Integration

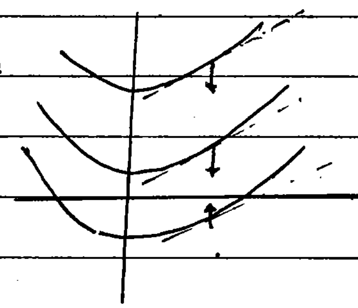
#1. $\int$ is reverse of $\frac{d}{dx}$

(Actual meaning of operators by area (definite $\int$))

#2. $\int$ as family of curves

$\int 2x\,dx = x^2 + C$ * family of Parabola.

Tangent at given x-coordinate on any family member is of same slope.



#3 $\int$ periodic f^n = Can't say

$\int \cos x\,dx = \sin x + C$ periodic
$\int (\cos x + 1)\,dx = \sin x + x + C$ aperiodic

#4. Differential of a f^n

$y = \tan x \quad dy = \sec^2 x\,dx = d(\tan x)$ $\quad \frac{d\tan x}{dx}\,dx = \cdots$
$y = x^2 \quad dy = 2x\,dx = d(x^2)$
$y = f(x) \quad dy = f'(x)\,dx = d(f(x))$

$\int x^4\,d(x^2) = \frac{x^6}{3} = \int x^4\, 2x\, dx$

Standard integration {V. Imp}

#1. $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

#2. $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{n+1} \times \frac{1}{a} + c$

#3. $\int \frac{1}{x} dx = \ln|x| + c$ → always

$$\ln|x| = \begin{cases} \ln x & x>0 \quad 1/x \\ \ln(-x) & x<0 \quad \frac{1}{-x} \times -1 \end{cases}$$

#4. $\int \frac{1}{ax+b} dx = \frac{\ln|ax+b|}{a} + c$

#5. $\int a^x dx = \frac{a^x}{\ln a} + c$

#6. $\int a^{px+q} dx = \frac{a^{px+q}}{p \ln a} + c$

#7. $\int \frac{dx}{1+x^2} = \tan^{-1} x + c$

★ #8 $\int \frac{dx}{a^2+x^2} = \frac{1}{a} \tan^{-1}(x/a) + c$

#9. $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + c$

★ #10. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + c$

#11. $\int \frac{dx}{x\sqrt{x^2-1}} = \frac{1}{a} \sec^{-1}\left(\frac{x}{a}\right) + c$

Standard Integration {V Imp}

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Ques: $\int \frac{x}{(x^2+2x+1)} dx$

Solu: $\int \frac{x+1-1}{(x+1)^2} dx = \int \frac{1}{x+1} - \frac{1}{(x+1)^2} dx = \ln|x+1| + \frac{1}{x+1} + C$

Ques: $\int \cos 2x \cos 3x \cos 5x \, dx$

Solu: $\frac{1}{2}\int (\cos^2 5x + \cos x \cos 5x)\, dx$

$= \frac{1}{2} \int \left[\frac{1+\cos 10x}{2} + \frac{1}{2}\cos 6x + \frac{1}{2}\cos 4x\right] dx$

$= \frac{1}{2}\left[\frac{1}{2}x + \frac{\sin 10x}{20} + \frac{\sin 6x}{12} + \frac{\sin 4x}{8}\right] + C$

$= \frac{x}{4} + \frac{\sin 10x}{20} + \frac{\sin 6x}{12} + \frac{\sin 4x}{8} + C$

Ques: $\int \frac{\cos x - \cos 2x}{1-\cos x} dx$

Solu: $\int \frac{\cos x - 2\cos^2 x + 1}{1-\cos x} dx = -\int \frac{(1-\cos x)(-2\cos x - 1)}{1-\cos x} dx$

$\cos x = 1$ $= \cancel{\text{test}} \int (2\cos x + 1)\, dx$

$1-1=0$ $= 2\sin x + x + C$

$1-2+1=0$

$1-\cos x$ is factor

Ques: $\int \sin^3 x \, dx$

Solu: $\int \frac{3\sin x - \sin 3x}{4} dx = \frac{-3}{4}\cos x + \frac{\cos 3x}{12} + C$

Ques: $\int \cos^4 x \, dx$

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Solu: $\frac{1}{4}\int (1+\cos 2x)^2 = \frac{1}{4}\int 1+2\cos 2x + \frac{1+\cos 4x}{2}\, dx$

$= \frac{1}{4}\left[x + \frac{2\sin 2x}{2} + \frac{1x}{2} + \frac{\sin 4x}{8}\right] + c$

$= \frac{x}{4} + \frac{\sin 2x}{4} + \frac{x}{8} + \frac{\sin 4x}{32} + c$

Note: $\int \sin^{odd} x\, dx \,/\, \int \cos^{odd} dx \rightarrow$ No presence of x.

$\int \sin^{even} x\, dx \,/\, \int \cos^{even} x\, dx \rightarrow$ x should be there in Integeration

Ques: $\int \frac{\cos 5x + \cos 4x}{1 - 2\cos 3x}\, dx$ — $3x$: 2 times of $3x/2$

Solu: $\int \frac{2\cos\left(\frac{9x}{2}\right)\cos\frac{x}{2}}{1-2\left(2\cos^2\frac{3x}{2}-1\right)}$ — $\frac{9x}{2}$: 3 times of $3x/2$

$= \int \frac{2\cos 9x \cos x/2}{3 - 4\cos^2 3x/2}$ $\qquad \cos 3\theta = \cos\theta(4\cos^2\theta - 3)$

$= -2\int \cos 3x/2 \cos x/2\, dx$

$= -\int \cos 2x + \cos x\, dx = -\frac{\sin 2x}{2} - \sin x + C$

Ques: $\int \frac{1-\cos x}{1+\cos x}\, dx = \int \tan^2 x/2\, dx = \int \sec^2 x/2 - 1\, dx$

$= 2\tan x/2 - x + C$

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Ques: $\int \frac{dx}{1-\sin 3x} = \int \frac{1+\sin 3x}{\cos^2 3x} = \int \sec^2 3x + \sec 3x \tan 3x \, dx$

$= \frac{\tan 3x}{3} + \frac{\sec 3x}{3} + c$

Ques: $\int \frac{\sin x}{\cos^2 x} (1 - 3\sin^3 x) dx$

$= \int \tan x \sec x - 3\frac{(1-\cos^2 x)^2}{\cos^2 x} dx$

$= \int \tan x \sec x - 3\left(\frac{1+\cos^4 x + 2\cos^2 x}{\cos^2 x}\right) dx$

$= \sec x - 3\tan x - \frac{3x}{2} - 3\frac{\sin 2x}{2} + 2x + c$

Ques: $\int \left(\frac{x^2+\cos^2 x}{x^2+1}\right) \operatorname{cosec}^2 x \, dx$

$\int \left(\frac{x^2+1-\sin^2 x}{x^2+1}\right) \operatorname{cosec}^2 x \, dx$

$= \int \left(\operatorname{cosec}^2 x - \frac{1}{x^2+1}\right) dx = -\cot x - \tan^{-1} x + C$

Ques: $\int \frac{x^4}{1+x^2} dx$

* $\int \frac{\text{Poly}}{\text{Poly}} dx \rightarrow$ If degree of $N^r \geq D_r$ divide

$\int \frac{x^4 - 1 + 1}{1+x^2} = \int x^2 - 1 + \frac{1}{x^2+1} dx = \frac{x^3}{3} - x + \tan^{-1} x + C$

* Quotient is x^2-1 * Remainder is 1

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Objective approach to find

Remainer Theorem

Coefficient of $Tan^{-1}x$ in last ques

poly $f(x)$ $(x-\alpha)$

Remainder $= f(\alpha)$

$(1+x^2) \to i$, $-i$

$f(i) \& f(-i) = 1$

$Poly = \frac{1}{(x^2+1)}$? $Tan^{-1}x$

Ques: $\frac{dx}{\sqrt{9-4x^2}}$

$*$ St. $f^n \int, x \to$ (Linear f^n of x)

Solu: $\int \frac{dx}{\sqrt{(3)^2-(2x)^2}} = \frac{1}{2} \sin^{-1}\left(\frac{2x}{3}\right) + c$

Ques: $\int \frac{dx}{x^2-x+1}$

Solu: $\int \frac{dx}{(x-1/2)^2+(\sqrt{3}/2)^2} = \frac{1}{\sqrt{3}/2} \tan^{-1}\left(\frac{x-1/2}{\sqrt{3}/2}\right) + c$

$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{(2x-1)}{3}\right) + c$

Ques: $\int \frac{3x\,dx}{7x+8}$

Solu: $\frac{3}{7}\int \frac{7x+8-8}{7x+8}\,dx = \frac{3}{7}\int 1 - \frac{8}{7x+8}\,dx$

$= \frac{3}{7}x - \frac{3\times 8}{7}\frac{\ln|7x+8|}{7} + c$

$*$ $\int \frac{dx}{x^2-5x+6} = \int \frac{(x-2)-(x-3)}{(x-2)(x-3)}\,dx = \int \frac{1}{x-3} - \frac{1}{x-2}\,dx$

$= \ln\left|\frac{x-3}{x-2}\right| + c$

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$\frac{1}{\text{Quad}}$ $D>0$ $\ln f^n$

$D=0$ Rational

$D<0$ $\tan^{-1}(?)$

Working rule:

★ Multiply divide by difference of root

$$\boxed{\bigstar \int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C}$$

$$\frac{1}{2a}\int \frac{(x+a)-(x-a)}{(x+a)(x-a)} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + c$$ Hence Proved.

★ $\overset{1}{\text{chota}}$ root $-$ $\overset{1}{\text{bda}}$ root

Q. $$\int \frac{dx}{(2x-7)\sqrt{(x-3)(x-4)}}$$

$$= \int \frac{2}{(2x-7)\sqrt{4x^2-28x+49-1}} = \int \frac{2}{(2x-7)\sqrt{(2x-7)^2-1}} = 2\sec^{-1}(2x-7) + c$$

Ques: $$\int \frac{dx}{(x^2-4x+4)(x^2-4x+5)}$$

$$= \int \frac{1}{(x^2-4x+4)} - \frac{1}{(x^2-4x+5)}\,dx = \int \frac{1}{(x-2)^2} - \frac{1}{(x-2)^2+1}\,dx$$

$$= \frac{-1}{(x-2)} - \tan^{-1}(x-2) + c$$

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Ques: $\int \frac{x^2+3}{x^6(x^2+1)}\,dx$

Solu: $\int \frac{x^2+1+2}{x^6(x^2+1)}\,dx = \int \left(\frac{1}{x^6} + \frac{2}{x^6(x^2+1)}\right)dx$

$= \int \frac{1}{x^6} + 2\int \frac{1}{x^4\, x^2(x^2+1)}\,dx$

$= \int \frac{1}{x^6} + 2\int \frac{1}{x^4}\left[\frac{1}{(x^2)} - \frac{1}{(x^2+1)}\right]dx$

$= \int \frac{3}{x^6} - 2\int \frac{1}{x^4(x^2+1)}\,dx$

$= 3\int \frac{1}{x^6} - 2\int \frac{1}{x^2\cdot x^2(x^2+1)}\,dx$

$= 3\int \frac{1}{x^6} - 2\int \frac{1}{x^2}\left[\frac{1}{x^2} - \frac{1}{(x^2+1)}\right]dx$

$= 3\int \frac{1}{x^6} - 2\int \frac{1}{x^4} + 2\int \frac{1}{x^2(x^2+1)}\,dx$

$= 3\int \frac{1}{x^6} - 2\int \frac{1}{x^4} + 2\int \frac{1}{x^2} - 2\int \frac{1}{x^2+1}\,dx$

$= \frac{-3'\times 5}{x^5} + \frac{2(3)}{x^3} \cancel{+} \frac{-2}{x} - 2\tan^{-1}x + C$

$= \frac{-15}{x^5} + \frac{6}{x^3} - \frac{2}{x} - 2\tan^{-1}x + C$

Ques: $\int \frac{x^8 - 3x^6 + 4x^4 - 2x^2 + 3}{x^2+1}\,dx$

Solw: $\int \frac{x^8 + x^6 - 4x^6 - 4x^4 + 8x^4 + 8x^2 - 10x^2 - 10 + 13}{x^2+1}\,dx$

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$$\int x^6 - 4x^4 + 8x^2 - 10 + \frac{13}{x^2+1}\, dx$$

$$= \frac{x^7}{7} - \frac{4x^5}{5} + \frac{8x^3}{3} - 10x + 13\tan^{-1}(x^2+1) + C_{4}$$

Ques: $f'(x^2) = \frac{1}{x} \quad \forall x > 0 \quad f(1) = 1 \quad f(x) = ?$

Solu: $x^2 = y \qquad 2x\,dx = dy \qquad x = \sqrt{y}$

$$f'(y) = \frac{1}{\sqrt{y}} \qquad f(y) = 2\sqrt{y} + c \qquad f(1) = 1$$

$$1 = 2 + c \Rightarrow c = -1$$

$$[f(y) = 2\sqrt{y} - 1] \text{ Ans}$$

Methods of Integration

i) By Substitution
ii) By Parts
iii) Partial fraction
iv) Miscellaneous (Kutur-Putur)

* $\int f^n(x) f'(x)\, dx$

$$f(x) = y \qquad f'(x)\,dx = dy$$

$$\int y^n dy = \frac{y^{n+1}}{n+1} + c = \frac{f^{n+1}(x)}{n+1} + c$$

* $\int \frac{f'(x)\,dx}{f(x)} = \ln|f(x)| + c$

* $\int \frac{f'(x)\,dx}{\sqrt{f(x)}} = 2\sqrt{f(x)} + c$

Imp

$\int \tan x\,dx = \ln|\sec x| = -\ln|\cos x|$

$$-\int \frac{-\sin x}{\cos x}dx = -\ln|\cos x| = \ln|\sec x|$$

$\int \cot x = \ln|\sin x| + C$

Ques: $\int \frac{\tan(\sin^{-1}x)}{\sqrt{1-x^2}}\,dx$

Solu:
$$\sin^{-1}x = y$$
$$\frac{1}{\sqrt{1-x^2}}dx = dy$$

$$\int \tan y\,dy = \cancel{xy}\ \ln|\sec y| + C$$

$$= \ln|\sec(\sin^{-1}x)| + C$$

Ques: $\int \frac{\cos x}{\cos(x-a)}\,dx$

Solu: $\int \frac{\cos(x-a+a)\,dx}{\cos(x-a)}$

$$= \int \frac{\cos(x-a)\cos a - \sin(x-a)\sin a}{\cos(x-a)}$$

$$= \int \cos a - \tan(x-a)\sin a$$

$$= x\cos a - \ln|\sec(x-a)|\sin a + C$$

Ques: $\int \frac{dx}{\cos(x-a)\cos(x-b)}$ = whenver there is sine or cos in d^r then multiply by its ~~sine~~ sine of angle diff.

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$$\frac{1}{\sin(b-a)}\int\frac{\sin[(x-b)-(x-a)]}{\cos(x-a)\cos(x-b)}$$

$$=\frac{1}{\sin(b-a)}\int(\tan(x-b)-\tan(x-a))\,dx$$

$$=\frac{1}{\sin(b-a)}\ln\left|\frac{\sec(x-b)}{\sec(x-a)}\right|+c$$

Ques: $\displaystyle\int\frac{x\cos x}{(x\sin x+\cos x)^5}\,dx$

Solu: $\displaystyle\int\frac{f'(x)}{[f(x)]^5}\,dx = -4[f(x)]^{-4}+c = -4(x\sin x+\cos x)^{-4}+c$

Ques: $\displaystyle\int\frac{x+(\cos^{-1}(3x))^2}{\sqrt{1-9x^2}}\,dx$

Solu: $\displaystyle\frac{1}{-18}\int\frac{-18x}{\sqrt{1-(3x)^2}}\,dx+\int\frac{(\cos^{-1}(3x))^2}{\sqrt{1-(9x^2)}}\,dx$

$$-\frac{1}{18}\times 2\times\sqrt{1-(3x)^2}+\frac{1}{3}\int y^2\,dy$$

$\cos^{-1}(3x)=y$

$\dfrac{1}{\sqrt{1-9x^2}}\times 3dx = dy$

$$-\frac{1}{9}\sqrt{1-9x^2}+\frac{1}{3\times 3}(\cos^{-1}(3x))^3+c$$

Ques: $\displaystyle\int\frac{x^5}{1+x^{12}}\,dx = \int\frac{6x^5}{1+(x^6)^2} = \frac{1}{6}\tan^{-1}(x^6)+c$

Ques: $\displaystyle\int\tan^3x-x\tan^2x\,dx=\int\tan^2x\{\tan x-x\}\,dx$ ($f'(x)$, $f(x)$)

$$=\frac{(\tan x-x)^2}{2}+c$$

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Q $\int \frac{dx}{x^{1/2}+x^{1/3}}$ * $\boxed{\text{LCM of } 2 \wedge 3}$

$x = y^6 \quad dx = 6y^5 dy$

$\int \frac{6y^5 \, dy}{y^3+y^2}$

$= \int \frac{6y^3}{y+1} = 6\int \frac{y^3+1-1}{y+1} = 6\int y^2 - y + 1 - \frac{1}{y+1} \, dy$

$= 6\frac{y^3}{3} - \frac{y^2}{2} + y - \ln|y+1| \quad + C$

$= 6x^{1/2} - \frac{x^{1/3}}{2} + x^{1/6} - \ln|x^{1/6}+1| + C$

Q: $\int \frac{\ln^3\left(\frac{x}{x+1}\right) dx}{x(x+1}$

Solu. $\ln(x) - \ln(x+1) = y$

$dy = \left(\frac{1}{x} - \frac{1}{x+1}\right)dx = \frac{1}{x(x+1)} dx$

$\int y^3 dy = \frac{y^4}{4} = \frac{\ln^4(x/x+1)}{4} + C$

$\int \operatorname{Sec}x \, dx = \ln|\operatorname{Sec}x + \operatorname{Tan}x| + C$

$= \ln\left|\operatorname{Tan}\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + C$

$\int \operatorname{Sec}x \, dx = \int \frac{\operatorname{Sec}x(\operatorname{Sec}x + \operatorname{Tan}x)}{\operatorname{Sec}x + \operatorname{Tan}x}$

$= \ln|\operatorname{Sec}x + \operatorname{Tan}x|$

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\# $\int \text{Cosec}x\,dx = \ln|(\text{Cosec}x - \text{Cot}x)| + c$

$= \ln|\text{Tan}(\frac{x}{2})| + c$

$\int \text{Cosec}x\,dx = -\int \frac{-\text{Cosec}x(\text{Cosec}x + \text{Cot}x)}{(\text{Cosec}x + \text{Cot}x)}$

$= -\ln|\text{Cosec}x + \text{Cot}x|$

$= \ln|\text{Cosec}x - \text{Cot}x| + c$

Ques: $\int \frac{\text{Tan}x + \text{Sec}x - 1}{\text{Tan}x - \text{Sec}x + 1}\,dx$

$= \int \frac{T + S - (S-T)(S+T)}{T - S + 1} = \int \frac{(T+S)(1 - \cancel{S} + T)}{(T - \cancel{S+1})} = \ln|\text{Sec}x| + \ln|\text{Sec}x + \text{Tan}x| + c$

Ques: $\int \frac{dx}{\text{Sin}x\,\text{Cos}^2 x}$

$\int \frac{\text{Sin}^2 x + \text{Cos}^2 x}{\text{Sin}x\,\text{Cos}^2 x}\,dx = \int \text{Sec}x\,\text{Tan}x + \text{Cosec}x\,dx$

$= \text{Sec}x + \ln|\text{Cosec}x - \text{Cot}x| + c$

Ques: $\int \frac{dx}{\text{Sec}x + \text{Cosec}x}$

Solu: $\int \frac{\text{Sin}x\,\text{Cos}x\,dx}{\text{Cos}x + \text{Sin}x}$

☆ $(\text{Sin}x + \text{Cos}x)^2 = 1 + 2\text{Sin}x\,\text{Cos}x$

☆ $(\text{Sin}x - \text{Cos}x)^2 = 1 - 2\text{Sin}x\,\text{Cos}x$

$= \frac{1}{2}\int \frac{(\text{Sin}x + \text{Cos}x)^2 - 1}{\text{Cos}x + \text{Sin}x}$

☆ $a\,\text{Sin}x + b\,\text{Cos}x = \sqrt{a^2+b^2}\,\text{Cos}(x - \theta)$

$= ~~\frac{1}{2}\int \text{Sin}x + \text{Cos}~~$

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$$\frac{1}{2}\int S+C\ dx - \frac{1}{2\sqrt{2}}\int \frac{dx}{\frac{S+C}{\sqrt{2}}}$$

$$\frac{1}{2}(-\cos x+\sin x) - \frac{1}{2\sqrt{2}}\int \sec(x-\pi/4)\,dx$$

$$\frac{1}{2}(\sin x-\cos x) - \frac{1}{2\sqrt{2}}\ln|\sec(x-\pi/4)+\tan(x-\pi/4)|+c$$

Ques: $\displaystyle\int \frac{dx}{\tan x+\cot x+\sec x+\operatorname{cosec} x}$

Solu: $\displaystyle\int \frac{dx}{\frac{S^2+C^2+S+C}{SC}} = \int \frac{SC}{1+S+C}\,dx$

$$= \frac{1}{2}\int \frac{(S+C)^2-1}{1+S+C}$$

$$= \frac{1}{2}\int \frac{(S+C-1)\cancel{(S+C+1)}}{\cancel{(S+C+1)}}$$

$$= \frac{1}{2}(-\cos x+\sin x-x)+c_1$$

Ques: $\displaystyle\int \frac{e^x-1}{e^x+1}\,dx$

multiplied & divided by e^{-x}

Solu: $\displaystyle\int \frac{e^x}{e^x+1}\,dx + \int \frac{-e^{-x}}{1+e^{-x}}\,dx$

$$= \ln|(e^x+1)| + \ln|(1+e^{-x})| + c$$

$$= \ln|(1+e^x)(1+e^{-x})| + c$$

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General Substitution

#1. $\sqrt{a^2-x^2}$ $\qquad x = a\sin\theta / a\cos\theta$

#2. $\sqrt{a^2+x^2} / a^2+x^2 / (a^2+x^2)^2 / (a^2+x^2)^3$ $\qquad x = a\tan\theta / a\cot\theta$

#3. $\sqrt{x^2-a^2}$ $\qquad x = a\sec\theta / a\,\mathrm{cosec}\,\theta$

#4. $\sqrt{\frac{a^2-x^2}{a^2+x^2}}$ or $\sqrt{\frac{1-x}{1+x}}$ $\qquad x^2 = a^2\cos 2\theta$

#5. $\sqrt{\frac{x-\alpha}{x-\beta}}$ / $\sqrt{(x-\alpha)(x-\beta)}$ $\qquad x = \alpha\sec^2\theta - \beta\tan^2\theta$

#6. $\sqrt{\frac{x-\alpha}{\beta-x}}$ / $\sqrt{(x-\alpha)(\beta-x)}$ $\qquad x = \alpha\cos^2\theta + \beta\sin^2\theta$

Or rationalise them. $\frac{x-\alpha}{\sqrt{(x-\alpha)(\beta-x)}}$

Ques: ⓐ $\int \frac{x\,dx}{(1+x^2)+\sqrt{(1+x^2)^3}}$ / ⓑ $\int \frac{x\,dx}{\sqrt{(1+x^2)+\sqrt{(1+x^2)^3}}}$

M-I $\quad x = \tan\theta$

M-II Substitution of square $\quad 1+x^2 = y^2$,

ⓐ M-I

$x = \tan\theta \qquad dx = \sec^2\theta\,d\theta$

$$\int \frac{\tan\theta\sec^2\theta\,d\theta}{\sec^2\theta+\sec^3\theta} = \int \frac{\tan\theta\,d\theta}{1+\sec\theta} = \int \frac{\sin\theta}{1+\cos\theta}\,d\theta = \int \tan\frac{\theta}{2}\,d\theta$$

$$= \frac{\ln|\sec\theta/2|}{1/2} + C = 2\ln|\sec\theta/2| + C$$

M-II $\qquad 1+x^2 = y^2 \qquad 2x = 2y\frac{dy}{dx}$

$$\int \frac{y\,dy}{y^2+y^3} = \int \frac{dy}{y^2+y} = \int \frac{dy}{y(y+1)} = \int \left(\frac{1}{y} - \frac{1}{y+1}\right)dy$$

$= \ln\left(\frac{y}{\ }\right) + c$

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(6) M-I

$$x = \tan\theta \qquad dx = \sec^2\theta\, d\theta$$

$$\int \frac{\tan\theta \sec^2\theta\, d\theta}{\sqrt{\sec^2\theta + \sec^3\theta}}$$

$$\int \frac{\tan\theta \sec\theta\, d\theta}{\sqrt{\sec\theta + 1}} = \int \frac{1\, dt}{\sqrt{t}}$$

$$= 2\sqrt{t} + c$$

$$= 2\sqrt{\sec\theta + 1} + c$$

M-II

$$x^2 + 1 = y^2 \qquad x\,dx = y\,dy$$

$$\int \frac{y\,dy}{\sqrt{y^2 + y^3}} = \int \frac{dy}{\sqrt{y+1}} = 2\sqrt{y+1} + c$$

Ques: $\int \cos\left(2\cot^{-1}\left(\sqrt{\frac{1-x}{1+x}}\right)\right) dx$

$$\cos 2\theta = \frac{1 - \tan^2\theta}{1 + \tan^2\theta}$$

$$\cot^{-1}\left(\sqrt{\frac{1-x}{1+x}}\right) = \theta$$

$$\left(\sqrt{\frac{1-x}{1+x}}\right) = \cot\theta \Rightarrow \tan\theta = \sqrt{\frac{1+x}{1-x}}$$

$$\int \frac{1 - \frac{1+x}{1-x}}{1 + \frac{1+x}{1-x}}\, dx = \int \frac{-2x}{2}\, dx = \frac{-x^2}{2} + c$$

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Ques $\displaystyle\int \sqrt{x}\,\mathrm{Tan}\left(2\,\mathrm{Tan}^{-1}\left(\frac{\sqrt{\sqrt{1+\sqrt{x}}+1}-\sqrt{\sqrt{1+\sqrt{x}}-1}}{\sqrt{\sqrt{1+\sqrt{x}}+1}+\sqrt{\sqrt{1+x}-1}}\right)\right)dx \qquad x\in(0,1)$

Solu: $x = \mathrm{Tan}^4\theta$

$$2\,\mathrm{Tan}^{-1}(\quad) = 2\,\mathrm{Tan}^{-1}\left(\frac{\sqrt{1+\sec\theta}-\sqrt{\sec\theta-1}}{\sqrt{1+\sec\theta}+\sqrt{\sec\theta-1}}\right)$$

$$= 2\,\mathrm{Tan}^{-1}\left(\frac{\sqrt{1+\cos\theta}-\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}+\sqrt{1-\cos\theta}}\right)$$

$$= 2\,\mathrm{Tan}^{-1}\left(\frac{\cos\theta/_2-\sin\theta/_2}{\cos\theta/_2+\sin\theta/_2}\right)$$

$$= 2\,\mathrm{Tan}^{-1}\left(\frac{1-\mathrm{Tan}\,\theta/_2}{1+\mathrm{Tan}\,\theta/_2}\right)$$

$$= 2\,\mathrm{Tan}^{-1}\left(\mathrm{Tan}\left(\frac{\pi}{4}-\theta/_2\right)\right) = \frac{\pi}{2}-\theta$$

$$\int \sqrt{\mathrm{Tan}^4\theta}\,\mathrm{Tan}\left(\frac{\pi}{2}-\theta\right) = \int \frac{\sqrt{x}}{x^{1/4}}\,dx = \int x^{1/4}\,dx = \frac{4}{5}x^{5/4}+c$$

Ques: $\displaystyle\int \mathrm{Cot}\left(2\,\mathrm{Tan}^{-1}\left(\sqrt{\frac{\sqrt{1+\sqrt{x}}-\sqrt[4]{x}}{\sqrt{1+\sqrt{x}}+\sqrt[4]{x}}}\right)\right)dx.$

$x = \mathrm{Tan}^4 x$ can be taken but best will be $x = \mathrm{Cot}^4\theta$.

$$\mathrm{Cot}\left(2\,\mathrm{Tan}^{-1}\sqrt{\frac{\mathrm{Cosec}\,\theta-\mathrm{Cot}\,\theta}{\mathrm{Cosec}\,\theta+\mathrm{Cot}\,\theta}}\right)$$

$$= \mathrm{Cot}\left(2\,\mathrm{Tan}^{-1}\left(\frac{1-\cos\theta}{1+\cos\theta}\right)^{1/2}\right)$$

$$= \mathrm{Cot}\left(2\,\mathrm{Tan}^{-1}\left(\tan\frac{\theta}{2}\right)\right) = \int \cot\theta = \int x^{1/4}\,dx = \frac{4}{5}x^{5/4}+c$$

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Ques $\int \frac{dx}{\sqrt{-2x^2+3x+5}}$

Sol^n $\int \frac{dx}{\sqrt{5+ -2(x^2+3/_2+\frac{9}{16}-\frac{9}{16})}} = \int \frac{dx}{\sqrt{\frac{49}{8}-2(x-3/_4)^2}}$

$= \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\left(\frac{7}{4}\right)^2-\left(x-\frac{3}{4}\right)^2}} = \frac{1}{\sqrt{2}} \sin^{-1}\left(\frac{x-3/4}{7/4}\right)$

or.

$\int \frac{dx}{((x+1)(5-2x))^{1/2}}$

~~(x&)~~ :

$x=(-1)\cos^2\theta+5/_2\sin^2\theta$

$dx= 2\sin\theta\cos\theta + 5\sin\theta\cos\theta$

$(x+1)(5-2x)$

$\int \frac{7\sin\theta \cos\theta\, d\theta}{\sqrt{7/_2\sin^2\theta\ 7\cos^2\theta}}$

$\int \sqrt{2}\, d\theta = \sqrt{2}\,\theta$ "

Ques. $\int \frac{\sqrt{(9-x^2)^3}}{x^6}$

Solu:

$x=3\sin\theta \quad dx=3\cos\theta\, d\theta$

$\int \frac{9^{3}\cos^3\theta}{(3)^6\sin^6\theta} 3\cos\theta\, d\theta = \int \cot^4\theta\,\mathrm{cosec}^2\theta\, d\theta$

$= -\int y^4 d\theta = \frac{y^5}{5}+c$

$= \frac{\cot^5\theta}{5}+c$

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Ques: $\int \frac{\sqrt{x}}{\sqrt{a^3 - x^3}}\,dx$

Sol^n

M I - $x = a\sin^{2/3}\theta$

M II $x^{3/2} = y$ $\quad \frac{3}{2}x^{1/2}\,dx = dy$

$$\frac{2}{3}\int \frac{dy}{\sqrt{a^3 - y^2}} = \frac{2}{3}\sin\left(\frac{y}{a^{3/2}}\right)$$

Ques: $\int \frac{dx}{(x^2+4)\sqrt{4x^2+1}}$

Sol^n

M - 1 $\quad x = \frac{1}{2}\tan\theta$

M - 2 $\quad \int \frac{dx}{x^3\left(1+\frac{4}{x^2}\right)\sqrt{4+1/x^2}}$ $\qquad \frac{1}{x^2} + 4 = y^2$

$$-\frac{2}{x^3}\,dx = 2y\,dy$$

$$-\int \frac{y\,dy}{y(4y^2-15)} = -\int \frac{dy}{(4y^2-15)} = \cancel{\frac{-1}{2(15)}\ln\left|\frac{4y-18}{4y+15}\right|}$$

$$-\int \frac{dy}{(2y)^2 - (\sqrt{15})^2} = \frac{-1}{2\sqrt{15}}\ln\left|\frac{2y-\sqrt{15}}{2y+\sqrt{15}}\right|\frac{1}{2} + c$$

V. Imp

Note: $\boxed{\tan(x+y)\tan x\tan y = \tan(x+y) - \tan x - \tan y}$

Q. $\int \tan(100x)\tan(73x)\tan(27x)\,dx$

$$= \int \tan 100x - \tan(73x) - \tan(27x)\,dx$$

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$$= \frac{\ln|Sec(100x)|}{100} - \frac{\ln|Sec(73x)|}{73} - \frac{\ln|Sec(27x)|}{27}$$

Ques: $\int Tan^9 x\,dx$

Solu^n $\int Tan^7 x\,(Sec^2 x - 1)\,dx$

$$= \frac{Tan^8 x}{8} - \int Tan^5 x\,(Sec^2 x - 1)\,dx$$

$$= \frac{Tan^8 x}{8} - \frac{Tan^6 x}{6} + \frac{Tan^4 x}{4} - \frac{Tan^2 x}{2} + \underbrace{\ln|Sec x|}_{\text{due to odd power}} + C$$

Important

$$\boxed{\begin{aligned} \# \quad & \int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left(x + \sqrt{x^2+a^2}\right) + C \\ \# \quad & \int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left(x + \sqrt{x^2-a^2}\right) + C \end{aligned}}$$

$x = a\,Tan\theta$

$x = a\,Sec\theta$

Ques: $\int \frac{e^x\,dx}{\sqrt{e^{2x}-1}}$

$e^x = y$

$e^x dx = dy$

$$\int \frac{dy}{\sqrt{y^2-1}} = \ln\left(y + \sqrt{y^2-1}\right) + C$$

Ques: $\int \frac{e^x}{\sqrt{5 - 4e^x + e^{2x}}}$

Sol^n $e^x - 2 = y$ $\int \frac{dy}{\sqrt{y^2+1}} = \ln\left(y + \sqrt{y^2+1}\right) + c$

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$\int \sin^n x \cos^n x \, dx$

I. m,n both even use Trigonometric Identities like $\sin 2x$, $\cos 2x$..

II. ———— Odd Substitute Higher Powers i.e. $s = y$ or $c = y$

III. One odd & One even substitute given power

IV. $\boxed{m+n = -\text{ve even substitute } \tan x = y \text{ or } \cot x}$

Ques: $\int \sin^2 x \cos^4 x \, dx = \frac{1}{4}\int \sin^2 2x \cos^2 x \, dx$

$= \frac{1}{8}\int \sin^2 2x\,(1+\cos 2x)\, dx$

$= \frac{1}{16}\int (1-\cos 4x)(1-\cos 2x)\, dx$

$= \frac{1}{16}\int 1 - \cos 4x - \cos 2x - \frac{1}{2}(\cos 6x + \cos 2x)\, dx$

$= \frac{1}{16}\left[x - \frac{\sin 4x}{4} - \frac{\sin 2x}{2} - \frac{1}{2}\left(\frac{\sin 6x}{6} + \frac{\sin 2x}{2}\right)\right] + C$

Ques: $\int \sin^3 x \cos^5 x \, dx$

$y = \cos x \quad dy = -\sin x \, dx$

$-\int y^5(1-y^2)\,dy = \int (y^7 - y^5)\,dy = \frac{y^8}{8} - \frac{y^6}{6} = \frac{\cos^8 x}{8} - \frac{\cos^6 x}{6} + C$

Ques: $\int \sin^4 x \cos^5 x \, dx$

$y = \sin x \quad dy = \cos x \, dx$

$\int y^4(1-y^2)^2\,dy = \int y^4(1+y^4-2y^2)\,dy = \int (y^4 + y^8 - 2y^6)\,dy$

$= \frac{\sin^5 x}{5} + \frac{\sin^9 x}{9} - \frac{2\sin^7 x}{7} + C$

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Ques: $\int \frac{dx}{\sin^{7/2}x \cos^{13/2}x}$

$\int \frac{\sec^2 x\, dx}{\sin^{7/2}x \cos^{7/2}x \cos^{13/2}x \sec^2 x}$

$= \int \frac{(1+T^2)^4\, dx \sec^2 x}{T^{7/2}}$

Binomial Theorem

$= \int \frac{1 + 4y^2 + 6y^4 + 4y^6 + y^8}{y^{7/2}}\, dy$

$= -\frac{2}{5} y^{-5/2} + 4y^{-1/2} \times (-2) + y^{7/2}\frac{2}{7} + y^{11/2}\left(\frac{2}{11}\right) + c$

Ques: $\int \frac{\sec^2 x\, dx}{(\sec x + \tan x)^{9/2}}$

Imp IIT 2 time asked on concept

Solⁿ

$\begin{bmatrix} \sec x + \tan x = y \\ \sec x - \tan x = \frac{1}{y} \end{bmatrix}$ (reciprocal wo hota jise multiply krne pe value 1)

$\int \frac{\sec x (\sec x + \tan x) \sec x\, dx}{(\sec x + \tan x)^{11/2}}$

$\int \frac{\frac{1}{2}\left(y + \frac{1}{y}\right) dy}{y^{11/2}} = \frac{1}{2}\int y^{-9/2} + y^{-13/2}\, dy$

$= \frac{1}{2}\left[\frac{-2}{7} y^{-7/2} + \frac{-2}{11} y^{-11/2}\right] + c$

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$\int \frac{\text{Linear}}{\text{Quad.}}$ // $\int \frac{\text{Linear}}{\sqrt{\text{Quad.}}}$

$$\int \frac{(ax+b)\,dx}{px^2+qx+r} \quad // \quad \int \frac{(ax+b)dx}{\sqrt{px^2+qx+r}}$$

Working rule:	$ax+b = A\frac{d}{dx}(px^2+qx+r)+B$	Identify in x Find A, B

Ques: $\int \frac{(2x-1)dx}{\sqrt{4x^2+4x+2}}$

Sol^n $2x-1 = A(8x+4)+B$

$A = \frac{1}{4}$ $\quad -1 = 4A+B \Rightarrow B = -2$

$$\frac{1}{4}\int \frac{8x+4}{\sqrt{4x^2+4x+2}} - 2\int \frac{dx}{\sqrt{(2x+1)^2+1^2}}$$

$$\frac{2}{4}\sqrt{4x^2+4x+2} - \frac{2}{2}\ln\left|2x+1+\sqrt{(2x+1)^2+1}\right| + C$$

Ques: $\int \frac{(4x+3)\,dx}{3x^2+3x+1}$

Sol^n $4x+3 = A(6x+3)+B$

$A = \frac{2}{3}$ $\quad B = 1$

$$\frac{2}{3}\int \frac{6x+3}{3x^2+3x+1} + \frac{1}{3}\int \frac{dx}{(x+\frac{1}{2})^2+\left(\frac{1}{\sqrt{12}}\right)^2}$$

$$\frac{2}{3}\ln|3x^2+3x+1| + \frac{1}{3}\cdot\frac{1}{\frac{1}{\sqrt{12}}}\tan^{-1}\left(\frac{x+1/2}{1/\sqrt{12}}\right) + C$$

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$\int$ by parts

$$\frac{d}{dx} fg = fg' + f'g$$

$$fg' = \frac{d}{dx} fg - f'g$$

Intigrating both sides.

$$\int fg' \, dx = fg - \int f'g \, dx$$

$$\boxed{\int uv \, dx = u\int v - \int u'\left(\int v\right) dx}$$

#1 $\underset{\frac{d}{dx}}{\text{I}}$ LAT$\underset{\int}{\text{E}}$

#2. Sometimes 1 is taken one of the function & integrated

#3 2nd term of by parts should be less complicated as compared to original integration.

#4 Many time two expression are given - Apply biparts in one expression in its 2nd term it get cancelled by original integration.

$$\int uv + gh \, dx.$$

Que: $\int x \tan^{-1} x \, dx$

$$= \tan^{-1} x \left(\frac{x^2}{2}\right) - \int \frac{1}{x^2+1} \frac{x^2}{2} \, dx$$

$$= \tan^{-1} x \left(\frac{x^2}{2}\right) - \frac{1}{2}\int \frac{x^2+1}{x^2+1} - \frac{1}{x^2+1} \, dx$$

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$= \tan^{-1}x \dfrac{x^2}{2} - \dfrac{1}{2}x + \dfrac{1}{2}\tan^{-1}x$

Ques: $\displaystyle\int \frac{\sin^{-1}x}{(1-x^2)^{3/2}}\,dx$ $\qquad x = \sin\theta$

$dx = \cos\theta\, d\theta$

$\displaystyle\int \frac{\theta\cos\theta\, d\theta}{\cos^3\theta} = \int \theta\sec^2\theta\, d\theta = \theta\tan\theta - \int \tan\theta\, d\theta$

$= \theta\tan\theta - \ln|\sec\theta| + c$

Ques: $\displaystyle\int e^x(1+x)\ln(xe^x)\,dx$

$xe^x = y$

$\Rightarrow$

$\displaystyle\int \ln y\, dy = y\ln y - \int y\frac{1}{y}\,dy = y\ln y - y$

$$\boxed{\begin{aligned}
&\int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + c\\
&\int\sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ln|x+\sqrt{x^2+a^2}| + c\\
&\int\sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln|x+\sqrt{x^2-a^2}| + c
\end{aligned}}$$

$\displaystyle\int\sqrt{a^2-x^2}\,dx = x\sqrt{a^2-x^2} + \int\frac{x^2-a^2}{\sqrt{a^2-x^2}} + a^2\int\frac{1}{\sqrt{a^2-x^2}}\,dx$

$I = x\sqrt{a^2-x^2} - I + a^2\sin^{-1}\left(\frac{x}{a}\right)$

$I = \dfrac{x}{2}\sqrt{a^2-x^2} + a^2\sin^{-1}\left(\dfrac{x}{a}\right) + c$

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Ques: $\int \sec^3 x \, dx$

$$= \int \sec^2 x \sqrt{1+\tan^2 x} \, dx$$

$$= \int \sqrt{1+y^2} \, dy$$

or

Sol^n $I = \int \sec x \cdot \sec^2 x \, dx$

$$I = \sec x \tan x - \int \sec x \tan^2 x \, dx$$

$$I = \sec x \tan x - \int \sec x (\sec^2 x - 1) \, dx$$

$$2I = \sec x \tan x + \ln|\sec x + \tan x|$$

$$I = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln|\sec x + \tan x| + c$$

Ques: $\int \sin^{-1} \sqrt{\frac{x}{a+x}} \, dx$

Sol^n

$$x = a \tan^2\theta$$

$$dx = 2a \tan\theta \sec^2\theta \, d\theta$$

$$2a \int \theta \tan\theta \sec^2\theta \, d\theta$$

$$= \frac{\theta \tan^2\theta}{2} - \frac{1}{2} \int (\sec^2\theta - 1) \, d\theta$$

$$= \frac{\theta \tan^2\theta}{2} - \frac{1}{2} \tan\theta + \frac{\theta}{2} + c$$

2 Classic Intergrands

$$\boxed{\begin{aligned} &\int e^x (f(x) + f'(x)) \, dx = e^x f(x) + c \\ &\int f(x) + x f'(x) \, dx = x f(x) + c \end{aligned}}$$

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Ques: $\int e^x \frac{x}{(1+x)^2} dx$

$\int e^x \left(\frac{x+1-1}{(x+1)^2}\right) dx$

$\int e^x \left(\frac{1}{x+1} - \frac{1}{(x+1)^2}\right) = \frac{e^x}{x+1} + C$

Ques $\int e^x \left(\frac{1+\sin x}{1+\cos x}\right) dx$

$e^x \left(\frac{1}{2}\sec^2\left(\frac{x}{2}\right) + \tan x/_2\right) dx$

$= e^x \tan\frac{x}{2} + C$

Ques $\int e^x x^4 dx$

$\int e^x \{x^4 + 4x^3 - 4x^3 - 12x^2 + 12x^2 + 24x - 24x - 24 + 24\} dx$

$= e^x (x^4 - 4x^3 + 12x^2 - 24x + 24) + C$

Ques $\int (x^3 + 3x + 1) e^{3x} dx$

$\int$ by $\frac{d}{dx}$ We can visualise that answer will be

$\int (x^3 + 3x + 1) e^{3x} dx = e^{3x}(Ax^3 + Bx^2 + Cx + D) + \lambda$

Now differentiating both sides

$\cancel{e^{3x}}(x^3 + 3x + 1) = 3\cancel{e^{3x}}(Ax^3 + Bx^2 + Cx + D) + \cancel{e^{3x}}(3Ax^2 + 2Bx + C)$

$1 = 3A$ $\quad$ $3 = 3C + 2B$

$0 = 3B + 3A$ $\quad$ $1 = 3D + C$

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can be applied by part

$$\int e^{g(x)}\left(f(x)\,g'(x) + f'(x)\right)dx = e^{g(x)} f(x) + c$$

By Part

$$f(x)\,e^{g(x)} - \cancel{\int f'(x)\,e^{g(x)}} + \cancel{\int f'(x)\,e^{gx}} =$$

Ques: $\displaystyle\int e^{\cos^{-1}x}\left(\frac{x+1+\sqrt{1-x^2}}{(x+1)^2\sqrt{1-x^2}}\right)$

$$\int e^{\cos^{-1}x}\left(\frac{1}{(x+1)\sqrt{1-x^2}} + \frac{1}{(x+1)^2}\right)$$

$$\frac{-1}{\sqrt{1-x^2}} = g' \qquad f = \frac{-1}{(1+x)}$$

$$\text{Ans} = e^{\cos^{-1}x}\left(\frac{-1}{1+x}\right)$$

Ques: $\displaystyle\int a^x\left(\ln x + \ln a\left(\ln\left(\frac{x}{e}\right)^x\right)\right)dx$

Sol^n.

$$\int \underset{f}{a^x\ln x} + \underbrace{x a^x \ln a \ln x - a^x \ln a\, x + a^x}_{x f'(x)} - a^x$$

$$= x a^x \ln x - x a^x + C$$

Ques: $\displaystyle\int\left(\frac{\ln x - 1}{\ln^2 x + 1}\right)^2 dx$

$$= \int \overset{f}{\frac{1}{\ln^2 x + 1}} - \overset{x f'(x)}{\frac{2\ln x}{(\ln^2 x + 1)^2}} = \frac{x}{(\ln^2 x + 1)} + C$$

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Ques: $\int x^x(\ln^2 x + \ln x + \frac{1}{x})\,dx$

$y = x^x \qquad dy = x^x(\ln x + 1)$

$\int \underbrace{x^x(\ln x+1)}_{g'}\underbrace{\ln x}_{f} + \frac{\overset{g}{x^x}}{\underset{f'}{x}}\,dx$

$= x^x \ln x$

Ques: $\int \sin(101x)\sin^{99}x\,dx$

$\int \underbrace{(\sin 100x\cos x + \cos 100x\sin x)\sin^{99}x}_{\text{By parts}}\,dx$

$\sin(100x)\frac{\sin^{100}x}{100} - \int \frac{\sin^{100}x}{\cancel{100}}\cos(100x)\,\cancel{100}\,dx$

$+ \int \sin^{100}x\cos(100x)\,dx$

$= \sin(100x)\frac{\sin^{100}x}{100} + c$

$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$ *

$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$

Part I

M-I By part

$I = e^{ax}\left(-\frac{\cos bx}{b}\right) + \int \cos bx\, e^{ax}\cdot\frac{a}{b}\,dx.$

$I = -\frac{1}{b}e^{ax}\cos bx + \frac{a}{b}\left[e^{ax}\frac{\sin bx}{b} - \frac{a}{b}\int \sin bx\, e^{ax}\,dx\right].$

$I\left(1+\frac{a^2}{b^2}\right) = \frac{(-b\cos bx + a\sin bx)}{b^2}\,e^{ax}$

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M–II

$$\int e^{ax}\sin bx\,dx = e^{ax}(A\sin bx + B\cos bx) + \lambda$$

$$e^{ax}\sin bx = a e^{ax}(A\sin bx + B\cos bx)$$
$$+ e^{ax}(Ab\cos bx - Bb\sin bx)$$

$$1 = aA - bB$$
$$0 = aB + Ab$$

M–III

By Complex $\boxed{e^{idx} = \cos bx + i\sin bx}$

required is Img

$$\mathrm{Im}\left(\int e^{ax}\cdot e^{ibx}\,dx\right)$$

$$\mathrm{Im}\left(\frac{e^{x(a+ib)}}{a+ib} \times \frac{a-ib}{a-ib}\right)$$

$$\mathrm{Im}\left(\frac{e^{x(a+ib)}(a-ib)}{a^2+b^2}\right)$$

$$\mathrm{Im}\left(\frac{e^{ax}(\cos x + i\sin bx)(a-ib)}{a^2+b^2}\right)$$

$$-\frac{e^{ax}\cos xb + a\sin bx\, e^{ax}}{a^2+b^2}$$

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Partial Fraction

Theory of partial fractions

$\int \frac{P_1}{P_2} dx$ P_1 & P_2 are Polynomials.

Case I Degree of $Nr \geq Dr$ & proceed to Case II

Case II Degree $Nr < Dr$

Form.	Form of Partial fraction
$\frac{px+q}{(x-a)(x-b)}$	$\frac{A}{(x-a)} + \frac{B}{(x-b)}$
$\frac{px+q}{(x-a)^2}$	$\frac{A}{(x-a)} + \frac{B}{(x-a)^2}$
$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{(x-a)} + \frac{B}{(x-b)} + \frac{C}{(x-c)}$ is an Identity in x.
$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$	$\frac{A}{(x-a)} + \frac{Bx+C}{x^2+bx+c}$

x^2+bx+c can't be factorised.

Ques: $\int \frac{(x^2+2x+4)}{(x-1)(x-2)(x+1)} dx$

$$\frac{(x^2+2x+4)}{(x-1)(x-2)(x+1)} = \frac{A}{(x-1)} + \frac{B}{(x-2)} + \frac{C}{(x+1)}$$

$$x^2+2x+4 = A(x-2)(x+1) + B(x-1)(x+1) + C\,\cancel{(x+1)}(x-1)(x-2)$$

Sub. $x=2$

$4+4+4 = 3B \quad B=4$

$x=1 \quad 7 = -2A \quad A = -7/2$

$x=-1 \quad 3 = 6C$

$$\int \frac{A}{x-1} + \boxed{\frac{B}{x-2} + \frac{C}{x+1}}\, dx$$

$$\frac{A}{\cancel{x-1}} = \frac{x^2+2x+4}{\cancel{(x-1)}(x-2)\cancel{(x+1)}}$$

Putting $x=1$

$$A = \frac{1+2+4}{(-1)(2)} = \frac{-7}{2}$$

Ques: $\int \frac{dx}{\sin x\,(3+2\cos x)}$

$$\int \frac{\sin x\, dx}{(1-\cos^2 x)(3+2\cos x)} = -\int \frac{\sin x\, dx}{(\cos x-1)(\cos x+1)(3+2\cos x)}$$

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$\cos x - 1 = y$

$$\int \frac{dy}{y(y+2)(2y+5)} = \frac{A}{y} + \frac{B}{y+2} + \frac{C}{2y+5}$$

$A = \frac{1}{10}$ $B = \frac{1}{-2(1)} = \frac{-1}{2}$ $C = \frac{1}{-5/2(-5/2+2)} = \frac{1}{-5/2(-1/2)}$

$C = \frac{4}{5}$

Ans: $\int \frac{x\,dx}{(x-1)(x^2+4)}$

$$\int \frac{A}{(x-1)} + \frac{Bx+C}{x^2+4}\,dx$$

$x = A(x^2+4) + (Bx+c)(x-1)$

$x = 1$

$5A = 1$ $A = \frac{1}{5}$

$x = \frac{1}{5}(x^2+4) + (Bx+c)(x-1)$

$x = 0$

$0 = \frac{4}{5} - c \Rightarrow c = 4/5$

$x = 2$

$2 = 8A + (2B+c)$

$2 = \frac{8}{5} + 2B + \frac{4}{5}$

$2 - \frac{12}{5} = 2B$ $B = 1 - \frac{6}{5} = -1/5$

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$$\frac{1}{5}\int \frac{-x+4}{x^2+4}\,dx = \frac{1}{10}\int \frac{-2x+8}{x^2+4}\,dx$$

$$= \frac{-1}{10}\ln|x^2+4| + \frac{8}{10}\times\frac{1}{2}\tan^{-1}\frac{x}{2}$$

Ans: $$\int \frac{(x-1)(x-2)(x-3)}{(x-4)(x-5)(x-6)}\,dx$$

$$= \int \left(1 + \frac{(x-1)(x-2)(x-3) - (x-4)(x-5)(x-6)}{(x-4)(x-5)(x-6)}\right) dx$$

$$= \int 1 + \frac{A}{x-4} + \frac{B}{x-5} + \frac{C}{x-6}\,dx$$

$$A = 3 \qquad B = -24 \qquad C = 30$$

Ans: $$\int \frac{\sin x}{\sin(4x)}\,dx$$

$$\frac{1}{4}\int \frac{\cancel{\sin x}}{\cancel{\sin x}\cos x\cos 2x}\,dx = \frac{1}{4}\int \frac{\cos x\,dx}{(1-\sin^2 x)(1-2\sin^2 x)}$$

$$= \frac{1}{4}\int \frac{dy}{(1-y^2)(1-2y^2)} = \frac{1}{8}\int \frac{dy}{(y^2-1)(y^2-1/2)}$$

$$= \frac{\frac{1}{8}}{1/2}\int \frac{(y^2-1/2) - (y^2-1/2)}{(y^2-1)(y^2-1/2)}\,dy$$

$$= \frac{1}{4}\int \frac{1}{y^2-1}\,dy - \frac{1}{4}\int \frac{1}{y^2-1/2}\,dy$$

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$\frac{1}{4}\left\{ \frac{1}{2}\ln\left|\frac{y-1}{y+1}\right| - \frac{1}{2\frac{1}{\sqrt{2}}} \ln\left|\frac{y-\frac{1}{\sqrt{2}}}{y+\frac{1}{\sqrt{2}}}\right| + c \right.$

Ques: $\int \frac{(1+x\cos x)}{x(1-x^2e^{2\sin x})}dx$.

$y = xe^{\sin x}$

$dy = e^{\sin x}(1+x\cos x)$

$\int \frac{e^{\sin x}(1+x\cos x)}{xe^{\sin x}(1-x^2e^{2\sin x})}dx$

$\int \frac{dy}{y(1-y^2)} = \frac{1}{-2}\int \frac{-2dy}{y^3\left(\frac{1}{y^2}-1\right)}$

$f' \quad f$

$= \frac{-1}{2}\ln\left|\frac{1}{y^2}-1\right|$

Ques: $\int \frac{x^3+3x+2}{(x^2+1)^2(x+1)}$

$\int \frac{x^3+x+2x+2}{(x^2+1)^2(x+1)} = \int \frac{x}{(x^2+1)(x+1)} + 2\int \frac{1}{(x^2+1)^2}dx$

$\int \frac{1}{(x^2+1)^2} = 2\int \cos^2\theta d\theta = 2\int (1+\cos 2\theta)d\theta = \theta + \frac{\sin 2\theta}{2}$

$x = \tan\theta$

$f'/f \quad \tan^{-1}x \quad \log$

$\int \frac{x}{(x^2+1)(x+1)} = \frac{1}{2}\int \frac{(x+1)}{x^2+1} - \frac{1}{x+1}$

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Kimur Pitur (Manipulating Integrands)

Ques: $\int \frac{x^4 + x^8}{(1-x^4)^{7/2}}$

Solu: * Eventh roots / Take even power of x out.

$$\int \frac{x^4 + x^8\, dx}{x^7 (x^{-2} + x^2)^{7/2}}$$

$$= \frac{1}{2}\int \frac{-2(x^{-3} + x)dx}{(x^{-2} + x^2)^{7/2}} = \frac{-1}{2}\frac{(x^{-2} - x^2)^{-5/2}}{-5/2} + C$$

Ques: $\int \frac{(2+\sqrt{x})\,dx}{(x+1+\sqrt{x})^2}$

Solu: Hint: Most of the time highest power is taken out

$$-2\int \frac{\frac{-1}{2}\left\{\frac{2}{x^2} - \frac{1}{x^{3/2}}\right\}dx}{\left(1 + 1/x + \frac{1}{\sqrt{x}}\right)^2}$$

$$= -2\left(\frac{-1}{1+\frac{1}{x}+\frac{1}{\sqrt{x}}}\right) + C$$

Ques: $\int \frac{(x\cos\alpha + 1)\;dx}{(x^2 + 2x\cos\alpha + 1)^{3/2}}$

$$\frac{-1}{2}\int \frac{-2\left(\frac{\cos\alpha}{x^2} + \frac{1}{x^3}\right)}{\left(1 + \frac{2\cos\alpha}{x} + \frac{1}{x^2}\right)^{3/2}} = \frac{-1}{2}\frac{\left(1 + \frac{2\cos\alpha}{x} + \frac{1}{x^2}\right)^{-1/2}}{-1/2} C$$

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Ques: $\int (x^6+x^4+x^2)(2x^4+3x^2+6)^{1/2}\,dx$

Sol^n $\frac{1}{12}\int 12(x^5+x^3+x)(2x^6+3x^4+6x^2)^{1/2}\,dx$

$\frac{1}{12}\int y^{1/2}\,dx = \frac{1}{12}\frac{y^{3/2}}{3/2} = \frac{(2x^6+3x^4+6x^2)^{3/2}}{18}+c$

Ques: $\int \frac{x^2\,dx}{(x\sin x+\cos x)^2}$

Solu $\int \frac{x\sec x \cdot x\cos x}{(x\sin x+\cos x)^2}\,dx$ $\frac{d}{dx}$ by part

$= \frac{-x\sec x}{(x\sin x+\cos x)} + \int \frac{1}{x\sin x+\cos x}\,\frac{\sec x(1+\cancel{\tan x})}{\cos x}\,dx$

$= \frac{-x\sec x}{x\sin x+\cos x} + \tan x + c$

Ques: $\int \frac{dx}{\sqrt[4]{(x-1)^3(x+2)^5}}$

Sol^n $\int \frac{dx}{\sqrt[4]{y^3(y+3)^5}} = \frac{1}{-3}\int \frac{-3}{y^2\sqrt[4]{\left(1+\frac{3}{y}\right)^5}} = -\frac{1}{3}\int \frac{1}{t^{5/4}} = -\frac{1}{3}\frac{t^{-1/4}}{-1/4}$

$= \frac{-4}{3}\left(1+\frac{3}{y}\right)^{-1/4}+c$

Ques: $f(x)$ is a quadratic function $f(0)=1$ $f(-1)=4$.

$\int \frac{f(x)\,dx}{x^2(x+1)^2}$ is rational f^n find $f(10)$

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Sol^n rational $\rightarrow \frac{P}{Q}$

$$\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2}$$

? ?

$A = 0$ $C = 0$

bcz on

$\int$ we will get log f^n not P/Q type.

$f(x) = B(x+1)^2 + D(x^2)$

$x = 0$ $1 = B$

$x = -1$ $4 = D$.

$f(x) = (x+1)^2 + 4x^2$

$f(10) = 121 + 400 = 521$

Ans: $\int \frac{\tan(\pi/4 - x)}{\cos^2 x\sqrt{\tan^3 x + \tan^2 x + \tan x}} dx$

$\tan x = y$

$$\int \frac{(1-y)\,dy}{1+y\sqrt{y^3+y^2+y}} = \int \frac{(1-y^2)}{(1+y)^2\sqrt{y^3+y^2+y}} dy$$

$$= \frac{\left(\frac{1}{y^2} - 1\right) dy}{\left(\frac{1}{y} + 2 + y\right)\sqrt{y + 1 + \frac{1}{y}}}$$

$$y + \frac{1}{y} + 1 = z^2$$

$$1 - \frac{1}{y^2} = 2z\,dz$$

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$$\int \frac{-2\cancel{z}\,dz}{(z^2+1)\cancel{z}} = -2\tan^{-1} z$$

Ques: $\displaystyle\int \frac{(5x^2-12)\,dx}{(x^2-6x+13)^2}$

Soln. $5x^2-12 = A(x^2-6x+13) + B(2x-6) + C$

$A = 5 \qquad D = -6A+2B \qquad B = 15 \qquad C = 13$

$$\int \frac{5}{(x-3)^2+2^2} + \int \frac{15\,(2x-6)\,dx}{(x^2-6x+13)^2} + \int \frac{13}{((x-3)^2+2^2)^2}$$

$$5\tan^{-1}\left(\frac{x-3}{2}\right) + \frac{15(-1)}{(x^2-6x+13)} + \cdot \quad \Downarrow$$

$x-3 = 2\tan\theta$

$$13\int \frac{2\sec^2\theta\,d\theta}{16\sec^4\theta} = \frac{13}{8}\int \cos^2\theta\,d\theta = \frac{13}{16}\int 1+\cos 2\theta\,d\theta \ldots$$

Integration of Trigonometric function

Type 1: $\displaystyle\int \frac{dx}{a+b\sin^2 x} \Big/ \int \frac{dx}{a+b\cos^2 x} \Big/\Big/ \int \frac{dx}{a\sin^2 x + b\cos^2 x + c\sin x\cos x}$ $\quad 0\,\&\,2^\circ$

Working rule: Substitute $\tan x = y$ & Multiply divide by $\sec^2 x$.

Type 2: $\displaystyle\int \frac{dx}{a+b\sin x} \Big/\Big/ \int \frac{dx}{a+b\cos x} \Big/\Big/ \int \frac{dx}{a+b\sin x + c\cos x}$ $\quad 0\,\&\,1^\circ$

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Working rule: Substitute $\tan\left(\frac{x}{2}\right) = y$ $\qquad$ $\times \div$ by $\sec^2 x/2$

Ques: $\displaystyle\int \frac{dx}{(\cos x)(5+3\cos x)}$

Sol^n $\displaystyle\frac{1}{5}\int \frac{1}{\cos x} - \frac{3}{5+3\cos x}\, dx$

$\displaystyle\frac{1}{5}\int \sec x - 3\int \frac{1}{(5+3\cos x)(\sec^2 x/2)} \times \sec^2 x/2 \, dx$

$\displaystyle\frac{1}{5}\, \ln|\sec x + \tan x| \quad -3\int \frac{\sec^2 x/2 \; dx}{5+5\tan^2 x/2 + 3 - 3\tan^2 x/2}$

$\tan x/2 = y$

$\frac{1}{2}\sec^2 x/2 \, dx = dy$

$\displaystyle -6\int \frac{dy}{8+2y^2} = -3\int \frac{dy}{4+y^2} = -\frac{3}{2}\tan^{-1}\left(\frac{y}{2}\right)$

Ques: $\displaystyle\int \frac{dx}{\sin^4 x + \cos^4 x}$

Sol^n $\displaystyle\int \frac{dx}{1 - \frac{\sin^2 2x}{2}} = \int \frac{\sec^2 2x \; dx}{(1 - \frac{1}{2}\sin^2 2x)\sec^2 2x}$

$\displaystyle= \int \frac{\sec^2 2x\, dx}{1+\tan^2 2x - \frac{1}{2}\tan^2 2x}$ $\qquad$ $y = \tan 2x$

$dy = \sec^2 2x \cdot 2dx$

$\displaystyle= \frac{1}{2}\int \frac{dy}{1 + \frac{y^2}{2}} = \int \frac{dy}{2+y^2} = \frac{1}{\sqrt{2}}\tan^{-1}(y/\sqrt{2})$

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Ans: $\int \frac{\sin 2x}{\sin^4 x + \cos^4 x} dx$

$$\int \frac{\frac{2\sin x \cos x}{\cos^2 x} \sec^2 x}{\tan^4 x + 1} dx = \int \frac{2\tan x \sec^2 x}{\tan^4 x + 1}$$

$$y = \tan^2 x \qquad dy = 2\tan x \sec^2 x \, dx$$

$$\int \frac{dy}{y^2+1} = \tan^{-1} \frac{y}{1} + c$$

Type-3. $\int \frac{a\sin x + b\cos x + c}{l\sin x + m\cos x + n} dx$

$$a\sin x + b\cos x + c = A(Dx) + B\frac{d}{dx}(Dx) + C.$$

Find A, B & C comparing coff of Sinx, Cosx & Absolute Term.

Ans. $\int \frac{6 + 3\sin x + 14\cos x}{3 + 4\sin x + 5\cos x} dx$

$$6 + 3\sin x + 14\cos x = A(3 + 4\sin x + 5\cos x) + B(4\cos x - 5\sin x) + C$$

$$6 = 3A + C$$

$$3 = 4A - 5B \quad \times 4$$

$$14 = 5A + 4B \quad \times 5$$

$$A = \frac{82}{41} = 2 \qquad B = 1 \qquad C = 0$$

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$2x + \ln|(3+4\sin x + 5\cos x)| + C$

Ques: $\int (\cos 2x) \ln(1+\tan x)\, dx$

Sol^n: Two different type f^n — by parts

$\ln(1+\tan x)\frac{\sin 2x}{2} - \int \frac{1}{1+\tan x} \sec^2 x \cdot \frac{\cancel{2}\sin x \cos x}{\cancel{2}}$

$- \int \frac{\tan x}{1+\tan x}\, dx$

$- \int \frac{\sin x}{\cos x + \sin x}\, dx$

$-\frac{1}{2}\int \frac{\sin x + \cos x}{\cos x + \sin x} + -\frac{1}{2}\int \frac{\overset{-f'(x)}{\sin x - \cos x}}{\underset{f}{\cos x + \sin x}}$

$-\frac{1}{2}x - \frac{1}{2}(-1)\ln|\cos x + \sin x| + c.$

Ques: $\int \frac{\cos^2 x + \sin 2x}{(2\cos x - \sin x)^2}\, dx$

Sol^n: $\frac{1 - \sin^2 x + 2\sin x \cos x}{(2\cos x - \sin x)^2}\, dx$

$\int \frac{\sin x\,(2\cos x - \sin x)}{(2\cos x - \sin x)^2}\, dx = \int \frac{\sin x}{2\cos x - \sin x}\, dx.$

$\sin x = A(2\cos x - \sin x) + B(-2\sin x - \cos x)$

$1 = -A - 2B \qquad 0 = 2A - B$

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$A = -\frac{1}{5}$ $B = -\frac{2}{5}$

$$\int \frac{\sin x}{2\cos x - \sin x}\,dx = -\frac{1}{5}x - \frac{2}{5}\ln|2\cos x - \sin x|$$

$$\int \frac{1}{(2\cos x - \sin x)^2}\,dx = -\int \frac{-\sec^2 x}{(2-\tan x)^2}\,dx = \frac{1}{2-\tan x}$$

$$y = 2 - \tan x$$

Ans $\frac{1}{2-\tan x} - \frac{1}{5}x - \frac{2}{5}\ln|2\cos x - \sin x| + C$

Ques: $\int \frac{3e^x + 5e^{-x}}{4e^x - 5e^{-x}}\,dx$

$$3e^x + 5e^{-x} = A(4e^x - 5e^{-x}) + B(4e^x + 5e^{-x})$$

$3 = 4A + 4B$ $\quad A + B = 3/4$

$5 = -5A + 5B$ $\quad B - A = 1$

$2B = \frac{7}{4}$ $\quad B = 7/8$ $\quad A = -1/8$

$$-\frac{1}{8}x + \frac{7}{8}\ln|4e^x - 5e^{-x}|$$

* Type $\int \frac{x^2+1}{x^4+Kx^2+1}$ / $\int \frac{x^2 \pm 1}{x^4+Kx^2+1}$

divide by x^2 — divide by x^2

Sub $x - \frac{1}{x} = y$ — $x + \frac{1}{x} = 2$

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Ques: $\int \frac{x^2+1}{x^4+7x^2+1}\,dx$

Solu: $\int \frac{1+\frac{1}{x^2}}{x^2+7+\frac{1}{x^2}}\,dx$ $\qquad y = x-\frac{1}{x} \qquad y^2 = x^2+\frac{1}{x^2}-2$

$dy = 1+\frac{1}{x^2}\,dx$

$\int \frac{dy}{y^2+2+7}\,dx = \frac{1}{3}\tan^{-1}\left(\frac{y}{3}\right)$

Ques $\int \frac{dx}{x^4+1}$

Sol^n. $\frac{1}{2}\int \frac{(x^2+1)-(x^2-1)}{x^4+1}$

$\frac{1}{2}\int \frac{x^2+1}{x^4+1} - \frac{1}{2}\int \frac{x^2-1}{x^4+1}\,dx$.

$\frac{1}{2}\int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\,dx - \frac{1}{2}\int \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\,dx$.

$\frac{1}{2}\int \frac{dy}{y^2+2} - \frac{1}{2}\int \frac{dz}{\cancel{y}^2-2}$

$\frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{y}{\sqrt{2}}\right) - \frac{1}{2}\ln\left|\frac{z-\sqrt{2}}{z+\sqrt{2}}\right|$

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Ques: $\int \frac{x^2}{x^4+1}\,dx$

Sol^n

$$\frac{1}{2}\int \frac{\cancel{2}\,x^2+1}{x^4+1} + \frac{x^2-1}{x^4+1}\,dx$$

$$\frac{1}{2\sqrt{2}}\tan^{-1}\frac{y}{\sqrt{2}} \quad + \frac{1}{2}\ln\left(\frac{z-\sqrt{2}}{z+\sqrt{2}}\right)$$

Ques; $\int \sqrt{\operatorname{Tan}\theta}\,d\theta$

Sol^n

$\operatorname{Tan}\theta = y^2$

$\operatorname{Sec}^2\theta\,d\theta = 2y\,dy$

$$\int \frac{y \cdot 2y\,dy}{1+y^4} = \int \frac{2y^2}{1+y^4}$$

$$\int \frac{y^2+1}{1+y^4}\,dy + \int \frac{y^2-1}{1+y^4}\,dy$$

$$\int \frac{1+\frac{1}{y^2}}{\frac{1}{y^2}+y^2}\,dy \quad + \int \frac{1-\frac{1}{y^2}}{\frac{1}{y^2}+y^2}\,dy$$

$u = x - \frac{1}{x}$ $\qquad$ $v = x + \frac{1}{x}$

$$\int \frac{du}{u^2+2} \quad + \int \frac{dv}{v^2-2}$$

$$\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) \quad + \ln\left|\frac{v-\sqrt{2}}{v+\sqrt{2}}\right|$$

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* $\cos x + \sin x$ or $\cos x - \sin x$ is lowing in N_r if D_r contains $a + b\sin 2x$, $\sqrt{a+b\sin 2x}$

Q. $\int \frac{\cos x}{\sqrt{8-\sin 2x}}$

Soln. $\frac{1}{2}\int \frac{\cos x+\sin x}{\sqrt{8-\sin 2x}} + \frac{1}{2}\int \frac{\cos x-\sin x}{\sqrt{8-\sin 2x}}$

sub $\sin x - \cos x = y$ sub $\sin x + \cos x = z$

$1 - \sin 2x = y^2$ $1 + \sin 2x = z^2$

$\frac{1}{2}\int \frac{dy}{\sqrt{y^2+7}} + \frac{1}{2}\int \frac{dz}{\sqrt{9-z^2}}$

$\frac{1}{2}\ln\left(y + \sqrt{y^2+7}\right) + \frac{1}{2}\sin^{-1}\left(\frac{z}{3}\right)$

Q. $\int \sqrt{\tan\theta} + \sqrt{\cot\theta}\, d\theta$

Soln $\int \frac{\sin\theta + \cos\theta}{\sqrt{\sin\theta\cos\theta}}\, d\theta$

$\sin\theta - \cos\theta = y$ $y^2 = 1 - 2\sin\theta\cos\theta$

$\int \frac{\sqrt{2}\, dy}{\sqrt{1-y^2}} = \sqrt{2}\sin^{-1}(y)$.

$I_1 = \int\sqrt{\tan\theta}\, d\theta$ $I_2 = \int\sqrt{\cot\theta}\, d\theta$

$I_1 + I_2 =$ ✓

$I_1 - I_2 =$ ✓

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Integration of Irrational Algebraic functions

Type: $\displaystyle\int \frac{dx}{(x-\alpha)\sqrt{(x-\alpha)(\beta-x)}}$ $(\beta > \alpha)$

M-I. $x = \alpha\cos^2\theta + \beta\sin^2\theta$.

$dx = (\beta-\alpha)\,2\sin\theta\cos\theta\,d\theta$

$$\int \frac{\cancel{(\beta-\alpha)}\,2\sin\cancel{\theta}\cos\cancel{\theta}\,d\theta}{(\beta-\alpha)\sin^2\theta\sqrt{\cancel{(\beta-\alpha)}\sin^2\theta\,\cancel{(\beta-\alpha)}\cos^2\theta}}$$

$$\frac{2}{(\beta-\alpha)}\int \operatorname{cosec}^2\theta\,d\theta = \frac{-2\cot\theta}{(\beta-\alpha)} + c$$

M-II $x-\alpha = y^2$

$dx = 2y\,dy$

$$\int \frac{2\cancel{y}\,dy}{y^2\cancel{y}\sqrt{\beta-\alpha-y^2}} = \int \underbrace{\frac{2}{y^3}}_{f'}\frac{dy}{\sqrt{\underbrace{\frac{\beta-\alpha}{y^2}-1}_{f}}}$$ $\Rightarrow$ Best Method.

Type-2: $\displaystyle\int \frac{dx}{\sqrt{(x-\alpha)(x-\beta)}}$

M.I- $x = \alpha\sec^2\theta - \beta\tan^2\theta$

$dx = 2(\alpha-\beta)\sec^2\theta\tan\theta\,d\theta$

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$$\int \frac{2(\alpha-\beta)\sec^2\theta \tan\theta\, d\theta}{\sqrt{(\alpha-\beta)\tan^2\theta(\alpha-\beta)\sec^2\theta}} = \int 2\sec\theta\, d\theta$$

$$= 2\ln|\sec\theta + \tan\theta|$$

M-II $$\int \frac{dx}{\sqrt{\left(x-\frac{\alpha+\beta}{2}\right)^2 - \left\{\left(\frac{\alpha+\beta}{2}\right)^2 - \alpha\beta\right\}}}$$

$$\int \frac{dx}{\sqrt{\left(x-\frac{\alpha+\beta}{2}\right)^2 - \left(\frac{\alpha-\beta}{2}\right)^2}}$$

$$= \ln\left|\left(x-\left(\frac{\alpha+\beta}{2}\right)\right) + \sqrt{(x-\alpha)(x-\beta)}\right|$$

Type 2: $$\int \frac{dx}{(x-\alpha)\sqrt{(x-\alpha)(x-\beta)}}$$

M-I. $x = \alpha\sec^2\theta - \beta\tan^2\theta$.

M-II $(x-\alpha) = y^2$

\# $$\int \frac{dy}{\text{Linear}\sqrt{\text{Linear}}} = \int \frac{dy}{L_1\sqrt{L_2}} \qquad L_2 = y^2$$

Q $$\int \frac{dx}{2x+1\,\sqrt{4x+3}}$$

$$4x+3 = y^2$$
$$4dx = 2y\,dy$$

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$$\frac{1}{2}\int \frac{\cancel{y}\,dy}{\left(\frac{y^2-1}{2}\right)\cancel{y}} = \int \frac{dy}{y^2-1} = \frac{1}{2}\ln\left|\frac{y-1}{y+1}\right| + C$$

\# $\displaystyle\int \frac{dx}{(ax+b)\sqrt{px^2+qx+r}}$ $\qquad ax+b = \frac{1}{t}$

$$\int \frac{dx}{L\sqrt{Q}} \qquad L = \frac{1}{t}$$

Ques: $\displaystyle\int \frac{dx}{(x+1)\sqrt{1+x-x^2}}$

Solu: $\qquad 1+x = \frac{1}{y} \qquad dx = -\frac{1}{y^2}\,dy$

$$\int \frac{-\frac{1}{y^2}\,dy}{\frac{1}{y}\sqrt{\frac{1}{y}-\left(\frac{1}{y}-1\right)^2}}$$

$$= -\int \frac{dy}{\sqrt{y-(1-y)^2}} = \int \frac{-dy}{\sqrt{-y^2+3y-1}} = \int \frac{-dy}{\sqrt{-1-\left(\left(y-\frac{3}{2}\right)^2-\frac{9}{4}\right)}}$$

$$= \int \frac{-dy}{\sqrt{\left(\frac{\sqrt{5}}{2}\right)^2-\left(y-\frac{3}{2}\right)^2}}$$

\# $\displaystyle\int \frac{dx}{(ax^2+bx+c)\sqrt{px+q}}$ $\qquad px+q = \frac{y^3}{6}$

Ques: $\displaystyle\int \frac{dx}{(x^2+5x+2)\sqrt{x-2}}$

Solu: $x-2=y^2 \qquad dx=2y\,dy$

$$\int \frac{2\cancel{y}\,dy}{\cancel{y}\{(2+y^2)^2+5(2+y^2)+2\}}$$

$$=\int \frac{2\,dy}{y^4+9y^2\cancel{+16}+16}$$ Imp step ↓

$$=\frac{1}{4}\int \frac{y^2+4}{y^4+9y^2+16}\,dy - \int \frac{y^2-4}{y^4+9y^2+16}\,dy$$

⇓

$$y-\frac{4}{y}=z$$

$$dz = 1+\frac{4}{y^2}$$

& $y^2+\frac{16}{y^2}-8=z^2$

$$=\frac{1}{4}\int \frac{1+\frac{4}{y^2}}{y^2+9+\frac{16}{y^2}}$$

$$=\frac{1}{4}\int \frac{dz}{z^2+17} + \frac{1}{4}\int \frac{dk}{k^2+1}$$

$\cancel{\dfrac{dx}{(x^2-x-2)\sqrt{x^2+\sqrt{x}+1}}}$

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$\displaystyle\int \frac{dx}{(ax^2+bx+c)\sqrt{px^2+qx+r}}$

Case I When $D>0$ of ax^2+bx+c

Q. $\displaystyle\int \frac{dx}{(x^2-x-2)\sqrt{x^2+x+1}}$ $\qquad (x-2)(x+1) = x^2-x-2$

$\displaystyle\frac{1}{3}\int \frac{1}{(x-2)\sqrt{x^2+x+1}} \qquad -\frac{1}{3}\int \frac{1}{(x+1)\sqrt{x^2+x+1}}$

$x-2 = 1/y \qquad x+1 = \frac{1}{z}$

Case II If $D=0$ $ax^2+bx+c = (lx+m)^2$ Then substitute $lx+m = 1/y$.

Skip
Case III : $b=0,\ q=0$ $\qquad \displaystyle\int \frac{dx}{(x^2+4)\sqrt{4x^2+1}}$

$x = 1/y$ & $dx = -1/y^2\,dy$

$$\int \frac{\frac{-1}{y^2}\,dy}{\left(1/y^2+4\right)\sqrt{\frac{4}{y^2}+1}} = \int \frac{-y\,dy}{(1+4y^2)\sqrt{(4+y^2)}}$$

$4+y^2 = z^2$

$$\int \frac{-\cancel{z}\,dz}{[1+4(z^2-4)]\cancel{z}} = \int \frac{dz}{4z^2-15}$$

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Ques. $\int \sqrt{\frac{1+x^2}{x^2-x^4}}$

Sol^n $\int \frac{1}{x} \frac{\sqrt{1+x^2}}{\sqrt{1-x^2}}\, dx \quad = \int \frac{1}{x} \frac{1+x^2}{\sqrt{1-x^4}}\, dx$

$\int \frac{1}{x\sqrt{1-x^4}} + \int \frac{x}{\sqrt{1-x^4}}\, dx$

$\int \frac{x^3}{x^4\sqrt{1-x^4}} + \int \frac{x}{\sqrt{1-x^4}}\, dx$

$\downarrow \qquad\qquad \downarrow$

$1-x^4 = y^2 \qquad\qquad x^2 = z$

$-4x^3 dx = 2y\,dy \qquad\qquad 2x\,dx = dz$

$\downarrow$

$\frac{-1}{2}\int \frac{y\,dy}{(1-y^2)y} \qquad\qquad \frac{1}{2}\int \frac{dz}{\sqrt{1-z^2}}$

n^{th} Integral.

$I_n = \int \tan^n x\, dx \qquad n \geq 3$

$I_n = \int \tan^{n-2} x \,\{\sec^2 x - 1\}\, dx$

$$\boxed{I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2}}$$

$I_n = \int \sin^n x\, dx$

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$$I_n = \int \sin^{n-1}x \cdot \sin x \, dx$$

$$I_n = \sin^{n-1}x \cdot (-\cos x) + \int (n-1) \sin^{n-2}x \cos^2 x \, dx$$

$$I_n = -\cos x \sin^{n-1}x + (n-1)\int \sin^{n-2}x \,(1-\sin^2 x)\, dx$$

$$nI_n = -\cos x \sin^{n-1}x + (n-1)I_{n-2}$$

Indefinite & Definite Integration has highest weightage explicitly & can easily be solved bcz no tough question can be formed in them.