Series Formulas

1. Arithmetic and Geometric Series

Definitions:

First term: a_1
Nth term: a_n
Number of terms in the series: n
Sum of the first n terms: S_n
Difference between successive terms: d
Common ratio: q
Sum to infinity: S

Arithmetic Series Formulas:

$$a_n = a_1 + (n-1)d$$

$$a_i = \frac{a_{i-1} + a_{i+1}}{2}$$

$$S_n = \frac{a_1 + a_n}{2} \cdot n$$

$$S_n = \frac{2a_1 + (n-1)d}{2} \cdot n$$

Geometric Series Formulas:

$$a_n = a_1 \cdot q^{n-1}$$

$$a_i = \sqrt{a_{i-1} \cdot a_{i+1}}$$

$$S_n = \frac{a_n q - a_1}{q-1}$$

$$S_n = \frac{a_1 (q^n - 1)}{q-1}$$

$$S = \frac{a_1}{1-q} \qquad for \quad -1 < q < 1$$

2. Special Power Series

Powers of Natural Numbers

$$\sum_{k=1}^{n} k = \frac{1}{2} n(n+1)$$

$$\sum_{k=1}^{n} k^2 = \frac{1}{6} n(n+1)(2n+1)$$

$$\sum_{k=1}^{n} k^3 = \frac{1}{4} n^2 (n+1)^2$$

Special Power Series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots \qquad (for: -1 < x < 1)$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots \qquad (for: -1 < x < 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} \ldots \qquad (for: -1 < x < 1)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} \ldots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} \ldots$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \ldots \qquad \left(for: -\frac{\pi}{2} < x < \frac{\pi}{2}\right)$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} \ldots$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} \ldots$$

$$\tan x = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + \ldots \qquad \left(for: -\frac{\pi}{2} < x < \frac{\pi}{2}\right)$$

3. Taylor and Maclaurin Series

Definition:

$$f(x)=f(a)+f'(a)(x-a)+\frac{f''(a)(x-a)^2}{2!}+\ldots+\frac{f^{(n-1)}(a)(x-a)^{n-1}}{(n-1)!}+R_n$$

$$R_n=\frac{f^{(n)}(\xi)(x-a)^n}{n!} \qquad \textit{Lagrange's form} \qquad a\le\xi\le x$$

$$R_n=\frac{f^{(n)}(\xi)(x-\xi)^{n-1}(x-a)}{(n-1)!} \qquad \textit{Cauch's form} \qquad a\le\xi\le x$$

This result holds if f(x) has continuous derivatives of order n at last. If $\lim_{n\to\infty}R_n=0$, the infinite series obtained is called Taylor series for f(x) about x = a. If a = 0 the series is often called a Maclaurin series.

Binomial series

$$(a+x)^n=a^n+na^{n-1}x+\frac{n(n-1)}{2!}a^{n-2}x^2+\frac{n(n-1)(n-2)}{3!}a^{n-3}x^3+\ldots$$

$$=a^n+\binom{n}{1}a^{n-1}x+\binom{n}{2}a^{n-2}x^2+\binom{n}{3}a^{n-3}x^3+\ldots$$

Special cases:

$$(1+x)^{-1}=1-x+x^2-x^3+x^4-\ldots \qquad -1<x<1$$

$$(1+x)^{-2}=1-2x+3x^2-4x^3+5x^4-\ldots \qquad -1<x<1$$

$$(1+x)^{-3}=1-3x+6x^2-10x^3+15x^4-\ldots \qquad -1<x<1$$

$$(1+x)^{-\frac{1}{2}}=1-\frac{1}{2}x+\frac{1\cdot 3}{2\cdot 4}x^2-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3+\ldots \qquad -1<x\le 1$$

$$(1+x)^{\frac{1}{2}}=1+\frac{1}{2}x-\frac{1}{2\cdot 4}x^2+\frac{1\cdot 3}{2\cdot 4\cdot 6}x^3+\ldots \qquad -1<x\le 1$$

Series for exponential and logarithmic functions

$$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots$$

$$a^x=1+x\ln a+\frac{(x\ln a)^2}{2!}+\frac{(x\ln a)^3}{3!}+\ldots$$

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}\ldots \qquad -1<x\le 1$$

$$\ln(1+x)=\left(\frac{x-1}{x}\right)+\frac{1}{2}\left(\frac{x-1}{x}\right)^2+\frac{1}{3}\left(\frac{x-1}{x}\right)^3+\ldots \qquad x\ge\frac{1}{2}$$

Series for trigonometric functions

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \ldots + \frac{2^{2n}\left(2^{2n}-1\right)B_n x^{2n-1}}{(2n)!} \qquad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

$$\cot x = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} - \ldots - \frac{2^{2n} B_n x^{2n-1}}{(2n)!} \qquad 0 < x < \pi$$

$$\sec x = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \ldots + \frac{E_n x^{2n}}{(2n)!} + \ldots \qquad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

$$\csc x = \frac{1}{x} + \frac{x}{6} + \frac{7x^3}{360} + \ldots + \frac{2\left(2^{2n}-1\right)E_n x^{2n}}{(2n)!} + \ldots \qquad 0 < x < \pi$$

$$\sin^{-1} x = x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7} + \ldots \qquad -1 < x < 1$$

$$\cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x = \frac{\pi}{2} - \left(x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{x^5}{5} + \ldots\right) \qquad -1 < x < 1$$

$$\tan^{-1} x = \begin{cases} x - \dfrac{x^3}{3} + \dfrac{x^2}{5} - \dfrac{x^3}{7} + \ldots & if \ -1 < x < 1 \\[2ex] \dfrac{\pi}{2} - \dfrac{1}{x} + \dfrac{1}{3x^3} - \dfrac{1}{5x^5} + \ldots & if \ x \ge 1 \\[2ex] -\dfrac{\pi}{2} - \dfrac{1}{x} + \dfrac{1}{3x^3} - \dfrac{1}{5x^5} + \ldots & if \ x < 1 \end{cases}$$

$$\cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x = \begin{cases} \dfrac{\pi}{2} - \left(x - \dfrac{x^3}{3} + \dfrac{x^2}{5} - \dfrac{x^3}{7} + ..\right). & if \ -1 < x < 1 \\[2ex] \dfrac{1}{x} - \dfrac{1}{3x^3} + \dfrac{1}{5x^5} - \dfrac{1}{7x^7} + \ldots & if \ x \ge 1 \\[2ex] \pi + \dfrac{1}{x} - \dfrac{1}{3x^3} + \dfrac{1}{5x^5} - \dfrac{1}{7x^7} + \ldots & if \ x < 1 \end{cases}$$

Series for hyperbolic functions

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots$$

$$\tanh x = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + \ldots + \frac{(-1)^{n-1} 2^{2n}\left(2^{2n}-1\right)B_n x^{2n-1}}{(2n)!} + \ldots \qquad if \ -\frac{\pi}{2} < x < \frac{\pi}{2}$$

$$\coth x = \frac{1}{x} + \frac{x}{3} - \frac{x^3}{45} + \frac{2x^7}{945} + \ldots + \frac{(-1)^{n-1} 2^{2n} B_n x^{2n-1}}{(2n)!} + \ldots \qquad if \ 0 < |x| < \pi$$