \times 1 = -2$
$C(3,2)$	$Z = -3 + 2 \times 2 = 1$

Since the feasible region is unbounded, therefore, $Z = 1$ may or may not be the maximum value. For this, we graph the inequality $-x + 2y > 1$ and check weather the resulting region has points in common with the feasible region or not.

The resulting feasible region has points in common with the feasible region. Hence, $Z = 1$ is not the maximum value or Z has no maximum value.

24. Maximize $Z = x + y$, subject to $x - y \leq -1$, $x + y \leq 0$, x, $y \geq 0$.

Sol : Comp 2015, Delhi 2009

We have the following LPP,

Maximize $\quad Z = x + y$

Subject to $\quad x - y \leq -1 \quad$ (i)

$$x + y \leq 0$$

and $\quad x \geq 0,\ y \geq 0$

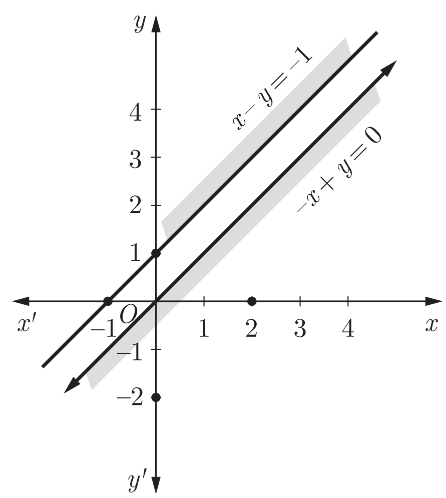
(i) Region corresponding to $x - y \leq -1$:

Line $x-y=-1$			
x	0	-1	Point (0,0) is false for $x-y \leq -1$. So, the region is away from the origin.
y	1	0	

(ii) Region corresponding to $x + y \leq 0$:

Line $x+y=0$			
x	0	2	Point $(0,-2)$ is true for $x+y \leq 0$. So, the region includes point $(0,-2)$.
y	0	2	

Since $x \geq 0$, $y \geq 0$ the feasible region lies in the first quadrant.



Now we draw all line on the graph and find the common area. Here we get no feasible region (constraints are inconsistent). Hence, Z has no maximum value.

25. Solve the following LPP graphically :

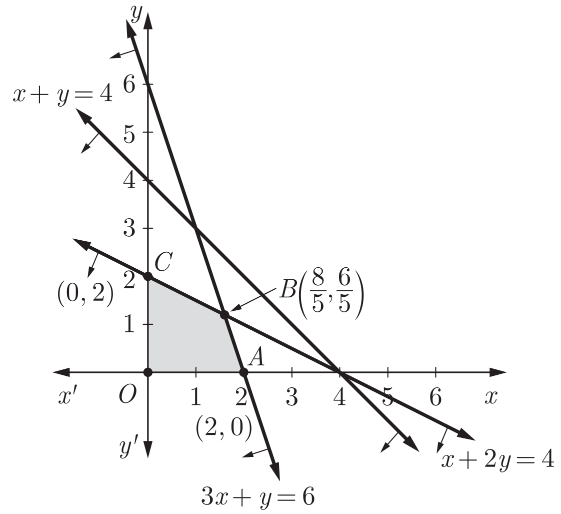
Minimise $\quad Z = 5x + 10y$

Subject to the constraints

$$x + 2y \leq 120$$

$$x + y \geq 60,$$

the feasible region of the given LPP. The point of intersection of the lines $x+2y=4$ and $3x+y=6$ is $B(\frac{8}{5},\frac{6}{5})$. The corner points are $O(0,\ 0)$, $A(2,\ 0)$ $B(\frac{8}{5},\frac{6}{5})$ and $C(0,2)$.



Since, the feasible region is a bounded region, we can check the objective function at all the vertices to find the maxima. The values of P at corner points are given below.

Corner Points	$Z=2x+5y$
$O(0,\ 0)$	$Z=0+0=0$
$A(2,\ 0)$	$Z=2\times 2+5\times 0=4$
$B(\frac{8}{5},\frac{6}{5})$	$Z=2\times\frac{8}{5}+5\times\frac{6}{5}=\frac{46}{5}=9.2$
$C(0,2)$	$Z=2\times 0+5\times 2=10$

Hence, the maximum value of Z is 10.

27. Maximise and minimise $Z=x+2y$ subject to the constraints

$$x+2y\geq 100$$
$$2x-y\leq 0$$
$$2x+y\leq 200$$
$$x,\ y\geq 0$$

Solve the above LPP graphically.

Sol : OD 2014, Delhi 2011

We have to minimise and maximise

$$Z=x+2y$$

Subject to constraints,

$$x+2y\geq 100 \quad ...(i)$$
$$2x-y\leq 0 \quad ...(ii)$$
$$2x+y\leq 200 \quad ...(iii)$$

and $x\geq 0,\ y\geq 0$

(i) Region corresponding to $x+2y\geq 100$:

Line $x+2y=100$			
x	0	100	Point $(0,0)$ is false for $x+2y\geq 100$. So, the region is away from the origin.
y	50	0	

(ii) Region corresponding to $2x-y\leq 0$:

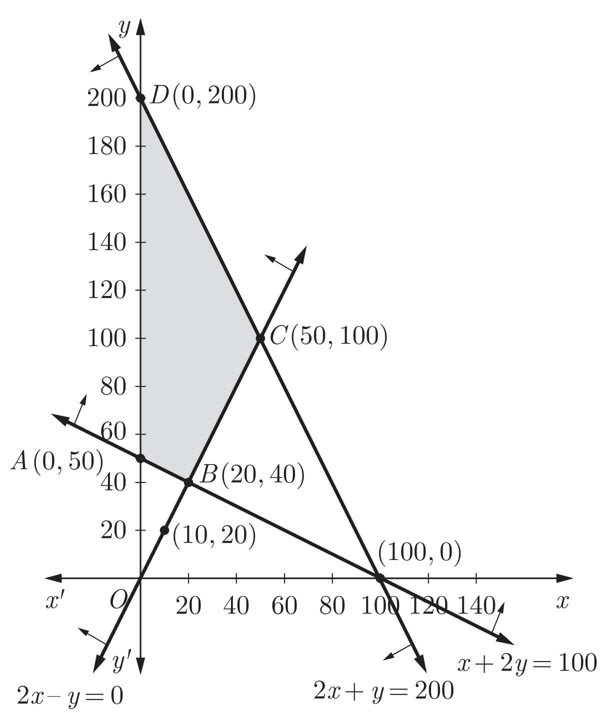
Line $2x-y=0$			
x	0	10	Point $(0,5)$ is true for $2x-y\leq 0$, thus region includes $(0,5)$ and towards y axis.
y	0	20	

(iii) Region corresponding to $2x+y\leq 200$:

Line $2x+y=200$			
x	0	100	Point $(0,0)$ is true for $2x+y\leq 200$. So, the region is towards the origin.
y	200	0	

Since $x\geq 0,\ y\geq 0$ the feasible region lies in the first quadrant.

Now we draw all line on the graph and find the common area. Clearly, feasible region is $ABCDA$. Solving equations $2x-y=0$ and $x+2y=100$, we get $B(20,40)$. Again, solving equations $2x-y=0$ and $2x+y=200$, we get $C(50,100)$. The corner points of the feasible region are $A(0,50)$, $B(20,40)$, $C(50,100)$ and $D(0,200)$.



Clearly $x \geq 0$ and $y \geq 0$

(i) Region corresponding to $2x + y \leq 40$:

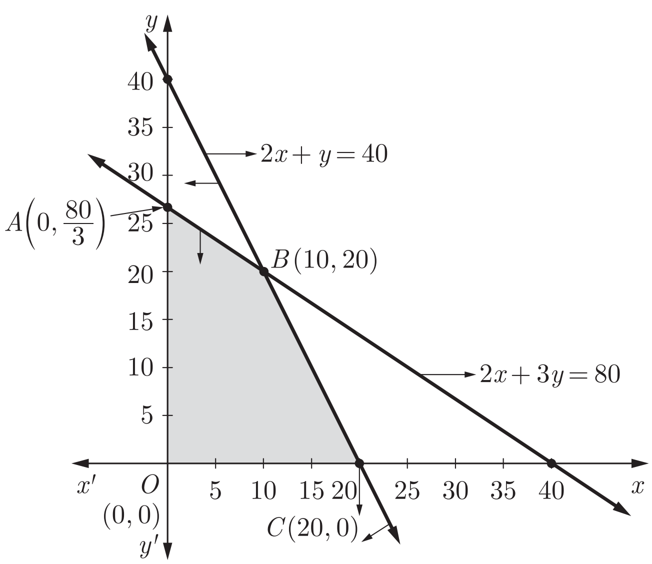
Line $2x + y = 40$			
x	0	20	Point $(0,0)$ is true for $2x + y \leq 40$. So, the region is towards the origin.
y	40	0	

(ii) Region corresponding to $2x + 3y \leq 80$:

Line $2x + 3y = 80$			
x	0	40	Point $(0,0)$ is true for $2x + 3y \leq 80$, So, the region is towards the origin.
y	$\frac{80}{3}$	0	

Since $x \geq 0$, $y \geq 0$ the feasible region lies in the first quadrant.

Now we draw all line on the graph and find the common area. Clearly, feasible region is $OABCO$. Solving equation $2x + y = 40$ and $2x + 3y = 80$ we get $B(10,20)$. The corner points of the feasible region are $A(0,\frac{80}{3})$, $B(10,20)$, $C(20,0)$, $O(0,0)$.



Since, the feasible region is a bounded region, we can check the objective function at all the vertices to find the maxima. The values of Z at corner points are given below

Corner points	$Z = x + 2y$
$A(0,\frac{80}{3})$	$Z = 15(0) + 10(\frac{80}{3}) = \frac{800}{3}$
$B(10,20)$	$Z = 15(10) + 10(20) = 350$
$C(20,0)$	$Z = 15(20) + 10(0) = 300$
$O(0,0)$	$Z = 15(0) + 10(0) = 0$

Thus, the maximum value is 350 is at point $B(10,20)$.

30. Maximise $P = 40x + 50y$ subject to the constraints

$$3x + y \leq 9$$

$$x + 2y \leq 8$$

and $x \geq 0,\ y \geq 0$

Sol : Delhi 2016, Comp 2012

We have the following LPP,

Maximize $P = 40x + 50y$

$$3x + y \leq 9 \qquad \text{(i)}$$

$$x + 2y \leq 8 \qquad \text{(ii)}$$

and $x \geq 0,\ y \geq 0$

(i) Region corresponding to $3x + y \leq 9$:

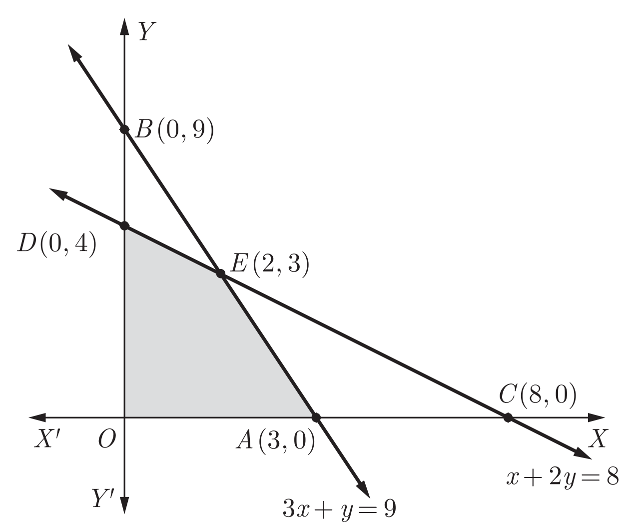
Line $3x + y = 9$			
x	0	3	Point $(0,0)$ is true for $3x + y \leq 9$. So, the region is towards the origin.
y	9	0	

(ii) Region corresponding to $x + 2y \leq 8$:

Line $x + 2y = 8$			
x	8	0	Point $(0,0)$ is true for $x + 2y \leq 8$, thus region includes $(0,0)$ and towards the origin.
y	0	4	

Since $x \geq 0$, $y \geq 0$ the feasible region lies in the first quadrant.

Now we draw all line on the graph and find the common area. Clearly, feasible region is $ODEAO$. Solving equation $3x + y = 9$ and $x + 2y = 8$ we get $E(2,3)$. The coordinates of the corner points of this region are $O(0,0)$, $A(3,0)$, $E(2,3)$ and $D(0,4)$.

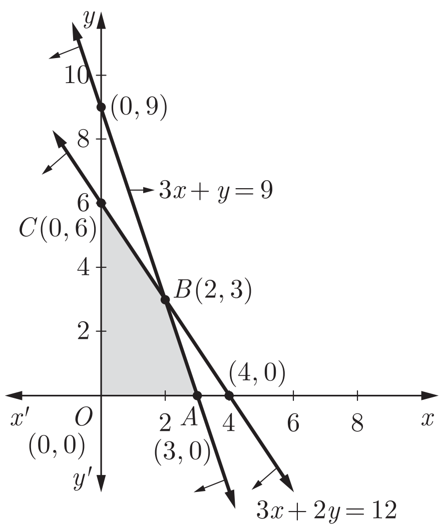


Since, the feasible region is a bounded region, we can check the objective function at all the vertices to find

Line $3x+y=9$			
x	0	3	Point $(0,0)$ is true for $3x+y\le 9$, thus region is towards the origin.
y	9	0	

Since $x\ge 0$, $y\ge 0$ the feasible region lies in the first quadrant.

Now we draw all line on the graph and find the common area. Clearly, feasible region is $OCBAO$. Solving equation $3x+2y=12$ and $3x+y=9$ we get $B(2,3)$. The coordinates of the corner points of this region are $O(0,0)$, $A(3,0)$, $B(2,3)$ and $C(0,6)$.



Since, the feasible region is a bounded region, we can check the objective function at all the vertices to find the maxima. The values of P at corner points are given below

Corner points	$Z=7x+4y$
$O(0,0)$	$Z=0+0=0$
$A(3,0)$	$Z=7\times 3+0=21$
$B(2,3)$	$Z=7\times 2+4\times 3=26$
$C(0,6)$	$Z=7\times 0+4\times 6=24$

The maximum value of P is 26 at $B(2,\ 3)$.

33. Maximise $R=0.1x+0.09y$ subject to the constraints

$$x+y=50000$$
$$x-y\ge 0$$
$$x\ge 20000$$
and
$$y\ge 10000$$

Sol : SQP 2020

We have the following LPP,

Maximize $R=0.1x+0.09y$

subject to $x+y=50000$ ...(i)

$x-y\ge 0$ ...(ii)

$x\ge 20000$ ...(iii)

and $y\ge 10000$ ...(iv)

(i) Region corresponding to $x+y=50000$:

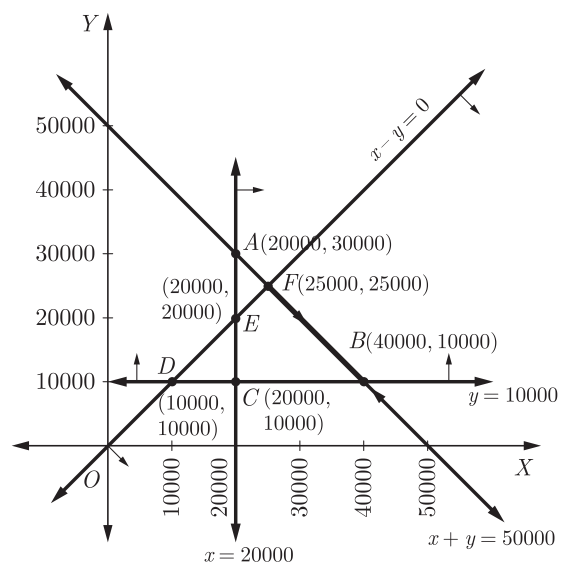
Line $x+y=50000$			
x	0	50000	This is not in equation but equation of line and solution lie on line $x+y=50000$ only
y	50000	0	

(ii) Region corresponding to $x-y\ge 0$:

Line $x-y=0$			
x	0	50000	Point $(50000,0)$ is true for $x-y\ge 0$, thus region include $(50000,0)$ and towards x axis.
y	0	50000	

Since $x\ge 20000$, $y\ge 10000$ the feasible region lies in the first quadrant.

Now we draw all line on the graph. Solving equation $x+y=50000$ and $x-y=0$ we get $F(25000,25000)$. From the graph, it is clear that feasible region lies on the line segment BF. The corner points of feasible region are $B(40000,10000)$ and $F(25000,25000)$.



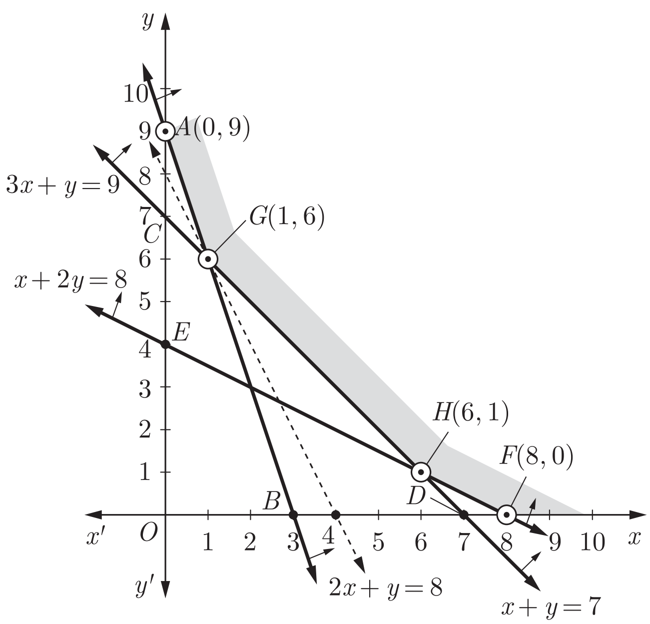
The values of R at corner points are given below

Corner Points	$R=0.1x+0.09y$
$B(40000,10000)$	$R=0.1\times 40000+0.09\times 10000$ $=4900$
$F(25000,25000)$	$R=0.1\times 25000+0.09\times 25000$ $=4750$

Here the maximum value of R is 4900 at $B(40000,\ 10000)$.

quadrant.

Now we draw all line on the graph and find the common area. The shaded region $AGHF$ represents the feasible region of the given LPP. The point of intersection of the lines $3x+y=9$ and $x+y=7$ is $G(1,6)$. Also, the point of intersection of the lines $x+y=7$ and $x+2y=8$ is $H(6,1)$. The corner points of the feasible region are $A(0,9)$, $G(1,6)$, $H(6,1)$ and $F(8,0)$.



The values of the objective function Z at these points are given in the following table :

Corner Points	$Z=2x+y$
$A(0,9)$	$Z=2\times 0+9=9$
$G(1,6)$	$Z=2\times 1+6=8$
$H(6,1)$	$Z=2\times 6+1=13$
$F(8,0)$	$Z=2\times 8+0=16$

From the table, we find that minimum value of Z is 8 at point $G(1,6)$. Since, the region is unbounded, therefore, 8 may or may not be the minimum value of Z. For this we have to check, that the open region $2x+y<8$ has any point common or not with the feasible region.

Since, it has no point in common with the feasible region. So, Z is minimum at $G(1,6)$ and the minimum value of Z is 8.

36. Maximise $Z=24x+18y$ subject to the constraints

$$2x+3y \le 10$$

$$3x+2y \le 10$$

and $\quad x \ge 0,\ y \ge 0$

Sol : OD 2015, SQP 2013

We have the following LPP,

Maximize $\quad Z = 24x+18y$

Subject to constraints

$$2x+3y \le 10$$

$$3x+2y \le 10$$

Also $\quad x \ge 0,\ y \ge 0$

(i) Region corresponding to $2x+3y \le 10$:

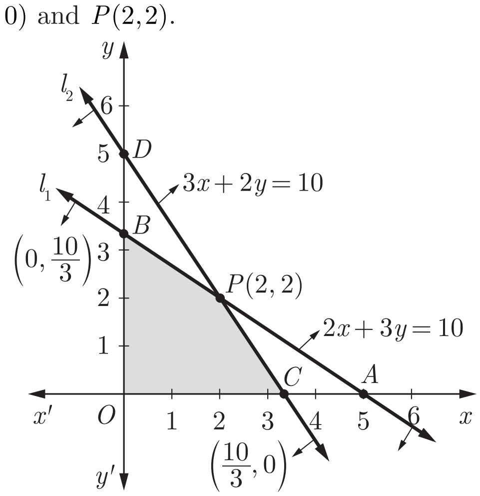
Line $2x+3y=10$			
x	0	5	Point $(0,0)$ is true for $2x+3y \le 10$ So, the region is towards the origin.
y	$\frac{10}{3}$	0	

(ii) Region corresponding to $3x+2y \le 10$:

Line $3x+2y=10$			
x	0	$\frac{10}{3}$	Point $(0,0)$ is true for $3x+2y \le 10$ So, the region is towards the origin.
y	5	0	

Since $x \ge 0,\ y \ge 0$ the feasible region lies in the first quadrant.

Now we draw all line on the graph and find the common area. The shaded region $OBPCO$ represents the feasible region of the given LPP. The point of intersection of the lines $3x+2y=10$ and $2x+3y=10$ is $P(2,2)$. The corner points are $O(0,\ 0)$, $B(0,\ \frac{10}{3})$, $C(\frac{10}{3},\ 0)$ and $P(2,2)$.



Since, the feasible region is a bounded region, we can check the objective function at all the vertices to find

38. Maximise $Z = x + y$ subject to the constraints

$$2x + y \le 50$$

$$x + 2y \le 40$$

and $x \ge 0,\ y \ge 0$

Sol : OD 2008

We have the following LPP,

Maximize $Z = x + y$

Subject to $2x + y \le 50$ ...(i)

$x + 2y \le 40$ ...(ii)

and $x \ge 0,\ y \ge 0$

(i) Region corresponding to $2x + y \le 50$:

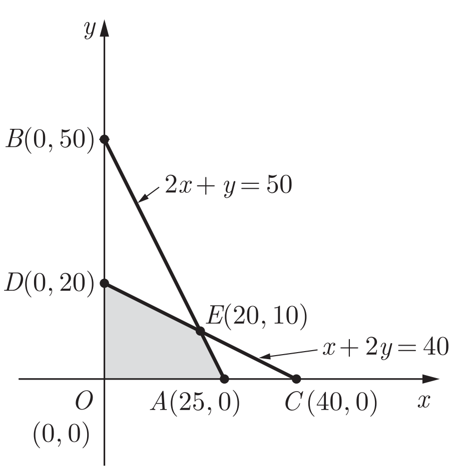
Line $2x + y = 50$			
x	0	25	Point $(0,0)$ is true for $2x + y \le 50$ So, the region is towards the origin.
y	50	0	

(ii) Region corresponding to $x + 2y \le 40$:

Line $x + 2y = 40$			
x	0	40	Point $(0,0)$ is true for $x + 2y \le 40$ So, the region is towards the origin.
y	20	0	

Since $x \ge 0,\ y \ge 0$ the feasible region lies in the first quadrant.

Now we draw all line on the graph and find the common area. The shaded region $OAEDO$ represents the feasible region of the given LPP. The point of intersection of the lines $2x + y = 50$ and $x + 2y = 40$ is $E(20,10)$. The corner points are $O(0,\ 0)$, $A(25,\ 0)$, $E(20,10)$ and $D(0,20)$.



Since, the feasible region is a bounded region, we can check the objective function at all the vertices to find the maxima. The values of Z at corner points are given below.

Corner points	$Z = x + y$
$O(0,\ 0)$	$Z = 0 + 0 = 0$
$A(25,\ 0)$	$Z = 25 + 0 = 25$
$D(0,20)$	$Z = 0 + 20 = 20$
$E(20,10)$	$Z = 20 + 10 = 30$

From the table, maximum value is 30 at point $E(20,10)$.

39. Maximise $Z = 22x + 18y$ subject to the constraints

$$3x + 2y \le 48$$

$$x + y \le 20$$

and $x \ge 0,\ y \ge 0$

Sol : Foreign 2012

We have the following LPP,

Maximize $Z = 22x + 18y$

Subject to $3x + 2y \le 48$ ...(i)

$x + y \le 20$ ...(ii)

and $x \ge 0,\ y \ge 0$

(i) Region corresponding to $3x + 2y \le 48$:

Line $3x + 2y = 48$			
x	0	16	Point $(0,0)$ is true for $3x + 2y \le 48$ So, the region is towards the origin.
y	24	0	

(ii) Region corresponding to $x + y \le 20$:

Line $x + y = 20$			
x	0	20	Point $(0,0)$ is true for $x + y \le 20$ So, the region is towards the origin.
y	20	0	

Since $x \ge 0,\ y \ge 0$ the feasible region lies in the first quadrant.

Now we draw all line on the graph and find the common area. The shaded region $OABCO$ represents the feasible region of the given LPP. The point of intersection of the lines $3x + 2y = 48$ and $x + y = 20$ is $B(8,12)$. The corner points are $O(0,\ 0)$, $A(16,\ 0)$, $B(8,12)$ and $C(0,20)$.

Sol : OD 2010

We have the following LPP,

Maximise, $P = 25x + 15y$

Subject to $2x + y \leq 12$

$3x + 2y \leq 20$

and $x \geq 0,\ y \geq 0$

(i) Region corresponding to $2x + y \leq 12$:

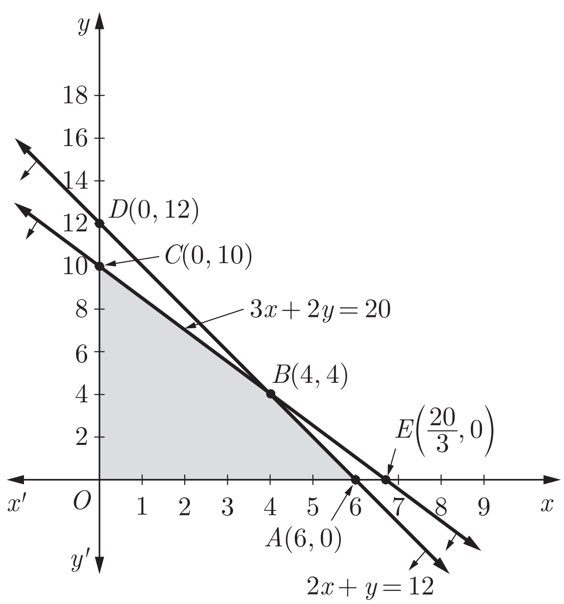
Line $2x + y = 12$			
x	0	6	Point $(0,0)$ is true for $2x + y \leq 12$ So, the region is towards the origin.
y	12	0	

(ii) Region corresponding to $3x + 2y \leq 20$:

Line $3x + 2y = 20$			
x	0	$\frac{20}{3}$	Point $(0,0)$ is true for $3x + 2y \leq 20$ So, the region is towards the origin.
y	10	0	

Since $x \geq 0,\ y \geq 0$ the feasible region lies in the first quadrant.

Now we draw all line on the graph and find the common area. The shaded region $OABCO$ represents the feasible region of the given LPP. The point of intersection of the lines $2x + y = 12$ and $3x + 2y = 20$ is $B(4,4)$. The corner points are $O(0,\ 0)$, $A(6,\ 0)$, $B(4,4)$ and $C(0,10)$.



Since, the feasible region is a bounded region, we can check the objective function at all the vertices to find the maxima. The values of P at corner points are given below.

Corner Points	$P = 25x + 15y$
$O(0,\ 0)$	$P = 0 + 0 = 0$
$A(6,\ 0)$	$P = 25 \times 6 + 0 = 150$
$B(4,4)$	$P = 25 \times 4 + 15 \times 4 = 160$
$C(0,10)$	$P = 0 + 15 \times 10 = 150$

From table, maximum value of P is 160 at $B(4,4)$.

42. Maximise $P = 50x + 28y$ subject to the constraints

$2x + y \leq 100$

$x + y \leq 80$

and $x \geq 0,\ y \geq 0$

Sol : Foreign 2016, OD 2012

We have the following LPP,

Maximise $P = 50x + 28y$

Subject to $2x + y \leq 100$ (i)

$x + y \leq 80$ (ii)

and $x \geq 0,\ y \geq 0$

(i) Region corresponding to $2x + y \leq 100$:

Line $2x + y = 100$			
x	0	50	Point $(0,0)$ is true for $2x + y \leq 100$ So, the region is towards the origin.
y	100	0	

(ii) Region corresponding to $x + y \leq 80$:

Line $x + y = 80$			
x	0	80	Point $(0,0)$ is true for $x + y \leq 80$ So, the region is towards the origin.
y	80	0	

Since $x \geq 0,\ y \geq 0$ the feasible region lies in the first quadrant.

Now we draw all line on the graph and find the common area. The shaded region $OBECO$ represents the feasible region of the given LPP. The point of intersection of the lines $2x + y = 100$ and $x + y = 80$ is $E(20,60)$. The corner points are $O(0,\ 0)$, $B(50,\ 0)$, $E(20,60)$ and $C(0,80)$.

44. Maximize $Z = 4500x + 5000y$ subject to the constraints

Subject to $5x + 8y \leq 1400$ (i)

$x + y \leq 250$ (ii)

and $x \geq 0,\ y \geq 0$

Sol : Delhi 2015

We have the following LPP,

Maximize $Z = 4500x + 5000y$

Subject to $5x + 8y \leq 1400$ (i)

$x + y \leq 250$ (ii)

and $x \geq 0,\ y \geq 0$

(i) Region corresponding to $5x + 8y \leq 1400$:

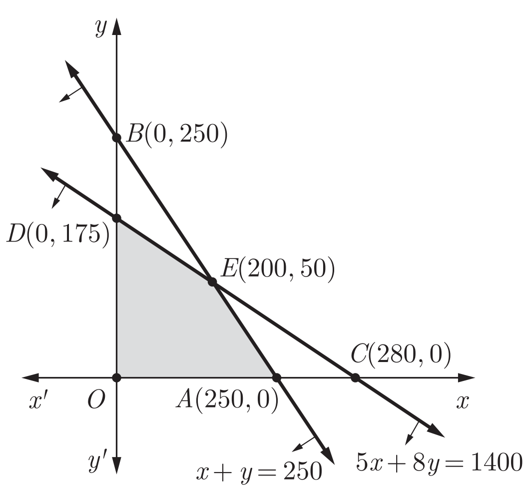
Line $5x + 8y = 1400$			
x	280	0	Point $(0,0)$ is true for $5x + 8y \leq 1400$. So, the region is towards the origin.
y	0	175	

(ii) Region corresponding to $x + y \leq 250$:

Line $x + y = 250$			
x	0	250	Point $(0,0)$ is true for $x + y \leq 250$ So, the region is towards the origin.
y	250	0	

Since $x \geq 0,\ y \geq 0$ the feasible region lies in the first quadrant.

Now we draw all line on the graph and find the common area. The shaded region $OAEDO$ represents the feasible region of the given LPP. The point of intersection of the lines $5x + 8y = 1400$ and $x + y = 250$ is $E(200, 50)$. The corner points are $O(0,\ 0)$, $A(250,\ 0)$, $E(200, 50)$ and $D(0, 175)$.



Since, the feasible region is a bounded region, we can check the objective function at all the vertices to find the maxima. The values of Z at corner points are given below.

Corner Points	$Z = 4500x + 5000y$
$O(0,\ 0)$	$Z = 0 + 0 = 0$
$A(250,\ 0)$	$Z = 4500 \times 250 + 0 = 1125000$
$E(200, 50)$	$Z = 4500 \times 200 + 5000 \times 50$ $= 1150000$
$D(0, 175)$	$Z = 0 + 5000 \times 175 = 875000$

From the table, maximum value of Z is 1150000 at $E(200, 50)$.

45. Maximize $Z = 300x + 190y$ subject to the constraints

$2x + y \leq 32$

$x + y \leq 24$

and $x \geq 0,\ y \geq 0$

Sol : Foreign 2014, Delhi 2008

We have the following LPP,

Maximize $Z = 300x + 190y$

Subject to $2x + y \leq 32$

$x + y \leq 24$

and $x \geq 0,\ y \geq 0$

(i) Region corresponding to $2x + y \leq 32$:

Line $2x + y \leq 32$			
x	16	0	Point $(0,0)$ is true for $2x + y \leq 32$. So, the region is towards the origin.
y	0	32	

(ii) Region corresponding to $x + y \leq 24$:

Line $x + y = 24$			
x	0	24	Point $(0,0)$ is true for $x + y \leq 24$ So, the region is towards the origin.
y	24	0	

Since $x \geq 0,\ y \geq 0$ the feasible region lies in the first quadrant.

Now we draw all line on the graph and find the common area. The shaded region $OCBAO$ represents the feasible region of the given LPP. The point of intersection of the lines $2x + y \leq 32$ and $x + y = 24$ is $B(8,\ 16)$. The corner points are $O(0,\ 0)$, $A(0,\ 24)$, $B(8,\ 16)$ and $C(16, 0)$.

Sol : OD 2012

We have the following LPP,

Minimize $Z = 2.5x + 1.5y + 410$

Subject to the constraints:

$$x + y \leq 100 \quad ...(1)$$

$$x + y \geq 60 \quad ...(2)$$

$$x \leq 60,\ y \leq 50$$

$$x, y \geq 0$$

(i) Region corresponding to $x + y \leq 100$:

Line $x+y=100$			
x	0	100	Point $(0,0)$ is true for $x+y \leq 100$. So, the region is towards the origin.
y	100	0	

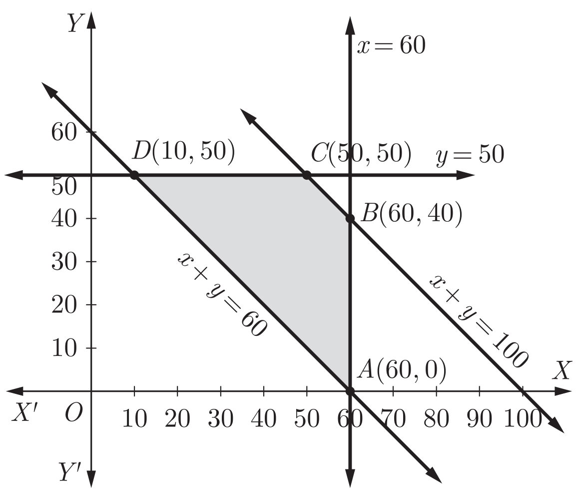
(ii) Region corresponding to $x + y \geq 60$:

Line $x+y=60$			
x	0	60	Point $(0,0)$ is false for $x+y \geq 60$ So, the region is away from the origin.
y	60	0	

(iii) Region corresponding to $y \leq 50$ and $x \leq 60$ is the region corresponding to the rectangle made by both line.

(iv) Since $x \geq 0$, $y \geq 0$ the feasible region lies in the first quadrant.

Now we draw all line on the graph and find the common area. The shaded region $ABCDA$ represents the feasible region of the given LPP. The corner points are $A(60,\ 0)$, $B(60,\ 40)$, $C(50,\ 50)$ and $D(10, 50)$.



Since, the feasible region is a bounded region, we can check the objective function at all the vertices to find the maxima. The values of Z at corner points are given below.

Corner Points	$Z = 2.5x + 1.5y + 410$
$A(60,\ 0)$	$Z = 2.5 \times 60 + 410 = 560$
$B(60,\ 40)$	$Z = 2.5 \times 60 + 1.5 \times 40 + 410 = 620$
$C(50,\ 50)$	$Z = 2.5 \times 50 + 1.5 \times 50 + 410 = 610$
$D(10, 50)$	$Z = 2.5 \times 10 + 1.5 \times 50 + 410 = 510$

Thus Z is minimum at $D(10, 50)$ and the minimum value is 510.

48. Minimize $Z = 0.3x + 0.1y + 3950$ subject to the constraints

$$x + y \leq 7000$$

$$x + y \geq 3500$$

$$x \leq 4500$$

$$y \leq 3000$$

and $\quad x, y \geq 0$

Sol : SQP 2017, OD 2013

We have the following LPP,

Minimize $Z = 0.3x + 0.1y + 3950$

Subject to the constraints:

$$x + y \leq 7000 \quad ...(1)$$

$$x + y \geq 3500 \quad ...(2)$$

$$x \leq 4500 \quad ...(3)$$

$$y \leq 3000 \quad ...(4)$$

$$x, y \geq 0$$

(i) Region corresponding to $x + y \leq 7000$:

Line $x+y=7000$			
x	0	7000	Point $(0,0)$ is true for $x+y \leq 7000$. So, the region is towards the origin.
y	7000	0	

(ii) Region corresponding to $x + y \geq 3500$:

Line $x+y=3500$			
x	0	3500	Point $(0,0)$ is false for $x+y \geq 3500$ So, the region is away from the origin.
y	3500	0	

(iii) Region corresponding to $y \leq 4000$ and $x \leq 4500$ is the region corresponding to the rectangle made by both line.

(iv) Since $x \geq 0$, $y \geq 0$ the feasible region lies in the first quadrant.

Corner Points	$Z = 3x + 4y + 370$
$A(10,\ 0)$	$Z = 3 \times 10 + 0 + 370 = 400$
$B(40,\ 0)$	$Z = 3 \times 40 + 0 + 370 = 490$
$C(40,\ 20)$	$Z = 3 \times 40 + 4 \times 20 + 370 = 570$
$D(20,\ 40)$	$Z = 3 \times 20 + 4 \times 40 + 370 = 590$
$E(0, 40)$	$Z = 0 + 4 \times 40 + 370 = 530$
$F(0, 10)$	$Z = 0 + 4 \times 10 + 370 = 410$

Thus Z is minimum at $(10, 0)$ and the minimum value is 400.

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CHAPTER 13

PROBABILITY

OBJECTIVE QUESTIONS

1. If A and B are events such that $P(A/B) = P(B/A) \neq 0$, then :

(a) $A \subset B$, but $A \neq B$
(b) $A = B$
(c) $A \cap B = \phi$
(d) $P(A) = P(B)$

Sol : OD 2024

$$\begin{aligned} P\left(\frac{A}{B}\right) &= P\left(\frac{B}{A}\right) \neq 0 \\ \frac{P(A \cap B)}{P(B)} &= \frac{P(A \cap B)}{P(A)} \neq 0 \\ P(A) &= P(B) \neq 0 \end{aligned}$$

Thus (d) is correct option.

2. If A and B are two events such that $P(\frac{A}{B}) = 2 \times P(\frac{B}{A})$ and $P(A) + P(B) = \frac{2}{3}$, then $P(B)$ is equal to :

(a) $\frac{2}{9}$
(b) $\frac{7}{9}$
(c) $\frac{4}{9}$
(d) $\frac{5}{9}$

Sol : OD 2023

We have $$\begin{aligned} P\left(\frac{A}{B}\right) &= 2 \times P\left(\frac{B}{A}\right) \\ \frac{P(A \cap B)}{P(B)} &= 2 \times \frac{P(A \cap B)}{P(A)} \\ P(A) &= 2 \times P(B) \end{aligned}$$

$$\begin{aligned} P(A) + P(B) &= \frac{2}{3} \\ 3P(B) &= \frac{2}{3} \\ P(B) &= \frac{2}{9} \end{aligned}$$

Thus (a) is correct option.

3. A card is picked at random from a pack of 52 playing cards. Given that the picked card is a queen, the probability of this card to be a card of spade is

(a) $\frac{1}{3}$
(b) $\frac{4}{13}$
(c) $\frac{1}{4}$
(d) $\frac{1}{2}$

Sol : OD 2020

Let A be the event that card drawn is a spade and B be the event that card drawn is a queen. We have a total of 13 spades and 4 queen and one queen is from spade.

$$\begin{aligned} P(A) &= \frac{13}{52} = \frac{1}{4}, \\ P(B) &= \frac{4}{52} = \frac{1}{13} \end{aligned}$$

and $$P(A \cap B) = \frac{1}{52}$$

$$\begin{aligned} P\left(\frac{A}{B}\right) &= \frac{P(A \cap B)}{P(B)} \\ &= \frac{\frac{1}{52}}{\frac{1}{13}} = \frac{1}{4} \end{aligned}$$

Thus (s) is correct option.

4. A die is thrown once. Let A be the event that the number obtained is greater than 3. Let B be the event that the number obtained is less than 5. Then $P(A \cup B)$ is

(a) $\frac{2}{5}$
(b) $\frac{3}{5}$
(c) 0
(d) 1

Sol : OD 2020

We have, $$\begin{aligned} A &= \{4,5,6\} \\ B &= \{1,2,3,4\} \\ A \cup B &= \{1,2,3,4,5,6\} \end{aligned}$$

Thus $$P(A \cup B) = \frac{6}{6} = 1$$

Alternative :

Now, $$A \cap B = \{4\}$$

Now, $$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= \frac{3}{6} + \frac{4}{6} - \frac{1}{6} \\ &= \frac{6}{6} = 1 \end{aligned}$$

Thus (d) is correct option.

5. Out of 50 tickets numbered 00, 01, 02, ..., 49, one ticket is drawn randomly, the probability of the ticket having the product of its digits 7, given that the sum of the digits is 8, is

10. It is given that the events A and B are such that $P(A)=\frac{1}{4}$, $P(\frac{A}{B})=\frac{1}{2}$ and $P(\frac{B}{A})=\frac{2}{3}$. Then $P(B)$ is equal to

(a) $\frac{1}{2}$ (b) $\frac{1}{6}$

(c) $\frac{1}{3}$ (d) $\frac{2}{3}$

Sol : SQP 2020

We have
$$P(B) = \frac{P\left(\frac{B}{A}\right)\cdot P(A)}{P\left(\frac{A}{B}\right)} = \frac{\frac{2}{5}\cdot\frac{1}{4}}{\frac{1}{2}} = \frac{1}{3}$$

Thus (c) is correct option.

11. For two events A and B, if $P(A)=P(\frac{A}{B})=\frac{1}{4}$ and $P(\frac{B}{A})=\frac{1}{2}$, then

(a) A and B are independent events

(b) $P\left(\frac{A'}{B}\right)=\frac{3}{4}$

(c) $P\left(\frac{B'}{A}\right)=\frac{1}{2}$

(d) All of the above

Sol : Comp 2017

We have
$$P\left(\frac{B}{A}\right) = \frac{1}{2}$$
$$P(B\cap A) = P\left(\frac{B}{A}\right)P(A) = \frac{1}{2}\times\frac{1}{4} = \frac{1}{8}$$

and
$$P\left(\frac{A}{B}\right) = \frac{1}{4}$$
$$P(B) = \frac{P(B\cap A)}{P\left(\frac{A}{B}\right)} = \frac{\frac{1}{8}}{\frac{1}{4}} = \frac{1}{2}$$
$$P(A\cap B) = \frac{1}{8}$$
$$P(A)\cdot P(B) = \frac{1}{4}\times\frac{1}{2} = \frac{1}{8}$$

Since $P(A\cap B)$ $P(A)\cdot P(B)$ events are independent.

Now,
$$P\left(\frac{A'}{B}\right) = \frac{P(A'\cap B)}{P(B)} = \frac{P(A\cap B)}{P(B)} = \frac{3}{4}$$

and
$$P\left(\frac{B'}{A}\right) = \frac{P(A\cap B')}{P(A)} = \frac{P(A)-P(A\cap B)}{P(A)} = \frac{1}{2}$$

Thus (d) is correct option.

12. If $P(A)=\frac{1}{12}$, $P(B)=\frac{5}{12}$ and $P(\frac{B}{A})=\frac{1}{15}$, then $P(A\cup B)$ is equal to

(a) $\frac{89}{180}$ (b) $\frac{90}{180}$

(c) $\frac{91}{180}$ (d) $\frac{92}{180}$

Sol : OD 2014

We have
$$P\left(\frac{B}{A}\right) = \frac{P(A\cap B)}{P(A)}$$
$$\frac{1}{15} = \frac{P(A\cap B)}{\frac{1}{12}}$$
$$P(A\cap B) = \frac{1}{180}$$

Now
$$P(A\cup B) = P(A)+P(B)-P(A\cap B) = \frac{1}{12}+\frac{5}{12}-\frac{1}{180} = \frac{89}{180}$$

Thus (a) is correct option.

13. If A and B are any two events, then the probability that exactly one of them occur is :

(a) $P(A)+P(B)-P(A\cap B)$

(b) $P(A)+P(B)-2P(A\cap B)$

(c) $P(A)+P(B)-P(A\cup B)$

(d) $P(A)+P(B)-2P(A\cup B)$

Sol : Delhi 2018, Comp 2015

$$P = P(A\cap \overline{B})+P(\overline{A}\cap B)$$
$$= P(A)-P(A\cap B)+P(B)-P(A\cap B)$$
$$= P(A)+P(B)-2P(A\cap B)$$

Thus (b) is correct option.

14. If $P(A)=\frac{4}{5}$, and $P(A\cap B)=\frac{7}{10}$, then $P(\frac{B}{A})$ is equal to

(a) $\frac{1}{10}$ (b) $\frac{1}{8}$

(c) $\frac{7}{8}$ (d) $\frac{17}{20}$

Sol : OD 2012

We have,
$$P(A) = \frac{4}{5},$$
$$P(A\cap B) = \frac{7}{10}$$
$$P\left(\frac{B}{A}\right) = \frac{P(A\cap B)}{P(A)} = \frac{\frac{7}{10}}{\frac{4}{5}} = \frac{7}{8}$$

$$P(B \cap A) = P\left(\frac{B}{A}\right) \cdot P(A)$$
$$= 0.6 \times 0.4$$
$$= 0.24$$
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$= 0.4 + 0.8 - 0.24$$
$$= 0.96$$

Thus (d) is correct option.

20. If A and B are two events and $A \neq \phi$, $B \neq \phi$, then

(a) $P\left(\frac{A}{B}\right) = P(A) \cdot P(B)$

(b) $P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)}$

(c) $P\left(\frac{A}{B}\right) \cdot P\left(\frac{B}{A}\right) = 1$

(d) $P\left(\frac{A}{B}\right) = \frac{P(A)}{P(B)}$

Sol : OD 2018, Delhi 2014

(b) If $A \neq \phi$ and $B \neq \phi$, then

$$P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)}$$

Thus (b) is correct option.

21. A and B events such that $P\left(\frac{A}{B}\right) = 0.4$, $P(B) = 0.3$ and $P(A \cup B) = 0.5$. Then $P(B' \cap A)$ equals

(a) $\frac{2}{3}$ (b) $\frac{1}{2}$

(c) $\frac{3}{10}$ (d) $\frac{1}{5}$

Sol : OD 2017

We have, $P(A) = 0.4$,

$$P(B) = 0.3$$

and $P(A \cup B) = 0.5$

Now, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$P(A \cap B) = 0.4 + 0.3 - 0.5 = 0.2$$
$$P(B' \cap A) = P(A) - P(A \cap B)$$
$$= 0.4 - 0.2 = 0.2 = \frac{1}{5}$$

Thus (d) is correct option.

22. You are given that A and B are two events such that $P(B) = \frac{3}{5}$, $P\left(\frac{A}{B}\right) = \frac{1}{2}$ and $P(A \cup B) = \frac{4}{5}$, then $P(A)$ equals

(a) $\frac{3}{10}$ (b) $\frac{1}{5}$

(c) $\frac{1}{2}$ (d) $\frac{3}{5}$

Sol : Comp 2015

We have $P(B) = \frac{3}{5}$,

$$P\left(\frac{A}{B}\right) = \frac{1}{2}$$

and $P(A \cup B) = \frac{4}{5}$

$$P(A \cap B) = P\left(\frac{A}{B}\right) \cdot P(B)$$
$$= \frac{1}{2} \cdot \frac{3}{5} = \frac{3}{10}$$

Now $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$\frac{4}{5} = P(A) + \frac{3}{5} - \frac{3}{10}$$
$$P(A) = \frac{4}{5} - \frac{3}{5} + \frac{3}{10} = \frac{1}{2}$$

Thus (c) is correct option.

23. If $P(B) = \frac{3}{5}$, $P\left(\frac{A}{B}\right) = \frac{1}{2}$ and $P(A \cup B) = \frac{4}{5}$, then $P(A \cap B)' + P(A' \cup B) =$

(a) $\frac{1}{5}$ (b) $\frac{4}{5}$

(c) $\frac{1}{2}$ (d) 1

Sol : Delhi 2008

We have $P(B) = \frac{3}{5}$

$$P\left(\frac{A}{B}\right) = \frac{1}{2}$$
$$P(A \cap B) = P\left(\frac{A}{B}\right) \cdot P(B)$$
$$= \frac{1}{2} \cdot \frac{3}{5} = \frac{3}{10}$$

Now, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$P(A) = \frac{4}{5} - \frac{3}{5} + \frac{3}{10} = \frac{1}{2}$$
$$P(A \cup B)' = 1 - P(A \cup B)$$
$$= 1 - \frac{4}{5} = \frac{1}{5}$$

and $P(A' \cup B) = 1 - P(A - B)$

$$= 1 - [P(A) - P(A \cap B)]$$
$$= 1 - \left(\frac{1}{2} - \frac{3}{10}\right) = \frac{4}{5}$$
$$P(A \cup B)' + P(A' \cup B) = \frac{1}{5} + \frac{4}{5} = 1$$

Thus (b) is correct option.

24. Let $P(A) = \frac{7}{15}$, $P(B) = \frac{9}{13}$ and $P(A \cap B) = \frac{4}{13}$. Then $P\left(\frac{A'}{B}\right)$ is equal to

(a) $\frac{6}{13}$ (b) $\frac{4}{13}$

29. Two events E and F are independent. If $P(E) = 0.3$, $P(E \cup F) = 0.5$, then $P(\frac{E}{F}) - P(\frac{F}{E})$ equals

(a) $\frac{2}{7}$ (b) $\frac{3}{35}$

(c) $\frac{1}{70}$ (d) $\frac{1}{7}$

Sol : Comp 2016, OD 2010

We have, $P(E) = 0.3$ and $P(E \cup F) = 0.5$

Also E and F are independent.

Now $P(E \cup F) = P(E) + P(F) - P(E \cap F)$

$$0.5 = 0.3 + P(F) - 0.3P(F)$$

$$P(F) = \frac{0.5 - 0.3}{0.7} = \frac{2}{7}$$

As E and F are independent, we have

$$P\left(\frac{E}{F}\right) - P\left(\frac{F}{E}\right) = P(E) - P(F)$$

$$= \frac{3}{10} - \frac{2}{7} = \frac{1}{70}$$

Thus (c) is correct option.

30. Assume that in a family, each child is equally likely to be a boy or a girl. A family with three children is chosen at random. The probability that the eldest child is a girl given that the family has at least one girl is

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$

(c) $\frac{2}{3}$ (d) $\frac{4}{7}$

Sol : Foreign 2015

We have $n(S) = 8$

where S is {(B, B, B), (G, B, B), (B, G, B), (B, B, G), (G, G, B), (G, B, G), (B, G, G), (G, G, G)}

Let E_1 be the event that a family has at-least one girl, then and E_2 be the event that the eldest child is a girl.

and E_2 is {(G, B, B), (G, G, B), (G, B, G), (G, G, G)}

$E_1 \cap E_2 =$ {(G, B, B), (G, G, B), (G, B, G), (G, G, G)}

$$P\left(\frac{E_2}{E_1}\right) = \frac{P(E_1 \cap E_2)}{P(E_1)}$$

$$= \frac{\frac{4}{8}}{\frac{7}{8}} = \frac{4}{7}$$

Thus (d) is correct option.

31. A dice is thrown and a card is selected at random from a deck of 52 playing cards. The probability of getting an even number on the die and a spade card is

(a) $\frac{1}{2}$ (b) $\frac{1}{4}$

(c) $\frac{1}{8}$ (d) $\frac{3}{4}$

Sol : Delhi 2007

Let E_1 be the event for getting an event number on the die and E_2 be the event that a spade card is selected.

Now $P(E_1) = \frac{3}{6} = \frac{1}{2}$

and $P(E_2) = \frac{13}{52} = \frac{1}{4}$

Then, $P(E_1 \cap E_2) = P(E_1) \cdot P(E_2)$

$$= \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$$

Thus (c) is correct option.

32. Two dice are thrown. If it is known that the sum of numbers on the dice was less than 6, the probability of getting a sum 3, is

(a) $\frac{1}{18}$ (b) $\frac{5}{18}$

(c) $\frac{1}{5}$ (d) $\frac{2}{5}$

Sol : OD 2013, Delhi 2009

Let E_1 be the event that the sum of numbers on the dice was less than 6 and E_2 be the event that the sum of numbers on the dice is 3.

$$E_1 = \{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (3,1)$$
$$(3,2), (4,1)\}$$

$$n(E_1) = 10$$

$$E_2 = \{(1,2), (2,1)\}$$

$$n(E_2) = 2$$

Required probability,

$$P\left(\frac{E_2}{E_1}\right) = \frac{2}{10} = \frac{1}{5}$$

Thus (c) is correct option.

33. If A' and B' are independent events then

(a) $P(A'B') = P(A) \cdot P(B)$

(b) $P(A'B') = P(A') + P(B')$

(c) $P(A'B') = P(A') \cdot P(B')$

(d) $P(A'B') = P(A') - P(B')$

Sol : SQP 2017

Two events A and B are said to be independent only and only if happening of B will have no effect on A. Conversely, happening of B' will have no effect on A'

$$P(AB) = P(A) \cdot P(B)$$

$$P(A'B') = P(A') \cdot P(B')$$

Thus (c) is correct option.

34. If events A and B are mutually exclusive then

(a) $P(A \cap B) = P(A) \cdot P(B)$

$$= \frac{3}{8} + \frac{1}{2} - \frac{1}{4}$$

$$= \frac{9+12-6}{24} = \frac{15}{24}$$

Now $\quad P\left(\frac{B'}{A'}\right) = P\frac{(B' \cap A')}{P(A')}$

$$= P\frac{(\overline{A \cup B})}{P(A')}$$

$$= \frac{1 - P(A \cup B)}{1 - P(A)} \qquad ...(1)$$

$$= \frac{1 - \frac{11}{24}}{1 - \frac{3}{8}} = \frac{3}{5}$$

Thus (a) is correct option.

45. If $P(A) = \frac{3}{8}$, $P(B) = \frac{1}{3}$ and $P(A \cap B) = \frac{1}{4}$ then $P(A' \cap B') =$

(a) $\frac{1}{4}$ (b) $\frac{1}{3}$

(c) $\frac{3}{4}$ (d) $\frac{3}{8}$

Sol : Foreign 2017, Delhi 2010

We have $\quad P(A) = \frac{3}{8}$

$$P(B) = \frac{1}{3}$$

$$P(A \cap B) = \frac{1}{4}$$

Now $\quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= \frac{3}{8} + \frac{1}{3} - \frac{1}{4} = \frac{11}{24}$$

Now $\quad P(A' \cap B') = P(\overline{A \cup B})$

$$= 1 - P(A \cup B)$$

$$= 1 - \frac{11}{24} = \frac{13}{24}$$

Thus (c) is correct option.

46. If $P(A) = \frac{3}{8}$, $P(B) = \frac{1}{2}$ and $P(A \cap B) = \frac{1}{4}$ then $P\left(\frac{A'}{B'}\right) =$

(a) $\frac{1}{4}$ (b) $\frac{1}{3}$

(c) $\frac{3}{4}$ (d) $\frac{3}{8}$

Sol : SQP 2019

We have $\quad P(A) = \frac{3}{8}$

$$P(B) = \frac{1}{2}$$

$$P(A \cap B) = \frac{1}{4}$$

Now $\quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= \frac{3}{8} + \frac{1}{2} - \frac{1}{4} = \frac{5}{8}$$

$$P\left(\frac{A'}{B'}\right) = \frac{P(A' \cap B')}{P(B')}$$

$$= \frac{1 - P(A \cup B)}{1 - P(B)}$$

$$= \frac{1 - \frac{5}{8}}{1 - \frac{1}{2}} = \frac{\frac{3}{8}}{\frac{1}{2}} = \frac{3}{4}$$

Thus (c) is correct option.

47. If A and B are any two events such that $P(A) + P(B) - P(A \cap B) = P(A)$ then

(a) $P\left(\frac{B}{A}\right) = 1$ (b) $P\left(\frac{B}{A}\right) = 0$

(c) $P\left(\frac{A}{B}\right) = 1$ (d) $P\left(\frac{A}{B}\right) = 0$

Sol : SQP 2020

We have $\quad P(A) = P(A) + P(B) - P(A \cap B)$

$$P(B) = P(A \cap B)$$

$$\frac{P(A \cap B)}{P(B)} = 1$$

$$P\left(\frac{A}{B}\right) = 1$$

Thus (c) is correct option.

48. If A and B are events such that $P(A \cup B) = \frac{3}{4}$, $P(A \cap B) = \frac{1}{4}$, $P(A) = \frac{2}{3}$, then $P(A' \cap B')$ is

(a) $\frac{3}{8}$ (b) $\frac{5}{8}$

(c) $\frac{5}{12}$ (d) $\frac{1}{4}$

Sol : OD 2010

We have $\quad P(A \cup B) = \frac{3}{4}$

$$P(A \cap B) = \frac{1}{4}$$

$$P(A) = \frac{2}{3}$$

$$P(A' \cap B') = P(\overline{A \cup B})$$

$$= 1 - P(A \cup B)$$

$$= 1 - \frac{3}{4} = \frac{1}{4}$$

Thus (d) is correct option.

49. If $P(A) = \frac{3}{8}$, $P(B) = \frac{5}{8}$ and $P(A \cup B) = \frac{3}{4}$, then $P\left(\frac{B}{A}\right)$ is

(a) $\frac{1}{4}$ (b) $\frac{1}{3}$

(c) $\frac{2}{3}$ (d) $\frac{1}{2}$

Reason (R) : $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ and $P(A \cap B) = P(A)P(B)$

(a) Both (A) and (R) are true and (R) is the correct explanation of (A).

(b) Both (A) and (R) are true but (R) is not the correct explanation of (A).

(c) (A) is true but (R) is false.

(d) (A) is false but (R) is true

Sol : Delhi 2012, OD 2009

Since A and B are two independent event

$$P(A \cap B) = P(A) \cdot P(B)$$

We know that

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= P(A) + P(B) - P(A) \cdot P(B) \\ &= P(A) + P(B) - \left(1 - P(\overline{A})\right)\left(1 - P(\overline{B})\right) \\ &= P(A) + P(B) - \big[1 - P(\overline{B}) - P(\overline{A}) \\ &\qquad + P(\overline{A}) \cdot P(\overline{B})\big] \\ &= P(A) + P(B) - 1 + P(\overline{B}) + P(\overline{A}) \\ &\qquad - P(\overline{A}) \cdot P(\overline{B}) \\ &= 1 - 1 + 1 - P(\overline{A}) \cdot P(\overline{B}) \\ &\qquad (\text{Since } P(A) + P(\overline{A}) = 1 \\ &\qquad \text{and } P(B) + P(\overline{B}) = 1) \\ &= 1 - P(\overline{A}) \cdot P(\overline{B}) \\ &= \text{R.H.S.} \end{aligned}$$

Thus both (A) and (R) are true and (R) is the correct explanation of (A).

Thus (a) is correct option.

54. If A and B be two events such that $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$, $P(A \cup B) = \frac{2}{3}$

Assertion (A) : A and B are independent.

Reason (R) : $P(A \cap B) = P(A) \cdot P(B)$

(a) Both (A) and (R) are true and (R) is the correct explanation of (A).

(b) Both (A) and (R) are true but (R) is not the correct explanation of (A).

(c) (A) is true but (R) is false.

(d) (A) is false but (R) is true

Sol : Comp 2017

We have
$$\begin{aligned} P(A) &= \frac{1}{2} \\ P(B) &= \frac{1}{3} \\ P(A \cup B) &= \frac{2}{3} \\ P(A \cap B) &= P(A) + P(B) - P(A \cup B) \\ &= \frac{1}{2} + \frac{1}{3} - \frac{2}{3} = \frac{1}{6} \end{aligned}$$

Moreover if A and B are two independent event, then

$$\begin{aligned} P(A \cap B) &= P(A) \cdot P(B) \\ &= \frac{1}{2} \cdot \frac{1}{3} \\ &= \frac{1}{6} \end{aligned}$$

Thus both (A) and (R) are true and (R) is the correct explanation of (A).

Thus (a) is correct option.

55. **Assertion (A) :** If $2P(A) = P(B) = \frac{5}{13}$ and $P\left(\frac{A}{B}\right) = \frac{2}{5}$, then $P(A \cup B)$ is $\frac{11}{20}$.

Reason (R) : E_1 and E_2 are two events. then

$$P\left(\frac{E_1}{E_2}\right) = \frac{P(E_1 \cup E_2)}{P(E_2)},\ 0 < P(E_2) \le 1$$

(a) Both (A) and (R) are true and (R) is the correct explanation of (A).

(b) Both (A) and (R) are true but (R) is not the correct explanation of (A).

(c) (A) is true but (R) is false.

(d) (A) is false but (R) is true

Sol : Foreign 2010, Delhi 2008

Given that,

$$\begin{aligned} 2P(A) &= P(B) = \frac{5}{13} \\ P(A) &= \frac{P(A \cup B)}{P(B)} \\ \frac{2}{5} &= \frac{P(A \cup B)}{\frac{5}{13}} \\ P(A \cap B) &= \frac{2}{5} \times \frac{5}{13} = \frac{2}{13} \end{aligned}$$

Again,

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= \frac{5}{26} + \frac{5}{13} - \frac{2}{13} \\ &= \frac{5 + 10 - 4}{26} \\ &= \frac{11}{26} \end{aligned}$$

If E_1 and E_2 are two events, then

$$P\left(\frac{E_1}{E_2}\right) = \frac{P(E_1 \cap E_2)}{P(E_2)},\ 0 \le P(E_2) \le 1$$

Hence, Assertion is true, reason is true and reason is a correct explanation for Assertion.

SHORT ANSWER QUESTIONS

59. Bag I contains 3 red and 4 black balls, Bag II contains 5 red and 2 black balls. Two balls are transferred at random from Bag I to Bag II and then a ball is drawn at random from Bag II. Find the probability that the drawn ball is red in colour.

Sol : OD 2024

Let the following event be defined

$E_1 =$ one red and one black ball is transferred

$E_2 =$ two red balls are transferred

$E_3 =$ two black balls are transferred

$E =$ drawn ball is red.

Now $$P(E_1) = \frac{{}^3C_1 \times {}^4C_1}{{}^7C_2} = \frac{3 \times 4}{21} = \frac{4}{7}$$

$$P(E_2) = \frac{{}^3C_2}{{}^7C_2} = \frac{3}{21} = \frac{1}{7},$$

$$P(E_3) = \frac{{}^4C_2}{{}^7C_2} = \frac{6}{21} = \frac{2}{7}$$

$$P\left(\frac{E}{E_1}\right) = \frac{6}{9},$$

$$P\left(\frac{E}{E_2}\right) = \frac{7}{9},$$

$$P\left(\frac{E}{E_3}\right) = \frac{5}{9}$$

Now required probability

$$P(E) = P(E_1) \cdot P\left(\frac{E}{E_1}\right) + P(E_2) \cdot P\left(\frac{E}{E_2}\right) + P(E_3) \cdot P\left(\frac{E}{E_2}\right)$$

$$= \frac{4}{7} \times \frac{6}{9} + \frac{1}{7} \times \frac{7}{9} + \frac{2}{7} \times \frac{5}{9}$$

$$= \frac{4 \times 6 + 1 \times 7 + 2 \times 5}{7 \times 9}$$

$$= \frac{24 + 7 + 10}{7 \times 9}$$

$$= \frac{41}{63}$$

60. A and B throw a die alternatively till one of them gets a '6' and wins the game. Find their respective probabilities of winning, if A starts the game first.

Sol : OD 2023

Probability of getting a 6 when a dice is rolled $= \frac{1}{6}$

Let S denote the success (getting a '6') and F denote the failure (not getting a '6').

$$P(S) = \frac{1}{6} = p \text{ and } P(F) = \frac{5}{6} = q$$

$$P(A \text{ wins in first throw}) = P(S) = p$$

$$P(A \text{ wins in third throw}) = P(FFS) = qqp$$

$$P(A \text{ wins in fifth throw}) = P(FFFFS)$$

$$= qqqqp \text{ and so on..}$$

Probability that A wins the game

$$P(A) = \frac{1}{6} + \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{1}{6} + \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{1}{6} + \dots$$

$$= \frac{\frac{1}{6}}{1 - \left(\frac{5}{6}\right)^2} = \frac{6}{11}$$

Probability that B wins the game

$$P(B) = 1 - P(A) = \frac{5}{11}$$

61. Given two independent events A and B such that $P(A) = 0.3$ and $P(B) = 0.6$, find $P(A' \cap B')$.

Sol : OD 2020

Since A and B are independent events, we have

$$P(A \cap B) = P(A)P(B)$$

We have $P(A) = 0.3,\ P(B) = 0.6$

Now

$$P(A' \cap B') = P(A \cup B)' = 1 - P[A \cup B]$$

$$= 1 - [P(A) + P(B) - P(A \cap B)]$$

$$= 1 - [P(A) + P(B) - P(A)P(B)]$$

$$= 1 - \{0.3 + 0.6 - 0.3 \times 0.6\}$$

$$= 1 - \{0.9 - 0.18\}$$

$$= 1 - \{0.72\} = 0.28$$

62. In a shop X, 30 tins of ghee of type A and 40 tins of ghee of type B which look alike, are kept for sale. While in shop Y, similar 50 tins of ghee of type A and 60 tins of ghee of type B are there. One tin of ghee is purchased from one of the randomly selected shop and is found to be of type B. Find the probability that it is purchased from shop Y.

Sol : OD 2020

Let the following event be defined

$E_1 =$ Getting ghee from shop X

$E_2 =$ Getting ghee from shop Y

$A =$ Getting type B ghee

Since both shop have equal chances, we have

$$P(E_1) = P(E_2) = \frac{1}{2}$$

Probability that type B ghee is purchased from shop X

$$P\left(\frac{A}{E_1}\right) = \frac{40}{70} = \frac{4}{7}$$

Probability that type B ghee is purchased from shop Y

Now $P(E_1) = \frac{1}{2}$,

$P(E_2) = \frac{1}{2}$,

$P\left(\frac{W}{E_1}\right) = \frac{4}{7}$,

and $P\left(\frac{W}{E_2}\right) = \frac{3}{10}$,

Now, required probability,

$$P\left(\frac{E_1}{W}\right) = \frac{P(E_1)\cdot P\left(\frac{W}{E_1}\right)}{P(E_1)\cdot P\left(\frac{W}{E_1}\right) + P(E_2)\cdot P\left(\frac{W}{E_2}\right)}$$

$$= \frac{\frac{1}{2}\cdot\frac{4}{7}}{\frac{1}{2}\cdot\frac{4}{7}+\frac{1}{2}\cdot\frac{3}{10}}$$

$$= \frac{\frac{4}{7}}{\frac{4}{7}+\frac{3}{10}} = = \frac{4}{7}\times\frac{70}{61}$$

$$= \frac{40}{61}$$

67. If $P(\overline{A}) = 0.7$, $P(B) = 0.7$ and $P(B/A) = 0.5$, then find $P(A/B)$.

Sol : OD 2019

We have $P(A') = 0.7$, $P(B) = 0.7$ and $P\left(\frac{B}{A}\right) = 0.5$

Now $P(A) = 1 - P(A')$

$= 1 - 0.7 = 03$

Now, $P\left(\frac{B}{A}\right) = \frac{P(A \cap B)}{P(A)}$

$0.5 = \frac{P(A \cap B)}{0.3}$

$P(A \cap B) = 0.15$

$P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)} = \frac{0.15}{0.7}$

Thus $P\left(\frac{A}{B}\right) = \frac{3}{14}$

68. A die marked 1, 2, 3 in red and 4, 5, 6 in green is tossed. Let A be the event 'number is tossed' and B be the event 'number is marked red'. Find whether the events A and B are independent or not.

Or

A die, whose faces are marked 1, 2, 3 in red and 4, 5, 6 in green, is tossed. Let A be the event "number obtained is even" and B be the event "number obtained is red". Find if A and B are independent events.

Sol : Delhi 2019, OD 2017

When a die is thrown, the sample space is

$$S = \{1,2,3,4,5,6\}$$

Thus $n(S) = 6$

Let, A Event that number is even and

B Event that number of red

Now $A = \{2,4,6\}$ and $B = \{1,2,3\}$ and

$A \cap B = \{2\}$

$n(A) = 3$, $n(B) = 3$ and $n(A \cap B) = 1$

Now, $P(A) = \frac{n(A)}{n(S)} = \frac{1}{2}$

$P(B) = \frac{n(B)}{n(S)} = \frac{3}{6} = \frac{1}{2}$

and $P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{1}{6}$

Now, $P(A)\times P(B) = \frac{1}{2}\times\frac{1}{2} = \frac{1}{4} \neq \frac{1}{6} = P(A \cap B)$

Thus $P(A \cap B) \neq P(A)\times P(B)$

Hence A and B are not independent events.

69. A black and a red die are rolled together. Find the conditional probability of obtaining the sum 8, given that the red die resulted in a number less than 4.

Sol : OD 2018

Let the numbers on black die is denoted by $B_1, B_2, ..,$ B_6 and the numbers on red die is denoted by R_1, R_2 ,..., R_6.

Now we have following sample space.

$$S = \begin{Bmatrix}(B_1,R_1),(B_1,R_2),\ldots,(B_1,R_6),(B_2,R_1),(B_2,R_2)\\ ,\ldots,(B_2,R_6),\ldots,(B_6,R_1),(B_6,R_2),\ldots,(B_6,R_6)\end{Bmatrix}$$

Now $n(S) = 36$

Let A be the event that sum of number obtained on the die is 8 and B be the event that red die shows a number less than 4.

$$A = \{(B_2,R_6),(B_6,R_2),(B_3,R_5),(B_5,R_3),(B_4,R_4)\}$$

and

$$B = \begin{Bmatrix}(B_1,R_1),(B_1,R_2),(B_1,R_3),(B_2,R_1),(B_2,R_2)\\ (B_2,R_3),\ldots,(B_6,R_1),(B_6,R_2),(B_6,R_2),(B_6,R_3)\end{Bmatrix}$$

$$A \cap B = \{(B_6,R_2),(B_5,R_3)\}$$

Now, required probability,

$$P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{2}{36}}{\frac{18}{36}} = \frac{2}{18} = \frac{1}{9}$$

70. Evaluate $P(A \cup B)$, if $2P(A) = P(B) = \frac{5}{13}$ and $P\left(\frac{A}{B}\right) = \frac{2}{5}$.

Sol : Delhi 2017

We have $2P(A) = P(B) = \frac{5}{13}$

$$P(A) = \frac{5}{26},\ P(B) = \frac{5}{13} \text{ and } P\left(\frac{A}{B}\right) = \frac{2}{5}$$

Now $$P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)}$$

$$\frac{2}{5} = \frac{P(A \cap B)}{\frac{5}{13}}$$

$$P(A \cap B) = \frac{2}{5} \times \frac{5}{13} = \frac{2}{13}$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{5}{26} + \frac{5}{13} - \frac{2}{13}$$

$$= \frac{5 + 10 - 4}{26} = \frac{11}{26}$$

71. Prove that if E and F are independent events, then the events E and F' are also independent.

Sol : Delhi 2017, OD 2013

Since event E and F are independent, we have

$$P(E \cap F) = P(E) \cdot P(F)$$

Now, $$P(E \cap F') + P(E \cap F) = P(E)$$

$$P(E \cap F') = P(E) - P(E \cap F)$$

$$= P(E) - P(E)P(F)$$

$$= P(E)[1 - P(F)]$$

or $$P(E \cap F') = P(E)P(F')$$

Thus E and F' are also independent events.

Hence proved.

LONG ANSWER QUESTIONS

72. A and B throw a pair of dice alternately. A wins the game, if he gets a total of 7 and B wins the game, if he gets a total of 10. If A starts the game, then find the probability that B wins.

Sol : Delhi 2016

Here $$n(S) = 6 \times 6 = 36$$

Let A be event of getting a sum of 7 in pair of dice

$$A = \{(1,6),(2,5),(3,4),(6,1),(5,2),(4,3)\}$$

$$n(A) = 6$$

and B be event of getting a sum of 10 in pair of dice

$$A = \{(4,6),(5,5),(6,4)\}$$

$$n(B) = 3$$

$$P(A) = \frac{n(A)}{n(S)} = \frac{6}{36} = \frac{1}{6}$$

and $$P(B) = \frac{n(B)}{n(S)} = \frac{3}{36} = \frac{1}{12}$$

$$P(\overline{A}) = 1 - \frac{1}{6} = \frac{5}{6}$$

and $$P(\overline{B}) = 1 - \frac{1}{12} = \frac{11}{12}$$

Now, the probability that if A start the game, then B wins,

$$P(B\, wins) = P(\overline{A} \cap \overline{B}) + P(\overline{A} \cap \overline{B} \cap \overline{A} \cap B)$$
$$+ P(\overline{A} \cap \overline{B} \cap \overline{A} \cap \overline{B} \cap \overline{A} \cap B) + \ldots$$
$$= P(\overline{A})P(B) + P(\overline{A})P(\overline{B})P(\overline{A})P(B)$$
$$+ P(\overline{A})P(\overline{B})P(\overline{A})P(\overline{B})P(\overline{A})P(B) + \ldots$$

[events are independent]

$$= \frac{5}{6} \times \frac{1}{12} + \frac{5}{6} \times \frac{11}{12} \times \frac{5}{6} \times \frac{1}{12}$$
$$+ \frac{5}{6} \times \frac{11}{12} \times \frac{5}{6} \times \frac{11}{12} \times \frac{5}{6} \times \frac{1}{12} + \ldots$$
$$= \frac{5}{72} + \frac{5}{72} \times \frac{55}{72} + \frac{5}{72} \times \left(\frac{55}{72}\right)^2 + \ldots$$
$$= \frac{5}{72}\left[1 + \frac{55}{72} + \left(\frac{55}{72}\right)^2 + \ldots\right]$$
$$= \frac{5}{72}\left(\frac{1}{1 - \frac{55}{72}}\right)$$
$$= \frac{5}{72}\left(\frac{1}{\frac{17}{72}}\right) = \frac{5}{17}$$

73. A and B throw a pair of dice alternately, till one of them gets a total of 10 and wins the game. Find their respectively probabilities of winning, if A starts first.

Sol : OD 2016

Here, $$n(S) = 6 \times 6 = 36$$

Let E be event of getting a total 10

$$E = \{(4,6),(5,5),(6,4)\}$$

$$n(E) = 3$$

Probability of getting a total of 10,

$$P(E) = \frac{n(E)}{n(S)} = \frac{3}{36} = \frac{1}{12}$$

Probability of not getting a total of 10,

$$P(\overline{E}) = 1 - P(E)$$
$$= 1 - \frac{1}{12} = \frac{11}{12}$$

Thus, $P(A$ getting 10$) = P(B$ getting 10$)$

$$= \frac{1}{12}$$

and $P(A$ is not getting 10$) = P(B$ is not getting 10$)$

$$= \frac{11}{12}$$

Let B be event that at-least one of the children is a boy

$$B = \{Bb, Gg, Gb\} \text{ and } n(B) = 3$$

Now, $$P(B) = \frac{n(B)}{n(S)} = \frac{3}{4} \quad \text{...(i)}$$

Here, $A \cap B = \{Bb\}$, then $n(A \cap B) = 1$

$$P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{1}{4} \quad \text{...(ii)}$$

Now $$P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3} \quad \text{...(iii)}$$

Hence, the required probability is $\frac{1}{3}$.

(ii) the older child is a boy.

Let C be the event that the older child is a boy.

Then, $C = \{Bb, Bg\}$

$n(C) = 2$

and $$P(C) = \frac{n(C)}{n(S)} = \frac{2}{4} = \frac{1}{2} \quad \text{...(iv)}$$

Here, $A \cap C = \{Bb\}$, then $n(A \cap C) = 1$

and $$P(A \cap C) = \frac{n(A \cap C)}{n(C)} = \frac{1}{4} \quad \text{...(v)}$$

Now, $$P\left(\frac{A}{C}\right) = \frac{P(A \cap C)}{P(C)} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$$

Hence, the required probability is $\frac{1}{2}$.

76. Assume that each born child is equally likely to be a boy or a girl. If a family has two children, then what is the conditional probability that both are girls? Given that

(i) the youngest is a girls?

(ii) atleast one a girl?

Sol : Delhi 2014

Let B and b represent elder and younger boy child. Also, G and g represent elder and younger girl child. If a family has two children, then all possible cases are

$$S = \{Bb, Bg, Gg, Gb\}$$

$$n(S) = 4$$

Let us define event A Both children are girls,

then $A = \{Gg\} \Rightarrow n(A) = 1$

(i) the youngest is a girls?

Let E_1 be the event that youngest child is a girl.

Now $E_1 = \{Gg, Gg\}$ and $= n(E_1) = 2$

so $$P(E_1) = \frac{n(E_1)}{n(S)} = \frac{2}{4} = \frac{1}{2}$$

and $A \cap E_1 = \{Gg\} \Rightarrow n(A \cap E_1) = 1$

so $$P(A \cap E_1) = \frac{n(A \cap E_1)}{n(S)} = \frac{1}{4}$$

Now, $$P\left(\frac{A}{E_1}\right) = \frac{P(A \cap E_1)}{P(E_1)} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$$

Thus required probability is $\frac{1}{2}$.

(ii) atleast one a girl?

Let E_2 be the event that atleast one is girl.

$$E_2 = \{Bg, Gg, Gb\} \Rightarrow n(E_2) = 3,$$

so $$P(E_2) = \frac{n(E_2)}{n(S)} = \frac{3}{4}$$

and $(A \cap E_2) = \{Gg\}$

$n(A \cap E_2) = 1$

so $$P(A \cap E_2) = \frac{P(A \cap E_2)}{n(S)} = \frac{1}{4}$$

Now, $$P\left(\frac{A}{E_2}\right) = \frac{P(A \cap E_2)}{P(E_2)} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

Thus required probability is $\frac{1}{3}$.

77. A speaks truth in 75% of the cases, while B in 90% of the cases. In what percent of cases are they likely to contradict each other in stating the same fact? Do you think that statement of B is true?

Sol : Delhi 2013, OD 2007

Let A be event that A speaks truth and B be event that B speaks truth.

We have $P(A) = \frac{75}{100}$,

then $$P(\overline{A}) = 1 - P(A) = 1 - \frac{75}{100} = \frac{25}{100}$$

and $P(B) = \frac{90}{100}$,

Then, $$P(\overline{B}) = 1 - \frac{90}{100} = \frac{10}{100}$$

Here event A and B are independent events.

$P(A$ and B are contradict to each other)

$$= P(A \cap \overline{B}) + P(\overline{A} \cap B)$$

$$= P(A) \cdot P(\overline{B}) + P(\overline{A}) \cdot P(B)$$

$$= \frac{75}{100} \times \frac{10}{100} + \frac{25}{100} \times \frac{90}{100}$$

$$= \frac{750 + 2250}{10000} = \frac{3000}{10000} = \frac{3}{10}$$

Percentage of P (A and B are contradict to each other) $= \frac{3}{10} \times 100 = 30\%$

Since, B speaks truth in only 90% (i.e. not 100%) of

Now, If $P(B) = \frac{1}{6}$, then

$$P(A) = \frac{1}{30} + \frac{1}{6} = \frac{1+5}{30} = \frac{6}{30} = \frac{1}{5}.$$

and if $P(B) = \frac{4}{5}$, then $P(A) = \frac{1}{30} + \frac{4}{5}$

$$= \frac{1+24}{30} = \frac{25}{30} = \frac{5}{6}$$

81. Consider the experiment of tossing a coin. If the coin shows head, toss it again, but if is shows tail, then throw a die. Find the conditional probability of the event that 'the die shows a number greater than 4', given that 'there is atleast one tail'.

Sol : Comp 2014

The sample space S of the experiment is given as

$S = \{(H,H),(H,T),(T,1),(T,2),(T,3),(T,4),(T,5),(T,6)\}$

The probabilities of these elementary events are

$$P\{(H,H)\} = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4},$$

$$P\{(H,T)\} = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4},$$

$$P\{(H,1)\} = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12},$$

$$P\{(T,2)\} = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12},$$

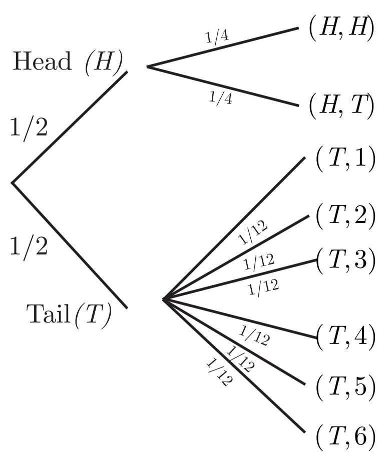
$$P\{(T,3)\} = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12},$$

$$P\{(T,4)\} = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12},$$

$$P\{(T,5)\} = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12} \text{ and}$$

$$P\{(T,6)\} = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$$

The outcomes of the experiment can be represented in the following tree diagram.



Let the following event be defined

$A =$ the die shows a number greater than 4 and

$B =$ there is atleast one tail.

Now,

$$A = \{(T,5),(T,6)\},$$

$$B = \{(H,T),(T,1),(T,2),(T,3),(T,4),(T,5),(T,6)\}$$

and $\quad A \cap B = \{(T,5),(T,6)\}$

$$P(B) = P((H,T) + P(T,1) + P(T,2))$$

$$+ P((T,3)) + P((T,4)) + P((T,5)) + P((T,6))$$

$$P(B)$$

$$= \frac{1}{4} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} = \frac{3}{4}$$

and $\quad P(A \cap B) = P((T,5)) + P((T,6))$

$$= \frac{1}{12} + \frac{1}{12} = \frac{1}{6}$$

Required probability

$$P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{6}}{\frac{3}{4}} = \frac{4}{18} = \frac{2}{9}$$

82. Suppose a girls throws a die. If she gets 1 or 2, she tosses a coin three times and notes the number of tails. If she gets 3, 4, 5 or 6, she tosses a coin once gets notes whether a 'head' or 'tail' is obtained. If she obtained exactly one 'tail', what is the probability that she threw 3, 4 ,5 or 6 with the die?

Sol : OD 2018

Let E_1 be the event that the girl gets 1 or 2, E_2 be the event that the girl gets 3, 4, 5 or 6 and A be the event that the girl gets exactly a tail.

Then, $\quad P(E_1) = \frac{2}{6} = \frac{1}{3}$

and $\quad P(E_2) = \frac{4}{6} = \frac{2}{3}$

Probability of getting exactly one tail when a coin is tossed three times)

$$P\left(\frac{A}{E_1}\right) = \frac{3}{8}$$

Probability of getting exactly a tail when a coin is tossed once

$$P\left(\frac{A}{E_2}\right) = \frac{1}{2}$$

Now, required probability

$$P\left(\frac{E_2}{A}\right) = \frac{P(E_2) \cdot P\left(\frac{A}{E_2}\right)}{P(E_1) \cdot P\left(\frac{A}{E_1}\right) + P(E_2) \cdot P\left(\frac{A}{E_2}\right)}$$

$$= \frac{\frac{2}{3} \cdot \frac{1}{2}}{\frac{1}{3} \cdot \frac{3}{8} + \frac{2}{3} \cdot \frac{1}{2}} = \frac{\frac{1}{3}}{\frac{1}{8} + \frac{1}{3}} = \frac{8}{11}$$

83. Three persons A, B and C apply for a job of manager in a private company.

Chances of their selection $(A, B \text{ and } C)$ are in the ratio 1 2 4. The probabilities that A, B and

$$8k = 1 \Rightarrow k = \frac{1}{8}$$

Now, $$P(X = x) = \begin{cases} \frac{1}{8}x, & \text{if } x = 0 \text{ or } 1 \\ \frac{1}{4}x, & \text{if } x = 2 \\ \frac{1}{8}(5 - x), & \text{if } x = 3 \text{ or } 4 \\ 0, & \text{if } x > 4 \end{cases} \quad ...(i)$$

(i) Probability of getting admission in exactly one college,

$$P(X = 1) = \frac{1}{8}$$

(ii) Probability of getting admission in at most 2 colleges,

$$P(X \le 2) = P(X = 0) + P(X = 1) + P(X = 2)$$
$$= 0 + \frac{1}{8} + \frac{2}{4} = \frac{5}{8}$$

(iii) Probability of getting admission in at least 2 colleges)

$$P(X \ge 2) = 1 - P(X < 2)$$
$$= 1 - [P(X = 0) + P(X = 1)]$$
$$= 1 - \left[0 + \frac{1}{8}\right] = 1 - \frac{1}{8} = \frac{7}{8}$$

86. A bag A is contains 4 black and 6 red balls and bag B contains 7 black and 3 red balls. A die is thrown. If 1 or 2 appears on it, then bag A is chosen, otherwise bag B. It two balls are drawn at random (without replacement) from the selected bag, find the probability of one them being red and another black.

Sol : Delhi 2015

Bag A contains 4 black and 6 red balls and bag B contains 7 black and 3 red balls.

Let the following event be defined

$E_1 =$ Event that die show 1 or 2

$E_2 =$ Event that die show 3 or 4 or 5 or 6

$E =$ Event that among two drawn balls, one of them is red and other is black

Here, $P(E_1) = \frac{2}{6}$, $P(E_2) = \frac{4}{6}$

Probability of getting one red and one black from bag A

$$P\left(\frac{E}{E_1}\right) = \frac{{}^4C_1 \times {}^6C_1}{{}^{10}C_2} = \frac{4 \times 6 \times 2}{10 \times 9}$$

Probability of getting one red and one black from bag B

$$P\left(\frac{E}{E_2}\right) = \frac{{}^7C_1 \times {}^3C_1}{{}^{10}C_2} = \frac{7 \times 3 \times 2}{10 \times 9}$$

Now, by theorem of total probability,

$$P(E) = P(E_1) \cdot P\left(\frac{E}{E_1}\right) + P(E_2) \cdot P\left(\frac{E}{E_2}\right)$$
$$= \frac{2}{6} \cdot \left(\frac{4 \times 6 \times 2}{10 \times 9}\right) + \frac{4}{6} \cdot \left(\frac{7 \times 3 \times 2}{10 \times 9}\right)$$
$$= \frac{4 \times 6}{6 \times 10 \times 9}(4 + 7)$$
$$= \frac{4 \times 6 \times 11}{6 \times 10 \times 9} = \frac{22}{45}$$

87. Three machines E_1, E_2 and E_3 in a certain factory producing electric bulbs, produce 50%, 25% and 25% respectively, of the total daily output of electric bulbs. it is known that 4% of the bulbs produced by each of machines E_1 and E_2 are defective E_3 are defective. If one bulb is picked up at random from a day's production, calculate the probability that it is defective.

Sol : Foreign 2015

Let the following event be defined

A_1 Event that the bulb is produced by E_1

A_2 Event that the bulb is produced by E_2

A_3 Event that the bulb is produced by E_3

A Event that the picked up bulb is defective

Here, $$P(A_1) = 50\% = \frac{50}{100} = \frac{1}{2}$$
$$P(A_2) = 25\% = \frac{25}{100} = \frac{1}{4},$$
$$P(A_3) = 25\% = \frac{25}{100} = \frac{1}{4},$$

Also, $$P\left(\frac{A}{A_1}\right) = 4\% = \frac{4}{100} = \frac{1}{25}$$
$$P\left(\frac{A}{A_2}\right) = 4\% = \frac{4}{100} = \frac{1}{25}$$

and $$P\left(\frac{A}{A_3}\right) = 5\% = \frac{5}{100} = \frac{1}{20}$$

The probability that the picked bulb is defective,

$$P(A) = P(A_1) \times P\left(\frac{A}{A_1}\right) + P(A_2) \times P\left(\frac{A}{A_2}\right) + P(A_3) \times P\left(\frac{A}{A_3}\right)$$
$$= \frac{1}{2} \times \frac{1}{25} + \frac{1}{4} \times \frac{1}{25} + \frac{1}{4} \times \frac{1}{20}$$
$$= \frac{1}{50} + \frac{1}{100} + \frac{1}{80}$$
$$= \frac{8 + 4 + 5}{400} = \frac{17}{400} = 0.0425$$

88. There are three coins. One is two-headed coin, another is biased coin that comes up heads 75% of the time and the third is an unbiased coin. One of three coin is chosen at random and tossed. If it shows heads, what is the probability that it is the two-headed coin?

Sol : OD 2019, Delhi 2012

Let the following event be defined

E_1 Event of selecting two headed coin

Now, the probability that selected bolt which is defective, is manufactured by machine B

$$P\left(\frac{E_2}{E}\right) = \frac{P(E_2)\cdot P\left(\frac{E}{E_2}\right)}{P(E_1)\cdot P\left(\frac{E}{E_1}\right)+P(E_2)\cdot P\left(\frac{E}{E_2}\right)+P(E_3)\cdot P\left(\frac{E}{E_3}\right)}$$

$$= \frac{\frac{50}{100}\times\frac{4}{100}}{\frac{30}{100}\times\frac{3}{100}+\frac{50}{100}\times\frac{4}{100}+\frac{20}{100}\times\frac{1}{100}}$$

$$= \frac{200}{90+200+20} = \frac{200}{310}$$

The probability that selected bolt which is defective, is not manufactured by machine B

$$= 1 - P\left(\frac{E_2}{E}\right)$$

$$= 1 - \frac{200}{310} = \frac{110}{310} = \frac{11}{31}$$

94. In answering a question on a sample choice test, a student either knows the answer or guesses. Let $\frac{3}{5}$ be the probability that he knows the answer and $\frac{2}{5}$ be the probability that he guesses. Assuming that a student who guesses at the answer will be correct with probability $\frac{1}{3}$, what is the probability that the student knows the answer given that he answered it correctly?

Sol : Comp 2015, OD 2007

Let the following event be defined

E_1 Event that the student knows the answer

E_2 Event that the student guesses the answer

E Event that the answer is correct

Here E_1 and E_2 are mutually exclusive and exhaustive events.

Thus $P(E_1) = \frac{3}{5}$ and $P(E_2) = \frac{2}{5}$

Probability that the student answered correctly, given he knows the answer

$$P\left(\frac{E}{E_1}\right) = 1$$

Probability that the student answered correctly, given he guesses

$$P\left(\frac{E}{E_2}\right) = \frac{1}{3}$$

The probability that the student knows the answer given that the answered it correctly is given by $P\left(\frac{E_1}{E_2}\right)$.

$$P\left(\frac{E_1}{E}\right) = \frac{P\left(\frac{E}{E_1}\right)\cdot P(E_1)}{P\left(\frac{E}{E_1}\right)P(E_1)+P\left(\frac{E}{E_2}\right)\cdot P(E_2)}$$

$$= \frac{1\times\frac{3}{5}}{1\times\frac{3}{5}+\frac{1}{3}\times\frac{2}{5}} = \frac{\frac{3}{5}}{\frac{3}{5}+\frac{2}{15}}$$

$$= \frac{\frac{3}{5}}{\frac{15+10}{15}} = \frac{3\times 3}{25} = \frac{9}{25}$$

95. A bag contains 4 red and 4 black balls, another bag contains 2 red and 6 black balls. One of the two bags is selected at random and two balls bags is selected at random without replacement from the bag and are found to be both red. Find the probability that the balls are drawn from the first bag.

Sol : Foreign 2015

Let the following event be defined

$E_1 =$ Event that first bag is chosen,

$E_2 =$ Event that second bag is chosen

$R =$ Event that two balls drawn at random are red.

Since, one of the bag is chosen at random and probability of selecting the bag is equal,

$$P(E_1) = P(E_2) = \frac{1}{2}$$

Let E_1 has already occurred, i.e. first bag is chosen, the probability of drawing two red balls in this case

$$P\left(\frac{R}{E_1}\right) = \frac{{}^4C_2}{{}^8C_2} = \frac{\frac{4\times 3}{2\times 1}}{\frac{8\times 7}{2\times 1}} = \frac{3}{14}$$

Similarly, $P\left(\frac{R}{E_2}\right) = \frac{{}^2C_2}{{}^8C_2} = \frac{1}{\frac{8\times 7}{2\times 1}} = \frac{1}{28}$

By Baye's theorem,

$$P\left(\frac{E_1}{R}\right) = \frac{P(E_1)\cdot P\left(\frac{R}{E_1}\right)}{P(E_1)\cdot P\left(\frac{R}{E_1}\right)+P(E_2)\cdot P\left(\frac{R}{E_2}\right)}$$

$$= \frac{\frac{1}{2}\times\frac{3}{14}}{\frac{1}{2}\times\frac{3}{14}+\frac{1}{2}\times\frac{1}{28}}$$

$$= \frac{\frac{3}{14}}{\frac{3}{14}+\frac{1}{28}} = \frac{\frac{3}{14}}{\frac{7}{28}} = \frac{6}{7}$$

96. There are three coins. One is a two headed coin (having head on both faces), another is a biased coin that comes up heads 75% of the times and third is also a biased coin that comes up tails 40% of the times. One of the three coins is chosen at random and tossed and it shows head. What is the probability that it was the two headed coin?

Sol : OD 2014

Let the following event be defined

$E_1 =$ Event of selecting two headed coin

$E_2 =$ Event of selecting second biased coin

$E_3 =$ Event of selecting third biased coin

$$= \frac{\frac{1}{6} \times \frac{3}{5}}{\frac{1}{6} \times \frac{3}{5} + \frac{5}{6} \times \frac{2}{5}} = \frac{3}{13}$$

99. A card from a pack of 52 playing cards is lost. From the remaining cards of the pack three cards are drawn at random (without replacement) and are found to be all spades. Find the probability of the lost card being a spade.

Sol : Delhi 2014, OD 2010

Let the following event be defined

$E_1 =$ Event that lost card is a spade card

$E_2 =$ Event that lost card is not a spade card

$A =$ Event that drawn cards are spade cards

Now $P(E_1) = \frac{13}{52} = \frac{1}{4},$

$$P(E_2) = \frac{39}{52} = \frac{3}{4},$$

$$P\left(\frac{A}{E_1}\right) = \frac{{}^{12}C_3}{{}^{51}C_3} = \frac{220}{20825}$$

and $P\left(\frac{A}{E_2}\right) = \frac{{}^{13}C_3}{{}^{51}C_3} = \frac{286}{20825}$

Now,required probability

$$P\left(\frac{E_1}{A_1}\right) = \frac{P(E_1)\cdot P\left(\frac{A}{E_1}\right)}{P(E_1)\cdot P\left(\frac{A}{E_1}\right) + P(E_2)\cdot P\left(\frac{A}{E_2}\right)}$$

$$= \frac{\frac{1}{4}\cdot\frac{220}{20825}}{\frac{1}{4}\cdot\frac{220}{20825} + \frac{3}{4}\cdot\frac{286}{20825}}$$

$$= \frac{220}{220+858} = \frac{220}{1078} = \frac{20}{98} = \frac{10}{49}$$

100. Assume that the changes of a patient having a heart attack is 40%. Assuming that a meditation and yoga course reduces the risk of heart attack by 30% and prescription of certain drug reduces its chance by 25%. At a time, a patient can choose anyone of the two options with equal probabilities. It is given that after going through one of two options, the patient selected at random suffers a heart attack. Find the probability that the patient followed a course of meditation and yoga. Interpret the result and state which of the above stated methods, is more beneficial for the patient?

Sol : Delhi 2013

Let the following event be defined

E_1: The patient follows meditation and yoga.

E_2: The patient uses drug.

E: The selected patient suffers a heart attack.

Here E_1 and E_2 are mutually exclusive

Now $P(E_1) = P(E_2) = \frac{1}{2}$

Then, $P\left(\frac{E}{E_1}\right) = \frac{40}{100}\left(1 - \frac{30}{100}\right)$

$$= \frac{40}{100} \times \frac{70}{100} = \frac{28}{100}$$

and $P\left(\frac{E}{E_2}\right) = \frac{40}{100}\left(1 - \frac{25}{100}\right)$

$$= \frac{40}{100} \times \frac{75}{100} = \frac{30}{100}$$

Probability of event that patient who suffers heart attack follows meditation and yoga) $= P(E_1/E)$

$$P\left(\frac{E_1}{E}\right) = \frac{P\left(\frac{E}{E_1}\right)\cdot P(E_1)}{P\left(\frac{E}{E_1}\right)\cdot P(E_1) + P\left(\frac{E}{E_2}\right)\cdot P(E_2)}$$

$$= \frac{\frac{28}{100}\times\frac{1}{2}}{\frac{28}{100}\times\frac{1}{2} + \frac{30}{100}\times\frac{1}{2}} = \frac{28}{58} = \frac{14}{29}$$

Yoga course and meditation are more beneficial for the heart patient.

101. An insurance company insured 2000 scooter drivers, 4000 car drivers and 6000 truck drivers, the probability of their meeting an accident respectively are 0.01, 0.03 and 0.15. One of the insured persons meets with an accident. What is the probability that he is a car driver?

Sol : Comp 2012

Let us define the events as

E_1 Event that insured person is a scooter driver

E_2 Event that insured person is a car driver

E_3 Event that insured person is a truck driver

A Event that insured person meets with an accident

Now $n(E_1) = 2000$, $n(E_2) = 4000$ and $n(E_3) = 6000$

Here, total insured person, $n(S) = 12000$

Probability that the insured person is a scooter driver

$$P(E_1) = \frac{n(E_1)}{n(S)} = \frac{2000}{12000} = \frac{1}{6}$$

Probability that the insured person is a car driver,

$$P(E_2) = \frac{n(E_2)}{n(S)} = \frac{4000}{12000} = \frac{1}{3}$$

Probability that the insured person is a truck driver,

$$P(E_3) = \frac{n(E_3)}{n(S)} = \frac{6000}{12000} = \frac{1}{2}$$

Probability that scooter driver meets with an accident

$$P\left(\frac{A}{E_1}\right) = 0.01$$

Probability that car driver meets with an accident

Probability of getting 5 or 6 on a die,

$$P(E_1) = \frac{2}{6} = \frac{1}{3}$$

Probability of getting 1, 2, 3 or 4 on a die

$$P(E_2) = \frac{4}{6} = \frac{2}{3}$$

Probability that girl gets exactly on head when she throws coin thrice

$$P\left(\frac{A}{E_1}\right) = \frac{3}{8}$$

Probability that girl gets exactly one head when she shows coin one

$$P\left(\frac{A}{E_2}\right) = \frac{1}{2}$$

The probability that she throws 1, 2, 3 or 4 with the die for getting exactly one head,

$$P\left(\frac{E_2}{A}\right) = \frac{P(E_2)\cdot P\left(\frac{A}{E_2}\right)}{P(E_1)\cdot P\left(\frac{A}{E_1}\right) + P(E_2)\cdot P\left(\frac{A}{E_2}\right)}$$

$$= \frac{\frac{2}{3}\times\frac{1}{2}}{\left(\frac{1}{3}\times\frac{1}{8}\right)+\left(\frac{2}{3}\times\frac{1}{2}\right)}$$

$$= \frac{\frac{1}{3}}{\frac{1}{8}+\frac{1}{3}} = \frac{\frac{1}{3}}{\frac{3+8}{243}} = \frac{1}{3}\times\frac{24}{11} = \frac{8}{11}$$

105. A girl throws a die. If she gets a 5 or 6, she tosses a coin three times and notes the number of heads. If she gets 1, 2, 3 or 4, she tosses a coin two times and notes the number of heads obtained. If she obtained exactly two heads, what is the probability that she threw 1, 2, 3 with the die?

Sol : Delhi 2012

Let the following event be defined

E_1 Girl gets 5 or 6 on a die

E_2 Girl gets 1, 2, 3 or 4 on a die

A She gets exactly one head

Probability of getting 5 or 6 on a die,

$$P(E_1) = \frac{2}{6} = \frac{1}{3}$$

Probability of getting 1, 2, 3 or 4 on a die

$$P(E_2) = \frac{4}{6} = \frac{2}{3}$$

Probability that girl gets exactly two head when she throws coin thrice

$$P\left(\frac{A}{E_1}\right) = \frac{3}{8}$$

Probability that girl gets exactly two head when she shows coin two times

$$P\left(\frac{A}{E_2}\right) = \frac{1}{4}$$

The probability that she throws 1, 2, 3 or 4 with the die for getting exactly one head,

$$P\left(\frac{E_2}{A}\right) = \frac{P(E_2)\cdot P\left(\frac{A}{E_2}\right)}{P(E_1)\cdot P\left(\frac{A}{E_1}\right) + P(E_2)\cdot P\left(\frac{A}{E_2}\right)}$$

$$= \frac{\frac{2}{3}\times\frac{1}{4}}{\frac{1}{3}\times\frac{3}{8}+\frac{2}{3}\times\frac{1}{4}}$$

$$= \frac{2\times 2}{3+2\times 2} = \frac{4}{7}$$

106. Suppose 5% of men and 0.25% of women have grey hair. A grey haired person is selected at random. What is the probability of this person being male? Assume that there are equal number of males and females.

Sol : Delhi 2011, Foreign 2010

Let the following event be defined

E_1 Event that person selected is a male

E_2 Event that person selected is a female

A Event that person selected has grey hair

Since there are equal number of males and females, probability that person selected is a male,

$$P(E_2) = \frac{1}{2}$$

Probability of selecting a person having grey hair is male

$$P\left(\frac{A}{E_1}\right) = \frac{5}{100}$$

Probability of selecting a person having grey hair is female

and $$P\left(\frac{A}{E_2}\right) = \ = \frac{0.25}{100}$$

The probability of selecting person is a male having grey hair,

$$P\left(\frac{E_1}{A}\right) = \frac{P(E_1)\cdot P\left(\frac{A}{E_1}\right)}{P(E_1)\cdot P\left(\frac{A}{E_1}\right) + P(E_2)\cdot P\left(\frac{A}{E_2}\right)}$$

$$= \frac{\frac{1}{2}\times\frac{5}{100}}{\left(\frac{1}{2}\times\frac{5}{100}\right)+\left(\frac{1}{2}\times\frac{0.25}{100}\right)}$$

$$= \frac{\frac{5}{100}}{\frac{5}{200}+\frac{0.25}{200}} = \frac{5}{5+0.25}$$

$$= \frac{5}{5.25} = \frac{500}{525} = \frac{100}{105} = \frac{20}{21}$$

Hence, the required probability is $\frac{20}{21}$.

E_1 Event that box I is selected

E_2 Event that box II is selected

E_3 Event that box II is selected

A Event that the drawn coin is gold coin

Since events E_1, E_2 and E_3 are mutually exclusive and exhaustive events.

$$P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$$

Since box I contain both gold coins, probability that a gold coin is drawn from box I,

$$P\left(\frac{A}{E_1}\right) = \frac{2}{2} = 1$$

Since box II contain both silver coins, probability that a gold coin is drawn from box II,

$$P\left(\frac{A}{E_2}\right) = \frac{2}{2} = 0$$

Since box III contain one silver coin and one gold, probability that a gold coin is drawn from box III,

$$P\left(\frac{A}{E_3}\right) = \frac{1}{2}$$

Probability that the drawing gold coin from bag I,

$$P\left(\frac{E_1}{A}\right) = \frac{P(E_1)\cdot P\left(\frac{E_1}{A}\right)}{P(E_1)\cdot P\left(\frac{A}{E_1}\right) + P(E_2)\cdot P\left(\frac{A}{E_2}\right) + P(E_3)\cdot P\left(\frac{A}{E_3}\right)}$$

$$= \frac{\frac{1}{3}\times 1}{\left(\frac{1}{3}\times 1\right) + \left(\frac{1}{3}\times 0\right) + \left(\frac{1}{3}\times\frac{1}{2}\right)}$$

$$= \frac{1}{1+0+\frac{1}{2}} = \frac{1}{\frac{3}{2}} = \frac{2}{3}$$

Hence, the required probability is $\frac{2}{3}$.

111. There are three coins. One is a two tailed coin (having tail on both faces), another is a biased coin that comes up heads 60% of the times and third is an unbiased coin. One of the three coins is chosen at random and tossed and it shows tail. What is the probability that it is a two tailed coin?

Sol : Comp 2011

Let the following event be defined

$E_1 =$ Event of selecting two tailed coin

$E_2 =$ Event of selecting biased coin

$E_3 =$ Event of selecting unbiased coin

$T =$ Event of getting tail

Since probability of selecting the coin is equal, we have

$$P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$$

Now $P\left(\frac{T}{E_1}\right) = 1$

$$P\left(\frac{T}{E_2}\right) = \frac{100-60}{100} = \frac{40}{100} = \frac{2}{5}$$

$$P\left(\frac{T}{E_3}\right) = \frac{1}{2}$$

By Baye's theorem

$$P\left(\frac{E_1}{T}\right) = \frac{P(E_1)\cdot P\left(\frac{T}{E_1}\right)}{P(E_1)\cdot P\left(\frac{T}{E_1}\right) + P(E_2)\cdot P\left(\frac{T}{E_2}\right) + P(E_3)\cdot P\left(\frac{T}{E_3}\right)}$$

$$= \frac{\frac{1}{3}\times 1}{\frac{1}{3}\times 1 + \frac{1}{3}\times\frac{2}{5} + \frac{1}{3}\times\frac{1}{2}} = \frac{1}{1+\frac{2}{5}+\frac{1}{2}}$$

$$= \frac{10}{19}$$

112. In a class, 5% of boys and 10% of girls have an IQ of more than 150. In the class, 60% are boys and rest are girls. If a student is selected at random and found to have an IQ of more than 150, then find the probability that the student is a boy.

Sol : OD 2010, Delhi 2008

Let the following event be defined

B Event that boy is selected

G Event that girl is selected

I Event that the student has an IQ of more than 150

Since in the class 60% students are boys, so 40% are girls

$$P(B) = 60\% = \frac{60}{100}$$

$$P(G) = 40\% = \frac{40}{100}$$

Now, probability that boys has an IQ of more than 150,

$$P\left(\frac{I}{B}\right) = 5\% = \frac{5}{100}$$

Now, probability that girls has an IQ of more than 150,

$$P\left(\frac{I}{G}\right) = 10\% = \frac{10}{100}$$

The probability that the selected boy having IQ more than 150 is

$$P\left(\frac{B}{I}\right) = \frac{P(B)\cdot P(\frac{I}{B})}{P(B)\cdot P(\frac{I}{B}) + P(G)\cdot P(\frac{I}{G})}$$

$$= \frac{\frac{60}{100}\times\frac{5}{100}}{\left(\frac{60}{100}\times\frac{5}{100}\right)+\left(\frac{40}{100}\times\frac{10}{100}\right)}$$

$$= \frac{300}{300+400} = \frac{300}{700} = \frac{3}{7}$$

Hence, the required probability is 3/7.

113. In a bolt factory, machines A, B and C manufacturer 25%, 35% and 40% of total production, respectively. Out of their total output, 5%, 4% and 2% are defective bolts. A bolt is drawn at random and is found to be defective. What is the probability that it is manufactured by machine B?

Sol : Delhi 2010, Foreign 2008

Let the following event be defined

E_1 Selected bolt is manufactured by machine A,

E_2 Selected bolt is manufactured by machine B,

E_3 Selected bolt is manufactured by machine C,

E Selected bolt is defective.

Now $P(E_1) = 25\% = \frac{25}{100}$

$P(E_2) = 35\% = \frac{35}{100}$

and $P(E_3) = 40\% = \frac{40}{100}$

Also, given that 5%, 4% and 2% bolts manufactured by machine A, B and C respectively are defective. So,

$$P\left(\frac{E}{E_1}\right) = 5\% = \frac{5}{100}$$
$$P\left(\frac{E}{E_2}\right) = 4\% = \frac{4}{100}$$
$$P\left(\frac{E}{E_3}\right) = 2\% = \frac{2}{100}$$

Now, the probability that selected bolt which is defective, is manufactured by machine B

$$P\left(\frac{E_2}{E}\right) = \frac{P(E_2)\cdot P\left(\frac{E}{E_2}\right)}{P(E_1)\cdot P\left(\frac{E}{E_1}\right) + P(E_2)\cdot P\left(\frac{E}{E_2}\right) + P(E_3)\cdot P\left(\frac{E}{E_3}\right)}$$
$$= \frac{\frac{35}{100}\times\frac{4}{100}}{\frac{25}{100}\times\frac{5}{100}+\frac{35}{100}\times\frac{4}{100}+\frac{40}{100}\times\frac{2}{100}}$$
$$= \frac{140}{125+140+80} = \frac{140}{345} = \frac{28}{69}$$

114. A card from a pack of 52 cards is lost. From the remaining cards of the pack, two cards are drawn at random and are found to both of clubs. Find the probability of the lost card being of clubs.

Sol : Delhi 2010

Let the following event be defined

$E_1 =$ Event that lost card is a club card

$E_2 =$ Event that lost card is not a club card

$A =$ Event that drawn cards are club cards

Now $P(E_1) = \frac{13}{52} = \frac{1}{4}$,

$$P(E_2) = \frac{39}{52} = \frac{3}{4},$$

$$P\left(\frac{A}{E_1}\right) = \frac{{}^{12}C_2}{{}^{51}C_2} = \frac{\frac{12\times 11}{2\times 1}}{\frac{51\times 50}{2\times 1}}$$
$$= \frac{12\times 11}{51\times 50} = \frac{22}{425}$$

and $$P\left(\frac{A}{E_2}\right) = \frac{{}^{13}C_2}{{}^{51}C_2} = \frac{\frac{13\times 12}{2\times 1}}{\frac{51\times 50}{2\times 1}}$$
$$= \frac{13\times 12}{51\times 50} = \frac{26}{425}$$

Now,required probability

$$P\left(\frac{E_1}{A_1}\right) = \frac{P(E_1)\cdot P\left(\frac{A}{E_1}\right)}{P(E_1)\cdot P\left(\frac{A}{E_1}\right) + P(E_2)\cdot P\left(\frac{A}{E_2}\right)}$$
$$= \frac{\frac{1}{4}\cdot\frac{22}{425}}{\frac{1}{4}\cdot\frac{22}{425}+\frac{3}{4}\cdot\frac{26}{425}}$$
$$= \frac{22}{22+78} = \frac{22}{100} = \frac{11}{50}$$

CASE BASED QUESTIONS

115. A departmental store sends bills to charge its customers once a month. Past experience shows that 70% of its customers pay their first month bill in time. The store also found that the customer who pays the bill in time has the probability of 0.8 of paying in time next month and the customer who doesn't pay in time has the probability of 0.4 of paying in time the next month.

Based on the above information, answer the following questions:

(i) Let E_1 and E_2 respectively denote the event of customer paying or not paying the first month bill in time. Find $P(E_1)$, $P(E_2)$

(ii) Let A denotes the event of customer paying second month's bill in time, then find $P(A \mid E_1)$ and $P(A \mid E_2)$.

(iii) Find the probability of customer paying second month's bill in time.

(iv) Find the probability of customer paying first month's bill in time if it is found that customer has paid the second month's bill in time.

Sol : OD 2024

Let the following event be defined

E_1 = customer paying the first month bill on time.

E_2 = customer not paying the first month bill on time.

(i) $P(E_1)$ and $P(E_2)$

achieved a vaccination coverage rate of at least 80% for the 4:3:1:3:3:1 series.26 The probability that a randomly selected toddler in Alabama has received a full set of inoculations is 0.792, for a toddler in Georgia, 0.839, and for a toddler in Utah, 0.711.27 Suppose a toddler from each state is randomly selected.

(i). Find the probability that all three toddlers have received these inoculations.

(ii) Find the probability that none of the three has received these inoculations.

Sol :

We define the following three events:

A = toddler A from Alabama has received these inoculations;

G = toddler G from Georgia has received these inoculations; and

U = toddler U from Utah has received these inoculations.

(i) Assume these three events are independent

$$P(A \cap G \cap U) = P(A) \cdot P(G) \cdot P(U)$$
$$= (0.792)(0.839)(0.711)$$
$$= 0.4725$$

(ii) The probability that all three toddlers have received these inoculations is 0.4725.

None of the three has received inoculations means toddler A has not received the inoculations and toddler G has not received the inoculations and toddler U has not received the inoculations.

$$P(A' \cap G' \cap U') = P(A') \cdot P(G') \cdot P(|U')$$
$$= [1 - P(A)] \cdot [1 - P(G)] \cdot [1 - P(U)]$$
$$= (1 - 0.792)(1 - 0.839)(1 - 0.711)$$
$$= (0.208(0.161)(0.289)$$
$$= 0.0097$$

118. Quality assurance (QA) testing is the process of ensuring that manufactured product is of the highest possible quality for customers. QA is simply the techniques used to prevent issues with product and to ensure great user experience for customers.

A manufactured component has its quality graded on its performance, appearance, and cost. Each of these three characteristics is graded as either pass or fail. There is a probability of 0.40 that a component passes on both appearance and cost. There is a probability of 0.35 that a component passes on both performance and appearance. There is a probability of 0.31 that a component passes on all three characteristics. There is a probability of 0.64 that a component passes on performance. There is a probability of 0.19 that a component fails on all three characteristics. There is a probability of 0.06 that a component passes on appearance but fails on both performance and cost.

(i) What is the probability that a component passes on cost but fails on both performance and appearance?

(ii) If a component passes on both appearance and cost, what is the probability that it passes on all three characteristics?

(iii) If a component passes on both performance and appearance, what is the probability that it passes on all three characteristics?

Sol :

Let E be the event that the 'component passes on performance',

Let A be the event that the 'component passes on appearance', and let C be the event that the 'component passes on cost'.

$$P(A \cap C) = 0.4$$

probability 0.6, whereas this probability is 0.2 for a person who is not accident prone. The company knows that 20% of the population is accident prone.

On the basis of above information, answer the following questions.

(i) What is the probability that a new policyholder will have an accident within a year of purchasing a policy?

(ii) Suppose that a new policy holder has an accident within a year of purchasing a policy. What is the probability that he or she is accident prone?

Sol :

Let the following event be defined

$E_1 =$ The policy holder is accident prone

$E_2 =$ The policy holder is not accident prone

$E =$ The new policy holder has an accident within a year of purchasing a policy.

$$P(E_1) = \frac{20}{100}$$

and $$P(E_2) = \frac{80}{100}$$

$$P\left(\frac{E}{E_1}\right) = 0.6 = \frac{6}{10}$$

and $$P\left(\frac{E}{E_2}\right) = 0.2 = \frac{2}{10}$$

(i) $$P(E) = P(E_1) \times P(E/E_1) + P(E_2) \times P(E/E_2)$$

$$= \frac{20}{100} \times \frac{6}{10} + \frac{80}{100} \times \frac{2}{10}$$

$$= \frac{280}{1000} = \frac{7}{25}$$

(ii) By Baye's theorem,

$$P(E_1/E) = \frac{P(E_1) \times P(E/E_1)}{P(E)}$$

$$= \frac{\frac{20}{100} \times \frac{6}{10}}{\frac{7}{25}} = \frac{\frac{3}{25}}{\frac{7}{25}} = \frac{3}{7}$$

121. At its simplest, a fair die states that each of the faces has a similar probability of landing facing up. A standard fair six-sided die, for example, can be regarded as "fair" if each of the faces consists of a probability of 1/6.

A fair die is rolled. Consider the events $A = \{1,3,5\}$, $B = \{2,3\}$, and $C = \{2,3,4,5,\}$

On the basis of above information, answer the following questions.

(i) Find the probability $P(A/B)$ and $P(B/A)$.

(ii) Find the probability $P(A/C)$, $P(A \cap B/C)$ and $P(A \cup B/C)$

Sol :

(i) We have,

$$S = \{1, 2, 3, 4, 5, 6\}$$

$$A = \{1, 3, 5\}$$

$$B = \{2, 3\}$$

and $$C = \{2, 3, 4, 5\}$$

$$n(S) = 6$$

$$n(A) = 3$$

$$n(B) = 2$$

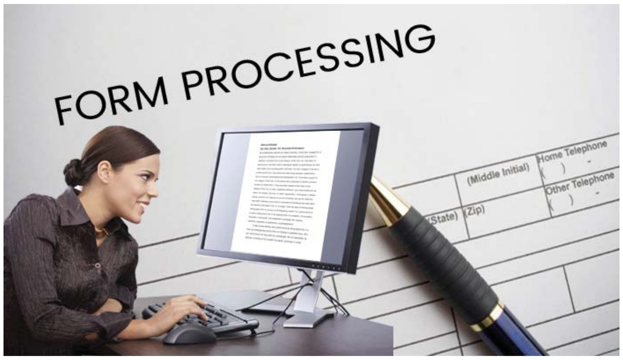
and $$n(C) = 4$$

$$P(A) = \frac{3}{6} = \frac{1}{2}, P(B)$$

$$= \frac{2}{6} = \frac{1}{3} P(C) = \frac{4}{6} = \frac{2}{3}$$

$$P(A \cap B) = \frac{1}{6}$$

and $$P(A \cap C) = \frac{2}{6} = \frac{1}{3}$$



Based on the above information answer the following questions.

(i) The total probability of committing an error in processing the form.

(ii) The manager of the company wants to do a quality check. During inspection he selects a form at random from the days output of processed forms. If the form selected at random has an error, the probability that the form is not processed by Vikas.

Sol :

(i) Required probability,

$$P(A) = P(E_1)\,P\Big(\frac{A}{E_1}\Big) + P(E_2)\,P\Big(\frac{A}{E_2}\Big) + P(E_3)\,P\Big(\frac{A}{E_3}\Big)$$

$$= 0.5 \times 0.06 + 0.2 \times 0.4 + 0.3 \times 0.3$$

$$= 0.030 + 0.008 + 0.009$$

$$= 0.047$$

(ii) Required probability,

$$P\Big(\frac{\overline{E_1}}{A}\Big) = 1 - P\Big(\frac{\overline{E_1}}{A}\Big)$$

$$= 1 - \left[\frac{P(E_1)\,P\Big(\frac{A}{E_1}\Big)}{P(E_1)\,P\Big(\frac{A}{E_1}\Big) + P(E_2)\,P\Big(\frac{A}{E_2}\Big) + P(E_3)\,P\Big(\frac{A}{E_3}\Big)}\right]$$

$$= 1 - \Big[\frac{0.5 \times 0.6}{0.5 \times 0.06 + 0.2 \times 0.04 + 0.3 \times 0.03}\Big]$$

$$= 1 - \frac{0.030}{0.047} = 1 - \frac{30}{47} = \frac{17}{47}$$

124. Joint Entrance Examination – Advanced, is an academic examination held annually in India. It is organised by one of the seven zonal IITs under the guidance of the Joint Admission Board on a round-robin rotation pattern for the qualifying candidates of the JEE-Main. A candidate can attempt JEE (Advanced) maximum of two times in two consecutive years. A successful candidates get the admission in any IITs of India based on merit.

Applicants have a 0.26 probability of passing IIT advanced test when they take it for the first time, and if they pass it they can get admission in IIT. However, if they fail the test the first time, they must take the test a second time, and when applicants take the test for the second time there is a 0.43 chance that they will pass and be allowed to get admission. Applicants are rejected if the test is failed on the second attempt.

(i) What is the probability that an applicant gets admission in IIT but needs two attempts at the test?

(ii) What is the probability that an applicant gets admission in IIT?

(iii) If an applicant gets admission in IIT, what is the probability that he or she passed the test on the first attempt?

Sol :

Let F be the event that applicant pass the advanced in first time.

Let S be the event that applicant pass the advanced in second time.

Let A be the event that applicant pass the advanced.

(i) Now $\quad P(F) = 0.26$

$$P(S) = 0.43$$

(i) Probability that an applicant gets admission in IIT but needs two attempts at the test.

$$P(F' \cup S) = P(F') \times P(S)$$

$$= (1 - 0.26) \times 0.43$$

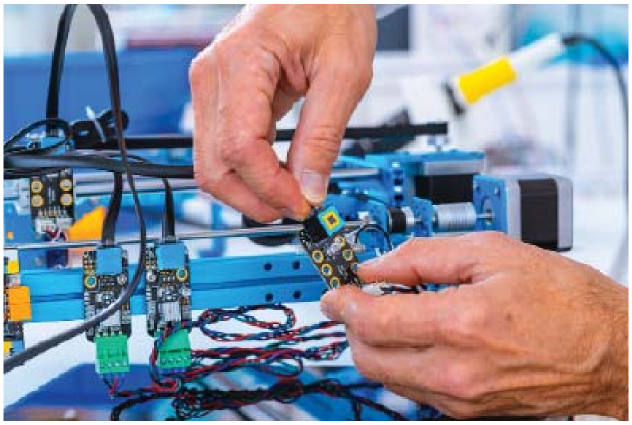
$$= 0.3182$$

(ii) Probability that an applicant gets admission in IIT,

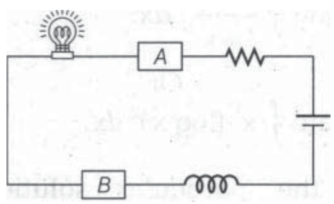
$$P(A) = 1 - P(F' \cup S')$$

$$= 1 - (1 - 0.26) \times (1 - 0.43)$$

$$= 0.5782$$



An electronic assembly consists of two sub-systems say A and B as shown below.



From previous testing procedures, the following probabilities are assumed to be known $P(A \text{ fails}) = 0.2$, $P(B \text{ fails alone}) = 0.15$, $P(A \text{ and } B\ fail) = 0.15$.

On the basis of above information, answer the following questions.

(i) Find the probability P(B fails) and the probability P(A fails alone).

(ii) Find the probability P(whole system fail) and the probability P(A fails/B has failed).

Sol :

Consider the following events

Let E be the event that sub-system A fails and F be the event that sub-system B fails.

(i) We have,

$$P(\overline{E} \cap F) = 0.15$$

$$P(F) - P(E \cap F) = 0.15$$

$$P(F) = 0.15 + P(E \cap F)$$

$$= 0.15 + 0.15 = 0.30$$

$$P(B \text{ fails}) = P(F) = 0.30$$

Now, $$P(E \cap \overline{F}) = P(E) - P(E \cap F)$$

$$= 0.2 - 0.15 = 0.05$$

$$P\ (A \text{ fails alone}) = P(E \cap \overline{F}) = 0.05$$

(ii) P (whole system fail)

$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

$$= 0.2 + 0.3 - 0.15$$

$$= 0.5 - 0.15 = 0.35$$

Now, P (A fails/B has failed)

$$P(E/F) = \frac{P(E \cap F)}{P(F)}$$

$$= \frac{0.15}{0.30}$$

$$= \frac{1}{2} = 0.5$$

128. OYO Rooms, also known as OYO Hotels & Homes, is an Indian multinational hospitality chain of leased and franchised hotels, homes and living spaces. Founded in 2012 by Ritesh Agarwal, OYO initially consisted mainly of budget hotels.

Data analyst at OYO say that during frequent trips to a certain city, a traveling salesperson stays at hotel A 50% of the time, at hotel B 30% of the time, and at hotel C 20% of the time. When checking in, there is some problem with the reservation 3% of the time at hotel A, 6% of the time at hotel B, and 10% of the time at hotel C. Suppose the salesperson travels to this city.

(i) Find the probability that the salesperson stays at hotel A and has a problem with the reservation.

(ii) Find the probability that the salesperson has a problem with the reservation.

(iii) Suppose the salesperson has a problem with the reservation; what is the probability that the salesperson is staying at hotel A?

Based on the above information answer the following questions.

(i) The total probability of committing an error in processing the form.

(ii) The manager of the company wants to do a quality check. During inspection he selects a form at random from the days output of processed forms. If the form selected at random has an error, the probability that the form is not processed by Vikas.

Sol :

(i) Required probability,

$$P(A) = P(E_1)P\left(\frac{A}{E_1}\right) + P(E_2)P\left(\frac{A}{E_2}\right) + P(E_3)P\left(\frac{A}{E_3}\right)$$

$$= 0.5 \times 0.06 + 0.2 \times 0.4 + 0.3 \times 0.3$$

$$= 0.030 + 0.008 + 0.009$$

$$= 0.047$$

(ii) Required probability,

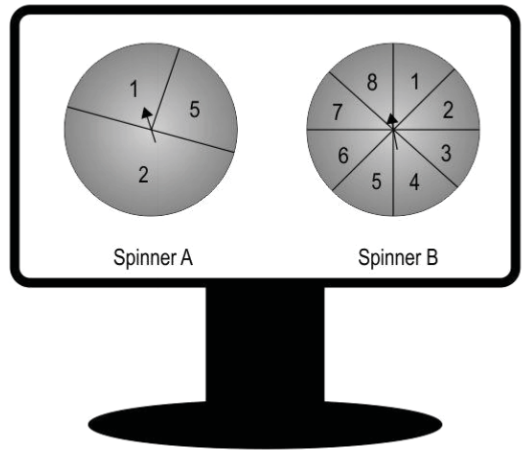
$$P\left(\frac{\overline{E_1}}{A}\right) = 1 - P\left(\frac{\overline{E_1}}{A}\right)$$

$$= 1 - \left[\frac{P(E_1)P\left(\frac{A}{E_1}\right)}{P(E_1)P\left(\frac{A}{E_1}\right) + P(E_2)P\left(\frac{A}{E_2}\right) + P(E_3)P\left(\frac{A}{E_3}\right)}\right]$$

$$= 1 - \left[\frac{0.5 \times 0.6}{0.5 \times 0.06 + 0.2 \times 0.04 + 0.3 \times 0.03}\right]$$

$$= 1 - \frac{0.030}{0.047} = 1 - \frac{30}{47} = \frac{17}{47}$$

131. Rubiya, Thaksh, Shanteri, and Lilly entered a spinning zone for a fun game, but there is a twist: they don't know which spinner will appear on their screens until it is their turn to play. They may encounter one of the following spinners, or perhaps even both:



Different combinations of numbers will lead to exciting prizes. Below are some of the rewards they can win:

Get the number 5, from Spinner A and 8 from Spinner B, and you'll win a music player!

You win a photo frame if Spinner A lands on a value greater than that of Spinner B!

Based on the above information answer the following:

(i) Thaksh spun both the spinners, A and B in one of his turns. What is the probability that Thaksh wins a music player in that turn?

(ii) Lilly spun spinner B in one of her turns. What is the probability that the number she got is even given that it is a multiple of 3 ?

(iii) Rubiya spun both the spinners. What is the probability that she wins a photo ?

(vi) As Shanteri steps up to the screen, the game administrator reveals that for her turn, the probability of seeing Spinner A on the screen is 65%, while that of Spinner B is 35%. What is the probability that Shanteri gets the number 2?

Sol :

$$P(5 \text{ from spinner A}) = \frac{1}{4}$$

$$P(8 \text{ from spinner B}) = \frac{1}{8}$$

(i) Probability that Thaksh wins a music player

Thaksh will win the music player if he get 5 from spinner A and 8 from spinner B.

$$P(5 \text{ from spinner A}) \cap P(8 \text{ from spinner B})$$

$$= \frac{1}{4} \times \frac{1}{8} = \frac{1}{32}$$

(ii) Probability that the number she got is even given that it is a multiple of 3 ?

$$P(\text{Multiple of } 3) = \frac{2}{8}$$

$$P(\text{Even} \cap \text{Multiple of } 3) = \frac{1}{8}$$

(iv) Probability that student is irregular given that he attains A grade. Required Probability

$$P\left(\frac{E_2}{A}\right) = \frac{P(E_2)\,P(\frac{A}{E_2})}{P(E_1)\,P(\frac{A}{E_1}) + P(E_2)\,P(\frac{A}{E_2})}$$

$$= \frac{\frac{70}{100}\cdot\frac{10}{100}}{\frac{30}{100}\cdot\frac{80}{100}+\frac{70}{100}\cdot\frac{10}{100}}$$

$$= \frac{7}{31}$$

133. In an Office three employees Jayant, Sonia and Oliver process incoming copies of a certain form. Jayant processes 50% of the forms, Sonia processes 20% and Oliver the remaining 30% of the forms. Jayant has an error rate of 0.06 , Sonia has an error rate of 0.04 and Oliver has an error rate of 0.03 .

Based on the above information answer the following:

(i) Find the probability that Sonia processed the form and committed an error.

(ii) Find the total probability of committing an error in processing the form.

(iii) The manager of the Company wants to do a quality check. During inspection, he selects a form at random from the days output of processed form. If the form selected at random has an error, find the probability that the form is not processed by Jayant.

(iv) Let E be the event of committing an error in processing the form and let J, S and O be the events the Jayant, Sonia and Oliver processed the form. Find the value of

$P(J|E)+P(S|E)+P(O|E)$

Sol :

Let J, S and O be the events that Jayant, Sonia and Oliver processed the form, which are clearly pairwise mutually exclusive and exhaustive st of events.

Then, $$P(J) = \frac{50}{100} = \frac{5}{10}$$

$$P(S) = \frac{20}{100} = \frac{1}{5}$$

and $$P(O) = \frac{30}{100} = \frac{3}{10}.$$

Also, let E be the event of committing an error.

We have, $$P(E|J) = 0.06,$$

$$P(E|S) = 0.04,$$

$$P(E|O) = 0.03.$$

(i) Probability that Sonia processed the form and committed an error is given by

$$P(E\cap S) = P(S)\cdot P(E|S)$$

$$= \frac{1}{5}\times 0.04$$

$$= 0.008.$$

(ii) Total probability of committing an error in processing the form is given by

$$P(E) = P(J)\cdot P(E|J) + P(S)\cdot P(E|S) + $$
$$+P(O)\cdot P(E|O)$$

$$= \frac{5}{10}\times 0.06 + \frac{1}{5}\times 0.04 + \frac{3}{10}\times 0.03$$

$$= 0.047.$$

(iii) Probability that the form is processed by Jayant given that form has an error is given by

$$P(J|E) = \frac{P(J)(E|J)}{P(J)P(E|J)+P(S)P(E|S)+P(O)P(E|O)}$$

$$= \frac{0.06\times\frac{50}{100}}{0.06\times\frac{50}{100}+0.04\times\frac{20}{100}+0.03\times\frac{30}{47}}$$

$$= \frac{30}{47}$$

Therefore, the required probability that the form is not processed by Jayant given that form has an error

$$P(\overline{J}\mid E) = 1 - P(J|E)$$

$$= 1 - \frac{30}{47} = \frac{17}{47}$$

(iv) $P(J|E)+P(S|E)+P(O|E)$

$P(J|E)+P(S|E)+P(O|E) = 1$

Because sum of the posterior probabilities is 1 .

134. There are two anti-aircraft guns, named as A and B . The probabilities that the shell fired from them hits an airplane are 0.3 and 0.2 respectively. Both of them fired one shell at an airplane at the same time.

Now $\quad P(S) = P(C)P(S \mid C) + P(T)P(S \mid T)$

$$= 0.6 \times 0.9 + 0.4 \times 0.8$$

$$= 0.86 \text{ or } \frac{86}{100}$$

(ii) Probability that a person selected at random prefers coffee given that it is without sugar

$$P(C \mid S') = \frac{P(C)\,P(S' \mid C)}{P(C)\,P(S' \mid C) + P(T)\,P(S' \mid T)}$$

$$= \frac{0.6 \times 0.1}{0.6 \times 0.1 + 0.4 \times 0.2}$$

$$= \frac{6}{6+8} = \frac{3}{7}$$

136. Ratna has two boxes I and II. Box I contains 3 red and 6 black balls. Box II contains 5 red and 5 black balls. Her friend Shivani selects one of the two boxes randomly and draws a ball out of it. The ball drawn by Shivani is found not be red. Let E_1, E_2 and A denote the following events:

E_1: box I is selected by Shivani

E_2: box II is selected by Shivani

R : Red ball is drawn by Shivani.

(i) Find $P(E_1)$ and $P(E_2)$

(ii) Find $P(R \mid E_1)$ and $P(R \mid E_2)$

(iii) Find $P(E_2 \mid R)$

Sol :

(i) $P(E_1)$ and $P(E_2)$

Probability of selecting Box I by Shivani,

$$P(E_1) = \frac{1}{2}$$

Probability of selecting Box II by Shivani,

$$P(E_2) = \frac{1}{2}$$

(ii) $P(R \mid E_1)$ and $P(R \mid E_2)$

Probability of selecting a red ball when box I has been already selected

$$P(R \mid E_1) = \frac{3}{3+6} = \frac{1}{3}$$

Probability of selecting a red ball when box II has been already selected

$$P(R \mid E_2) = \frac{5}{5+5} = \frac{1}{2}$$

(iii) $P(E_2 \mid R)$

By Bayes' Theorem

$$P(E_2 \mid R) = \frac{P(E_2)P(R \mid E_1)}{P(E_1)P(R \mid E_1) + P(E_2)P(R \mid E_2)}$$

$$= \frac{\frac{1}{2} \times \frac{1}{2}}{\frac{1}{2} \times \frac{1}{3} + \frac{1}{2} \times \frac{1}{2}}$$

$$= \frac{\frac{1}{2}}{\frac{1}{3} + \frac{1}{2}} = \frac{3}{5}$$

137. In an office three employees Vinay, Sonia and Iqbal process incoming copies of a certain form. Vinay process 50% of the forms. Sonia processes 20% and Iqbal the remaining 30% of the forms. Vinay has an error rate of 0.06, Sonia has an error rate of 0.04 and Iqbal has an error rate of 0.03.

Based on the above information answer the following questions:

(i) The total probability of committing an error in processing the form.

(ii) The manager of the company wants to do a quality check. During inspection he selects a form at random from the days output of processed forms. If the form selected at random has an error, the probability that the form is not processed by Vinay.

Sol :

(i) Total probability of committing an error

Let V be the event of processing form by Vinay, S be the event of processing form by Soniya and I be the event of processing form by Iqbal.

Let E be the event of error.

Now $\quad P(V) = \frac{50}{100} = \frac{5}{10},$

$$P\left(\frac{E}{V}\right) = 0.06,$$

$$P(S) = \frac{20}{100} = \frac{2}{10},$$

$$P\left(\frac{E}{S}\right) = 0.04,$$

$$P(I) = \frac{30}{100} = \frac{3}{10},$$

$$P\left(\frac{E}{I}\right) = 0.03$$

Required Probability

$$P(E) = P(V)P\left(\frac{E}{V}\right) + P(S)P\left(\frac{E}{S}\right) + P(I)P\left(\frac{E}{I}\right)$$

$$= \frac{5}{10}(0.06) + \frac{2}{10}(0.04) + \frac{3}{10}(0.03)$$

$$= 0.03 + 0.008 + 0.009$$

$$= 0.047$$

(ii) Probability that the form is not processed by Vinay

$$P(\overline{V}|E) = 1 - P(V|E)$$

By Bayes' Theorem

$$P(V|E) = \frac{P(V)P(E|V)}{P(V)(E|V) + P(S)(E|S) + P(I)(E|I)}$$

$$P(V|E) = \frac{P(V)P(E|V)}{P(E)}$$

$$= \frac{\frac{5}{10}(0.06)}{0.047}$$

$$= \frac{0.03}{0.047} = \frac{30}{47}$$

$$P(\overline{V}|E) = 1 - P(V|E)$$

$$= 1 - \frac{30}{47} = \frac{17}{47}$$

138. A doctor is to visit a patient. From the past experience, it is known that the probabilities that he will come by train, bus, scooter or by other means of transport are respectively $\frac{3}{10}, \frac{1}{5}, \frac{1}{10}$ and $\frac{2}{5}$. The probabilities that he will be late are $\frac{1}{4}, \frac{1}{3}$ and $\frac{1}{12}$ if he comes by train, bus and scooter respectively, but if he comes by other means of transport, then he will not be late.

On the basis of above information, answer the following questions.

(i) Find the probability that he is late.

(ii) When he arrives, he is late. What is the probability that he comes by train?

(iii) When he arrives, he is late. What is the probability that he comes by bus?

Sol :

Let E_1, E_2, E_3 and E_4 be the events that the doctor comes by train, bus, scooter and other means of transport respectively.

It is given that $P(E_1) = \frac{3}{10}$,

$$P(E_2) = \frac{1}{5},$$

$$P(E_3) = \frac{1}{10}$$

and $P(E_4) = \frac{2}{5}$

Let A be the event that doctor visit the patient late.

It is given that

$$P\left(\frac{A}{E_1}\right) = \frac{1}{4},$$

$$P\left(\frac{A}{E_2}\right) = \frac{1}{3},$$

$$P\left(\frac{A}{E_3}\right) = \frac{1}{12},$$

and $P\left(\frac{A}{E_4}\right) = 0$,

(i) Probability that doctor is late

$$P(A) = P(E_1)P\left(\frac{A}{E_1}\right) + P(E_2)P\left(\frac{A}{E_2}\right) + P(E_3)P\left(\frac{A}{E_3}\right) + P(E_4)P\left(\frac{A}{E_4}\right)$$

$$= \left(\frac{3}{10}\right)\left(\frac{1}{4}\right) + \left(\frac{1}{5}\right)\left(\frac{1}{3}\right) + \left(\frac{1}{10}\right)\left(\frac{1}{12}\right) + \left(\frac{2}{5}\right)(0)$$

$$= \frac{3}{40} + \frac{1}{15} + \frac{1}{120}$$

$$= \frac{9+8+1}{120}$$

$$= \frac{18}{120} = \frac{3}{20}$$

(ii) Probability that he comes by train given that he is late

$$P\left(\frac{E_1}{A}\right) = \frac{P(E_1)P\left(\frac{A}{E_1}\right)}{P(A)}$$

$$= \frac{\left(\frac{3}{10}\right)\left(\frac{1}{4}\right)}{\frac{3}{20}}$$

$$= \frac{3}{40} \times \frac{20}{3}$$

$$= \frac{1}{2}$$

(iii) Probability that he comes by train given that he is late

$$P\left(\frac{E_2}{A}\right) = \frac{P(E_2)P\left(\frac{A}{E_2}\right)}{P(A)}$$

$$= \frac{\left(\frac{1}{5}\right)\left(\frac{1}{3}\right)}{\frac{3}{20}}$$

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