Chapter-5

Continuity and Differentiability

Fill in the blanks

1). f is a real function on a subset of the real numbers and let c be a point in the domain f. Then f is continuous at c if $\lim_{x \to c} f(x) = f(c)$.

2). A function is continuous at $x=c$ if the function is defined at $x=c$ and if value of the function $x=c$ equals the limit of the function at $x=c$.

3). If f is not continuous at c, we say f is discontinuous at c and c is called point of discontinuity of f

4) f is continuous at $x=c$ if the left hand limit equal to the right hand limit

5). Left hand side limit is written as $\lim_{x \to c^-} f(x)$ at $x=c$

6) Right hand side limit is written as $\lim_{x \to c^+} f(x)$. at $x=c$

(6) Let f ad g be two real functions continuous at $x=c$, then

7) $(f+g)(x) = \cancel{f(x)+g(x)}$ continuous at $x=c$

8) $(f-g)(x) = \underline{f(x)-g(x)}$ "

9) $(fg)(x) = \underline{f(x).g(x).}$ "

10). $\left(\frac{f}{g}\right)(x) = \underline{\frac{f(x)}{g(x)}}$ " . $g(x) \neq 0$.

11). If $f(x) = \lambda$ then $\lambda \cdot g(x)$ is also continuous at $x=c$

12). If $f(x) = \lambda$ then $\frac{\lambda}{g(x)}$ is also continuous at $x=c$

13) f be a real function ad c is a point in its domain. The derivative of f at c is defined as

$$\underline{\lim_{h\to 0} \frac{f(c+h)-f(c)}{h} = f'(c)}$$

14) $f'(x) = \underline{\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}}$

15) Every differentiable function is continuous.

16). Every continuous function ________ differentiable.

17) If f is not continuous at c, then we say f is discontinuous ________ at c and c is called a point of ________ discontinuity of f

18). If u and v be the functions of x then

(a) $(u \pm v)' = \underline{u' \pm v'}$

(b) $(uv)' = \underline{u'v + uv'}$

(c) $\left(\frac{u}{v}\right)' = \underline{\frac{u'v - uv'}{v^2}, v \neq 0}$

19) ~~If t = vou~~, and $u(x) = t$, $f = v(t)$, t is a real valued function which is a composite of two functions u and v i.e; $f = vou$, then

$$\frac{df}{dx} = \underline{\frac{dv}{dt} \cdot \frac{dt}{dx}}.$$

20) Chain rule is $\underline{\frac{dt}{dx} = \frac{dt}{ds} \cdot \frac{ds}{dt} \cdot \frac{dt}{dx}}$ where f is a composite function of u, v, and w $u(x) = t$, $v(t) = s$, $f = w(s)$.

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21) $\frac{d}{dx}\sin^{-1}(x)\,dx = \frac{1}{\sqrt{1-x^2}}$

22) $\frac{d}{dx}\cos^{-1}(x)\,dx = -\frac{1}{\sqrt{1-x^2}}$

23) $\frac{d}{dx}\tan^{-1}(x)\,dx = \frac{1}{1+x^2}$

24) $\frac{d}{dx}\cot^{-1}x\,dx = -\frac{1}{1+x^2}$

25) $\frac{d}{dx}\sec^{-1}x\,dx = \frac{1}{x\sqrt{x^2-1}}$

26) $\frac{d}{dx}\operatorname{cosec}^{-1}x\,dx = -\frac{1}{x\sqrt{x^2-1}}$

27) $\frac{d}{dx}(a^x)\,dx = a^x \log_e a$

28) $\frac{d}{dx}(\log_{a} x)\,dx = \frac{1}{x}$.

29) $\frac{d}{dx}(e^x)\,dx = e^x$

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Properties of logarithm.

(30) $\log(ab) = \underline{\log a + \log b}$.

(31) $\log\left(\frac{p}{q}\right) = \underline{\log p - \log q}$.

(32) $\log a^{n} = \underline{n \log a}$.

(33) $\log_a b = \underline{\frac{\log_b b}{\log_b a}}$

(34) The functions are given in the form of $y = [u(x)]^{v(x)}$ and $y = \frac{u(x)v(x)}{w(x)}$, then we ~~do so~~ use

<u>Logarithmic</u> differentiation.

35). Let $x = f(t)$, $y = g(t)$ where t, is the parameter

$\frac{dx}{dt} = \underline{f'(t)}$, $\frac{dy}{dt} = \underline{g'(t)}$, $\frac{dy}{dx} = \underline{\frac{g'(t)}{f'(t)}}$.

(36). If $f : [a\ b] \to R$ is continuous on $[a\ b]$ and differentiable on (a, b) such that $f(a) = f(b)$. Then there exists some c in $(a\ b)$ such that $f'(c) = 0$. is known as <u>Rolle's</u> theorem

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37) If $f:[a\ b] \to R$ is continuous on $[a\ b]$ and differentiable on $(a\ b)$. Then there exists some $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$ is

know as Mean value theorem.

Short Answer Type

~~CONTINUITY~~

(38) Check the continuity of the following functions f given by.

(38) $f(x) = 2x+3$ at $x=1$ Ans cont at $x=1$

(39) $f(x) = |x|$ at $x=0$. Ans Cont at $x=0$.

(40) $f(x) = 5x-3$ at $x=0$, at $x=-3$. Ans Cont at $x=0$ and $x=-3$

Discuss the continuity of the following functions

(41) $f(x) = x^3 + x^2 - 1$. Ans cont function

(42) $f(x) = \begin{cases} x & \text{if } x \geq 0 \\ x^2 & \text{if } x < 0 \end{cases}$ Ans cont function

(43) Show that every polynomial function is continuous.

44) Examine if $\sin|x|$ is a continuous function

45). Prove that the function $f(x) = 5x - 0$ is continuous at $x = -3$.

46) Show that the function defined by $f(x) = \sin(x^2)$ and $f(x) = \cos(x^2)$ is a continuous function

47) Examine the following functions for continuity.

(a) $f(x) = x - 5$ (b) $f(x) = \frac{1}{x-5}$, $x \neq 5$

(c) $f(x) = \frac{x^2 - 25}{x+5}$, $x \neq -5$ (d) $f(x) = |x-5|$

48) find all points of discontinuity of f when f is defined by

(a) $f(x) = \begin{cases} 2x+3 & \text{if } x \leq 2 \\ 2x-3 & \text{if } x > 2 \end{cases}$

(e) $f(x) = \begin{cases} x^{10} - 1 & \text{if } x \leq 1 \\ x^2 & \text{if } x > 1. \end{cases}$

(b) $f(x) = \begin{cases} |x| + 3 & \text{if } x \leq -3. \\ -2x & \text{if } -3 < x < 3. \\ 6x + 2 & \text{if } x \geq 3 \end{cases}$

(c) $f(x) = \begin{cases} \frac{x}{|x|}, & \text{if } x < 0 \\ -1, & \text{if } x \geq 0 \end{cases}$

(d) $f(x) = \begin{cases} x + 1 & \text{if } x \geq 1 \\ x^2 + 1, & \text{if } x < 1. \end{cases}$

49). Discuss the continuity of the function f, if

a) $f(x) = \begin{cases} 3 & \text{if } 0 \leq x \leq 1 \\ 4 & \text{if } 1 < x < 3 \\ 5 & \text{if } 3 \leq x \leq 10 \end{cases}$

(b) $f(x) = \begin{cases} 2x & \text{if } x < 0 \\ 0 & \text{if } 0 \leq x \leq 1 \\ 4x & \text{if } x > 1 \end{cases}$

50). S.T the function defined by $g(x) = x - [x]$ is discontinuous at all integral points. ~~Here~~ Here $[x]$ denotes the greatest integer less than or equal to x.

51) Find all points of discontinuity of f where

$$f(x) = \begin{cases} \frac{\sin x}{x}. & \text{if } x < 0. \\ x+1 & \text{if } x \geq 0. \end{cases}$$

52). Examine the continuity of f where f is defined by

$$f(x) = \begin{cases} \sin x - \cos x & \text{if } x \neq 0 \\ -1. & \text{if } x = 0 \end{cases}$$

53) Find the values of K so that f is continuous at the indicated point.

ⓐ $f(x) = \begin{cases} \frac{K\cos x}{\pi - 2x} & \text{if } x \neq \frac{\pi}{2} \\ 3 & \text{if } x = \frac{\pi}{2} \end{cases}$ at $x = \frac{\pi}{2}$

(b) $f(x) = \begin{cases} Kx^2 & \text{if } x \leq 2 \\ 3 & \text{if } x > 2 \end{cases}$ at $x = 2$

(c) $f(x) = \begin{cases} Kx+1 & \text{if } x \leq \pi \\ \cos x & \text{if } x > \pi \end{cases}$ at $x = \pi$

54) Show that $f(x) = \cos(x^2)$ ad $f(x) = |\cos x|$ is continuous function

Differentiability

Chain Rule

~~(55)~~ Differentiate the functions wrt x

(55) $\cos(\sin x)$

(56) $\sin(ax+b)$

(57) $\sec(\tan\sqrt{x})$

58) $\cos x^3 \cdot \sin^2(x^5)$.

59) $\cos(\sqrt{x})$

Derivatives of Implicit functions

60) Find $\frac{dy}{dx}$, if $y + \sin y = \cos x$.

61) Find $\frac{dy}{dx}$ if $2x+3y = \sin y$

(62) " " $ax+by^2 = \cos y$

(63) " " $x^3+x^2y+xy^2+y^3 = 81$

64) " $xy + y^2 = \tan x + y$.

Derivative of Inverse Trigonometry function

(65) Find $\frac{dy}{dx}$ if $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$.

(66) Find $\frac{dy}{dx}$ when $y = \sin^{-1}\left[\frac{5x+12\sqrt{1-x^2}}{13}\right]$ Ans $\frac{1}{\sqrt{1-x^2}}$

(67). Find $\frac{dy}{dx}$ when $y = \tan^{-1}\left[\frac{a\cos x - b\sin x}{b\cos x + a\sin x}\right]$ Ans $= -1$

(68) Find $\frac{dy}{dx}$ if $y = \sin^{-1}x + \sin^{-1}\sqrt{1-x^2}$, $-1 \le x \le 1$.

(69) Find $\frac{dy}{dx}$ when $y = \tan^{-1}\left[\frac{4x}{1+5x^2}\right] + \tan^{-1}\left[\frac{2+3x}{3-2x}\right]$

Ans $\frac{5}{1+25x^2}$

70) Find $\frac{dy}{dx}$ when $y = \cot^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right]$, $0 < x < \frac{\pi}{2}$.

71) Find $\frac{dy}{dx}$ when $y = \sin^{-1}\left[x^2\sqrt{1-x^2} + x\sqrt{1-x^4}\right]$.

Ans $\frac{2x}{\sqrt{1-x^4}} + \frac{1}{\sqrt{1-x^2}}$.

Exponential and Logarithmic function

Differentiate the following w.r.t x.

72) $e^{\sin^{-1}x}$

73) $\sqrt{e^{\sqrt{x}}}$

74) $\sin(\tan^{-1}e^{-x})$

75) $\log(\cos e^x)$

76) $\frac{\cos x}{\log x}$, $x > 0$

77) $\cos(\log x + e^x)$ $x > 0$

Logarithmic Differentiation

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78) Find $\frac{dy}{dx}$ if $y^x + x^y + x^x = a^b$.

$$\frac{dy}{dx} = \frac{-\left[y^x \log y + y \cdot x^{y-1} + x^x(1+\log x)\right]}{x \cdot y^{x-1} + x^y \log x}$$

79) ~~Find $\frac{dy}{dx}$~~.

Differentiate wrt x. $\cos x \cdot \cos 2x \cdot \cos 3x$.

80) Differentiate wrt x $\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$

(81) Differentiate wrt x. $(x+3)^2 \cdot (x+4)^3 \cdot (x+5)^4$.

82) $(x\cos x)^x + (x \sin x)^{\frac{1}{x}}$.

Parametric Form

83) Find $\frac{dy}{dx}$ if $x = a\cos\theta$, $y = a\sin\theta$, Ans $\frac{dy}{dx} = -\cot\theta$

84) Find $\frac{dy}{dx}$ if $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$. An $\frac{dy}{dx} = -\sqrt[3]{\frac{y}{x}}$.

85). Find $\frac{dy}{dx}$ if $x = 4t$, $y = \frac{4}{t}$

86) " " $x = a(\theta - \sin\theta)$, $y = a(1+\cos\theta)$ $\frac{dy}{dx}$ at $\theta = \frac{\pi}{3}$ Ans $\frac{\sin^2(a+y)}{\sin a}$.

87) $x = \frac{\sin^3 t}{\sqrt{\cos 2t}}$, $y = \frac{\cos^3 t}{\sqrt{\cos 2t}}$

88) $x = a\left(\cos t + \log \tan \frac{t}{2}\right);\ y = a \sin t$

Second Order Derivatives

~~89)~~ Find the second order derivative

89) $x^3 \log x$

92) $e^{6x} \cos 3x$

90) $\log(\log x)$.

91) $\sin(\log x)$

Mean value theorems

(92) State the mean value theorem

(93) State Rolle's theorem

(94) Verify Rolle's theorem for the function

$f(x) = \sin 2x$ in $\left\{0, \frac{\pi}{2}\right\}$ Ans $c = \frac{\pi}{4}$

(95) Verify mean value theorem for $f(x) = x^2$ in $[2, 4]$

Ans $c = 3,\ f'(c) = 6$.

Long Answer Type

① Find the values of a and b such that the function defined by

$$f(x) = \begin{cases} 5 & \text{if } x \le 2 \\ ax+b & \text{if } 2 < x < 10 \\ 21 & \text{if } x \ge 10 \end{cases}$$

is a continuous function

② Find all the points of discontinuity of f defined

$f(x) = |x| - |x+1|$.

③ Differentiate $\sin(\cos(x^2))$ with respect to x.

④ Prove that the greatest integer function defined by $f(x) = [x]$, $0 < x < 3$. is not differentiable at $x = 1$ and $x = 2$.

⑤ S.T the function $f(x) = \begin{cases} \dfrac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1}, & x \neq 0 \\ 0, & \text{when } x = 0 \end{cases}$ is discontinuous at $x = 0$

⑥ S.T the function $f(x) = |\sin x + \cos x|$ is continuous at $x = \pi$

⑦ Discuss the continuity of the following function at $x = 0$.

$$f(x) = \begin{cases} \dfrac{x^4 + x^3 + 2x^2}{\tan^{-1} x}, & x \neq 0 \\ 10, & x = 0 \end{cases}$$

Find $\frac{dy}{dx}$ in the following functions.

8) $y = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right), \; -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$.

9) $y = \sin^{-1}\left(2x\sqrt{1-x^2}\right), \; -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$

10) $y = \sec^{-1}\left(\frac{1}{2x^2-1}\right), \; 0 < x < \frac{1}{\sqrt{2}}$.

11) $y = \log\sqrt{\frac{1+x\cos x}{1-x\cos x}}$. Ans: $\frac{\cos x - x\sin x}{1-x^2\cos^2 x}$

12) $y = x^{x\cos x} + \frac{x^2+1}{x^2-1}$. Ans $\frac{4x}{(x^2-1)^2}$

13) If $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$ find $\frac{dy}{dx}$. Ans $\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$

14) $y = (\sin x - \cos x)^{(\sin x - \cos x)}, \; \frac{\pi}{4} < x < \frac{3\pi}{4}$. Ans $\frac{dy}{dx} = \log[e(\sin x - \cos x)]$

15) If $y = \cos^{-1}\left(\frac{3x+4\sqrt{1-x^2}}{5}\right)$. Ans $\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$

16) If $y = x^{\sin x} + \cos x^{x}$. Ans $\frac{dy}{dx} = (\cos x)^x(\log\cos x - x\tan x)$

17) If $(\cos x)^y = (\sin y)^x$. Ans $\frac{dy}{dx} = \frac{y\tan x + \log(\sin y)}{\log\cos y - x\cot y}$

18) If $y = \log\sqrt{\frac{1-\cos 2x}{1+\cos 2x}}$ Ans $\frac{dy}{dx} = 2\operatorname{cosec} 2x$.

19). $y = \tan^{-1}\left[\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right]$. Ans $\frac{dy}{dx} = \frac{1}{2\sqrt{1-x^2}}$

20) $y = \sqrt{x^2+1} - \log\left(\frac{1}{x}+\sqrt{1+\frac{1}{x^2}}\right)$ Ans $\frac{dy}{dx} = \frac{x^2+\sqrt{1+x^2}+1}{x(1+\sqrt{1+x^2})}$

21). $x\sqrt{1+y} \pm y\sqrt{1+x}$ Ans $\frac{dy}{dx} = \frac{-1}{(1+x)^2}$

22) $\sqrt{1-x^2}+\sqrt{1-y^2} = a(x-y)$ Ans $\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$

23). Find the derivative of the function given by $f(x) = (1+x)(1+x^2)(1+x^4)(1+x^8)$ and hence find $f'(1)$.

24). If $y = \sin^{-1}x$ Show that $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0$

25). If $y = Ae^{mx}+Be^{nx}$ Show that $y'' - (m+n)y' + mny = 0$

26) If $e^y(x+1) = 1$ S.T $y'' = (y')^2$

27) If $y = (\tan^{-1}x)^2$. S.T $(x^2+1)^2 y_2 - 2x(x^2+1)y_1 = 2$

28). If $y = \sin(\log x)$ P.T $x^2y'' + xy' + y = 0$

29). If $y = \frac{\sin^{-1} x}{\sqrt{1-x^2}}$ S.T $(1-x^2)\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} - y = 0$

30) $y = e^{x}\sin x$ P.T $y'' - 2y' + 2y = 0$

31) $y = e^{\tan x}$ P.T $\cos^2 x \frac{d^2y}{dx^2} - (1+\sin 2x)\frac{dy}{dx} = 0$.

32). $y = \operatorname{cosec}^{-1} x$, $x > 1$ S.T $x(x^2-1)y'' + (2x^2-1)y' = 0$.

33) If $y = (x+\sqrt{x^2-1})^m$ S.T $(x^2+1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - m^2 y = 0$.

34) If $\log(\sqrt{1+x^2} - x) = y\sqrt{1+x^2}$ S.T $(1+x^2)\frac{dy}{dx} + xy + 1 = 0$.

35). If $y = 3\cos(\log x) + 4\sin(\log x)$ S.T $x^2y'' + xy' + y = 0$

(36) If $x = a(\cos t + \log \tan t/2)$

$y = a\sin t$

find $\frac{d^2y}{dt^2}$ and $\frac{d^2y}{dx^2}$ An $\frac{d^2y}{dt^2} = -a\sin t$, $\frac{d^2y}{dx^2} = \frac{1}{a}\sec^4 t \sin t$

37) If $x = a(\theta - \sin\theta)$

$y = a(1+\cos\theta)$

Find $\frac{d^2y}{dx^2}$. An $= \frac{1}{4a}\operatorname{cosec}^4\frac{\theta}{2}$.

38Q) Find dy/dx at $t = \pi/4$.

$x = a\sin 2t(1+\cos 2t)$.

$y = b\cos 2t(1-\cos 2t)$.

39) If $x = a\left(\cos\theta + \log\tan\frac{\theta}{2}\right)$

$y = a\sin\theta$ find the value of $\frac{dy}{dx}$ at $\theta = \pi/4$.

Ans $\frac{dy}{dx} = 1$.

40) If $x = \sqrt{a^{\sin^{-1}t}}$, $y = \sqrt{a^{\cos^{-1}t}}$ find $\frac{dy}{dx}$. Ans $\frac{dy}{dx} = \frac{-y}{x}$.

41) If $f(x)$ defined by the following, is continuous at $x=0$, find the values of a, b, c.

$$f(x) = \begin{cases} \dfrac{\sin(a+1)x + \sin x}{x} & \text{if } x < 0 \\ c & \text{if } x = 0 \\ \dfrac{\sqrt{x+bx^2}-\sqrt{x}}{bx^{3/2}} & \text{if } x > 0 \end{cases}$$

Ans $a = -\frac{3}{2}$, $c = \frac{1}{2}$

$b =$ any non zero real no.

42) Find all points of discontinuity of f where

$$f(x) = \begin{cases} |x|+3 & , x \le -3 \\ -2x & , -3 < x < 3 \\ 6x+2 & , x \ge 3. \end{cases}$$

Ans $x = 3$ (dis cont)

43). S.T the function f defined as follows is continuous at $x=2$ but not differentiable at that point.

44) find the value of k for which the function f defined below is continuous

$$f(x)=\begin{cases} \dfrac{k\cos x}{\pi-2x}, & x<\pi/2 \\ 3, & x=\pi/2 \\ \dfrac{3\tan 2x}{2x-\pi}, & x>\pi/2 \end{cases}$$

45) For what values of a and b the function f defined as

$$f(x)=\begin{cases} 3ax+b & \text{if } x<1 \\ 4 & \text{if } x=1 \\ 5ax-2b & \text{if } x>1 \end{cases}$$ is continuous at $x=1$

46) Verify Rolle's theorem for the function

(a) $f(x)=x^2+2x-8$, $x\in[-4,2]$.

(b) $f(x)=[x]$ for $x\in[-2,2]$.

(47) Verify mean value theorem if $f(x)=x^2-4x-3$ in $[a,b]$ when $a=1$ $b=4$

48) Verify MVT $f(x) = x^3 - 5x^2 - 3x$ in $[a, b]$ where $a = 1$ and $b = 3$. Find all $c \in (1, 3)$ for which $f'(c) = 0$.

49) Verify Rolle's theorem for the function
$f(x) = \sin x - \cos x$ in $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$. Ans $c = \frac{3\pi}{4}$

50) It is given that $f(x) = x^3 - 6x^2 + px + q$ on $[1, 3]$.
Rolle's theorem hold with $c = 2 + \frac{1}{\sqrt{3}}$.
Find values of p and q

51) If $(x-a)^2 + (y-b)^2 = c^2$ for some $c > 0$ P.T

$$\frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$ is a constant independent of a and b.

52) If $\cos y = x \cos(a+y)$ with $\cos a \neq \pm 1$.
P.T $\frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$.

53) If $x = a(\cos t + t \sin t)$ and $y = a(\sin t - t \cos t)$ find $\frac{d^2y}{dx^2}$

54) If $y = e^{a \cos^{-1} x}$, $-1 \leq x \leq 1$ S.T $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2 y = 0$