CBCS Scheme

15MAT11

1st Semester B.E Degree Examination Dec 2016/Jan-17

ENGINEERING MATHEMATICS - I

① ⓐ If $y = e^{-3x}\cos^3 x$, find y_n

$$\cos^3 x = \frac{1}{4}[\cos 3x + 3\cos x]$$

$$y = \frac{1}{4}e^{-3x}\cos 3x + \frac{3}{4}e^{-3x}\cos x$$

$$y_n = \frac{d^n x}{dx^n} = D^n y_n = (a^2+b^2)^{n/2} e^{ax}\cos\{(bx+c)+n\tan^{-1}(b/a)\}$$

$$\frac{d^n}{dx^n}\left[e^{ax}\cos(bx+c)\right]$$

$a = -3 \quad b = 3 \quad \& \quad a = -3 \quad b = 1$

$$(18)^{n/2} e^{-3x}\cos\{3x + n\tan^{-1}(-1)\} + \frac{3}{4}(10)^{n/2} e^{-3x}\cos\{x + n\tan^{-1}(-1/3)\}$$

Not required ×

$$\boxed{(18)^{n/2} e^{-3x}\cos\{3x - \frac{n\pi}{4}\} + \frac{3}{4}(10)^{n/2} e^{-3x}\cos\{x + n\tan^{-1}(-1/3)\}}$$

ⓑ Find the angle b/w the curves

$$r = \frac{a}{1+\cos\theta} \quad \& \quad r = \frac{b}{1-\cos\theta}$$

$$\frac{dr}{d\theta} = \frac{a.\sin\theta}{(1+\cos\theta)^2} \qquad \frac{dr}{d\theta} = \frac{-b\sin\theta}{(1-\cos\theta)^2}$$

$$\tan\phi_1 = r.\frac{d\theta}{dr} = \left(\frac{a}{1+\cos\theta}\right)\left(\frac{(1+\cos\theta)^2}{a.\sin\theta}\right) \qquad \tan\phi_2 = \frac{-b}{(1-\cos\theta)} \times \frac{(1-\cos\theta)^2}{b\sin\theta}$$

$$\tan\phi_1 \tan\phi_2 = \frac{-(1+\cos\theta)}{\sin\theta} \times \frac{(1-\cos\theta)}{\sin\theta}$$

$$= \frac{-(1-\cos^2\theta)}{\sin^2\theta} = -1$$

$$\tan\phi_1 \tan\phi_2 = -1$$

$$\boxed{\therefore |\phi_1 - \phi_2| = \pi/2}$$

we say that the curves intersect orthogonally

(C) Find the radius of curvature of the curve $x^4+y^4=2$ at the pt $(1,1)$.

Sol

$x^4+y^4=2$

diff w.r.to x we get

$$4x^3+4y^3\cdot\frac{dy}{dx}=0 \rightarrow (1)$$

$$\frac{dy}{dx}=-x^3/y^3$$

$$y'=\left(\frac{dy}{dx}\right)_{at\,(1,1)}=-1 \rightarrow (2)$$

diff. w.r.to x (1) we get

$$12x^2+12y^2\cdot\left(\frac{dy}{dx}\right)^2+4y^3\cdot\frac{d^2y}{dx^2}=0$$

at $(1,1)$

$$12+12+4\frac{d^2y}{dx^2}=0$$

$$y''=\frac{d^2y}{dx^2}=-6 \rightarrow (3)$$

we have

$$\rho=\frac{(1+(y')^2)^{3/2}}{y''}$$

$$\rho=\frac{(1+(-1)^2)^{3/2}}{-6}=-\frac{\sqrt{2}}{3}$$

$\therefore$ The reqr radius of curvature

$$\boxed{|\rho|=\frac{\sqrt{2}}{3}}$$

2 (a) If $x = \tan(\log y)$, find the value of

$$(1+x^2)\, y_{n+1} + (2nx-1)\, y_n + (n)(n-1)\, y_{n-1}$$

Sol

$$x = \tan(\log y)$$

$$y = e^{\tan^{-1}x} \rightarrow (1)$$

$$\frac{dy}{dx} = e^{\tan^{-1}x} \cdot \frac{1}{(1+x^2)} \qquad \frac{dy}{dx} = y_1$$

$$(1+x^2)\, y_1 = e^{\tan^{-1}x}$$

$$(1+x^2)\, y_1 - y = 0 \rightarrow (2)$$

Applying Leibnitz rule, we get

$$[(1+x^2)\, y_1]_n - [y]_n = 0$$

$$^nC_0 (1+x^2)\, y_{n+1} + {}^nC_1\, 2x\, y_n + {}^nC_2\, 2 \cdot y_{n-1} - y_n = 0$$

$$1 \cdot (1+x^2)\, y_{n+1} + 2nx\, y_n + \frac{n(n-1)}{2} \cdot 2\, y_{n-1} - y_n = 0$$

$$(1+x^2)\, y_{n+1} + (2nx-1)\, y_n + (n)(n-1)\, y_{n-1} = 0$$

(b) Find the pedal equation of $\frac{2a}{r} = 1 + \cos\theta \rightarrow (1)$

Sol Taking logarithms on both sides of the equal

$$\log\left[\frac{2a}{r}\right] = \log[1+\cos\theta]$$

$$\log 2a - \log r = \log(1+\cos\theta)$$

diff w.r.to θ we get

$$-\frac{1}{r} \cdot \frac{dr}{d\theta} = \frac{-\sin\theta}{(1+\cos\theta)} \qquad \left[\tan\varphi = r \cdot \frac{d\theta}{dr}\right]$$

$$-\cot\varphi = \frac{-2\sin\theta/2\, \cos\theta/2}{2\cos^2\theta/2} = -\tan\theta/2$$

$$+\cot\varphi = \cot(\pi/2 - \theta/2)$$

$$\varphi = \pi/2 - \theta/2$$

we have, $p = r\sin\phi = r\sin(\pi/2 - \theta/2)$

$= r\cos\theta/2$

$p^2 = r^2\cos^2\theta/2 = r^2\left[\frac{1+\cos\theta}{2}\right]$

$p^2 = \frac{r^2}{2}\cdot\frac{2a}{r}$ using ①

$\boxed{p^2 = ar}$ is the required pedal eq.

ⓒ Find the radius of curvature of the curve $r^n = a^n\cos n\theta$

Sol

$r^n = a^n\cos n\theta \rightarrow ①$

Taking logarithm on both sides

$n\log r = n\log a + \log\cos n\theta$

Diff w.r.to θ.

$n\cdot\frac{1}{r}\cdot\frac{dr}{d\theta} = 0 - \frac{\sin n\theta}{\cos n\theta}\cdot n$

$1/r\cdot\frac{dr}{d\theta} = -\frac{\sin n\theta}{\cos n\theta}$

$\cot\phi = -\tan n\theta$

$\cot\phi = \cot(\pi/2 + n\theta)$

$\phi = \pi/2 + n\theta$

we have $p = r\sin\phi$

$= r\sin(\pi/2 + n\theta)$

$p = r\cos n\theta \rightarrow ②$

$\therefore$ ② becomes $p = r\cdot\frac{r^n}{a^n} = \frac{r^{n+1}}{a^n}$

$\boxed{p = \frac{r^{n+1}}{a^n}} \rightarrow ③$

Diff w.r.to r, we have

$\frac{dp}{dr} = (n+1)\frac{r^n}{a^n}$

$\rho = r\cdot\frac{dr}{dp} = r\cdot\frac{a^n}{r^n}\frac{1}{(n+1)} = \frac{a^n}{(n+1)}\times\frac{1}{r^{n-1}}$

(3)

Module-2

3 (a) Explain log(cos x) about the point $x = \pi/3$ upto 3^{rd} degree terms using Taylor's series.

Sol Here, $f(x) = \log(\cos x)$ in powers $x - \pi/3$ upto 3^{rd} degree term.

$$f(x) = f(\pi/3) + (x-\pi/3) f'(\pi/3) + \frac{(x-\pi/3)^2}{2!} f''(\pi/3) + \frac{(x-\pi/3)^3}{3!} f'''(\pi/3)$$

Now, $f(x) = \log(\cos x)$

$f(\pi/3) = \log(\cos \pi/3) = \log 1/2$

$f'(x) = \frac{-1}{\cos x} \sin x = -\tan x$

$f'(\pi/3) = -\tan(\pi/3) = -\sqrt{3}$

$f''(x) = -\sec^2 x$

$f''(\pi/3) = -\sec^2(\pi/3) = -4$

$f'''(x) = -(2\sec x)(\sec x \cdot \tan x) = -2\sec^2 x \tan x$

$f'''(\pi/3) = -2\sec^2(\pi/3)\tan(\pi/3) = -2(4)(\sqrt{3}) = -8\sqrt{3}$

$$f(x) = \log(\cos x) = \log 1/2 + (x-\pi/3)(-\sqrt{3}) + \frac{(x-\pi/3)^2}{2!}(-4) + \frac{(x-\pi/3)^3}{3!}(-8\sqrt{3})$$

is the required expansion

(b) Eval $\lim_{x \to 0} \left(\frac{\tan x}{x}\right)^{1/x^2}$

Applying logarithms on both sides

$$\log L = \lim_{x\to 0} 1/x^2 \log\left(\frac{\tan x}{x}\right)$$

$$= \lim_{x\to 0} \frac{\log\left(\frac{\tan x}{x}\right)}{x^2} \quad \ldots = \frac{0}{0} \text{ form indeterminant}$$

Applying L'Hospital's rule, we get

$$= \lim_{x\to 0} \frac{\frac{1}{\frac{\tan x}{x}} \cdot \frac{x\sec^2 x - \tan x}{x^2}}{2x}$$

$$= \lim_{x\to 0} \frac{x}{\tan x} \cdot \lim_{x\to 0} \left(\frac{x\sec^2 x - \tan x}{2x^3} \right) \rightarrow 0/0.$$

$$= \lim_{x\to 0} \frac{x}{\tan x} \cdot \lim_{x\to 0} \left(\frac{2x\sec x \cdot \sec x\tan x + \sec^2 x - \sec^2 x}{6x^2} \right) \rightarrow 0/0.$$

$$= \lim_{x\to 0} \frac{x}{\tan x} \lim_{x\to 0} \frac{2x\sec^2 x\tan x}{6x^2}$$

Applying L'Hospital's Rule

$$= \lim_{x\to 0} \frac{x}{\tan x} \lim_{x\to 0} \left(\frac{2\sec^2 x\tan x + 4x\sec^2 x\tan^2 x + 2x\sec^4 x}{12x} \right) \rightarrow 0/0$$

$$= \lim_{x\to 0} \frac{x}{\tan x} \lim_{x\to 0} \frac{[2\sec^4 x + 4\sec^2 x\tan^2 x + 4\sec^2 x\tan^2 x + 8x\sec^2 x\tan^4 x + 8x\sec^3 x\tan x + 2\sec^4 x + 8x\sec^3 x\sec x\tan x]}{12}$$

$$= 1 \times \frac{4\sec^4 0}{12}$$

$$\log L = 1/3$$

$$\boxed{L = e^{1/3}}$$

3 ⓒ State Euler's theorem. & use it to find $x\frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y}$ when, $u = \tan^{-1}\left(\frac{x^2+y^2}{x+y}\right)$

Sol <u>statement</u> If u is homogeneous fn of x & y with degree n, then

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = nu$$

④

Sol

$$u = \tan^{-1}\left(\frac{x^2+y^2}{x+y}\right)$$

(c)

$$\tan u = \frac{x^2+y^2}{x+y} = f \rightarrow (\text{say})$$

$$f = \frac{x^2(1+y^2/x^2)}{x(1+y/x)} = x\cdot\left[\frac{1+(y/x)^2}{1+y/x}\right]$$

f is homogeneous fn of x & y with degree 1.

∴ Using Euler's theorem

$$x\cdot\frac{\partial f}{\partial x} + y\cdot\frac{\partial f}{\partial y} = 1\cdot f$$

$$x\cdot\frac{\partial}{\partial x}(\tan u) + y\cdot\frac{\partial}{\partial y}(\tan u) = \tan u$$

$$x\cdot\sec^2 u\,\frac{\partial u}{\partial x} + y\cdot\sec^2 u\cdot\frac{\partial u}{\partial y} = \tan u$$

$$x\cdot\frac{\partial u}{\partial x} + y\cdot\frac{\partial u}{\partial y} = \frac{\tan u}{\sec^2 u} = \sin u\cos u = \tfrac{1}{2}\sin 2u$$

4 (a) Expand, $\frac{e^x}{1+e^x}$ using Maclaurin's series upto & including 3rd degree terms.

Sol

$$y = f(x) = \frac{e^x}{1+e^x}, \quad x = 0.$$

Using Maclaurin's series expansion,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0)$$

$$f(x) = \frac{e^x}{1+e^x}$$

$$y(0) = f(0) = \frac{1}{2}$$

$$f'(x) = \frac{e^x(1+e^x) - e^x\cdot e^x}{(1+e^x)^2}$$

$$f'(x) = \frac{1}{e^x}\left(\frac{e^x}{1+e^x}\right)^2$$

$$y'(0) = f'(0) = 1/4$$

Scanned by CamScanner

$$y'(0) = \quad f'(x) = \frac{1}{e^x} y^2 = e^{-x} y^2$$

$$f''(x) = -e^{-x} y^2 + e^{-x} 2y \cdot y'$$

$$= -e^{-x}(y^2 - 2yy_1')$$

$$= -e^{-x}\left(\left(\frac{e^x}{1+e^x}\right)^2 - 2\left(\frac{e^x}{1+e^x}\right)\left(\frac{1}{e^x}\cdot\left(\frac{e^x}{1+e^x}\right)^2\right)\right)$$

$$y''(0) = f''(0) = -\left(\frac{1}{4} - 2\left(\frac{1}{2}\right)\left(\frac{1}{4}\right)\right) = 0$$

$$f'''(x) = e^{-x}(y^2 - 4yy' + 2(y_1')^2 + 2yy'')$$

$$f'''(0) = 1(1/4 - 4(1/2)(1/4) + 2(1/4)^2 + 2(1/2)(0))$$

$$= (1/4 - 1/2 + 2/16) = -1/8$$

$$f(x) = \frac{e^x}{1+e^x} = 1/2 + 1/4\, x - 1/8\, x^3/3!$$

(b) Find, $\frac{du}{dt}$ when, $u = x^3y^2 + x^2y^3$ with $x = at^2$, $y = 2at$.
Use partial derivatives

Sol

$$u = x^3y^2 + x^2y^3$$

$$\frac{\partial u}{\partial x} = 3x^2y^2 + 2xy^3 \qquad \frac{\partial u}{\partial y} = 2x^3y + 3x^2y^2$$

$$x = at^2 \qquad y = 2at$$

$$\frac{\partial x}{\partial t} = 2at \qquad \frac{\partial y}{\partial t} = 2a$$

We knw $\quad \frac{du}{dt} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}$

$$= (3x^2y^2 + 2xy^3)(2at) + (2x^3y + 3x^2y^2)(2a)$$

$$= 6atx^2y^2 + 4atxy^3 + 4ax^3y + 6ax^2y^2$$

$$= 6at(at^2)^2(2at)^2 + 4at(at^2)(2at)^3 + 4a(at^2)^3(2at) + 6a(at^2)^2(2at)^2$$

$$= 32a^5t^7 + 56a^5t^6$$

$$= 8a^5t^6(4t+7)$$

Scanned by CamScanner

(c) If $u = \frac{x_2 x_3}{x_1}$, $v = \frac{x_1 x_3}{x_2}$, $w = \frac{x_1 x_2}{x_3}$, find the value of Jacobian $J\left(\frac{u, v, w}{x_1, x_2, x_3}\right)$

Sol

$$J\left(\frac{u,v,w}{x_1,x_2,x_3}\right) = \begin{vmatrix} \frac{\partial u}{\partial x_1} & \frac{\partial u}{\partial x_2} & \frac{\partial u}{\partial x_3} \\ \frac{\partial v}{\partial x_1} & \frac{\partial v}{\partial x_2} & \frac{\partial v}{\partial x_3} \\ \frac{\partial w}{\partial x_1} & \frac{\partial w}{\partial x_2} & \frac{\partial w}{\partial x_3} \end{vmatrix}$$

$$= \begin{vmatrix} -\frac{x_2 x_3}{x_1^2} & \frac{x_3}{x_1} & \frac{x_2}{x_1} \\ \frac{x_3}{x_2} & -\frac{x_1 x_3}{x_2^2} & \frac{x_1}{x_2} \\ \frac{x_2}{x_3} & \frac{x_1}{x_3} & -\frac{x_1 x_2}{x_3^2} \end{vmatrix}$$

$$= 4$$

Module-3

5 (a) A particle moves on the curve $x = 2t^2$, $y = t^2 - 4t$, $z = 3t - 5$ where, t is the time, find the components of vel. & acc. at time $t = 1$, in the direction of $i - 3j + 2k$.

Sol

$$r = xi + yj + zk$$
$$= 2t^2 + (t^2 - 4t)j + (3t-5)k$$

$$\vec{V} = \frac{dr}{dt} = 4ti + (2t-4)j + 3k$$

$$\vec{a} = \frac{d^2r}{dt^2} = 4i + 2j$$

at $t = 1$, $\vec{V} = 4i + (2-4)j + 3k = 4i - 2j + 3k$

given, $\vec{d} = i - 3j + 2k$

direction

unit vector in the direction $i - 3j + 2k$

$$\hat{n} = \frac{i - 3j + 2k}{\sqrt{1 + 9 + 4}} = \frac{i - 3j + 2k}{\sqrt{14}}$$

Vel component at $t = 1$ in the direction of vector $i - 3j + 2k$ is

$$\frac{1}{\sqrt{14}}(i - 3j + 2k)(4i - 2j + 3k) = \frac{4 + 6 + 6}{\sqrt{14}} = \frac{16}{\sqrt{14}}$$

acc. is $\frac{1}{\sqrt{14}}(i - 3j + 2k)(4i + 2j)$

$$= \frac{4 - 6}{\sqrt{14}} = -2/\sqrt{14}$$

A (b) Find the divergence & curl of the vector $\vec{V} = (xyz)i + (3x^2y)j + (xz^2 - y^2z)k$ at the pt. $(2, -1, 1)$

Sol

$$\operatorname{div} \vec{V} = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right)\left(xyz\, i + (3x^2y)j + (xz^2 - y^2z)k\right)$$

$$\partial/\partial x\,(xyz) + \frac{\partial}{\partial y}(3x^2y) + \frac{\partial}{\partial z}(xz^2 - y^2z)$$

$$= yz + 3x^2 + 2xz - y^2$$

at $(2, -1, 1)$ $\operatorname{div} \vec{V} = -1 + 12 + 4 - 1 = 14$

$$\operatorname{curl} \vec{V} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xyz & 3x^2y & (xz^2 - y^2z) \end{vmatrix} = -2yz\, i - (z^2 - xy)j + (6xy - xz)k$$

at $(2, -1, 1)$

$$\operatorname{curl} \vec{V} = 2i - 3j + 4k$$

(c) A vector field is given by $\vec{A} = (x^2+xy)\,i + (y^2+x^2y)\,j$ S.T the field is irrotational & find the scalar potential.

Sol

A vector $\vec{A}$ is said to be irrotational if curl $\vec{A} = 0$.

$$\text{curl } \vec{A} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2+xy^2 & y^2+x^2y & 0 \end{vmatrix}$$

$$= i(0-0) - j\left(\partial/\partial x\,(\;) - \partial/\partial z\,(x^2+xy)\right)$$

$$+ k\left(\frac{\partial}{\partial x}(y^2+x^2y) - \partial/\partial y\,(x^2+xy^2)\right)$$

$$= 0 - j(-0) + k(2xy - 2xy) = 0$$

$\therefore \vec{A}$ is irrotational

Consider, $\nabla\varphi = \vec{A}$

$$\frac{\partial\varphi}{\partial x} = x^2+xy^2 \qquad \frac{\partial\varphi}{\partial y} = y^2+x^2y$$

on integration, $\varphi = x^3/3 + \frac{x^2y^2}{2} + f(y) \rightarrow (1)$

$\varphi = y^3/3 + \frac{x^2y^2}{2} + g(x) \rightarrow (2)$

$\therefore \varphi = x^3/3 + y^3/3 + \frac{x^2y^2}{2}$ is the required scalar potential.

Scanned by CamScanner

6 (a) Find the grad ϕ when, $\phi = 3x^2y - y^3z^2$ at the pt $(1,-2,-1)$

Sol

$$\text{grad}\,\phi = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k$$

$$= 6xy i + (3x^2 - 3y^2z^2) j + (-y^3(2z)) k$$

$$\nabla\phi\,(1,-2,-1) = 12i - 9j - 16k$$

(b) Find 'a' for which $f = (x+3y)i + (y-2z)j + (x+az)k$ is solenoidal

Sol

$$\text{div}\,\vec{f} = 0$$

$$\left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right)\left[(x+3y)i + (y-2z)j + (x+az)k\right] = 0$$

$$1 + 1 + a = 0 \Rightarrow a = -2$$

(c) P.T Div (Curl $\vec{v}$) = 0.

Proof

$$\text{curl}\,\vec{v} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= i\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) - \left(\frac{\partial v_3}{\partial x} - \frac{\partial v_1}{\partial z}\right) j + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right) k$$

$$\text{div}\,(\text{curl}\,\vec{v}) = \frac{\partial}{\partial x}\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial v_3}{\partial x} - \frac{\partial v_1}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)$$

$$= \frac{\partial^2 v_3}{\partial x \partial y} - \frac{\partial^2 v_2}{\partial x \partial z} - \frac{\partial^2 v_3}{\partial y \partial x} + \frac{\partial^2 v_1}{\partial y \partial z} + \frac{\partial^2 v_2}{\partial z \partial x} - \frac{\partial^2 v_1}{\partial z \partial y}$$

$$= 0$$

Scanned by CamScanner

Module-4

7 (a) obtain the reduction formula of $\int \sin^m x \cos^n x \, dx$

Let, $I_{m,n} = \int \sin^m x \cos^n x \, dx$

where, m & n are +ve integers, writing,

$$I_{m,n} = \int (\sin^{m-1} x)(\sin x \cos^n x) \, dx$$

and Integrating by parts, we get

$$I_{m,n} = (\sin^{m-1} x) \int \sin x \cos^n x \cdot dx - \int (m-1) \sin^{m-2} x \cdot \cos x \int \sin x \cos^n x \, dx$$

$\cos x = t$ $\qquad -\int t^n dt = -\frac{t^{n+1}}{(n+1)}$

$-\sin x \, dx = dt$

$$= \sin^{m-1} x \left\{ \frac{-\cos^{n+1} x}{(n+1)} \right\} - \int (m-1) \sin^{m-2} x \cos x \left\{ \frac{-\cos^{n+1} x}{n+1} \right\} dx$$

$$= \frac{-(\sin^{m-1} x)(\cos^{n+1} x)}{(n+1)} + \frac{(m-1)}{(n+1)} \int \sin^{m-2} x \cos^n x \cdot \cos^2 x \, dx$$

$$= \frac{-(\sin^{m-1} x)(\cos^{n+1} x)}{(n+1)} + \frac{(m-1)}{(n+1)} \int \sin^{m-2} x \cos^n x (1 - \sin^2 x) \, dx$$

$$= \frac{-(\sin^{m-1} x)(\cos^{n+1} x)}{(n+1)} + \frac{(m-1)}{(n+1)} \int \sin^{m-2} x \cos^n x \, dx$$

$$+ \frac{(m-1)}{(n+1)} \int \sin^m x \cos^n x \, dx$$

$\boxed{\sin^{m-2} x \cdot \sin^2 x = \sin^m x}$

$$I_{m,n} = \frac{-(\sin^{m-1} x)(\cos^{n+1} x)}{(n+1)} + \frac{(m-1)}{(n+1)} I_{m-2,n} + \frac{(m-1)}{(n+1)} I_{m,n}$$

$$I_{m,n} + \frac{(m-1)}{(n+1)} I_{m,n} = \frac{-(\sin^{m-1} x)(\cos^{n+1} x)}{(n+1)} + \frac{(m-1)}{(n+1)} I_{m-2,n}$$

$$\frac{(m+n)}{(n+1)} I_{m,n} = \frac{-(\sin^{m-1} x)(\cos^{n+1} x)}{(n+1)} + \frac{(m-1)}{(n+1)} I_{m-2,n}$$

$$I_{m,n} = \frac{-(\sin^{m-1}x)(\cos^{n+1}x)}{(m+n)} + \frac{(m-1)}{(m+n)} I_{m-2,n}$$

(b) Eval $\int_0^{2a} x\sqrt{2ax-x^2}\,dx$

Sol $\int_0^{2a} x^n\sqrt{2ax-x^2}\,dx = \frac{(2n+1)!}{(n+2)!\,n!}\frac{a^{n+2}}{2^n}\pi$

$$= \frac{(2+1)!}{(1+2)!}\frac{a^3}{2}\pi = \frac{a^3\pi}{2}$$

(c) ~~Solve, $(2x\log x - xy)\,dy + 2y\,dx = 0$~~

$x = 2a\sin^2\theta$

$dx = 4a\sin\theta\cos\theta\,d\theta$

$\theta : 0 \to \pi/2$

$$\int_0^{\pi/2} 2a\sin^2\theta\sqrt{(2a\sin^2\theta)2a - 4a^2\sin^4\theta}\quad 4a\sin\theta\cos\theta\cdot d\theta$$

$$\int_0^{\pi/2} 8a^2\sin^3\theta\cos\theta\sqrt{4a^2\sin^2\theta(1-\sin^2\theta)}\,d\theta$$

$$\int_0^{\pi/2} 8a^2\sin^3\theta\cos\theta\,2a\sin\theta\cos\theta\,d\theta = 16a^3\int_0^{\pi/2}\sin^4\theta\cdot\cos^2\theta\cdot d\theta$$

$$= \frac{3\pi a^3}{6} = \frac{\pi a^3}{2}$$

(c) Solve, $(2x\log x - xy)\,dy + 2y\,dx = 0$

$M = 2y$ $\qquad N = 2x\log x - xy$

$\frac{\partial M}{\partial y} = 2$ $\qquad \frac{\partial N}{\partial x} = 2 + 2\log x - y$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ $\quad\therefore$ Eqn is not exact

8

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -2\log x + y$$

Multiplying $\frac{1}{N}$ on both sides

$$\frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = \frac{-1}{x}$$

$\therefore$ Integrating factor $= e^{\int P\,dx} = e^{-\int 1/x\,dx} = 1/x$

Multiplying I.F to the given equ

$$-\frac{1}{x}\left[2x\log x - xy\right] dx \cancel{+} 2\frac{y}{x}\,dx = 0$$

$$-2\log x\,dx + y\,dy - \frac{2y}{x}\,dx = 0$$

$$2y\log x - \frac{y^2}{2} = 6$$

8 (a) obtain the reduction formula of $\int \cos^n x \cdot dx$ where, n is +ve integer.

Let, $I_n = \int \cos^n x\,dx$

$$I_n = \int \underset{u}{(\cos^{n-1}x)}\ \underset{v}{(\cos x)}\,dx$$

Integrating by parts, we have

$$I_n = (\cos^{n-1}x)\int \cos x\,dx - \int (n-1)\cos^{n-2}x\,(-\sin x)\,(\sin x)\,dx$$

$\sin^2 x = 1-\cos^2 x$

$$I_n = (\cos^{n-1}x)(\sin x) + (n-1)\int (\cos^{n-2}x)(1-\cos^2 x)\,dx$$

$$= (\cos^{n-1}x)(\sin x) + (n-1)\int \cos^{n-2}x\,dx - (n-1)\int \cos^n x\,dx$$

$$I_n = (\cos^{n-1}x)(\sin x) + (n-1)I_{n-2} - (n-1)I_n$$

$$I_n + nI_n - I_n = \cos^{n-1}x\sin x + (n-1)I_{n-2}$$

$$I_n = \frac{(\cos^{n-1}x)\sin x}{n} + \frac{(n-1)}{n}I_{n-2}$$

Scanned by CamScanner

(b) Obtain the orthogonal trajectory of the family of curves $r^n \cos n\theta = a^n$. Hence, solve it.

Sol

$$r^n \cos n\theta = a^n$$

Taking logarithms on both sides

$$n \log r + \log \cos n\theta = n \log a$$

diff. w.r.to θ we get

$$n \frac{1}{r}\frac{dr}{d\theta} + \frac{1}{\cos n\theta}(-\sin n\theta)(n) = 0$$

replace $\frac{dr}{d\theta}$ as $-r^2\frac{d\theta}{dr}$

$$-r\frac{d\theta}{dr} - \tan n\theta = 0$$

variable separable method

$$\frac{d\theta}{\tan n\theta} = \frac{dr}{-r}$$

$$\frac{1}{r}dr + \cot n\theta \, d\theta = 0$$

integrating

$$\log r + \frac{1}{n}\log \sin n\theta = \log c$$

$$n \log r + \log \sin n\theta = n \log c$$

$$\boxed{r^n \sin n\theta = c^n}$$

(c) A body originally at 80°c cools down at 60°c in 20 minutes, the temp of the air being 40°c. What will be the temp. of the body after 40 min from the original?

Sol From Newton's law of cooling, we have

$$T = t_2 + (t_1 - t_2)e^{-kt}$$

$t_1 = 80°c$ $t_2 = 40°c$

$T = 60°c$ when $t = 20$ min

$$T = 40 + (80-40)e^{-kt}$$

60°c, 20 min
↓T ↓t

$$60 = 40 + 40e^{-20k}$$

$$k = 0.0346$$

$T = 40 + 40 e^{-0.0346t}$

After 40 min
$\hookrightarrow E$

$T = 40 + 40 e^{-0.0340(40)} = 50$

9(a) Find the rank of the matrix $A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

Sol $R_1 \to R_2$ (interchanging)

$$A \sim \begin{bmatrix} 1 & -2 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

$R_4 \to -(R_1 + R_2 + R_3) + R_4$ $\boxed{R_4 \to -(1+2+3)+6 = 0}$

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \to -2R_1 + R_2$, $R_3 \to -3R_1 + R_3$ $\boxed{\begin{array}{l} R_2 \to -2(1)+2 \\ \to 0 \\ R_3 \to -3(1)+3 \\ \to 0 \end{array}}$

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_3 \to -4R_2 + 5R_3$ $\boxed{\begin{array}{l} R_3 \to -4(0)+5(0) \\ \to 0. \end{array}}$

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ $R(A)$ = no of non-zero rows $= 3$

(b) By Solve by Gauss - Jordan method the system of linear eqn $2x + y + z = 10$, $3x + 2y + 3z = 18$, $x + 4y + 9z = 16$

Sol The Augmented matrix is

$$[A : B] = \begin{bmatrix} 2 & 1 & 1 & : & 10 \\ 3 & 2 & 3 & : & 18 \\ 1 & 4 & 9 & : & 16 \end{bmatrix}$$

$R_2 \to R_2 - 3/2 R_1 \qquad R_3 \to R_3 - 1/2 R_1$

$$\boxed{\begin{array}{ll} R_2 \to 3 - 3/2(2) & R_3 \to 1 - 1/2\ 2 \\ \to 0 & \to 0 \end{array}}$$

$$\sim \left[\begin{array}{ccc:c} 2 & 1 & 1 & 10 \\ 0 & 1/2 & 3/2 & 3 \\ 0 & 7/2 & 17/2 & 11 \end{array}\right]$$

$R_1 \to R_1 - 2R_2 \ , \quad R_3 \to R_3 - 7R_2$

$$\boxed{\begin{array}{ll} R_1 \to 2 - 2(0) & R_3 \to 0 - 7(0) \\ \to 2 & \to 0 \end{array}}$$

$$\sim \left[\begin{array}{ccc:c} 2 & 0 & -2 & 4 \\ 0 & 1/2 & 3/2 & 3 \\ 0 & 0 & -2 & -10 \end{array}\right]$$

$R_1 \to R_1 - R_3 \ , \quad R_2 \to 2R_2 + 3/2 R_3$

$$\boxed{\begin{array}{ll} R_1 \to 2 - 0 & R_2 \to 2(0) + 3/2(0) \\ \to 2 & \to 0 \end{array}}$$

$$\sim \left[\begin{array}{ccc:c} 2 & 0 & 0 & 14 \\ 0 & 1 & 0 & -9 \\ 0 & 0 & -2 & -10 \end{array}\right]$$

$\Rightarrow \quad 2x = 14 \ , \quad y = -9 \qquad -2z = -10$

$\therefore \ x = 7 \ , \quad y = -9 , \quad z = 5$

(c) Find the largest eigen value & the corresponding eigen vector by power method given that $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ use $[1\ 0\ 0]^T$ as initial vector. Apply 4 iterations.

Sol. $\quad x^{(0)} = [1\ 0\ 0]^T$

$$\therefore \ A x^{(0)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ 0.5 \end{bmatrix} = \lambda^{(1)} x^{(1)}$$

$$A x^{(1)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 0 \\ 2.0 \end{bmatrix} = 2.5 \begin{bmatrix} 1 \\ 0 \\ 0.8 \end{bmatrix} = \lambda^{(2)} x^{(2)}$$

$$A x^{(2)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.8 \end{bmatrix} = \begin{bmatrix} 2.8 \\ 0 \\ 2.6 \end{bmatrix} = 2.8 \begin{bmatrix} 1 \\ 0 \\ 0.93 \end{bmatrix} = \lambda^{(3)} x^{(3)}$$

$$A x^{(3)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.93 \end{bmatrix} = \begin{bmatrix} 2.93 \\ 0 \\ 2.86 \end{bmatrix} = 2.93 \begin{bmatrix} 1 \\ 0 \\ 0.98 \end{bmatrix} = \lambda^{(4)} x^{(4)}$$

$$\Rightarrow \lambda = 2.93 \qquad X = [1 \quad 0 \quad 0.98]$$

10 (a) use Gauss-Seidel method to solve the equations

$20x + y - 2z = 17$, $3x + 20y - z = 18$

$2x - 3y + 20z = 25$ carry out 2 iterations

$x_0 = y_0 = z_0 = 0$.

Sol we can write $x = 1/20[17 - y + 2z]$

$y = 1/20[18 - 3x + z]$

$z = 1/20[25 - 2x + 3y]$

Using, $x_0 = y_0 = z_0 = 0$, we get

1st iteration

$x^{(1)} = \frac{17}{20} = 0.85$, $y^{(1)} = 1/20[18 - x^{(1)} + z_0]$

$y^{(1)} = 1/20[18 - 3(0.85)] = 0.7725$

$z^{(1)} = 1/20[25 - 2x^{(1)} + 3y^{(1)}] = 1.2808$

2nd Iteration

$x^{(2)} = 1/20[17 - (0.7725) + 2(1.2808)] = 0.9395$

$y^{(2)} = 1/20[18 - 3(0.9395) + 1.2808] = 0.8231$

$z^{(2)} = 1/20[25 - 2(0.9395) + 3(0.8231)] = 1.2795$

$\therefore x = 0.9395 \quad y = 0.8231 \quad z = 1.2795$

(5) Reduce the matrix $A = \begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$ to the diagonal form

Sol The characteristic eqn is

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} -1-\lambda & 2 & -2 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^3 - \lambda^2 - 5\lambda + 5 = 0$$

$\therefore$ Eigen values are $1, \sqrt{5}, -\sqrt{5}$

Let, $[A - \lambda I][X] = [0]$

$$\begin{bmatrix} -1-\lambda & 2 & -2 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & -\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} (-1-\lambda)x + 2y - 2z = 0 \\ x + (2-\lambda)y + z = 0 \\ -x - y - \lambda z = 0 \end{aligned}\right\} \rightarrow (1)$$

Let, $\lambda = 1$

$-x - y - z = 0$

$-2x + 2y - 2z = 0$

$x + y + z = 0$

$$\Rightarrow \frac{x}{\begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -2 & -2 \\ 1 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -2 & 2 \\ 1 & 1 \end{vmatrix}}$$

$$\therefore \frac{x}{4} = \frac{-y}{0} = \frac{z}{-4}$$

$$\therefore x_1 = [1, 0, 1]^T$$

IIly for $\lambda = \sqrt{5}$ & $\lambda = -\sqrt{5}$ we can obtain

$$x_2 = [6 - 2\sqrt{5}, \sqrt{5} - 1, 1 - \sqrt{5}]^T$$

$$x_3 = [6 + 2\sqrt{5}, -\sqrt{5} - 1, 3 + \sqrt{5}]^T$$

$\therefore$ Modal matrix $P = \begin{bmatrix} 1 & 6-2\sqrt{5} & 6+2\sqrt{5} \\ 0 & \sqrt{5}-1 & -\sqrt{5}-1 \\ -1 & 1-\sqrt{5} & 3+\sqrt{5} \end{bmatrix}$

They diagonalization

$$D = P^{-1}AP$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{5} & 0 \\ 0 & 0 & -\sqrt{5} \end{bmatrix}$$

(c) Reduce the quadratic form $3x^2+5y^2+3z^2-2yz+2zx-2xy$ to canonical form

Sol The matrix of the given quadratic form is

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

The characteristic eqn of A is $\begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$

$$\Rightarrow \lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$$

$$\lambda = 2, 3, 6$$

$\therefore$ The required Canonical form is

$$\lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2$$

$$= 2y_1^2 + 3y_2^2 + 6y_3^2$$